Introduction to Financial Data Analysis

Week 9: Risk Beyond Volatility

Measurements of Risk Beyond Volatility

- Target: Understand the measurements of downside risk of a portfolio: VaR, Expected Shortfall (ES), Drawdowns.
- Risk: The most popular and traditional measure of risk is volatility.
- Problems of volatility: It does not care about the direction of an investment's movement: stock can be volatile because it suddenly jumps higher.

Downside Risk is Important

- 1. Loss is painful: Investors care more about loss than gains since loss are more painful.
- 2. Recovery from a loss is difficult

Loss is Painful

You have been given $1,000.
Now choose between gambles A and B

- A = (1000, 0.5; 0, 0.5)
- B = (500; 1)

You have been given $2,000,
Now choose between C and D

- C = (-1000,0.5; 0, 0.5)
- D = (-500; 1)

Loss is Painful

- More people chose B over A.
- More people chose C over D.
- The two problems are identical in terms of their the final wealth positions:
  - $1,500 of expected wealth from the risky choices (choices A and C), and $1,500 from the riskless choices (choices B and D).
- The traditional theory predicts that rational risk-averse investors always prefer the riskless over the risky choices for the latter do not offer extra compensation.

Loss is Painful

- **Daniel Kahneman** (Nobel Prize Laureate in Economics and one of the most cited author in Economics):

People are loss averse

- **Loss aversion** refers to people's tendency to prefer avoiding losses to acquiring equivalent gains: it is better to not lose $5 than to find $5.
- In reality, institutional investors might prefer a portfolio with Sharpe Ratio of 0.5 with 5% maximum loss to a portfolio of SR 2 which 20% loss.

Recovery from a Loss is Difficult

- It is more difficult to achieve a 10% gain than a 10% loss:
  - A stock drop from a previous peak price of 100 to 80 achieved a loss of 20%. To recover to the previous price 100 from 80, the stock need to increase by 25% rather than 20%.
  - A 50% drop from the peak has been seen during the 2008 to 2009 Great Recession, requires a whopping 100% increase to recover the former peak.

Measurements of Risk Beyond Volatility

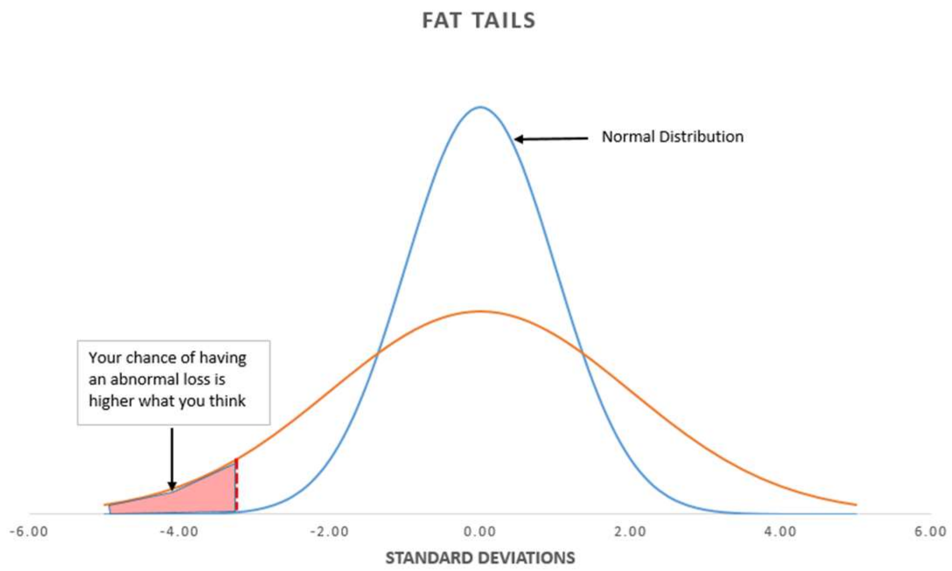
- Conclusion: Measuring downside risk is important.
- What is a Downside Risk Measurement?
  - Loss is a Random Variable: X.
  - Example: Purchase 1 share of a stock today at price 100, your potential loss tomorrow is $X = Y - 100$. Since Y is unknown today, X is a random variable.
- Downside Risk Measurements: Summary statistics that are used to quantify the loss distributions.

Value at Risk

- **Value at risk** (**VaR**) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability), in a set time period such as a day.
- The measure can be used by financial institutions to assess their risks or by a regulatory committee to set margin requirements.
- Example 1: An asset has a 3% one-month VaR of 2%, representing a 3% chance of the asset declining in value by 2% or more than 2% during the one-month time frame.
- Example 2: A portfolio of stocks has a one-day 5% VaR of $1 million, that means that there is a 5% probability that the portfolio will fall in value by more than $1 million over a one-day period if there is no trading.

Problems of VaR

- Problem of VaR: VaR can not capture how large the loss we should expect if the loss exceeds VaR or what is known as **tail risk**.
- Tail Risk: The risk that an investment's value moves more than three standard deviations from the mean is greater than what is shown by a normal distribution.



Expected Shortfall (ES)

- Expected Shortfall (ES): ES calculates how much we should expect to lose if our losses exceed VaR.
- ES: also called conditional value at risk (CVaR)
- The Basel Committee on Banking Supervision is proposing to change the system banks use to calculate losses from VaR to ES.

Two Methods of VaR and ES

- VaR/ES: Gaussian VaR/ES and Historical VaR/ES.
  - Gaussian VaR/ES: Assumes that the data follow Normal Distribution
  - Historical VaR/ES: Uses the distributional properties of the actual data.

- Example of VaR and ES: See “L3_Var_ES.html”

Gaussian VaR Example

- Gaussian VaR: Assume the portfolio returns follow a normal distribution.
- Market convention of VaR:
  - 1. Positive number
  - 2. Measured at 1% or 5% probability. That is, the lowest 1% and 5% returns for the given period of time.
- In the example, we measure daily VaR over one year period.

Formal Definition of Gaussian VaR

$$Pr[r_p \times I < -VaR_\alpha] \leq \alpha \qquad (*)$$

- The probability that an investment of value I would encounter a loss -$r_p \times I$ that is greater than VaR_α is at most of probability α.
- r_p is normally distributed with mean μ and standard deviation σ
- (*) is equivalent to
- $Pr[\frac{r_p - \mu}{\sigma} < \frac{-VaR_\alpha/I - \mu}{\sigma}] \leq \alpha \qquad (**)$
- Define $z \equiv \frac{r_p - \mu}{\sigma}$,then $z \sim N(0,1)$
- (**) is equivalent to $F\left(\frac{-\frac{VaR_\alpha}{I} - \mu}{\sigma}\right) = \alpha$
- where $F(\cdot)$ is the cumulative distribution function of a standard normal variable.
- $\frac{-VaR_\alpha/I - \mu}{\sigma} = F^{-1}(\alpha) \equiv Z_\alpha$
- We have $VaR_\alpha = -(\sigma Z_\alpha + \mu) \times I$

Coherent Risk Measurement

- A reasonable risk measure in finance must be consistent with the basic theory in finance.
- 1. Monotonicity
- 2. Positive Homogeneity
- 3. Translation Invariance
- 4. Subadditivity

Coherent Risk Measurement

- Monotonicity: If one financial position always has greater losses than another position under all circumstances, then its risk measure should always be greater.
- Example:
- Three regimes in economy: Crash, Recovery, Good Times
- Two assets: A, B

Loss	Crash	Recovery	Good Times
Asset A	10%	0%	0%
Asset B	20%	5%	1%

- M(B)>M(A)

Coherent Risk Measurement

- Positive homogeneity:
  - 1. Doubling a financial position should also double its risk.
  - 2. The risk does not depend on the currency in which the risk is measured.

	Loss in %	$1 Investment	$100 Investment
Investment A	10%	$0.1	$10
Investment B	20%	$0.2	$20
Risk		M(B) = 2* M(A)	M(B) = 2* M(A)

$1 = ￥7	$1 Investment	￥7 Investment
Investment A	$0.1	￥0.7
Investment B	$0.2	￥1.4
Risk Measure	M(B) = 2*M(A)	M(B) = 2*M(A)

Coherent Risk Measurement

- Translation invariance: Adding cash (deterministic portfolio with guaranteed return) to your portfolio will reduce the portfolio risk

Weight	Equity $ 100	Cash $ 0	Portfolio A Risk M(A)
Expected Return	-20	0	M(A) = M(-20)

Weight	Equity $ 100	Cash $ 1	Portfolio B Risk M(B)
Expected Return	-20	1	M(B) = M(-(20-1))

- M(B) < M(A)

Coherent Risk Measurement

- Subadditivity: the risk measure for a combined position should not be greater than risks of the two positions treated separately.
  - Diversification: The risk of a diversified portfolio should not be greater than risks of the individual components
  - M(w1*A+w2*B) is smaller or equal to w1*M(A) + w2*M(B)

Return	t1	t2	t3
Asset A	10%	5%	-20%
Asset B	-10%	1%	-5%
Portfolio(0.5A+0.5B)	0%	3%	-12.5%

Loss	t1	t2	t3
A	0%	0%	20%
B	10%	0%	5%
0.5*M(A) + 0.5*M(B)	0.5*M(-0%, -0%, -20%)+ 0.5*M(-10%,0%,-5%)		
Portfolio(0.5A+0.5B)	M(-0%, -0%, -12.5%)		

Coherent Risk Measurement

Let η be a risk measure.

- η is coherent if it satisfies the following four conditions for any two loss random variables X and Y :
  1. Monotonicity: If $X \leqslant Y$ for all possible scenarios, then $\eta(X) \leqslant \eta(Y)$.
  2. Positive homogeneity: For any positive constant c, $\eta(cX) = c\eta(X)$.
  3. Translation invariance: For any positive constant c, $\eta(X + C) = \eta(X) - c$,
     where C is the investment in deterministic portfolio with guaranteed return c.
  4. Subadditivity: $\eta(X + Y) \leqslant \eta(X) + \eta(Y)$.

Coherence of Gaussian VaR

- Monotonicity, positive homogeneity, and translation invariance hold obviously
- Claim: Subadditivity holds under Gaussian distribution but in general not hold under other distributions.

Subadditivity of Gaussian VaR

- $VaR_\alpha = -(\sigma Z_\alpha + \mu) \times I$
- Individual Asset: $VaR_\alpha^X = -\sigma_x Z_\alpha$ and $VaR_\alpha^Y = -\sigma_y Z_\alpha$
- Remove the constant μ and I for simplicity. The results below still hold with μ.
- Portfolio: $VaR_\alpha^{X+Y} = -\sigma_{x+y} Z_\alpha$
- $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$
- $\quad = \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y$
- $\quad \leq \sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y$
- $\quad = (\sigma_x + \sigma_y)^2$
- ρ is the correlation between two returns X and Y
- Therefore, $\sigma_{x+y} \leq \sigma_x + \sigma_y$
- $-\sigma_{x+y} Z_\alpha \leq -\sigma_x Z_\alpha - \sigma_y Z_\alpha$ with $Z_\alpha < 0$

Gaussian VaR Example

- Step 1: Import Daily EW Portfolio Returns we used in Lecture 2.
  - Load the data using load("crsp.ew.RData")
  - Filter data to year 2013
  - Assuming we placed $ 1 million at the start of the period into the portfolio
- Step 2: Calculate Mean and Standard Deviation of Historical Daily Portfolio Returns
- Step 3: Calculate 1% and 5% VaR
  - $VaR_\alpha = -(\sigma Z_\alpha + \mu) \times I$
  - Z_α is calculated using the qnorm(α) command, which returns the inverse cumulative density function.

Historical VaR

- Historical VaR: The negative of the α quantile of returns observed over some historical periods is VaR_α.
  - Step 1: Rank observed returns from low to high.
  - Step 2: Calculate the integer k that is most close to $\alpha \times n$ where n is the number of returns observed.
  - Step 3: VaR_α is the negative of the return ranked at k.
- Assumes future returns follow the same distribution as the historical return.

Historical VaR

We present VaR as a positive number in convention.

Historical VaR works better when we have lots of data.

Three to five years of daily data is recommended for this approach.

- Example: returns over the past 100 days are ranked from smallest to the largest $r_1 < r_2 < r_3 <\ldots< r_{100}$,(the $VaR_{0.05} = -r_5$.)
- Example: returns over the past 252 days(1 year) are ranked from smallest to the largest $r_1 < r_2 < r_3 <\ldots< r_{252}$,(the $VaR_{0.05} = -r_{int(252\times 0.05)} = -r_{13}$ as $252 \times 0.05 \approx 13$.
- Use quantile(data, probs, type=3) to get the value in the data ranked at round (probs $*$ n)

Historical VaR Violates the Subadditivity

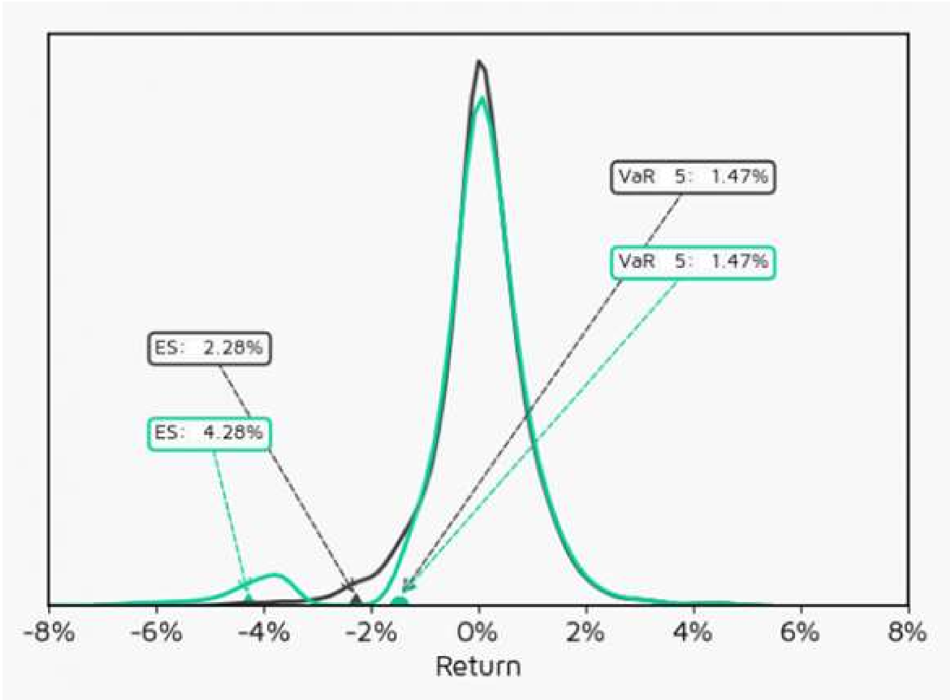
- We consider 100 days of returns for two assets A and B.
- Asset A's largest two losses are 1 million and 0.5 million happened on t=1, and t = 2.
- Asset B's largest two losses are 0.9 million and 0.1 million happened on t=2, and t = 1.
- Assume the returns of two assets are simply 0 on other days.
- $VaR_{0.02}^{A} = \mathbf{0.5}$ and $VaR_{0.02}^{B} = \mathbf{0.1}$
- $VaR_{0.02}^{A+B} = 1.1$
- $VaR_{0.02}^{A+B} > VaR_{0.02}^{A} + VaR_{0.02}^{B}$
- Violate the subadditivity

Historical VaR Example

- Step 1: Import Daily EW Portfolio Returns we used in Lecture 2.
  - Load the data using load("crsp.ew.RData")
  - Filter data to year 2013
  - Assuming we placed $1 million at the start of the period into the portfolio
- Step 2: Calculate the 0.01 and 0.05 historical VaR
- Step 3: Plot the VaR in relation to return density
  - The density of the portfolio returns
  - The normal distribution of returns based on the mean and standard deviation of portfolio
  - The estimates of the 1% and 5% 1-Day Historical VaR

Expected Shortfall

- Problem with VaR: VaR does not capture the shape of the tail of the distribution.
- Q: If losses exceed VaR, how bad should we expect that loss to be?
- A: Expected Shortfall, also tail VaR and CVaR.
- ES is a coherent risk measurement. (HW)
- Types of ES: Gaussian ES, Historical ES



Gaussian ES

- Denote $X = r_p$ as the return of a portfolio
- X is normally distributed with CDF F(X)
- The calculation of Gaussian ES is
- $ES_\alpha = E(-X|X \leq -VaR_\alpha(X)) = \frac{\int_{-\infty}^{-VaR_\alpha(X)} xf(x)dx}{\alpha}$
- Under Gaussian Distribution
- $ES_\alpha = \mu + \sigma \times \frac{f(F^{-1}(\alpha))}{\alpha}$
- In R: $ES_\alpha = \mu + \sigma \times dnorm(qnorm(\alpha))/\alpha$

Example of Gaussian ES:

- Step 1: Use “crsp.ew.RData”
  - Filter data to year 2013.
  - Assuming we placed $ 1 million at the start of the period into the portfolio
- Step 2: Calculate the ES at 1% and 5% using the formula

Historical ES

- The Historical ES is calculated by taking the average portfolio loss that falls short of the Historical VaR estimate.
- Example of Historical ES:
  - Step 1: Identify Historical VaR Limit for Portfolio using data in 2013
    - Let us first create variables that hold the historical 1 and 5%VaR estimates.
  - Step 2: Identify Portfolio Losses in Excess of VaR
    - Use dummy variables to indicate what returns fall short of the threshold for 1 and 5% Historical ES.
  - Step 3: Compute Average of Losses in Excess of VaR

Drawdown

- VaR and ES are measuring losses using the current or exist price relative to the purchase price.
- Investors also care about the loss of an investment from its peak value, the frequency of losses, the average size of losses, and the period of staying in loss.
- Drawdown: is a peak-to-trough decline during a specific period for an investment, trading account, or fund. It refers to how much an investment or trading account is down from the peak before it recovers back to the peak.

Key Points of Drawdown

- Drawdowns are a measure of downside volatility.
- A drawdown is usually quoted as the percentage between the peak and the subsequent trough, even though dollar terms may also be used.
  - If a trading account has $10,000 in it, and the funds drop to $9,000 before moving back above $10,000, then the trading account witnessed a 10% drawdown.
- Investors usually also consider the time it takes to recover a drawdown.
- A drawdown and loss aren't necessarily the same thing. Most traders view a drawdown as a peak-to-trough metric, while losses typically refer to the purchase price relative to the current or exit prices.

Example of a Drawdown

- Assume a trader decides to buy Apple stock at $100. The price rises to $110 (peak) but then swiftly falls to $80 (trough) and then climbs back above $110.
- Drawdowns measure peak to subsequent trough. The peak price for the stock was $110, and the subsequent trough was $80. The Drawdown is $30 / $110 = 27.3%.
- This shows that a drawdown isn't necessarily the same as a loss. The stocks drawdown was 27.3%, yet the trader would be showing an unrealized loss of 20% when the stock was at $80. This is because most traders view losses in terms of their purchase price ($100 in this case), and not the peak price the investment reached after entry.
- Continuing with the example, the price then rallies to $120 (peak) and then falls back to $105 before rallying to $125.
- The new peak is now $120 and the newest trough is $105. This is a $15 drawdown, or $15 / $120 = 12.5%.

Example of Drawdown

- Example: SSE Composite Index
- See "L3_Drawdown.html"
- SSE Composite Index: also known as SSE Index is a stock market index of all stocks (A shares and B shares) that are traded at the Shanghai Stock Exchange.
  - China A-shares are the stock shares of mainland China-based companies that trade on the two Chinese stock exchanges, the Shanghai Stock Exchange (SSE) and the Shenzhen Stock Exchange (SZSE).
  - China B-shares are equity share investments in companies based in China. They trade in foreign currency on two different Chinese exchanges. On the Shanghai Exchange, B-shares trade in U.S. dollars. On the Shenzhen Exchange, B-shares trade in Hong Kong dollars.
  - China-H shares are the shares of companies incorporated in mainland China that are traded on the Hong Kong Stock Exchange. Many companies float their shares simultaneously on the Hong Kong market and one of the two mainland Chinese stock exchanges in Shanghai or Shenzhen, they are known as A+H companies.
- SSE Composite Index: Value weighted. The base day for SSE Composite Index is December 19, 1990. The Base Value is 100.

Example of Drawdown

- Step 1. Load data and visualize the historical performance of returns of SSE Composite Index
- Step 2. Calculate the drawdowns
  - Use function cummax() from base R to calculate the cumulative maxima of a vector
  - Drawdown(t) = (Price_t-Cummax_t)/Cummax_t
- Step 3. Summarize the drawdowns
- 1. The maximum drawdown
- 2. The longest period of loss and average period of loss
  - Use rle() to compute the lengths and values of runs of equal values in a vector.

Example of Drawdown

- Step 4. Calmar Ratio/Drawdown ratio:
- **Calmar Ratio = Average Annual Rate of Return / Maximum Drawdown**
  - The higher the Calmar ratio the better.
  - Anything over 0.50 is considered to be good.
  - A Calmar ratio of 3.0 to 5.0 is really good.

Co-drawdown

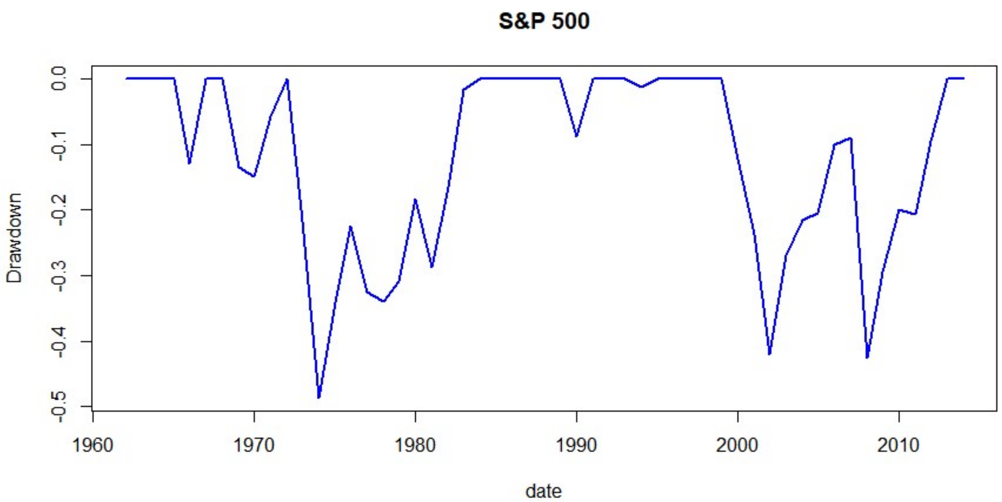
- Investors holding multiple assets to diversify away the risk of each asset.
- Correlation between assets captures investors' ability in diversification.
- A negative correlation is preferred since it means when one asset is performing bad, the other asset is performing good.
- Problem: Downside correlation is more important than upside correlation. Investors prefer two assets not having drawdowns simultaneously but would like to see two assets giving good returns simultaneously.

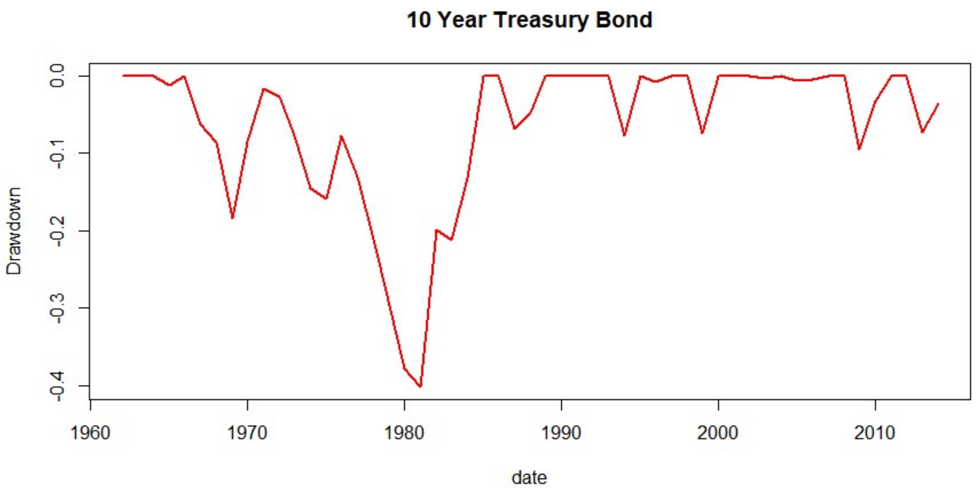
Co-drawdown

- Co-drawdown measures the probability that two assets simultaneously experience drawdowns taking into account of the size of drawdown.
- Two Asset A, B
- DDA(t) is absolute value of drawdown of asset A at t
- DDB(t) is absolute value of drawdown of asset B at t.
- Co-drawdown = $\frac{\sum_t \min(DDA(t), DDB(t))}{\sum_t \max(DDA(t), DDB(t))}$

Co-drawdown

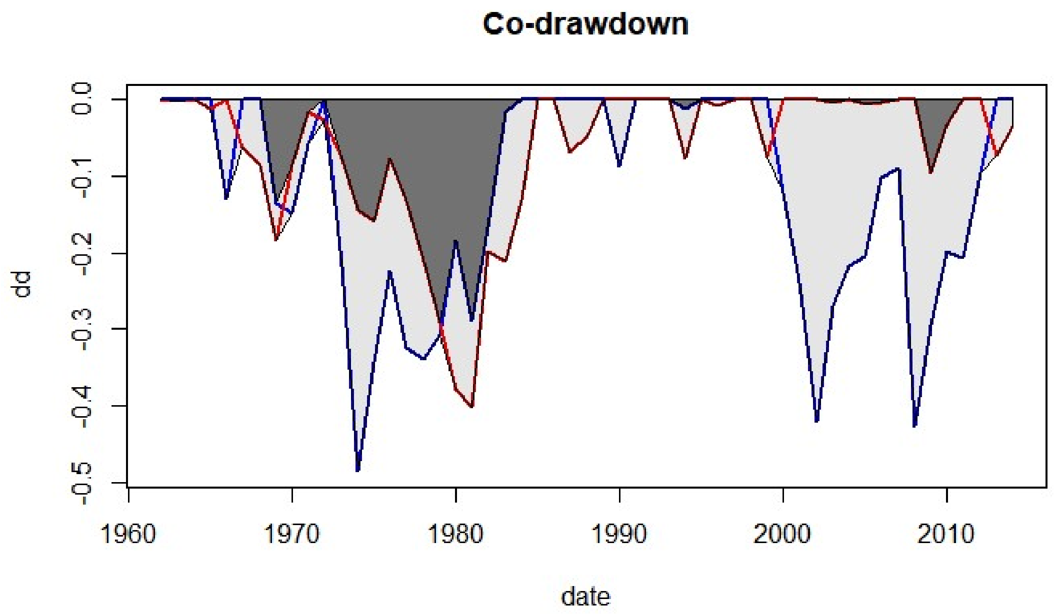
- Red and blue lines are two assets drawdowns separately
- DDA(t) is absolute value of drawdown of asset A at t
- DDB(t) is absolute value of drawdown of asset B at t.





Co-Drawdown

- The light shaded area marks when at least one asset is underwater:
  max(DDA(t),DDB(t))
- The dark grey shaded area marks when both assets are under water:
  min(DDA(t),DDB(t))
- Co-drawdown = $\frac{\sum_t \text{ min(DDA(t),DDB(t))}}{\sum_t \text{ max(DDA(t),DDB(t))}}$ = dark area/light area
- Co-drawdown "probability" increases in the overlay area, measuring the chance that two assets simultaneously having a bad performance and the depth of the simultaneous loss.



Co-drawdown Example

- The Co-drawdown between bond and equity
- See “L3_Drawdown.html”
- Previously we mentioned that a lot of institutional investors hold bond and equity simultaneously.
- In crisis, bond and equity tend to move in the opposite direction. This is called “fly to safety”.
- A very bad situation is when bond and equity are both performing bad.

Co-drawdown Example

- Step 1. Load the 10-year Treasury bond and S&P500 data from L2: “eq.bond.RData”
- Step 2. Calculate the cumulative returns of S&P500 and 10year Treasuary Bond, using them as the level of index so that we can calculate the drawdown of each asset. We work on the yearly data over the long history.
- Step 3. Calculate drawdowns of each individual asset and visualize the drawdowns of each asset.
- Step 4. Calculate the total area under water, that is, the area when either asset is underwater.

Co-drawdown Example

- Step 5. Calculate the total area when both assets are under water.
- Step 6. Visualize the co-drawdown
- Step 7. Calculate the probability of co-drawdown