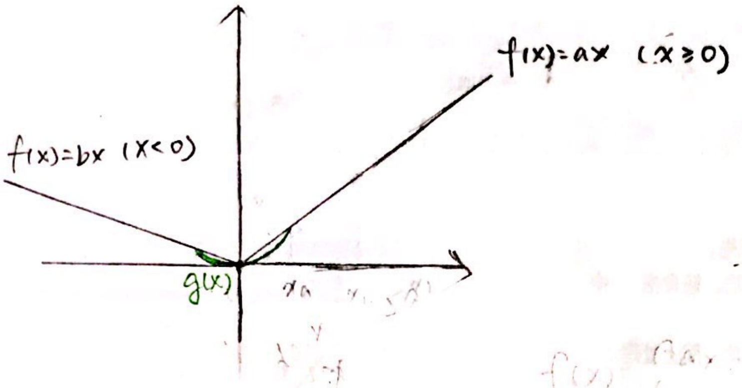


- Loss function for quantile estimation

$$L(x;\tau)=\begin{cases}\tau x & x>0\\(\tau-1)x & \text{otherwise.}\end{cases}\xrightarrow{\text{generalise}} ① \ f(x)=\begin{cases}ax & x\geq 0\\ bx & x<0\end{cases}\quad \text{where } a>0,\ b<0$$

- Why require $y=cx$: Piecewise-defined function $f(x)$ is the combination of two linear functions in form of $y=cx$.

Let $g(x)$ be the differentiable function to smooth $f(x)$, and $h(x)=\begin{cases}g(x) & (\text{when } x \text{ is around } 0)\\ f(x) & \text{otherwise}\end{cases}$

It's impossible for $h(x)$ to satisfy both requirements: ① $h(x)$ is a convex function

② $h(x)$ passes through $(0,0)$.

- A very informal proof:

For $x\geq 0$: Let's assume $h(x)=\begin{cases}g(x) & 0\leq x<c\\ f(x) & x>c\end{cases}$ and $g(x)$ passes through $(0,0)$.

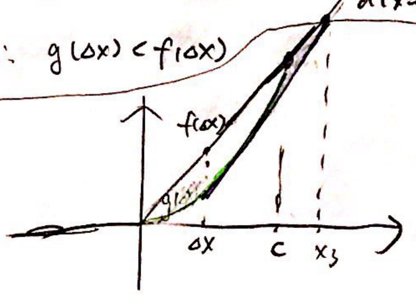
Since $h(x)$ is convex, $g(x)$ is convex. Thus we have

$$\begin{cases}g(0)=0\\ \frac{dg(x)}{x}=0 \text{ at } x=0\end{cases}$$

$$\therefore\ g(\Delta x)=g(0)+\Delta x\cdot\frac{dg(x=0)}{d(x=0)}=g(0)+\Delta x\cdot 0=0.$$

$$f(\Delta x)=f(0)+\Delta x\cdot\frac{df(x=0)}{d(x=0)}=0+\Delta x\cdot a$$

$$\therefore\ g(\Delta x)<f(\Delta x)$$



let $x_3>c$.

Connect $(\Delta x, g(x))$ and $(x_3, f(x_3))$, we see $h(x)$ is not convex.

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