Bayesian Conversations

Submission 2977

Abstract

We initiate the study of Bayesian conversations, which model interactive communication between two strategic agents without a mediator. We compare this to communication through a mediator and investigate the settings in which a mediation can expand the range of implementable outcomes.

We look into the eventual outcome of two-player games after interactive communication. We focus on games where only one agent has a non-trivial action and examine the performance of communication protocols that are individually rational (IR) for both parties. We characterize the structure of the social-welfare optimal protocol of a given number of rounds and thus show a separation between Bayesian conversation and mediated protocols. We demonstrate an example where the optimal conversation protocol requires infinitely many rounds of communication, and further show that for settings with binary actions and binary types, any optimal protocol either is finite (with at most 6 rounds) or requires infinitely many rounds of communication.

CCS Concepts

• **Theory of computation → Algorithmic game theory**.

Keywords

Mechanism Design, Information Design, Game Theory

ACM Reference Format:
Submission 2977. 2025. Bayesian Conversations. In *Proceedings of The ACM Web Conference 2026 (WWW '2026)*. ACM, New York, NY, USA, 38 pages. https://doi.org/XXXXXXX.XXXXXXX

1 Introduction

Much attention has been paid to the topic of strategic communication in recent years, in both Computer Science and Economics. An example is Bayesian persuasion ([26], [21]). In most previous work, the communication is unidirectional: it flows from senders to receivers. In this paper we focus on settings that resemble conversations: two agents talk back and forth, where each message depends not only on their private information but on the transcript of the communication so far. Such conversations arise in various settings of interest: a customer negotiating a price with a merchant, a worker negotiating their salary with a firm, prosecution and defense attorneys in a courtroom trial, political debates between two candidates, job interviews, etc. All of these settings are characterized by a long interactive conversation whose outcome will determine the payoffs of the parties involved.

We propose the framework of *Bayesian conversations*. The notion of interactive communication is not new and has been studied in many contexts, such as Communication Complexity [28, 36], Information Complexity [11] and Secure Multiparty Computation [8] in Computer Science and Cheap Talk [6, 18] and Bargaining [32] in Economics.

While we are inspired and draw tools from those fields, we take a different viewpoint here characterized by the following aspects. We consider strategic agents and their incentives while communicating in contrast to communication and information complexity. Unlike cheap-talk, we assume that agents commit to a communication protocol, which is the key assumption in Bayesian Persuasion. In a sense, our model can be viewed as an interactive form of Bayesian persuasion.

Formally our setting consists of two agents (Alice and Bob) each of whom has a private type drawn from a known prior distribution: $\theta_A \sim \mathbb{P}(\theta_A)$ and $\theta_B \sim \mathbb{P}(\theta_B)$. Our central object of study will be an *unmediated Bayesian conversation* which is a protocol that specifies messages each agent sends in each round. The protocol has a finite length T and each round $t = 1 \dots T$ is associated with a space of messages A_t and B_t. A conversation specifies for each round,

- a randomized mapping from Alice's type θ_A and the history transcript $(a_1, b_1, \dots, a_{t-1}, b_{t-1})$ to a message in A_t;
- a randomized mapping from Bob's type θ_B and the history transcript $(a_1, b_1, \dots, a_{t-1}, b_{t-1}, a_t)$ to a message in B_t.

With each message sent by Bob, Alice updates her belief about Bob's type and vice versa. Conversations, therefore, can be viewed as ways to split the initial pair of beliefs $(\mathbb{P}(\theta_A), \mathbb{P}(\theta_B))$ to refined beliefs (q_A, q_B). See Example 2.1 for concrete examples.

When exploring the potential of Bayesian conversations, two fundamental questions arise: What are the optimal outcomes that can be achieved through these conversations? And how powerful are Bayesian conversations as a class of communication protocols? Answering these questions is particularly challenging due to the inherent complexity of Bayesian conversations, which involve multiple rounds of signaling.

We study the power of communication protocols in terms of the eventual outcomes that can be reached. To that end, we will assume that a Bayesian game will be played after the conversation. In order to avoid issues like equilibrium selection, we will further assume that only one player (Alice) has a non-trivial action, and the payoffs depend on three things: (i) Alice's chosen action; (ii) Alice's type; and (iii) Bob's type. In this scenario, Alice's chosen strategy depends on both her type and her belief about Bob's type. Given that her belief about his type is a function of the communication protocol, the conversation will directly affect the payoff of both parties.

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WWW '2026, April 13–17, 2026, Dubai, UAE

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ACM ISBN 978-x-xxxx-xxxx-x/YY/MM
https://doi.org/XXXXXXX.XXXXXXX

A protocol is designed by an external party and proposed to the agents. Since Alice is the only agent with a non-trivial action, she always weakly prefers to participate in the communication protocol – after all, she can always ignore the communication and choose an action based on her prior. Bob, however, needs to be incentivized to participate in the protocol. This will take the form of an ex-post IR constraint: for any realization of types and transcripts of the communication protocol, Bob prefers the protocol outcome to the outcome when Alice plays only based on her type and her prior belief about Bob's type (i.e., when no conversation or signaling occurs).

Our choice of ex-post IR as a participation constraint is motivated by its naturalness and the fact it leads to a rich mathematical structure. In the appendix, we explore alternative notions such as ex-ante and interim IR constraints and also consider settings where Bob can deviate from participating in the protocol mid-conversation. See Appendix B for a discussion of additional results.

1.1 Our Results

In our first main result, we provide a polynomial-time algorithm for computing the social-welfare maximizing r-round Bayesian conversation when Alice and Bob have binary types (Theorem 3.1). This algorithm is a dynamic program based on the structural observation that the optimal social welfare of any r-round Bayesian conversation as a function of the starting prior can be described by its value on a fixed number of points polynomial in the size of the game and independent of r. Using the algorithm structure, we find that there is a strict separation between mediator protocols and Bayesian conversations; we construct an example where the maximal social welfare achievable by an ex-post IR mediator protocol is strictly larger than the maximal social welfare achievable by a conversation (Theorem 4.1). We additionally provide an exponential-time algorithm for determining whether a specific posterior belief distribution is realizable by an r-round Bayesian conversation in the setting with more than two types (Theorem E.1).

We also consider the value of additional rounds of communication in Bayesian conversations. Do multiple rounds of interaction help when designing Bayesian conversations? We show that, although there exist examples where the optimal ex-post Bayesian conversation requires only finitely many rounds of interaction (Section 4.1), there also exist examples where the social welfare of the optimal Bayesian conversation strictly increases as the number of rounds of interaction approaches infinity (Theorem 4.2).

Finally, we analyze the structure of optimal welfare Bayesian conversations in the setting where both types and actions are binary, we prove a dichotomy, showing that either the optimal protocol requires infinitely many rounds (the optimal welfare is only achieved asymptotically), or there is a finite protocol that requires at most 6 rounds. In cases where the optimal welfare is only approached asymptotically, we prove that the convergence rate is linear. Additionally, we provide a polynomial-time algorithm that decides whether a given game admits a Bayesian conversation that achieves the optimal social welfare within finitely many rounds (Theorem 4.3).

One key technical tool necessary to establish the above structural and algorithmic results is the reduction of the problem of computing ex-post IR Bayesian conversations to the question of understanding the behavior of *alternating concavification* of piecewise linear functions. In particular, given a two dimensional piecewise linear function $f(x, y)$ (with pieces given by sub-rectangles of the grid), we can consider the process of repeatedly taking the upper concave hull first of each of the horizontal slices of this function (fixing y and letting x vary), then of each the vertical slices (fixing x and letting y vary), and repeating this process until there is some convergence. For example, the above 6 round versus infinite-round dichotomy follows from the fact that when this process is iterated on a 4-by-5 grid, it either stabilizes after at most 6 rounds or continues indefinitely (Lemma K.3).

1.2 Related Work

Economics literature. There exists a vast body of literature on information transmission in economics, with Bayesian persuasion [26] and cheap talk [6, 22] being two extensively researched models of communication. While our model shares connections with these two frameworks, it presents a fundamentally distinct framework: we consider *interactive* communication under *commitment* where *both* players hold private information.

Our paper is part of the growing body of literature that extends the Bayesian persuasion model of [26] to richer communication models. In the original model of persuasion, there is a sender and a receiver where the sender is more informed but the receiver is the one that takes a payoff-relevant action. This basic model is very well understood, both its strategic aspects [9, 26] and computational [20] aspects. The model has been extended to allow the possibility of multiple senders [10, 23], multiple receivers [1, 2, 4, 35], information intermediaries [3, 30, 31, 37] as well as communication along a network [7, 13, 15, 16, 29]. There is a wide variation of the type of communication in those papers: public vs private, constrained vs unconstrained, etc. A common element is that information flows in one direction: from more informed to less informed agents. It is not uncommon for agents to act both as senders and receivers, however in the network/mediator/intermediary model the communication typically flows $A \to B \to C$, i.e., Alice sends a message to Bob who sends a message to Charlie. Bob acts both as a sender and a receiver, but the communication still flows only in one direction. In our model, we consider two agents but neither is strictly more informed than the other. Their communication is both bi-directional and interactive.

There is a substantial body of literature that examines long interactive communication, such as the Long Cheap Talk model proposed by [6]. One key distinction of our work is that we assume players will commit to a communication protocol if they choose to participate, and our focus lies in exploring the outcomes achievable through interactive communication under commitment. To the best of our knowledge, previous studies do not make the assumption of player commitment to a protocol but instead analyze communication strategies within equilibria. The most relevant aspect connecting our research to the preceding Long Cheap Talk literature is our utilization of the "dimartingale" concept [5, 6, 24], which is defined as a martingale with three component where one of the first two components remains the same at each time step. Dimartingale plays an important role in the identification of Nash equilibria in

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long interactive communication (see [6] and subsequent literature). However, in this work, we employ dimartingales for a distinct purpose: finding the feasible distributions of two players' posterior beliefs about each other's type after a Bayesian conversation. To achieve this objective, we pinpoint a particular dimartingale that will characterize the set of feasible posterior belief distributions.

Another instance of interactive communication is the interactive bilateral trade model developed by [32]. However, in contrast to our approach, Mao's study adopts a mechanism design perspective, investigating communication protocols emerging as equilibria of a specific game. Additionally, [27] investigates information design in a long cheap talk game, but they do not assume player commitment as we do.

Computer science literature. The interactive communication is studied in Computer Science in settings such as Yao's communication complexity model [36], information complexity [11] and secure multi party computation [8]. Unlike our model, these settings do not have a strategic component.

Feasible posterior distributions. Finally, there is a line of work that investigates the set of possible joint posterior belief distributions of a group of agents [4, 14, 17, 19, 25]. In this work, we are also interested in the joint posterior belief distributions that can be generated by communication protocols. Our study differs from previous work in two ways: (1) previous studies have focused on multiple agents' beliefs about a common state, whereas our focus is on the beliefs of two agents about each other's private type; (2) previous studies consider arbitrary information structures, whereas we specifically consider the communication protocol of Bayesian conversations.

2 Problem Description and Preliminaries

We consider a two-player setting where both players (Alice and Bob) have private types, denoted by θ_A and θ_B respectively. We assume that their types are drawn independently from commonly known distributions with finite support, i.e., θ_A and θ_B are drawn from $\mathbb{P}(\theta_A) \in \Delta(\Theta_A)$ and $\mathbb{P}(\theta_B) \in \Delta(\Theta_B)$ independently, where $|\Theta_A|, |\Theta_B|$ are finite and these prior distributions are common knowledge.

2.1 Communication protocols

Before delving into the game-theoretic aspects of this problem, we begin by establishing the communication models via which Alice and Bob can share information about their types. In general, we will consider settings where Alice and Bob agree on the precise protocol of communication ahead of time; later, we will impose constraints on this protocol so that Alice and Bob are incentivized to participate in the protocol and not defect.

Broadly, we consider two different classes of protocols. In the first, we assume Alice and Bob alternate revealing information about their private types to one another via a pre-determined protocol. This could represent Alice and Bob communicating via a platform which sets the communication rules and monitors the communication process.

Definition 2.1 (Bayesian conversations). *For two players with private types θ_A and θ_B, a T-round Bayesian conversation $\pi = \langle \mathbf{A}, \mathbf{B}, \mathbf{f}, \mathbf{g} \rangle$ specifies how the agents communicate in multiple rounds. Here, $\mathbf{f} = (f_1, \ldots, f_T)$, where each f_i is a function mapping Alice's type and the current transcript of the protocol to the randomized distribution over messages that Alice will send in round i. Likewise, $\mathbf{g} = (g_1, \ldots, g_T)$, contains the functions g_i which describe how Bob will send the message in round i. We will assume that at any round, Alice and Bob only ever send a finite number of possible distinct signals (i.e., f_i and g_i all have finite support), and write A_i and B_i to denote the supports of f_i and g_i respectively. More explicitly, the Bayesian conversation defined above proceeds as follows:*

- *At round i, based on her private type θ_A and the observed history, Alice randomly sends to Bob a signal $a_i \sim f_i(\theta_A, a_1, b_1, \ldots, a_{i-1}, b_{i-1}) \in \Delta(A_i)$; then, based on θ_B and the observed history, Bob randomly sends a signal $b_i \sim g_i(\theta_B, a_1, b_1, \ldots, a_{i-1}, b_{i-1}, a_i) \in \Delta(B_i)$. Here a_i and b_i are random variables taking values in A_i and B_i respectively.*
- *The process repeats for T rounds.*

We denote by Π_{BC} the class of all Bayesian conversations. Bayesian conversations involve multiple rounds of information disclosure. We focus on the case where full participation is enforced, meaning that agents are committed to completing the entire protocol once they agree to join. In Appendix B.1, we also discuss *non-committed* Bayesian conversations, where agents have the option to quit midway. In both settings, we assume that agents will adhere to the pre-determined protocol as long as they choose to proceed.

Definition 2.2 (Committed Bayesian Conversation protocols). *A committed Bayesian Conversation protocol π is where Alice and Bob decide at the beginning whether to communicate via a Bayesian conversation π or not, and once agreed, they must complete the full protocol without quitting.*

To formally analyze Bayesian conversations, we define random variable $h^{(t)} = (a_1, b_1, \ldots, a_t, b_t)$ as the history up to round t. As a slight abuse of notation, we use $f_i(a_i|\theta_A, h^{(t-1)})$ and $g_i(b_i|\theta_B, h^{(t-1)}, a_i)$ to represent the probability of sending a_i and b_i given the true type and the current history. After each round of communication, the players update their beliefs about the other player's type using Bayes' rule. Let $\mathbb{P}_\pi(\theta_A, \theta_B, a_1, b_1, \ldots, a_T, b_T)$ be the joint distribution of all types and signals over a complete execution of the protocol π is used, i.e.,

$$\begin{aligned}&\mathbb{P}_\pi(\theta_A, \theta_B, a_1, b_1, \ldots, a_T, b_T)\\ =&\mathbb{P}(\theta_A)\mathbb{P}(\theta_B) f_1(a_1|\theta_A) \cdot g_1(b_1|\theta_B, a_1) \cdots g_T(b_T|\theta_B, a_1, b_1, \ldots, a_T).\end{aligned}$$

After T rounds, Alice's belief about θ_B becomes $q_B^{(T)} = \mathbb{P}_\pi(\theta_B|\theta_A, h^{(T)}) \in \Delta(\Theta_B)$ and Bob's belief about θ_A becomes $q_A^{(T)} = \mathbb{P}_\pi(\theta_A|\theta_B, h^{(T)}) \in \Delta(\Theta_A)$. We will see that a specific choice of $q_A^{(T)}, q_B^{(T)}, \theta_A$, and θ_B uniquely determine the outcome of the game, so we will be particularly interested in the distribution $\mathbb{P}_\pi(\theta_A, \theta_B, q_B^{(T)}, q_A^{(T)})$ of the types and posteriors induced by π (e.g., which such distributions we can achieve with a Bayesian conversation). We call this distribution the *(joint) posterior distribution* induced by π. Note that although θ_A, θ_B are the players' private types, the posterior distribution induced by π is commonly known by both players. This is because we assume the priors $\mathbb{P}(\theta_A)$ and

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$\mathbb{P}(\theta_B)$ are common knowledge and the protocol π is commonly known.

Second, we consider communication that involves a trusted-third party who we refer to as *the mediator*. In this setting, we consider "revelation-style" protocols, where Alice and Bob begin by revealing their full type to the mediator, who then sends a single public signal to both players. We assume that the mediator is not strategic and will faithfully execute the protocol.

Definition 2.3 (Mediator protocols). *For two players with private type θ_A and θ_B, a mediator's protocol π is a signaling scheme $\langle S, \{\pi(\cdot|\theta_A, \theta_B)\}_{\theta_A \in \Theta_A, \theta_B \in \Theta_B}\rangle$ that specifies how the mediator sends the public signal: when the players' types are θ_A and θ_B, the mediator sends signal $s \in S$ with probability $\pi(s|\theta_A, \theta_B)$. We denote by Π_M the class of all mediator protocols.*

After observing the public signal, the players update their beliefs about the other player's type according to Bayes' rule. For a signaling scheme π, let the joint distribution of the types and the signals be $\mathbb{P}_\pi(\theta_A, \theta_B, s)$. Then Alice's posterior about θ_B after seeing s is $q_B = \mathbb{P}_\pi(\theta_B|\theta_A, s)$, and Bob's posterior about θ_A after seeing s is $q_A = \mathbb{P}_\pi(\theta_A|\theta_B, s)$. Again, we are interested in the joint distribution of players' types and beliefs after seeing the public signal, and we say that the distribution of the posteriors $\mathbb{P}(\theta_A, \theta_B, q_B, q_A)$ is the distribution of posteriors induced by π.

2.2 Actions and incentives

After Alice and Bob communicate, the two players will play a "game". As in the case of Bayesian persuasion, in this paper we primarily focus on the very simple class of games with a single action-taker. In particular, we assume that Alice takes an action $r \in R$ (for some finite action set R) and that by taking this action Alice receives utility $u_A(\theta_A, \theta_B, r)$ and Bob receives utility $u_B(\theta_B, \theta_A, r)$.[1] Then Alice's optimal strategy in such a setting is clear: if she has a posterior belief q_B for Bob's type, Alice will take the action that maximizes her expected utility given this posterior, namely

$$r^*(\theta_A, q_B) = \arg\max_{r \in R} \mathbb{E}_{\theta_B \sim q_B} u_A(\theta_A, \theta_B, r).$$

Example 2.1 (Bilateral trade). *One specific Stackelberg game we use as a running example will be the case of* bilateral trade. *In this game, Alice and Bob's types θ_A and θ_B belong to $\Theta_A = \Theta_B = [0, 1]$ and represent their valuation of an item Alice is attempting to sell to Bob. After communicating about their values, Alice sets a price $r \in [0, 1]$, which Bob accepts iff $r \le \theta_B$. In terms of the notation above, we have that*

$$\begin{aligned} u_A(\theta_A, \theta_B, r) &= (r - \theta_A) \cdot \mathbf{1}[r \le \theta_B] \\ u_B(\theta_A, \theta_B, r) &= (\theta_B - r) \cdot \mathbf{1}[r \le \theta_B]. \end{aligned}$$

[1]We briefly note that our model extends slightly beyond the setting where only one player takes an action and captures the case where Alice and Bob play a two-player *Stackelberg game* G, where Alice takes an action r_A, Bob responds (after seeing r) with an action r_B, Alice receives utility $G_A(r_A, r_B, \theta_A)$ and Bob receives utility $G_B(r_A, r_B, \theta_B)$ (note that in contrast to the model above, here we assume their utilities in this game only depend on the actions taken and their individual private types). If we let $r_B(r_A, \theta_B) = \arg\max_{r_B} G_B(r_A, r_B, \theta_B)$ (i.e., r_B is the best response for Bob to r_A), then our model captures this by letting $u_A(\theta_A, \theta_B, r) = G_A(r, r_B(r, \theta_B), \theta_A)$ and $u_B(\theta_A, \theta_B, r) = G_B(r, r_B(r, \theta_B), \theta_B)$.

Alice's optimal action may change depending on the communication between the two players, and this communication may modify both players' expected utility. In particular, Bob can easily find himself in a situation where he would prefer not to share a specific piece of information with Alice (e.g., in the bilateral trade game, if Bob fully reveals his type θ_B, Alice's optimal strategy is to set a price of θ_B which leaves Bob with no net utility).

We formally the participation constraints below. Before we do, however, we point out one important observation which is central to our model: since Alice is the sole action taker, **Alice always (weakly) benefits from additional communication**. In other words, we only need to ensure that the protocols are individually rational from the perspective of Bob.

Definition 2.4 (Ex-post players). *An ex-post player does not want to regret joining/proceeding after seeing their types and completing the protocol.*

We say that a committed protocol is ex-post IR if ex-post players will not regret joining after seeing their types and completing the protocol.

Definition 2.5 (Ex-post IR for committed protocols). *A committed protocol π is ex-post individually rational if after completing the protocol (and before Alice taking an action), Bob never regrets joining the protocol. For Bayesian conversations, we should have: for any $y \in \Theta_B$ and any outcome $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{a}_T, \widetilde{b}_T)$, suppose Bob has type y and Bob's posterior belief becomes $\widetilde{q}_A$ after seeing $\widetilde{h}^{(T)}$ and let $\mathbb{P}_\pi(\theta_A, q_B|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)})$ be the conditional distribution of (θ_A, q_B) when Bob has type y and the protocol ends at $\widetilde{h}^{(T)}$. Then it requires*

$$\mathbb{E}_{(\theta_A, q_B) \sim \mathbb{P}_\pi}[u_B(\theta_A, y, r^*(\theta_A, q_B))|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)}] \quad (1)$$

$$\ge \mathbb{E}_{\theta_A \sim \widetilde{q}_A}[u_B(\theta_A, y, r^0)], \ \forall \widetilde{h}^{(T)}, y \in \Theta_B, \quad (2)$$

where random variable $r^0 = \arg\max_{r \in R} \mathbb{E}_{\theta_B \sim \mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. For mediator protocols, we simply replace the outcome $\widetilde{h}^{(T)}$ with the realized public signal $\widetilde{s} \in S$. Again, we only need this inequality for Bob because it always holds for Alice.

For the ex-post IR mediator protocol, we have an exponential-size LP algorithm that can compute the protocol achieving optimal social welfare:

Theorem 2.1. *The optimal ex-post IR mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ can be solved by a linear program with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|^{|\Theta_A|})$.*

We defer the proof of this theorem to Appendix C.

In addition to the ex-post and committed protocol, we also examine Bayesian conversations and mediator protocols under ex-ante, interim or non-committed conditions; see Appendix A and Appendix B for details.

3 Optimal ex-post IR Bayesian conversations

In contrast to the case of mediator protocols, finding a Bayesian conversation that achieves optimal social welfare is significantly more challenging due to multiple rounds of communication.

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In this subsection, we introduce a dynamic programming algorithm to compute the optimal ex-post IR T-round Bayesian conversation in which both players have two types.

3.1 Algorithm for finding the optimal ex-post IR Bayesian Conversation

We will focus on the case where each agent has two types $\Theta_A = \Theta_B = \{L, H\}$ and we aim to maximize the expected social welfare. We will refer to the prior that agents have about each other's types before any communication as q_A^0, q_B^0. We will also assume that Alice's choice of action $r^*(\theta_A, q_B)$ breaks ties in favor of Bob, i.e., between two actions that lead to the same payoff for her, she chooses the best action for Bob, which is also the action that maximizes social welfare.

In that case, the beliefs of Alice and Bob can be described by a pair $(q_B, q_A) \in [0,1]^2$ representing the belief that the other agent has type H.

Overall, the algorithm follows a dynamic programming procedure from the final round to the first round: At the final round, for any posterior belief (q_B, q_A) that can be induced by a Bayesian conversation, we can compute the corresponding social welfare $W_0(q_B, q_A)$.

During each round of interaction, Alice and Bob effectively refine the belief about her/himself q into new believes q'. Consequently, the optimal social welfare attainable at a belief in the previous round can be derived from the convex hull of the values in the next round along the x- or y-direction (Equation (3), Equation (5)). Hence, starting from W_0 (the welfare at the final round) and proceeding backward, we can thus compute the optimal social welfare at any belief and any earlier round via dynamic programming.

Note that the ex-post IR condition (equation 2) is a condition on the state of the agent's beliefs after last round of the game. To establish the ex-post IR property, we first define the ex-post IR region, based on which the algorithm is initialized. This guarantees that the resulting outcome of our algorithm satisfies Bob's ex-post IR constraint in the final round.

Ex-post IR region. The region of the space of beliefs $[0,1]^2$ that lead to ex-post IR in the last period is given by:

$$\mathsf{IR}_0 = \{(q_B, q_A) \in [0,1]^2 \mid \forall \theta_B \in \mathrm{supp}(q_B),$$
$$\mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) \geq \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0))\}$$

where $\mathrm{supp}(q_B) \subseteq \Theta_B$ are the types that occur with non-zero probability in q_B (also, recall that q_B^0 is Alice's prior belief about Bob's type – see equation 2). Using this notion we can define the starting point of the recursion as:

$$W_0(q_B, q_A) = \begin{cases} W^*(q_B, q_A) & \text{if } (q_B, q_A) \in \mathsf{IR}_0 \\ -\infty & \text{o.w.} \end{cases}$$

where W^* is the optimal welfare given the final beliefs (q_A, q_B) without taking IR into account. Notice that since Alice breaks ties in favor of Bob, it has the following form:

$$W^*(q_B, q_A) = \mathbb{E}_{\theta_B, \theta_A \sim (q_B, q_A)} [u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))]$$

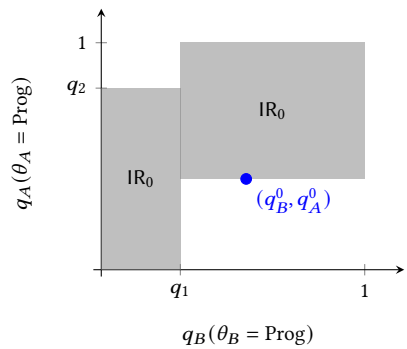

Figure 1: A possible illustration of ex-post IR region IR_0. When $q_1 < q_B < 1$, Alice's behavior remains unchanged and Bob's IR holds only if $q_A \geq q_A^0$; when $0 < q_B < q_1$, Alice takes diffrent actions and Bob's IR holds only if $q_A \leq q_2$.

Recursive Step. Assuming we know W_k, we can compute W_{k+1} by a process of alternating concavification: i.e, we will take concave hulls of the function with respect to Alice's belief (for odd k) and Bob's belief (for even k). We can view each round of communication by e.g. Alice as splitting the belief q_A into a distribution of beliefs Q'_A supported on $[0,1]$ that preserves the average belief. From this perspective, we will update:

$$W_k(q_B, q_A) = \max \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}(q'_B, q_A)] \tag{3}$$
$$\text{s.t. } \mathbb{E}_{q'_B \sim Q'_B}[q'_B] = q_B \text{ for odd } k \tag{4}$$

$$W_k(q_B, q_A) = \max \mathbb{E}_{q'_A \sim Q'_A}[W_{k-1}(q_B, q'_A)] \tag{5}$$
$$\text{s.t. } \mathbb{E}_{q'_A \sim Q'_A}[q'_A] = q_A \text{ for even } k \tag{6}$$

Such functions provide enough information to compute the optimal social welfare:

Lemma 3.1. *Given the $W_k(q_B, q_A)$ functions defined above, the optimal social welfare obtained by a protocol with k rounds is $W_k(q_B^0, q_A^0)$, i.e., the function evaluated on the original prior.*

We defer the proof of this lemma to Appendix J.

These steps essentially implement a dynamic programming procedure, where each iteration backwardly updates values for earlier rounds based on the results from later ones.

For example, suppose we run this process for a 4-round Bayesian conversation. Then $W_1(0.6, 0.4)$ and $W_4(0.6, 0.4)$ correspond to the optimal social welfare achievable, respectively, in the 4-th and 1-st rounds when starting from the belief point $q_B = 0.6$, $q_A = 0.4$.

Exact discretization. The recursion above specifies continuous functions $W_k : [0,1]^2 \to \mathbb{R}$. However, directly computing the recursive steps in the continuous domain is not computationally feasible. Therefore, to address this issue, we show that it is possible to discretize the space $[0,1]^2$ into finitely many points and only compute the W_k restricted to this set. A crucial fact here is that the discretization is exact, i.e., it introduces no approximation errors. In this way, we transform the continuous recursion into a computationally tractable discrete algorithm.

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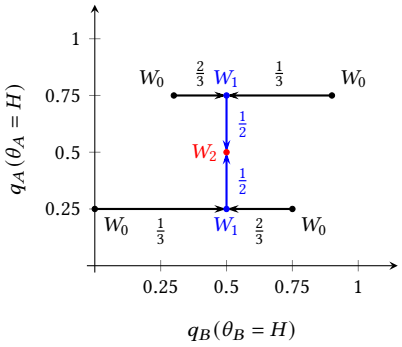

Figure 2: Illustration of the recursive steps: one possible computation path of $W_2(0.5, 0.5)$.

Lemma 3.2. *There are finite sets $X^* = \{0 = x_0 < x_1 < \ldots < x_m = 1\}$ and $Y^* = \{0 = y_0 < y_1 < \ldots < y_n = 1\}$ with size $O(|R|)$ such that W_0 restricted to the rectangle $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ is bilinear, where R is the set of Alice's actions.*

We will provide a detailed construction of X^* and Y^* in the appendix, and prove the correctness of this lemma therein.

Algorithm. A consequence of the existence of an *exact* discretization that has size polynomial in the size of the input is that it is possible to compute each entry of $W_k(x, y)$ for $(x, y) \in X^* \times Y^*$ in poly-time.

Lemma 3.3. *Given the values of $W_{k-1}(x, y)$ for $(x, y) \in X^* \times Y^*$ we can determine $W_k(x, y)$ by solving the following optimization programs over the simplex $\Delta_n = \{p \in [0, 1]^n; \sum_i p_i = 1\}$. For odd k:*

$$W_k(\hat{x}, \hat{y}) = \max \sum_{i=1}^{m} p_i W_{k-1}(x_i, \hat{y})$$
$$s.t \sum_{i=1}^{m} p_i x_i = \hat{x} \quad \text{and} \quad p \in \Delta_m$$

and for even k:

$$W_k(\hat{x}, \hat{y}) = \max \sum_{j=1}^{n} p_j W_{k-1}(\hat{x}, y_j)$$
$$s.t \sum_{j=1}^{m} p_j y_j = \hat{y} \quad \text{and} \quad p \in \Delta_n$$

Proof Sketch. We prove the claim by induction on k. Here we assume k is odd. The proof for the even case is entirely similar.

The main argument of the proof is as follows. When updating $W_k(\hat{x}, \hat{y})$ for any $\hat{x}, \hat{y}$, assume $x_i < x_0 < x_{i+1}$ and $x_0 = qx_i + (1 - q)x_{i+1}$. By the induction hypothesis, W_{k-2} is linear in x on segments $[x_i, x_{i+1}] \times y_j$ for all y_j's, combining the definition of W_{k-1}, we have

$$\begin{aligned} W_{k-1}(x_0, \hat{y}) &= \sum_{j=1}^{n} p_j W_{k-2}(x_0, y_j) \\ &= q \sum_{j=1}^{n} p_j W_{k-2}(x_i, y_j) + (1 - q) \sum_{j=1}^{n} p_j W_{k-2}(x_{i+1}, y_j) \\ &\le q W_{k-1}(x_i, \hat{y}) + (1 - q) W_{k-1}(x_{i+1}, \hat{y}). \end{aligned}$$

(Here we assume $k \ge 3$. The cases $k = 1$ and $k = 2$ are similar, with only minor technical differences (only requiring the application of Lemma 3.2), and are provided in the full proof in the appendix.)

Hence, when updating $W_k(\hat{x}, \hat{y})$, for any $x_0 \notin X^*$ we do not need to consider $W_{k-1}(x_0, \hat{y})$, since its value is (weakly) lower than the convex combination $q\, W_{k-1}(x_i, \hat{y}) + (1 - q)\, W_{k-1}(x_{i+1}, \hat{y})$.

Moreover, by the induction hypothesis, assume $y_j < \hat{y} < y_{j+1}$ and $\hat{y} = q' y_j + (1 - q') y_{j+1}$. Then W_{k-1} is linear in y on both segments $\{x_i\} \times [y_j, y_{j+1}]$ and $\{x_{i+1}\} \times [y_j, y_{j+1}]$. Consequently,

$$W_{k-1}(x_i, \hat{y}) = q'\, W_{k-1}(x_i, y_j) + (1 - q')\, W_{k-1}(x_i, y_{j+1}),$$
$$W_{k-1}(x_{i+1}, \hat{y}) = q'\, W_{k-1}(x_{i+1}, y_j) + (1 - q')\, W_{k-1}(x_{i+1}, y_{j+1}).$$

Therefore, throughout the recursive step, it suffices to track of the values on the grid $X^* \times Y^*$ only.

□

We observe that while it is useful to describe W_k as a linear program, there is a linear-time algorithm to compute $W_k(\cdot, \hat{y})$ as a function of $W_{k-1}(\cdot, \hat{y})$ as a convex hull computation problem.

We summarize and unify all the preceding steps in Algorithm 1.

Theorem 3.1. *Algorithm 1 runs in time poly$(t, |R|)$ and for each t, computes the highest welfare achievable by an ex-post IR Bayesian conversation with t rounds ($\lceil t/2 \rceil$ messages by Bob and $\lfloor t/2 \rfloor$ messages by Alice).*

Given all the preceding developments, the correctness of this theorem follows directly; a formal proof is deferred to Appendix J.

4 Properties of Ex-post Optimal ex-post Bayesian Conversations

We next study the properties of the optimal Bayesian conversations. We make two key observations: (1) The optimal ex-post IR Bayesian conversation is strictly weaker than the mediator protocol in terms of the ability to improve social welfare; (2) There are two types of ex-post IR Bayesian conversations where both Alice and Bob have two types and Alice has two available actions: in one, the optimal welfare is achieved within 6 rounds; in the other, it requires infinitely many rounds for convergence. We also provide a polynomial-time test that distinguishes between these two cases.

4.1 Gap between Bayesian conversations and mediator protocols

When ex-post IR is required, the two types of protocols are not equivalent. We show (via a game between an "employer" and "job candidate") that mediator protocols are more powerful than committed Bayesian conversations in this setting.

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ALGORITHM 1: Algorithm for optimal t-round ex-post IR Bayesian Conversation

Input: Prior distributions q_A^0, q_B^0, Utility functions u_A, u_B
Output: Optimal Social welfare $W_t(q_B^0, q_A^0)$
Compute IR_0, X^*, Y^*
foreach $(x^*, y^*) \in X^* \times Y^*$ **do**
  Compute $W_0(x^*, y^*)$
end
for $k = 1$ **to** t **do**
  foreach $(x^*, y^*) \in X^* \times Y^*$ **do**
    Compute $W_k(x^*, y^*)$
  end
end
if $q_B^0 \notin X^*$ **then**
  Compute $W_t(q_B^0, q_A^0)$:
  if *t is even* **then**

$$W_t(q_B^0, q_A^0) = \max \sum_{j=1}^{n} p_j W_{t-1}(q_B^0, y_j)$$
$$\text{s.t} \quad \sum_{j=1}^{m} p_j y_j = q_A^0 \quad \text{and} \quad p \in \Delta_n.$$

  end
  else

$$W_t(q_B^0, q_A^0) = qW_{t-1}(x_i, q_A^0) + (1-q)W_{t-1}(x_{i+1}, q_A^0)$$
$$\text{where } x_i < q_B^0 < x_{i+1} \text{ and } q_B^0 = qx_i + (1-q)x_{i+1}$$

  end
end
return $W_t(q_B^0, q_A^0)$

Theorem 4.1. *There exists a game where the highest social welfare that can be implemented by ex-post IR committed Bayesian conversations is lower than the highest social welfare that can be implemented by ex-post IR mediator protocols.*

We provide an example. Consider a two-player game between employers (Alice) and job candidates (Bob). Suppose there are two types of employers: employers who prefer candidates with good programming skills and employers who prefer candidates with good communication skills, so we have $\theta_A \in \{\text{Prog}, \text{Comm}\}$. The candidates also have two types: good at programming and good at communication $\theta_B \in \{\text{Prog}, \text{Comm}\}$. An employer can choose to hire or not hire a candidate $r \in \{\text{hire}, \text{not hire}\}$ and an employer's utility function is shown in Table 1. And a candidate simply wants

$u_A(\text{Prog}, \cdot)$	$\theta_B = \text{Prog}$	$\theta_B = \text{Comm}$
hire	10	−10
not hire	0	0

$u_A(\text{Comm}, \cdot)$	$\theta_B = \text{Prog}$	$\theta_B = \text{Comm}$
hire	−1	1
not hire	0	0

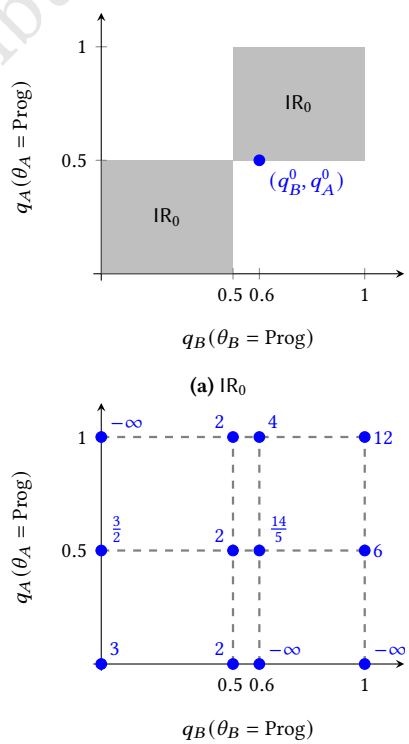
Table 1: The employer's utility function in the hiring problem.

to be hired, $u_B = 2 \cdot \mathbf{1}(r = \text{hire})$. Suppose $\mathbb{P}(\theta_A = \text{Prog}) = 0.5$ and $\mathbb{P}(\theta_B = \text{Prog}) = 0.6$, so without any communication a Prog-type employer always hires and a Comm-type employer never hires. Suppose we want to maximize the expected social welfare $\mathbb{E}[u_A + u_B]$.

We show that the optimal ex-post IR mediator protocol achieves a social welfare of 22/5 whereas the the optimal ex-post IR Bayesian conversation only achieves 21/5.

In the optimal mediator protocol, the mediator sends one of two public signals s_1 indicates "good communication skill" and s_2 indicates "good programming skill". If Alice wants to hire someone with good programming skills, the mediator sends a perfect signal. If the Alice wants to hire someone with good communication skills, the mediator sends an garbled signal: with some probability it will send signal s_1 when $\theta_b = \text{Prog}$. In Appendix H.1 we describe the details of the optimal mediator protocol and show it achieves welfare 22/5.

In the same appendix, we use the technique developed in the previous section to show compute optimal ex-post IR Bayesian conversation and show the optimal welfare is 21/5, providing a strict separation. We also show that the optimal conversation is a single message from Bob to Alice that is unconditioned on Alice's type, as illustrated in Figure 4b. The intuition is that if Bob were to condition on Alice's type, some types of Bob would have regretted the outcome of the conversation. The corresponding IR_0, W_0, and W_k are plotted in Figures 3a and 4a.



(a) IR_0

(b) W_0

Figure 3: Illustrations of IR_0 and W_0 for finding the optimal ex-post IR Bayesian protocol in the hiring problem.

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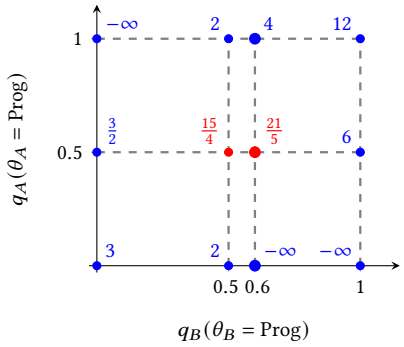
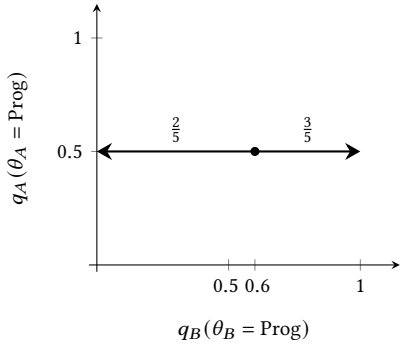


(a) $W_1 = W_2 = \ldots$



(b) Optimal ex-post IR Bayesian Conversation

Figure 4: Illustrations of W_k for all $k \geq 1$ and the final optimal ex-post IR Bayesian conversation in the hiring problem.

4.2 The Convergence Structure of Ex-post IR Bayesian Conversations

From the above example, we can see that some games can achieve the optimal social welfare through a finite number of ex-post IR Bayesian conversations. However, not all games exhibit this property. There exist certain games where achieving the optimal social welfare requires an infinite number rounds for ex-post IR Bayesian conversation protocols:

Theorem 4.2. *There exists a game where the highest social welfare achievable through ex-post IR committed Bayesian conversations requires infinitely many rounds to converge to the optimum.*

The proof of Theorem 4.2 involves again applying the methodology in Theorem 3.1 to the following game and then computing an explicit expression for $W_k(q_B^0, q_A^0)$.

In the two-player game Alice has types $\Theta_A = \{\theta_{A0}, \theta_{A1}\}$ and Bob has types $\Theta_B = \{\theta_{B0}, \theta_{B1}\}$. Alice can take two actions $r \in \{r_0, r_1\}$. Suppose $q_B^0 = \mathbb{P}(\theta_B = \theta_{B0}) = 0.4$, and $q_A^0 = \mathbb{P}(\theta_A = \theta_{A0}) = 0.6$. The utilities of the two players are given in the following table:

We write the equations in Lemma 3.3 and identify an explicit recursion for $W_k(x, y)$ at the breakpoints. The full analysis can be found in Appendix H.2. Then we derive that:

$$W_{4k+1}(0.4, 0.6) = -\frac{144}{325}\left(\frac{3}{16}\right)^k + \frac{4369}{325},$$

$u_A(\theta_{A0}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	7	5
r_1	5	7

$u_A(\theta_{A1}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	1	3
r_1	0	5

$u_B(\theta_{B0}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	5	10
r_1	10	0

$u_B(\theta_{B1}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	10	10
r_1	10	4

Table 2: Alice and Bob's utility function.

and in particular, the welfare of the optimal protocol strictly increases with the number of rounds of communication.

The above example illustrates a case where infinitely many rounds of conversation are required to approach the optimal social welfare, whereas the hiring problem converges within a finite number of rounds. The results reveal two distinct regimes: some conversations converge in finitely many rounds, while others require infinitely many rounds.

Naturally, this raises several fundamental questions: Does a universal upper bound exist on the number of rounds required for convergence in all finitely-converging games? What is the rate of convergence in the infinite-round regime? And is there an efficient algorithm to determine whether a given game achieves optimal social welfare in finitely many rounds?

We now consider Bayesian conversations where both Alice and Bob have two types, and Alice has two possible actions. We show that, in such games, the optimal social welfare is either attained within 6 rounds, or requires infinitely many rounds of communication for the welfare to converge linearly to the optimal value—thus answering all three questions raised above. In addition, we present a polynomial-time procedure that determines which of the two cases occurs.

Theorem 4.3. *Consider a Bayesian conversation in which both Alice and Bob have two types, i.e., $|\Theta_A| = |\Theta_B| = 2$, and Alice has two available actions, i.e., $|R| = 2$. Then the optimal social welfare is either achieved within 6 rounds, or it converges linearly to the optimal value over infinitely many rounds. Moreover, there exists a polynomial-time algorithm that determines which of the two cases occurs.*

Our overall proof strategy is as follows: we first show that for any game satisfying the stated conditions, the algorithm described in Theorem 3.1 is equivalent to performing alternating concavification along two dimensions on a mesh of size at most 4×5. We then analyze the properties of such meshes. The complete proof details are deferred to the appendix.

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References

[1] Ricardo Alonso and Odilon Câmara. 2016. Persuading voters. *American Economic Review* 106, 11 (2016), 3590–3605.

[2] Itai Arieli, Yakov Babichenko, and Manuel Mueller-Frank. 2019. Naive learning through probability matching. In *Proceedings of the 2019 ACM Conference on Economics and Computation*. 553–553.

[3] Itai Arieli, Yakov Babichenko, and Fedor Sandomirskiy. 2022. Bayesian Persuasion with Mediators. *arXiv preprint arXiv:2203.04285* (2022).

[4] Itai Arieli, Yakov Babichenko, Fedor Sandomirskiy, and Omer Tamuz. 2021. Feasible Joint Posterior Beliefs. *Journal of Political Economy* 129, 9 (2021), 2546–2594. https://doi.org/10.1086/714993 arXiv:https://doi.org/10.1086/714993

[5] Robert J Aumann and Sergiu Hart. 1986. Bi-convexity and bi-martingales. *Israel Journal of Mathematics* 54 (1986), 159–180.

[6] Robert J Aumann and Sergiu Hart. 2003. Long cheap talk. *Econometrica* 71, 6 (2003), 1619–1660.

[7] Yakov Babichenko, Inbal Talgam-Cohen, Haifeng Xu, and Konstantin Zabarnyi. 2021. Multi-channel bayesian persuasion. *arXiv preprint arXiv:2111.09789* (2021).

[8] Saugata Basu, Hamidreza Amini Khorasgani, Hemanta K. Maji, and Hai H. Nguyen. 2022. Geometry of Secure Two-party Computation. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*. 1035–1044. https://doi.org/10.1109/FOCS54457.2022.00101

[9] Dirk Bergemann and Stephen Morris. 2016. Information design, Bayesian persuasion, and Bayes correlated equilibrium. *American Economic Review* 106, 5 (2016), 586–91.

[10] Sourav Bhattacharya and Arijit Mukherjee. 2013. Strategic information revelation when experts compete to influence. *The RAND Journal of Economics* 44, 3 (2013), 522–544.

[11] Mark Braverman. 2012. Interactive information complexity. In *Proceedings of the forty-fourth annual ACM symposium on Theory of computing*. 505–524.

[12] Mark Braverman, Ankit Garg, Denis Pankratov, and Omri Weinstein. 2013. From Information to Exact Communication. In *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing* (Palo Alto, California, USA) *(STOC '13)*. Association for Computing Machinery, New York, NY, USA, 151–160. https://doi.org/10.1145/2488608.2488628

[13] Benjamin Brooks, Alexander Frankel, and Emir Kamenica. 2022. Information hierarchies. *Econometrica* 90, 5 (2022), 2187–2214.

[14] Krzysztof Burdzy and Jim Pitman. 2020. Bounds on the probability of radically different opinions. *Electronic Communications in Probability* 25, none (2020), 1 – 12. https://doi.org/10.1214/20-ECP296

[15] Ozan Candogan. 2019. Persuasion in networks: Public signals and k-cores. In *Proceedings of the 2019 ACM Conference on Economics and Computation*. 133–134.

[16] Ozan Candogan and Kimon Drakopoulos. 2020. Optimal signaling of content accuracy: Engagement vs. misinformation. *Operations Research* 68, 2 (2020), 497–515.

[17] Stanislaw Cichomski and Adam Oskekowski. 2021. The maximal difference among expert's opinions. *Electronic Journal of Probability* 26, none (2021), 1 – 17. https://doi.org/10.1214/21-EJP675

[18] Vincent P Crawford and Joel Sobel. 1982. Strategic information transmission. *Econometrica: Journal of the Econometric Society* (1982), 1431–1451.

[19] A. P. Dawid, M. H. DeGroot, J. Mortera, R. Cooke, S. French, C. Genest, M. J. Schervish, D. V. Lindley, K. J. McConway, and R. L. Winkler. 1995. Coherent combination of experts' opinions. *Test* 4, 2 (1995), 263–313. https://doi.org/10.1007/BF02562628

[20] Shaddin Dughmi, David Kempe, and Ruixin Qiang. 2016. Persuasion with limited communication. In *Proceedings of the 2016 ACM Conference on Economics and Computation*. 663–680.

[21] Shaddin Dughmi and Haifeng Xu. 2016. Algorithmic bayesian persuasion. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*. ACM, 412–425.

[22] Joseph Farrell and Matthew Rabin. 1996. Cheap Talk. *The Journal of Economic Perspectives* 10, 3 (1996), 103–118. http://www.jstor.org/stable/2138522

[23] Matthew Gentzkow and Emir Kamenica. 2016. Competition in persuasion. *The Review of Economic Studies* 84, 1 (2016), 300–322.

[24] Sergiu Hart. 1985. Nonzero-Sum Two-Person Repeated Games with Incomplete Information. *Mathematics of Operations Research* 10, 1 (1985), 117–153. http://www.jstor.org/stable/3689297

[25] Kevin He, Fedor Sandomirskiy, and Omer Tamuz. 2022. Private Private Information. In *Proceedings of the 23rd ACM Conference on Economics and Computation* (Boulder, CO, USA) *(EC '22)*. Association for Computing Machinery, New York, NY, USA, 1145. https://doi.org/10.1145/3490486.3538348

[26] Emir Kamenica and Matthew Gentzkow. 2011. Bayesian persuasion. *American Economic Review* 101, 6 (2011), 2590–2615.

[27] Frederic Koessler, Marie Laclau, Jérôme Renault, and Tristan Tomala. 2022. Long information design. *Theoretical Economics* 17, 2 (2022), 883–927. https://doi.org/10.3982/TE4557 arXiv:https://onlinelibrary.wiley.com/doi/pdf/10.3982/TE4557

[28] E. Kushilevitz and N. Nisan. 1997. *Communication Complexity*. Cambridge University Press. https://books.google.com.br/books?id=dHH7rdhKwzsC

[29] Marie Laclau, Ludovic Renou, and Xavier Venel. 2020. Robust communication on networks. *arXiv preprint arXiv:2007.00457* (2020).

[30] Fei Li and Peter Norman. 2021. Sequential persuasion. *Theoretical Economics* 16, 2 (2021), 639–675.

[31] Majid Mahzoon. 2022. Hierarchical Bayesian Persuasion: Importance of Vice Presidents. *arXiv preprint arXiv:2204.05304* (2022).

[32] Jieming Mao, Renato Paes Leme, and Kangning Wang. 2022. Interactive Communication in Bilateral Trade. In *13th Innovations in Theoretical Computer Science Conference (ITCS 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik.

[33] Jiří Matoušek and Petr Plecháč. 1998. On functional separately convex hulls. *Discrete & Computational Geometry* 19 (1998), 105–130.

[34] Sylvain Sorin. 2002. *A first course on zero-sum repeated games*. Vol. 37. Springer Science & Business Media.

[35] Yun Wang. 2013. Bayesian persuasion with multiple receivers. *Available at SSRN 2625399* (2013).

[36] Andrew C-C Yao. 1979. Some complexity questions related to distributed computing. In *Proc. 11th STOC*. 209–213.

[37] Andy Zapechelnyuk. 2022. Sequential obfuscation and toxic argumentation. *Available at SSRN 4142663* (2022).

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A Optimal ex-ante IR Bayesian conversations

We finally consider ex-ante players and ex-ante individual rationality.

Definition A.1 (Ex-ante players). *An ex-ante agent makes decisions on participating/quitting before seeing their type.*

Then we define a committed protocol as ex-ante individually rational if ex-ante players will choose to participate before seeing their types.

Definition A.2 (Ex-ante IR for committed protocols). *A committed protocol π is ex-ante individually rational if Bob's ex-ante expected utility of following the entire protocol is no lower than the expected utility of no communication. Formally, let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B)$ be the distribution of the agents' types and posteriors after completing the entire protocol, then it requires*

$$\mathbb{E}_{(\theta_B,\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))] \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B),\theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)], \tag{7}$$

where random variable $r^0 = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. Note that we only need this inequality for Bob because it always holds for the action taker Alice.

Our key finding is that if the players decide whether to participate ex-ante, Bayesian conversations are equivalent to mediator protocols in terms of the possible induced expected utilities in the base game. In fact, as we will see, Bob (the non-action-taker) can act as the mediator in a Bayesian conversation without violating any IR constraints. As a result, finding the optimal Bayesian conversation that maximizes the expectation of a utility function is equivalent to finding the optimal mediator protocol, which can be solved by a linear program.

A.1 Equivalence of protocols under ex-ante IR

We first show that when our protocols only need to be ex-ante IR, mediator protocols are no more powerful than committed Bayesian conversations when we consider the induced expected utilities in the base game. In fact, as we will see, Bob (the non-action-taker) can act as the mediator in such protocols without violating any IR constraints.

We introduce some notation before the main results. Suppose we are interested in the expectation of an arbitrary utility function $u(\theta_A, \theta_B, r)$. This utility function can be Alice's utility $u(\theta_A, \theta_B, r) = u_A(\theta_A, \theta_B, r)$ or Bob's utility or a designer's utility that depends on the outcome of the game. We may also have some constraints that restrict the valid belief distributions to a subset $\mathcal{P} \subseteq \Delta(\Theta_A \times \Theta_B \times \Delta\Theta_B \times \Delta\Theta_A)$. For example, we may want the protocol to be ex-ante IR (Definition A.2) for the committed players, in which we have $\mathcal{P}$ equal to the subset of distributions that satisfies equation (7). We then define the range of expected utilities that can be implemented by a class of protocol Π as

$$\text{Range}(u, \Pi, \mathcal{P}) = \{\mathbb{E}_P[u(\theta_A, \theta_B, r^*(\theta_A, q_B))] : P \in \mathcal{P} \text{ and } P \text{ can be induced by a protocol } \pi \in \Pi\},$$

which represents all possible values of the expectation of $u(\theta_A, \theta_B, r)$ when Alice chooses her best action after a valid communication protocol.

We first show that when we only need ex-ante, the mediator protocol and the committed BC protocols are equivalent in the sense that the ranges of implementable expected utilities are the same. This is due to the following observation: we actually only need to care about the marginal distribution $P(\theta_A, q_B)$ generated by a protocol in this case, and for any mediator protocol, there exists a one-round Bayesian conversation that generates the same $P(\theta_A, q_B)$ in which Alice fully reveals her type in the first round. Furthermore, all feasible marginal distributions $P(\theta_A, q_B)$ can be characterized by a simple equation $\mathbb{E}[q_B|\theta_A] = \mathbb{P}(\theta_B)$ for all θ_A, which says the conditional expectation of q_B must always equal the prior. For ex-ante IR, we have the following theorem.

Theorem A.1. *For any utility function $u(\theta_A, \theta_B, r)$, the range of expected utilities that can be implemented by ex-ante IR committed Bayesian conversation protocols is equivalent to the range of expected utilities that can be implemented by ex-ante IR mediator protocols,*

$$Range_{ex\text{-}ante}(u, \Pi_{BC}) = Range_{ex\text{-}ante}(u, \Pi_M),$$

where $Range_{ex\text{-}ante}(u, \Pi) = Range(u, \Pi, \mathcal{P})$ with $\mathcal{P}$ being the subset of distributions that guarantee ex-ante IR for committed protocols (satisfy equation (7)).

We defer the proof to Appendices G.1. In addition, Theorem A.1 and Theorem D.1 can be easily extended to a vector of utility functions $\mathbf{u} = (u_1, \ldots, u_L)$ with each $u_j = u_j(\theta_A, \theta_B, r)$. We give the full extended theorem in Appendix G.2. In particular, the extended theorem can be used to show that the utility pairs $(\mathbb{E}[u_A], \mathbb{E}[u_B])$ that can be generated for the two players are the same no matter which type of protocols we use, indicating that the two types of protocols generate the same *Pareto frontier*.

Corollary A.1. *The Pareto frontier of the two players' utilities induced by ex-ante/interim IR mediator protocols is the same as the Pareto frontier induced by ex-ante/interim IR committed Bayesian conversations.*

Discussion. The equivalence of these two models relies crucially on two assumptions: (1) the protocols are committed, and (2) there is only one action-taker (Alice). Both of the assumptions are necessary. We show in Section B.1 that (1) is necessary. We provide an example in Appendix G.3 that illustrates that, in general, when both players make a choice of action (even in the case where their action only affects their own reward), the two types of protocols are not equivalent.

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A.2 Optimal protocol by linear programming

As a result, solving the optimal ex-ante IR Bayesian conversation that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ reduces to finding the optimal mediator protocol, which can be solved by a linear program.

Theorem A.2. *The optimal ex-ante IR mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ can be solved by a linear program with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$.*[2]

We defer the details of the linear program to Appendix C. After determining the optimal mediator protocol, we can transform it into a Bayesian conversation by having Alice disclose her type in the first round and asking Bob to simulate the mediator's role.

B Interim IR and non-committed protocols

We define non-committed Bayesian conversations as follows.

Definition B.1 (Non-committed Bayesian Conversation protocols). *A non-committed Bayesian conversation protocol is where Alice and Bob agree to start a Bayesian conversation π but they can quit in the middle of the protocol.*

For non-committed protocols, the players can choose to quit at any step. We thus define a non-committed protocol as ex-ante IR if ex-ante players will choose to proceed at every step.

Definition B.2 (Ex-ante IR for non-committed protocols). *A non-committed protocol π is ex-ante IR if at any point of the protocol, the ex-ante expected utility of completing the protocol is no lower than quitting for both of the agents. Formally, for any time step t with history $\widetilde{h}^{(t)}$ and agents' posteriors $\widetilde{q}_A^{(t)}, \widetilde{q}_B^{(t)}$, let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B | h^{(t)} = \widetilde{h}^{(t)})$ be the distribution of $(\theta_A, \theta_B, q_A, q_B)$ after the players complete the protocol conditioning on they reach $\widetilde{h}^{(t)}$ at time t. Then it requires*

$$\mathbb{E}_{(\theta_A,\theta_B,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)|h^{(t)} = \widetilde{h}^{(t)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A^{(t)},\theta_B\sim\widetilde{q}_B^{(t)}}[u_B(\theta_B, \theta_A, r^{(t)})], \forall t, \widetilde{h}^{(t)}, \tag{8}$$

where random variable $r^{(t)} = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action facing $\widetilde{h}^{(t)}$. Again, we only need this inequality for Bob because it always holds for Alice.

Second, we consider players who make decisions after seeing their own types.

Definition B.3 (Interim players). *An interim player makes decisions on participating/quitting after seeing their types and before starting the protocol.*

We define a committed protocol as interim IR if interim players will choose to participate.

Definition B.4 (Interim IR for committed protocols). *A committed protocol π is interim individually rational if for any $y \in \Theta_B$, Bob's expected utility of following the entire protocol is no lower than the expected utility of no communication when Bob knows that his type is y. Formally, let $\mathbb{P}_\pi(\theta_A, q_A, q_B|\theta_B = y)$ be the conditional distribution of (θ_A, q_A, q_B) after completing the entire protocol when Bob's type is y, then it requires*

$$\mathbb{E}_{(\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B))|\theta_B = y] \geq \mathbb{E}_{\theta_A\sim\mathbb{P}(\theta_A)}[u_B(y, \theta_A, r^0)], \ \forall y \in \Theta_B, \tag{9}$$

where random variable $r^0 = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. Again, we only need this inequality for Bob because it always holds for Alice.

A non-committed protocol is interim IR if interim players will always choose to proceed.

Definition B.5 (Interim IR for non-committed protocols). *A non-committed protocol π is interim IR if at any point of the protocol, the expected utility of completing the protocol is no lower than quitting for both of the agents. Formally, for any type $y \in \Theta_B$ of Bob, for any time step t with history $\widetilde{h}^{(t)}$ and agents' posteriors $\widetilde{q}_A^{(t)}, \widetilde{q}_B^{(t)}$, let $\mathbb{P}_\pi(\theta_A, q_A, q_B|\theta_B = y, h^{(t)} = \widetilde{h}^{(t)})$ be the distribution of (θ_A, q_A, q_B) after the players complete the protocol conditioning on they reach $\widetilde{h}^{(t)}$ at time t and Bob has type y. Then it requires*

$$\mathbb{E}_{(\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B)|\theta_B = y, h^{(t)} = \widetilde{h}^{(t)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A^{(t)}}[u_B(y, \theta_A, r^{(t)})], \forall t, \widetilde{h}^{(t)}, y \in \Theta_B, \tag{10}$$

where random variable $r^{(t)} = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action facing $\widetilde{h}^{(t)}$. Again, we only need this inequality for Bob because it always holds for Alice.

And we define a non-committed protocol as ex-post IR if ex-post players will not regret proceeding at each step after seeing their types and completing the protocol.

[2]The optimal ex-post IR mediator protocol can be solved by an exponential-size LP. See details in Appendix C.

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Definition B.6 (Ex-post IR for non-committed protocols). *A non-committed protocol π is ex-post individually rational if after completing the protocol (and before Alice taking an action), Bob never regrets proceeding at every step. More specifically, for any $y \in \Theta_B$ and any outcome $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{a}_T, \widetilde{b}_T)$, suppose Bob has type y and Bob's posterior belief becomes $\widetilde{q}_A$ after seeing $\widetilde{h}^{(T)}$ and let $\mathbb{P}_\pi(\theta_A, q_B | \theta_B = y, h^{(T)} = \widetilde{h}^{(T)})$ be the conditional distribution of (θ_A, q_B) when Bob has type y and the protocol ends at $\widetilde{h}^{(T)}$. Then it requires*

$$\mathbb{E}_{(\theta_A, q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B)) | \theta_B = y, h^{(T)} = \widetilde{h}^{(T)}] \geq \mathbb{E}_{\theta_A \sim \widetilde{q}_A}[u_B(y, \theta_A, r^{(t)})], \ \forall \widetilde{h}^{(T)}, t \leq T, y \in \Theta_B, \tag{11}$$

where random variable $r^{(t)} = \arg\max_{r \in R} \mathbb{E}_{\theta_B \sim \widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action at time t.

B.1 Non-committed protocols

We now compare non-committed Bayesian conversation protocols with committed Bayesian conversations and mediator protocols, mainly focusing on the ex-ante IR and interim IR notions. As proved in Appendix A.1, committed Bayesian conversations are equivalent to mediator protocols under ex-ante IR and interim IR. Therefore, it suffices to compare non-committed Bayesian conversations with mediator protocols.

B.1.1 Gap between the protocols under ex-ante/interim IR. We first show by the example of employer-candidate game that non-committed Bayesian conversations are not so powerful as mediator protocols/committed Bayesian conversations when we consider ex-ante IR or interim IR.

Theorem B.1. *There exists a game where the highest social welfare that can be implemented by ex-ante IR non-committed Bayesian conversations is lower than the highest social welfare that can be implemented by ex-ante IR mediator protocols/committed Bayesian conversations, and the highest social welfare that can be implemented by interim IR non-committed Bayesian conversations is lower than the highest social welfare that can be implemented by interim IR mediator protocols/committed Bayesian conversations.*

Proof. We prove by the same employer-candidate game in Section 4.1. We show that no interim IR non-committed Bayesian conversation can achieve the same social welfare as the optimal mediator protocol given in the proof of Theorem 4.1.

We first consider interim IR and prove by contradiction, assume that there exists a non-committed Bayesian conversation π that achieves the same maximum social welfare as the mediator protocol, and the protocol is interim IR. Let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B)$ be the distribution of types and beliefs after completing π. As we show in the proof of Theorem 4.1, if π achieves the maximum social welfare, we must have all possible (q_B, q_A) lying in the region plotted in Figure 5c, and the protocol must end at the upper-left point $(q_B(\theta_B = \text{Prog}) = 0, q_A(\theta_A = \text{Prog}) = 1)$ with a non-zero probability. Suppose the protocol ends at the upper-left point with probability $\widetilde{p}$ with $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{a}_T, \widetilde{b}_T)$. Let t be the last round in $\widetilde{h}^{(T)}$ where there is still a non-zero probability that θ_A =Comm,

$$t = \max\{i : 1 \leq i \leq T, P(\theta_A = \text{Comm} | h^{(i)} = \widetilde{h}^{(i)}) > 0\}.$$

We claim that a Comm-type candidate will not want to continue the protocol after the employer sends $\widetilde{a}_{t+1}$. This means that once the employer reveals that her type is Prog, a Comm-type candidate will not want to continue. Denote by $q_B^{(t)}$ and $q_A^{(t)}$ the players' beliefs at the end of round t. Then an important observation is that employer's belief about the candidate $q_B^{(t)}$ must have $q_B^{(t)}(\theta_B = \text{Prog}) \in [\frac{1}{2}, 1]$. This is because the probability that $\theta_A = \text{Comm}$ is still non-zero $P(\theta_A = \text{Comm} | h^{(t)} = \widetilde{h}^{(t)}) > 0$, and if we want to guarantee that the employer's final belief is either $\frac{1}{2}$ or 1 whenever she has type Comm (or in other words, (q_B, q_A) lying in the region plotted in Figure 5c), we must have $q_B^{(t)}(\theta_B = \text{Prog}) \in [\frac{1}{2}, 1]$. Based on this observation, it is not difficult to see that a Comm-type candidate will not want to continue after seeing $\widetilde{a}_{t+1}$ (and knowing that the employer has type Prog), because based on the current belief $q_B^{(t)}$ with $q_B^{(t)}(\theta_B = \text{Prog}) \geq \frac{1}{2}$, the employer (with type Prog) always hires; but if the candidate follows the protocol until the end, there is a non-zero probability $\widetilde{p}$ that the protocol will end at $\widetilde{h}^{(T)}$ and the candidate will not be hired, which is strictly worse than quitting the protocol.

The proof is similar when we consider ex-ante IR. When the candidate (Bob) does not know his type, he will still want to quit after the employer sends $\widetilde{a}_{t+1}$. Because as we proved, the employer (with type Prog) always hires if the candidate chooses to quit, but if the candidate follows the protocol until the end, there is a non-zero probability that the candidate will not be hired. □

C Optimal mediator protocol by linear programming

We firstly show that the optimal ex-post IR mediator protocol can be solved by an exponential-size LP.

C.1 Optimal ex-post IR Mediator Protocol

Consider a game in which Alice has m possible types: $\Theta_A = \{\theta_{A1}, \ldots, \theta_{Am}\}$, and Bob has n possible types: $\Theta_B = \{\theta_{B1}, \ldots, \theta_{Bn}\}$. The set of actions that Alice can take is denoted by $R = \{r_1, r_2, \ldots, r_k\}$. We specify a signal using the joint probability distribution of player types and signals, as illustrated in Table **??**.

Theorem C.1. *The optimal ex-post IR mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ can be solved by a linear program with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|^{|\Theta_A|})$.*

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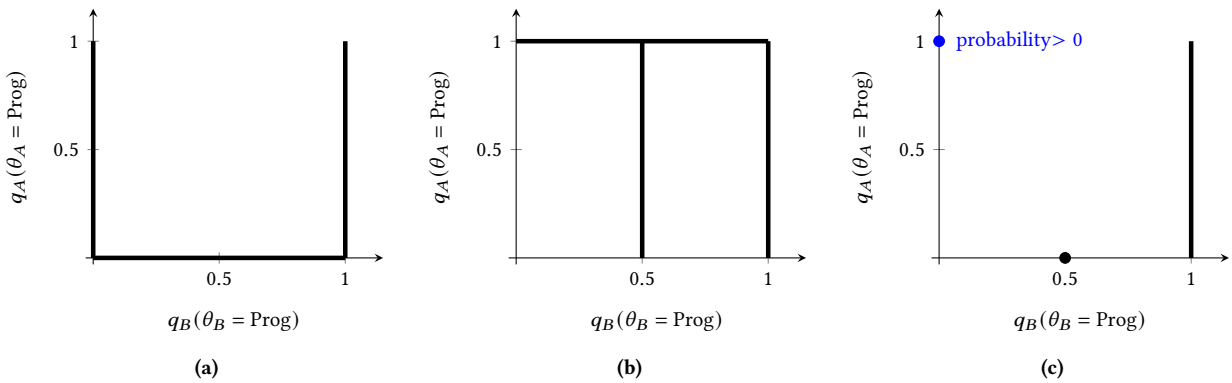

Figure 5: Regions that (q_B, q_A) must fall in after the communication in order to maximize the social welfare in the employer-candidate game. To make sure that the conditional expectation $\mathbb{E}[u_A + u_B | \theta_A = \textbf{Prog}]$ reaches the desired highest value, we must have (q_B, q_A) fall in the segments plotted in Figure 5a. To make sure that the conditional expectation $\mathbb{E}[u_A + u_B | \theta_A = \textbf{Comm}]$ reaches the desired highest value, we must have (q_B, q_A) fall in the segments plotted in Figure 5b. Taking the intersection of the two pictures, we must have (q_B, q_A) fall in the region plotted in Figure 5c in order to maximize the expected social welfare.

Unpublished working draft. Not for distribution.

$\Pr(\theta_A, \theta_B, s)$	θ_{B1}	θ_{B2}	$\dots$	θ_{Bn}
θ_{A1}	$p_{11}(s)$	$p_{12}(s)$	$\dots$	$p_{1n}(s)$
θ_{A2}	$p_{21}(s)$	$p_{22}(s)$	$\dots$	$p_{2n}(s)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
θ_{Am}	$p_{m1}(s)$	$p_{m2}(s)$	$\dots$	$p_{mn}(s)$

Table 3: Joint probability distribution for players' types θ_A, θ_B and signal s

To find the optimal ex post IR mediator protocol, we describe a signal by the actions taken by different types of Alice under that signal. For instance, in a game where $|\Theta_A| = 3$, a signal $s(r_3, r_1, r_2)$ represents that an Alice of type θ_{A1} will take action r_3 under this signal, an Alice of type θ_{A2} will take action r_1 under this signal, and so on.

Thus, we have a linear programming algorithm to find the optimal ex post IR mediator protocol. In this algorithm, we only need to enumerate all possible kinds of signals that exhaustively represent all combinations of actions that different types of Alice might take:

$$\mathcal{S} = \{s(\mathbf{r}) : \mathbf{r} \in R^m\}$$

We define the objective such that each kind of signal in the set $\mathcal{S}$ appears at most once in the optimal solution (it may also not appear, when all the corresponding probabilities $p_{ij}(s)$ are 0). The objective function maximizes social welfare, while constraints are imposed to ensure the existence of a mediator that can induce the target solution, guide different types of Alice to select the expected actions, and satisfy Bob's ex-post IR condition. The resulting linear programming formulation is as follows:

$$
\begin{aligned}
\max \quad & \sum_{s=s(r_1^*,r_2^*,\ldots,r_m^*)\in\mathcal{S}} \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}(s)u(\theta_{Ai},\theta_{Bj},r_i^*) \\
\text{subject to} \quad & \sum_{s\in\mathcal{S}} p_{ij}(s) = \mathbb{P}(\theta_{Ai})\cdot\mathbb{P}(\theta_{Bj}), \quad \forall i=1,2,\ldots,m, \forall j=1,2,\ldots,n \\
& \sum_{j=1}^{n} p_{ij}(s)u_A(\theta_{Ai},\theta_{Bj},r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s)u_A(\theta_{Ai},\theta_{Bj},r'), \\
& \qquad \forall s=s(r_1^*,r_2^*,\ldots,r_m^*)\in\mathcal{S}, \forall i=1,2,\ldots,m, \forall r'\neq r_i^* \\
& \sum_{i=1}^{m} p_{ij}(s)u_B(\theta_{Ai},\theta_{Bj},r_i^*) \geq \sum_{i=1}^{m} p_{ij}(s)u_B(\theta_{Ai},\theta_{Bj},r_i^0), \\
& \qquad \forall s=s(r_1^*,r_2^*,\ldots,r_m^*)\in\mathcal{S}, \forall j=1,2,\ldots,n
\end{aligned}
$$

where $r_i^0 = \arg\max_{r\in R}\mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_{Ai},\theta_B,r)]$ is the best action for Alice of type θ_{Ai} without any communication.

Theorem C.2. *The above linear programming method with size $O(|\Theta_A|\cdot|\Theta_B|\cdot|R|^{|\Theta_A|})$ can find a ex post mediator protocol that maximizes the expectation of a utility function $u(\theta_A,\theta_B,r)$.*

Proof. The first constraint in the linear programming formulation satisfies $\mathbb{E}[q_{AB}] = \mathbb{P}(\theta_A)\times\mathbb{P}(\theta_B)$, which guarantees the existence of a mediator protocol capable of implementing the identified probability distribution.

The second constraint ensures that the best response for each type of Alice corresponds to the action induced by the signal. The third constraint guarantees that each type of Bob satisfies the ex-post IR condition.

Therefore, what we need to prove is that the previously assumed set of signals $\mathcal{S}$ can be used to find the optimal mediator protocol.

To prove this, we only need to show that: given two signals s and s', under which the same type of Alice chooses the same action, and both satisfy Bob's ex-post IR condition, then the summation signal s_{sum} (defined such that $\Pr(\theta_{Ai},\theta_{Bj},s_{\text{sum}}) = p_{ij}(s_{\text{sum}}) = p_{ij}(s)+p_{ij}(s')$) can also induce the corresponding type of Alice to take the same action while maintaining Bob's ex-post IR condition.

This is easy to prove because the signal summation defined above does not change the expectation of q_{AB} induced by the protocol, thereby satisfying the first constraint.

The second and third constraints are equivalent to the following:

$$
\begin{aligned}
\sum_{j=1}^{n} p_{ij}(s)(u_A(\theta_{Ai},\theta_{Bj},r_i^*) - u_A(\theta_{Ai},\theta_{Bj},r')) \geq 0, \\
\sum_{i=1}^{m} p_{ij}(s)(u_B(\theta_{Ai},\theta_{Bj},r_i^*) - u_B(\theta_{Ai},\theta_{Bj},r_i^0)) \geq 0,
\end{aligned}
$$

If $p_{ij}(s)$ and $p_{ij}(s')$ both satisfy these constraints, then summing the corresponding inequalities gives $p_{ij}(s_{\text{sum}}) = p_{ij}(s)+p_{ij}(s')$ also satisfies these constraints, which completes the proof.

There are $|R|^{|\Theta_A|}$ signals in total, and each of them has $|\Theta_A|\cdot|\Theta_B|$ variables, so the size of this linear program is $O(|\Theta_A|\cdot|\Theta_B|\cdot|R|^{|\Theta_A|})$. □

For ex-ante case, the optimal mediator protocol can be foung using a polynomial-size LP algorithm:

C.2 Optimal ex-ante IR mediator protocol

To find the optimal ex ante IR mediator protocol, we focus on signals that may occur for only one type of Alice and classify them based on the action Alice would take in response. For example, the signal $s(i,l)$ indicates that for all $i'\neq i$ and $r=1,2,\ldots,n$, $p_{i'n}(s)=0$, and this signal induces Alice of type θ_{Ai} to take action r_l.

We denote the set of this new classification as $\mathcal{S}'$.

$$\mathcal{S}' = \{s(i,l) : i=1,2,\ldots,m, l=1,2,\ldots,k\}$$

The new linear program is similar, with only the last constraint having changed:

Unpublished working draft. Not for distribution.

$\Pr(\theta_A, \theta_B, s)$	θ_{B1}	θ_{B2}	$\dots$	θ_{Bn}
θ_{A1}	0	0	$\dots$	0
θ_{A2}	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$
θ_{Ai}(will do r_l)	$p_{i1}(s)$	$p_{i2}(s)$	$\dots$	$p_{in}(s)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
θ_{Am}	0	0	$\dots$	0

Table 4: Joint probability distribution for players' types θ_A, θ_B and signal $s(i,l)$

$$
\begin{aligned}
\max \quad & \sum_{s=s(i,l)\in\mathcal{S}'} \sum_{j=1}^{n} p_{ij}(s) u(\theta_{Ai}, \theta_{Bj}, r_l) \\
\text{subject to} \quad & \sum_{s\in\mathcal{S}'} p_{ij}(s) = \mathbb{P}(\theta_{Ai}) \cdot \mathbb{P}(\theta_{Bj}), \quad \forall i = 1,2,\dots,m, \forall j = 1,2,\dots,n \\
& \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r'), \quad \forall s = s(i,l) \in \mathcal{S}', \forall r' \neq r_l \\
& \sum_{s=s(i,l)\in\mathcal{S}'} \sum_{j=1}^{n} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bi}, r_l) \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B), \theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)]
\end{aligned}
$$

Theorem C.3. *The above linear programming method with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$ can find a ex ante mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$.*

Proof. First, the sum of two signals of the same kind still satisfies the constraints above. This is straightforward, similar to the proof provided earlier, and it also utilizes the linearity of the constraints. We can simply add the probabilities at corresponding positions to obtain a new solution that meets the conditions.

Next, we prove that decomposing any signal $s(r_1^*, r_2^*, \dots, r_m^*) \in \mathcal{S}$ into signals

$$\{s(1, r_1^*), s(2, r_2^*) \dots, s(m, r_m^*)\} \in \mathcal{S}'$$

(that is, taking each row as a separate signal and setting other positions to 0) can also satisfy constraints:

$$
\begin{aligned}
& \sum_{s} p_{ij}(s) = \mathbb{P}(\theta_{Ai}) \cdot \mathbb{P}(\theta_{Bj}), \quad \forall i = 1,2,\dots,m, \forall j = 1,2,\dots,n \\
& \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r'), \quad \forall s, \forall i = 1,2,\dots,m, \forall r' \neq r_i^* \\
& \sum_{s=s(r_1^*, r_2^*, \dots, r_m^*)} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B), \theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)]
\end{aligned}
$$

Proving that the first and second constraints are satisfied is easy. The first constraint requires that the relationships between corresponding positions remain unchanged, and the second constraint requires that the relationships within corresponding rows remain unchanged. Such decomposition does not break these relative relationships.

Since the actions taken by Alice at corresponding positions do not change before and after the decomposition, the left side of the third constraint remains unchanged. The right side is a constant. Therefore, the third constraint is also satisfied.

Notice that the signals resulting from the decomposition can be added to signals of the same kind that have been decomposed from other original signals. Therefore, each kind of signal only needs to appear at most once in the linear program.

There are $|R| \cdot |\Theta_A|$ signals in total, and each of them has $|\Theta_B|$ variables, so the size of this linear program is $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$. □

D Equivalence for interim IR

The result for interim IR is similar.

Unpublished working draft. Not for distribution.

Theorem D.1. *For any utility function $u(\theta_A, \theta_B, r)$, the range of expected utilities that can be implemented by interim IR committed Bayesian conversations is equivalent to the range of expected utilities that can be implemented by interim IR mediator protocols,*

$$Range_{interim}(u, \Pi_{BC}) = Range_{interim}(u, \Pi_M),$$

where $Range_{interim}(u, \Pi) = Range(u, \Pi, \mathcal{P})$ with $\mathcal{P}$ being the subset of distributions that guarantee interim IR for committed protocols (satisfy equation (9)).

Proof. The proof for interim IR is basically the same as the proof for ex-ante IR. The only difference is that we need to prove the constraint of interim IR (9) can be reduced to a constraint on $P(\theta_A, q_B)$ as well, which is true because interim IR can be written as

$$\sum_{\theta_A, q_B} P(\theta_A, q_B) q_B(\theta) u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)) \geq \mathbb{P}(\theta_B) \sum_{\theta_A} \mathbb{P}(\theta_A) u_B(\theta_B, \theta_A, r^*(\theta_A, \mathbb{P}(\theta_B))), \forall \theta_B \in \Theta_B.$$

□

E Feasible posterior distributions for Bayesian Conversations

In this section, we investigate the following problem: Given a belief distribution $P(\theta_A, \theta_B, q_B, q_A)$, or equivalently $P(q_B, q_A)$, we want to decide whether it can be generated by a Bayesian conversation π in T rounds.

According to [5], the belief-splitting process of a Bayesian conversation $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^{\infty}$ can be viewed as a *bimartingale*. Furthermore, [5] showed that given a set of final beliefs $A = \{(q_A^{(1)}, q_B^{(1)}), \ldots, (q_A^{(K)}, q_B^{(K)})\}$, the set of feasible priors $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ is the functional bi-convex hull ([33]) of A, where we say that a prior $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ is feasible if there exists a Bayesian conversation, possibly infinite-round, that starts with the prior $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ and generates a final belief distribution supported on A. However, their method does not decide whether a distribution $P(q_B, q_A)$ over final beliefs can be generated by a Bayesian conversation.

To decide the feasibility of a distribution over beliefs, we augment the bimartingale $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^{\infty}$ as a *dimartingale* (defined in [24]) which also includes *a bystander's belief about Alice and Bob's final beliefs*. The dimartingale is defined as follows.

Definition E.1 (dimartingale [24]). *A dimartingale $\{(\alpha_t, \beta_t, p_t)\}_{t=1}^{T}$ is a martingale that has either $\alpha_t = \alpha_{t+1}$ or $\beta_t = \beta_{t+1}$ at each time step t.*

Lemma E.1. *Consider any Bayesian conversation π. Let $q_A^{(t)}$ be Bob's (and a by-stander's belief about θ_A at the end of round t. Similarly, let $q_B^{(t)}$ be Alice's belief about θ_B at the end of round t. Let $\gamma^{(t)} = \mathbb{P}_\pi(q_A^{(T)}, q_B^{(T)} | h^{(t)})$ be a by-stander's belief about $q_A^{(T)}, q_B^{(T)}$ (the players' **final** beliefs) at the end of round t. Then for any Bayesian conversation π, $\{(q_A^{(t)}, q_B^{(t)}, \gamma^{(t)})\}_{t=1}^{T}$ form a dimartingale.*

Then we can decide whether a distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a finite-time Bayesian conversation by reversing this dimartingale. The reversing process can be formalized as follows.

Given a final belief distribution $P(\theta_A, \theta_B, q_B, q_A)$, we want to decide whether it can be generated by a Bayesian conversation π in T rounds. Consider the marginal distribution of the beliefs $P(q_B, q_A)$. Let Q be the support of $P(q_B, q_A)$, $Q = \{(q_B, q_A) : P(q_B, q_A) > 0\}$. Let $e_{q_B, q_A} \in \Delta(Q)$ be the deterministic distribution on Q that takes value (q_B, q_A) with probability 1. Then for every $(q_B, q_A) \in Q$, we append e_{q_B, q_A} and add the tuple into a set

$$\mathcal{S}_0 = \{(q_B, q_A, e_{q_B, q_A}) : (q_B, q_A) \in Q\}.$$

Define $\phi_1(q_B, q_A, z) = q_B$ and $\phi_2(q_B, q_A, z) = q_A$ for any $z \in \Delta(Q)$. Then for any points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{S}_i$ with the same q_A, we add all their convex combinations to set $\mathcal{T}_i$; and then for any points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{T}_i$ with the same q_B, we add all their convex combinations to set $\mathcal{S}_{i+1}$.

$$\mathcal{T}_i = \left\{ \begin{array}{ll} \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k : & \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{S}_i,\ \phi_2(\mathbf{x}_1) = \cdots = \phi_2(\mathbf{x}_k) \\ & \lambda_1 + \cdots + \lambda_k = 1,\ 0 \leq \lambda_i \leq 1 \end{array} \right\}$$

$$\mathcal{S}_{i+1} = \left\{ \begin{array}{ll} \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k : & \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{T}_i,\ \phi_1(\mathbf{x}_1) = \cdots = \phi_1(\mathbf{x}_k) \\ & \lambda_1 + \cdots + \lambda_k = 1,\ 0 \leq \lambda_i \leq 1 \end{array} \right\}$$

See Figure 6 for an example.

Theorem E.1. *A distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be implemented by a T-round Bayesian conversation if and only if*

(1) $P(\theta_A, \theta_B | q_B, q_A) = q_A(\theta_A) \cdot q_B(\theta_B)$, for all q_B, q_A;
(2) $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$, where $\mathbb{P}(\theta_B), \mathbb{P}(\theta_A)$ are the priors of θ_B and θ_A, and $P(q_B, q_A)$ is the marginal distribution of q_B, q_A based on P.

We give a proof for Theorem E.1 in Appendix F.4.

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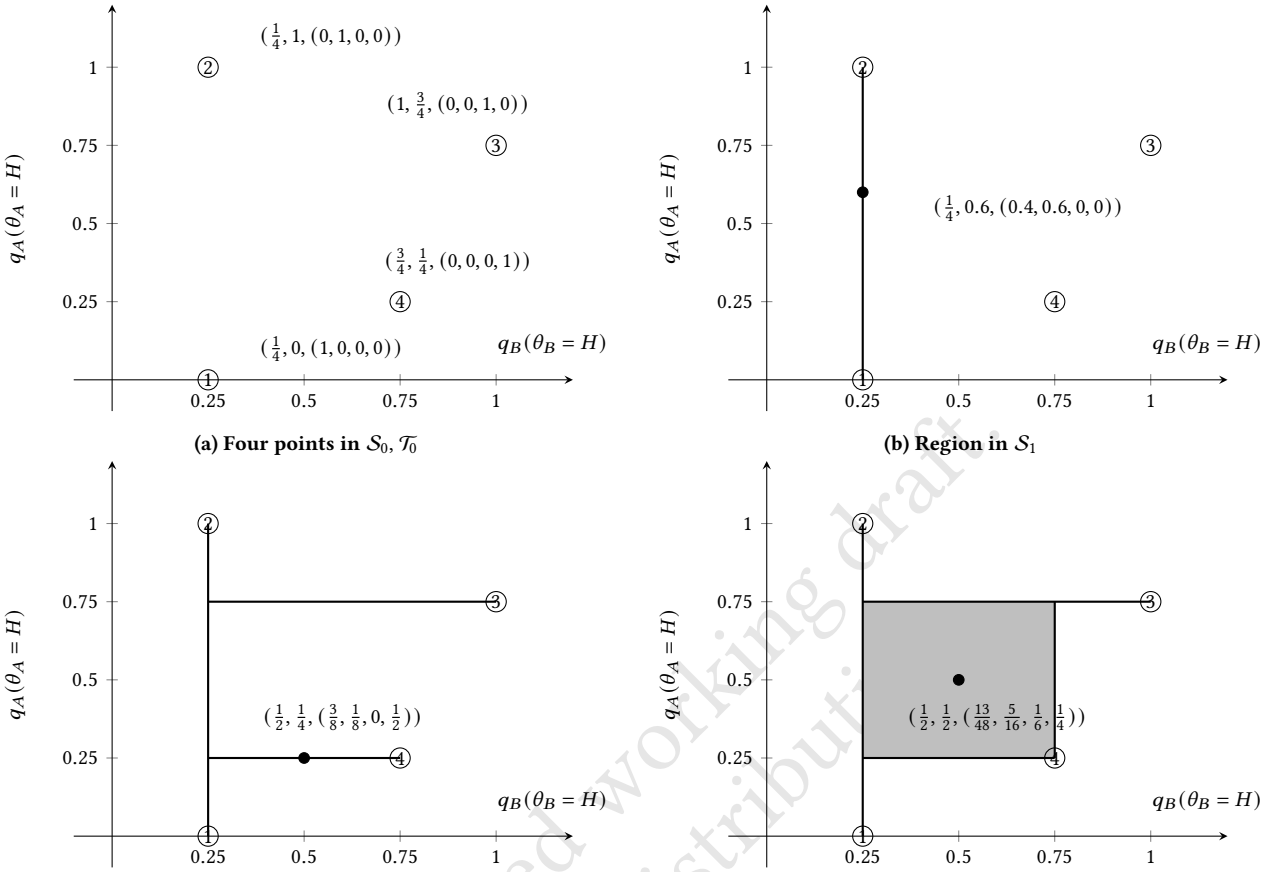

(a) Four points in $\mathcal{S}_0, \mathcal{T}_0$

(b) Region in $\mathcal{S}_1$

(c) Region in $\mathcal{T}_1$

(d) Region in $\mathcal{S}_2$.

Figure 6: An example of set $\mathcal{S}_i$. Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A) = \mathbb{P}(\theta_B) = 0.5$. Consider a posterior distribution that satisfies Condition (1) with marginal distribution $P(q_B, q_A)$ being the four points in Figure 6a with probabilities $(\frac{13}{48}, \frac{5}{16}, \frac{1}{6}, \frac{1}{4})$ respectively. Then we have S_0 plotted in the first picture with each point labeled as $(q_B(\theta_B = H), q_A(\theta_A = H), e_{q_B, q_A})$. Notice that no two points in S_0 share the same q_A, so we have $\mathcal{T}_0 = \mathcal{S}_0$. To get $\mathcal{S}_1$, we add all convex combinations of the points with the same q_B, which is just the segment plotted in Figure 6b. To get $\mathcal{T}_1$ we add all convex combinations of the points with the same q_A as in Figure 6c. Finally, we get the region of $\mathcal{S}_2$, and we have $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) = (0.5, 0.5, (\frac{13}{48}, \frac{5}{16}, \frac{1}{6}, \frac{1}{4})) \in \mathcal{S}_2$, so the posterior distribution is feasible.

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F Missing proofs in Appendix I

F.1 Proof of Proposition I.2

Proof. Let $|\text{supp}(P)| = Q$ and label the elements of the support $q_{AB}^{(s)}$ for $s \in [Q]$. The mediator will send a randomized signal $\pi(\theta_A, \theta_B) \in [Q]$ defined via

$$\Pr_{\pi}[\pi(\theta_A, \theta_B) = s] = \frac{q_{AB}^{(s)}(\theta_A, \theta_B) \cdot P\left(q_{AB}^{(s)}\right)}{\Pr[\theta_A] \cdot \Pr[\theta_B]}.$$

(If $\Pr[\theta_A] \cdot \Pr[\theta_B] = 0$, we can set the probabilities arbitrarily, since the mediator is guaranteed to never receive that pair of (θ_A, θ_B)). It follows that the posterior conditioned on receiving signal s equals $q_{AB}^{(s)}$ (and therefore must match this distribution). □

F.2 Proof for the Example in Figure 9

We show that it is not possible to start with $\theta_A, \theta_B \in \{L, H\}$, $\mathbb{P}(\theta_A = H) = \mathbb{P}(\theta_B = H) = 0.5$ and have $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5. Suppose to the contrary, we have a mediator protocol that gives $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5 in the end. This

means that the two players have to simultaneously hold the same belief 0.75 or 0.25. Then after seeing the public signal s, the conditional distribution of the two players' types (i.e., the observer's posterior) must be one of the following tables, and we must arrive at each of the

$P(\cdot\|s)$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	9/16	3/16
$\theta_A = L$	3/16	1/16

$P(\cdot\|s)$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	1/16	3/16
$\theta_A = L$	3/16	9/16

tables with probability 0.5. But this does not match the prior. Therefore it is not possible to have $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with

$0.5\cdot$

$P(\cdot\|s)$	H	L
H	9/16	3/16
L	3/16	1/16

$+\ 0.5\cdot$

$P(\cdot\|s)$	H	L
H	1/16	3/16
L	3/16	9/16

$\neq$

$\mathbb{P}(\cdot)$	H	L
H	1/4	1/4
L	1/4	1/4

probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5.

F.3 Observations about Bayesian conversations

In this section, we give several observations about the posterior distributions generated by Bayesian conversations.

Observation F.1 (Lemma 3 in [8]). *For a T-round Bayesian conversation π and any $1 \le t \le T$, let $q_B^{(t)} = P(\theta_B|\theta_A, h^{(t)})$ be Alice's belief about θ_B at the end of round t, and let $q_A^{(t)} = P(\theta_A|\theta_B, h^{(t)})$ be Bob's belief about θ_A at the end of round t. Then we have*

- *The players' beliefs $q_A^{(t)}, q_B^{(t)}$ are uniquely determined by the history $\widetilde{h}^{(t)}$ no matter what their types are, i.e., for any two possible types of Alice $x, x' \in \Theta_A$, we have $P(\theta_B|\theta_A = x, \widetilde{h}^{(t)}) = P(\theta_B|\theta_A = x', \widetilde{h}^{(t)})$, and the same holds for Bob.*
- *Alice's belief about Bob's type does not change before and after sending her signal a_t, and the same holds for Bob.*
- *Conditioning on any history $\widetilde{h}^{(t-1)}$ before Alice sending a_t, the expectation of Bob's belief does not change before and after Alice sends a_t. More specifically, let $\widetilde{q}_A^{(t-1)} = P(\theta_A|\theta_B, \widetilde{h}^{(t-1)})$ be Bob's unique belief after seeing $\widetilde{h}^{(t-1)}$ and let random variable $q_A^{(t)} = P(\theta_A|\theta_B, \widetilde{h}^{(t-1)}, a_t)$ be Bob's belief after seeing $\widetilde{h}^{(t-1)}$ and a_t. Then we have*
$$\mathbb{E}_{a_t}[q_A^{(t)}] = \widetilde{q}_A^{(t-1)}.$$
The same holds for Bob. The observation is illustrated in Figure 7.

Observation F.2 (Lemma 2 in [8]). *The joint distribution of the players' types and beliefs $P(\theta_A, \theta_B, q_A^{(t)}, q_B^{(t)})$ can be fully determined by the marginal distribution of their beliefs $P(q_A^{(t)}, q_B^{(t)})$ as*
$$P(\theta_A, \theta_B|q_A^{(t)}, q_B^{(t)}) = q_A^{(t)}(\theta_A) \cdot q_B^{(t)}(\theta_B).$$

Observation F.1 and Observation F.2 are implied by the following Observation F.3 and the fact that θ_A and θ_B are independently drawn from $\mathbb{P}(\theta_A)$ and $\mathbb{P}(\theta_B)$.

We consider $P(\theta_A, \theta_B, h^{(t)})$, the probability of reaching a history $h^{(t)}$ while the players' types being θ_A, θ_B, and characterize how $P(\theta_A, \theta_B, h^{(t)})$ evolves. For any round t and any realization $\widetilde{h}^{(t)}$, we represent $P(\theta_A, \theta_B, h^{(t)} = \widetilde{h}^{(t)})$ by a $|\Theta_A| \times |\Theta_B|$ matrix M_t with
$$M_t[x, y] = P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}),\ \forall x \in \Theta_A, y \in \Theta_B.$$

Observation F.3. *For any realization $\widetilde{h}^{(t+1)} = (\widetilde{h}^{(t)}, \widetilde{a}_{t+1}, \widetilde{b}_{t+1})$, after Alice sending $\widetilde{a}_{t+1}$, the probability matrix becomes*
$$D_{\widetilde{a}_{t+1}} \cdot M_t,$$
where $D_{\widetilde{a}_{t+1}}$ is a $|\Theta_A| \times |\Theta_A|$ diagonal matrix with $D_{\widetilde{a}_{t+1}}[x, x] = \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1})$ for all $x \in \Theta_A$. And after Bob sending $\widetilde{b}_{t+1}$, the probability matrix becomes
$$M_{t+1} = D_{\widetilde{a}_{t+1}} \cdot M_t \cdot Z,$$
where Z is a $|\Theta_B| \times |\Theta_B|$ diagonal matrix with $Z[y, y] = \Pr(g_{t+1}(y, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}) = \widetilde{b}_{t+1})$ for all $y \in \Theta_B$.

Proof. The proof follows directly from the formulas
$$P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}, a_{t+1} = \widetilde{a}_{t+1}) = P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}) \cdot \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1})$$
and
$$\begin{aligned} &P(\theta_A = x, \theta_B = y, h^{(t+1)} = \widetilde{h}^{(t+1)}) \\ =&P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}) \cdot \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1}) \Pr(g_{t+1}(y, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}) = \widetilde{b}_{t+1}). \end{aligned}$$
□

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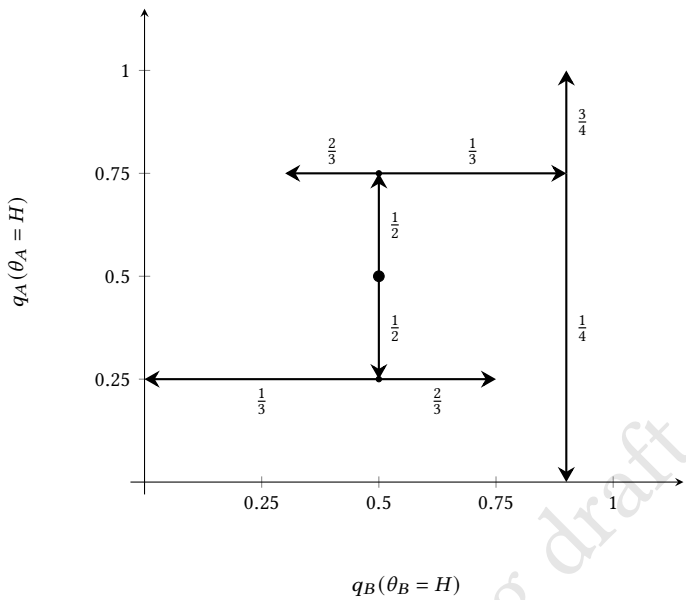

Figure 7: Illustration of Observation F.1. Suppose $\theta_A, \theta_B \in \{L, H\}$. At any step of the protocol, the status of the protocol can be represented as a two-dimensional point $(q_B(\theta_B = H), q_A(\theta_A = H))$. When Alice sends a signal, q_B remains unchanged and q_A is decomposed along y-axis while preserving the expectation. The same holds for Bob.

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F.4 Proof of Theorem E.1

We first prove that if a distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a Bayesian conversation π, i.e., $P(\theta_A, \theta_B, q_B, q_A) = \mathbb{P}_\pi(\theta_A, \theta_B, q_B^{(T)}, q_A^{(T)})$, then it must satisfy (1) and (2). For (1), it is directly proved by Observation F.2 because we must have $\mathbb{P}_\pi(\theta_A, \theta_B | q_A^{(T)}, q_B^{(T)}) = q_A^{(T)}(\theta_A) \cdot q_B^{(T)}(\theta_B)$. For (2), we prove that $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$ by showing that

$$(\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)})) \in \mathcal{S}_{T-t}, \text{ for all } t, \widetilde{h}^{(t)},$$

$$(\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)}, \widetilde{a}_{t+1})) \in \mathcal{T}_{T-t-1}, \text{ for all } t, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}$$

where $\widetilde{h}^{(t)}$ is any realization of history up to round t, and $\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t)}$ are the players' beliefs at the end of round t when the history is $\widetilde{h}^{(t)}$, and $\mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)}))$ is the distribution of the players' final beliefs conditioning on the history up to round t is $\widetilde{h}^{(t)}$. We prove by induction. First, it is clear that the statement holds for $t = T$,

$$(\widetilde{q}_B^{(T)}, \widetilde{q}_A^{(T)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(T)})) \in \mathcal{S}_0$$

because $\mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(T)})) = e_{\widetilde{q}_B^{(T)}, \widetilde{q}_A^{(T)}}$ and it is the definition of $\mathcal{S}_0$. Then assume that the statement is always true for $t \geq k+1$, it must also hold for $t = k$. According to Observation F.1, when Bob sending a signal, q_A does not change and q_B is decomposed into a convex combination of points. This means that $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(k)}, \widetilde{a}_{k+1}))$ must be a convex combination of the points in the next round $(q_B^{(k+1)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(k)}, \widetilde{a}_{k+1}, b_{k+1}))$ with the same $\widetilde{q}_A^{(k+1)}$. By induction, $(q_B^{(k+1)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(k)}, \widetilde{a}_{k+1}, b_{k+1}))$ are in the set $\mathcal{S}_{T-k-1}$ and by the definition of $\mathcal{T}_i$, we must have $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(k)}, \widetilde{a}_{k+1})) \in \mathcal{T}_{T-k-1}$ because it is a convex combination of points in $\mathcal{S}_{T-k-1}$ with the same q_A. Similarly, we must have $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(k)})) \in \mathcal{S}_{T-k}$.

We then prove if a distribution $P(\theta_A, \theta_B, q_B, q_A)$ satisfies (1) and (2), it can be generated by a Bayesian conversation. For any $P(\theta_A, \theta_B, q_B, q_A)$ that satisfies (1) and (2), we construct a Bayesian conversation by reversing the merging path. We start from $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$ and be definition, there exists a convex combination of points in $\mathcal{T}_{T-1}$ with the same q_B that gives $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$. Then we can define signal distribution $f_1(\theta_A)$ to be the distribution that decomposes $q_A = \mathbb{P}(\theta_A)$ to the q_A's in this convex combination. Similarly, for each point in the decomposition, we can find its convex combination of points in $\mathcal{S}_{T-1}$ with the same q_A and we can define the corresponding $g_1(\theta_A, a_1)$. Repeating this process for T rounds, we get a Bayesian conversation that generates the correct marginal distribution $P(q_B, q_A)$. Then by Observation F.2, we must have the induced $\mathbb{P}_\pi(\theta_A, \theta_B | q_B, q_A) = q_A(\theta) q_B(\theta_B)$, which matches $P(\theta_A, \theta_B | q_B, q_A)$ because of Condition (1).

G Missing proofs in Section A

G.1 Proof of Theorem A.1

Proof. For a utility function u and a class of protocol Π, $\text{Range}_{\text{ex-ante}}(u, \Pi)$ can be represented as the set of

$$\begin{aligned} &\sum_{\theta_A,\theta_B,q_A,q_B} P(\theta_A,\theta_B,q_A,q_B)u(\theta_A,\theta_B,r^*(\theta_A,q_B)) \\ \text{s.t.} \quad & P(\theta_A,\theta_B,q_A,q_B) \text{ satisfies (7)} \\ & P(\theta_A,\theta_B,q_A,q_B) \text{ can be induced by a protocol } \pi \in \Pi \end{aligned}$$

The key observation is the following: if Alice's posterior belief is q_B after the communication, then the conditional probability of $\theta_B = x$ must be equal to $q_B(x)$ for $x \in \Theta_B$. We can thus simplify the expected utility so that it only depends on θ_A and q_B. Define $U(\theta_A, q_B) = \sum_{x\in\Theta_B} q_B(x)u(\theta_A, x, r^*(\theta_A, q_B))$ to be the expected utility when Alice's type is θ_A and her posterior belief is q_B. Then the expected utility is equal to $\sum_{\theta_A,q_B} P(\theta_A, q_B)U(\theta_A, q_B)$. For the same reason, the constraint of ex-ante IR (7) can also be reduced to a constraint on $P(\theta_A, q_B)$,

$$\sum_{\theta_B,\theta_A,q_B} P(\theta_A,q_B)q_B(\theta_B)u_B(\theta_B,\theta_A,r^*(\theta_A,q_B)) \geq \sum_{\theta_B,\theta_A} \mathbb{P}(\theta_B)\mathbb{P}(\theta_A)u_B(\theta_B,\theta_A,r^*(\theta_A,\mathbb{P}(\theta_B))). \tag{12}$$

Then when finding the range of implementable expected utilities, we only need to consider the marginal distributions of θ_A and q_B that can be induced by the class of protocols,

$$\begin{aligned} &\sum_{\theta_A,q_B} P(\theta_A,q_B)U(\theta_A,q_B) \\ \text{s.t.} \quad & P(\theta_A,q_B) \text{ satisfies (12)} \\ & P(\theta_A,q_B) \text{ can be induced by a protocol } \pi \in \Pi \end{aligned}$$

We then show that the set of marginal distributions $P(\theta_A, q_B)$ that can be induced by mediator protocols is the same as the set of $P(\theta_A, q_B)$ that can be induced by one-round Bayesian conversations. We prove this by an exact characterization of feasible marginal distributions: $P(\theta_A, q_B)$ can be induced by a mediator protocol/Bayesian conversation if and only if

$$\sum_{q_B} P(\theta_A,q_B)q_B(\theta_B) = \mathbb{P}(\theta_A)\mathbb{P}(\theta_B), \quad \forall \theta_A, \theta_B. \tag{13}$$

This means that for all θ_A, the conditional expectation of q_B, that is $\mathbb{E}[q_B|\theta_A]$, must be equal to the prior $\mathbb{P}(\theta_B)$. By Proposition I.2, the equation is necessary. And it is also sufficient because any marginal distribution that satisfies the equation can be implemented by a one-round Bayesian conversation: Alice first fully reveals θ_A, and then based on the observed θ_A, Bob sends a signal so that Alice's posterior belief will follow the distribution $P(q_B|\theta_A)$. It is always possible for Bob to generate $P(q_B|\theta_A)$ because we have

$$\sum_{q_B} P(q_B|\theta_A)q_B(\theta_B) = \mathbb{P}(\theta_B), \quad \forall \theta_A, \theta_B,$$

then Bob can just send a signal b_{q_B} with probability $P(q_B|\theta_A)q_B(\theta_B)/\mathbb{P}(\theta_B)$ when his type is θ_B and Alice's observed type is θ_A, so that when b_{q_B} is sent, Alice's belief becomes q_B and that happens with probability $P(q_B|\theta_A)\sum_{\theta_B}\mathbb{P}(\theta_B)q_B(\theta_B)/\mathbb{P}(\theta_B) = P(q_B|\theta_A)$. This completes our proof. □

G.2 Extension of Theorem A.1

Suppose now we consider a vector of utility functions $\mathbf{u} = (u_1, \ldots, u_L)$. Define

$$\begin{aligned} \text{Range}(\mathbf{u}, \Pi, \mathcal{P}) = \{&(\mathbb{E}_P[u_1(\theta_A,\theta_B,r^*(\theta_A,q_B))], \ldots, \mathbb{E}_P[u_L(\theta_A,\theta_B,r^*(\theta_A,q_B))]) \\ &: P \in \mathcal{P} \text{ and } P \text{ can be induced by a protocol } \pi \in \Pi\}. \end{aligned}$$

Then when we only need ex-ante/interim IR, the two classes of protocols induce the same range of utilities.

Theorem G.1. *When the players are committed and the base game has only one action-taker, the utility vectors that can be implemented by ex-ante/interim IR mediator protocols is the same as the utility vectors that can be implemented by ex-ante/interim IR Bayesian conversations,*

$$\begin{aligned} \textit{Range}_{\textit{ex-ante}}(\mathbf{u}, \Pi_{\textit{two-way}}) &= \textit{Range}_{\textit{ex-ante}}(\mathbf{u}, \Pi_{\textit{mediator}}), \\ \textit{Range}_{\textit{interim}}(\mathbf{u}, \Pi_{\textit{two-way}}) &= \textit{Range}_{\textit{interim}}(\mathbf{u}, \Pi_{\textit{mediator}}). \end{aligned}$$

Proof. The key idea is basically the same as in the proof of Theorem A.1. For any θ_A and q_B, define $U_j(\theta_A, q_B) = \sum_{x\in\Theta_B} q_B(x)u_j(\theta_A, x, r^*(\theta_A, q_B))$ to be the expected u_j when Alice's type is θ_A and her posterior belief is q_B. Then the utility vectors that

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can be implemented by ex-ante IR mediator/Bayesian conversations can be equally represented as the set of

$$\left(\sum_{\theta_A,q_B} P(\theta_A,q_B)U_1(\theta_A,q_B),\ldots,\sum_{\theta_A,q_B} P(\theta_A,q_B)U_L(\theta_A,q_B)\right)$$

$$\text{s.t.}\quad \sum_{q_B} P(\theta_A,q_B)q_B(\theta_B) = \mathbb{P}(\theta_A)\mathbb{P}(\theta_B),\quad \forall\theta_A,\theta_B \tag{feasibility}$$

$$\sum_{\theta_B,\theta_A,q_B} P(\theta_A,q_B)q_B(\theta_B)u_B(\theta_B,\theta_A,r^*(\theta_A,q_B)) \geq \sum_{\theta_B,\theta_A} \mathbb{P}(\theta_B)\mathbb{P}(\theta_A)u_B(\theta_B,\theta_A,r^*(\theta_A,\mathbb{P}(\theta_B))), \tag{ex-ante IR}$$

For interim IR, we just replace the inequality for ex-ante IR with the inequality for interim IR.

Setting $u_1 = u_A$ and $u_2 = u_B$, we know that the implementable pairs of the two players' utilities are the same, which implies that the induced Pareto frontiers are the same. □

G.3 Two action-takers

Suppose Alice and Bob each holds a private random bit $\theta_A, \theta_B \in \{0,1\}$ with $\mathbb{P}(\theta_A) = \mathbb{P}(\theta_B) = 0.5$. The designer wants to reveal minimum information so that both of them know the AND of their bits. The designer's utility is

$$u(\theta_A,\theta_B,q_B,q_A) = \begin{cases} 0, & \text{if } \theta_A = \theta_B = 1 \\ H(q_B), & \text{if } \theta_A = 0, \theta_B = 1, \text{ and } q_A(\theta_A = 0) = 1 \\ H(q_A), & \text{if } \theta_A = 1, \theta_B = 0, \text{ and } q_B(\theta_B = 0) = 1 \\ H(q_A) + H(q_B), & \text{if } \theta_A = \theta_B = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

where $H(\cdot)$ is the entropy function $H(q) = -q\log q - (1-q)\log(1-q)$. Then a mediator protocol can achieve the highest expected utility for the designer by directly revealing the value of θ_A AND θ_B, which give $\mathbb{E}[u] = 1$. [12] prove that no Bayesian conversation can achieve this expected utility.

Theorem G.2 (Theorem 7.1 and Theorem 7.9 in [12]). *No Bayesian conversation (a.k.a. two-way communication protocol) can achieve an expected utility of* 1 *for the designer. In addition, the optimal Bayesian conversation takes infinitely many rounds.*

We can turn the utility function $u(\theta_A,\theta_B,q_B,q_A)$ into a utility function $u(\theta_A,\theta_B,r_A,r_B)$ that depends on the players' types and actions by setting $R_A = R_B = [0,1]$ and define the players' utility functions to be the value of the log scoring rule,

$$u_A(r_A,\theta_B) = \log(r_A)\cdot \mathbf{1}[\theta_B = 1] + \log(1-r_A)\cdot \mathbf{1}[\theta_B = 0].$$

so that Alice's best action when she holds belief q_B is just $r_A^* = q_B(\theta_B = 1)$, and Bob's best action when he holds belief q_A is just $r_B^* = q_A(\theta_A = 1)$.

H Examples

H.1 Gap between ex-post IR mediator protocols and Bayesian conversations

We provide the optimal mediator protocol and Bayesian conversation for the hiring problem in Section 4.1.

Optimal mediator protocol. The optimal ex-post IR mediator protocol π that achieves the highest possible social welfare is as follows. Consider a mediator who sends two possible public signals s_1 ="good communication skill" and s_2 ="good programming skill" using the following signaling scheme:

- when θ_A = Prog, fully reveal θ_B, that is, send "good communication skill" when θ_B =Comm and send "good programming skill" when θ_B =Prog.
- when θ_A = Comm, partially reveal θ_B: send "good communication skill" when θ_B =Comm; and when θ_B =Prog, send "good communication skill" with probability 2/3 and send "good programming skill" with probability 1/3.

The joint distribution of the signal and the players' types are shown in Table 5.

$P(s_1,\cdot)$	θ_B = Prog	θ_B = Comm
θ_A = Prog	0	0.2
θ_A = Comm	0.2	0.2

$P(s_2,\cdot)$	θ_B = Prog	θ_B = Comm
θ_A = Prog	0.3	0
θ_A = Comm	0.1	0

Table 5: The joint distribution of the signal and the types.

An employer's best actions are shown in Table 6. A Prog-type employer hires when receiving the "good programming skill" signal because the candidate's type is fully revealed for them. A Comm-type employer hires when receiving the "good communication skill" signal, because

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the posterior probability of θ_B =Comm, that is $P(\theta_B = \text{Comm}|s_1, \theta_A = \text{Comm})$, becomes 0.5 and we assume that the employer breaks ties by choosing the action that is more favorable for the candidate.

best action	s_1	s_2
$\theta_A = \text{Prog}$	not hire	hire
$\theta_A = \text{Comm}$	hire	not hire

Table 6: The employer's best action.

We first show that the protocol is ex-post IR. The protocol is ex-post IR for a Prog-type candidate because they still get hired by the Prog-type employer and they only get hired more by the Comm-type employer. The protocol is ex-post IR for a Comm-type candidate, because based on a Comm-type candidate's belief, the probability of getting hired does not change. When we use the protocol for a Comm-type candidate, the mediator will only send the "good communication skill" signal and the candidate will only be hired by the Comm-type employer, whereas the candidate only gets hired by the Prog-type employer when there's no communication. A Comm-type candidate's posterior belief about the employer's type is $P(\theta_A = \text{Prog}|s_1, \theta_B = \text{Comm}) = 0.5$ (the protocol reveals no information about θ_A to a type-Comm candidate; see Table 5), so the probability of getting hired does not change for a Comm-type candidate.

We then show that the protocol achieves the highest possible social welfare. The expected social welfare can be decomposed as

$$\begin{aligned}\mathbb{E}[u_A + u_B] = {} & P(\theta_A = \text{Prog}, \theta_B = \text{Prog}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Prog}] \\ & + P(\theta_A = \text{Prog}, \theta_B = \text{Comm}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Comm}] \\ & + P(\theta_A = \text{Comm}, \theta_B = \text{Prog}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}] \\ & + P(\theta_A = \text{Comm}, \theta_B = \text{Comm}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Comm}].\end{aligned}$$

It is not difficult to see that $\mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Prog}]$, $\mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Comm}]$, and $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Comm}]$ induced by the mediator protocol have reached their maximum value, because after seeing the signal sent by the mediator, the employer always hires when $\theta_A = \theta_B$ and the employer never hires when $\theta_A = \text{Prog}$ and $\theta_B = \text{Comm}$. So we only need to prove that $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}]$ has also reached its maximum. To simplify the notation, we denote by $\mathcal{E}$ the event that $\theta_A = \text{Comm}$ and $\theta_B = \text{Prog}$. Then the conditional social welfare $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}]$ can be written as

$$\begin{aligned}& \mathbb{E}[u_A + u_B|\mathcal{E}] \\ = {} & \mathbb{P}(r = \text{hire}|\mathcal{E}) \cdot \mathbb{E}[u_A + u_B|r = \text{hire}, \mathcal{E}] + \mathbb{P}(r = \text{not hire}|\mathcal{E}) \cdot \mathbb{E}[u_A + u_B|r = \text{not hire}, \mathcal{E}] \\ = {} & \mathbb{P}(r = \text{hire}|\mathcal{E})(-1 + 2) + \mathbb{P}(r = \text{not hire}|\mathcal{E})(0 + 0).\end{aligned}$$

This means that the higher $\mathbb{P}(r = \text{hire}|\mathcal{E})$ is, the higher the conditional social welfare is. We show that $\mathbb{E}[u_A + u_B|\mathcal{E}]$ cannot exceed $\frac{2}{3}$ by showing that $\mathbb{P}(r = \text{hire}|\mathcal{E})$ cannot exceed $\frac{2}{3}$. This is because a Comm-type employer only hires when the candidate is more likely to have type Comm, and it implies

$$P(\theta_B = \text{Prog}|\theta_A = \text{Comm}, r = \text{hire}) \leq P(\theta_B = \text{Comm}|\theta_A = \text{Comm}, r = \text{hire}).$$

Multiply both sides by $P(r = \text{hire}|\theta_A = \text{Comm})$, we get

$$\begin{aligned}P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm}) & \leq P(\theta_B = \text{Comm}, r = \text{hire}|\theta_A = \text{Comm}) \\ & \leq P(\theta_B = \text{Comm}|\theta_A = \text{Comm}) \\ & = 0.4.\end{aligned}$$

The second inequality trivially holds and the last equality is because θ_A and θ_B are independent. Finally,

$$\begin{aligned}\mathbb{P}(r = \text{hire}|\theta_B = \text{Prog}, \theta_A = \text{Comm}) & = \frac{P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm})}{P(\theta_B = \text{Prog}|\theta_A = \text{Comm})} \\ & = \frac{P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm})}{0.6} \\ & \leq \frac{0.4}{0.6} = \frac{2}{3}.\end{aligned}$$

Therefore we must have $\mathbb{P}(r = \text{hire}|\mathcal{E}) \leq \frac{2}{3}$, and this upper bound is reached by the mediator protocol because we have $\mathbb{P}(s_1|\mathcal{E}) = \frac{2}{3}$ and a Comm-type employer always hires after receiving s_1.

By employing the linear program in C.1, we can obtain the same mediator protocol as in Table 5.

And the optimal social welfare by ex-post IR mediator protocol is

$$W = 0.2 \times (0 + 0) + 0.2 \times (-1 + 2) + 0.2 \times (1 + 2) + 0.3 \times (10 + 2) + 0.1 \times (0 + 0) = \frac{22}{5}.$$

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Optimal Bayesian conversation. We next employ our algorithm to find the optimal ex-post IR Bayesian conversation. First of all, the ex-post IR region IR_0 is plotted as in Figure 3a with

$$\mathsf{IR}_0 = [0, 0.5] \times [0, 0.5] \cup [0.5, 1] \times [0.5, 1],$$
$$X^* = \{0, 0.5, 0.6, 1\}, \quad Y^* = \{0, 0.5, 1\}.$$

And the corresponding $W_0(q_A, q_B)$ is plotted in Appendix H.2 with $W_0(0, 1) = W_0(1, 0) = -\infty$, $W_0(0.5, 0.5) = W_0(0.5, 0) = W(0.5, 1) = 2$, and so on. Then, we perform the recursive updates to find $W_k(\cdot)$ for $k > 0$. The resulting $W_1(q_B, q_A)$ is plotted in Figure 4a, and we have $W_2(q_B, q_A) = W_1(q_B, q_A)$, for all $(q_B, q_A) \in X^* \times Y^*$. This means that our algorithm has converged after just one step, so the Bayesian conversation after one round has already achieved the optimal social welfare, which is equal to $\frac{21}{5}$.

We can see from the IR_0 region that if Alice splits belief q_A, there must be one of final believes fall outside the IR_0 region. Therefore, a 1-round Bayesian conversation can achieve the optimum.

H.2 Missing proof of theorem 4.2

Proof. Consider a two-player game between Alice and Bob. Alice can be one of two types: $\Theta_A = \{\theta_{A0}, \theta_{A1}\}$, and Bob can also be one of two types: $\Theta_B = \{\theta_{B0}, \theta_{B1}\}$. Alice can take two actions $r \in \{r_0, r_1\}$. Suppose $\mathbb{P}(\theta_A = \theta_{A0}) = 0.6$, and $\mathbb{P}(\theta_B = \theta_{B0}) = 0.4$. The utilities of the two players are given in the following table:

$u_A(\theta_{A0}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	7	5
r_1	5	7

$u_A(\theta_{A1}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	1	3
r_1	0	5

$u_B(\theta_{B0}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	5	10
r_1	10	0

$u_B(\theta_{B1}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	10	10
r_1	10	4

Table 7: Alice and Bob's utility function.

According to the algorithm in Section 3.1, calculate W_0 and plot it on the coordinate system as shown in the following figure:
We denote $W_k\left(\frac{2}{5}, \frac{4}{5}\right)$ as $m_1^{(k)}$, $W_k\left(\frac{1}{2}, \frac{4}{5}\right)$ as $m_2^{(k)}$, $W_k\left(\frac{2}{3}, \frac{4}{5}\right)$ as $m_3^{(k)}$, $W_k\left(\frac{2}{5}, \frac{3}{5}\right)$ as $m_4^{(k)}$, $W_k\left(\frac{1}{2}, \frac{3}{5}\right)$ as $m_5^{(k)}$, and $W_k\left(\frac{2}{3}, \frac{3}{5}\right)$ as $m_6^{(k)}$.
After calculations, we have discovered the following iterative pattern: For all $k > 0$, When k is even,

$$\begin{aligned}
m_1^{(k+1)} &= m_1^{(k)}, \\
m_2^{(k+1)} &= m_2^{(k)}, \\
m_3^{(k+1)} &= \frac{2}{3} m_2^{(k)} + \frac{1}{3} W_k\left(1, \frac{4}{5}\right), \\
m_4^{(k+1)} &= \frac{3}{5} m_6^{(k)} + \frac{2}{5} W_k\left(0, \frac{3}{5}\right), \\
m_5^{(k+1)} &= \frac{3}{4} m_6^{(k)} + \frac{1}{4} W_k\left(0, \frac{3}{5}\right), \\
m_6^{(k+1)} &= m_6^{(k)}.
\end{aligned}$$

when k is odd,

$$\begin{aligned}
m_1^{(k+1)} &= \frac{1}{2} m_4^{(k)} + \frac{1}{2} W_k\left(\frac{2}{5}, 1\right), \\
m_2^{(k+1)} &= \frac{1}{2} m_5^{(k)} + \frac{1}{2} W_k\left(\frac{1}{2}, 1\right), \\
m_3^{(k+1)} &= m_3^{(k)}, \\
m_4^{(k+1)} &= m_4^{(k)}, \\
m_5^{(k+1)} &= m_5^{(k)}, \\
m_6^{(k+1)} &= \frac{3}{4} m_3^{(k)} + \frac{1}{4} W_k\left(\frac{2}{3}, 0\right).
\end{aligned}$$

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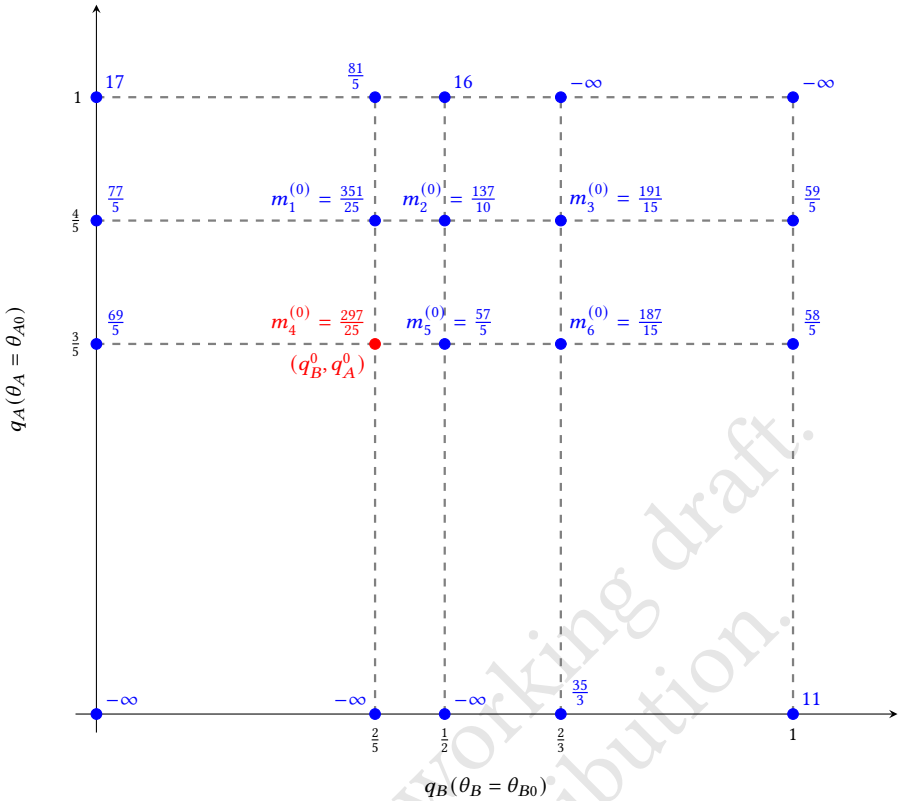

Figure 8: Illustration of W_0 in the two-player game where the highest social welfare requires infinite rounds of communication to achieve.

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And in all iterations, the values of other points remain unchanged: For all $k > 0$, for all $(x, y) \in X^* \times Y^* - \{\left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{1}{2}, \frac{4}{5}\right), \left(\frac{2}{3}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{3}{5}\right), \left(\frac{2}{3}, \frac{3}{5}\right)\}$,

$$W_{k+1}(x, y) = W_k(x, y).$$

Since for even k,

$$m_4^{(k+1)} = \frac{3}{5} m_6^{(k)} + \frac{2}{5} W_k\left(0, \frac{3}{5}\right), \qquad m_5^{(k+1)} = \frac{3}{4} m_6^{(k)} + \frac{1}{4} W_k\left(0, \frac{3}{5}\right),$$

and for odd k, m_4 and m_5 remain unchanged, thus we know that $W_k(\cdot, \frac{3}{5})$ is linear in $(0, \frac{1}{2})$ for all k.

For all k, $W_k(0, \frac{3}{5})$, $W_k(0, \frac{4}{5})$, $W_k(0, 1)$ remain unchanged, and their values are linear to y. And for all k, $W_k(0, 1)$, $W_k(\frac{2}{5}, 1)$, $W_k(\frac{1}{2}, 1)$ remain unchanged, and their values are linear to x.

For odd k,

$$m_1^{(k+1)} = \frac{1}{2} m_4^{(k)} + \frac{1}{2} W_k\left(\frac{2}{5}, 1\right), \qquad m_2^{(k+1)} = \frac{1}{2} m_5^{(k)} + \frac{1}{2} W_k\left(\frac{1}{2}, 1\right)$$

So

$$\begin{aligned} m_1^{(k+1)} =& \frac{1}{2} m_4^{(k)} + \frac{1}{2} W_k\left(\frac{2}{5}, 1\right) \\ =& \frac{1}{2}\left(\frac{4}{5} m_5^{(k)} + \frac{1}{5} W_k\left(0, \frac{3}{5}\right)\right) + \frac{1}{2}\left(\frac{4}{5} W_k\left(\frac{1}{2}, 1\right) + \frac{1}{5} W_k\left(0, 1\right)\right) \\ =& \frac{4}{5} m_2^{(k+1)} + \frac{1}{5} W_{k+1}\left(0, \frac{4}{5}\right). \end{aligned}$$

and for even k, m_1 and m_2 remain unchanged, making $W_k(\cdot, \frac{4}{5})$ linear in $(0, \frac{1}{2})$ for all k, so:

$$m_2^{(k)} = \frac{5}{4} m_1^{(k)} - \frac{1}{4} W_k\left(0, \frac{4}{5}\right) = \frac{5}{4} m_1^{(k)} - \frac{77}{20}, \quad \forall k.$$

For odd k,

$$\begin{aligned}
m_4^{(k+4)} &= \frac{3}{5} m_6^{(k+3)} + \frac{138}{25}, \\
m_6^{(k+3)} &= \frac{3}{4} m_3^{(k+2)} + \frac{35}{12}, \\
m_3^{(k+2)} &= \frac{2}{3} m_2^{(k+1)} + \frac{59}{15} = \frac{5}{6} m_1^{(k+1)} + \frac{41}{30}, \\
m_1^{(k+1)} &= \frac{1}{2} m_4^{(k)} + \frac{81}{10}.
\end{aligned}$$

By recursively applying the above equations, we have for odd k:

$$m_4^{(k+4)} = \frac{3}{16} m_4^{(k)} + \frac{4369}{400}.$$

Since $m_4^{(1)} = 13$, we get

$$W_{4k+1}(0.4, 0.6) = m_4^{(4k+1)} = -\frac{144}{325}\left(\frac{3}{16}\right)^k + \frac{4369}{325}.$$

And from this expression, we can see that the optimal welfare is never achieved by any finite value of k. □

I Posterior distributions of the two protocols

Communication protocols change the outcome of the game by changing the players' beliefs about the other player's type. It is therefore natural to ask: what are the belief distributions that can be generated by a communication protocol? Determining whether a distribution can be generated by a communication protocol is not as straightforward as one might assume. For instance, in Figure 9, we present a belief distribution that cannot be generated by any communication protocol.

In this section, we present some preliminary observations from the existing literature for understanding the space of feasible posterior distributions, both in the mediator model and the Bayesian conversation model. One important consequence of the observations in this section is the following fact: mediator protocols possess (strictly) greater power than Bayesian conversations, as there exist belief distributions that can only be generated by a mediator protocol and not by a Bayesian conversation (note that conversely, Bayesian conversations are a subset of mediator protocols, as the transcript of a Bayesian conversation can be used as the public signal in a mediator protocol). We summarize this in the following proposition.

Proposition I.1. *There exists a posterior distribution $P(\theta_A, \theta_B, q_B, q_A)$ that can be generated by mediator protocol that cannot be generated by Bayesian conversations (demonstrated in Table 8). In particular, there exists a distribution that can be generated by mediator protocol that does not satisfy Condition (1) in Theorem E.1, which must be satisfied by a Bayesian conversation.*

$\mathbb{P}(\cdot)$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	0.3	0.2
$\theta_A = L$	0.3	0.2

$= 0.6$

$q_{AB}^{(s_1)}$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	0	1/3
$\theta_A = L$	1/3	1/3

$\downarrow$

$P(H, L, (0,1), (\frac{1}{2}, \frac{1}{2})) = 0.2$
$P(L, H, (\frac{1}{2}, \frac{1}{2}), (0,1)) = 0.2$
$P(L, L, (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})) = 0.2$

$+ 0.4$

$q_{AB}^{(s_2)}$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	3/4	0
$\theta_A = L$	1/4	0

$\downarrow$

$P(H, H, (1,0), (\frac{3}{4}, \frac{1}{4})) = 0.3$
$P(L, H, (1,0), (\frac{3}{4}, \frac{1}{4})) = 0.1$

Table 8: Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A = H) = 0.5$ and $\mathbb{P}(\theta_B = H) = 0.6$. The plot shows how a distribution of posteriors $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a mediator protocol with $S = \{s_1, s_2\}$ with $q_{AB}^{(s_1)}$ shown in the second table with $q_{AB}^{(s_2)}$ shown in the third table. See Section H.1 for a concrete application of this example.

I.0.1 The observer's posterior. Our goal is to characterize the possible distribution of the possible tuples $(\theta_A, \theta_B, q_B, q_A)$ of types and posteriors induced by a given protocol π (what we have called *joint posterior distributions*). To analyze this distribution, we consider the *observer's posterior distribution*; the posterior distribution about (θ_A, θ_B) a third-party observer (who cannot observe Alice or Bob's types directly, but does know their priors) arrives at after seeing the public signal s.

$$q_{AB}(\theta_A, \theta_B) = \mathbb{P}_\pi(\theta_A, \theta_B | s).$$

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And we denote by $\mathbb{P}_\pi(q_{AB})$ the induced distribution of the observer's posterior where $\mathbb{P}_\pi(q_{AB}) = \mathbb{P}_\pi(s)$. Note that it is possible to recover the posteriors of Alice and Bob from the observer's posteriors along with Alice and Bob's realized types. Therefore, we can recover a joint posterior distribution uniquely from the observer's posterior distribution as shown in Table 8.

I.0.2 Mediator protocols. Then the observer's posterior distribution induced by a mediator protocol can be characterized by the Splitting Lemma [34]: sending the public signal can be thought of as "splitting" the prior $\mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$ into $|S|$ different posteriors q_{AB} each with probability $\Pr(s)$ of occurring. Note that this splitting operation is *mean-preserving*: the expected posterior of the observer at the conclusion of the protocol must equal the original prior: $\mathbb{E}_\pi[q_{AB}] = \mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$.For the case of mediator protocols, the Splitting Lemma condition is sufficient as well as necessary: any mean-preserving distribution over posteriors is incentivizable via a single signal.

Proposition I.2. *Let $P(q_{AB})$ be a distribution over posterior distributions $q_{AB} \in \Delta(\Theta_A \times \Theta_B)$ with the property that $\mathbb{E}[q_{AB}] = \mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$. Then there exists a mediator protocol π such that $\mathbb{P}_\pi(q_{AB}) = P(q_{AB})$.*

One remark: a version of the splitting lemma holds for the (marginals of the) joint posterior distributions, in the sense that $\mathbb{E}[q_A] = \mathbb{P}(\theta_A)$ and $\mathbb{E}[q_B] = \mathbb{P}(\theta_B)$. However, the obvious analogue of Proposition I.2 does not hold: in the example shown in Figure 9, the expectations of the posteriors equal the priors, but it cannot be induced by a mediator protocol.

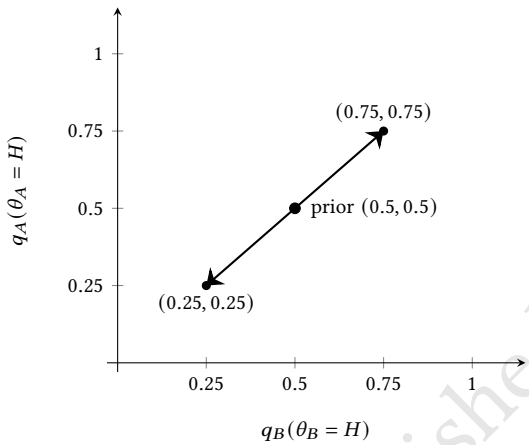

Figure 9: This example shows a posterior distribution that cannot be induced by any communication protocols. Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A = H) = \mathbb{P}(\theta_B = H) = 0.5$. Then we cannot have $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5. We give the proof in Appendix F.2.

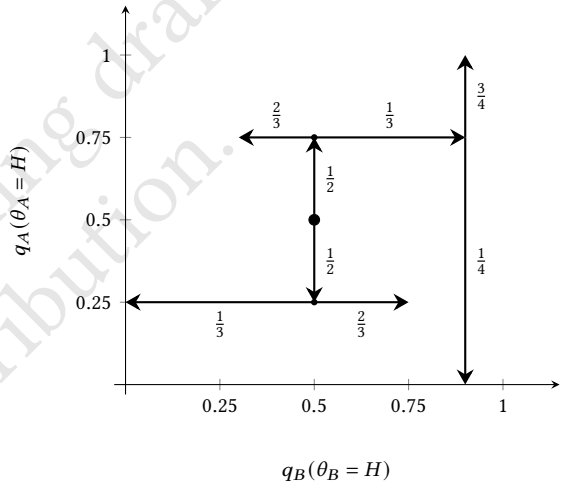

Figure 10: An illustration of a posterior distribution generated by a Bayesian conversation. Suppose $\theta_A, \theta_B \in \{L, H\}$. At any step of the protocol, the status of the protocol can be represented as a two-dimensional point $(q_B(\theta_B = H), q_A(\theta_A = H))$. When Alice sends a signal, q_B remains unchanged and q_A is decomposed along y-axis while preserving the expectation. The same holds for Bob.

I.0.3 Bayesian conversations. In Bayesian conversations, we have the additional restriction that each signal sent either contains information only about θ_A (if Alice is sending the signal) or only about θ_B (if Bob is sending the signal); in other words, at every step t of the protocol, either we have $\Pr[s|\theta_A, \theta_B, h^{(t)}] = \Pr[s|\theta_A, h^{(t)}]$ or $\Pr[s|\theta_A, \theta_B, h^{(t)}] = \Pr[s|\theta_B, h^{(t)}]$. This has following two important consequences. First, since the observer's posterior starts as a product distribution (since the priors for θ_A and θ_B are independent), the observer's posterior will always remain a product distribution (i.e. in each q_{AB}, θ_A and θ_B are independent). Secondly, this lets us relate the observer's posterior and the joint posteriors much more directly: we always have that $q_{AB}(\theta_A, \theta_B) = q_A(\theta_A)q_B(\theta_B)$. Pictorially, we can represent this procedure as in Figure 10; at each step of the protocol, each posterior can be split either "horizontally" (along dimensions in $\Delta(\Theta_A)$) or "vertically" (along dimensions in $\Delta(\Theta_B)$) but not both. This belief-splitting process is defined as a *bimartingale* in [5, 6].

Proposition I.3 ([5, 6]). *The belief-splitting process $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^\infty$ can be viewed as a* bimartingale. *A bimartingale is a martingale $\{(\alpha_t, \beta_t)\}_{t=1}^\infty$ that has either $\alpha_t = \alpha_{t+1}$ or $\beta_t = \beta_{t+1}$ at each time step t.*

Note that $P(\theta_A, \theta_B | q_B, q_A) = q_A(\theta_A)q_B(\theta_B)$ does *not* have to be true for mediator protocols (where the mediator can correlate the observer's posterior for θ_A and θ_B), and hence this provides a proof of Proposition I.1. An explicit counter-example is given in Table 8.

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J Missing Proofs in Section 3

J.1 Proof of Lemma 3.1

Proof. This can be seen by induction on k. It holds for $k = 0$ by definition. For odd k, let W_k^* be the welfare of the optimal protocol and W_k the function computed above. Assume that Bob's first signal splits Alice's prior belief q_B^0 into a distribution Q'_B. For each $q'_B \sim Q'_B$ what follows is a $k-1$ round protocol starting at (q'_B, q_A) where Bob doesn't regret being at the prior. Hence:

$$W_k^*(q_B, q_A) = \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}^*(q'_B, q_A)] = \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}(q'_B, q_A)] \leq W_k(q'_B, q_A)$$

since W_k is the maximum. Since W_k corresponds to a valid protocol, $W_k(q'_B, q_A) = W_k^*(q_B, q_A)$. The argument for even k is the same, swapping the roles of Alice and Bob. □

J.2 Proof of Lemma 3.2

Proof. Define X^* as:

$$X^* = \{0, 1\} \cup \{q_B \in [0, 1] : \exists \theta_A \in \Theta_A, \text{such that} r^*(\theta_A, q_B) \text{ differs on the left and right of } q_B\}$$

This lists all the q_B values at which Alice's decision changes. Within the interval (x_i^*, x_{i+1}^*), Alice's decisions remain unchanged. Therefore, we define

$$X_{\text{int}} = \left\{\frac{x_i + x_{i+1}}{2} \mid i = 0, 1, \ldots, m-1\right\}$$

by selecting the midpoints $\frac{x_i + x_{i+1}}{2}$ as representatives of the intervals (x_i, x_{i+1}). Consequently, all x in $X^* \cup X_{\text{int}}$ correspond to Alice's strategies, collectively covering all possible strategies she may adopt over the entire range $x \in [0, 1]$.

As q_B varies, Alice will take different strategies. Under different strategies of Alice, whether Bob's IR condition is satisfied will change depending on q_A. We therefore define the set Y^*, which enumerates the boundaries at which Bob's IR condition is satisfied or not, under different q_B, considering all possible strategies of Alice.

$$Y^* = \{0, 1\} \cup \{q_A \in [0, 1] \mid \exists q_B \in X^* \cup X_{\text{int}}, \exists \theta_B \in \text{supp}(q_B), \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) = \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0))\}$$

Now, recall the definition of W^*:

$$\begin{aligned}
&W^*(q_A, q_B) \\
=&\mathbb{E}_{\theta_B, \theta_A \sim (q_B, q_A)}[u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))] \\
=&\sum_{\theta_A \in \Theta_A} \sum_{\theta_B \in \Theta_B} q_A(\theta_A) q_B(\theta_B)(u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)))
\end{aligned}$$

For each rectangle, either it is not in IR_0, or it is in IR_0 and $r^*(\theta_A, q_B)$ is a constant within it, hence W_0 restricted to each rectangle is bilinear.

Since the number of actions that Alice can take is $|R|$, the number of beliefs q_B where Alice changes her action at a specific type is also $O(|R|)$. Given that Alice has two types, the size of X^* is $O(|R|)$. Furthermore, we know that the size of X_{int} is also $O(|R|)$.

For each $q_B \in X^* \cup X_{\text{int}}$, there are at most two θ_B in $\text{supp}(q_B)$. For each θ_B, there is at most one $q_A \in [0, 1]$ that satisfies

$$\mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) = \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0))\},$$

and therefore, $|Y^*| \leq 1 \cdot 2 \cdot |X^* \cup X_{\text{int}}| = O(|R|)$. □

J.3 Proof of Lemma 3.3

Proof. We prove the claim by induction on k. The proof for cases $k = 1$ and $k = 2$ are special and then $k \geq 3$ follows a general argument.

Case $k = 1$: We know by Lemma 3.2 that W^* is bilinear. Hence once we fix a $\hat{y}$, $W_0(x, \hat{y})$ is piecewise linear on $[0, 1]$ with segments bounded by $x_i \in X^*$. The function $x \mapsto u_A(x, \hat{y})$ is continuous at x_i but $x \mapsto u_B(x, \hat{y})$ may be discontinuous. Since Alice breaks ties in favor of Bob, the value of $u_B(x_i, \hat{y})$ is the maximum of its left and right limits, hence:

$$W_0(x_i, \hat{y}) = \max\left(\lim_{x \to x_i^-} W_0(x, \hat{y}), \lim_{x \to x_i^+} W_0(x, \hat{y})\right).$$

We need to show that given a point $(\hat{x}, \hat{y})$, if X' is the distribution in the solution of the optimization problem (3), then:

$$\begin{aligned}
W_1(\hat{x}, \hat{y}) &= \max \mathbb{E}_{x' \sim X'}[W_0(x', \hat{y})] \text{ s.t. } \mathbb{E}_{x' \sim X'}[x'] = \hat{x} \\
&= \max \sum_{i=1}^{m} p_i W_0(x_i, \hat{y}) \quad \text{ s.t } \sum_{i=1}^{m} p_i x_i = \hat{x} \text{ and } p \in \Delta_m
\end{aligned}$$

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We prove this by contradiction. Suppose there exists $x^* \in \text{supp}(X')$, and $x^* \notin X^*$. Assume $x_i < x^* < x_{i+1}$ and $x^* = qx_i + (1-q)x_{i+1}$. We know that $W_0(x, \hat{y})$ restricted to (x_i, x_{i+1}) is linear in x, and therefore,

$$\begin{aligned} W_0(x^*, \hat{y}) =& W_0(qx_i + (1-q)x_{i+1}, \hat{y}) \\ \leq& q \lim_{x \to x_i^+} W_0(x, \hat{y}) + (1-q) \lim_{x \to x_{i+1}^-} W_0(x, \hat{y}) \\ \leq& qW_0(x_i, \hat{y}) + (1-q)W_0(x_{i+1}, \hat{y}). \end{aligned}$$

Hence we can substitute x^* with x_i w.p. q and x_{i+1} w.p. $1-q$.

Case $k = 2$: Fix $\hat{x}$ and consider the function $y \mapsto W_0(\hat{x}, y)$. For a fixed $\hat{x}$, Alice's action is uniquely determined, and Bob's ex-post IR condition is satisfied within a closed interval. Therefore, we conclude that $W_0(\hat{x}, y)$ restricted to $[y_j, y_{j+1}]$ is linear. Now, let Y' be the distribution in the solution of the optimization problem (**??**). We want to show that:

$$\begin{aligned} W_2(\hat{x}, \hat{y}) =& \max \mathbb{E}_{y' \sim Y'}[W_1(\hat{x}, y')] \text{ s.t. } \mathbb{E}_{y' \sim Y'}[y'] = y \\ =& \max \sum_{j=1}^{n} p_j W_1(\hat{x}, y_j) \quad \text{ s.t } \sum_{j=1}^{m} p_j y_j = \hat{y} \text{ and } p \in \Delta_n \end{aligned}$$

Suppose there exists $y^* \in \text{supp}(Y')$, and $y^* \notin Y^*$. Assume $y_j < y^* < y_{j+1}$ and $y^* = qy_i + (1-q)y_{i+1}$. We know that $W_0(\hat{x}, y)$ restricted to $[y_j, y_{j+1}]$ is linear in y, therefore,

$$\begin{aligned} W_1(\hat{x}, y^*) =& \sum_{i=1}^{m} p_i W_0(x_i, y^*) \\ =& q \sum_{i=1}^{m} p_i W_0(x_i, y_j) + (1-q) \sum_{i=1}^{m} p_i W_0(x_i, y_{j+1}) \\ \leq& qW_1(\hat{x}, y_j) + (1-q)W_1(\hat{x}, y_{j+1}). \end{aligned}$$

Hence we can substitute y^* with y_j w.p. q and y_{j+1} w.p. $1-q$.

Case $k \geq 3$: We now consider the case where $k \geq 3$. We take the case where k is an odd number as an example. The proof for the even case is entirely similar. We assume the base condition:

$$\begin{aligned} W_{k-1}(\hat{x}, \hat{y}) =& \max \sum_{j=1}^{n} p_j W_{k-2}(\hat{x}, y_j) \\ & \text{s.t} \quad \sum_{j=1}^{m} p_j y_j = \hat{y} \quad \text{and} \quad p \in \Delta_n \end{aligned}$$

$$\begin{aligned} W_{k-2}(\hat{x}, \hat{y}) =& \max \sum_{i=1}^{m} p_i W_{k-3}(x_i, \hat{y}) \\ & \text{s.t} \quad \sum_{i=1}^{m} p_i x_i = \hat{x} \quad \text{and} \quad p \in \Delta_m \end{aligned}$$

From the property of concave closure, the second condition gives $W_{k-2}(x, y_j)$ restrict to $[x_i, x_{i+1}]$ is linear. We want to prove

$$\begin{aligned} W_k(\hat{x}, \hat{y}) =& \max \mathbb{E}_{x' \sim X'}[W_{k-1}(x', \hat{y})] \text{ s.t. } \mathbb{E}_{x' \sim X'}[x'] = \hat{x} \\ =& \max \sum_{i=1}^{m} p_i W_{k-1}(x_i, \hat{y}) \quad \text{ s.t } \sum_{i=1}^{m} p_i x_i = \hat{x} \text{ and } p \in \Delta_m \end{aligned}$$

We prove by contradiction. Suppose there exists $x^* \in \text{supp}(X')$, and $x^* \notin X^*$. Assume $x_i < x^* < x_{i+1}$ and $x^* = qx_i + (1-q)x_{i+1}$. The key observartion of the proof is:

$$\begin{aligned} W_{k-1}(x^*, \hat{y}) =& \sum_{j=1}^{n} p_j W_{k-2}(x^*, y_j) \\ =& q \sum_{j=1}^{n} p_j W_{k-2}(x_i, y_j) + (1-q) \sum_{j=1}^{n} p_j W_{k-2}(x_{i+1}, y_j) \\ \leq& qW_{k-1}(x_i, \hat{y}) + (1-q)W_{k-1}(x_{i+1}, \hat{y}). \end{aligned}$$

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Hence we can substitute x^* with x_i w.p. q and x_{i+1} w.p. $1-q$.

By the induction hypothesis, assume $y_j < \hat{y} < y_{j+1}$ and $\hat{y} = q' y_j + (1-q') y_{j+1}$. Then W_{k-1} is linear in y on both segments $\{x_i\} \times [y_j, y_{j+1}]$ and $\{x_{i+1}\} \times [y_j, y_{j+1}]$. Consequently,

$$W_{k-1}(x_i, \hat{y}) = q' \, W_{k-1}(x_i, y_j) + (1-q') \, W_{k-1}(x_i, y_{j+1}),$$
$$W_{k-1}(x_{i+1}, \hat{y}) = q' \, W_{k-1}(x_{i+1}, y_j) + (1-q') \, W_{k-1}(x_{i+1}, y_{j+1}).$$

Therefore, throughout the recursive step, it suffices to track of the values on the grid $X^* \times Y^*$ only. □

J.4 Proof of Theorem 3.1

Proof. Using Lemma 3.2 we can construct the sets X^* and Y^* in polynomial time, where each set has size $O(|R|)$.Using the definition of W_0 we can construct a table that stores the value of W_0 on $X^* \times Y^*$. Note that $q_A^0 \in Y^*$ (See proof of Lemma K.1). If $q_B^0 \in X^*$ then we can simply use the algorithm in Lemma 3.3 we can construct each W_{k-1} from W_k in this set: $X^* \times Y^*$.

If $q_B \notin X^*$ and t is even (which means the updation in the final round is along vertical direction), we only need to modify the computation of the algorithm in the final step (the t-th round) as follows:

$$W_t(q_B^0, q_A^0) = \max \sum_{j=1}^{n} p_j W_{t-1}(q_B^0, y_j)$$
$$\text{s.t} \quad \sum_{j=1}^{m} p_j y_j = q_A^0 \quad \text{and} \quad p \in \Delta_n.$$

If t is odd, by the computation of W_t, we know that W_t is linear in x on segment $[x_i, x_{i+1}] \times q_A^0$ (assume $x_i < q_B^0 < x_{i+1}$ and $q_B^0 = qx_i + (1-q)x_{i+1}$), hence:

$$W_t(q_B^0, q_A^0) = qW_{t-1}(x_i, q_A^0) + (1-q)W_{t-1}(x_{i+1}, q_A^0).$$

□

K Proof of Theorem 4.3

Prior to the proof, we establish a few preliminary lemmas.

Lemma K.1. *Consider a Bayesian conversation in which both Alice and Bob have two types, i.e., $|\Theta_A| = |\Theta_B| = 2$, and Alice has two available actions, i.e., $|R| = 2$. Then the algorithm described in Theorem 3.1 is equivalent to performing alternating concavification on a mesh with at most 4 columns and at most 5 rows.*

Proof. We consider $|X^*|$ and $|Y^*|$.

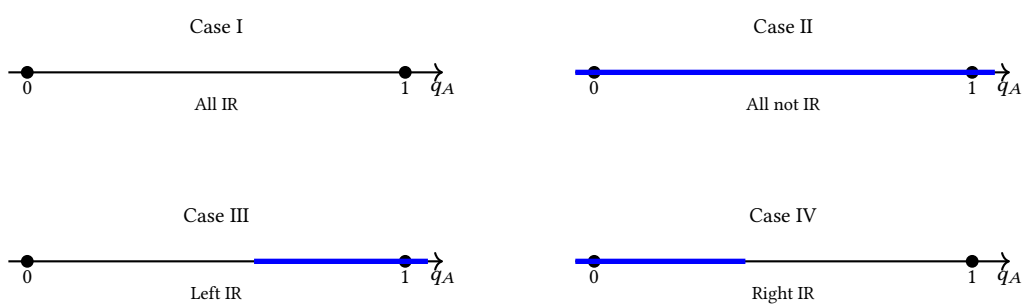
Since Alice has only 2 available actions, this implies that for each type of Alice, her chosen action can switch at most once as q_B varies from 0 to 1. As there are two types of Alice and we include both endpoints 0 and 1, it follows that $|X^*| \leq 4$.

We consider the case where $|X^*| = 4$, meaning that each of the two types of Alice switches her action at a different value of q_B. In this case, X^* partitions the interval $q_B \in [0, 1]$ into 3 **segments**. We analyze a fixed type of Bob and examine the structure of his IR region within each of these 3 segments.

For a given type θ_B of Bob, let v_0 denote his expected payoff from not participating in the conversation. Within a single segment, the actions of both types of Alice are fixed; suppose that θ_{A1} and θ_{A2} choose actions r_1 and r_2, respectively. Denote $u_B(\theta_{A1}, \theta_B, r_1)$ as v_1, $u_B(\theta_{A2}, \theta_B, r_2)$ as v_2, then Bob's IR region in this segment is:

$$\{q_A \mid q_A \cdot v_1 + (1-q_A) \cdot v_2 \geq v_0\}$$

By marking the non-IR regions in blue, the IR region can be classified into the following 4 cases:



Lemma K.2. *For a fixed type of Bob, it is impossible for the IR regions in two adjacent segments to correspond to Case III and Case IV, respectively.*

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Proof. Case III corresponds to $v_1 < v_0$ and $v_2 > v_0$, while Case IV corresponds to $v_1 > v_0$ and $v_2 < v_0$. However, between two adjacent segments, only one type of Alice changes her action, so it is not possible for both v_1 and v_2 to change. □

We now show that $|Y^*| \leq 5$.

In each segment, the final IR region is given by the intersection of the IR regions corresponding to the two types of Bob. The two endpoints of this region both belong to Y^*. The 3 segments contribute at most 6 elements to Y^*.

We begin by focusing on the segment that contains the prior (q_B^0, q_A^0). In this segment, q_A^0 must belong to Y^*, since for both type of Bob, the value v_0 is defined based on this point:

$$v_0 = q_A^0 \cdot u_B(\theta_{A1}, \theta_B, r_{10}) + (1 - q_A^0) \cdot u_B(\theta_{A2}, \theta_B, r_{20}).$$

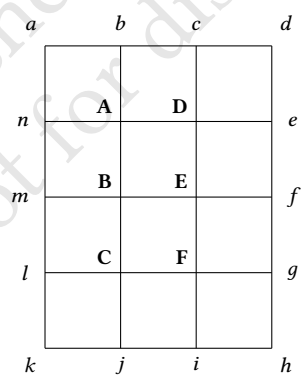
- If in this segment, the IR regions for the two types of Bob correspond to Case III and Case IV, respectively, then the final IR region reduces to the single point q_A^0. In this case, we clearly have $|Y^*| \leq 5$.
- If the IR regions for the two types of Bob in this segment are of the same type, without loss of generality, suppose they correspond to Case III. Then 0 is one endpoint of the IR interval in that segment. We consider a segment adjacent to this one.
  - If the IR region of one type of Bob in the adjacent segment corresponds to Case II, then the entire segment is non-IR and contributes nothing to Y^*, so we have $|Y^*| \leq 4$.
  - If the IR regions of both types of Bob in the adjacent segment do not correspond to Case II, then they must correspond to either Case I or Case III. In either case, $q_A = 0$ is guaranteed to be in the IR region, so 0 is one endpoint of the IR interval in that segment. Since both the segment in which the prior falls and this adjacent segment have 0 as an endpoint of their IR regions, it follows that $|Y^*| \leq 5$.

At $q_B = 0$ and $q_B = 1$, the IR region is not the intersection of the IR regions of the two types of Bob, but rather the IR region of the corresponding type alone. However, this does not affect the result, since the points in the boundary columns become fixed after the 2nd round and no longer influence the updates of other points.

□

Lemma K.3. *Consider a 4×5 mesh of points. Suppose we perform alternating concavification along the two coordinate directions. Then the process either terminates in at most 6 steps, or there exist points whose values strictly increase with each iteration and require infinitely many updates to reach their optimal values. In the latter case, the convergence is linear.*

The complete proof will be given later in the paper. We first establish several basic properties concerning alternating concavification on a 4×5 mesh. For convenience, we label each point on the 4×5 grid with a letter as shown below.



Lemma K.4. *Starting from the 3rd iteration of the concavification process, only the values of the points in the central 2×3 subgrid are updated.*

Proof. This is because the updates of the boundary rows and columns are independent of the values of the interior points. As a result, the values on the boundary rows and columns remain unchanged after the first 2 concavification steps. □

Corollary K.1. *Starting from the 4th concavification step,*

- *During each odd-numbered (horizontal) update, it is impossible for both 2 interior points in any row to be updated simultaneously;*
- *During each even-numbered (vertical) update, it is impossible for all 3 interior points in any column to be updated simultaneously.*

Proof. Otherwise, this would imply that at least one of the two endpoints of that row or column was updated in the previous iteration. □

Definition K.1 (Update Arrow). *For convenience of exposition, we introduce the notion of* update arrows. *For each concavification step, we represent an update by a directed arrow from point A to point B, where B is a point whose value is updated in the current step, and A is a point whose value was updated in the previous step and directly caused the update at B.*

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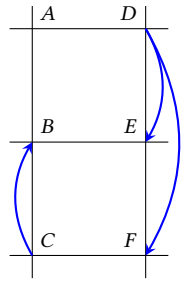
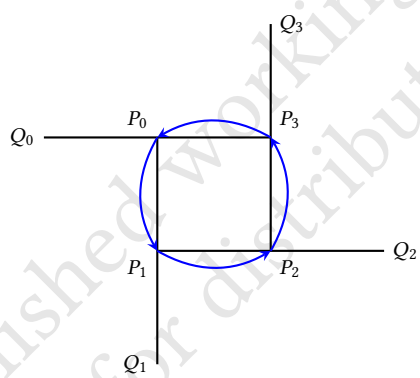


Figure 11: Example of update arrows for the 4th (vertical) concavification step. This indicates that in the 3rd round, the values at points C and D were updated, and in the 4th round, the update at D caused updates at E and F, while the update at C caused an update at B.

Lemma K.5. *Let a sequence of concavification steps be applied to a 4×5 mesh. Suppose that for some integer $k \geq 4$ and positive integer n, the update diagrams from round k to round $k + n - 1$ form a directed cycle. Then:*

- *There exist points whose values require infinitely many iterations to approach their maximal values.*
- *The convergence of these values is linear.*

Proof. We first examine how the values of the points evolve after the cycle is formed. We begin by analyzing the following simple case.



Case $n = 4$. Firstly we consider the case where the cycle is formed from round k to round $k + 3$; specifically, for integer $p \in \{0, 1, 2, 3\}$, only the arrow $P_{p \bmod 4} \to P_{(p+1) \bmod 4}$ is formed in round $k + p$.

Consider round k. After this round, the values at the points P_0, P_1, and Q_1 become linear in the y-direction. This linear relationship remains unchanged during rounds $k + 1$ and $k + 2$. In round $k + 3$, the value at P_0 increases. Therefore, in round $k + 4$, there will still be an arrow from P_0 to P_1. By repeating this argument, we see that for any integer $p \geq 0$, arrow $P_{p \bmod 4} \to P_{(p+1) \bmod 4}$ is formed in round $k + p$. This indicates that an infinite number of iterations is required.

In terms of quantitative relations, let $v(M, k)$ denote the value of point M at round k. Since the value at each point Q_x remains constant throughout the process, we use the simplified notation $v(Q_x)$ to denote its value. Then we have, for any integer m,

$$\begin{aligned}
v(P_1, k + 4m + 1) &= t_0 v(P_0, k + 4m) + (1 - t_0) v(Q_1), \\
v(P_2, k + 4m + 2) &= t_1 v(P_1, k + 4m + 1) + (1 - t_1) v(Q_2), \\
v(P_3, k + 4m + 3) &= t_2 v(P_2, k + 4m + 2) + (1 - t_2) v(Q_3), \\
v(P_0, k + 4m + 4) &= t_3 v(P_3, k + 4m + 3) + (1 - t_3) v(Q_0).
\end{aligned}$$

Here, for all $x \in \{0, 1, 2, 3\}$, the value of t_x is only determined by the grid spacing, and $t_x \in (0, 1)$.

Multiplying the above four equations, we obtain for any integer m:

$$v(P_x, k + 4m + 4) = T_x v(P_x, k + 4m) + c_x.$$

Here, for all $x \in \{0, 1, 2, 3\}$, the value of T_x and c_0 is only determined $t_0 \sim t_3$ and $c_0 \sim c_3$, and $T_x \in (0, 1)$.

Since

$$v(P_0, \infty) = T_0 v(P_0, \infty) + c_0 \Rightarrow v(P_0, \infty) = \frac{c_0}{1 - T_0}.$$

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we know the error

$$\begin{aligned} & v(P_0, k+4m+4) - v(P_0, \infty) \\ =& T_0 v(P_0, k+4m) + c_0 - v(P_0, \infty) \\ =& T_0(v(P_0, k+4m) - v(P_0, \infty)). \end{aligned}$$

Therefore, the convergence rate is linear. The same applies to the other points P_1 through P_3.

Case $n = 2$. We consider the case where the cycle is formed within 2 rounds, namely rounds k and $k+1$. Specifically, in round k, the arrows $P_0 \to P_1$ and $P_2 \to P_3$ are formed, and in round $k+1$, the arrows $P_1 \to P_2$ and $P_3 \to P_0$ are formed.

After round k, the values at P_0, P_1, Q_1 and at P_2, P_3, Q_3 satisfy a linear relationship in the y-direction. In round $k+1$, the values of P_0 and P_2 are the only ones that increase. Therefore, in round $k+2$, the arrows $P_0 \to P_1$ and $P_2 \to P_3$ will appear again.

By iterating the above reasoning, it follows that for every integer $m \geq 0$, the update diagram at round $k+2m$ includes the arrows $P_0 \to P_1$ and $P_2 \to P_3$, and the diagram at round $k+2m+1$ includes the arrows $P_1 \to P_2$ and $P_3 \to P_0$. This indicates that an infinite number of iterations is required.

As for quantitative relations, we have, for any integer m,

$$\begin{aligned} v(P_1, k+2m+1) &= t_0 v(P_0, k+2m) + (1-t_0) v(Q_1), \\ v(P_3, k+2m+1) &= t_2 v(P_2, k+2m) + (1-t_2) v(Q_3), \\ v(P_0, k+2m+2) &= t_3 v(P_3, k+2m+1) + (1-t_3) v(Q_0), \\ v(P_2, k+2m+2) &= t_1 v(P_1, k+2m+1) + (1-t_1) v(Q_2). \end{aligned}$$

Here, for all $x \in \{0,1,2,3\}$, the value of t_x is only determined by the grid spacing, and $t_x \in (0,1)$.

We also have, for any integer $x \in \{0,1,2,3\}$, any integer m,

$$v(P_x, k+4m+4) = T_x v(P_x, k+4m) + c_x.$$

Here, for all $x \in \{0,1,2,3\}$, the value of T_x and c_x is only determined $t_0 \sim t_3$ and $c_0 \sim c_3$, and $T_x \in (0,1)$.

Therefore, we also conclude that the convergence rate in this case is linear.

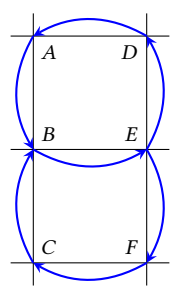
Remark K.1. *The case $n = 2$ can be viewed as two simultaneous cycles of the type in the $n = 4$ case, with starting points P_0 and P_2, respectively.*

Based on the analysis of the preceding simple cases, we proceed to consider the case of the 4×5 grid.

According to Lemma 0.2, starting from the 4th round, the boundary rows and columns can not be updated. Therefore, any vertex involved in a cycle must lie in the interior—that is, among the points labeled with uppercase letters.

We classify the discussion according to the vertices of the directed cycle. Up to symmetry, there are 3 distinct cases to consider:

Case I: Two cycles with the shared edge $B \to E$.



We show that this case is impossible. It suffices to show that the following configuration cannot occur: in round k ($k \geq 4$), the arrows $A \to B$ and $C \to B$ appear, and in round $k+1$, the arrow $B \to E$ appears.

Suppose that in round k , the arrows $A \to B$ and $C \to B$ appear. Then, at the end of round $k-1$, the values at n, A, D and those at l, C, F each satisfy a linear relationship in the x-direction. The update at round k can be viewed as computing a "new value" by taking a linear combination of the values from the n–e row and the l–g row at the end of round $k-1$, and then comparing it to the "old value" on the m–f row, keeping the maximum of the two.

Since the values along the a–k and e–g columns are concave in the y-direction, the "new value" at points m and f cannot exceed the corresponding "old value". Moreover, the m–f row is concave in the x-direction at the end of round $k-1$.

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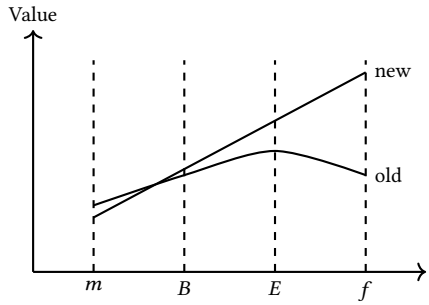

On the interval from m to E, since point B is updated, the new value at B is greater than the old one, while the new value at m is no greater than the old value. As the new values form a linear segment and the old values are concave, it follows that E is also updated to the new value. This implies that round k also contains the arrows $D \to E$ and $F \to E$, which indicates that, prior to round k, the values along the n–e and l–g rows are both linear. As a result, point f would also be updated. This leads to a contradiction.

Case II: A single cycle $A \to B \to E \to D \to A$.

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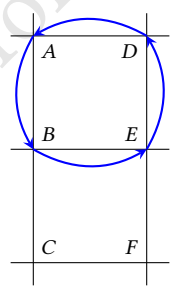
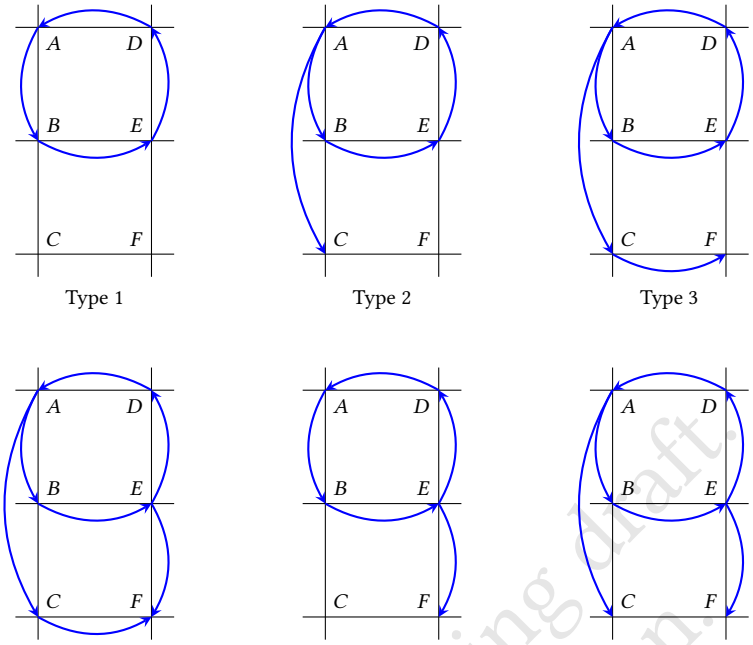


We firstly show that, in all rounds after the cycle has formed, the arrow $C \to B$ do not occur. We simply need to show that the route $A \to B \to E \to F \to C$ does not exist. Otherwise, when $A \to B$, the value from A to C is linear with respect to y-axis. Then when $F \to C$, there must be $C \to B$ in the next step, forming cycle $B \to E \to F \to C \to B$, which contradicts there exists only 1 cycle $A \to B \to E \to D \to A$.

We consider the update diagrams over one full cycle (4 rounds) in the case where $n = 4$. There are six possible configurations, categorized as follows:



Type 1 Type 2 Type 3

Type 4 Type 5 Type 6

We analyze the possible types that may arise in the next cycle after completing one full cycle.

- **Type 1:** When the iteration follows Type 1, the value at point B increases with each iteration. As the cycle progresses, it is possible that a convex combination of the values at A and point j eventually exceeds the value at C, leading to the appearance of the arrow $A \to C$. The increase at point C may then immediately trigger $C \to F$ in the next round. Similar cases may occur for other arrows as well.
  Therefore, after evolving according to Type 1 in one cycle, the next cycle may either remain in Type 1 or transition to one of Types $2 \sim 6$. In particular, if $A \to C$ and $C \to F$ is triggered, and then the combination of the updated values at F and C surpasses that at E, the iteration transitions to Type 7 in Case III.
- **Type 2 & 5:** By the same reasoning as in Type 1, a cycle of Type 2 or Type 5 may transition in the next round to Type 3, 4, 6, or 7. In particular, the transition from Type 5 to Type 3 can be explained as follows: the appearance of the arrow $C \to F$ increases the value at point F, which may in turn prevent the arrow $E \to F$ from appearing. If this occurs, the cycle transitions to Type 3; otherwise, it becomes Type 4.
- **Type 3:** We prove that if a cycle is of Type 3, then the next cycle must also be of Type 3 and cannot transition to any other type. Specifically, we show that neither $E \to F$ nor $F \to E$ can occur in the next cycle.
  We focus on the current cycle and let the step in which $A \to B$ appears be round $k+1$. Then we have:

$$\begin{aligned} v(B,k+1) &= t_{AB}v(A,k) + (1-t_{AB})v(j), \\ v(C,k+1) &= t_{AC}v(A,k) + (1-t_{AC})v(j), \\ v(E,k+2) &= t_{BE}v(B,k+1) + (1-t_{BE})v(f), \\ v(F,k+2) &= t_{CF}v(C,k+1) + (1-t_{CF})v(g). \end{aligned}$$

  Then

$$\begin{aligned} v(E,k+2) &= t_{AB}t_{BE}v(A,k) + c_1, \\ v(F,k+2) &= t_{AC}t_{CF}v(A,k) + c_2. \end{aligned}$$

  Since in the current cycle, there is no $E \to F$ and $F \to E$, we have

$$\begin{aligned} & v(E,k+2) \geq t_{FE}v(F,k+2) + (1-t_{FE})v(d) \\ \Rightarrow & (t_{AB}t_{BE} - t_{AC}t_{CF}t_{FE})v(A,k) \geq C_1 \\ & v(F,k+2) \geq t_{EF}v(E,k+2) + (1-t_{EF})v(i) \\ \Rightarrow & (t_{AC}t_{CF} - t_{AB}t_{BE}t_{EF})v(A,k) \geq C_2, \end{aligned}$$

  where C_i is a constant determined by the values at outer points, and for any t, $t \in (0,1)$.

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Since $t_{AB} > t_{AC}, t_{AB}t_{EF} = t_{AB}t_{BC} > t_{AC}$, and $t_{BE} = t_{CF}$, we know

$$t_1 := t_{AB}t_{BE} - t_{AC}t_{CF}t_{FE} > 0$$
$$t_2 := t_{AC}t_{CF} - t_{AB}t_{BE}t_{EF} > 0$$

and because for any integer m, $v(A, k+4m) > v(A, k)$, we have $t_1 v(A, k+4m) \geq C_1$ and $t_2 v(A, k+4m) \geq C_2$ holds for any integer m, hence $E \to F$ and $F \to E$ cannot occur in the following cycles, which means the cycle remains Type 3.

- **Type 6:** We prove that if a cycle is of Type 6, then the next cycle must also be of Type 6 and cannot transition to any other type. Specifically, we show that $C \to F$ can not occur in the next cycle. ($F \to C$ can not occur either, as is proven above.)
  We focus on the current cycle and let the step in which $E \to F$ and $E \to D$ appears be round $k+1$. Then we have:

$$v(F, k+1) = t_{EF}v(E, k) + (1 - t_{EF})v(i),$$
$$v(D, k+1) = t_{ED}v(E, k) + (1 - t_{ED})v(c),$$
$$v(A, k+2) = t_{DA}v(D, k+1) + (1 - t_{DA})v(n),$$
$$v(C, k+3) = t_{AC}v(A, k+2) + (1 - t_{AC})v(j).$$

  Then

$$v(F, k+1) = t_{EF}v(E, k) + c_1$$
$$v(C, k+3) = t_{ED}t_{DA}t_{AC}v(E, k) + c_2$$

  Since in the current cycle, there is no $C \to F$, we have

$$v(F, k+1) \geq t_{CF}v(C, k+3) + (1 - t_{CF})v(g)$$
$$\Rightarrow (t_{EF} - t_{ED}t_{DA}t_{AC}t_{CF})v(E, k) \geq C.$$

  , where C is a constant determined by the values at outer points, and for any t, $t \in (0, 1)$.
  Since $t_{EF} > t_{AC}$, we know

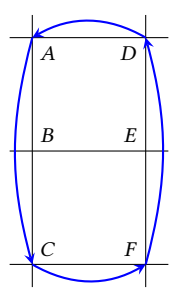
$$t := t_{EF} - t_{ED}t_{DA}t_{AC}t_{CF} > 0$$

  and because for any integer m, $v(E, k+4m) > v(E, k)$, we have $tv(A, k+4m) \geq C$ holds for any integer m, hence $C \to F$ cannot occur in the following cycles, which means the cycle remains Type 6.
- **Type 4:** For Type 4, the alternating appearance of $C \to F$ and $E \to F$ may result in one preventing the other from forming. As a result, Type 4 may evolve into Type 3 or Type 6.

We now turn to the case of $n = 2$. In the case of $n = 4$, the process can essentially be viewed as the value at point A being updated once every four rounds. Based on the amount by which A increases, one can determine the subsequent evolution. The case of $n = 2$ can be interpreted as two four-round cycles running in parallel, each occupying half of the full loop.

This means that the value at A is updated once every two rounds, which in turn drives the evolution. For each individual four-round cycle, the possible update diagrams remain the same as the cases listed above, and the directionality of the evolution remains unchanged, since the value at point A always increases monotonically.

Case III: A single cycle $A \to C \to F \to D \to A$.



Type 7

If no arrow appears along the edge BE during the update process, then the evolution proceeds exactly as in the initial simple case.

If an update does occur along BE, without loss of generality, suppose it is $B \to E$. Since E is updated, the arrow $F \to D$ in the next round may be blocked as a result.

- If the arrow is not blocked, the cycle proceeds as before.
- If it is blocked, then the next round of updates can lead to two possibilities:

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- If only $E \to D$ appears, the system transitions to Type 3;
- If both $E \to D$ and $E \to F$ appear, it transitions to Type 4.

In conclusion, we can summarize the evolutionary relationships of the 7 types in Figure 12.

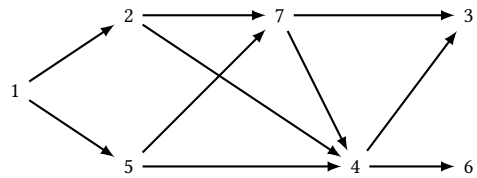

Figure 12: Transition diagram of types. An arrow $i \to j$ indicates that Type i can evolve into Type j. During the evolution process, a type may transition to any other type reachable via a directed path. (Except that Type 7 cannot directly evolve into Type 6.)

□

With the above groundwork in place, we are now ready to prove Lemma K.3.

Proof of Lemma K.3. It suffices to show that if some point is still updated in round 7, then the update diagram must form a cycle, and hence the process cannot terminate in finite steps.

By symmetry, it suffices to consider the cases in which point A or point B is updated in round 7.

Case 1: Point A is updated in round 7. By Lemma 0.2, we know that starting from round 4, if a point is updated, it must be caused by an update to some point in the central 2×3 region in the previous round. In other words, the updated point in the current round must have an incoming arrow from a point in the central 2×3 region.

Taking into account that odd-numbered rounds apply horizontal updates and even-numbered rounds apply vertical updates, it follows that in round 7 there must be an arrow $D \to A$.

(1) $E \to D$ in round 6.
This implies that $B \to E$ is in round 5.
  A. $A \to B$ in round 4.
  Then cycle $A \to B \to E \to D \to A$ is formed.
  B. $C \to B$ in round 4.
  From the proof of Theorem 0.5, we know that such a path $C \to B \to D \to E \to A$ directly leads to the formation of the arrow $A \to B$ in round 8. As a result, the cycle $B \to E \to D \to A \to B$ is formed.

(2) $F \to D$ in round 6.
This implies that $F \to E$ in round 6, and $C \to F$ in round 5.
  A. $A \to C$ in round 4.
  Then cycle $A \to C \to F \to D \to A$ is formed.
  B. $B \to C$ in round 4.
  This indicates B is updated in round 3, hence the value of m, B, E is linear in x-direction after round 3.
    a. B is not updated in round 5.
    Since in round 6 E is updated, then $E \to B$ must be in round 7, therefore cycle $B \to C \to F \to E \to B$ is formed.
    b. B is updated in round 5, i.e. $E \to B$ in round 5.
    Since $B \to C$ in round 4, B is updated in round 5, $B \to C$ must be in round 6. Because $C \to F$ is in round 5, $F \to E$ is in round 6, hence cycle $B \to C \to F \to E \to B$ is formed.

Case 2: Point B is updated in round 7. This indicates $E \to B$ in round 7. From the proof of Theorem 0.5, we know that $D \to E$ and $F \to E$ cannot exist simultaneously in round 6. WLOG we assume $D \to E$ in round 6. Hence $A \to D$ is in round 5.

(1) $B \to A$ in round 4.
Then cycle $B \to A \to D \to E \to B$ is formed.

(2) $C \to A$ in round 4.
The value of B, A, b is linear in y-direction after round 4.
  A. A is not updated in round 6.
  Since in round 7, B is updated, $B \to A$ must be in round 8. Therefore cycle $A \to D \to E \to B \to A$ is formed.
  B. A is updated in round 6.
    a. $B \to A$ in round 6.
    In this case $E \to B$ must be in round 5. Therefore a cycle $A \to D \to E \to B \to A$ is formed.

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b. $C \to A$ in round 6.
In this case $F \to C$ must be in round 5, which indicates the value of E, F, i is linear in y-direction after round 5. Since $D \to E$ is in round 6, $D \to F$ must be in round 6, too. Therefore a cycle $A \to D \to F \to C \to A$ is formed.

In conclusion, in both cases, a cycle is formed, and hence the process cannot terminate in finitely many steps, which completes the proof. □

Lemma K.6. *For a given 4×5 mesh, if the update diagrams form a cycle during the alternating concavification process, then there exists a polynomial-time algorithm that determines the type to which the update diagram stabilizes.*

Proof. Given the current type, one can compute a closed-form expression for the value at each point as a function of the iteration number. Once these expressions are obtained, they can be used to check whether the conditions for a type transition are satisfied.

- If there doesn't exist an iteration satisfies the transition condition, then the current type is the final stable type.
- If there exists an iteration in which a transition occurs (i.e., a new arrow inconsistent with the current type appears), then a full cycle is simulated starting from that round to determine the resulting type. The process then restarts from the beginning with this new type.

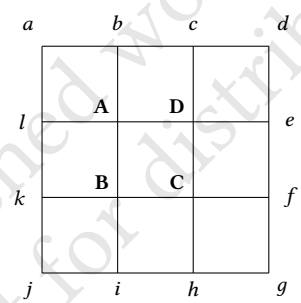
Since the total number of type transitions is bounded (as shown in Figure 12), the number of iterations is finite, and the final stable type can be determined in polynomial time.

□

Analogous to the 4×5 mesh, the 4×4 mesh exhibits a similar property as follows:

Lemma K.7. *Consider a 4×4 mesh of points. Suppose we perform alternating concavification along the two coordinate directions. Then the process either terminates in at most 5 steps, or there exist points whose values strictly increase with each iteration and require infinitely many updates to reach their optimal values. In the latter case, the convergence is linear.*

Proof. Similar to the 4×5 case, the boundary points stop updating after the 3rd round. Therefore, starting from the 3rd round, it suffices to consider only the interior 2×2 points.



We show that if any point is updated in the 6th round, then a cycle must have formed—identical to the simple case analyzed in the proof of Lemma K.5—which implies that the iteration cannot terminate in finitely many rounds.

Since points A, B, C, and D are symmetric, we may assume without loss of generality that point D is updated in round 6. That is, the update $C \to D$ occurs in round 6, which implies $B \to C$ in round 5 and $A \to B$ in round 4.

This means that prior to round 3, the values at points l, A, and D must lie on a line in the x-direction.

- If point A is not updated in round 5, then the values at l, A, and D remain linear in the x-direction before round 6. Since D is updated in round 6, it follows that $D \to A$ occurs in round 7. Therefore, a cycle $A \to B \to C \to D \to A$ is formed over rounds 4–7.
- If point A is updated in round 5, i.e., $D \to A$ occurs, then $C \to D$ must have occurred in round 4. Thus, a cycle $A \to B \to C \to D \to A$ is formed over rounds 4 and 5.

□

For finite meshes with at most 3 rows or columns, we show that no instance requires infinitely many rounds to converge.

Lemma K.8. *Consider a finite mesh of points with at most 3 columns or rows. Suppose we perform alternating concavification along the two coordinate directions. Then the process terminates in at most 4 steps.*

Proof. As argued previously, the boundary points stop changing after 2 rounds of updates.

Therefore, when the mesh has only 1 or 2 columns (or rows), the claim is immediate.

If the mesh has 3 columns (or rows), then at round 5, no point in the middle column can be updated. Otherwise, it would imply that some point in the boundary columns (rows) was updated in round 4, which is impossible.

□

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Finally, we use all the foundational results above to prove the main theorem stated at the beginning of this section.

Proof of Theorem 4.3. According to the algorithm in Theorem 3.1, we only need to track the value changes of a single point in the mesh as alternating concavification is applied. By prior analysis, q_A^0 must lie in Y^*, meaning the point $P(q_B^0, q_A^0)$ representing the prior is located on one of the rows of the constructed mesh.

If P lies on the boundary of the mesh, then the value converges to the maximum within two rounds.

If P is not on the boundary:

- If the mesh has at most 3 rows or columns, the process converges within 4 rounds.
- If the mesh is 4×4:
  - If alternating concavification terminates within 5 rounds, then the social welfare also achieves its maximum.
  - If it does not terminate in finitely many rounds, then the value along every edge adjacent to points A–D grows indefinitely, and the value at point P converges linearly to the maximum.
- If the mesh is 4×5:
  - If alternating concavification terminates within 6 rounds, then the social welfare achieves its maximum.
  - If it does not terminate in finitely many rounds, we apply the algorithm from Theorem 0.9 to determine the eventual stable update type.
    - If the stable type is Type 3, 4, 6, or 7, then since all six interior points (A–F) are updated in every round, all edges connected to them are updated as well. Since P must lie on one of these edges, the social welfare converges linearly to the maximum.
    - If the stable type is Type 1, 2, or 5, then certain edges never get updated:
      - Type 1: edges lC, CF, and Fg,
      - Type 2: edge Fg,
      - Type 5: edge lC.

      These edges remain unchanged even after cycles form.
      - If P lies on one of these non-updating edges, then the social welfare reaches its maximum within 6 rounds.
      - Otherwise, the social welfare converges linearly to the maximum.

In summary, in all of the above cases, the value at point P either stops updating within 6 rounds or converges linearly to its maximum value over infinitely many rounds. Note that all the cases in the above case analysis can be checked in polynomial time, and therefore this constitutes a polynomial-time algorithm. Hence, the proposition is proven.

□

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