Bayesian Conversations

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Abstract

We initiate the study of Bayesian conversations, which model interactive communication between two strategic agents without a mediator. We compare this to communication through a mediator and investigate the settings in which mediation can expand the range of implementable outcomes.

We look into the eventual outcome of two-player games after interactive communication. We focus on games where only one agent has a non-trivial action and examine the performance of communication protocols that are individually rational (IR) for both parties. Our key findings reveal that for ex-ante IR the expected social welfare achievable through a mediator protocol are equivalent to that achievable through unmediated Bayesian conversations. For ex-post IR, we observe a gap in the achievable welfare of the two protocols. We also establish that the optimal welfare under ex-post IR Bayesian conversation may require infinitely many rounds of communication. Additionally, we provide characterizations of which distributions over posteriors are achievable via Bayesian conversations.

1 Introduction

Much attention has been paid to the topic of strategic communication in recent years, in both Computer Science and Economics. An example is Bayesian persuasion ([26], [21]). In most previous work, the communication is unidirectional: it flows from senders to receivers. In this paper we focus on settings that resemble conversations: two agents talk back and forth, where each message depends not only on their private information but on the transcript of the communication so far. Such conversations arise in various settings of interest: a customer negotiating a price with a merchant, a worker negotiating their salary with a firm, prosecution and defense attorneys in a courtroom trial, political debates between two candidates, job interviews, etc. All of these settings are characterized by a long interactive conversation whose outcome will determine the payoffs of the parties involved.

In addition, the widespread use of the Internet has led to an increase in online conversations, many of which take place on third-party platforms where monitoring and verification of information is possible. Examples of this include online rental platforms that connect landlords and renters, online marketplaces like Amazon and Etsy, and home-sharing platforms like Airbnb. These platforms typically have a verification process for users and listings, as well as systems in place to monitor interactions and reviews.

In light of these observations, we propose the framework of *Bayesian conversations*. This framework is characterized by three distinct features: they are strategic, committed, and designed. The notion of interactive communication is not new and has been studied in many contexts, such as Communication Complexity [28, 36], Information Complexity [11] and Secure Multiparty Computation [8] in Computer Science and Cheap Talk [18, 6] and Bargaining [32] in Economics. We take a different viewpoint here that is characterized by the following aspects:

- **Strategic**: We consider agents that have payoffs and respond to incentives when deciding how to communicate. This is different from communication and information complexity, for example, where there is a common goal and we are trying to find the most efficient way to accomplish it.

- **Committed**: We assume that agents can commit to a communication protocol (e.g. they can send verifiable messages about their type), which is the key assumption in Bayesian Persuasion. This is in sharp contrast with cheap talk, where the communication is not binding.

- **Designed:** We take the mechanism design philosophy and explicitly design a communication protocol that will be followed by the agents (with some goal in mind, like maximizing one or both players' expected utility). This is in contrast with the literature on bargaining and trade [32] where we study protocols that naturally emerge from the agents' own choices.

Formally our setting consists of two agents (Alice and Bob) each of whom has a private type drawn from a known prior distribution: $\theta_A \sim \mathbb{P}(\theta_A)$ and $\theta_B \sim \mathbb{P}(\theta_B)$. Our central object of study will be an *unmediated Bayesian conversation* which is a protocol that specifies messages each agent sends in each round. The protocol has a finite length T and each round $t = 1 \dots T$ is associated with a space of messages A_t and B_t. A conversation specifies for each round,

- a randomized mapping from Alice's type θ_A and the history transcript $(a_1, b_1, \dots, a_{t-1}, b_{t-1})$ to a message in A_t;

- a randomized mapping from Bob's type θ_B and the history transcript $(a_1, b_1, \dots, a_{t-1}, b_{t-1}, a_t)$ to a message in B_t.

With each message sent by Bob, Alice updates her belief about Bob's type and vice versa. Conversations, therefore, can be viewed as ways to split the initial pair of beliefs $(\mathbb{P}(\theta_A), \mathbb{P}(\theta_B))$ to refined beliefs (q_A, q_B). See Examples 2.1 and 2.2 for concrete examples.

When exploring the potential of Bayesian conversations, two fundamental questions arise: What are the optimal outcomes that can be achieved through these conversations? And how powerful are Bayesian conversations as a class of communication protocols? Answering these questions is particularly challenging due to the inherent complexity of Bayesian conversations, which involve multiple rounds of signaling.

To shed light on these problems, we find it helpful to compare them with a simpler class of protocols, *mediator protocols*. In a mediator protocol, Alice and Bob both report their types to a trusted third-party: Martha the Mediator. Martha then outputs a signal in a set S using a randomized mapping from the pair of types (θ_A, θ_B) to S, and Alice and Bob will then update their beliefs based on the signal s. Mediator protocols are more general than Bayesian conversations in the sense that every distribution of beliefs that can be obtained by a Bayesian conversation can also be obtained using a mediator, since the mediator can always simulate the conversation and send the transcript as the signal. Therefore, mediated protocols serve as an upper bound of the usefulness of Bayesian conversations: if a Bayesian conversation can achieve the same outcome as the optimal mediated protocol, it must be an optimal Bayesian conversation. From a broader perspective, understanding the relationship between Bayesian conversations and mediated protocols provides valuable insights into the power of both protocols.

We thus investigate the following main question: When are mediator protocols more effective than Bayesian conversations? In particular, when is it worth hiring a mediator? It turns out that the answer to this question is subtle and it depends on how incentives are defined. Our paper is an exploration of this topic under different incentive models which we describe below.

We compare the power of communication protocols in terms of the eventual outcomes that can be reached. To that end, we will assume that a Bayesian game will be played after the conversation. In order to avoid issues like equilibrium selection, we will further assume that only one player (Alice) has a non-trivial action, and the payoffs depend on three things: (i) Alice's chosen action; (ii) Alice's type; and (iii) Bob's type. In this scenario, Alice's chosen strategy depends on both her type and her belief about Bob's type. Given that her belief about his type is a function of the communication protocol, the conversation will directly affect the payoff of both parties.

A protocol is designed by an external party and proposed to the agents. Since Alice is the only agent with a non-trivial action, she always weakly prefers to participate in the communication protocol – after all, she can always ignore the communication and choose an action based on her prior. We will consider two different notions[1] of voluntary participation for Bob. Both notions compare the outcome of the communication protocol with the outcome obtained when Alice plays only based on her type and her prior belief about Bob's type (i.e., when no conversation or signaling occurs):

- **Ex-ante individually rational (ex-ante IR)**: Before learning his type, Bob prefers executing the protocol. In this model, Bob takes expectations over his type, Alice's type and the randomness of the protocol.

[1] In the Appendix, we additionally consider an intermediate notion of individual rationality where the protocol must remain IR after Bob learns his type, but before participating in the conversation (*interim IR*). We also consider settings where Bob can deviate from participating in the protocol mid-conversation. See Section A for a discussion of additional results.

- **Ex-post individually rational (ex-post IR)**: Bob does not regret executing the protocol even after observing its final transcript. In this model, Bob takes expectations over Alice's type conditioned on his type and the final transcript of the protocol.

Note that these IR notions apply equally to both mediated protocols and Bayesian conversations.

1.1 Our Results

We now return to our original question: When are mediator protocols more effective than Bayesian conversations?

One way to answer this question is to notice that both mediator protocols and Bayesian conversations act by changing the players' belief about the other player's type – in particular, it causes each player's original prior over the other player's type to evolve into a distribution of posterior beliefs. Therefore, we can ask whether mediator protocols and Bayesian conversations can implement the same sets of distributions of posterior beliefs. It is not too hard to verify here (and follows from well-established results in the fields of information design and interactive communication) that these sets of posteriors are very different. In particular, mediator protocols can implement any mean-preserving spread of the original prior distribution (Proposition 2.2), whereas the posterior belief distributions implementable by Bayesian conversations are those than can be written as the limiting distribution of a far more constrained process known as a "bimartingale" (Proposition 2.3).

Despite this, in our first main result (Theorem 3.1), we show that when we restrict ourselves to the problem of designing ex-ante IR protocols, mediator protocols are *equivalent* to Bayesian conversations, in the sense that given any ex-ante IR mediator protocol, it is always possible to design an ex-ante IR Bayesian conversation (in fact, one with at most two rounds) where Alice and Bob achieve the same expected utilities. On the other hand, for ex-post IR protocols, there is a strict separation between mediator protocols and Bayesian conversations; we construct an example where the maximal social welfare achievable by an ex-post IR mediator protocol is strictly larger than the maximal social welfare achievable by an ex-post IR Bayesian conversation (Theorem 4.2).

This gap raises the algorithmic question of how to design optimal Bayesian conversations. For designing ex ante IR Bayesian conversations, the characterization in Proposition 2.2 of possible posterior belief distributions for mediator protocols allows us to efficiently construct the optimal such conversation via linear programming (Theorem 3.2). However, no such clean characterization exists for posterior beliefs that can be induced by a bimartingale. In our second main result, we provide a polynomial-time algorithm for computing the social-welfare maximizing r-round ex post IR Bayesian conversation when Alice and Bob have binary types (Theorem 4.1). This algorithm is a dynamic program based on the structural observation that the optimal social welfare of any r-round Bayesian conversation as a function of the starting prior can be described by its value on a fixed number of points polynomial in the size of the game and independent of r. We additionally provide an exponential-time algorithm for determining whether a specific posterior belief distribution is realizable by an r-round Bayesian conversation in the setting with more than two types (Theorem D.1).

Finally, we consider the value of additional rounds of communication in Bayesian conversations. As we have already mentioned, for the design of ex-ante IR Bayesian conversations, we need at most two rounds of communication to implement a protocol that achieves the optimal social welfare (or any other utility metric of our choice). Do multiple rounds of interaction help when designing ex-post IR Bayesian conversations? We show that, although there exist examples where the optimal ex-post Bayesian conversation requires only finitely many rounds of interaction (Section 4.2), there also exist examples where the social welfare of the optimal Bayesian conversation strictly increases as the number of rounds of interaction approaches infinity (Theorem 4.3).

1.2 Related Work

Economics literature. There exists a vast body of literature on information transmission in economics, with Bayesian persuasion [26] and cheap talk [22, 6] being two extensively researched models of communication. While our model shares connections with these two frameworks, it presents a fundamentally distinct framework: we consider *interactive* communication under *commitment* where *both* players hold private information.

Our paper is part of the growing body of literature that extends the Bayesian persuasion model of [26] to richer communication models. In the original model of persuasion, there is a sender and a receiver where the sender is more informed but the receiver is the one that takes a payoff-relevant action. This basic model is very well understood, both its strategic aspects [26, 9] and computational [20] aspects. The model has been extended to allow the possibility of multiple senders [23, 10], multiple receivers [2, 1, 4, 35], information intermediaries [31, 3, 37, 30] as well as communication along a network [13, 7, 29, 15, 16]. There is a wide variation of the type of communication in those papers: public vs private, constrained vs unconstrained, etc. A common element is that information flows in one direction: from more informed to less informed agents. It is not uncommon for agents to act both as senders and receivers, however in the network/mediator/intermediary model the communication typically flows $A \to B \to C$, i.e., Alice sends a message to Bob who sends a message to Charlie. Bob acts both as a sender and a receiver, but the communication still flows only in one direction. In our model, we consider two agents but neither is strictly more informed than the other. Their communication is both bi-directional and interactive.

There is a substantial body of literature that examines long interactive communication, such as the Long Cheap Talk model proposed by [6]. One key distinction of our work is that we assume players will commit to a communication protocol if they choose to participate, and our focus lies in exploring the outcomes achievable through interactive communication under commitment. To the best of our knowledge, previous studies do not make the assumption of player commitment to a protocol but instead analyze communication strategies within equilibria. The most relevant aspect connecting our research to the preceding Long Cheap Talk literature is our utilization of the "dimartingale" concept [24, 5, 6], which is defined as a martingale with three component where one of the first two components remains the same at each time step. Dimartingale plays an important role in the identification of Nash equilibria in long interactive communication (see [6] and subsequent literature). However, in this work, we employ dimartingales for a distinct purpose: finding the feasible distributions of two players' posterior beliefs about each other's type after a Bayesian conversation. To achieve this objective, we pinpoint a particular dimartingale that will characterize the set of feasible posterior belief distributions.

Another instance of interactive communication is the interactive bilateral trade model developed by [32]. However, in contrast to our approach, Mao's study adopts a mechanism design perspective, investigating communication protocols emerging as equilibria of a specific game. Additionally, [27] investigates information design in a long cheap talk game, but they do not assume player commitment as we do.

Computer science literature. The interactive communication is studied in Computer Science in settings such as Yao's communication complexity model [36], information complexity [11] and secure multi party computation [8]. Unlike our model, these settings do not have a strategic component.

Feasible posterior distributions. Finally, there is a line of work that investigates the set of possible joint posterior belief distributions of a group of agents [19, 4, 25, 17, 14]. In this work, we are also interested in the joint posterior belief distributions that can be generated by communication protocols. Our study differs from previous work in two ways: (1) previous studies have focused on multiple agents' beliefs about a common state, whereas our focus is on the beliefs of two agents about each other's private type; (2) previous studies consider arbitrary information structures, whereas we specifically consider the communication protocol of Bayesian conversations.

2 Problem Description and Preliminaries

We consider a two-player setting where both players (Alice and Bob) have private types, denoted by θ_A and θ_B respectively. We assume that their types are drawn independently from commonly known distributions with finite support, i.e., θ_A and θ_B are drawn from $\mathbb{P}(\theta_A) \in \Delta(\Theta_A)$ and $\mathbb{P}(\theta_B) \in \Delta(\Theta_B)$ independently, where $|\Theta_A|, |\Theta_B|$ are finite and these prior distributions are common knowledge.

2.1 Communication protocols

Before delving into the game-theoretic aspects of this problem, we begin by establishing the communication models via which Alice and Bob can share information about their types. In general, we will consider settings where Alice and Bob agree on the precise protocol of communication ahead of time; later, we will impose constraints on this protocol so that Alice and Bob are incentivized to participate in the protocol and not defect.

Broadly, we consider two different classes of protocols. In the first, we assume Alice and Bob alternate revealing information about their private types to one another via a pre-determined protocol. This could represent Alice and Bob communicating via a platform which sets the communication rules and monitors the communication process. See Example 2.1.

Definition 2.1 (Bayesian conversations)**.** *For two players with private types θ_A and θ_B, a T-round Bayesian conversation $\pi = \langle \mathbf{A}, \mathbf{B}, \mathbf{f}, \mathbf{g} \rangle$ specifies how the agents communicate in multiple rounds. Here, $\mathbf{f} = (f_1, \ldots, f_T)$, where each f_i is a function mapping Alice's type and the current transcript of the protocol to the randomized distribution over messages that Alice will send in round i. Likewise, $\mathbf{g} = (g_1, \ldots, g_T)$, contains the functions g_i which describe how Bob will send the message in round i. We will assume that at any round, Alice and Bob only ever send a finite number of possible distinct signals (i.e., f_i and g_i all have finite support), and write A_i and B_i to denote the supports of f_i and g_i respectively. More explicitly, the Bayesian conversation defined above proceeds as follows:*

- *At round i, based on her private type θ_A and the observed history, Alice randomly sends to Bob a signal $a_i \sim f_i(\theta_A, a_1, b_1, \ldots, a_{i-1}, b_{i-1}) \in \Delta(A_i)$; then, based on θ_B and the observed history, Bob randomly sends a signal $b_i \sim g_i(\theta_B, a_1, b_1, \ldots, a_{i-1}, b_{i-1}, a_i) \in \Delta(B_i)$. Here a_i and b_i are random variables taking values in A_i and B_i respectively.*
- *The process repeats for T rounds.*

We denote by Π_{BC} the class of all Bayesian conversations. Bayesian conversations involve multiple rounds of information disclosure. We focus on the case where full participation is enforced, meaning that agents are committed to completing the entire protocol once they agree to join. In Appendix A.1, we also discuss *non-committed* Bayesian conversations, where agents have the option to quit midway. In both settings, we assume that agents will adhere to the pre-determined protocol as long as they choose to proceed.

Definition 2.2 (Committed Bayesian Conversation protocols). *A committed Bayesian Conversation protocol π is where Alice and Bob decide at the beginning whether to communicate via a Bayesian conversation π or not, and once agreed, they must complete the full protocol without quitting.*

We give an example of a Bayesian conversation between two players as follows.

Example 2.1 (House searching). *Consider a landlord (Alice) who wants to lease out a house and a renter (Bob) who is searching for a home. The landlord has a private type θ_A which may include the features of the house and her valuation of the house. The renter has a private type θ_B which may include his preferred features and budget. Without any communication, they are both uncertain about one another's type and the trade may not happen. They may hold priors $\mathbb{P}(\theta_B)$ and $\mathbb{P}(\theta_A)$ based on past experience. A rental search platform can design a Bayesian conversation protocol as follows:*

1. *The landlord first posts some basic information about the house, e.g., location, floor plan, etc.*
2. *If the renter is interested in the house, he needs to send a request which includes some basic information about his preferences, e.g., preferred floor plans, lease dates, etc.*
3. *If the landlord wants to continue, she needs to send more detailed information, e.g., photos or videos.*
4. *If the renter wants to continue, he needs to provide more specific preference, e.g., budget range.*

The information sent at each round can be represented as a random signal (that is correlated with the players' private type and the history). For example, the floor plan can be represented as $a_1 \in A_1 = \{studio, 1B1B, 2B2B\}$ and a photo may be represented as $a_2 \in A_2 = \{a\ good\ photo, a\ bad\ photo\}$. The signal can be randomized. For example, suppose $\theta_A \in \{a\ good\ house, a\ bad\ house\}$ and with probability 0.9, a photo will truly reflects the house's quality (it may depend on photography skills). In this case, we have a randomized a_2 with

$$f_2(a_2 = a\ good\ photo|\theta_A = a\ good\ house) = 0.9, \quad f_2(a_2 = a\ bad\ photo|\theta_A = a\ bad\ house) = 0.9.$$

Therefore specifying what information to send in round t is basically specifying a signal set A_t with an associated distribution $f_t(\cdot)$ (that may depend on the private type and the history). When the protocol is committed, the agents must complete all of the steps; when non-committed, the agents can quit after seeing partial information. Note that this setup is: (a) strategic, since both the landlord and prospective renter have clear incentives; (b) verifiable, since the renter will eventually see the house and the landlord can run checks to verify the information about the renter; (c) designed by a platform matching the two parties.

To formally analyze Bayesian conversations, we define random variable $h^{(t)} = (a_1, b_1, \dots, a_t, b_t)$ as the history up to round t. As a slight abuse of notation, we use $f_i(a_i|\theta_A, h^{(t-1)})$ and $g_i(b_i|\theta_B, h^{(t-1)}, a_i)$ to represent the probability of sending a_i and b_i given the true type and the current history. After each round of communication, the players update their beliefs about the other player's type using Bayes' rule. Let $\mathbb{P}_\pi(\theta_A, \theta_B, a_1, b_1, \dots, a_T, b_T)$ be the joint distribution of all types and signals over a complete execution of the protocol π is used, i.e.,

$$\mathbb{P}_\pi(\theta_A, \theta_B, a_1, b_1, \dots, a_T, b_T) = \mathbb{P}(\theta_A)\mathbb{P}(\theta_B) \cdot f_1(a_1|\theta_A) \cdot g_1(b_1|\theta_B, a_1) \cdots g_T(b_T|\theta_B, a_1, b_1, \dots, a_T).$$

After T rounds, Alice's belief about θ_B becomes $q_B^{(T)} = \mathbb{P}_\pi(\theta_B|\theta_A, h^{(T)}) \in \Delta(\Theta_B)$ and Bob's belief about θ_A becomes $q_A^{(T)} = \mathbb{P}_\pi(\theta_A|\theta_B, h^{(T)}) \in \Delta(\Theta_A)$. We will see that a specific choice of $q_A^{(T)}$, $q_B^{(T)}$,

θ_A, and θ_B uniquely determine the outcome of the game, so we will be particularly interested in the distribution $\mathbb{P}_\pi(\theta_A, \theta_B, q_B^{(T)}, q_A^{(T)})$ of the types and posteriors induced by π (e.g., which such distributions we can achieve with a Bayesian conversation). We call this distribution the *(joint) posterior distribution* induced by π. Note that although θ_A, θ_B are the players' private types, the posterior distribution induced by π is commonly known by both players. This is because we assume the priors $\mathbb{P}(\theta_A)$ and $\mathbb{P}(\theta_B)$ are common knowledge and the protocol π is commonly known.

Second, we consider communication that involves a trusted-third party who we refer to as *the mediator*. In this setting, we consider "revelation-style" protocols, where Alice and Bob begin by revealing their full type to the mediator, who then sends a single public signal to both players. We assume that the mediator is not strategic and will faithfully execute the protocol.

Definition 2.3 (Mediator protocols). *For two players with private type θ_A and θ_B, a mediator's protocol π is a signaling scheme $\langle S, \{\pi(\cdot|\theta_A, \theta_B)\}_{\theta_A \in \Theta_A, \theta_B \in \Theta_B}\rangle$ that specifies how the mediator sends the public signal: when the players' types are θ_A and θ_B, the mediator sends signal $s \in S$ with probability $\pi(s|\theta_A, \theta_B)$. We denote by Π_M the class of all mediator protocols.*

After observing the public signal, the players update their beliefs about the other player's type according to Bayes' rule. For a signaling scheme π, let the joint distribution of the types and the signals be $\mathbb{P}_\pi(\theta_A, \theta_B, s)$. Then Alice's posterior about θ_B after seeing s is $q_B = \mathbb{P}_\pi(\theta_B|\theta_A, s)$, and Bob's posterior about θ_A after seeing s is $q_A = \mathbb{P}_\pi(\theta_A|\theta_B, s)$. Again, we are interested in the joint distribution of players' types and beliefs after seeing the public signal, and we say that the distribution of the posteriors $\mathbb{P}(\theta_A, \theta_B, q_B, q_A)$ is the distribution of posteriors induced by π.

2.2 Actions and incentives

After Alice and Bob communicate, the two players will play a "game". As in the case of Bayesian persuasion, in this paper we primarily focus on the very simple class of games with a single action-taker. In particular, we assume that Alice takes an action $r \in R$ (for some finite action set R) and that by taking this action Alice receives utility $u_A(\theta_A, \theta_B, r)$ and Bob receives utility $u_B(\theta_B, \theta_A, r)$.[2] Then Alice's optimal strategy in such a setting is clear: if she has a posterior belief q_B for Bob's type, Alice will take the action that maximizes her expected utility given this posterior, namely

$$r^*(\theta_A, q_B) = \arg\max_{r \in R} \mathbb{E}_{\theta_B \sim q_B} u_A(\theta_A, \theta_B, r).$$

Example 2.2 (Bilateral trade). *One specific Stackelberg game we use as a running example will be the case of* bilateral trade. *In this game, Alice and Bob's types θ_A and θ_B belong to $\Theta_A = \Theta_B = [0, 1]$ and represent their valuation of an item Alice is attempting to sell to Bob. After communicating about their values, Alice sets a price $r \in [0, 1]$, which Bob accepts iff $r \leq \theta_B$. In terms of the notation above, we have that*

$$\begin{aligned} u_A(\theta_A, \theta_B, r) &= (r - \theta_A) \cdot \mathbf{1}[r \leq \theta_B] \\ u_B(\theta_A, \theta_B, r) &= (\theta_B - r) \cdot \mathbf{1}[r \leq \theta_B]. \end{aligned}$$

[2] We briefly note that our model extends slightly beyond the setting where only one player takes an action and captures the case where Alice and Bob play a two-player *Stackelberg game* G, where Alice takes an action r_A, Bob responds (after seeing r) with an action r_B, Alice receives utility $G_A(r_A, r_B, \theta_A)$ and Bob receives utility $G_B(r_A, r_B, \theta_B)$ (note that in contrast to the model above, here we assume their utilities in this game only depend on the actions taken and their individual private types). If we let $r_B(r_A, \theta_B) = \arg\max_{r_B} G_B(r_A, r_B, \theta_B)$ (i.e., r_B is the best response for Bob to r_A), then our model captures this by letting $u_A(\theta_A, \theta_B, r) = G_A(r, r_B(r, \theta_B), \theta_A)$ and $u_B(\theta_A, \theta_B, r) = G_B(r, r_B(r, \theta_B), \theta_B)$.

Alice's optimal action may change depending on the communication between the two players, and this communication may modify both players' expected utility. In particular, Bob can easily find himself in a situation where he would prefer not to share a specific piece of information with Alice (e.g., in the bilateral trade game, if Bob fully reveals his type θ_B, Alice's optimal strategy is to set a price of θ_B which leaves Bob with no net utility). As mechanism designers, our goal is therefore to design conversation protocols that benefit both parties – i.e., protocols that are individually rational (IR). Due to the dynamic and randomized nature of the communication between Alice and Bob, there are several nuances in how we define this. At a high level:

- We can impose different restrictions on the strength of the individual rationality: do we expect communication to help both parties only in expectation ("ex ante" IR), or always help regardless of the outcome of the protocol ("ex post" IR).

We formally define all relevant notions below. Before we do, however, we point out one important observation which is central to our model: since Alice is the sole action taker, **Alice always (weakly) benefits from additional communication**. In other words, we only need to ensure that the protocols are individually rational from the perspective of Bob.

We first consider ex-ante players and ex-ante individual rationality.

Definition 2.4 (Ex-ante players)**.** *An ex-ante agent makes decisions on participating/quitting before seeing their type.*

Then we define a committed protocol as ex-ante individually rational if ex-ante players will choose to participate before seeing their types.

Definition 2.5 (Ex-ante IR for committed protocols)**.** *A committed protocol π is ex-ante individually rational if Bob's ex-ante expected utility of following the entire protocol is no lower than the expected utility of no communication. Formally, let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B)$ be the distribution of the agents' types and posteriors after completing the entire protocol, then it requires*

$$\mathbb{E}_{(\theta_B,\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))] \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B),\theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)], \tag{1}$$

where random variable $r^0 = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. Note that we only need this inequality for Bob because it always holds for the action taker Alice.

Finally, we consider players who do not want to regret joining.

Definition 2.6 (Ex-post players)**.** *An ex-post player does not want to regret joining/proceeding after seeing their types and completing the protocol.*

We say that a committed protocol is ex-post IR if ex-post players will not regret joining after seeing their types and completing the protocol.

Definition 2.7 (Ex-post IR for committed protocols)**.** *A committed protocol π is ex-post individually rational if after completing the protocol (and before Alice taking an action), Bob never regrets joining the protocol. For Bayesian conversations, we should have: for any $y \in \Theta_B$ and any outcome $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \dots, \widetilde{a}_T, \widetilde{b}_T)$, suppose Bob has type y and Bob's posterior belief becomes $\widetilde{q}_A$ after seeing $\widetilde{h}^{(T)}$ and let $\mathbb{P}_\pi(\theta_A, q_B|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)})$ be the conditional distribution of (θ_A, q_B) when Bob has type y and the protocol ends at $\widetilde{h}^{(T)}$. Then it requires*

$$\mathbb{E}_{(\theta_A,q_B)\sim\mathbb{P}_\pi}[u_B(\theta_A, y, r^*(\theta_A, q_B))|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A}[u_B(\theta_A, y, r^0)], \ \forall\widetilde{h}^{(T)}, y \in \Theta_B, \tag{2}$$

where random variable $r^0 = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. For mediator protocols, we simply replace the outcome $\widetilde{h}^{(T)}$ with the realized public signal $\widetilde{s} \in S$. Again, we only need this inequality for Bob because it always holds for Alice.

2.3 Posterior distributions of the two protocols

Communication protocols change the outcome of the game by changing the players' beliefs about the other player's type. It is therefore natural to ask: what are the belief distributions that can be generated by a communication protocol? Determining whether a distribution can be generated by a communication protocol is not as straightforward as one might assume. For instance, in Figure 1, we present a belief distribution that cannot be generated by any communication protocol.

In this section, we present some preliminary observations from the existing literature for understanding the space of feasible posterior distributions, both in the mediator model and the Bayesian conversation model. One important consequence of the observations in this section is the following fact: mediator protocols possess (strictly) greater power than Bayesian conversations, as there exist belief distributions that can only be generated by a mediator protocol and not by a Bayesian conversation (note that conversely, Bayesian conversations are a subset of mediator protocols, as the transcript of a Bayesian conversation can be used as the public signal in a mediator protocol). We summarize this in the following proposition.

Proposition 2.1. *There exists a posterior distribution $P(\theta_A, \theta_B, q_B, q_A)$ that can be generated by mediator protocol that cannot be generated by Bayesian conversations (demonstrated in Table 1). In particular, there exists a distribution that can be generated by mediator protocol that does not satisfy Condition (1) in Theorem D.1, which must be satisfied by a Bayesian conversation.*

2.3.1 The observer's posterior

$$
\begin{array}{|c|c|c|}\hline \mathbb{P}(\cdot) & \theta_B = H & \theta_B = L \\ \hline \theta_A = H & 0.3 & 0.2 \\ \hline \theta_A = L & 0.3 & 0.2 \\ \hline \end{array} = 0.6 \begin{array}{|c|c|c|}\hline q_{AB}^{(s_1)} & \theta_B = H & \theta_B = L \\ \hline \theta_A = H & 0 & 1/3 \\ \hline \theta_A = L & 1/3 & 1/3 \\ \hline \end{array} + 0.4 \begin{array}{|c|c|c|}\hline q_{AB}^{(s_2)} & \theta_B = H & \theta_B = L \\ \hline \theta_A = H & 3/4 & 0 \\ \hline \theta_A = L & 1/4 & 0 \\ \hline \end{array}
$$

$\downarrow$

$P(H, L, (0, 1), (\frac{1}{2}, \frac{1}{2})) = 0.2$
$P(L, H, (\frac{1}{2}, \frac{1}{2}), (0, 1)) = 0.2$
$P(L, L, (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})) = 0.2$

$\downarrow$

$P(H, H, (1, 0), (\frac{3}{4}, \frac{1}{4})) = 0.3$
$P(L, H, (1, 0), (\frac{3}{4}, \frac{1}{4})) = 0.1$

Table 1: Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A = H) = 0.5$ and $\mathbb{P}(\theta_B = H) = 0.6$. The plot shows how a distribution of posteriors $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a mediator protocol with $S = \{s_1, s_2\}$ with $q_{AB}^{(s_1)}$ shown in the second table with $q_{AB}^{(s_2)}$ shown in the third table. See Section G.1 for a concrete application of this example.

Our goal is to characterize the possible distribution of the possible tuples $(\theta_A, \theta_B, q_B, q_A)$ of types and posteriors induced by a given protocol π (what we have called *joint posterior distributions*). To analyze this distribution, we consider the *observer's posterior distribution*; the posterior distribution about (θ_A, θ_B) a third-party observer (who cannot observe Alice or Bob's types directly, but does know their priors) arrives at after seeing the public signal s.

$$q_{AB}(\theta_A, \theta_B) = \mathbb{P}_\pi(\theta_A, \theta_B | s).$$

And we denote by $\mathbb{P}_\pi(q_{AB})$ the induced distribution of the observer's posterior where $\mathbb{P}_\pi(q_{AB}) = \mathbb{P}_\pi(s)$. Note that it is possible to recover the posteriors of Alice and Bob from the observer's posteriors along with Alice and Bob's realized types. Therefore, we can recover a joint posterior distribution uniquely from the observer's posterior distribution as shown in Table 1.

2.3.2 Mediator protocols

Then the observer's posterior distribution induced by a mediator protocol can be characterized by the Splitting Lemma [34]: sending the public signal can be thought of as "splitting" the prior $\mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$ into $|S|$ different posteriors q_{AB} each with probability $\Pr(s)$ of occurring. Note that this splitting operation is *mean-preserving*: the expected posterior of the observer at the conclusion of the protocol must equal the original prior: $\mathbb{E}_\pi[q_{AB}] = \mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$.For the case of mediator protocols, the Splitting Lemma condition is sufficient as well as necessary: any mean-preserving distribution over posteriors is incentivizable via a single signal.

Proposition 2.2. *Let $P(q_{AB})$ be a distribution over posterior distributions $q_{AB} \in \Delta(\Theta_A \times \Theta_B)$ with the property that $\mathbb{E}[q_{AB}] = \mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$. Then there exists a mediator protocol π such that $\mathbb{P}_\pi(q_{AB}) = P(q_{AB})$.*

One remark: a version of the splitting lemma holds for the (marginals of the) joint posterior distributions, in the sense that $\mathbb{E}[q_A] = \mathbb{P}(\theta_A)$ and $\mathbb{E}[q_B] = \mathbb{P}(\theta_B)$. However, the obvious analogue of Proposition 2.2 does not hold: in the example shown in Figure 1, the expectations of the posteriors equal the priors, but it cannot be induced by a mediator protocol.

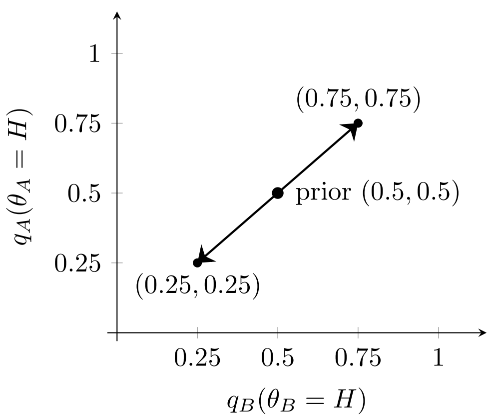


Figure 1: This example shows a posterior distribution that cannot be induced by any communication protocols. Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A = H) = \mathbb{P}(\theta_B = H) = 0.5$. Then we cannot have $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5. We give the proof in Appendix E.2.

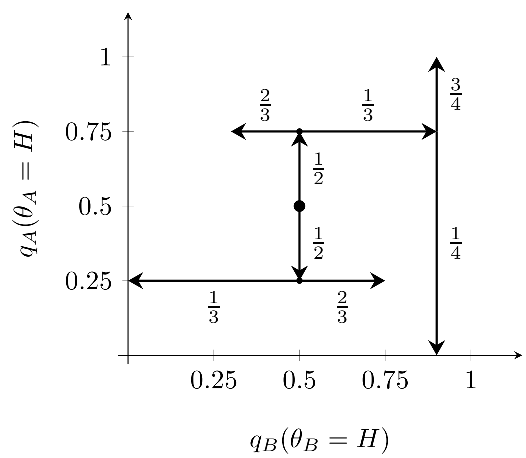


Figure 2: An illustration of a posterior distribution generated by a Bayesian conversation. Suppose $\theta_A, \theta_B \in \{L, H\}$. At any step of the protocol, the status of the protocol can be represented as a two-dimensional point $(q_B(\theta_B = H), q_A(\theta_A = H))$. When Alice sends a signal, q_B remains unchanged and q_A is decomposed along y-axis while preserving the expectation. The same holds for Bob.

2.3.3 Bayesian conversations

In Bayesian conversations, we have the additional restriction that each signal sent either contains information only about θ_A (if Alice is sending the signal) or only about θ_B (if Bob is sending the signal); in other words, at every step t of the protocol, either we have $\Pr[s|\theta_A, \theta_B, h^{(t)}] = \Pr[s|\theta_A, h^{(t)}]$

or $\Pr[s|\theta_A, \theta_B, h^{(t)}] = \Pr[s|\theta_B, h^{(t)}]$. This has following two important consequences. First, since the observer's posterior starts as a product distribution (since the priors for θ_A and θ_B are independent), the observer's posterior will always remain a product distribution (i.e. in each q_{AB}, θ_A and θ_B are independent). Secondly, this lets us relate the observer's posterior and the joint posteriors much more directly: we always have that $q_{AB}(\theta_A, \theta_B) = q_A(\theta_A)q_B(\theta_B)$. Pictorially, we can represent this procedure as in Figure 2; at each step of the protocol, each posterior can be split either "horizontally" (along dimensions in $\Delta(\Theta_A)$) or "vertically" (along dimensions in $\Delta(\Theta_B)$) but not both. This belief-splitting process is defined as a *bimartingale* in [5, 6].

Proposition 2.3 ([5, 6])**.** *The belief-splitting process* $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^{\infty}$ *can be viewed as a* bimartingale. *A bimartingale is a martingale* $\{(\alpha_t, \beta_t)\}_{t=1}^{\infty}$ *that has either* $\alpha_t = \alpha_{t+1}$ *or* $\beta_t = \beta_{t+1}$ *at each time step* t*.*

Note that $P(\theta_A, \theta_B|q_B, q_A) = q_A(\theta_A)q_B(\theta_B)$ does *not* have to be true for mediator protocols (where the mediator can correlate the observer's posterior for θ_A and θ_B), and hence this provides a proof of Proposition 2.1. An explicit counter-example is given in Table 1.

3 Optimal ex-ante IR Bayesian conversations

We start by introducing the optimal ex-ante IR Bayesian conversations. Our key finding is that if the players decide whether to participate ex-ante, Bayesian conversations are equivalent to mediator protocols in terms of the possible induced expected utilities in the base game. In fact, as we will see, Bob (the non-action-taker) can act as the mediator in a Bayesian conversation without violating any IR constraints. As a result, finding the optimal Bayesian conversation that maximizes the expectation of a utility function is equivalent to finding the optimal mediator protocol, which can be solved by a linear program.

3.1 Equivalence of protocols under ex-ante IR

We first show that when our protocols only need to be ex-ante IR/interim IR, mediator protocols are no more powerful than committed Bayesian conversations when we consider the induced expected utilities in the base game. In fact, as we will see, Bob (the non-action-taker) can act as the mediator in such protocols without violating any IR constraints.

We introduce some notation before the main results. Suppose we are interested in the expectation of an arbitrary utility function $u(\theta_A, \theta_B, r)$. This utility function can be Alice's utility $u(\theta_A, \theta_B, r) = u_A(\theta_A, \theta_B, r)$ or Bob's utility or a designer's utility that depends on the outcome of the game. We may also have some constraints that restrict the valid belief distributions to a subset $\mathcal{P} \subseteq \Delta(\Theta_A \times \Theta_B \times \Delta\Theta_B \times \Delta\Theta_A)$. For example, we may want the protocol to be ex-ante IR (Definition 2.5) for the committed players, in which we have $\mathcal{P}$ equal to the subset of distributions that satisfies equation (1). We then define the range of expected utilities that can be implemented by a class of protocol Π as

$$\mathrm{Range}(u, \Pi, \mathcal{P}) = \{\mathbb{E}_P[u(\theta_A, \theta_B, r^*(\theta_A, q_B))] : P \in \mathcal{P} \text{ and } P \text{ can be induced by a protocol } \pi \in \Pi\},$$

which represents all possible values of the expectation of $u(\theta_A, \theta_B, r)$ when Alice chooses her best action after a valid communication protocol.

We first show that when we only need ex-ante, the mediator protocol and the committed BC protocols are equivalent in the sense that the ranges of implementable expected utilities are the same. This is due to the following observation: we actually only need to care about the marginal distribution $P(\theta_A, q_B)$ generated by a protocol in this case, and for any mediator protocol, there exists a one-round Bayesian conversation that generates the same $P(\theta_A, q_B)$ in which Alice fully reveals her type in the first round. Furthermore, all feasible marginal distributions $P(\theta_A, q_B)$ can be characterized by a simple equation $\mathbb{E}[q_B|\theta_A] = \mathbb{P}(\theta_B)$ for all θ_A, which says the conditional expectation of q_B must always equal the prior. For ex-ante IR, we have the following theorem.

Theorem 3.1. *For any utility function $u(\theta_A, \theta_B, r)$, the range of expected utilities that can be implemented by ex-ante IR committed Bayesian conversation protocols is equivalent to the range of expected utilities that can be implemented by ex-ante IR mediator protocols,*

$$Range_{\text{ex-ante}}(u, \Pi_{BC}) = Range_{\text{ex-ante}}(u, \Pi_M),$$

where $Range_{\text{ex-ante}}(u, \Pi) = Range(u, \Pi, \mathcal{P})$ with $\mathcal{P}$ being the subset of distributions that guarantee ex-ante IR for committed protocols (satisfy equation (1)).

We defer the proof to Appendices F.1. In addition, Theorem 3.1 and Theorem C.1 can be easily extended to a vector of utility functions $\mathbf{u} = (u_1, \dots, u_L)$ with each $u_j = u_j(\theta_A, \theta_B, r)$. We give the full extended theorem in Appendix F.2. In particular, the extended theorem can be used to show that the utility pairs $(\mathbb{E}[u_A], \mathbb{E}[u_B])$ that can be generated for the two players are the same no matter which type of protocols we use, indicating that the two types of protocols generate the same *Pareto frontier*.

Corollary 3.1. *The Pareto frontier of the two players' utilities induced by ex-ante/interim IR mediator protocols is the same as the Pareto frontier induced by ex-ante/interim IR committed Bayesian conversations.*

Discussion. The equivalence of these two models relies crucially on two assumptions: (1) the protocols are committed, and (2) there is only one action-taker (Alice). Both of the assumptions are necessary. We show in Section A.1 that (1) is necessary. We provide an example in Appendix F.3 that illustrates that, in general, when both players make a choice of action (even in the case where their action only affects their own reward), the two types of protocols are not equivalent.

3.2 Optimal protocol by linear programming

As a result, solving the optimal ex-ante IR Bayesian conversation that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ reduces to finding the optimal mediator protocol, which can be solved by a linear program.

Theorem 3.2. *The optimal ex-ante IR mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ can be solved by a linear program with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$.*[3]

We defer the details of the linear program to Appendix B. After determining the optimal mediator protocol, we can transform it into a Bayesian conversation by having Alice disclose her type in the first round and asking Bob to simulate the mediator's role.

4 Optimal ex-post IR Bayesian conversations

When designing ex-post IR protocols, the equivalence between Bayesian conversations and mediator protocols no longer holds, and the proposed linear program (LP) no longer solves the optimal

[3]The optimal ex-post IR mediator protocol can be solved by an exponential-size LP. See details in Appendix B.

Bayesian conversation. In this section, we introduce a dynamic programming algorithm to compute the optimal ex-post IR T-round Bayesian conversation. Using this algorithm, we present an example where the optimal ex-post IR protocol and the optimal ex-post Bayesian conversation yield different expected utilities. Additionally, we construct a game in which the optimal ex-post IR Bayesian conversation necessitates infinitely many rounds of communication.

4.1 Algorithm for finding the optimal ex-post IR protocol

We will focus on the case where each agent has two types $\Theta_A = \Theta_B = \{L, H\}$ and we aim to maximize the expected social welfare. We will refer to the prior that agents have about each other's types before any communication as q_A^0, q_B^0. We will also assume that Alice's choice of action $r^*(\theta_A, q_B)$ breaks ties in favor of Bob, i.e., between two actions that lead to the same payoff for her, she chooses the best action for Bob, which is also the action that maximizes social welfare.

In that case, the beliefs of Alice and Bob can be described by a pair $(q_B, q_A) \in [0,1]^2$ representing the belief that the other agent has type H. We will show how to recursively compute a function $W_k(q_B, q_A)$ which corresponds to the expected optimal welfare of a protocol with $\lceil k/2 \rceil$ messages by Bob and $\lfloor k/2 \rfloor$ messages by Alice such that at the last step, Bob prefers Alice's final posterior to the prior q_A^0 (i.e., the conversation satisfies the ex-post IR condition in (2)). We will set $W_k(q_B, q_A) = -\infty$ if this is not feasible. It is important to notice that the ex-post IR condition in $W_k(q_B, q_A)$ is with respect to the fixed prior q_A^0 and not q_A in the k-th round.

Ex-post IR region We start by noticing that the ex-post IR condition (equation 2) is a condition on the state of the agent's beliefs in the last stage of the game. The region of the space of beliefs $[0,1]^2$ that lead to ex-post IR in the last period is given by:

$$\mathsf{IR}_0 = \{(q_B, q_A) \in [0,1]^2 \mid \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) \geq \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0)), \forall \theta_B \in \operatorname{supp}(q_B)\}$$

where $\operatorname{supp}(q_B) \subseteq \Theta_B$ are the types that occur with non-zero probability in q_B (also, recall that q_B^0 is Alice's prior belief about Bob's type – see equation 2). Using this notion we can define the starting point of the recursion as:

$$W_0(q_B, q_A) = W^*(q_B, q_A) \text{ if } (q_B, q_A) \in \mathsf{IR}_0 \quad \text{and} \quad W_0(q_B, q_A) = -\infty \text{ o.w.}$$

where W^* is the optimal welfare given the final beliefs (q_A, q_B) without taking IR into account. Notice that since Alice breaks ties in favor of Bob, it has the following form:

$$W^*(q_B, q_A) = \mathbb{E}_{\theta_B, \theta_A \sim (q_B, q_A)}[u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))]$$

Recursive Step Assuming we know W_k, we can compute W_{k+1} by a process of alternating concavification: i.e, we will take concave hulls of the function with respect to Alice's belief (for odd k) and Bob's belief (for even k). Following Proposition 2.3, we can view each round of communication by e.g. Alice as splitting the belief q_A into a distribution of beliefs Q'_A supported on $[0,1]$ that preserves the average belief. From this perspective, we will update:

$$W_k(q_B, q_A) = \max \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}(q'_B, q_A)] \text{ s.t. } \mathbb{E}_{q'_B \sim Q'_B}[q'_B] = q_B \text{ for odd } k \tag{3}$$

$$W_k(q_B, q_A) = \max \mathbb{E}_{q'_A \sim Q'_A}[W_{k-1}(q_B, q'_A)] \text{ s.t. } \mathbb{E}_{q'_A \sim Q'_A}[q'_A] = q_A \text{ for even } k \tag{4}$$

Such functions provide enough information to compute the optimal social welfare:

Lemma 4.1. *Given the $W_k(q_B, q_A)$ functions defined above, the optimal social welfare obtained by a protocol with k rounds is $W_k(q_B^0, q_A^0)$, i.e., the function evaluated on the original prior.*

Proof. This can be seen by induction on k. It holds for $k = 0$ by definition. For odd k, let W_k^* be the welfare of the optimal protocol and W_k the function computed above. Assume that Bob's first signal splits Alice's prior belief q_B^0 into a distribution Q'_B. For each $q'_B \sim Q'_B$ what follows is a $k-1$ round protocol starting at (q'_B, q_A) where Bob doesn't regret being at the prior. Hence:

$$W_k^*(q_B, q_A) = \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}^*(q'_B, q_A)] = \mathbb{E}_{q'_B \sim Q'_B}[W_{k-1}(q'_B, q_A)] \leq W_k(q'_B, q_A)$$

since W_k is the maximum. Since W_k corresponds to a valid protocol, $W_k(q'_B, q_A) = W_k^*(q_B, q_A)$. The argument for even k is the same, swapping the roles of Alice and Bob.

□

Exact discretization The recursion above specifies continuous functions $W_k : [0,1]^2 \to \mathbb{R}$. To convert this into a discrete algorithm, we show that it is possible to discretize the space $[0,1]^2$ into finitely many points and only compute the W_k restricted to this set. A crucial fact here is that the discretization is exact, i.e., it introduces no approximation errors.

Lemma 4.2. *There are sets $X^* = \{0 = x_0 < x_1 < \ldots < x_m = 1\}$ and $Y^* = \{0 = y_0 < y_1 < \ldots < y_n = 1\}$ such that W_0 restricted to the rectangle $(x_i, x_{i+1}) \times (y_j, y_{j+1})$ is bilinear.*

Proof. Define X^* as:

$$X^* = \{0,1\} \cup \{q_B \in [0,1] : \exists \theta_A \in \Theta_A, \text{such that } r^*(\theta_A, q_B) \text{ differs on the left and right of } q_B\}$$

This lists all the q_B values at which Alice's decision changes. Within the interval (x_i^*, x_{i+1}^*), Alice's decisions remain unchanged. Therefore, we define

$$X_{\text{int}} = \left\{ \frac{x_i + x_{i+1}}{2} \mid i = 0, 1, \ldots, m-1 \right\}$$

by selecting the midpoints $\frac{x_i + x_{i+1}}{2}$ as representatives of the intervals (x_i, x_{i+1}). Consequently, all x in $X^* \cup X_{\text{int}}$ correspond to Alice's strategies, collectively covering all possible strategies she may adopt over the entire range $x \in [0,1]$.

As q_B varies, Alice will take different strategies. Under different strategies of Alice, whether Bob's IR condition is satisfied will change depending on q_A. We therefore define the set Y^*, which enumerates the boundaries at which Bob's IR condition is satisfied or not, under different q_B, considering all possible strategies of Alice.

$$\begin{aligned} Y^* = \{0,1\} \cup \{q_A \in [0,1] \mid &\exists q_B \in X^* \cup X_{\text{int}}, \exists \theta_B \in \text{supp}(q_B), \\ &\mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) = \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0))\} \end{aligned}$$

Now, recall the definition of W^*:

$$\begin{aligned} W^*(q_A, q_B) =& \mathbb{E}_{\theta_B, \theta_A \sim (q_B, q_A)}[u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))] \\ =& \sum_{\theta_A \in \Theta_A} \sum_{\theta_B \in \Theta_B} q_A(\theta_A) q_B(\theta_B)(u_A(\theta_A, \theta_B, r^*(\theta_A, q_B)) + u_B(\theta_A, \theta_B, r^*(\theta_A, q_B))) \end{aligned}$$

For each rectangle, either it is not in IR_0, or it is in IR_0 and $r^*(\theta_A, q_B)$ is a constant within it, hence W_0 restricted to each rectangle is bilinear. □

Lemma 4.3. *The sets X^* and Y^* are finite and have size $O(|R|)$, where R is the set of Alice's actions.*

Proof. Since the number of actions that Alice can take is $|R|$, the number of beliefs q_B where Alice changes her action at a specific type is also $O(|R|)$. Given that Alice has two types, the size of X^* is $O(|R|)$. Furthermore, we know that the size of X_{int} is also $O(|R|)$.

For each $q_B \in X^* \cup X_{\text{int}}$, there are at most two θ_B in $\text{supp}(q_B)$. For each θ_B, there is at most one $q_A \in [0,1]$ that satisfies

$$\mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B)) = \mathbb{E}_{\theta_A \sim q_A} u_B(\theta_A, \theta_B, r^*(\theta_A, q_B^0))\},$$

and therefore, $|Y^*| \leq 1 \cdot 2 \cdot |X^* \cup X_{\text{int}}| = O(|R|)$. □

Algorithm A consequence of the existence of an *exact* discretization that has size polynomial in the size of the input is that it is possible to compute each entry of $W_k(x,y)$ for $(x,y) \in X^* \times Y^*$ in poly-time.

Lemma 4.4. *Given the values of $W_{k-1}(x,y)$ for $(x,y) \in X^* \times Y^*$ we can determine $W_k(x,y)$ by solving the following optimization programs over the simplex $\Delta_n = \{p \in [0,1]^n; \sum_i p_i = 1\}$. For odd k:*

$$W_k(\hat{x},\hat{y}) = \max \sum_{i=1}^{m} p_i W_{k-1}(x_i,\hat{y}) \quad s.t \quad \sum_{i=1}^{m} p_i x_i = \hat{x} \quad and \quad p \in \Delta_m$$

and for even k:

$$W_k(\hat{x},\hat{y}) = \max \sum_{j=1}^{n} p_j W_{k-1}(\hat{x},y_j) \quad s.t \quad \sum_{j=1}^{m} p_j y_j = \hat{y} \quad and \quad p \in \Delta_n$$

Proof. We prove the claim by induction on k. The proof for cases $k=1$ and $k=2$ are special and then $k \geq 3$ follows a general argument.

Case $k=1$: We know by Lemma 4.2 that W^* is bilinear. Hence once we fix a $\hat{y}$, $W_0(x,\hat{y})$ is piecewise linear on $[0,1]$ with segments bounded by $x_i \in X^*$. The function $x \mapsto u_A(x,\hat{y})$ is continuous at x_i but $x \mapsto u_B(x,\hat{y})$ may be discontinuous. Since Alice breaks ties in favor of Bob, the value of $u_B(x_i,\hat{y})$ is the maximum of its left and right limits, hence:

$$W_0(x_i,\hat{y}) = \max\left(\lim_{x \to x_i^-} W_0(x,\hat{y}), \lim_{x \to x_i^+} W_0(x,\hat{y})\right).$$

We need to show that given a point $(\hat{x},\hat{y})$, if X' is the distribution in the solution of the optimization problem (3), then:

$$\begin{aligned} W_1(\hat{x},\hat{y}) &= \max \mathbb{E}_{x' \sim X'}[W_0(x',\hat{y})] \text{ s.t.} \quad \mathbb{E}_{x' \sim X'}[x'] = \hat{x} \\ &= \max \sum_{i=1}^{m} p_i W_0(x_i,\hat{y}) \quad \text{s.t} \quad \sum_{i=1}^{m} p_i x_i = \hat{x} \quad \text{and} \quad p \in \Delta_m \end{aligned}$$

We prove this by contradiction. Suppose there exists $x^* \in \text{supp}(X')$, and $x^* \notin X^*$. Assume $x_i < x^* < x_{i+1}$ and $x^* = qx_i + (1-q)x_{i+1}$. We know that $W_0(x,\hat{y})$ restricted to (x_i, x_{i+1}) is linear in x, and therefore,

$$\begin{aligned}W_0(x^*,\hat{y}) =&W_0(qx_i+(1-q)x_{i+1},\hat{y})\\ \leq& q\lim_{x\to x_i^+}W_0(x,\hat{y})+(1-q)\lim_{x\to x_{i+1}^-}W_0(x,\hat{y})\\ \leq& qW_0(x_i,\hat{y})+(1-q)W_0(x_{i+1},\hat{y}).\end{aligned}$$

Hence we can substitute x^* with x_i w.p. q and x_{i+1} w.p. $1-q$.

Case $k=2$: Fix $\hat{x}$ and consider the function $y\mapsto W_0(\hat{x},y)$. For a fixed $\hat{x}$, Alice's action is uniquely determined, and Bob's ex-post IR condition is satisfied within a closed interval. Therefore, we conclude that $W_0(\hat{x},y)$ restricted to $[y_j,y_{j+1}]$ is linear. Now, let Y' be the distribution in the solution of the optimization problem (4). We want to show that:

$$\begin{aligned}W_2(\hat{x},\hat{y}) =& \max \mathbb{E}_{y'\sim Y'}[W_1(\hat{x},y')] \text{ s.t. } \quad \mathbb{E}_{y'\sim Y'}[y']=y\\ =& \max\sum_{j=1}^{n}p_jW_1(\hat{x},y_j) \quad \text{s.t} \quad \sum_{j=1}^{m}p_jy_j=\hat{y} \quad \text{and} \quad p\in\Delta_n\end{aligned}$$

Suppose there exists $y^*\in\text{supp}(Y')$, and $y^*\notin Y^*$. Assume $y_j<y^*<y_{j+1}$ and $y^*=qy_i+(1-q)y_{i+1}$. We know that $W_0(\hat{x},y)$ restricted to $[y_j,y_{j+1}]$ is linear in y, therefore,

$$\begin{aligned}W_1(\hat{x},y^*) =& \sum_{i=1}^{m}p_iW_0(x_i,y^*)\\ =& q\sum_{i=1}^{m}p_iW_0(x_i,y_j)+(1-q)\sum_{i=1}^{m}p_iW_0(x_i,y_{j+1})\\ \leq& qW_1(\hat{x},y_j)+(1-q)W_1(\hat{x},y_{j+1}).\end{aligned}$$

Hence we can substitute y^* with y_j w.p. q and y_{j+1} w.p. $1-q$.

Case $k\geq 3$: We now consider the case where $k\geq 3$. We take the case where k is an odd number as an example. The proof for the even case is entirely similar. We assume the base condition:

$$W_{k-1}(\hat{x},\hat{y}) = \max\sum_{j=1}^{n}p_jW_{k-2}(\hat{x},y_j) \quad \text{s.t} \quad \sum_{j=1}^{m}p_jy_j=\hat{y} \quad \text{and} \quad p\in\Delta_n$$

$$W_{k-2}(\hat{x},\hat{y}) = \max\sum_{i=1}^{m}p_iW_{k-3}(x_i,\hat{y}) \quad \text{s.t} \quad \sum_{i=1}^{m}p_ix_i=\hat{x} \quad \text{and} \quad p\in\Delta_m$$

From the property of concave closure, the second condition gives $W_{k-2}(x,y_j)$ restrict to $[x_i,x_{i+1}]$ is linear. We want to prove

$$\begin{aligned}W_k(\hat{x},\hat{y}) =& \max \mathbb{E}_{x'\sim X'}[W_{k-1}(x',\hat{y})] \text{ s.t. } \mathbb{E}_{x'\sim X'}[x']=\hat{x}\\ =& \max\sum_{i=1}^{m}p_iW_{k-1}(x_i,\hat{y}) \quad \text{s.t} \quad \sum_{i=1}^{m}p_ix_i=\hat{x} \quad \text{and} \quad p\in\Delta_m\end{aligned}$$

We prove by contradiction. Suppose there exists $x^*\in\text{supp}(X')$, and $x^*\notin X^*$. Assume $x_i<x^*<x_{i+1}$ and $x^*=qx_i+(1-q)x_{i+1}$. The key observartion of the proof is:

$$
\begin{aligned}
W_{k-1}(x^*, \hat{y}) =& \sum_{j=1}^{n} p_j W_{k-2}(x^*, y_j) \\
=& q \sum_{j=1}^{n} p_j W_{k-2}(x_i, y_j) + (1-q) \sum_{j=1}^{n} p_j W_{k-2}(x_{i+1}, y_j) \\
\leq& q W_{k-1}(x_i, \hat{y}) + (1-q) W_{k-1}(x_{i+1}, \hat{y}).
\end{aligned}
$$

Hence we can substitute x^* with x_i w.p. q and x_{i+1} w.p. $1-q$, which completes the proof. □

We observe that while it is useful to describe W_k as a linear program, there is a linear time algorithm to compute $W_k(\cdot, \hat{y})$ as a function of $W_{k-1}(\cdot, \hat{y})$ as a convex hull computation problem.

Theorem 4.1. *There is algorithm running in time poly$(k, |R|)$ that for each k, computes the highest welfare achievable by an ex-post IR protocol with k rounds ($\lceil k/2 \rceil$ messages by Bob and $\lfloor k/2 \rfloor$ messages by Alice).*

Proof. Using Lemma 4.3 we can construct the sets X^* and Y^* in polynomial time, where each set has size $O(|R|)$. Note that we can also (without loss of generality) assume that $q_A^0 \in X^*$ and $q_B^0 \in Y^*$ – if they are not originally there, we can simply add one extra point to each set. Using the definition of W_0 we can construct a table that stores the value of W_0 on $X^* \times Y^*$ and then using the algorithm in Lemma 4.4 we can construct each W_{k-1} from W_k in this set: $X^* \times Y^*$. □

4.2 Gap between Bayesian conversations and mediator protocols

When ex-post IR is required, it is no longer true that the two types of protocols are equivalent. We show (via a game between an "employer" and "job candidate") that mediator protocols are more powerful than committed Bayesian conversations in this setting.

Theorem 4.2. *There exists a game where the highest social welfare that can be implemented by ex-post IR committed Bayesian conversations is lower than the highest social welfare that can be implemented by ex-post IR mediator protocols.*

We provide an example. Consider a two-player game between employers (Alice) and job candidates (Bob). Suppose there are two types of employers: employers who prefer candidates with good programming skills and employers who prefer candidates with good communication skills, so we have $\theta_A \in \{\text{Prog, Comm}\}$. The candidates also have two types: good at programming and good at communication $\theta_B \in \{\text{Prog, Comm}\}$. An employer can choose to hire or not hire a candidate $r \in \{\text{hire, not hire}\}$ and an employer's utility function is shown in Table 2.

$u_A(\text{Prog}, \cdot)$	$\theta_B = \text{Prog}$	$\theta_B = \text{Comm}$
hire	10	-10
not hire	0	0

$u_A(\text{Comm}, \cdot)$	$\theta_B = \text{Prog}$	$\theta_B = \text{Comm}$
hire	-1	1
not hire	0	0

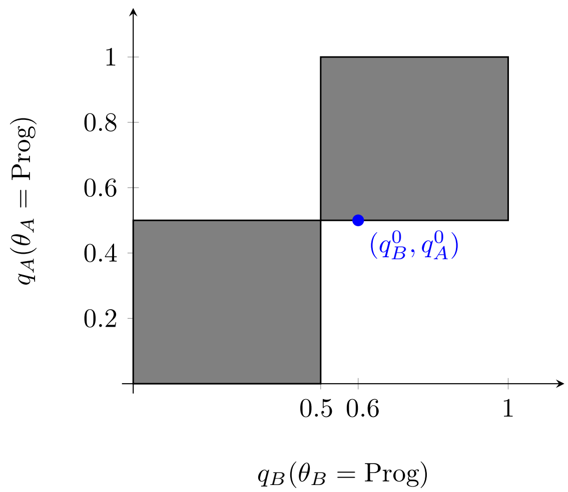
Table 2: The employer's utility function in the hiring problem.

And a candidate simply wants to be hired, $u_B = 2 \cdot \mathbf{1}(r = \text{hire})$. Suppose $\mathbb{P}(\theta_A = \text{Prog}) = 0.5$ and $\mathbb{P}(\theta_B = \text{Prog}) = 0.6$, so without any communication a Prog-type employer always hires and a Comm-type employer never hires. Suppose we want to maximize the expected social welfare $\mathbb{E}[u_A + u_B]$.

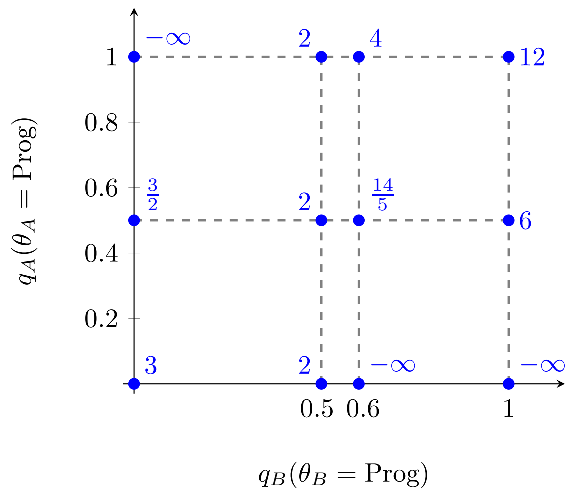
We show that the optimal ex-post IR mediator protocol achieves a social welfare of 22/5 whereas the the optimal ex-post IR Bayesian conversation only achieves 21/5.

In the optimal mediator protocol, the mediator sends one of two public signals s_1 indicates "good communication skill" and s_2 indicates "good programming skill". If Alice wants to hire someone with good programming skills, the mediator sends a perfect signal. If the Alice wants to hire someone with good communication skills, the mediator sends an garbled signal: with some probability it will send signal s_1 when $\theta_b = \text{Prog}$. In Appendix G.1 we describe the details of the optimal mediator protocol and show it achieves welfare 22/5.

In the same appendix, we use the technique developed in the previous section to show compute optimal ex-post IR Bayesian conversation and show the optimal welfare is 21/5, providing a strict separation. We also show that the optimal conversation is a single message from Bob to Alice that is unconditioned on Alice's type, as illustrated in Figure 4b. The intuition is that if Bob were to condition on Alice's type, some types of Bob would have regretted the outcome of the conversation. The corresponding IR_0, W_0, and W_k are plotted in Figures 3a and 4a.



(a) IR_0



(b) W_0

Figure 3: Illustrations of IR_0 and W_0 for finding the optimal ex-post IR Bayesian protocol in the hiring problem.

4.3 Instance of infinite round convergence

From the above example, we can see that some games can achieve the optimal social welfare through a finite number of ex-post IR Bayesian conversations. However, not all games exhibit this property. There exist certain games where achieving the optimal social welfare requires an infinite number rounds for ex-post IR Bayesian conversation protocols:

Theorem 4.3. *There exists a game where the highest social welfare achievable through ex-post IR committed Bayesian conversations requires infinitely many rounds to converge to the optimum.*

The proof of Theorem 4.3 involves again applying the methodology in Theorem 4.1 to the following game and then computing an explicit expression for $W_k(q_B^0, q_A^0)$.

In the two-player game Alice has types $\Theta_A = \{\theta_{A0}, \theta_{A1}\}$ and Bob has types $\Theta_B = \{\theta_{B0}, \theta_{B1}\}$. Alice can take two actions $r \in \{r_0, r_1\}$. Suppose $q_B^0 = \mathbb{P}(\theta_B = \theta_{B0}) = 0.4$, and $q_A^0 = \mathbb{P}(\theta_A = \theta_{A0}) = 0.6$. The utilities of the two players are given in Table 3:

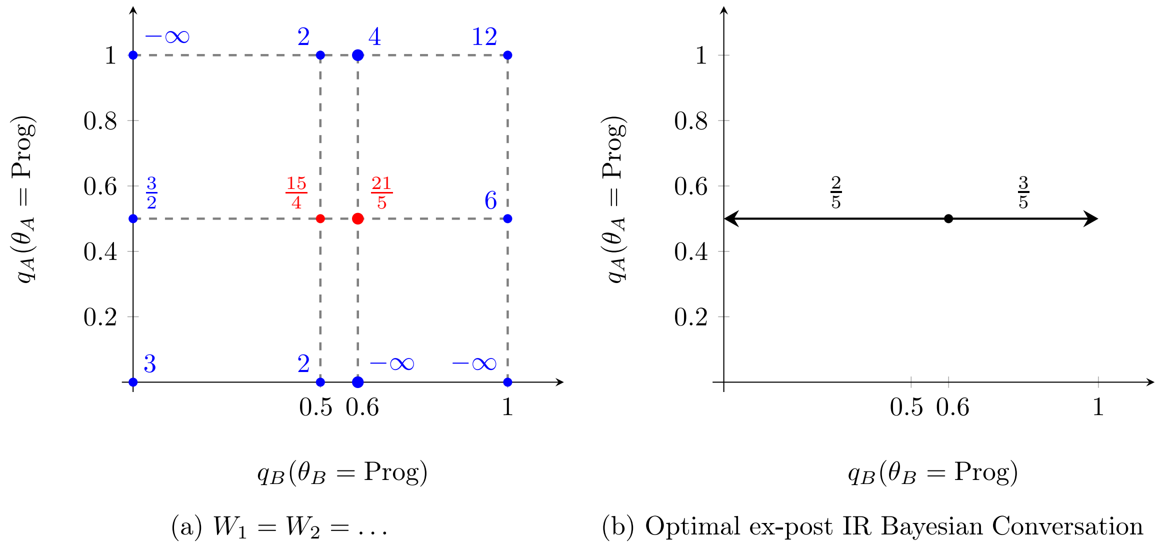


Figure 4: Illustrations of W_k for all $k \geq 1$ and the final optimal ex-post IR Bayesian conversation in the hiring problem.

$u_A(\theta_{A0}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	7	5
r_1	5	7

$u_A(\theta_{A1}, \cdot)$	$\theta_B = \theta_{B0}$	$\theta_B = \theta_{B1}$
r_0	1	3
r_1	0	5

$u_B(\theta_{B0}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	5	10
r_1	10	0

$u_B(\theta_{B1}, \cdot)$	$\theta_A = \theta_{A0}$	$\theta_A = \theta_{A1}$
r_0	10	10
r_1	10	4

Table 3: Alice and Bob's utility function.

We write the equations in Lemma 4.4 and identify an explicit recursion for $W_k(x, y)$ at the breakpoints. The full analysis can be found in Appendix G.2. Then we derive that:

$$W_{4k+1}(0.4, 0.6) = -\frac{144}{325}\left(\frac{3}{16}\right)^k + \frac{4369}{325},$$

and in particular, the welfare of the optimal protocol strictly increases with the number of rounds of communication.

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A Interim IR and non-committed protocols

We define non-committed Bayesian conversations as follows.

Definition A.1 (Non-committed Bayesian Conversation protocols)**.** *A non-committed Bayesian conversation protocol is where Alice and Bob agree to start a Bayesian conversation π but they can quit in the middle of the protocol.*

For non-committed protocols, the players can choose to quit at any step. We thus define a non-committed protocol as ex-ante IR if ex-ante players will choose to proceed at every step.

Definition A.2 (Ex-ante IR for non-committed protocols)**.** *A non-committed protocol π is ex-ante IR if at any point of the protocol, the ex-ante expected utility of completing the protocol is no lower than quitting for both of the agents. Formally, for any time step t with history $\widetilde{h}^{(t)}$ and agents' posteriors $\widetilde{q}_A^{(t)}, \widetilde{q}_B^{(t)}$, let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B | h^{(t)} = \widetilde{h}^{(t)})$ be the distribution of $(\theta_A, \theta_B, q_A, q_B)$ after the players complete the protocol conditioning on they reach $\widetilde{h}^{(t)}$ at time t. Then it requires*

$$\mathbb{E}_{(\theta_A,\theta_B,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)|h^{(t)} = \widetilde{h}^{(t)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A^{(t)}, \theta_B\sim\widetilde{q}_B^{(t)}}[u_B(\theta_B, \theta_A, r^{(t)})], \forall t, \widetilde{h}^{(t)}, \quad (5)$$

where random variable $r^{(t)} = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action facing $\widetilde{h}^{(t)}$. Again, we only need this inequality for Bob because it always holds for Alice.

Second, we consider players who make decisions after seeing their own types.

Definition A.3 (Interim players)**.** *An interim player makes decisions on participating/quitting after seeing their types and before starting the protocol.*

We define a committed protocol as interim IR if interim players will choose to participate.

Definition A.4 (Interim IR for committed protocols)**.** *A committed protocol π is interim individually rational if for any $y \in \Theta_B$, Bob's expected utility of following the entire protocol is no lower than the expected utility of no communication when Bob knows that his type is y. Formally, let $\mathbb{P}_\pi(\theta_A, q_A, q_B|\theta_B = y)$ be the conditional distribution of (θ_A, q_A, q_B) after completing the entire protocol when Bob's type is y, then it requires*

$$\mathbb{E}_{(\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B))|\theta_B = y] \geq \mathbb{E}_{\theta_A\sim\mathbb{P}(\theta_A)}[u_B(y, \theta_A, r^0)], \ \forall y \in \Theta_B, \quad (6)$$

where random variable $r^0 = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B)}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action without any communication. Again, we only need this inequality for Bob because it always holds for Alice.

A non-committed protocol is interim IR if interim players will always choose to proceed.

Definition A.5 (Interim IR for non-committed protocols)**.** *A non-committed protocol π is interim IR if at any point of the protocol, the expected utility of completing the protocol is no lower than quitting for both of the agents. Formally, for any type $y \in \Theta_B$ of Bob, for any time step t with history $\widetilde{h}^{(t)}$ and agents' posteriors $\widetilde{q}_A^{(t)}, \widetilde{q}_B^{(t)}$, let $\mathbb{P}_\pi(\theta_A, q_A, q_B|\theta_B = y, h^{(t)} = \widetilde{h}^{(t)})$ be the distribution of (θ_A, q_A, q_B) after the players complete the protocol conditioning on they reach $\widetilde{h}^{(t)}$ at time t and Bob has type y. Then it requires*

$$\mathbb{E}_{(\theta_A,q_A,q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B)|\theta_B = y, h^{(t)} = \widetilde{h}^{(t)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A^{(t)}}[u_B(y, \theta_A, r^{(t)})], \forall t, \widetilde{h}^{(t)}, y \in \Theta_B, \quad (7)$$

where random variable $r^{(t)} = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action facing $\widetilde{h}^{(t)}$. Again, we only need this inequality for Bob because it always holds for Alice.

And we define a non-committed protocol as ex-post IR if ex-post players will not regret proceeding at each step after seeing their types and completing the protocol.

Definition A.6 (Ex-post IR for non-committed protocols)**.** *A non-committed protocol π is ex-post individually rational if after completing the protocol (and before Alice taking an action), Bob never regrets proceeding at every step. More specifically, for any $y \in \Theta_B$ and any outcome $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{a}_T, \widetilde{b}_T)$, suppose Bob has type y and Bob's posterior belief becomes $\widetilde{q}_A$ after seeing $\widetilde{h}^{(T)}$ and let $\mathbb{P}_\pi(\theta_A, q_B|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)})$ be the conditional distribution of (θ_A, q_B) when Bob has type y and the protocol ends at $\widetilde{h}^{(T)}$. Then it requires*

$$\mathbb{E}_{(\theta_A,q_B)\sim\mathbb{P}_\pi}[u_B(y, \theta_A, r^*(\theta_A, q_B))|\theta_B = y, h^{(T)} = \widetilde{h}^{(T)}] \geq \mathbb{E}_{\theta_A\sim\widetilde{q}_A}[u_B(y, \theta_A, r^{(t)})], \ \forall \widetilde{h}^{(T)}, t \leq T, y \in \Theta_B, \tag{8}$$

where random variable $r^{(t)} = \arg\max_{r\in R} \mathbb{E}_{\theta_B\sim\widetilde{q}_B^{(t)}}[u_A(\theta_A, \theta_B, r)]$ is Alice's best action at time t.

A.1 Non-committed protocols

We now compare non-committed Bayesian conversation protocols with committed Bayesian conversations and mediator protocols, mainly focusing on the ex-ante IR and interim IR notions. As proved in Section 3.1, committed Bayesian conversations are equivalent to mediator protocols under ex-ante IR and interim IR. Therefore, it suffices to compare non-committed Bayesian conversations with mediator protocols.

A.1.1 Gap between the protocols under ex-ante/interim IR

We first show by the example of employer-candidate game that non-committed Bayesian conversations are not so powerful as mediator protocols/committed Bayesian conversations when we consider ex-ante IR or interim IR.

Theorem A.1. *There exists a game where the highest social welfare that can be implemented by ex-ante IR non-committed Bayesian conversations is lower than the highest social welfare that can be implemented by ex-ante IR mediator protocols/committed Bayesian conversations, and the highest social welfare that can be implemented by interim IR non-committed Bayesian conversations is lower than the highest social welfare that can be implemented by interim IR mediator protocols/committed Bayesian conversations.*

Proof. We prove by the same employer-candidate game in Section 4.2. We show that no interim IR non-committed Bayesian conversation can achieve the same social welfare as the optimal mediator protocol given in the proof of Theorem 4.2.

We first consider interim IR and prove by contradiction, assume that there exists a non-committed Bayesian conversation π that achieves the same maximum social welfare as the mediator protocol, and the protocol is interim IR. Let $\mathbb{P}_\pi(\theta_A, \theta_B, q_A, q_B)$ be the distribution of types and beliefs after completing π. As we show in the proof of Theorem 4.2, if π achieves the maximum social welfare, we must have all possible (q_B, q_A) lying in the region plotted in Figure 5c, and the protocol must end at the upper-left point $(q_B(\theta_B = \text{Prog}) = 0, q_A(\theta_A = \text{Prog}) = 1)$ with a non-zero probability. Suppose the protocol ends at the upper-left point with probability $\widetilde{p}$ with $\widetilde{h}^{(T)} = (\widetilde{a}_1, \widetilde{b}_1, \ldots, \widetilde{a}_T, \widetilde{b}_T)$. Let t be the last round in $\widetilde{h}^{(T)}$ where there is still a non-zero probability that θ_A =Comm,

$$t = \max\{i : 1 \leq i \leq T, P(\theta_A = \text{Comm}|h^{(i)} = \widetilde{h}^{(i)}) > 0\}.$$

We claim that a Comm-type candidate will not want to continue the protocol after the employer sends $\widetilde{a}_{t+1}$. This means that once the employer reveals that her type is Prog, a Comm-type candidate will not want to continue. Denote by $q_B^{(t)}$ and $q_A^{(t)}$ the players' beliefs at the end of round t. Then an important observation is that employer's belief about the candidate $q_B^{(t)}$ must have $q_B^{(t)}(\theta_B = \text{Prog}) \in [\frac{1}{2}, 1]$. This is because the probability that $\theta_A = \text{Comm}$ is still non-zero $P(\theta_A = \text{Comm}|h^{(t)} = \widetilde{h}^{(t)}) > 0$, and if we want to guarantee that the employer's final belief is either $\frac{1}{2}$ or 1 whenever she has type Comm (or in other words, (q_B, q_A) lying in the region plotted in Figure 5c), we must have $q_B^{(t)}(\theta_B = \text{Prog}) \in [\frac{1}{2}, 1]$. Based on this observation, it is not difficult to see that a Comm-type candidate will not want to continue after seeing $\widetilde{a}_{t+1}$ (and knowing that the employer has type Prog), because based on the current belief $q_B^{(t)}$ with $q_B^{(t)}(\theta_B = \text{Prog}) \geq \frac{1}{2}$, the employer (with type Prog) always hires; but if the candidate follows the protocol until the end, there is a non-zero probability $\widetilde{p}$ that the protocol will end at $\widetilde{h}^{(T)}$ and the candidate will not be hired, which is strictly worse than quitting the protocol.

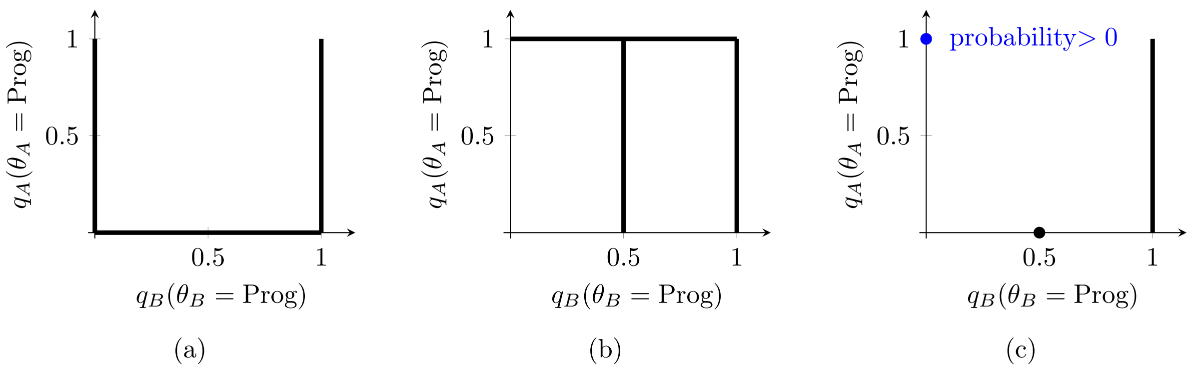


Figure 5: Regions that (q_B, q_A) must fall in after the communication in order to maximize the social welfare in the employer-candidate game. To make sure that the conditional expectation $\mathbb{E}[u_A + u_B|\theta_A = \text{Prog}]$ reaches the desired highest value, we must have (q_B, q_A) fall in the segments plotted in Figure 5a. To make sure that the conditional expectation $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}]$ reaches the desired highest value, we must have (q_B, q_A) fall in the segments plotted in Figure 5b. Taking the intersection of the two pictures, we must have (q_B, q_A) fall in the region plotted in Figure 5c in order to maximize the expected social welfare.

The proof is similar when we consider ex-ante IR. When the candidate (Bob) does not know his type, he will still want to quit after the employer sends $\widetilde{a}_{t+1}$. Because as we proved, the employer (with type Prog) always hires if the candidate chooses to quit, but if the candidate follows the protocol until the end, there is a non-zero probability that the candidate will not be hired. $\square$

B Optimal mediator protocol by linear programming

Consider a game where Alice has m possible types: $\Theta_A = \{\theta_{A1}, \ldots, \theta_{Am}\}$, and Bob has n possible types: $\Theta_B = \{\theta_{B1}, \ldots, \theta_{Bn}\}$. The set of actions Alice can take is denoted by $R = \{r_1, r_2, \ldots, r_k\}$. We specify a signal using the joint probability distribution of player types and signals, as illustrated in the table below:

B.1 Optimal ex post IR mediator protocol

We show that the optimal ex-post IR mediator protocol can be solved by an exponential-size LP.

$\Pr(\theta_A, \theta_B, s)$	θ_{B1}	θ_{B2}	$\ldots$	θ_{Bn}
θ_{A1}	$p_{11}(s)$	$p_{12}(s)$	$\ldots$	$p_{1n}(s)$
θ_{A2}	$p_{21}(s)$	$p_{22}(s)$	$\ldots$	$p_{2n}(s)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
θ_{Am}	$p_{m1}(s)$	$p_{m2}(s)$	$\ldots$	$p_{mn}(s)$

Table 4: Joint probability distribution for players' types θ_A, θ_B and signal s

Theorem B.1. *The optimal ex-post IR mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$ can be solved by a linear program with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|^{|\Theta_A|})$.*

To find the optimal ex post IR mediator protocol, we describe a signal by the actions taken by different types of Alice under that signal. For instance, in a game where $|\Theta_A| = 3$, a signal $s(r_3, r_1, r_2)$ represents that an Alice of type θ_{A1} will take action r_3 under this signal, an Alice of type θ_{A2} will take action r_1 under this signal, and so on.

Thus, we have a linear programming algorithm to find the optimal ex post IR mediator protocol. In this algorithm, we only need to enumerate all possible kinds of signals that exhaustively represent all combinations of actions that different types of Alice might take:

$$\mathcal{S} = \{s(\mathbf{r}) : \mathbf{r} \in R^m\}$$

We define the objective such that each kind of signal in the set $\mathcal{S}$ appears at most once in the optimal solution (it may also not appear, when all the corresponding probabilities $p_{ij}(s)$ are 0). The objective function maximizes social welfare, while constraints are imposed to ensure the existence of a mediator that can induce the target solution, guide different types of Alice to select the expected actions, and satisfy Bob's ex-post IR condition. The resulting linear programming formulation is as follows:

$$\begin{aligned}
\max \quad & \sum_{s=s(r_1^*, r_2^*, \ldots, r_m^*) \in \mathcal{S}} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(s) u(\theta_{Ai}, \theta_{Bj}, r_i^*) \\
\text{subject to} \quad & \sum_{s \in \mathcal{S}} p_{ij}(s) = \mathbb{P}(\theta_{Ai}) \cdot \mathbb{P}(\theta_{Bj}), \quad \forall i = 1, 2, \ldots, m, \forall j = 1, 2, \ldots, n \\
& \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r'), \\
& \qquad \forall s = s(r_1^*, r_2^*, \ldots, r_m^*) \in \mathcal{S}, \forall i = 1, 2, \ldots, m, \forall r' \neq r_i^* \\
& \sum_{i=1}^{m} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{i=1}^{m} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bj}, r_i^0), \\
& \qquad \forall s = s(r_1^*, r_2^*, \ldots, r_m^*) \in \mathcal{S}, \forall j = 1, 2, \ldots, n
\end{aligned}$$

where $r_i^0 = \arg\max_{r \in R} \mathbb{E}_{\theta_B \sim \mathbb{P}(\theta_B)}[u_A(\theta_{Ai}, \theta_B, r)]$ is the best action for Alice of type θ_{Ai} without any communication.

Theorem B.2. *The above linear programming method with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|^{|\Theta_A|})$ can find a ex post mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$.*

Proof. The first constraint in the linear programming formulation satisfies $\mathbb{E}[q_{AB}] = \mathbb{P}(\theta_A) \times \mathbb{P}(\theta_B)$, which guarantees the existence of a mediator protocol capable of implementing the identified probability distribution.

The second constraint ensures that the best response for each type of Alice corresponds to the action induced by the signal. The third constraint guarantees that each type of Bob satisfies the ex-post IR condition.

Therefore, what we need to prove is that the previously assumed set of signals $\mathcal{S}$ can be used to find the optimal mediator protocol.

To prove this, we only need to show that: given two signals s and s', under which the same type of Alice chooses the same action, and both satisfy Bob's ex-post IR condition, then the summation signal s_{sum} (defined such that $\Pr(\theta_{Ai}, \theta_{Bj}, s_{\text{sum}}) = p_{ij}(s_{\text{sum}}) = p_{ij}(s) + p_{ij}(s'))$ can also induce the corresponding type of Alice to take the same action while maintaining Bob's ex-post IR condition.

This is easy to prove because the signal summation defined above does not change the expectation of q_{AB} induced by the protocol, thereby satisfying the first constraint.

The second and third constraints are equivalent to the following:

$$\sum_{j=1}^{n} p_{ij}(s)(u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) - u_A(\theta_{Ai}, \theta_{Bj}, r')) \geq 0,$$
$$\sum_{i=1}^{m} p_{ij}(s)(u_B(\theta_{Ai}, \theta_{Bj}, r_i^*) - u_B(\theta_{Ai}, \theta_{Bj}, r_i^0)) \geq 0,$$

If $p_{ij}(s)$ and $p_{ij}(s')$ both satisfy these constraints, then summing the corresponding inequalities gives $p_{ij}(s_{\text{sum}}) = p_{ij}(s) + p_{ij}(s')$ also satisfies these constraints, which completes the proof.

There are $|R|^{|\Theta_A|}$ signals in total, and each of them has $|\Theta_A| \cdot |\Theta_B|$ variables, so the size of this linear program is $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|^{|\Theta_A|})$.

□

B.2 Optimal ex ante IR mediator protocol

To find the optimal ex ante IR mediator protocol, we focus on signals that may occur for only one type of Alice and classify them based on the action Alice would take in response. For example, the signal $s(i, l)$ indicates that for all $i' \neq i$ and $r = 1, 2, ..., n$, $p_{i'n}(s) = 0$, and this signal induces Alice of type θ_{Ai} to take action r_l.

$\Pr(\theta_A, \theta_B, s)$	θ_{B1}	θ_{B2}	$\dots$	θ_{Bn}
θ_{A1}	0	0	$\dots$	0
θ_{A2}	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$
θ_{Ai}(will do r_l)	$p_{i1}(s)$	$p_{i2}(s)$	$\dots$	$p_{in}(s)$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
θ_{Am}	0	0	$\dots$	0

Table 5: Joint probability distribution for players' types θ_A, θ_B and signal $s(i, l)$

We denote the set of this new classification as $\mathcal{S}'$.

$$\mathcal{S}' = \{s(i,l) : i = 1, 2, \dots, m, l = 1, 2, \dots, k\}$$

The new linear program is similar, with only the last constraint having changed:

$$\begin{aligned}
\max \quad & \sum_{s=s(i,l)\in\mathcal{S}'} \sum_{j=1}^{n} p_{ij}(s) u(\theta_{Ai}, \theta_{Bj}, r_l) \\
\text{subject to} \quad & \sum_{s\in\mathcal{S}'} p_{ij}(s) = \mathbb{P}(\theta_{Ai}) \cdot \mathbb{P}(\theta_{Bj}), \quad \forall i = 1, 2, \dots, m, \forall j = 1, 2, \dots, n \\
& \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r'), \quad \forall s = s(i,l) \in \mathcal{S}', \forall r' \neq r_l \\
& \sum_{s=s(i,l)\in\mathcal{S}'} \sum_{j=1}^{n} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bi}, r_l) \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B), \theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)]
\end{aligned}$$

Theorem B.3. *The above linear programming method with size $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$ can find a ex ante mediator protocol that maximizes the expectation of a utility function $u(\theta_A, \theta_B, r)$.*

Proof. First, the sum of two signals of the same kind still satisfies the constraints above. This is straightforward, similar to the proof provided earlier, and it also utilizes the linearity of the constraints. We can simply add the probabilities at corresponding positions to obtain a new solution that meets the conditions.

Next, we prove that decomposing any signal $s(r_1^*, r_2^*, \dots, r_m^*) \in \mathcal{S}$ into signals

$$\{s(1, r_1^*), s(2, r_2^*) \dots, s(m, r_m^*)\} \in \mathcal{S}'$$

(that is, taking each row as a separate signal and setting other positions to 0) can also satisfy constraints:

$$\begin{aligned}
& \sum_{s} p_{ij}(s) = \mathbb{P}(\theta_{Ai}) \cdot \mathbb{P}(\theta_{Bj}), \quad \forall i = 1, 2, \dots, m, \forall j = 1, 2, \dots, n \\
& \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \sum_{j=1}^{n} p_{ij}(s) u_A(\theta_{Ai}, \theta_{Bj}, r'), \quad \forall s, \forall i = 1, 2, \dots, m, \forall r' \neq r_i^* \\
& \sum_{s=s(r_1^*, r_2^*, \dots, r_m^*)} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(s) u_B(\theta_{Ai}, \theta_{Bj}, r_i^*) \geq \mathbb{E}_{\theta_B\sim\mathbb{P}(\theta_B), \theta_A\sim\mathbb{P}(\theta_A)}[u_B(\theta_A, \theta_B, r^0)]
\end{aligned}$$

Proving that the first and second constraints are satisfied is easy. The first constraint requires that the relationships between corresponding positions remain unchanged, and the second constraint requires that the relationships within corresponding rows remain unchanged. Such decomposition does not break these relative relationships.

Since the actions taken by Alice at corresponding positions do not change before and after the decomposition, the left side of the third constraint remains unchanged. The right side is a constant. Therefore, the third constraint is also satisfied.

Notice that the signals resulting from the decomposition can be added to signals of the same kind that have been decomposed from other original signals. Therefore, each kind of signal only needs to appear at most once in the linear program.

There are $|R| \cdot |\Theta_A|$ signals in total, and each of them has $|\Theta_B|$ variables, so the size of this linear program is $O(|\Theta_A| \cdot |\Theta_B| \cdot |R|)$. $\square$

C Equivalence for interim IR

The result for interim IR is similar.

Theorem C.1. *For any utility function $u(\theta_A, \theta_B, r)$, the range of expected utilities that can be implemented by interim IR committed Bayesian conversations is equivalent to the range of expected utilities that can be implemented by interim IR mediator protocols,*

$$Range_{interim}(u, \Pi_{BC}) = Range_{interim}(u, \Pi_M),$$

where $Range_{interim}(u, \Pi) = Range(u, \Pi, \mathcal{P})$ with $\mathcal{P}$ being the subset of distributions that guarantee interim IR for committed protocols (satisfy equation (6)).

Proof. The proof for interim IR is basically the same as the proof for ex-ante IR. The only difference is that we need to prove the constraint of interim IR (6) can be reduced to a constraint on $P(\theta_A, q_B)$ as well, which is true because interim IR can be written as

$$\sum_{\theta_A, q_B} P(\theta_A, q_B) q_B(\theta) u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)) \geq \mathbb{P}(\theta_B) \sum_{\theta_A} \mathbb{P}(\theta_A) u_B(\theta_B, \theta_A, r^*(\theta_A, \mathbb{P}(\theta_B))), \forall \theta_B \in \Theta_B.$$

$\square$

D Feasible posterior distributions for Bayesian Conversations

In this section, we investigate the following problem: Given a belief distribution $P(\theta_A, \theta_B, q_B, q_A)$, or equivalently $P(q_B, q_A)$, we want to decide whether it can be generated by a Bayesian conversation π in T rounds.

According to [5], the belief-splitting process of a Bayesian conversation $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^{\infty}$ can be viewed as a *bimartingale*. Furthermore, [5] showed that given a set of final beliefs $A = \{(q_A^{(1)}, q_B^{(1)}), \ldots, (q_A^{(K)}, q_B^{(K)})\}$, the set of feasible priors $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ is the functional bi-convex hull ([33]) of A, where we say that a prior $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ is feasible if there exists a Bayesian conversation, possibly infinite-round, that starts with the prior $(\mathbb{P}(\theta_a), \mathbb{P}(\theta_B))$ and generates a final belief distribution supported on A. However, their method does not decide whether a distribution $P(q_B, q_A)$ over final beliefs can be generated by a Bayesian conversation.

To decide the feasibility of a distribution over beliefs, we augment the bimartingale $\{(q_A^{(t)}, q_B^{(t)})\}_{t=1}^{\infty}$ as a *dimartingale* (defined in [24]) which also includes *a bystander's belief about Alice and Bob's final beliefs.* The dimartingale is defined as follows.

Definition D.1 (dimartingale [24])**.** *A dimartingale $\{(\alpha_t, \beta_t, p_t)\}_{t=1}^{T}$ is a martingale that has either $\alpha_t = \alpha_{t+1}$ or $\beta_t = \beta_{t+1}$ at each time step t.*

Lemma D.1. *Consider any Bayesian conversation π. Let $q_A^{(t)}$ be Bob's (and a by-stander's belief about θ_A at the end of round t. Similarly, let $q_B^{(t)}$ be Alice's belief about θ_B at the end of round t. Let $\gamma^{(t)} = \mathbb{P}_\pi(q_A^{(T)}, q_B^{(T)} | h^{(t)})$ be a by-stander's belief about $q_A^{(T)}, q_B^{(T)}$ (the players' **final** beliefs) at the end of round t. Then for any Bayesian conversation π, $\{(q_A^{(t)}, q_B^{(t)}, \gamma^{(t)})\}_{t=1}^{T}$ form a dimartingale.*

Then we can decide whether a distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a finite-time Bayesian conversation by reversing this dimartingale. The reversing process can be formalized as follows.

Given a final belief distribution $P(\theta_A, \theta_B, q_B, q_A)$, we want to decide whether it can be generated by a Bayesian conversation π in T rounds. Consider the marginal distribution of the beliefs $P(q_B, q_A)$. Let $\mathcal{Q}$ be the support of $P(q_B, q_A)$, $\mathcal{Q} = \{(q_B, q_A) : P(q_B, q_A) > 0\}$. Let $e_{q_B,q_A} \in \Delta(\mathcal{Q})$ be the deterministic distribution on $\mathcal{Q}$ that takes value (q_B, q_A) with probability 1. Then for every $(q_B, q_A) \in \mathcal{Q}$, we append e_{q_B,q_A} and add the tuple into a set

$$\mathcal{S}_0 = \{(q_B, q_A, e_{q_B,q_A}) : (q_B, q_A) \in \mathcal{Q}\}.$$

Define $\phi_1(q_B, q_A, z) = q_B$ and $\phi_2(q_B, q_A, z) = q_A$ for any $z \in \Delta(\mathcal{Q})$. Then for any points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{S}_i$ with the same q_A, we add all their convex combinations to set $\mathcal{T}_i$; and then for any points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{T}_i$ with the same q_B, we add all their convex combinations to set $\mathcal{S}_{i+1}$.

$$\mathcal{T}_i = \left\{ \begin{array}{rl} \lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k : & \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{S}_i,\ \phi_2(\mathbf{x}_1) = \cdots = \phi_2(\mathbf{x}_k) \\ & \lambda_1 + \cdots + \lambda_k = 1,\ 0 \le \lambda_i \le 1 \end{array} \right\}$$

$$\mathcal{S}_{i+1} = \left\{ \begin{array}{rl} \lambda_1\mathbf{x}_1 + \cdots + \lambda_k\mathbf{x}_k : & \mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathcal{T}_i,\ \phi_1(\mathbf{x}_1) = \cdots = \phi_1(\mathbf{x}_k) \\ & \lambda_1 + \cdots + \lambda_k = 1,\ 0 \le \lambda_i \le 1 \end{array} \right\}$$

See Figure 6 for an example.

Theorem D.1. *A distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be implemented by a T-round Bayesian conversation if and only if*

1. *$P(\theta_A, \theta_B | q_B, q_A) = q_A(\theta_A) \cdot q_B(\theta_B)$, for all q_B, q_A;*
2. *$(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$, where $\mathbb{P}(\theta_B), \mathbb{P}(\theta_A)$ are the priors of θ_B and θ_A, and $P(q_B, q_A)$ is the marginal distribution of q_B, q_A based on P.*

We give a proof for Theorem D.1 in Appendix E.4.

E Missing proofs in Section 2.3

E.1 Proof of Proposition 2.2

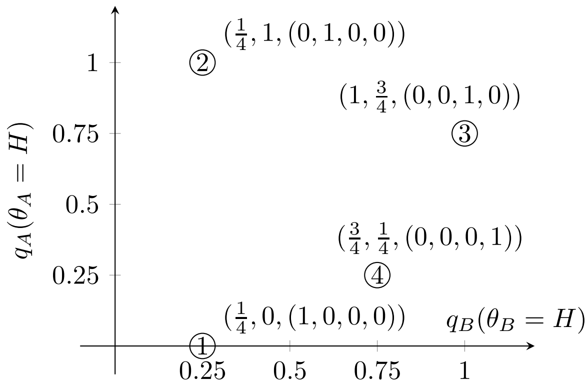
Proof. Let $|\text{supp}(P)| = Q$ and label the elements of the support $q_{AB}^{(s)}$ for $s \in [Q]$. The mediator will send a randomized signal $\pi(\theta_A, \theta_B) \in [Q]$ defined via

$$\Pr_\pi[\pi(\theta_A, \theta_B) = s] = \frac{q_{AB}^{(s)}(\theta_A, \theta_B) \cdot P\left(q_{AB}^{(s)}\right)}{\Pr[\theta_A] \cdot \Pr[\theta_B]}.$$

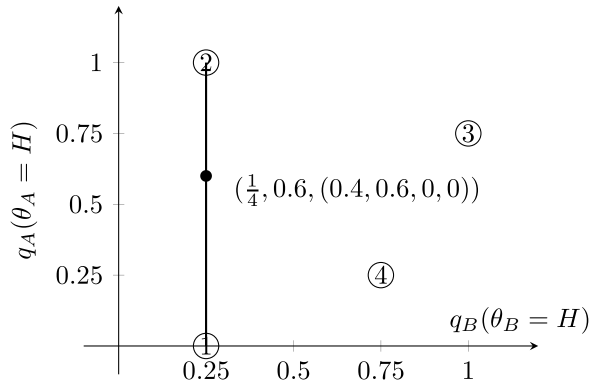
(If $\Pr[\theta_A] \cdot \Pr[\theta_B] = 0$, we can set the probabilities arbitrarily, since the mediator is guaranteed to never receive that pair of (θ_A, θ_B)). It follows that the posterior conditioned on receiving signal s equals $q_{AB}^{(s)}$ (and therefore must match this distribution). □

E.2 Proof for the Example in Figure 1

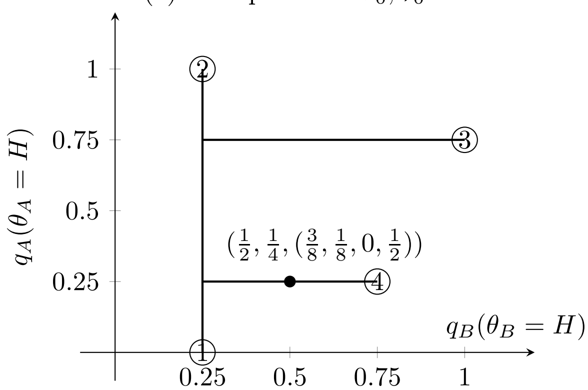
We show that it is not possible to start with $\theta_A, \theta_B \in \{L, H\}$, $\mathbb{P}(\theta_A = H) = \mathbb{P}(\theta_B = H) = 0.5$ and have $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5. Suppose to the contrary, we have a mediator protocol that gives $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability



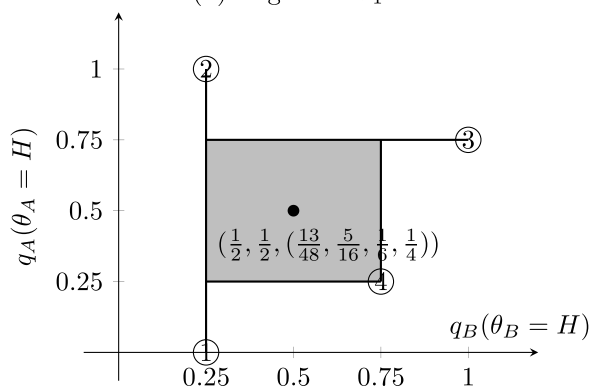
(a) Four points in $\mathcal{S}_0, \mathcal{T}_0$



(b) Region in $\mathcal{S}_1$



(c) Region in $\mathcal{T}_1$



(d) Region in $\mathcal{S}_2$.

Figure 6: An example of set $\mathcal{S}_i$. Suppose $\theta_A, \theta_B \in \{L, H\}$ and $\mathbb{P}(\theta_A) = \mathbb{P}(\theta_B) = 0.5$. Consider a posterior distribution that satisfies Condition (1) with marginal distribution $P(q_B, q_A)$ being the four points in Figure 6a with probabilities $(\frac{13}{48}, \frac{5}{16}, \frac{1}{6}, \frac{1}{4})$ respectively. Then we have S_0 plotted in the first picture with each point labeled as $(q_B(\theta_B = H), q_A(\theta_A = H), e_{q_B, q_A})$. Notice that no two points in S_0 share the same q_A, so we have $\mathcal{T}_0 = \mathcal{S}_0$. To get $\mathcal{S}_1$, we add all convex combinations of the points with the same q_B, which is just the segment plotted in Figure 6b. To get $\mathcal{T}_1$ we add all convex combinations of the points with the same q_A as in Figure 6c. Finally, we get the region of $\mathcal{S}_2$, and we have $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) = (0.5, 0.5, (\frac{13}{48}, \frac{5}{16}, \frac{1}{6}, \frac{1}{4})) \in \mathcal{S}_2$, so the posterior distribution is feasible.

$P(\cdot\|s)$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	9/16	3/16
$\theta_A = L$	3/16	1/16

$P(\cdot\|s)$	$\theta_B = H$	$\theta_B = L$
$\theta_A = H$	1/16	3/16
$\theta_A = L$	3/16	9/16

$0.5\cdot$

$P(\cdot\|s)$	H	L
H	9/16	3/16
L	3/16	1/16

$+\ 0.5\cdot$

$P(\cdot\|s)$	H	L
H	1/16	3/16
L	3/16	9/16

$\neq$

$\mathbb{P}(\cdot)$	H	L
H	1/4	1/4
L	1/4	1/4

0.5 in the end. This means that the two players have to simultaneously hold the same belief 0.75 or 0.25. Then after seeing the public signal s, the conditional distribution of the two players' types (i.e., the observer's posterior) must be one of the following tables, and we must arrive at each of the tables with probability 0.5. But this does not match the prior. Therefore it is not possible to have

$q_B(\theta_B = H) = q_A(\theta_A = H) = 0.75$ with probability 0.5 and $q_B(\theta_B = H) = q_A(\theta_A = H) = 0.25$ with probability 0.5.

E.3 Observations about Bayesian conversations

In this section, we give several observations about the posterior distributions generated by Bayesian conversations.

Observation E.1 (Lemma 3 in [8]). *For a T-round Bayesian conversation π and any $1 \le t \le T$, let $q_B^{(t)} = P(\theta_B|\theta_A, h^{(t)})$ be Alice's belief about θ_B at the end of round t, and let $q_A^{(t)} = P(\theta_A|\theta_B, h^{(t)})$ be Bob's belief about θ_A at the end of round t. Then we have*

- *The players' beliefs $q_A^{(t)}, q_B^{(t)}$ are uniquely determined by the history $\widetilde{h}^{(t)}$ no matter what their types are, i.e., for any two possible types of Alice $x, x' \in \Theta_A$, we have $P(\theta_B|\theta_A = x, \widetilde{h}^{(t)}) = P(\theta_B|\theta_A = x', \widetilde{h}^{(t)})$, and the same holds for Bob.*
- *Alice's belief about Bob's type does not change before and after sending her signal a_t, and the same holds for Bob.*
- *Conditioning on any history $\widetilde{h}^{(t-1)}$ before Alice sending a_t, the expectation of Bob's belief does not change before and after Alice sends a_t. More specifically, let $\widetilde{q}_A^{(t-1)} = P(\theta_A|\theta_B, \widetilde{h}^{(t-1)})$ be Bob's unique belief after seeing $\widetilde{h}^{(t-1)}$ and let random variable $q_A^{(t)} = P(\theta_A|\theta_B, \widetilde{h}^{(t-1)}, a_t)$ be Bob's belief after seeing $\widetilde{h}^{(t-1)}$ and a_t. Then we have*

  $$\mathbb{E}_{a_t}[q_A^{(t)}] = \widetilde{q}_A^{(t-1)}.$$

  The same holds for Bob. The observation is illustrated in Figure 7.

Observation E.2 (Lemma 2 in [8]). *The joint distribution of the players' types and beliefs $P(\theta_A, \theta_B, q_A^{(t)}, q_B^{(t)})$ can be fully determined by the marginal distribution of their beliefs $P(q_A^{(t)}, q_B^{(t)})$ as*

$$P(\theta_A, \theta_B|q_A^{(t)}, q_B^{(t)}) = q_A^{(t)}(\theta_A) \cdot q_B^{(t)}(\theta_B).$$

Observation E.1 and Observation E.2 are implied by the following Observation E.3 and the fact that θ_A and θ_B are independently drawn from $\mathbb{P}(\theta_A)$ and $\mathbb{P}(\theta_B)$.

We consider $P(\theta_A, \theta_B, h^{(t)})$, the probability of reaching a history $h^{(t)}$ while the players' types being θ_A, θ_B, and characterize how $P(\theta_A, \theta_B, h^{(t)})$ evolves. For any round t and any realization $\widetilde{h}^{(t)}$, we represent $P(\theta_A, \theta_B, h^{(t)} = \widetilde{h}^{(t)})$ by a $|\Theta_A| \times |\Theta_B|$ matrix M_t with

$$M_t[x, y] = P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}),\ \forall x \in \Theta_A, y \in \Theta_B.$$

Observation E.3. *For any realization $\widetilde{h}^{(t+1)} = (\widetilde{h}^{(t)}, \widetilde{a}_{t+1}, \widetilde{b}_{t+1})$, after Alice sending $\widetilde{a}_{t+1}$, the probability matrix becomes*

$$D_{\widetilde{a}_{t+1}} \cdot M_t,$$

where $D_{\widetilde{a}_{t+1}}$ is a $|\Theta_A| \times |\Theta_A|$ diagonal matrix with $D_{\widetilde{a}_{t+1}}[x, x] = \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1})$ for all $x \in \Theta_A$. And after Bob sending $\widetilde{b}_{t+1}$, the probability matrix becomes

$$M_{t+1} = D_{\widetilde{a}_{t+1}} \cdot M_t \cdot Z,$$

where Z is a $|\Theta_B| \times |\Theta_B|$ diagonal matrix with $Z[y, y] = \Pr(g_{t+1}(y, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}) = \widetilde{b}_{t+1})$ for all $y \in \Theta_B$.

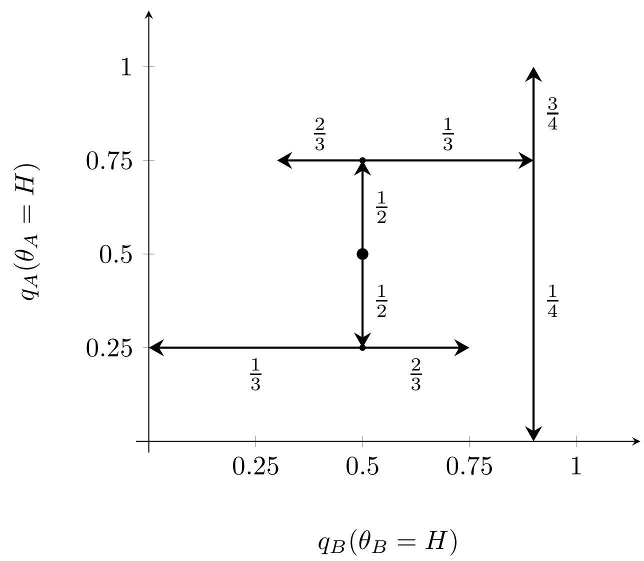


Figure 7: Illustration of Observation E.1. Suppose $\theta_A, \theta_B \in \{L, H\}$. At any step of the protocol, the status of the protocol can be represented as a two-dimensional point $(q_B(\theta_B = H), q_A(\theta_A = H))$. When Alice sends a signal, q_B remains unchanged and q_A is decomposed along y-axis while preserving the expectation. The same holds for Bob.

Proof. The proof follows directly from the formulas

$$P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}, a_{t+1} = \widetilde{a}_{t+1}) = P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}) \cdot \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1})$$

and

$$\begin{aligned} &P(\theta_A = x, \theta_B = y, h^{(t+1)} = \widetilde{h}^{(t+1)}) \\ =&P(\theta_A = x, \theta_B = y, h^{(t)} = \widetilde{h}^{(t)}) \cdot \Pr(f_{t+1}(x, \widetilde{h}^{(t)}) = \widetilde{a}_{t+1}) \Pr(g_{t+1}(y, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}) = \widetilde{b}_{t+1}). \end{aligned}$$

□

E.4 Proof of Theorem D.1

We first prove that if a distribution $P(\theta_A, \theta_B, q_B, q_A)$ can be generated by a Bayesian conversation π, i.e., $P(\theta_A, \theta_B, q_B, q_A) = \mathbb{P}_\pi(\theta_A, \theta_B, q_B^{(T)}, q_A^{(T)})$, then it must satisfy (1) and (2). For (1), it is directly proved by Observation E.2 because we must have $\mathbb{P}_\pi(\theta_A, \theta_B | q_A^{(T)}, q_B^{(T)}) = q_A^{(T)}(\theta_A) \cdot q_B^{(T)}(\theta_B)$. For (2), we prove that $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$ by showing that

$$(\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)})) \in \mathcal{S}_{T-t}, \text{ for all } t, \widetilde{h}^{(t)},$$

$$(\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)}, \widetilde{a}_{t+1})) \in \mathcal{T}_{T-t-1}, \text{ for all } t, \widetilde{h}^{(t)}, \widetilde{a}_{t+1}$$

where $\widetilde{h}^{(t)}$ is any realization of history up to round t, and $\widetilde{q}_B^{(t)}, \widetilde{q}_A^{(t)}$ are the players' beliefs at the end of round t when the history is $\widetilde{h}^{(t)}$, and $\mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(t)}))$ is the distribution of the players' final beliefs conditioning on the history up to round t is $\widetilde{h}^{(t)}$. We prove by induction. First, it is clear that the statement holds for $t = T$,

$$(\widetilde{q}_B^{(T)}, \widetilde{q}_A^{(T)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)} | \widetilde{h}^{(T)})) \in \mathcal{S}_0$$

because $\mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(T)})) = e_{\widetilde{q}_B^{(T)}, \widetilde{q}_A^{(T)}}$ and it is the definition of $\mathcal{S}_0$. Then assume that the statement is always true for $t \geq k+1$, it must also hold for $t = k$. According to Observation E.1, when Bob sending a signal, q_A does not change and q_B is decomposed into a convex combination of points. This means that $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(k)}, \widetilde{a}_{k+1}))$ must be a convex combination of the points in the next round $(q_B^{(k+1)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(k)}, \widetilde{a}_{k+1}, b_{k+1}))$ with the same $\widetilde{q}_A^{(k+1)}$. By induction, $(q_B^{(k+1)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(k)}, \widetilde{a}_{k+1}, b_{k+1}))$ are in the set $\mathcal{S}_{T-k-1}$ and by the definition of $\mathcal{T}_i$, we must have $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k+1)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(k)}, \widetilde{a}_{k+1})) \in \mathcal{T}_{T-k-1}$ because it is a convex combination of points in $\mathcal{S}_{T-k-1}$ with the same q_A. Similarly, we must have $(\widetilde{q}_B^{(k)}, \widetilde{q}_A^{(k)}, \mathbb{P}_\pi(q_B^{(T)}, q_A^{(T)}|\widetilde{h}^{(k)})) \in \mathcal{S}_{T-k}$.

We then prove if a distribution $P(\theta_A, \theta_B, q_B, q_A)$ satisfies (1) and (2), it can be generated by a Bayesian conversation. For any $P(\theta_A, \theta_B, q_B, q_A)$ that satisfies (1) and (2), we construct a Bayesian conversation by reversing the merging path. We start from $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$ and be definition, there exists a convex combination of points in $\mathcal{T}_{T-1}$ with the same q_B that gives $(\mathbb{P}(\theta_B), \mathbb{P}(\theta_A), P(q_B, q_A)) \in \mathcal{S}_T$. Then we can define signal distribution $f_1(\theta_A)$ to be the distribution that decomposes $q_A = \mathbb{P}(\theta_A)$ to the q_A's in this convex combination. Similarly, for each point in the decomposition, we can find its convex combination of points in $\mathcal{S}_{T-1}$ with the same q_A and we can define the corresponding $g_1(\theta_A, a_1)$. Repeating this process for T rounds, we get a Bayesian conversation that generates the correct marginal distribution $P(q_B, q_A)$. Then by Observation E.2, we must have the induced $\mathbb{P}_\pi(\theta_A, \theta_B|q_B, q_A) = q_A(\theta)q_B(\theta_B)$, which matches $P(\theta_A, \theta_B|q_B, q_A)$ because of Condition (1).

F Missing proofs in Section 3

F.1 Proof of Theorem 3.1

Proof. For a utility function u and a class of protocol Π, $\text{Range}_{\text{ex-ante}}(u, \Pi)$ can be represented as the set of

$$\begin{aligned} &\sum_{\theta_A, \theta_B, q_A, q_B} P(\theta_A, \theta_B, q_A, q_B) u(\theta_A, \theta_B, r^*(\theta_A, q_B)) \\ \text{s.t.} \quad & P(\theta_A, \theta_B, q_A, q_B) \text{ satisfies (1)} \\ & P(\theta_A, \theta_B, q_A, q_B) \text{ can be induced by a protocol } \pi \in \Pi \end{aligned}$$

The key observation is the following: if Alice's posterior belief is q_B after the communication, then the conditional probability of $\theta_B = x$ must be equal to $q_B(x)$ for $x \in \Theta_B$. We can thus simplify the expected utility so that it only depends on θ_A and q_B. Define $U(\theta_A, q_B) = \sum_{x \in \Theta_B} q_B(x) u(\theta_A, x, r^*(\theta_A, q_B))$ to be the expected utility when Alice's type is θ_A and her posterior belief is q_B. Then the expected utility is equal to $\sum_{\theta_A, q_B} P(\theta_A, q_B) U(\theta_A, q_B)$. For the same reason, the constraint of ex-ante IR (1) can also be reduced to a constraint on $P(\theta_A, q_B)$,

$$\sum_{\theta_B, \theta_A, q_B} P(\theta_A, q_B) q_B(\theta_B) u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)) \geq \sum_{\theta_B, \theta_A} \mathbb{P}(\theta_B)\mathbb{P}(\theta_A) u_B(\theta_B, \theta_A, r^*(\theta_A, \mathbb{P}(\theta_B))). \quad (9)$$

Then when finding the range of implementable expected utilities, we only need to consider the marginal distributions of θ_A and q_B that can be induced by the class of protocols,

$$\begin{aligned} &\sum_{\theta_A, q_B} P(\theta_A, q_B) U(\theta_A, q_B) \\ \text{s.t.} \quad & P(\theta_A, q_B) \text{ satisfies (9)} \\ & P(\theta_A, q_B) \text{ can be induced by a protocol } \pi \in \Pi \end{aligned}$$

We then show that the set of marginal distributions $P(\theta_A, q_B)$ that can be induced by mediator protocols is the same as the set of $P(\theta_A, q_B)$ that can be induced by one-round Bayesian conversations. We prove this by an exact characterization of feasible marginal distributions: $P(\theta_A, q_B)$ can be induced by a mediator protocol/Bayesian conversation if and only if

$$\sum_{q_B} P(\theta_A, q_B) q_B(\theta_B) = \mathbb{P}(\theta_A)\mathbb{P}(\theta_B), \quad \forall \theta_A, \theta_B. \tag{10}$$

This means that for all θ_A, the conditional expectation of q_B, that is $\mathbb{E}[q_B|\theta_A]$, must be equal to the prior $\mathbb{P}(\theta_B)$. By Proposition 2.2, the equation is necessary. And it is also sufficient because any marginal distribution that satisfies the equation can be implemented by a one-round Bayesian conversation: Alice first fully reveals θ_A, and then based on the observed θ_A, Bob sends a signal so that Alice's posterior belief will follow the distribution $P(q_B|\theta_A)$. It is always possible for Bob to generate $P(q_B|\theta_A)$ because we have

$$\sum_{q_B} P(q_B|\theta_A) q_B(\theta_B) = \mathbb{P}(\theta_B), \quad \forall \theta_A, \theta_B,$$

then Bob can just send a signal b_{q_B} with probability $P(q_B|\theta_A)q_B(\theta_B)/\mathbb{P}(\theta_B)$ when his type is θ_B and Alice's observed type is θ_A, so that when b_{q_B} is sent, Alice's belief becomes q_B and that happens with probability $P(q_B|\theta_A)\sum_{\theta_B} \mathbb{P}(\theta_B)q_B(\theta_B)/\mathbb{P}(\theta_B) = P(q_B|\theta_A)$. This completes our proof. □

F.2 Extension of Theorem 3.1

Suppose now we consider a vector of utility functions $\mathbf{u} = (u_1, \ldots, u_L)$. Define

$$\begin{aligned}\text{Range}(\mathbf{u}, \Pi, \mathcal{P}) = \{&(\mathbb{E}_P[u_1(\theta_A, \theta_B, r^*(\theta_A, q_B))], \ldots, \mathbb{E}_P[u_L(\theta_A, \theta_B, r^*(\theta_A, q_B))]) \\ &: P \in \mathcal{P} \text{ and } P \text{ can be induced by a protocol } \pi \in \Pi\}.\end{aligned}$$

Then when we only need ex-ante/interim IR, the two classes of protocols induce the same range of utilities.

Theorem F.1. *When the players are committed and the base game has only one action-taker, the utility vectors that can be implemented by ex-ante/interim IR mediator protocols is the same as the utility vectors that can be implemented by ex-ante/interim IR Bayesian conversations,*

$$\begin{aligned}\mathit{Range}_{\textit{ex-ante}}(\mathbf{u}, \Pi_{\textit{two-way}}) &= \mathit{Range}_{\textit{ex-ante}}(\mathbf{u}, \Pi_{\textit{mediator}}), \\ \mathit{Range}_{\textit{interim}}(\mathbf{u}, \Pi_{\textit{two-way}}) &= \mathit{Range}_{\textit{interim}}(\mathbf{u}, \Pi_{\textit{mediator}}).\end{aligned}$$

Proof. The key idea is basically the same as in the proof of Theorem 3.1. For any θ_A and q_B, define $U_j(\theta_A, q_B) = \sum_{x \in \Theta_B} q_B(x) u_j(\theta_A, x, r^*(\theta_A, q_B))$ to be the expected u_j when Alice's type is θ_A and her posterior belief is q_B. Then the utility vectors that can be implemented by ex-ante IR mediator/Bayesian conversations can be equally represented as the set of

$$\left(\sum_{\theta_A, q_B} P(\theta_A, q_B) U_1(\theta_A, q_B), \ldots, \sum_{\theta_A, q_B} P(\theta_A, q_B) U_L(\theta_A, q_B) \right)$$

$$\text{s.t.} \quad \sum_{q_B} P(\theta_A, q_B) q_B(\theta_B) = \mathbb{P}(\theta_A)\mathbb{P}(\theta_B), \quad \forall \theta_A, \theta_B \tag{feasibility}$$

$$\sum_{\theta_B, \theta_A, q_B} P(\theta_A, q_B) q_B(\theta_B) u_B(\theta_B, \theta_A, r^*(\theta_A, q_B)) \geq \sum_{\theta_B, \theta_A} \mathbb{P}(\theta_B)\mathbb{P}(\theta_A) u_B(\theta_B, \theta_A, r^*(\theta_A, \mathbb{P}(\theta_B))), \tag{ex-ante IR}$$

For interim IR, we just replace the inequality for ex-ante IR with the inequality for interim IR.

Setting $u_1 = u_A$ and $u_2 = u_B$, we know that the implementable pairs of the two players' utilities are the same, which implies that the induced Pareto frontiers are the same. □

F.3 Two action-takers

Suppose Alice and Bob each holds a private random bit $\theta_A, \theta_B \in \{0,1\}$ with $\mathbb{P}(\theta_A) = \mathbb{P}(\theta_B) = 0.5$. The designer wants to reveal minimum information so that both of them know the AND of their bits. The designer's utility is

$$u(\theta_A, \theta_B, q_B, q_A) = \begin{cases} 0, & \text{if } \theta_A = \theta_B = 1 \\ H(q_B), & \text{if } \theta_A = 0, \theta_B = 1, \text{ and } q_A(\theta_A = 0) = 1 \\ H(q_A), & \text{if } \theta_A = 1, \theta_B = 0, \text{ and } q_B(\theta_B = 0) = 1 \\ H(q_A) + H(q_B), & \text{if } \theta_A = \theta_B = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

where $H(\cdot)$ is the entropy function $H(q) = -q \log q - (1-q)\log(1-q)$. Then a mediator protocol can achieve the highest expected utility for the designer by directly revealing the value of θ_A AND θ_B, which give $\mathbb{E}[u] = 1$. [12] prove that no Bayesian conversation can achieve this expected utility.

Theorem F.2 (Theorem 7.1 and Theorem 7.9 in [12])**.** *No Bayesian conversation (a.k.a. two-way communication protocol) can achieve an expected utility of* 1 *for the designer. In addition, the optimal Bayesian conversation takes infinitely many rounds.*

We can turn the utility function $u(\theta_A, \theta_B, q_B, q_A)$ into a utility function $u(\theta_A, \theta_B, r_A, r_B)$ that depends on the players' types and actions by setting $R_A = R_B = [0,1]$ and define the players' utility functions to be the value of the log scoring rule,

$$u_A(r_A, \theta_B) = \log(r_A) \cdot \mathbf{1}[\theta_B = 1] + \log(1 - r_A) \cdot \mathbf{1}[\theta_B = 0].$$

so that Alice's best action when she holds belief q_B is just $r_A^* = q_B(\theta_B = 1)$, and Bob's best action when he holds belief q_A is just $r_B^* = q_A(\theta_A = 1)$.

G Examples

G.1 Gap between ex-post IR mediator protocols and Bayesian conversations

We provide the optimal mediator protocol and Bayesian conversation for the hiring problem in Section 4.2.

Optimal mediator protocol. The optimal ex-post IR mediator protocol π that achieves the highest possible social welfare is as follows. Consider a mediator who sends two possible public signals s_1 ="good communication skill" and s_2 ="good programming skill" using the following signaling scheme:

- when $\theta_A =$ Prog, fully reveal θ_B, that is, send "good communication skill" when $\theta_B =$Comm and send "good programming skill" when $\theta_B =$Prog.
- when $\theta_A =$ Comm, partially reveal θ_B: send "good communication skill" when $\theta_B =$Comm; and when $\theta_B =$Prog, send "good communication skill" with probability $2/3$ and send "good programming skill" with probability $1/3$.

$P(s_1, \cdot)$	$\theta_B =$ Prog	$\theta_B =$ Comm
$\theta_A =$ Prog	0	0.2
$\theta_A =$ Comm	0.2	0.2

$P(s_2, \cdot)$	$\theta_B =$ Prog	$\theta_B =$ Comm
$\theta_A =$ Prog	0.3	0
$\theta_A =$ Comm	0.1	0

Table 6: The joint distribution of the signal and the types.

The joint distribution of the signal and the players' types are shown in Table 6.

An employer's best actions are shown in Table 7. A Prog-type employer hires when receiving the "good programming skill" signal because the candidate's type is fully revealed for them. A Comm-type employer hires when receiving the "good communication skill" signal, because the posterior probability of θ_B =Comm, that is $P(\theta_B = \text{Comm}|s_1, \theta_A = \text{Comm})$, becomes 0.5 and we assume that the employer breaks ties by choosing the action that is more favorable for the candidate.

best action	s_1	s_2
$\theta_A =$ Prog	not hire	hire
$\theta_A =$ Comm	hire	not hire

Table 7: The employer's best action.

We first show that the protocol is ex-post IR. The protocol is ex-post IR for a Prog-type candidate because they still get hired by the Prog-type employer and they only get hired more by the Comm-type employer. The protocol is ex-post IR for a Comm-type candidate, because based on a Comm-type candidate's belief, the probability of getting hired does not change. When we use the protocol for a Comm-type candidate, the mediator will only send the "good communication skill" signal and the candidate will only be hired by the Comm-type employer, whereas the candidate only gets hired by the Prog-type employer when there's no communication. A Comm-type candidate's posterior belief about the employer's type is $P(\theta_A = \text{Prog}|s_1, \theta_B = \text{Comm}) = 0.5$ (the protocol reveals no information about θ_A to a type-Comm candidate; see Table 6), so the probability of getting hired does not change for a Comm-type candidate.

We then show that the protocol achieves the highest possible social welfare. The expected social welfare can be decomposed as

$$\begin{aligned}\mathbb{E}[u_A + u_B] = {} & P(\theta_A = \text{Prog}, \theta_B = \text{Prog}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Prog}] \\ & + P(\theta_A = \text{Prog}, \theta_B = \text{Comm}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Comm}] \\ & + P(\theta_A = \text{Comm}, \theta_B = \text{Prog}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}] \\ & + P(\theta_A = \text{Comm}, \theta_B = \text{Comm}) \cdot \mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Comm}].\end{aligned}$$

It is not difficult to see that $\mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Prog}], \mathbb{E}[u_A + u_B|\theta_A = \text{Prog}, \theta_B = \text{Comm}]$, and $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Comm}]$ induced by the mediator protocol have reached their maximum value, because after seeing the signal sent by the mediator, the employer always hires when $\theta_A = \theta_B$ and the employer never hires when $\theta_A =$ Prog and $\theta_B =$ Comm. So we only need to prove that $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}]$ has also reached its maximum. To simplify the notation, we denote by $\mathcal{E}$ the event that $\theta_A =$ Comm and $\theta_B =$ Prog. Then the conditional social welfare $\mathbb{E}[u_A + u_B|\theta_A = \text{Comm}, \theta_B = \text{Prog}]$ can be written as

$$\begin{aligned}& \mathbb{E}[u_A + u_B|\mathcal{E}] \\ = {} & \mathbb{P}(r = \text{hire}|\mathcal{E}) \cdot \mathbb{E}[u_A + u_B|r = \text{hire}, \mathcal{E}] + \mathbb{P}(r = \text{not hire}|\mathcal{E}) \cdot \mathbb{E}[u_A + u_B|r = \text{not hire}, \mathcal{E}] \\ = {} & \mathbb{P}(r = \text{hire}|\mathcal{E})(-1 + 2) + \mathbb{P}(r = \text{not hire}|\mathcal{E})(0 + 0).\end{aligned}$$

This means that the higher $\mathbb{P}(r = \text{hire}|\mathcal{E})$ is, the higher the conditional social welfare is. We show that $\mathbb{E}[u_A + u_B|\mathcal{E}]$ cannot exceed $\frac{2}{3}$ by showing that $\mathbb{P}(r = \text{hire}|\mathcal{E})$ cannot exceed $\frac{2}{3}$. This is because a Comm-type employer only hires when the candidate is more likely to have type Comm, and it implies

$$P(\theta_B = \text{Prog}|\theta_A = \text{Comm}, r = \text{hire}) \le P(\theta_B = \text{Comm}|\theta_A = \text{Comm}, r = \text{hire}).$$

Multiply both sides by $P(r = \text{hire}|\theta_A = \text{Comm})$, we get

$$\begin{aligned} P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm}) &\le P(\theta_B = \text{Comm}, r = \text{hire}|\theta_A = \text{Comm}) \\ &\le P(\theta_B = \text{Comm}|\theta_A = \text{Comm}) \\ &= 0.4. \end{aligned}$$

The second inequality trivially holds and the last equality is because θ_A and θ_B are independent. Finally,

$$\begin{aligned} \mathbb{P}(r = \text{hire}|\theta_B = \text{Prog}, \theta_A = \text{Comm}) &= \frac{P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm})}{P(\theta_B = \text{Prog}|\theta_A = \text{Comm})} \\ &= \frac{P(\theta_B = \text{Prog}, r = \text{hire}|\theta_A = \text{Comm})}{0.6} \\ &\le \frac{0.4}{0.6} = \frac{2}{3}. \end{aligned}$$

Therefore we must have $\mathbb{P}(r = \text{hire}|\mathcal{E}) \le \frac{2}{3}$, and this upper bound is reached by the mediator protocol because we have $\mathbb{P}(s_1|\mathcal{E}) = \frac{2}{3}$ and a Comm-type employer always hires after receiving s_1.

By employing the linear program in B.1, we can obtain the same mediator protocol as in Table 6.

And the optimal social welfare by ex-post IR mediator protocol is

$$W = 0.2 \times (0 + 0) + 0.2 \times (-1 + 2) + 0.2 \times (1 + 2) + 0.3 \times (10 + 2) + 0.1 \times (0 + 0) = \frac{22}{5}.$$

Optimal Bayesian conversation. We next employ our algorithm to find the optimal ex-post IR Bayesian conversation. First of all, the ex-post IR region IR_0 is plotted as in Figure 3a with

$$\mathsf{IR}_0 = [0, 0.5] \times [0, 0.5] \cup [0.5, 1] \times [0.5, 1],$$
$$X^* = \{0, 0.5, 0.6, 1\}, \quad Y^* = \{0, 0.5, 1\}.$$

And the corresponding $W_0(q_A, q_B)$ is plotted in Figure 3b with $W_0(0, 1) = W_0(1, 0) = -\infty$, $W_0(0.5, 0.5) = W_0(0.5, 0) = W(0.5, 1) = 2$, and so on. Then, we perform the recursive updates to find $W_k(\cdot)$ for $k > 0$. The resulting $W_1(q_B, q_A)$ is plotted in Figure 4a, and we have $W_2(q_B, q_A) = W_1(q_B, q_A)$, for all $(q_B, q_A) \in X^* \times Y^*$. This means that our algorithm has converged after just one step, so the Bayesian conversation after one round has already achieved the optimal social welfare, which is equal to $\frac{21}{5}$.

We can see from the IR_0 region that if Alice splits belief q_A, there must be one of final believes fall outside the IR_0 region. Therefore, a 1-round Bayesian conversation can achieve the optimum.

G.2 Missing proof of theorem 4.3

Proof. Consider a two-player game between Alice and Bob. Alice can be one of two types: $\Theta_A = \{\theta_{A0}, \theta_{A1}\}$, and Bob can also be one of two types: $\Theta_B = \{\theta_{B0}, \theta_{B1}\}$. Alice can take two actions

$u_A(\theta_{A0},\cdot)$	$\theta_B=\theta_{B0}$	$\theta_B=\theta_{B1}$
r_0	7	5
r_1	5	7

$u_A(\theta_{A1},\cdot)$	$\theta_B=\theta_{B0}$	$\theta_B=\theta_{B1}$
r_0	1	3
r_1	0	5

$u_B(\theta_{B0},\cdot)$	$\theta_A=\theta_{A0}$	$\theta_A=\theta_{A1}$
r_0	5	10
r_1	10	0

$u_B(\theta_{B1},\cdot)$	$\theta_A=\theta_{A0}$	$\theta_A=\theta_{A1}$
r_0	10	10
r_1	10	4

Table 8: Alice and Bob's utility function.

$r \in \{r_0, r_1\}$. Suppose $\mathbb{P}(\theta_A=\theta_{A0})=0.6$, and $\mathbb{P}(\theta_B=\theta_{B0})=0.4$. The utilities of the two players are given in the following table:

According to the algorithm in Section 4.1, calculate W_0 and plot it on the coordinate system as shown in Figure 8.

We denote $W_k\left(\frac{2}{5},\frac{4}{5}\right)$ as $m_1^{(k)}$, $W_k\left(\frac{1}{2},\frac{4}{5}\right)$ as $m_2^{(k)}$, $W_k\left(\frac{2}{3},\frac{4}{5}\right)$ as $m_3^{(k)}$, $W_k\left(\frac{2}{5},\frac{3}{5}\right)$ as $m_4^{(k)}$, $W_k\left(\frac{1}{2},\frac{3}{5}\right)$ as $m_5^{(k)}$, and $W_k\left(\frac{2}{3},\frac{3}{5}\right)$ as $m_6^{(k)}$.

After calculations, we have discovered the following iterative pattern: For all $k>0$, When k is even,

$$\begin{aligned}
m_1^{(k+1)} =& m_1^{(k)},\\
m_2^{(k+1)} =& m_2^{(k)},\\
m_3^{(k+1)} =& \frac{2}{3}m_2^{(k)} + \frac{1}{3}W_k\left(1,\frac{4}{5}\right),\\
m_4^{(k+1)} =& \frac{3}{5}m_6^{(k)} + \frac{2}{5}W_k\left(0,\frac{3}{5}\right),\\
m_5^{(k+1)} =& \frac{3}{4}m_6^{(k)} + \frac{1}{4}W_k\left(0,\frac{3}{5}\right),\\
m_6^{(k+1)} =& m_6^{(k)}.
\end{aligned}$$

when k is odd,

$$\begin{aligned}
m_1^{(k+1)} =& \frac{1}{2}m_4^{(k)} + \frac{1}{2}W_k\left(\frac{2}{5},1\right),\\
m_2^{(k+1)} =& \frac{1}{2}m_5^{(k)} + \frac{1}{2}W_k\left(\frac{1}{2},1\right),\\
m_3^{(k+1)} =& m_3^{(k)},\\
m_4^{(k+1)} =& m_4^{(k)},\\
m_5^{(k+1)} =& m_5^{(k)},\\
m_6^{(k+1)} =& \frac{3}{4}m_3^{(k)} + \frac{1}{4}W_k\left(\frac{2}{3},0\right).
\end{aligned}$$

And in all iterations, the values of other points remain unchanged: For all $k>0$, for all $(x,y) \in X^* \times Y^* - \{\left(\frac{2}{5},\frac{4}{5}\right),\left(\frac{1}{2},\frac{4}{5}\right),\left(\frac{2}{3},\frac{4}{5}\right),\left(\frac{2}{5},\frac{3}{5}\right),\left(\frac{1}{2},\frac{3}{5}\right),\left(\frac{2}{3},\frac{3}{5}\right)\}$,

$$W_{k+1}(x,y) = W_k(x,y).$$

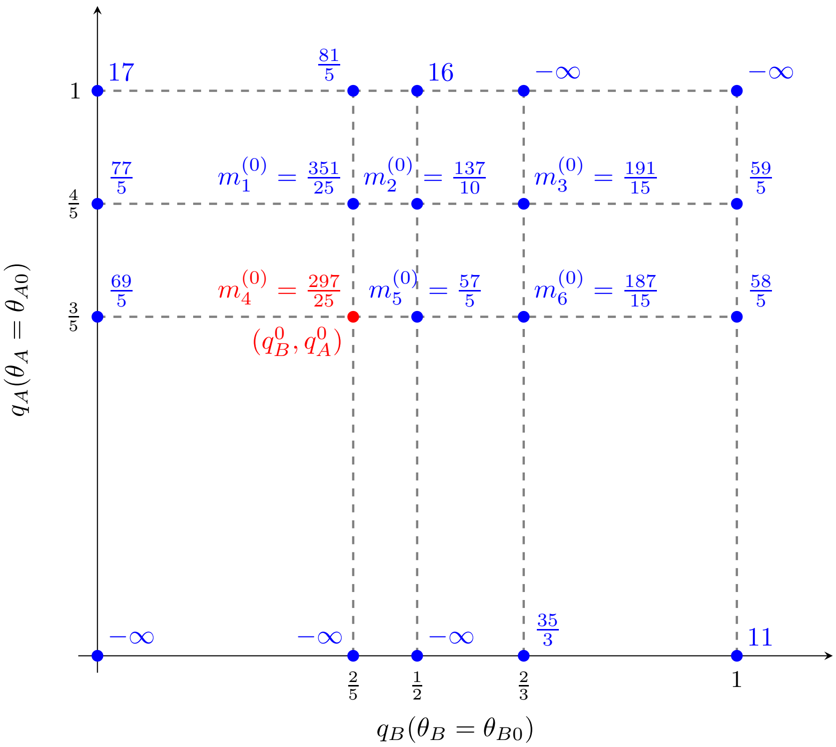


Figure 8: Illustration of W_0 in the two-player game where the highest social welfare requires infinite rounds of communication to achieve.

Since for even k,

$$m_4^{(k+1)} = \frac{3}{5}m_6^{(k)} + \frac{2}{5}W_k\left(0, \frac{3}{5}\right), \qquad m_5^{(k+1)} = \frac{3}{4}m_6^{(k)} + \frac{1}{4}W_k\left(0, \frac{3}{5}\right),$$

and for odd k, m_4 and m_5 remain unchanged, thus we know that $W_k(\cdot, \frac{3}{5})$ is linear in $(0, \frac{1}{2})$ for all k.

For all k, $W_k(0, \frac{3}{5}), W_k(0, \frac{4}{5}), W_k(0, 1)$ remain unchanged, and their values are linear to y. And for all k, $W_k(0, 1), W_k(\frac{2}{5}, 1), W_k(\frac{1}{2}, 1)$ remain unchanged, and their values are linear to x.

For odd k,

$$m_1^{(k+1)} = \frac{1}{2}m_4^{(k)} + \frac{1}{2}W_k\left(\frac{2}{5}, 1\right), \qquad m_2^{(k+1)} = \frac{1}{2}m_5^{(k)} + \frac{1}{2}W_k\left(\frac{1}{2}, 1\right)$$

So

$$\begin{aligned}
m_1^{(k+1)} =& \frac{1}{2}m_4^{(k)} + \frac{1}{2}W_k\left(\frac{2}{5}, 1\right) \\
=& \frac{1}{2}\left(\frac{4}{5}m_5^{(k)} + \frac{1}{5}W_k\left(0, \frac{3}{5}\right)\right) + \frac{1}{2}\left(\frac{4}{5}W_k\left(\frac{1}{2}, 1\right) + \frac{1}{5}W_k\left(0, 1\right)\right) \\
=& \frac{4}{5}m_2^{(k+1)} + \frac{1}{5}W_{k+1}\left(0, \frac{4}{5}\right).
\end{aligned}$$

and for even k, m_1 and m_2 remain unchanged, making $W_k(\cdot, \frac{4}{5})$ linear in $(0, \frac{1}{2})$ for all k, so:

$$m_2^{(k)} = \frac{5}{4}m_1^{(k)} - \frac{1}{4}W_k\left(0, \frac{4}{5}\right) = \frac{5}{4}m_1^{(k)} - \frac{77}{20}, \quad \forall k.$$

For odd k,

$$\begin{aligned}
m_4^{(k+4)} &= \frac{3}{5} m_6^{(k+3)} + \frac{138}{25}, \\
m_6^{(k+3)} &= \frac{3}{4} m_3^{(k+2)} + \frac{35}{12}, \\
m_3^{(k+2)} &= \frac{2}{3} m_2^{(k+1)} + \frac{59}{15} = \frac{5}{6} m_1^{(k+1)} + \frac{41}{30}, \\
m_1^{(k+1)} &= \frac{1}{2} m_4^{(k)} + \frac{81}{10}.
\end{aligned}$$

By recursively applying the above equations, we have for odd k:

$$m_4^{(k+4)} = \frac{3}{16} m_4^{(k)} + \frac{4369}{400}.$$

Since $m_4^{(1)} = 13$, we get

$$W_{4k+1}(0.4, 0.6) = m_4^{(4k+1)} = -\frac{144}{325} \left(\frac{3}{16}\right)^k + \frac{4369}{325}.$$

And from this expression, we can see that the optimal welfare is never achieved by any finite value of k. □