Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 4

To be uploaded before the exercise group on November 20, 2024

1 The Logistic Map

Consider the map:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n)$$

1. Prove that, for $0 \leq x_n \leq 1$ and $0 \leq r \leq 4$, we have $0 \leq f(x_n) \leq 1$.
2. Given a trajectory $x_1, x_2, \ldots x_T$, here's a recipe to make a cobweb plot of a map:
   (a) Plot (x_1, x_2)
   (b) Connect that point to (x_2, x_2)
   (c) Connect this to (x_2, x_3), and so forth

   Starting from some $x_1 \in (0, 1)$, plot a cobweb plot for $r = 0.5, 1.5, 2.5, 3.5$ and 3.9 with 30 steps.
3. Show that the logistic map has a cycle of order 2 for $r > 3$. Use that a 2-cycle requires $f(q) = p$ and $f(p) = q$. What is the stability of the 2-cycle? Does the stability change for some $r > 3$?
4. For each $r \in \{0.001, 0.002, 0.003 \ldots 3.998, 3.999\}$, produce 1000 trajectories starting at random initial conditions with 100 steps. Plot only the endpoints of the trajectories according to their respective r in a 2D-scatter plot.
   Hint: Since you only need to plot the last points of the trajectories, don't store the trajectories in order to save memory.
   Hint 2: Make your code flexible so you can use it with other maps as well
5. Redo the plot for $r \in \{3.44500, 3.44501, 3.44502, \ldots, 3.56999, 3.57000\}$ and use it to find another r such that the logistic map has a cycle of order $p \geq 3$. Then show the existence of the cycle mathematically.
6. The qualitative changes in the dynamics that you can see in the diagram for certain values of r are called bifurcations. This is where the topology of the dynamics change. Develop a procedure to numerically compute the r values for the following m bifurcations, starting from $r = 3.44$. Describe the algorithm.
   Hint 1: You might have to increase floating-point precision.
   Hint 2: Make your algorithm independent of the map, then you can reuse it.

Hint 3: This algorithm may take a long time to run, depending on your hardware. Let it run for a time that's feasible for you and see how many bifurcations you can find. That implies your code has to save a bifurcation point as soon as it finds it, because you will interrupt the code in the end.

7. Let $2 \leq r_n \leq m - 1$ be a bifurcation point. Calculate the relative distance between successive bifurcations points
$$\delta = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$
for all $n \in \{1, \ldots 100\}$. What do you find?

8. Bonus exercise: Consider the map:
$$x_{n+1} = r \sin(\pi x_n)$$
Compute the first bifurcations again for $r > 0.82$ and calculate δ from the exercise above. Make a bifurcation diagram (as with the logistic map) and see what it looks like.

2 Poincaré Maps

The Poincaré Map can be used to find closed orbits and classify their stability. When you have an n-dimensional system $\dot{x} = f(x)$, a Surface of Section S is an $(n-1)$-dimensional subspace, chosen such that for a trajectory of interest, there are $t_0, t_1, t_2, \ldots$ with $x(t_0), x(t_1), x(t_2), \ldots \in S$, but for all other t, $x(t) \notin S$. Also, S is not tangent to the trajectory. That means, that the trajectory crosses the Surface of Section, but does not just touch it or move within it. A Poincaré Map P for the Surface of Section S and the dynamical system $\dot{x} = f(x)$ is then defined via $P(x(t_i)) = x(t_{i+1})$, i.e. the Map "samples" from a trajectory at the times it crosses S.

1. Let $\dot{x} = f(x)$ be a dynamical system and P a corresponding Poincaré map with surface of section S. If there exists $y \in S$ such that $P(y) = y$, then there exists a closed cycle in the system. Is that also a necessary condition? Why/why not?

2. Consider the following system:
$$\dot{r} = r(1 - r), \quad \dot{\theta} = 2.$$
Explicitly find a Poincaré Map for the surface of section $S = \{(r, \theta) : r > 0, \theta = 0\}$. Use it to prove that there exists a closed cycle at $r = 1$. Is it stable, unstable or half-stable? (Hint: to find the map, take an initial condition $(r_1, 0) \in S$. After one revolution, the trajectory intersects S at $(r_2, 0)$. To get r_2, note that you can read off the time T the revolution takes from the equations.)