Dynamical Systems Theory in Machine Learning & Data Science

lecturers: Daniel Durstewitz
tutors: Christoph Hemmer, **Alena Brändle**, Lukas Eisenmann, Florian Hess
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email to: alena.braendle@zi-mannheim.de

Exercise 5

To be uploaded before the exercise group on November 27, 2024

1 Coupled Oscillators

Recall the coupled oscillators

$$\dot{\varphi}_1 = \omega_1 + C_1 \sin(\varphi_2 - \varphi_1), \quad \dot{\varphi}_2 = \omega_2 + C_2 \sin(\varphi_1 - \varphi_2)$$

from the lecture.

1. Transform it into a 1D system that represents the oscillators' phase difference by setting $\theta = \varphi_1 - \varphi_2$ and write down the differential equation.
2. Show that the system undergoes a saddle-node bifurcation at $|\omega_1 - \omega_2| = C_1 + C_2$. What does this mean qualitatively for the coupled oscillators?

2 Sustainable Fishing Bifurcation

Let x be proportionate to the number of fish in a lake. They reproduce with rate r, but the more fish there are, the more the limited resources are strained, so their reproduction equation is $\dot{x} = rx(1 - x)$. Recently, people discovered that these fish are also very tasty, which is why they catch them at rate c. This makes the total fish reproduction equation

$$\dot{x} = rx(1 - x) - cx.$$

Argue that people should restrain their fishing activities to a certain level, if they want to continue eating fish in the future, by showing that the system undergoes a bifurcation (of which type?).

3 Teaser on the Lorenz System

The Lorenz equations

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{aligned}$$

(with parameters $\sigma, r, b \geq 0$) are an important system in the study of bifurcations and chaos (which will probably be covered next lecture). They often serve as a benchmark for dynamical systems reconstruction (which will be covered some lectures after that). The following exercises are meant to provide some first insight.

1. Find three fixed points. Some of them exist only for certain parameter configurations, to be determined.

2. Classify the fixed points' stability depending on parameter configurations. What kind of bifurcation(s) do they undergo? Draw a bifurcation diagram (by hand or computer). (Hint: determining stability only means to distinguish between stable, unstable or half-stable points. Their exact shape doesn't matter)

3. Bonus exercise: Show that there can be no cycles for $r < 1$. (Hint: The function $V(x, y, z) = \frac{x^2}{\sigma} + y^2 + z^2$ might be of help.)