Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 3

To be uploaded before the exercise group on November 13, 2024

1 FitzHugh-Nagumo Model

This system imitates the generation of neuronal action potentials:

$$\dot{v} = v - \frac{1}{3}v^3 - w + I, \quad \tau\dot{w} = v - a - bw$$

with $a \in \mathbb{R}, b \in (0,1), I \geq 0, \tau \gg 1$. v mimics the membrane potential and w an outward current.

1. Write down equations for the nullclines for $a = 0, b = 1/2$. Calculate the equilibrium point and determine its properties as a function of the input current I.
2. Use your code from last exercise sheet to make a phase plane plot (including nullclines) for $I = 0, \tau = 10$. Plot into it a trajectory starting from any initial condition except the origin. It's not necessary to attach the code, just the plot.
3. You should see a limit cycle, which partly follows the v nullcline. Show in a plot that the cycle vanishes for some $a < 0$, but returns if you increase I.

2 More Cycles

A cycle or closed orbit is a trajectory $x(t)$ in a dynamical system such that there are $t_2 > t_1$ with $x(t_2) = x(t_1)$, and for all $t \in (t_1, t_2)$, $x(t) \neq x(t_1)$. A center is an equilibrium point such that all trajectories sufficiently close to it are cycles.
For each of the following cases, decide if the system either has no cycle or at least one (sometimes depending on parameters). Explain.

1. (Easy) Consider the system
$$\dot{x} = y, \quad \dot{y} = -2x^3$$
Hint: examine the properties of the fixed point, and show that $E(x, y) = x^4 + y^2$ is constant in time (i.e. along every trajectory).
2. (Medium) Consider the system:
$$\dot{x} = (x^2 + y^2)(ax - y) - y$$
$$\dot{y} = (x^2 + y^2)(x + ay) + x$$
for $a \in \mathbb{R}$. Hint: transfer the system into polar coordinates.

3. (Hard) Consider the system

$$\dot{x} = -x + ay + x^2 y, \quad \dot{y} = b - ay - x^2 y$$

where $a, b > 0$. This is a model of glycolysis, the transformation of sugar (y) to ADP (x), which fuels cells. This is why we only consider $x, y \geq 0$. Hint: Use the Poincasé-Bendixson Theorem. For this, you need to construct a compact connected set C, which contains no equilibrium points, and where the flow is directed inwards on the boundary ∂C. Then there must be a limit cycle inside. Try to find an area with a pentagon-shaped outline that fulfils these conditions. Use the x and y axes as parts of the boundary. We call C a trapping region. Sketching this system is very helpful.

3 Hartman-Grobman Theorem

Given a differential equation system $\dot{x} = f(x), f \in C^\infty$ with a hyperbolic equilibrium point $x^\star$, the Hartman-Grobman theorem states that in a neighbourhood N around $x^\star$, we can linearize the system around $x^\star$ and obtain a topologically conjugate flow. By giving a counterexample, show that indeed $x^\star$ must be hyperbolic for the theorem to hold.

4 Example Systems

For each of the following conditions, provide an example of a dynamical system defined in $\mathbb{R}^2$ which fulfills them (with proof that it does). The examples have to be nontrivial, i.e. no derivative can be 0 everywhere. If you think that no such system exists, explain why.

1. An equilibrium point at (a, a^2) and an equilibrium point at (b, b^2) for $a, b \neq 0$. One of them is stable, the other unstable. No other equilibrium points.

2. A linear system with an equilibrium point other than the origin, and no line attractor (a.k.a. non-isolated equilibrium point).

3. A saddle node, a stable and an unstable node. The system may have more equilibrium points.

4. Infinitely many equilibrium points, but no line attractor.