Dynamical Systems Theory in Machine Learning & Data Science

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Exercise 6

To be uploaded before the exercise group on December 4, 2024

1 Hopf Bifurcation

In the 1800s, Australia had the idea to combat the invasive rabbits R by introducing another foreign species: foxes F. The following system of population growth rate of both species (kind of) shows why this was a bad idea:

$$\dot{R} = (1-R)R - \frac{RF}{0.3+\alpha R}, \quad \dot{F} = -0.5F + \frac{RF}{0.3+\alpha R}$$

1. Show that the system undergoes two supercritical Hopf bifurcations at $\alpha = 0.5$ and $\alpha = 1.2$. The bifurcation point is *not* the origin. Draw a bifurcation diagram (by hand or computer). Also classify the origin's stability.

   Hint: It is strongly recommended to use a symbolic computing library (for example sympy, for python or julia). No calculations by hand are required for this exercise. If you do not manage please write your ideas down anyways.

2 Cantor Set as a Bernoulli process

In the lecture we discussed the Cantor Set. You can model it through a Bernoulli process by drawing from a Bernoulli distribution with $p = \frac{1}{2}$ and defining the following mapping:

$$\begin{aligned} \textit{if } q = 0 : x_{n+1} &= \frac{x_n}{3} \\ \textit{if } q = 1 : x_{n+1} &= \frac{x_n + 2}{3} \end{aligned}$$

1. Simulate this mapping for an appropriate number of iterations N for different initial conditions in the interval between 0 and 1 and plot the resulting set of points. Investigate the self-similar behavior of the set by plotting different slices of the x-axis.

3 Cantor Set and the Tent Map

We went on hinting at an interesting relationship between the Cantor Set and the tent map, defined by

$$x_{n+1} = f_\mu(x_n) = \begin{cases} \mu x_n & \text{for } x_n < \frac{1}{2} \\ \mu(1-x_n) & \text{for } \frac{1}{2} \leq x_n \end{cases}$$

This mapping can be interpreted as a simple piecewise discrete-time dynamical system, which will play a crucial role in the rest of the lecture.

1. Generate trajectories from the tent map for $\mu = 3$, starting from different initial conditions. When do you observe divergent behavior? How can you make use of the results of the previous exercise to find initial states that do not lead to divergences?