Lecture 9: Value Function Methods

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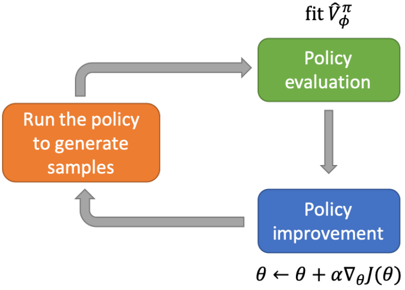
Table of Contents

Contents and Goals

- What if we just use a critic, without an actor?
- Extracting a policy from a value function
- The fitted value iteration, fitted Q-iteration algorithms
- Goals
    - Understand how value functions give rise to policies
    - Understand the Q-learning with function approximation algorithm
    - Understand practical considerations for Q-learning

Review: the actor-critic algorithm

- Loop:
  1. sample $\{s_i, a_i, r_i, s'_i\}$ from $\pi_\theta(a|s)$ (run it on the robot)
  2. policy evaluation: fit $\hat{V}^\pi_\phi(s)$ to sampled reward sums
  3. evaluate $\hat{A}^\pi(s_i, a_i) = r_i + \gamma \hat{V}^\pi_\phi(s'_i) - \hat{V}^\pi_\phi(s_i)$
  4. policy improvement: $\nabla_\theta J(\theta) \approx \sum_i \nabla_\theta \log \pi_\theta(a_i|s_i) \hat{A}^\pi(s_i, a_i)$
  5. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

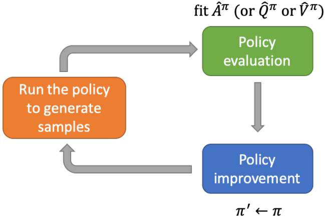


Can we omit policy gradient completely?

- The advantage function $A^\pi(s_t, a_t)$:
  - how much better is a_t than the average action according to π
- $\arg\max_{a_t} A^\pi(s_t, a_t)$: best action from s_t, if we then follow π
  - forget about approximating policies directly
  - just derive policies from value functions

- Is π' better than π, i.e., $\pi' \geq \pi$?
  - **The policy improvement theorem!**

$$\pi'(a_t|s_t) = \begin{cases} 1 & \text{if } a_t = \arg\max_{a_t} A^\pi(s_t, a_t) \\ 0 & \text{otherwise} \end{cases}$$



Review: Policy Improvement Theorem

- Let π and π' be any pair of deterministic policies such that,

$$Q^{\pi}(s, \pi'(s)) \geq V^{\pi}(s), \quad \forall s \in \mathcal{S}.$$

  Then the policy π' must be as good as, or better than, π.

Review: Policy improvement theorem

$$
\begin{aligned}
V^{\pi}(s) &\leq Q^{\pi}(s, \pi'(s)) \\
&= \mathbb{E}[R_{t+1} + \gamma V^{\pi}(S_{t+1}) | S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma V^{\pi}(S_{t+1}) | S_t = s] \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma Q_{\pi}(S_{t+1}, \pi'(S_{t+1})) | S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma V^{\pi}(S_{t+2}) | S_{t+1}, A_{t+1} = \pi'(S_{t+1})] | S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 V^{\pi}(S_{t+2}) | S_t = s] \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V^{\pi}(S_{t+3}) | S_t = s] \\
&\leq ... \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + ... | S_t = s] \\
&= V^{\pi'}(s)
\end{aligned}
$$

Review: Policy improvement theorem

- Consider the new **greedy** policy, π', selecting at each state the action that appears best according to $Q^\pi(s,a)$

$$\begin{aligned}\pi'(s) &= \arg\max_a Q^\pi(s,a) \\ &= \arg\max_a \mathbb{E}[R_{t+1} + \gamma V^\pi(S_{t+1}) | S_t = s, A_t = a] \\ &= \arg\max_a \sum_{s',a} p(s',r|s,a)[r + \gamma V^\pi(s')]\end{aligned}$$

- The process of making a new policy that improves on an original policy, by making greedy w.r.t. the value function of the original policy, is called **policy improvement**
  - The greedy policy meets the conditions of the policy improvement theorem

The Generalized Policy Iteration framework

- High-level idea: **policy iteration** algorithm. Loop:
  1. Policy evaluation: evaluate $A^\pi(s,a)$
  2. Policy improvement: set $\pi \leftarrow \pi'$

$$\pi'(a_t|s_t) = \begin{cases} 1 & \text{if } a_t = \arg\max_{a_t} A^\pi(s_t,a_t) \\ 0 & \text{otherwise} \end{cases}$$

- **Question**: How to evaluate A^π?

$$A^\pi(s,a) = r(s,a) + \gamma V^\pi(s') - V^\pi(s)$$

- As we did in actor-critic algorithms, just evaluate V^π!

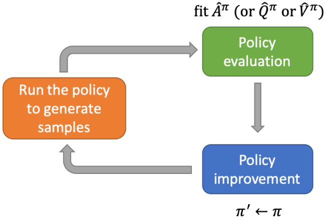
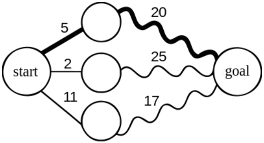


Table of Contents

Review: Dynamic Programming (DP)

- It refers to simplifying a complicated problem by breaking it down into simpler sub-problems in a **recursive** manner.



- Finding the shortest path in a graph using optimal substructure
- A straight line: a single edge,
  a wavy line: a shortest path
- The bold line: the overall shortest path from start to goal

Review: Dynamic Programming (DP)

- A collection of algorithms that can be used to compute optimal policies given a perfect model of the environment (MDP)
  - Of limited utility in RL both because of their assumption of a perfect model and because of their great computational expense
  - Important theoretically, provide an essential foundation for the understanding of RL methods
  - **RL methods can be viewed as attempts to achieve much the same effect as DP**, only with less computation and without assuming a perfect model of the environment

Review: Policy evaluation in DP

- Compute the state-value function V^π for an arbitrary policy π

$$
\begin{aligned}
V^\pi(s) &= \mathbb{E}_\pi[G_t | S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} | S_t = s] \\
&= \sum_a \pi(a|s) \sum_{s'} \sum_r p(s', r | s, a)[r + \gamma \mathbb{E}_\pi[G_{t+1} | S_{t+1} = s']] \\
&= \sum_a \pi(a|s) \sum_{s', r} p(s', r | s, a)[r + \gamma V^\pi(s')]
\end{aligned}
$$

0.5	0.8	0.3	0.4
0.4	0.3	0.8	0.5
0.7	0.6	0.6	0.7
0.9	0.5	0.1	0.2

- 16 states, 4 actions per state
- can store full $V^\pi(s)$ in a table
- iterative sweeping over the state space

Review: Policy improvement in DP

- Consider the new **greedy** policy, π', selecting at each state the action that appears best according to $Q^\pi(s, a)$

$$\begin{aligned}
\pi'(s) &= \arg\max_a Q^\pi(s, a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma V^\pi(S_{t+1}) | S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s', r} p(s', r | s, a)[r + \gamma V^\pi(s')]
\end{aligned}$$

- The process of making a new policy that improves on an original policy, by making greedy w.r.t. the value function of the original policy, is called **policy improvement**
  - The greedy policy meets the conditions of the policy improvement theorem

Review: Policy iteration

- Using policy improvement theorem, we can obtain a sequence of monotonically improving policies and value functions

$$\pi_0 \xrightarrow{E} V^{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} V^{\pi_1} \xrightarrow{I} \pi_2 \xrightarrow{E} \ldots \xrightarrow{I} \pi^* \xrightarrow{E} V^*$$

- This process is guaranteed to converge to an optimal policy and optimal value function in a finite number of iterations
  - Each policy is guaranteed to be a strictly improvement over the previous one unless it is already optimal
  - A finite MDP has only a finite number of policies

Review: Policy iteration algorithm

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   - $\Delta \leftarrow 0$
   - Loop for each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V(s')\right]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma V(s')\right]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

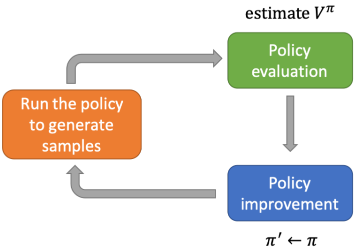
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Dynamic programming with policy iteration

- Policy iteration. Loop:
  1. Policy evaluation: evaluate $V^\pi(s)$
  2. Policy improvement: set $\pi \leftarrow \pi'$

$$\pi'(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_a Q^\pi(s,a) \\ 0 & \text{otherwise} \end{cases}$$

$$Q^\pi(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma V^\pi(s')]$$



0.5	0.8	0.3	0.4
0.4	0.3	0.8	0.5
0.7	0.6	0.6	0.7
0.9	0.5	0.1	0.2

- 16 states, 4 actions per state
- can store full $V^\pi(s)$ in a table
- iterative sweeping over the state space

- In policy iteration, stop policy evaluation after just one sweep

$$V_{k+1}(s) = \sum_{s',r} p(s', r|s, \pi_k(s))[r + \gamma V_k(s')]$$

$$\pi_{k+1}(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)[r + \gamma V_{k+1}(s')]$$

- Combine into one operation, called **value iteration** algorithm

$$V_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a)[r + \gamma V_k(s')]$$

  - For arbitrary V_0, the sequence $\{V_k\}$ converges to V^* under the same conditions that guarantee the existence of V^*

Review: Value iteration

- Bellman optimality equation

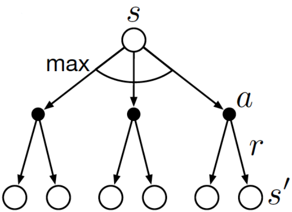
$$V^*(s) = \max_a \sum_{s',r} p(s', r|s, a)[r + \gamma V^*(s')]$$

- Value iteration

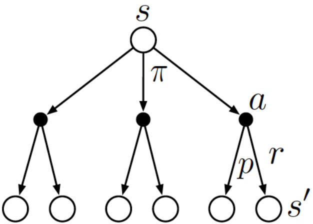
$$V_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a)[r + \gamma V_k(s')]$$

  - Turn Bellman optimality equation into an update rule
  - Directly approximate the optimal state-value function, V^*

Review: Value iteration vs. Policy evaluation



Backup diagram for value iteration



Backup diagram for policy evaluation

Review: Value iteration algorithm

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $\quad v \leftarrow V(s)$
| $\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
| $\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

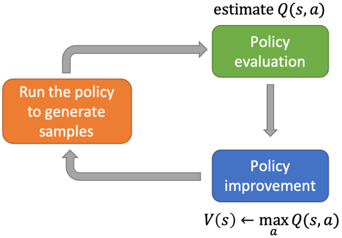
Output a deterministic policy, $\pi \approx \pi_*$, such that
$\quad \pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$

- One sweep = one sweep of policy evaluation + one sweep of policy improvement

Dynamic programming with value iteration

- Value iteration. Loop:
  1. Policy evaluation: evaluate $Q(s,a)$
  2. Implicit policy improvement: set $V(s) \leftarrow \max_a Q(s,a)$

$$Q(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$$



- Skip the policy and compute values directly!

$$V(s) \leftarrow \max_a Q(s,a) \quad \overset{\text{equivalent}}{\Rightarrow} \quad \pi'(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_a Q(s,a) \\ 0 & \text{otherwise} \end{cases}$$

Review: For large/continuous state/action spaces

- **Curse of dimensionality**: Computational requirements grow exponentially with the number of state variables
  - Theoretically, all state-action pairs need to be visited infinite times to guarantee an optimal policy
  - In many practical tasks, almost every state encountered will never have been seen before
- **Generalization**: How can experience with a limited subset of the state space be usefully generalized to produce a good **approximation** over a much larger subset?

Review: Curse of dimensionality

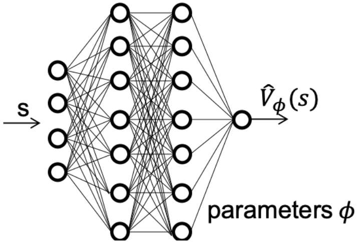
0.5	0.8	0.3	0.4
0.4	0.3	0.8	0.5
0.7	0.6	0.6	0.7
0.9	0.5	0.1	0.2

- In discrete case, represent $V(s)$ as a table
  - 16 states, 4 actions per state
  - can store full $V(s)$ in a table
  - iterative sweeping over the state space

- An image
  - $|\mathcal{S}| = (255^3)^{200\times 200}$
  - more than atoms in the universe
  - can we store such a large table?

Review: Function approximation

- It takes examples from a desired function (e.g., a value function) and attempts to generalize from them to construct an approximation to the entire function
  - Linear function approximation: $V(s) = \sum_i \phi_i(s) w_i$
  - Neural network approximation: $V(s) = V_\phi(s)$

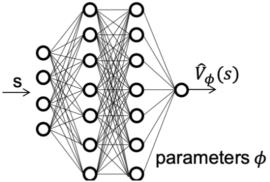


Review: Function approximation

- Function approximation is an instance of **supervised learning**, the primary topic studied in machine learning, artificial neural networks, pattern recognition, and statistical curve fitting
  - In theory, any of the methods studied in these fields can be used in the role of function approximator within RL algorithms
  - RL with function approximation involves a number of **new issues** that do not normally arise in conventional supervised learning, e.g., non-stationarity, bootstrapping, and delayed targets

Fitted value iteration with function approximation

- How do we represent $V(s)$?
  - Discrete: big table, one entry for each s
  - Continuous: neural network function $V_\phi : \mathcal{S} \to \mathbb{R}$



- Fitted value iteration. Loop:
  1. set target value: $y \leftarrow \max_a Q(s,a)$
  2. update neural net $\phi \leftarrow \arg\min_\phi ||V_\phi(s) - y||^2$

$$Q(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma V_\phi(s')]$$

- Loss function: Mean squared error (MSE)

$$\mathcal{L}(\phi) = \frac{1}{2}||V_\phi(s) - \max_a Q(s,a)||^2$$

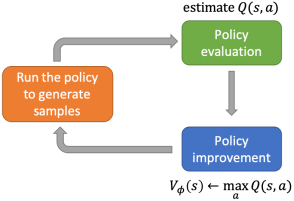


Table of Contents

Review: Determine policies from value functions

- For $V^*(s)$: a one-step search
  - Actions that appear best after one-step search will be optimal actions
  - $a^* = \arg\max_a Q^*(s,a) = \arg\max_a \sum_{s',r} p(s',r|s,a)(r + \gamma V^*(s'))$
- For $Q^*(s,a)$: no need to do a one-step-ahead search
  - $a^* = \arg\max_a Q^*(s,a)$
  - The optimal action-value function allows optimal actions to be selected without having to know anything about possible successor states and their values, i.e., without having to know anything about the environment's dynamics

Review: Value iteration and Q-learning

- From prediction problems to control problems
- From model-based to model-free
- From state-value functions $V(s)$ to action-value functions $Q(s,a)$

$$V_{k+1}(s) \leftarrow \boxed{\max_a \sum_{s',r} p(s',r|s,a)[r+\gamma V_k(s')]}$$

- Directly approximate the optimal state-value function, V^*
- Need to know outcomes for different actions

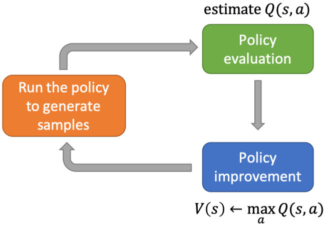
$$Q_{k+1}(s,a) \leftarrow r(s,a) + \gamma \max_{a'} Q_k(s',a')$$

- Directly approximate the optimal action-value function, Q^*
- Fit the value function using only samples (s,a,r,s')

Value iteration $\rightarrow$ Fitted value iteration

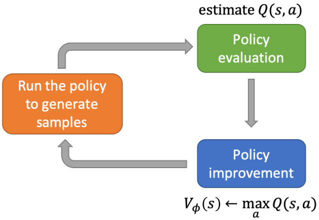
- Value iteration. Loop:
  1. Policy evaluation: evaluate $Q(s, a)$
  2. Implicit policy improvement:
     set $V(s) \leftarrow \max_a Q(s, a)$

$$Q(s,a) = \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$$



- Fitted value iteration. Loop:
  1. set target value: $y \leftarrow \max_a Q(s, a)$
  2. update neural net
     $\phi \leftarrow \arg\min_\phi ||V_\phi(s) - y||^2$

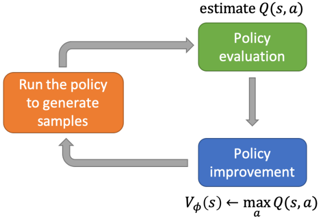
$$Q(s,a) = \sum_{s',r} p(s', r|s, a)[r + \gamma V_\phi(s')]$$



What if we don't know the transition dynamics?

- Fitted value iteration. Loop:
  1. set target value: $y \leftarrow \boxed{\max_a Q(s,a)}$
  2. update neural net $\phi \leftarrow \arg\min_\phi ||V_\phi(s) - y||^2$

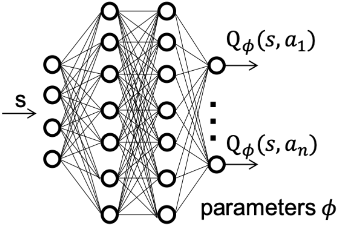
$$Q(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma V_\phi(s')]$$



- Need to know outcomes for different actions!
- **Question**: Can we extend the idea of fitted value iteration to a model-free learner?
  - The most popular model-free algorithm: Q-learning
  - Fitted value iteration → fitted Q-learning?

- Q-learning. Loop:
  1. Policy evaluation: evaluate $Q_k(s,a)$
  2. Implicit policy improvement, set: $Q_{k+1}(s,a) \leftarrow r(s,a) + \gamma \max_{a'} Q_k(s',a')$

- Fitted Q-iteration. Loop:
  1. set target value: $y \leftarrow r(s,a) + \gamma \max_{a'} Q_\phi(s',a')$
  2. update neural net $\phi \leftarrow \arg\min_\phi ||Q_\phi(s,a) - y||^2$



Q-learning → Fitted Q-iteration (FQI)

- Q-learning. Loop:
  1. Policy evaluation: evaluate $Q_k(s,a)$
  2. Implicit policy improvement, set: $Q_{k+1}(s,a) \leftarrow r(s,a) + \gamma \max_{a'} Q_k(s',a')$

- Temporal difference error (Bellman residual) for Q-learning:

$$\delta = r(s,a) + \gamma \max_{a'} Q(s',a') - Q(s,a)$$

- Fitted Q-iteration. Loop:
  1. set target value: $y \leftarrow r(s,a) + \gamma \max_{a'} Q_\phi(s',a')$
  2. update neural net $\phi \leftarrow \arg\min_\phi ||Q_\phi(s,a) - y||^2$

- Loss function (Bellman residual) for fitted Q-iteration:

$$\mathcal{L}(\phi) = ||r(s,a) + \gamma \max_{a'} Q_\phi(s',a') - Q_\phi(s,a)||^2$$

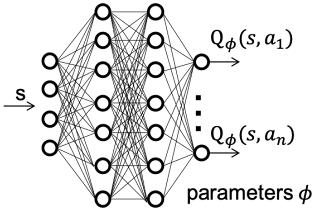
Full fitted Q-iteration algorithm

- Loop:
  1. collect dataset $\{(s_i, a_i, r_i, s'_i)\}$ using behavior policy π

     loop for K iterations:

     2. set $y_i \leftarrow r_i + \gamma \max_{a'_i} Q_\phi(s'_i, a'_i)$
     3. set $\phi \leftarrow \arg\min_\phi \sum_i ||Q_\phi(s_i, a_i) - y_i||^2$

- Hyper-parameters:
  - dataset size N
  - behavior policy π for data collection
  - iterations K
  - gradient steps S for minimizing $\mathcal{L}(\phi)$



Review: On-policy vs. Off-policy

Target policy $\pi(a|s)$	Behavior policy $b(a|s)$
To be evaluated or improved	To explore to generate data
Make decisions finally	Make decisions in training phase

- **On-policy** methods: $\pi(a|s) = b(a|s)$
  - Evaluate or improve the policy that is used to make decisions during training
  - e.g., SARSA
- **Off-policy** methods: $\pi(a|s) \neq b(a|s)$
  - Evaluate or improve a policy different from that used to generate the data
  - Separate exploration from control
  - e.g., Q-learning

Review: Q-learning vs. SARSA

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha[R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)]$$

- Q-learning approximates the optimal action-value function for an optimal policy, $Q \approx Q_* = Q_{\pi_*}$
  - The target policy is greedy w.r.t Q, $\pi(a|s) = \arg\max_a Q(s, a)$
  - The behavior policy can be others, e.g., $b(a|s) = \varepsilon$-greedy

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$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)]$$

- SARSA approximates the action-value function for the behavior policy, $Q \approx Q_\pi = Q_b$
  - The target and the behavior policy are the same, e.g., $\pi(a|s) = b(a|s) = \varepsilon$-greedy

Why FQI is off-policy?

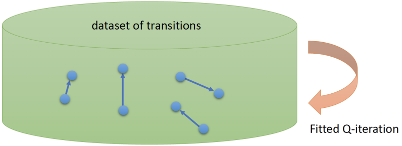
- Loop:
  1. collect dataset $\{(s_i, a_i, r_i, s_i')\}$ using behavior policy π

     loop for K iterations:

  2. set $y_i \leftarrow r_i + \gamma \max_{a_i'} Q_\phi(s_i', a_i')$
  3. set $\phi \leftarrow \arg\min_\phi \sum_i ||Q_\phi(s_i, a_i) - y_i||^2$

- The target greedy policy:

$$\pi(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_a Q_\phi(s, a) \\ 0 & \text{otherwise} \end{cases}$$



What is FQI optimizing?

- Loop:
  1. collect dataset $\{(s_i, a_i, r_i, s'_i)\}$ using behavior policy π

     loop for K iterations:

     2. set $y_i \leftarrow r_i + \gamma \max_{a'_i} Q_\phi(s'_i, a'_i)$
     3. set $\phi \leftarrow \arg\min_\phi \sum_i ||Q_\phi(s_i, a_i) - y_i||^2$

- Loss function (Bellman residual) for fitted Q-iteration:

$$\mathcal{L}(\phi) = ||r(s,a) + \gamma \max_{a'} Q_\phi(s', a') - Q_\phi(s,a)||^2$$

- If $\mathcal{L}(\phi) = 0$, then we have **Bellman optimality equation**:

$$Q_{\phi^*}(s,a) = r(s,a) + \gamma \max_{a'} Q_{\phi^*}(s', a')$$

Exploration strategies - Behavior policy

- Loop:
  1. collect dataset $\{(s_i, a_i, r_i, s_i')\}$ using some behavior policy $b(a|s)$

     loop for K iterations:

     2. set $y_i \leftarrow r_i + \gamma \max_{a_i'} Q_\phi(s_i', a_i')$
     3. set $\phi \leftarrow \arg\min_\phi \sum_i ||Q_\phi(s_i, a_i) - y_i||^2$

- The target policy is the greedy policy w.r.t. $Q_\phi(s, a)$:

$$\pi(a|s) = \begin{cases} 1 & \text{if } a = \arg\max_a Q_\phi(s, a) \\ 0 & \text{otherwise} \end{cases}$$

- **Question**: Is it a good idea to use the target greedy policy as the behavior policy to collect samples?

Exploitation-exploration dilemma

- ϵ-greedy strategy:

$$b(a_t|s_t) = \begin{cases} 1-\epsilon & \text{if } a_t = \arg\max_{a_t} Q_\phi(s_t, a_t) \\ \epsilon/(|\mathcal{A}|-1) & \text{otherwise} \end{cases}$$

  - $1-\epsilon$: the exploitation part
  - $\epsilon/(|\mathcal{A}|-1)$: the exploration part
  - balance/trade-off between exploitation and exploration: decrease ϵ as the learning proceeds

- Boltzmann strategy, softmax strategy:

$$b(a_t|s_t) = \frac{e^{Q(s_t,a_t)/\tau}}{\sum_a e^{Q(s_t,a)/\tau}}$$

  - τ: temperature parameter
  - balance/trade-off between exploitation and exploration: decrease τ as the learning proceeds

Batch-mode (offline) vs. online

- Batch-model (offline) algorithms
  - Collect a batch of samples using some policy
  - Fit the state- or action-value function iteratively
- Online algorithms
  - Take some action to collect one sample
  - Fit the value function
  - Iteratively alternate the above two steps

Online Q-learning algorithms

- Batch-mode fitted Q-iteration algorithm. Loop:
  1. collect dataset $\{(s_i, a_i, r_i, s'_i)\}$ using behavior policy π

     loop for K iterations:
     2. set $y_i \leftarrow r_i + \gamma \max_{a'_i} Q_\phi(s'_i, a'_i)$
     3. set $\phi \leftarrow \arg\min_\phi \sum_i ||Q_\phi(s_i, a_i) - y_i||^2$

- Online fitted Q-iteration algorithm. Loop:
  1. observe one sample (s_i, a_i, r_i, s'_i) using behavior policy π
  2. set $y_i \leftarrow r_i + \gamma \max_{a'_i} Q_\phi(s'_i, a'_i)$
  3. set $\phi \leftarrow \phi - \alpha \frac{\mathrm{d}Q_\phi(s_i,a_i)}{\mathrm{d}\phi}(Q_\phi(s_i, a_i) - y_i)$

Looking back from DRL

- Tabular RL algorithms
  - R. S. Sutton and A. G. Barto, **Reinforcement Learning: An Introduction**, 2nd Edition.
- RL with linear function approximation
  - M. G. Lagoudakis and R. Parr, **Least-Squares Policy Iteration**, *Journal of Machine Learning Research*, 2003.
- RL with nonlinear function approximation, e.g., neural network
  - J. A. Boyan, et al., **Generalization in reinforcement learning: Safely approximating the value function**, NIPS 1995.
  - R. S. Sutton, et al., **Policy gradient methods for reinforcement learning with function approximation**, NIPS 2000.
  - Martin Riedmiller, **Neural Fitted Q Iteration – First Experiences with a Data Efficient Neural Reinforcement Learning Method**, ECML 2005.
- RL with deep neural networks
  - Human-level performance, the milestone of artificial intelligence

Review

- Value-based methods
  - Don't learn a policy explicitly
  - Just learn state- or action-value function
- If we have value function, we have a policy
- Fitted value iteration
- Fitted Q-iteration
  - Batch mode, off-policy method

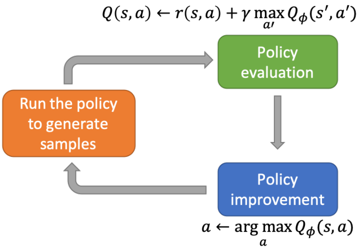


Table of Contents

Value iteration theory in tabular cases

- Bellman optimality equation

$$V^*(s) = \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V^*(s')]$$

- Value iteration

$$V_{k+1}(s) = \max_a \sum_{s',r} p(s',r|s,a)[r + \gamma V_k(s')]$$

  - Turn Bellman optimality equation into an update rule
  - Directly approximate the optimal state-value function, V^*
  - For arbitrary V_0, the sequence $\{V_k\}$ converges to V^* under the same conditions that guarantee the existence of V^*

Value iteration

- Value iteration algorithm. Loop:
  1. Policy evaluation: evaluate $Q(s,a)$
  2. Implicit policy improvement: set $V(s) \leftarrow \max_a Q(s,a)$

$$Q(s,a) = \sum_{s',r} p(s',r|s,a)[r + \gamma V(s')]$$

- Define a Bellman operator $\mathcal{B} : \mathcal{B}V = \max_a(r + \gamma \mathcal{T}_a V)$
  - $\mathcal{T}$: state transition function, $\mathcal{T}_{a,i.j} = p(s' = i|s = j, a)$
- V^* is a **fixed point** of $\mathcal{B}$, $V^* = \mathcal{B}V^*$, Bellman optimality equation:

$$V^*(s) = \sum_{s',r} p(s',r|s,a)[r + \gamma V^*(s')]$$

  - always exists, is always unique, always corresponds to the optimal policy
  - ...but will we reach it?

Bellman operator is a contraction

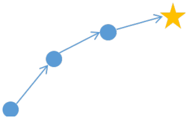
- V^* is a fixed point of $\mathcal{B}$, i.e., $V^* = \mathcal{B}V^*$, Bellman optimality equation:

$$V^*(s) = \sum_{s',r} p(s', r|s, a)[r + \gamma V^*(s')]$$

- We can prove that value iteration reaches V^* since $\mathcal{B}$ is a contraction
  - For any V and $\bar{V}$, we have $||\mathcal{B}V - \mathcal{B}\bar{V}||_\infty \leq \gamma ||V - \bar{V}||_\infty$
  - The gap always gets smaller by $\gamma \in (0, 1)$

- Choose V^* as $\bar{V}$, note that $\mathcal{B}V^* = V^*$, we have:

$$||\mathcal{B}V - V^*||_\infty \leq \gamma ||V - V^*||_\infty$$



Norm (mathematics)

- A norm is a function from a real or complex vector space to the nonnegative real numbers
  - behave in certain ways like the distance from the origin: it commutes with scaling, obeys a form of the triangle inequality, and is zero only at the origin
  - the length or size of a vector in a certain space

- p-norm: $||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{(1/p)}$
  - l_1 norm: $||x||_1 = \sum_{i}^{n} |x_i|$
  - l_2 norm, the Euclidean norm: $||x||_2 = \sqrt{x_1^2 + ... + x_n^2}$
  - l_∞ norm, the infinity norm: $||x||_\infty = \max_i |x_i|$

Non-tabular value function learning

- Define a new operator Π:

$$\Pi V = \arg\min_{V' \in \Omega} ||V'(s) - V(s)||^2$$

  - Π is a **projection** onto Ω in terms of l_2 norm

- Fitted value iteration. Loop:
  1. $\mathcal{B}$ operator: set $y \leftarrow \max_a \sum_{s',r} p(s', r|s, a)(r + \gamma V_\phi(s'))$
  2. Π operator: set $\phi \leftarrow \arg\min_\phi ||V_\phi(s) - y||^2$

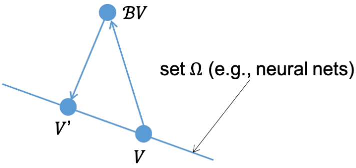
$$V' \leftarrow \arg\min_{V' \in \Omega} ||V'(s) - (\mathcal{B}V)(s)||^2$$

Non-tabular value function learning

- Fitted value iteration. Loop:
  1. $\mathcal{B}$ operator: set $y \leftarrow \max_a \sum_{s',r} p(s',r|s,a)(r+\gamma V_\phi(s'))$
  2. Π operator: set $\phi \leftarrow \arg\min_\phi ||V_\phi(s) - y||^2$

$$V' \leftarrow \arg\min_{V' \in \Omega} ||V'(s) - (\mathcal{B}V)(s)||^2$$

- $(\mathcal{B}V)(s)$: updated value function
- Ω: all value functions represented by, e.g., neural nets
- Π is a **projection** onto Ω in terms of l_2 norm

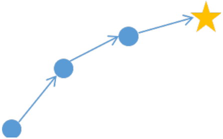


Non-tabular value function learning

- fitted value iteration using $\mathcal{B}$ and Π. Loop:
  1. $V \leftarrow \Pi \mathcal{B} V$

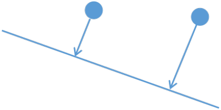
- $\mathcal{B}$ is a contraction w.r.t. l_∞ norm ("max" norm)

$$||\mathcal{B}V - \mathcal{B}\bar{V}||_\infty \leq \gamma ||V - \bar{V}||_\infty$$



- Π is a contraction w.r.t. l_2 norm (Euclidean distance)

$$||\Pi V - \Pi \hat{V}||^2 \leq ||V - \bar{V}||^2$$



Non-tabular value function learning

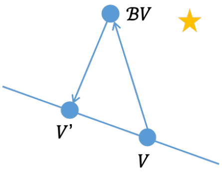
- $\mathcal{B}$ is a contraction w.r.t. l_∞-norm ("max" norm)

$$||\mathcal{B}V - \mathcal{B}\bar{V}||_\infty \leq \gamma||V - \bar{V}||_\infty$$

- Π is a contraction w.r.t. l_2 norm (Euclidean distance)

$$||\Pi V - \Pi\hat{V}||^2 \leq ||V - \bar{V}||^2$$

- But... $\Pi\mathcal{B}$ is not contraction of any kind!
- Conclusions:
  - value iteration converges (tabular case)
  - fitted value iteration does **not** converge
  - not in general
  - often not in practice



What about fitted Q-iteration?

- Define a Bellman operator $\mathcal{B}$: $\mathcal{B}Q = r + \gamma\mathcal{T}\max_a Q$
- Define an operator Π : $\Pi Q = \arg\min_{Q' \in \Omega} ||Q'(s,a) - Q(s,a)||^2$

- Fitted Q-iteration. Loop:
  1. $\mathcal{B}$ operator: set $y \leftarrow r(s,a) + \gamma \max_{a'} Q_\phi(s', a')$
  2. Π operator: set $\phi \leftarrow \arg\min_\phi ||Q_\phi(s,a) - y||^2$

- Using $\mathcal{B}$ and Π. Loop:
  1. $Q \leftarrow \Pi\mathcal{B}Q$

- Conclusions:
  - $\mathcal{B}$ is a contraction w.r.t. l_∞ norm ("max" norm)
  - Π is a contraction w.r.t. l_2 norm (Euclidean distance)
  - But... $\Pi\mathcal{B}$ is not contraction of any kind!

Fitted Q-iteration – Regression, but not gradient descent

- Online fitted Q-iteration algorithm. Loop:
  1. observe one sample (s_i, a_i, r_i, s_i') using behavior policy π
  2. set $y_i \leftarrow r_i + \gamma \max_{a_i'} Q_\phi(s_i', a_i')$
  3. set $\boxed{\phi \leftarrow \phi - \alpha \frac{\mathrm{d}Q_\phi(s_i, a_i)}{\mathrm{d}\phi}(Q_\phi(s_i, a_i) - y_i)}$

- isn't this just gradient descent? that converges, right?

- Fitted Q-iteration is not gradient descent!

$$\phi \leftarrow \phi - \alpha \frac{\mathrm{d}Q_\phi(s_i, a_i)}{\mathrm{d}\phi} \left(Q_\phi(s_i, a_i) - \underbrace{\left(r_i + \gamma \max_{a_i'} Q_\phi(s_i', a_i') \right)}_{\text{no gradient through target value!}} \right)$$

Value function learning for actor-critic

- Batch actor-critic algorithm. Loop:
  1. sample $\{s_i, a_i, r(s_i, a_i), s_i'\}$ from $\pi_\theta(a|s)$ (run it on the robot)
  2. policy evaluation: fit $\hat{V}_\phi^\pi(s)$ to sampled reward sums
  3. evaluate $\hat{A}^\pi(s_i, a_i) = r(s_i, a_i) + \gamma \hat{V}_\phi^\pi(s_i') - \hat{V}_\phi^\pi(s_i)$
  4. policy improvement: $\nabla_\theta J(\theta) \approx \sum_i \nabla_\theta \log \pi_\theta(a_i|s_i) \hat{A}^\pi(s_i, a_i)$
  5. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

- An aside regarding terminology
  - V^π: value function for policy π
    this is what the critic (policy evaluation) does
  - V^*: value function for optimal policy π^*
    this is what value iteration does

Value function learning for actor-critic

- Batch actor-critic algorithm. Loop:
  1. sample $\{s_i, a_i, r(s_i, a_i), s'_i\}$ from $\pi_\theta(a|s)$ (run it on the robot)
  2. policy evaluation: fit $\hat{V}^\pi_\phi(s)$ to sampled reward sums
  3. evaluate $\hat{A}^\pi(s_i, a_i) = r(s_i, a_i) + \gamma \hat{V}^\pi_\phi(s'_i) - \hat{V}^\pi_\phi(s_i)$
  4. policy improvement: $\nabla_\theta J(\theta) \approx \sum_i \nabla_\theta \log \pi_\theta(a_i|s_i) \hat{A}^\pi(s_i, a_i)$
  5. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

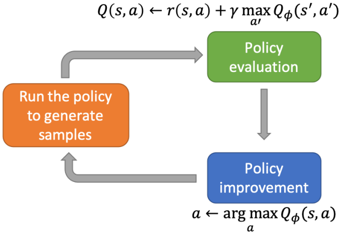
- Define a Bellman operator $\mathcal{B}$: $\mathcal{B}V = r + \gamma \mathcal{T} V$
  - l_∞ contraction, without the $\max$ operator; $y_i = r(s_i, a_i) + \gamma V^\pi_\phi(s'_i)$

- Define an operator Π : $\Pi V' = \arg\min_{V' \in \Omega} ||V'(s) - V(s)||^2$
  - l_2 contraction; $\mathcal{L}(\phi) = \sum_i ||V^\pi_\phi(s_i) - y_i||^2$

- Value function learning for the critic:
  - $V \leftarrow \Pi \mathcal{B} V$; fitted bootstrapped policy evaluation doesn't converge!

Review

- Value iteration theory
  - Linear operator for backup
  - Linear operator for projection
  - Backup is contraction
  - Value iteration converges
- Convergence with function approximation
  - Projection is also a contraction
  - Projection + backup is not a contraction
  - Fitted value iteration does not in general converge
- Implications for Q-learning
  - Q-learning, fitted Q-iteration, etc. does not converge with function approximation
- But we can make it work in practice!



Learning objectives of this lecture

- You should be able to...
  - Extend discrete value iteration to fitted value iteration with function approximation
  - Extend discrete Q-learning to fitted Q-iteration with function approximation
  - Be aware of some theories of value function methods

Suggested readings

- Lecture 7 of CS285 at UC Berkeley, **Deep Reinforcement Learning, Decision Making, and Control**
  - `http://rail.eecs.berkeley.edu/deeprlcourse/static/slides/lec-7.pdf`
- Classical papers
  - Lagoudakis. 2003. **Least-squares policy iteration**: linear function approximation
  - Riedmiller. 2005. **Neural fitted Q-iteration**: batch mode Q-learning with neural networks

THE END