Lecture 6: Policy Gradients

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What we'll cover

— Contents

- The policy gradient algorithm
- What does the policy gradient do?
- Basic variance reduction: causality
- Basic variance reduction: baselines
- Policy gradient examples

— Goals

- Understand policy gradient RL
- Understand practical considerations for policy gradients

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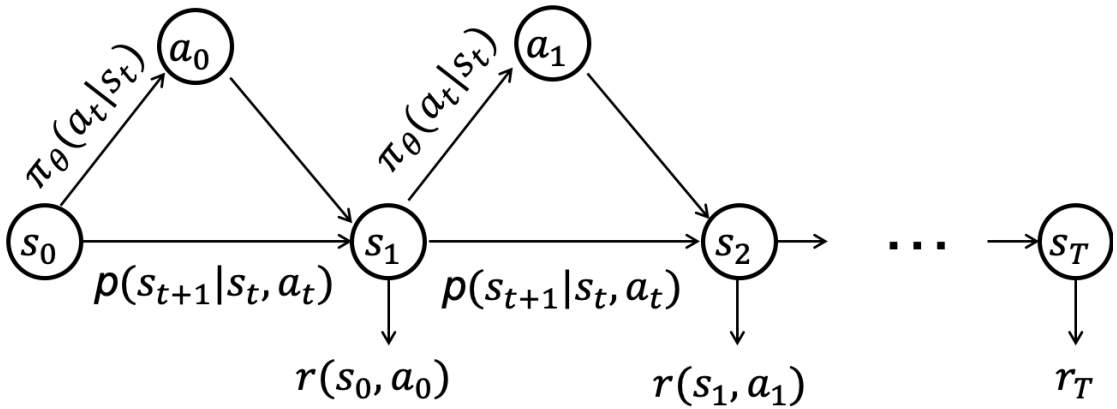
Review: the goal of RL

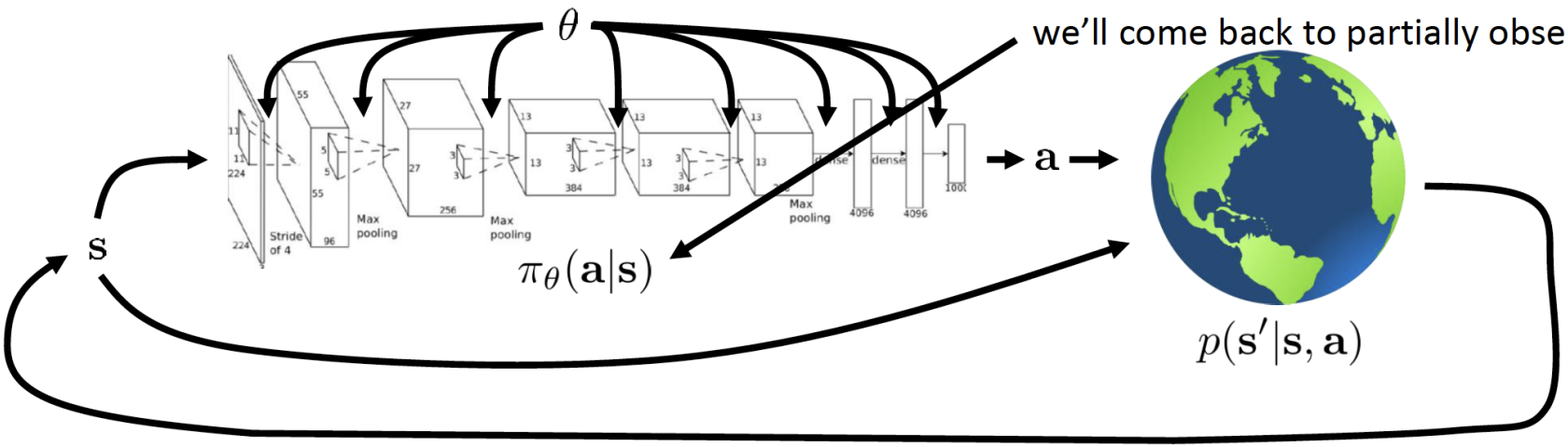
- Find **optimal policies** to maximize cumulative reward

$$\pi^* = \arg\max_{\pi} \mathbb{E}_{\pi} \left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \right]$$

  - In a **trial-and-error** manner
  - A general **optimization** framework for sequential decision-making



The goal of RL



$$p_\theta(\tau) = p_\theta(s_0, a_0, ..., s_T, a_T) = p(s_0)\prod_{t=0}^{T} \pi_\theta(a_t|s_t)p(s_{t+1}|s_t, a_t)$$

$$\theta^* = \arg\max_{\theta\in\mathbb{R}^d} \mathbb{E}_{\tau\sim p_\theta(\tau)}\left[\sum_t \gamma^t r(s_t, a_t)\right] = \arg\max_{\theta\in\mathbb{R}^d} \mathbb{E}_{\tau\sim \pi_\theta(\tau)}\left[\sum_t \gamma^t r(s_t, a_t)\right]$$

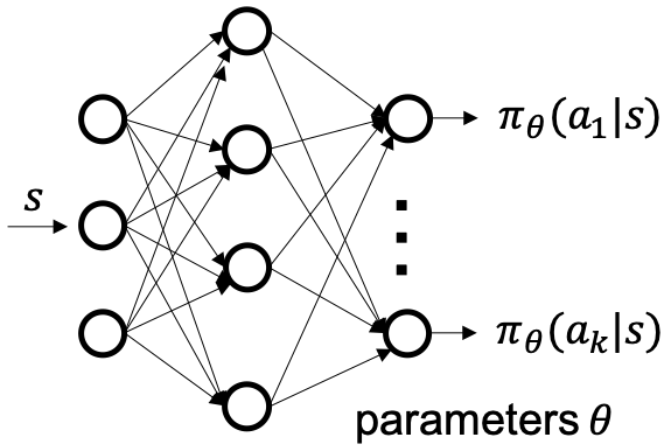
Discrete action space: Categorical distribution

- Categorical distribution for the finite number of actions
- What is Categorical distribution?

Discrete action space: Categorical distribution

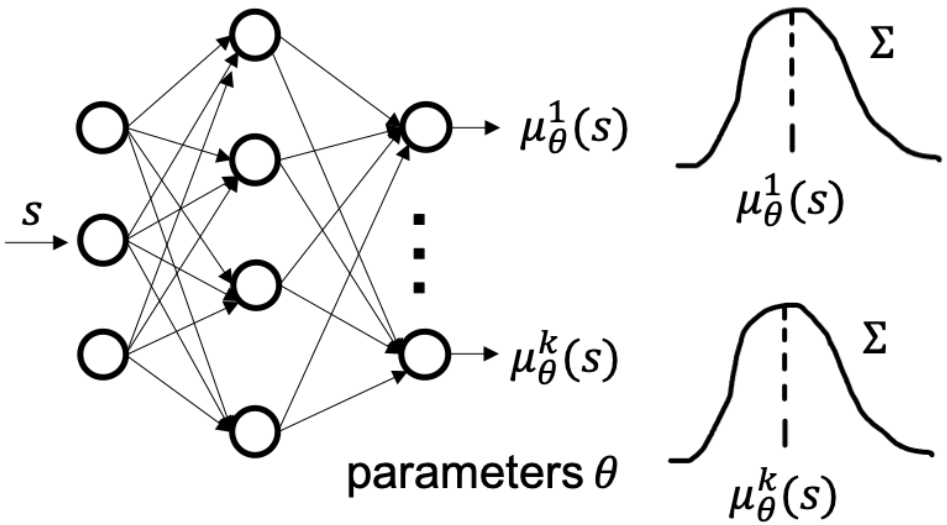
- A discrete probability distribution that describes the possible results of a random variable that can take on one of k possible categories, with the probability of each category separately specified

- $k > 0$: number of categories
- $p_1, ..., p_k$: event probabilities
- $p_i > 0, \sum_i p_i = 1$



Continuous action space: Gaussian distribution

- $k > 0$: the dimension of action space, $a = [a_1, ..., a_k] \in \mathbb{R}^k$
- The Gaussian policy: $\pi_\theta(a|s) = \mathcal{N}(\mu_\theta(s); \Sigma)$



The goal of RL

$$\theta^* = \arg\max_{\boldsymbol{\theta} \in \mathbb{R}^d} \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[\sum_t \gamma^t r(s_t, a_t) \right]$$

- Infinite horizon case

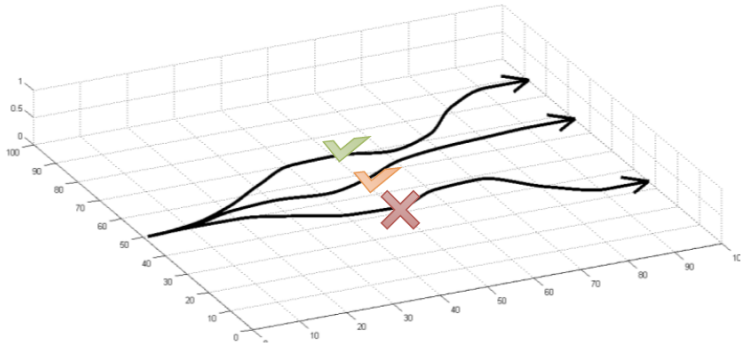
$$\theta^* = \arg\max_\theta \mathbb{E}_{(s,a) \sim \pi_\theta(s,a)} [r(s,a)]$$

- Finite horizon case

$$\theta^* = \arg\max_\theta \mathbb{E}_{(s_t,a_t) \sim \pi_\theta(s_t,a_t)} \left[\sum_{t=0}^{T} \gamma^t r(s_t, a_t) \right]$$

Evaluating the objective

$$\theta^* = \arg\max_{\theta \in \mathbb{R}^d} \underbrace{\mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[\sum_t \gamma^t r(s_t, a_t) \right]}_{J(\theta)}$$



$$J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[\sum_t \gamma^t r(s_t, a_t) \right] \approx \frac{1}{N} \underbrace{\sum_i}_{\uparrow} \sum_t \gamma^t r(s_t^i, a_t^i)$$

sum over samples from π_θ

Direct policy differentiation

- Objective function / cost function

$$J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\underbrace{r(\tau)}_{\sum_{t=0}^{T} r(s_t, a_t)}] = \int \pi_\theta(\tau) r(\tau) \, \mathrm{d}\tau$$

- The gradient – differentiate the objective function

$$\begin{aligned} \nabla_\theta J(\theta) &= \int \nabla_\theta \pi_\theta(\tau) r(\tau) \, \mathrm{d}\tau = \int \pi_\theta(\tau) \nabla_\theta \log \pi_\theta(\tau) r(\tau) \, \mathrm{d}\tau \\ &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)] \end{aligned}$$

- A convenient identity

$$\pi_\theta(\tau) \nabla_\theta \log \pi_\theta(\tau) = \pi_\theta(\tau) \frac{\nabla_\theta \pi_\theta(\tau)}{\pi_\theta(\tau)} = \nabla_\theta \pi_\theta(\tau)$$

Direct policy differentiation

- $\nabla_\theta J(\theta) = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)]$

$$\underbrace{\pi_\theta(s_0, a_0, ..., s_T, a_T)}_{\pi_\theta(\tau)} = p(s_0) \prod_{t=0}^{T} \pi_\theta(a_t|s_t) p(s_{t+1}|s_t, a_t)$$

$$\begin{aligned}\nabla_\theta \log \pi_\theta(\tau) &= \nabla_\theta \left[\underline{\log p(s_0)} + \sum_{t=0}^{T} [\log \pi_\theta(a_t|s_t) + \underline{\log p(s_{t+1}|s_t, a_t)}] \right] \\ &= \nabla_\theta \sum_{t=0}^{T} \log \pi_\theta(a_t|s_t)\end{aligned}$$

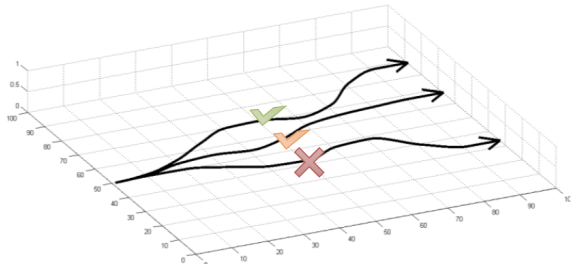
- $\nabla_\theta J(\theta) = \mathbb{E}_{\tau\sim\pi_\theta(\tau)} \left[\left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t, a_t) \right) \right]$

Evaluating the policy gradient

$$J(\theta) = \mathbb{E}_{\tau \sim p_\theta(\tau)} \left[\sum_t \gamma^t r(s_t, a_t) \right] \approx \frac{1}{N} \sum_i \sum_t \gamma^t r(s_t^i, a_t^i)$$

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[\left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t, a_t) \right) \right]$$

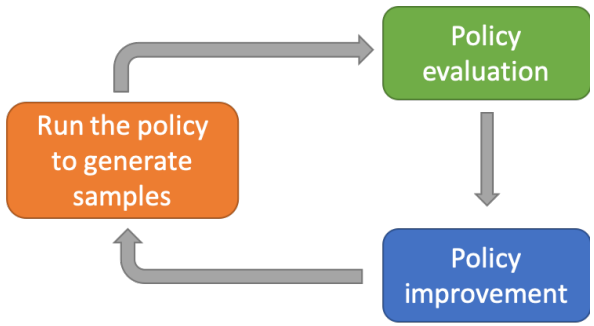
$$\approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i|s_t^i) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$$



$$\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$$

REINFORCE algorithm. Loop:

1. sample $\{\tau^i\}$ from $\pi_\theta(a_t|s_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(a_t^i|s_t^i) \right) \left(\sum_t \gamma^t r(s_t^i, a_t^i) \right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$



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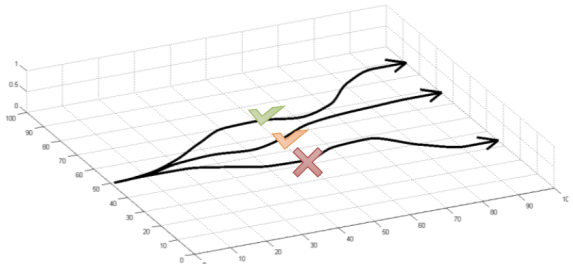
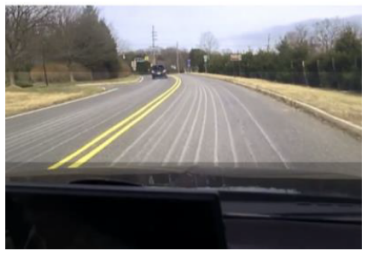
Evaluating the policy gradient

$$J(\theta) = \mathbb{E}_{\tau \sim p_\theta(\tau)} \left[\sum_t r(s_t, a_t) \right] \approx \frac{1}{N} \sum_i \sum_t \gamma^t r(s_t^i, a_t^i)$$

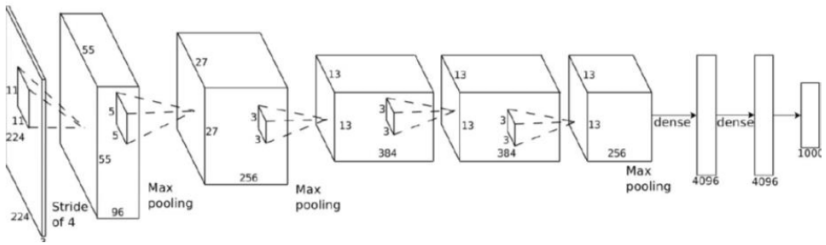
$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)} \left[\left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t|s_t) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t, a_t) \right) \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \underbrace{\left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i|s_t^i) \right)}_{\text{What is this?}} \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$$

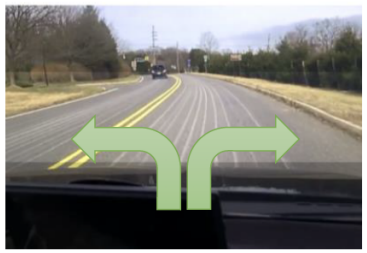




$\mathbf{s}_t$



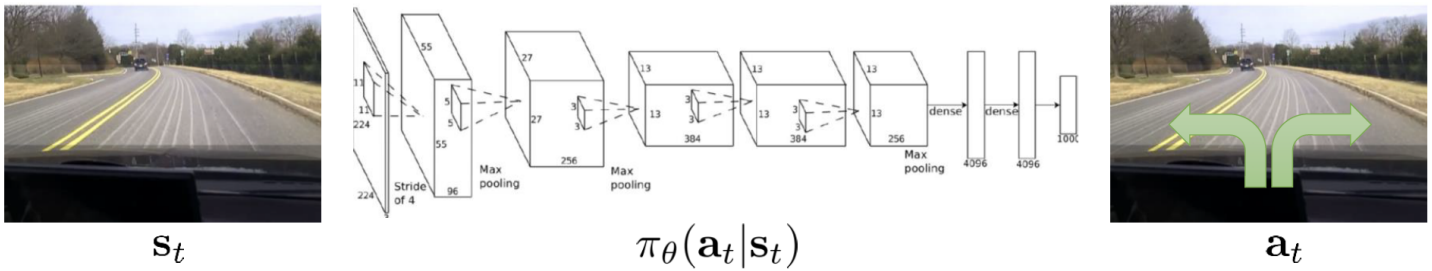
$\pi_\theta(\mathbf{a}_t|\mathbf{s}_t)$

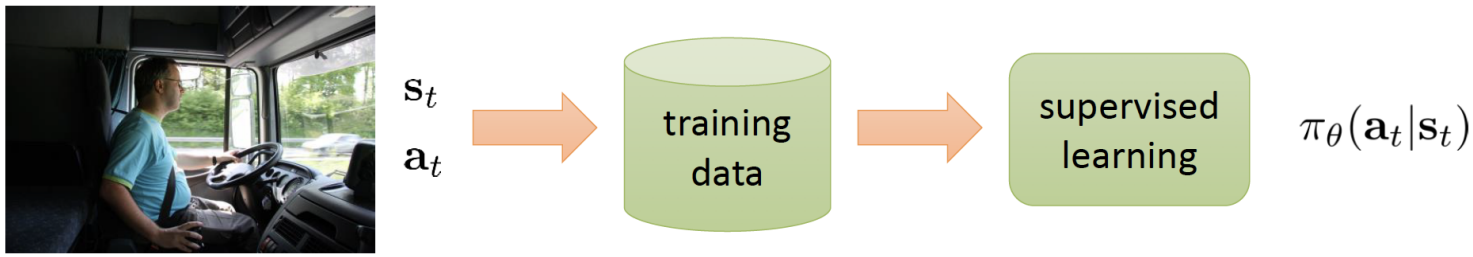


$\mathbf{a}_t$

Comparison to maximum likelihood

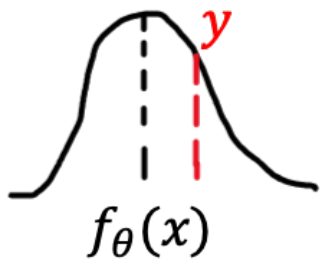
- Policy gradient: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$
- Maximum likelihood estimation: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right)$





Maximum likelihood estimation (MLE)

- A method of estimating the parameters of a probability distribution by maximizing a likelihood function
  - So that under the assumed statistical model, **the observed data is most probable**



- For regression problem, input-label (x, y), prediction $\hat{y} = f_\theta(x)$

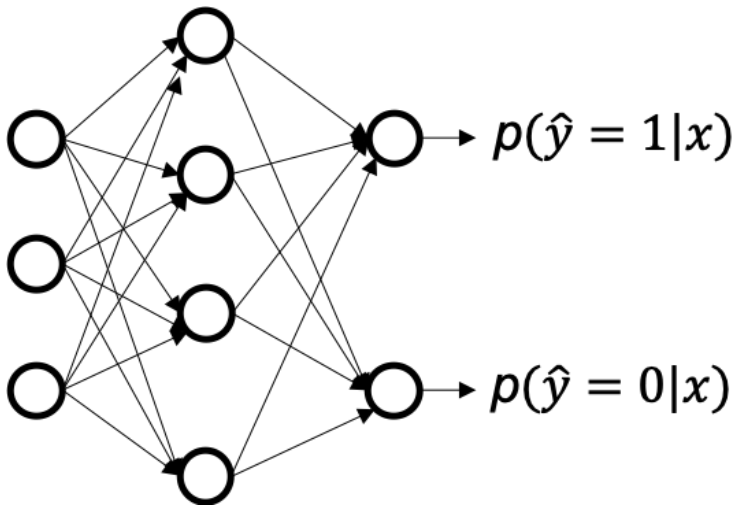
$$\text{likelihood:} \quad \theta^* = \arg\max_{\theta \in \mathbb{R}^d} \Pi_i \mathcal{N}(y_i; f_\theta(x_i), \sigma^2)$$

$$\text{negative log-likelihood:} \quad \theta^* = \arg\min_{\theta \in \mathbb{R}^d} \sum_i (y_i - f_\theta(x_i))^2$$

Maximum likelihood estimation (MLE)

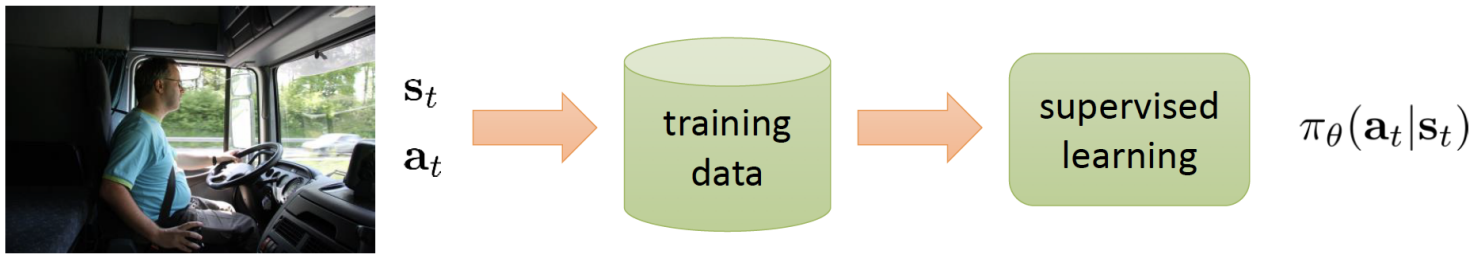
- For classification problem, input-label (x, y), prediction $p_\theta(y|x)$
- Cross-entropy loss, minimize the negative log-likelihood:

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^d} \sum_i -y_i \log p_\theta(\hat{y}_i = 1|x_i) - (1 - y_i) \log(1 - p_\theta(\hat{y}_i = 1|x_i))$$



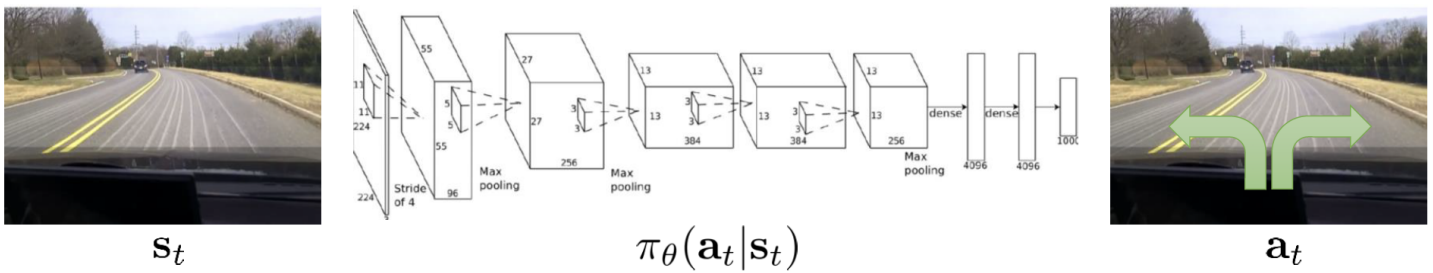
Maximum likelihood estimation – Imitation learning

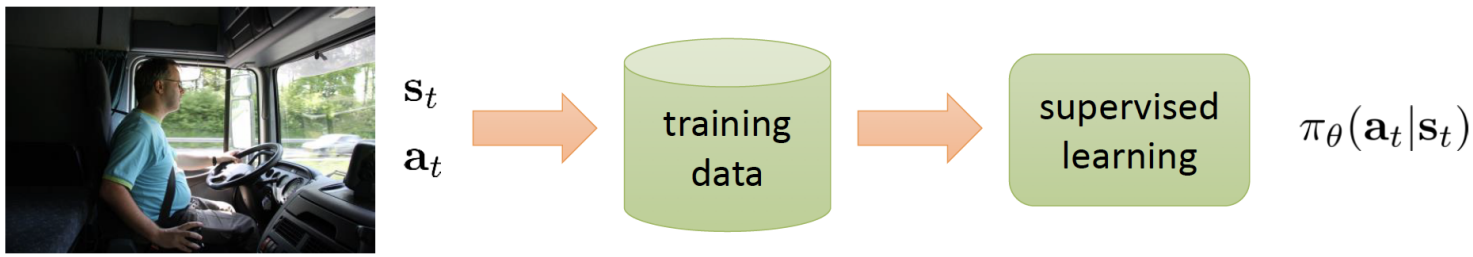
- Maximum likelihood estimation: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right)$
- Review: Imitation learning – Imitating expert demonstrations (s_t^i, a_t^i)
  - Maximizing the likelihood function $\pi_\theta(a_t^i | s_t^i)$
  - Under the assumed statistical model θ,
    the expert demonstrations are most probable



Comparison to maximum likelihood

- Policy gradient: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$
  - **Make what most probable?**
- Maximum likelihood estimation: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right)$
  - Make every demonstration most probable





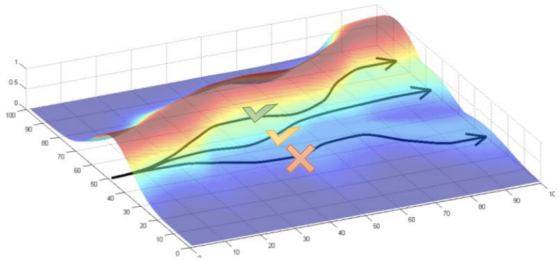
What did we just do?

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$$

- Policy gradient: $\nabla_\theta J(\theta) \approx \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(\tau^i) r(\tau^i)$
  - large $r(\tau^i)$: push $\pi_\theta(\tau^i)$ close to 1, good stuff is made more likely
  - small $r(\tau^i)$: push $\pi_\theta(\tau^i)$ close to 0, bad stuff is made less likely
  - simply formalizes the note of "trial and error"

REINFORCE algorithm: Loop:

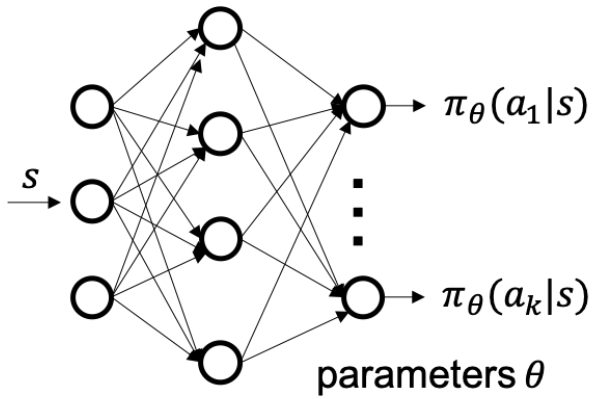
1. sample $\{\tau^i\}$ from $\pi_\theta(a_t|s_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(a_t^i|s_t^i) \right) \left(\sum_t \gamma^t r(s_t^i, a_t^i) \right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$



Discrete action space: Categorical distribution

- A discrete probability distribution that describes the possible results of a random variable that can take on one of k possible categories, with the probability of each category separately specified

- $k > 0$: number of categories
- $p_1, ..., p_k$: event probabilities
- $p_i > 0, \sum_i p_i = 1$

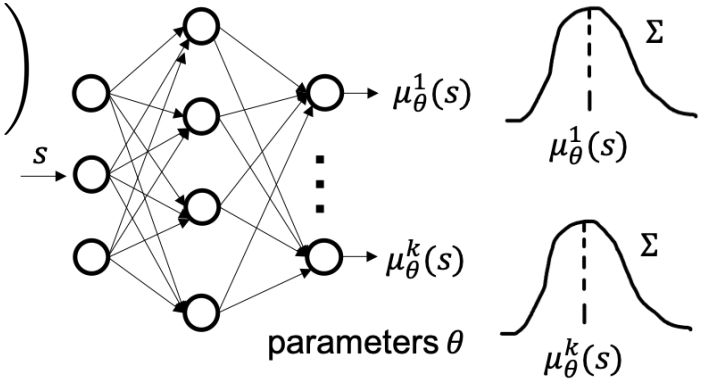


Discrete action space: Categorical distribution

```python
import torch
import torch.nn as nn
import torch.nn.functional as F
from torch.distributions import Categorical


class CategoricalMLPPolicy(Policy):
    def __init__(self, state_dim, num_actions):
        super(CategoricalMLPPolicy, self).__init__()

        self.input_layer = nn.Linear(state_dim, 512)
        self.hidden_layer = nn.Linear(512, 512)
        self.output_layer = nn.Linear(512, num_actions)

    def forward(self, input, params=None):
        # states: (time_horizon * batch_size * state_dim) tensor of states
        x = self.input_layer(states)
        x = F.relu(x)
        x = self.hidden_layer(x)
        x = F.relu(x)

        # probabilities for discrete actions
        logits = self.output_layer(x)

        return Categorical(logits=logits)
```

Continuous action space: Gaussian distribution

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{t=0}^{T}\nabla_\theta \log \pi_\theta(a_t^i|s_t^i)\right)\left(\sum_{t=0}^{T}\gamma^t r(s_t^i,a_t^i)\right)$$



- example: $\pi_\theta(a_t|s_t) = \mathcal{N}(\mu_\theta(s_t);\Sigma)$
- $\log \pi_\theta(a_t|s_t) = -\frac{1}{2}||\mu_\theta(s_t) - a_t||_\Sigma^2 + \text{const}$
- $\nabla_\theta \log \pi_\theta(a_t|s_t) = -\Sigma^{-1}(\mu_\theta(s_t) - a_t)\frac{\mathrm{d}\mu_\theta(s_t)}{\mathrm{d}\theta}$

REINFORCE algorithm. Loop:

1. sample $\{\tau^i\}$ from $\pi_\theta(a_t|s_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(a_t^i|s_t^i)\right)\left(\sum_t \gamma^t r(s_t^i,a_t^i)\right)$
3. $\theta \leftarrow \theta + \alpha\nabla_\theta J(\theta)$

Continuous action space: Gaussian distribution

```python
import torch
import torch.nn as nn
import torch.nn.functional as F
from torch.distributions import Normal


class NormalMLPPolicy(nn.Module):
    def __init__(self, state_dim, action_dim):
        super(NormalMLPPolicy, self).__init__()

        self.input_layer = nn.Linear(state_dim, 512)
        self.hidden_layer = nn.Linear(512, 512)
        self.output_layer = nn.Linear(512, action_dim)

        # Gaussian layer, add stochasticity to actions
        # the scale of the Gaussian distribution for the action
        self.sigma = nn.Parameter(torch.Tensor(action_dim))
        self.sigma.data.fill_(math.log(1.0))


    def forward(self, states):
        # states: (time_horizon * batch_size * state_dim) tensor of states
        x = self.input_layer(states)
        x = F.relu(x)
        x = self.hidden_layer(x)
        x = F.relu(x)

        # the mean of the Gaussian distribution for the action
        mu = self.output_layer(x)

        return Normal(loc=mu, scale=self.sigma)
```

Policy gradient with automatic differentiation

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \underline{\left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right)} \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$$

- pretty inefficient to compute these explicitly!
- How can we compute policy gradients with automatic differentiation?
- We need a graph such that its gradient is the policy gradient!

- Just implement "pseudo-loss" as a *weighted* maximum likelihood:

$$J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \underline{\log \pi_\theta(a_t^i | s_t^i)} Q^\pi(s_t^i, a_t^i)$$

cross-entropy loss (discrete)
mean squared error (continuous)

- maximum likelihood estimation

$$\nabla_\theta J_{ML}(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i)$$

$$J_{ML}(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \log \pi_\theta(a_t^i | s_t^i)$$

Policy gradient with automatic differentiation

- Pseudocode example (with discrete actions):

- Maximum likelihood:

```
# Given:
# states: (time_horizon * batch_size * state_dim) tensor of states
# actions: (time_horizon * batch_size * num_actions) tensor of actions

# Build the graph
logits = policy(states)
negative_likelihoods = torch.nn.CrossEntropyLoss(lables=actions, logits=logits)
loss = torch.mean(negative_likelihoods)   # The loss is a scalar
gradients = torch.autograd.grad(loss, policy.parameters())
```

Policy gradient with automatic differentiation

- Pseudocode example (with discrete actions):

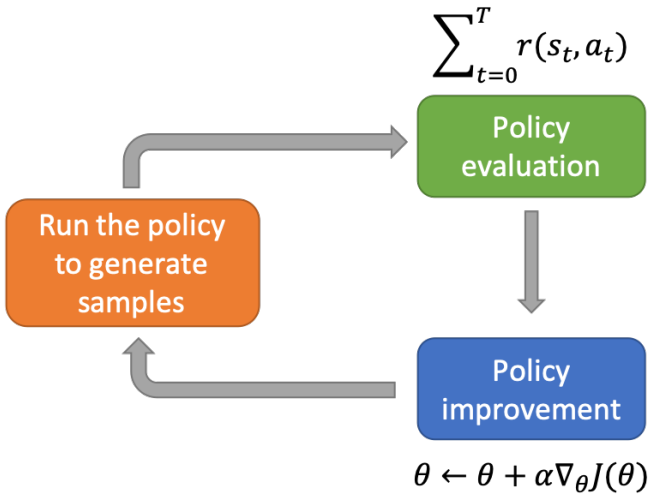
- Policy gradient:

```
# Given:
# states: (time_horizon * batch_size * state_dim) tensor of states
# actions: (time_horizon * batch_size * num_actions) tensor of actions
# q_values: (time_horizon * batch_size * 1) tensor of estimated state-action values

# Build the graph
logits = policy(states)
negative_likelihoods = torch.nn.CrossEntropyLoss(lables=actions, logits=logits)
weighted_negative_likelihoods = negative_likelihoods * q_values  # element-wise multiply
loss = torch.mean(weighted_negative_likelihoods)   # The loss is a scalar
gradients = torch.autograd.grad(loss, policy.parameters())
```

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \underbrace{\left(\sum_{t'=t}^{T} \gamma^{t'-t} r(s_{t'}^i, a_{t'}^i) \right)}_{Q^\pi(s_t^i, a_t^i)}$$

Review of the policy gradient

- Evaluating the RL objective
  - Generate samples
- Evaluating the policy gradient
  - Log gradient trick
  - Generate samples
- Understand policy gradient
  - Formalization of trial-and-error
- Can implement with automatic differentiation
  - need to know what to backpropagate



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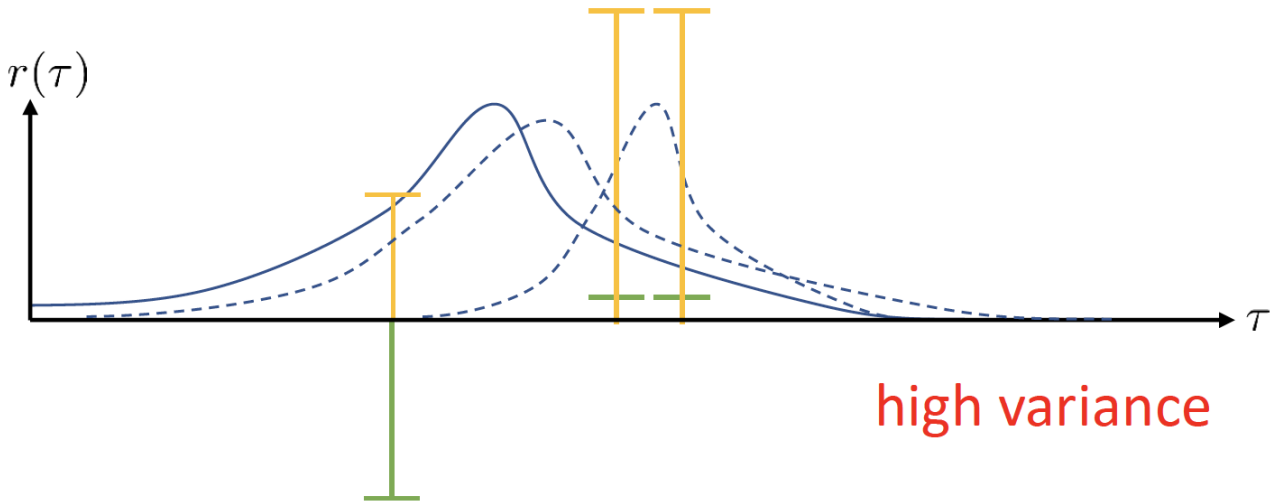
What is wrong with the policy gradient?

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log \pi_\theta(\tau) r(\tau)$$



- Even worse: what if the two "good" samples have $r(\tau) = 0$?

Reducing variance - Causality

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \right) \left(\sum_{t=0}^{T} \gamma^t r(s_t^i, a_t^i) \right)$$

$$\Downarrow$$

- **Causality**: policy at time t' cannot affect reward at time t when $t < t'$

$$\Downarrow$$

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_t^i | s_t^i) \underbrace{\left(\sum_{t'=t}^{T} \gamma^{t'-t} r(s_{t'}^i, a_{t'}^i) \right)}_{Q^\pi(s_t^i, a_t^i)}$$

"reward-to-go"

Review: Value functions

- V^π, the **state-value function** for policy π

$$V^\pi(s) = \mathbb{E}_\pi[G_t|S_t = s] = \mathbb{E}_\pi\left[\sum_{k=1}^{\infty} \gamma^k R_{t+k+1}|S_t = s\right], \forall s \in \mathcal{S}$$

- Q^π, the **action-value function** for policy π

$$\begin{aligned} Q^\pi(s,a) &= \mathbb{E}_\pi[G_t|S_t = s, A_t = a] \\ &= \mathbb{E}_\pi\left[\sum_{k=1}^{\infty} \gamma^k R_{t+k+1}|S_t = s, A_t = a\right], \forall s \in \mathcal{S}, a \in \mathcal{A}(s) \end{aligned}$$

Reducing variance - Baselines

$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log \pi_\theta(\tau)[r(\tau) - b]$$

$$b = \frac{1}{N} \sum_{i=1}^{N} r(\tau)$$

- But... are we allowed to do that?
- The key problem is

$$\mathbb{E}_{\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)] = \mathbb{E}_{\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau)(r(\tau) - b)] \quad \textbf{???}$$

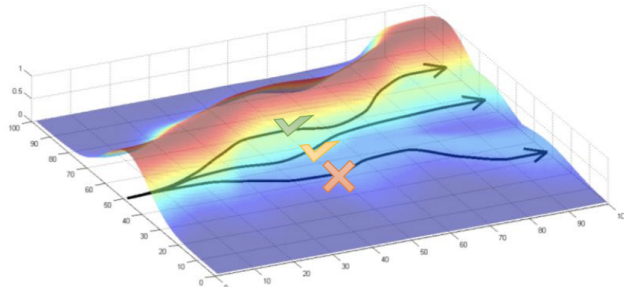
Reducing variance - Baselines

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N} \nabla_\theta \log \pi_\theta(\tau)[r(\tau) - b]$$

a convenient identity

$$\pi_\theta(\tau)\nabla_\theta \log \pi_\theta(\tau) = \nabla_\theta \pi_\theta(\tau)$$

$$b = \frac{1}{N}\sum_{i=1}^{N} r(\tau)$$



- But... are we allowed to do that?

$$\mathbb{E}_{\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) b] = \int \pi_\theta(\tau)\nabla_\theta \log \pi_\theta(\tau) b \, \mathrm{d}\tau = \int \nabla_\theta \pi_\theta(\tau) b \, \mathrm{d}\tau$$

$$= b\nabla_\theta \int \pi_\theta(\tau) \, \mathrm{d}\tau = b\nabla_\theta 1 = 0$$

- Subtracting a baseline is unbiased in expectation!
- Average reward is not the best baseline, but it's pretty good!

What is the best baseline?

- The best baseline makes the **variance** of $\nabla_\theta J(\theta)$ minimal

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau)(r(\tau) - b)] = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[g(\tau)(r(\tau) - b)]$$

- Compute the variance: $var[x] = \mathbb{E}[x^2] - \mathbb{E}[x]^2$

$$var = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[g(\tau)^2(r(\tau) - b)^2] - \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau)(r(\tau) - b)]^2$$

- **How to derive the best baseline b ???**

Analyzing the variance

$$var = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}[g(\tau)^2(r(\tau)-b)^2] - \underbrace{\mathbb{E}_{\tau\sim\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau)(r(\tau)-b)]^2}_{\mathbb{E}_{\tau\sim\pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)]^2}$$

(baselines are unbiased in expectation)

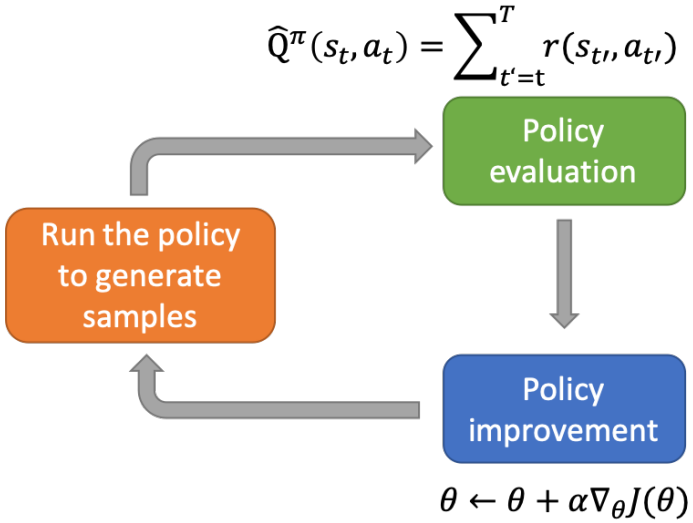
$$\begin{aligned}
\frac{\mathrm{d}var}{\mathrm{d}b} &= \frac{\mathrm{d}}{\mathrm{d}b}\mathbb{E}[g(\tau)^2(r(\tau)-b)^2] \\
&= \frac{\mathrm{d}}{\mathrm{d}b}(\mathbb{E}[g(\tau)^2 r(\tau)^2] \underbrace{-2\mathbb{E}[g(\tau)^2 r(\tau) b] + b^2\mathbb{E}[g(\tau)^2]}_{\text{dependent of } b}) \\
&= -2\mathbb{E}[g(\tau)^2 r(\tau)] + 2b\mathbb{E}[g(\tau)^2] = 0
\end{aligned}$$

$$b^* = \frac{\mathbb{E}[g(\tau)^2 r(\tau)]}{\mathbb{E}[g(\tau)^2]}$$

This is just expected reward, but weighted by gradient magnitudes!

Review

- The high variance of policy gradient
- Exploiting causality
  - Future doesn't affect the past
- Baselines
  - Unbiased!
- Analyzing variance
  - Can derive optimal baselines



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Review: On-policy vs. off-policy

Target policy $\pi(a\|s)$	Behavior policy $b(a\|s)$
To be evaluated or improved	To explore to generate data
Make decisions finally	Make decisions in training phase

- **On-policy** methods: $\pi(a|s) = b(a|s)$
  - Evaluate or improve the policy that is used to make decisions during training
  - e.g., SARSA
- **Off-policy** methods: $\pi(a|s) \neq b(a|s)$
  - Evaluate or improve a policy different from that used to generate the data
  - Separate exploration from control
  - e.g., Q-learning

Policy gradient is on-policy

$$\theta^* = \arg\max_{\theta \in \mathbb{R}^d} J(\theta)$$

$$J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[r(\tau)]$$

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)]$$

This is the trouble!

- Neural networks change only a little bit with each gradient step
- On-policy learning can be extremely inefficient!

REINFORCE algorithm. Loop:

1. sample $\{\tau^i\}$ from $\pi_\theta(a_t|s_t)$ (run the policy)
2. $\nabla_\theta J(\theta) \approx \sum_i \left(\sum_t \nabla_\theta \log \pi_\theta(a_t^i|s_t^i)\right)\left(\sum_t \gamma^t r(s_t^i, a_t^i)\right)$
3. $\theta \leftarrow \theta + \alpha \nabla_\theta J(\theta)$

can't just skip this!

The off-policy case

- How to reuse samples that are generated by another behavior policy $\bar{\pi}, \bar{\pi} \neq \pi_\theta$?

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)] \\ &\neq \mathbb{E}_{\tau \sim \bar{\pi}(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)]\end{aligned}$$

- How to estimate $\nabla_\theta J(\theta)$ using samples $\tau \sim \bar{\pi}(\tau)$?

Importance sampling

- A general technique for estimating properties of a particular distribution $p(x)$, while only having samples generated from a different distribution $q(x)$ than the distribution of interest $p(x)$

$$
\begin{aligned}
\mathbb{E}_{x\sim p(x)}[f(x)] &= \int p(x)f(x)dx \\
&= \int \frac{q(x)}{q(x)}p(x)f(x)\,\mathrm{d}x \\
&= \int q(x)\frac{p(x)}{q(x)}f(x)\,\mathrm{d}x \\
&= \mathbb{E}_{x\sim q(x)}\left[\frac{p(x)}{q(x)}f(x)\right]
\end{aligned}
$$

Importance sampling for estimating policy gradient

- We need to estimate the gradient $\nabla_\theta \log \pi_\theta(\tau) r(\tau)$ of a distribution $\tau \sim \pi_\theta(\tau)$, while only having samples generated from a different distribution $\tau \sim \bar{\pi}(\tau)$

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)] \\ &= \mathbb{E}_{\tau \sim \bar{\pi}(\tau)}[???]\end{aligned}$$

Off-policy learning with importance sampling

- We need to estimate the gradient $\nabla_\theta \log \pi_\theta(\tau) r(\tau)$ of a distribution $\tau \sim \pi_\theta(\tau)$, while only having samples generated from a different distribution $\tau \sim \bar{\pi}(\tau)$

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[\nabla_\theta \log \pi_\theta(\tau) r(\tau)] \\ &= \mathbb{E}_{\tau \sim \bar{\pi}(\tau)}\left[\frac{\pi_\theta(\tau)}{\bar{\pi}(\tau)} \nabla_\theta \log \pi_\theta(\tau) r(\tau)\right]\end{aligned}$$

$$\frac{\pi_\theta(\tau)}{\bar{\pi}(\tau)} = \frac{p(s_0)\prod_{t=0}^{T} \pi_\theta(a_t|s_t) p(s_{t+1}|s_t, a_t)}{p(s_0)\prod_{t=0}^{T} \bar{\pi}(a_t|s_t) p(s_{t+1}|s_t, a_t)} = \frac{\prod_{t=0}^{T} \pi_\theta(a_t|s_t)}{\prod_{t=0}^{T} \bar{\pi}(a_t|s_t)}$$

Policy gradient with importance sampling

- $\theta^* = \arg\max_{\theta \in \mathbb{R}^d} J(\theta), \quad J(\theta) = \mathbb{E}_{\tau \sim \pi_\theta(\tau)}[r(\tau)]$
- How to estimate the value of some *new* parameters θ'?

$$\nabla_{\theta'} J(\theta') = \mathbb{E}_{\tau \sim \pi_{\theta'}(\tau)}[\nabla_{\theta'} \log \pi_{\theta'}(\tau) r(\tau)] = \mathbb{E}_{\tau \sim \pi_\theta(\tau)}\left[\frac{\pi_{\theta'}(\tau)}{\pi_\theta(\tau)} \nabla_\theta \log \pi_\theta(\tau) r(\tau)\right]$$

$$= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}\left[\left(\frac{\prod_{t=0}^{T} \pi_{\theta'}(a_t|s_t)}{\prod_{t=0}^{T} \pi_\theta(a_t|s_t)}\right)\left(\sum_{t=0}^{T} \nabla_{\theta'} \log \pi_{\theta'}(a_t|s_t)\right)\left(\sum_{t=0}^{T} \gamma^t r(s_t, a_t)\right)\right]$$

$$= \mathbb{E}_{\tau \sim \pi_\theta(\tau)}\left[\sum_{t=0}^{T} \nabla_{\theta'} \log \pi_{\theta'}(a_t|s_t)\left(\prod_{t'=1}^{t} \frac{\pi_{\theta'}(a_{t'}|s_{t'})}{\pi_\theta(a_{t'}|s_{t'})}\right)\left(\sum_{t'=t}^{T} \gamma^{t'-t} r(s_{t'}, a_{t'})\right)\right]$$

Causality: future actions don't affect current weight

A first-order approximation for importance sampling

$$\nabla_{\theta'} J(\theta') = \mathbb{E}_{\tau\sim\pi_\theta(\tau)}\left[\sum_{t=0}^{T}\nabla_{\theta'}\log\pi_{\theta'}(a_t|s_t)\underbrace{\left(\prod_{t'=1}^{t}\frac{\pi_{\theta'}(a_{t'}|s_{t'})}{\pi_\theta(a_{t'}|s_{t'})}\right)}_{\text{exponential in } T}\left(\sum_{t'=t}^{T}\gamma^{t'-t}r(s_{t'},a_{t'})\right)\right]$$

- On-policy policy gradient: $(s_t^i, a_t^i) \sim \pi_\theta(s_t, a_t)$

$$\nabla_\theta J(\theta) \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\nabla_\theta\log\pi_\theta(a_t^i|s_t^i)Q^\pi(s_t^i,a_t^i)$$

write the objective a bit differently

- Off-policy policy gradient:

$$\nabla_{\theta'} J(\theta') \approx \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{\pi_{\theta'}(s_t^i,a_t^i)}{\pi_\theta(s_t^i,a_t^i)}\nabla_{\theta'}\log\pi_{\theta'}(a_t^i|s_t^i)Q^\pi(s_t^i,a_t^i)$$

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{\pi_{\theta'}(s_t^i)}{\pi_\theta(s_t^i)}\frac{\pi_{\theta'}(a_t^i|s_t^i)}{\pi_\theta(a_t^i|s_t^i)}\nabla_{\theta'}\log\pi_{\theta'}(a_t^i|s_t^i)Q^\pi(s_t^i,a_t^i)$$

ignore this part

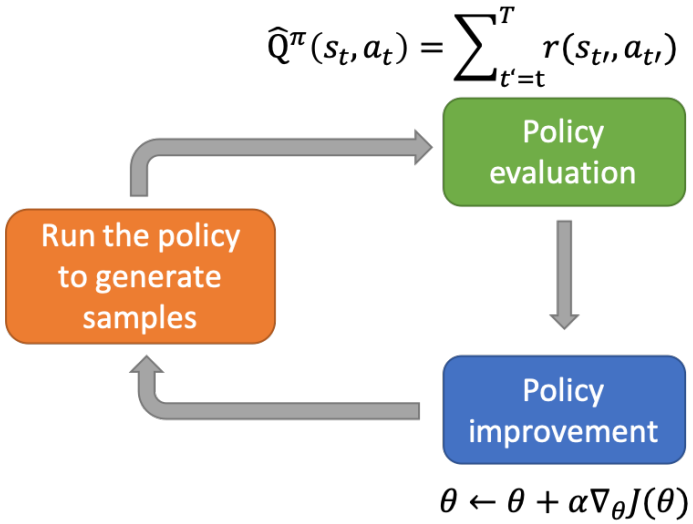
$$\frac{\pi_{\theta'}(s_t^i)}{\pi_\theta(s_t^i)} \approx 1$$

Policy gradient in practice

- Remember that the gradient has high variance
  - This isn't the same as supervised learning!
  - Gradients will be really noisy!
- Consider using much larger batches
- Tweaking learning rates is very hard
  - Using adaptive step size rules like ADAM
  - More policy gradient specific learning rate adjustment methods...

Review

- Policy gradient is on-policy
- Can derive off-policy variant
  - Use importance sampling
  - Exponential scaling in T
  - Can ignore state portion (first-order approximation)
- Practical considerations: batch size, learning rates, optimizers



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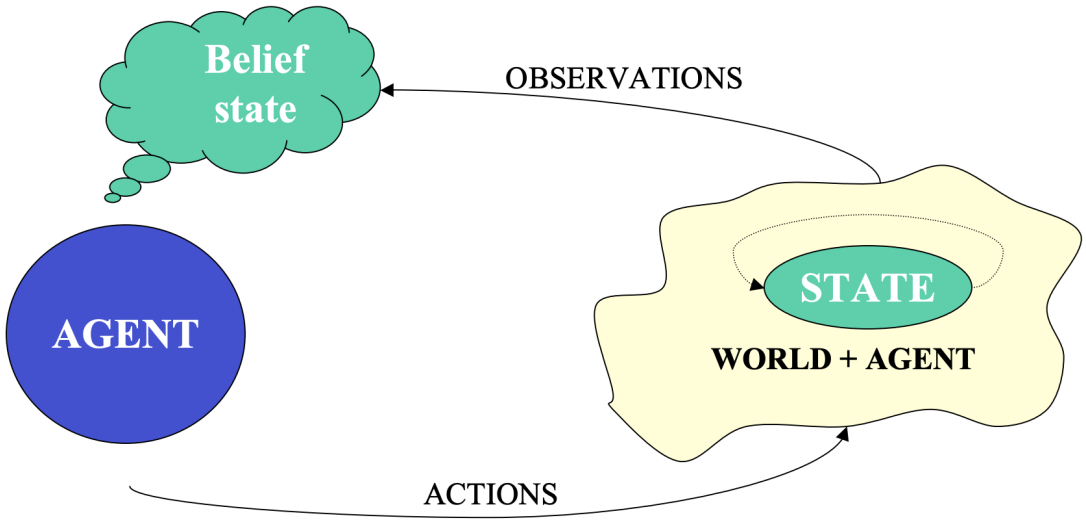
Recall: Partially Observable MDP (POMDP)

POMDP: Uncertainty

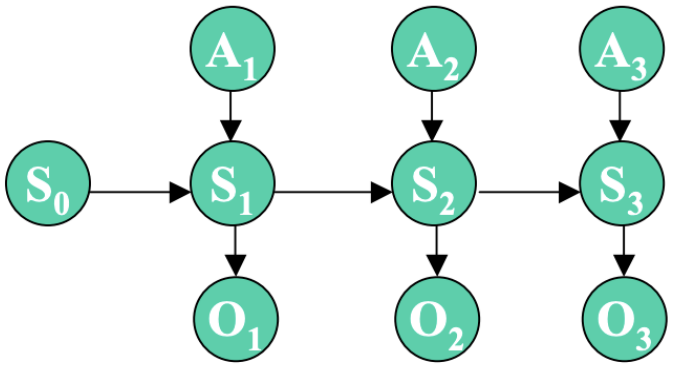
- Case 1: Uncertainty about the action outcome
- Case 2: Uncertainty about the world state due to imperfect (partial) information



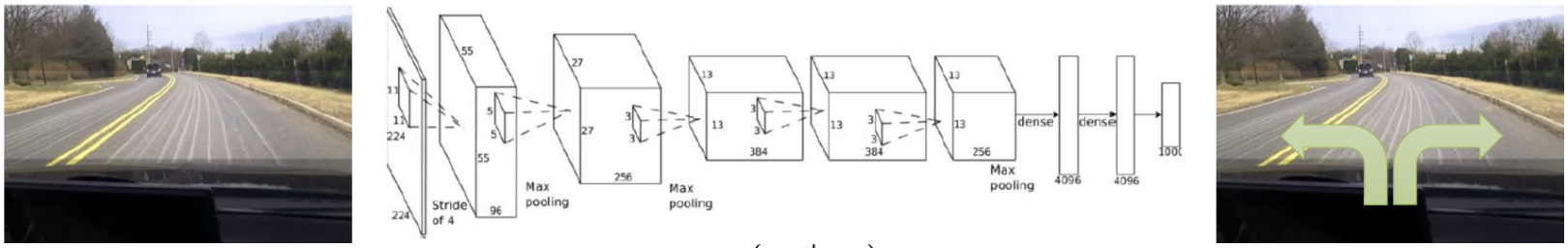
GOAL = Selecting appropriate actions

Recall: Partially Observable MDP (POMDP)

- A generalization of an MDP
  - the agent cannot directly observe the underlying state
  - it must maintain a probability distribution over the set of possible states, based on a set of observations and observation probabilities
- $M = \langle S, A, T, R, \Omega, O \rangle$
  - $\Omega = \{o_1, ..., o_k\}$ is a set of observations
  - $O(o|s', a)$ is a set of conditional observation probabilities



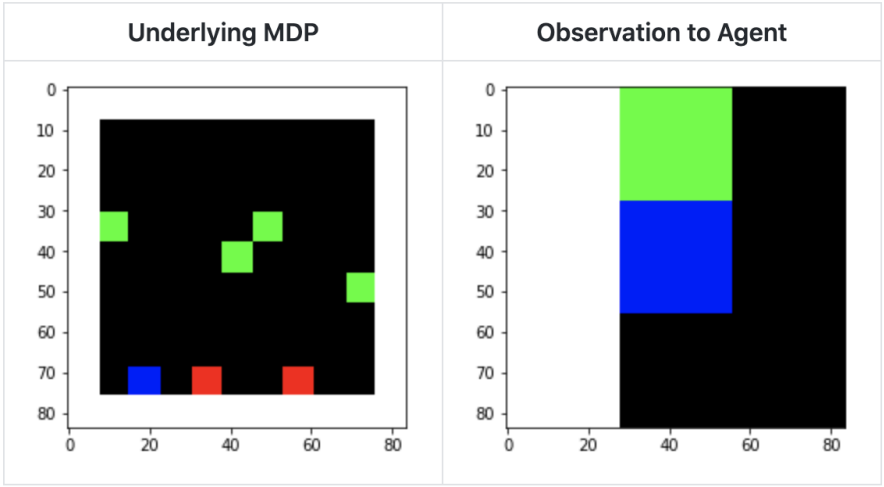
Partial Observability

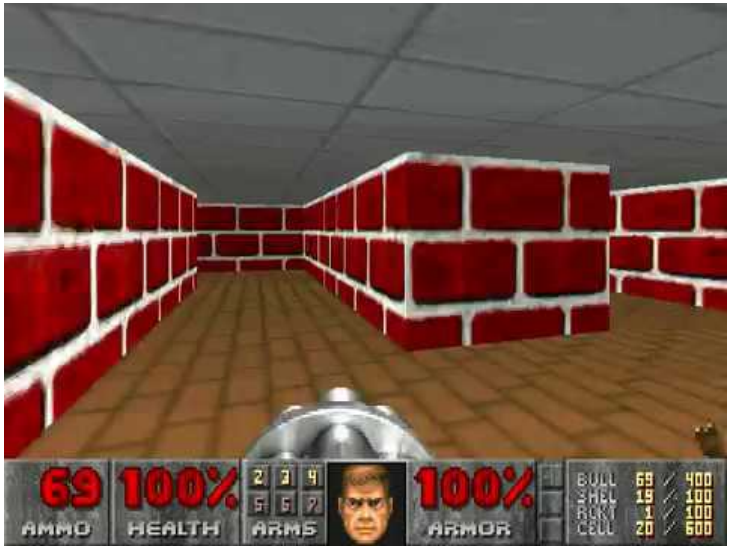


$$\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_{t=0}^{T} \nabla_\theta \log \pi_\theta(a_{i,t}|o_{i,t}) \sum_{t'=t}^{T} \gamma^{t'-t} r(s_{i,t}, a_{i,t})$$

- Markov property is not actually used!
- Can use policy gradient in partially observable MDPs without modification, empirically

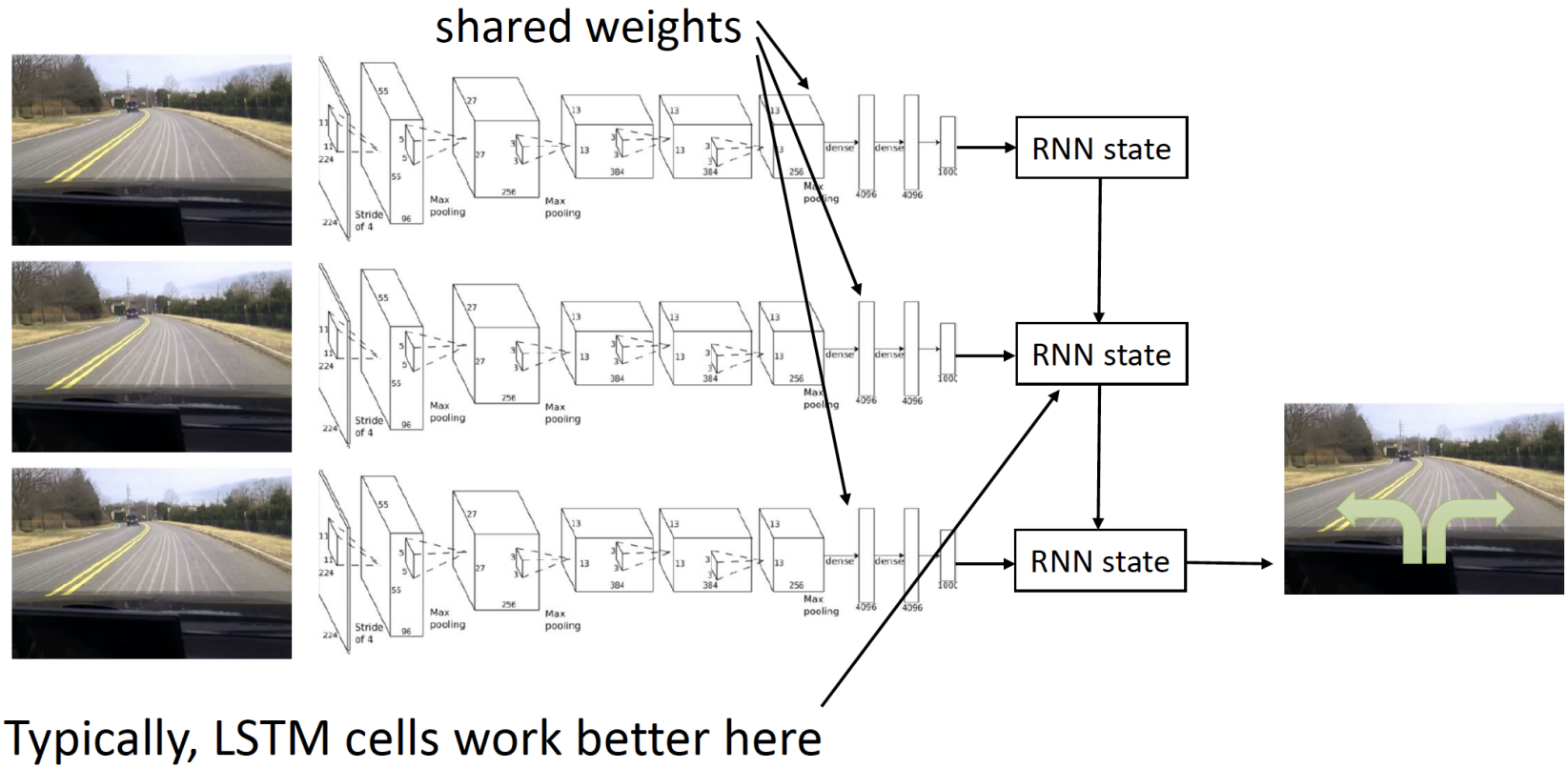
Examples: Partial Observability





Address partial observability: use the whole history

- Use recurrent neural network (RNN) as the encoder
- Embed the summary of past states into the internal **memory**



Learning objectives of this lecture

- You should be able to...
  - Understand and be able to use the vanilla policy gradient method
  - Be able to use the baseline to reduce the variance of policy gradient
  - Know the importance sampling technique for off-policy policy gradient
  - Know the implementation tricks in practice

References

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  - Williams (1992). **Simple statistical gradient following algorithms for connectionist reinforcement learning**: introduces REINFORCE algorithm.
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THE END