

Lecture 3: Martingale Theory

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3.1 Discrete-time Martingale

3.1.1 Background and Definitions

Given a stochastic process $\{X_n\}_{n \in \mathbb{N}_0}$, as the time n increases, so does our knowledge about what has happened in the past, which can be modelled using filtration.

Definition 3.1 (Filtration (Discrete)) *Given the measurable space (Ω, \mathcal{F}) , a sequence of σ -algebras $\mathcal{F}_1, \mathcal{F}_2, \dots$, on Ω such that*

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}$$

is called a filtration.

Here \mathcal{F}_n represents our knowledge at time n . It contains all events such that at time n it is possible to decide whether A has occurred or not. As n increases, there will be more such events A , i.e. the family \mathcal{F}_n representing our knowledge will become larger.

Based on the definition of filtration, we can further define the notion of martingale.

Definition 3.2 (Discrete-time Martingales) *Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a sequence M_1, M_2, \dots , of random variables is called a martingale (resp, supermartingale, submartingale) with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \dots$, if*

- 1) M_n is integrable for each $n = 1, 2, \dots$;
- 2) M_1, M_2, \dots is adapted to $\mathcal{F}_1, \mathcal{F}_2, \dots$;
- 3) $\mathbb{E}(M_{n+1} | \mathcal{F}_n) = (\text{resp, } \leq, \geq) M_n$ a.s. for each $n = 1, 2, \dots$

The concept of a martingale has its origin in gambling, namely, it describes a fair game of chance. Similarly, the notions of submartingale and supermartingale defined below are related to favourable and unfavourable games of chance.

3.1.2 Examples

We now introduce some widely-used examples of martingales.

Mean-corrected Sums of i.i.d. r.v.s & Simple Symmetric Random Walk

Let X_1, X_2, \dots be independent random variables each with mean μ . Let $S_0 = 0$ and for $n \geq 0$ let S_n be the partial sum $S_n = X_1 + \dots + X_n$. Then $M_n = S_n - n\mu$ is a martingale with respect to \mathcal{F}_n , the information contained in X_1, \dots, X_n .

If we now assume that X_i is the Rademacher variable, then this stochastic process is called a **simple symmetric random walk**.

Harmonic Functions of a Markov Chain

Consider a discrete-time Markov chain $\{X_n\}_{n \in \mathbb{N}_0}$ with finite sample space χ and transition matrix \mathcal{P} . We say a bounded function $f : \chi \rightarrow \mathbb{R}$ is **harmonic** if $\mathcal{P}f = f$, i.e.,

$$\mathbb{E}(f(X_1) | X_0 = i) = \sum_{j \in \chi} f(j) \mathcal{P}_{ij} = f(i),$$

then $M_n^f := f(X_n)$ is a martingale w.r.t. $\mathcal{F}_n := \sigma(\{X_0, \dots, X_n\})$.

Pólya Urn At time $t = 0$, an urn contains b black balls and w white balls. At each time $t = n$ ($n = 1, 2, \dots$) a ball is chosen at random from the urn, and is replaced together with another ball of the same color. After time $t = n$, there are thus $n + b + w$ balls in the urn, of which $b + B_n$ are black, where B_n is the number of black balls chosen by time n . Then $M_n := \frac{b + B_n}{b + w + n}$ the proportion of black balls in the urn right after $t = n$, is a positive martingale.

Remark: the limit of M_n , $M_\infty := \lim_{n \rightarrow \infty} M_n$ is $\mathcal{B}(b, w)$ -distributed.

Martingale transform

Let $\{X_n\}_{n \in \mathbb{N}_0}$ be a martingale (w.r.t. to the filtration \mathcal{F}_n), and let $\{Y_n\}_{n \in \mathbb{N}_0}$ be a predictable sequence. The **martingale transform** $\{M_n\}_{n \in \mathbb{N}_0}$ is defined as follows:

$$M_n = (Y \cdot X)_n = X_0 + \sum_{t=0}^{n-1} Y_t(\Delta X)_{t+1},$$

where $(\Delta X)_{t+1} = X_{t+1} - X_t$ is the martingale difference. And $\{M_n\}_{n \in \mathbb{N}_0}$ is a martingale w.r.t. \mathcal{F}_n .

Wald's Martingale

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. r.v.s with MGF $M(t) = \mathbb{E}(e^{tX_1})$ is finite for some $t > 0$, and set $S_n = X_1 + \dots + X_n$ with $S_0 = 1$. The process $\{W_n\}_{n \in \mathbb{N}}$ defined by

$$W_n = \frac{e^{tS_n}}{M^n(t)}$$

is a martingale known as **Wald's martingale**.

Likelihood-ratio Testing (Simple Hypotheses)

A random variable X is thought to be distributed according either to probability density f (null hypothesis) or to a different probability density g (alternative hypothesis). A random i.i.d. sample X_1, \dots, X_n is taken. Let L_n be the "likelihood ratio"

$$L_n = \frac{\mathcal{L}(g|x)}{\mathcal{L}(f|x)} = \prod_{i=1}^n \frac{g(X_i)}{f(X_i)},$$

and $\{L_n\}_{n \in \mathbb{N}}$ is a martingale under H_0 .

3.2 Continuous-time Martingale

3.2.1 Notions

Let us now, extend the above notions to general cases, which contain continuous-time situations.

Definition 3.3 (Filtration) *Given the measurable space (Ω, \mathcal{F}) , a family of σ -algebras \mathcal{F}_t on Ω , parametrized by $t \in T \subset \mathbb{R}$, is called a filtration if*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$$

for any $s, t \in T$ such that $s \leq t$.

Definition 3.4 (Continuous-time Martingales) *Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a stochastic process ξ_t ($t \in T$) is called a martingale (resp, supermartingale, submartingale) with respect to a filtration \mathcal{F}_t , if*

- 1) ξ_t is integrable for each $t \in T$;
- 2) ξ_t is adapted to \mathcal{F}_t for each $t \in T$;
- 3) $\mathbb{E}(\xi_t | \mathcal{F}_s) = (\text{resp, } \leq, \geq) \xi_s$ a.s. for every $s, t \in T$ and $s \leq t$.

3.3 Examples

A few examples of continuous-time martingale will appear in our lectures later. And I just provide some frequently-used processes here.

Time-homogeneous Poisson Process, Subtracted by Its Mean

Let us consider a time-homogeneous Poisson process $N(t)$, which may be the first continuous process you encountered in your probability study, with rate parameter λ . Then we directly find that the processes

$$M_t = N(t) - \lambda t$$

is a martingale, w.r.t. the filtration generated by $\{N(s)\}_{0 \leq s \leq t}$.

Brownian Motion

The **Wiener process** (or **Brownian motion**) is a stochastic process $W(t)$ with values in \mathbb{R} defined for $t \in [0, \infty)$ such that

- 1) $W(0) = 0$ a.s.;
- 2) the sample paths $t \mapsto W(t)$ are a.s. continuous;
- 3) for any finite sequence of times $0 < t_1 < \dots < t_n$ and Borel sets $A_1, \dots, A_n \subset \mathbb{R}$

$$\mathbb{P}(W(t_1) \in A_1, \dots, W(t_n) \in A_n) = \int_{A_1} \dots \int_{A_n} p(t_1, 0, x_1) p(t_2 - t_1, x_1, x_2) \dots p(t_n - t_{n-1}, x_{n-1}, x_n) dx_1 \dots dx_n,$$

where

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

defined for any $x, y \in \mathbb{R}$ and $t > 0$, called the transition density.