Advanced Probability Theory	**Fall 2023**

Lecture 3: Conditional Probabilities

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For any event $A \in \mathcal{F}$, and with $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$, we define

$$\mathbb{P}(A|\mathcal{G}) := \mathbb{E}(\mathbb{I}_A|\mathcal{G}), \tag{3.1}$$

the **conditional probability of A given** $\mathcal{G}$, as the conditional indicator random variable $\mathbb{I}_A$.

In other words, $\mathbb{P}(A\mathcal{G})$ is the $\mathbb{P}-$a.e. unique random variable with the property

$$\mathbb{P}(A \cap G) = \mathbb{E}(\mathbb{I}_A \mathbb{I}_G) = \mathbb{E}[\mathbb{P}(A|\mathcal{G}).\mathbb{I}_G], \tag{3.2}$$

for every $G \in \mathcal{G}$.

Let us specialize now to the discrete setting discussed as the start of the "Conditional Expectation" notes, to get some feel for this definition.

There, we have two random variables, X taking values in $\{x_1, ..., x_m\}$, and Z taking values in $\{z_1, ..., z_n\}$ and $\mathcal{G} = \sigma(Z)$ consisting of all 2^n possible union of the "atoms" $G_j := \{Z = z_j\}, \ \ , j = 1, ..., n.$

Let us fix a Borel set B, and consider the event $A := \{X \in B\}$. Here and there, we would like to characterize the random variable

$$\begin{aligned} H &:= \mathbb{E}(\mathbb{I}_A|\mathcal{G}) = \mathbb{P}(X \in B|\mathcal{G}) = \mathbb{P}(X \in B|Z) \\ &= \mathbb{P}(A|\mathcal{G}) = \mathbb{P}(A|Z). \end{aligned} \tag{3.3}$$

We do this as follows: elementary considerations give

$$\begin{aligned} \mathbb{P}(A|Z = z_j) &= \mathbb{P}(Z \in B|Z = z_j) \\ &= \frac{1}{\mathbb{P}(Z = z_j)} \sum_{i=1, x_i \in B}^{m} \mathbb{P}(X = x_i, Z = z_j) \\ &=: h_j \qquad\qquad j = 1, ...n \end{aligned}$$

and we define $H(\omega) = h_j$, on $\{Z = z_j\}$, i.e.,

$$H := \sum_{j=1}^{n} h_j \mathbb{I}_{\{Z=z_j\}}. \tag{3.4}$$

This is a simple random variable.

Then, the requirement (3.2) becomes

$$\mathbb{P}[\{x \in B\} \cap G] = \mathbb{E}[H.\mathbb{I}_G], \qquad \forall G \in \mathcal{G}$$

in the notation of (3.3); that is

$$\mathbb{P}[\{x \in B\} \cap G_j] = \mathbb{E}[H.\mathbb{I}_{G_j}], \qquad j = 1, ..., n \tag{3.5}$$

along the atoms, for the simple random variable in (3.4)

Let us check the claim (3.5): for every fixed j = 1,...,n, we have

$$\begin{aligned}
\mathbb{E}(H.\mathbb{I}_{G_j}) &= \int_{Z=z_j} H d\mathbb{P} \\
&= h_j \mathbb{P}(Z = z_j) \\
&= \sum_{i=1, x_i \in B}^{n} \mathbb{P}(X = x_i, Z = z_j) \\
&= \mathbb{P}(\{x \in B\} \cap \{Z = z_j\}) \qquad = \mathbb{P}[\{x \in B\} \cap G_j],
\end{aligned}$$

that is, (3.5).