Topic 7
Moments of Randomly Stopped Sums of Independent Variables

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Wald's Identity

We begin with Wald's equations, which constitute the cornerstone of the theory of sequential analysis.

Theorem (Wald, 1944 [5])

Let X_i be a sequence of i.i.d. random variables adapted to $\mathcal{F}_i = \sigma(X_1, ..., X_i)$, with $\mathbb{E}(X) = \mu$, $|\mu| < \infty$. Let T be a stopping time adapted to $\sigma(X_i)$. Set $S_n = X_1, + ... + X_n$. Then

$$\mathbb{E}(S_T) = \mu \mathbb{E}(T), \text{ whenever } \mathbb{E}(T) < \infty \tag{1}$$

Moreover, if $\mathbb{E}(X_1) = 0$, and $\mathbb{E}({X_1}^2) < \infty$, then

$$\mathbb{E}(S_T^2) = \mathbb{E}({X_1}^2)\mathbb{E}(T), \text{ whenever } \mathbb{E}(T) < \infty. \tag{2}$$

In this context, although the stopping time T is adapted to $\mathbb{F} := \{\mathcal{F}_i\}_{i\geq 1} := \{\sigma(X_1, ..., X_i)\}_{i\geq 1}$, we can decouple this structure and still keep the identity, the proof of which is even simpler than the Wald identity.

Theorem

Let X_i be a sequence of independent random variables adapted to $\mathcal{F}_i = \sigma(X_1, ..., X_i)$, with $\mathbb{E}(X_i) = \mu_i$, $|\mu_i| < \infty$ for all i. Let T be a stopping time adapted to $\mathbb{F}$. Let $\{\tilde{X}_i\}$ be the i.i.d. copy of $\{X_i\}$, and be independent of T. Set $S_n = X_1, + ... + X_n$, and $\tilde{S}_n = \tilde{X}_1, + ... + \tilde{X}_n$. Then

$$\mathbb{E}(S_T) = \mathbb{E}(\tilde{S}_T), \textit{ whenever } \mathbb{E}(T) < \infty \tag{3}$$

Moreover, if $\mathbb{E}(X_i) = 0$, and $\mathbb{E}({X_1}^2) < \infty$, then

$$\mathbb{E}(S_T^2) = \mathbb{E}(\tilde{S}_T^2), \textit{ whenever } \mathbb{E}(T) < \infty. \tag{4}$$

Bounding $\mathbb{E}S_T^2$ via $\mathbb{E}\tilde{S}_T^2$

We now consider the second moment of S_T, where X_i's are square-integrable but may not be mean-zero. Although the equation (4) for the mean-zero random variables no longer holds, we may still use the second moment of $\tilde{S}_T$ to bound $\mathbb{E}S_T^2$. In the following discussion, we presume that X_i's are independent (but may not be identically distributed) and square-integrable, and $\mathbb{E}T < \infty$.

Remark: We note that $\tilde{S}_T$ has the same distribution of $S_{\tilde{T}}$, where $\tilde{T}$ is an independent copy of T.

Lemma

Let $\mathbb{E}X_i = \mu_i$ *and* $\mathbb{V}ar(X_i) = \sigma_i^2$*, and let* $\tilde{T}$ *an independent copy of* T*. Then*

$$\mathbb{E}S_{\tilde{T}}^2 = \mathbb{E}\sum_{i=1}^{\tilde{T}}\sigma_i^2 + \mathbb{E}\left(\sum_{i=1}^{\tilde{T}}\mu_i\right)^2 = \mathbb{E}\sum_{i=1}^{T}\sigma_i^2 + \mathbb{E}\left(\sum_{i=1}^{T}\mu_i\right)^2. \tag{5}$$

$$\begin{aligned}\mathbb{E}S_{\tilde{T}}^2 &= \mathbb{E}\left(\sum_{i=1}^{\tilde{T}}(X_i - \mu_i) + \sum_{i=1}^{\tilde{T}}\mu_i\right)^2 \\ &= \mathbb{E}\left(\sum_{i=1}^{\tilde{T}}(X_i - \mu_i)\right)^2 + 2\mathbb{E}\left(\sum_{i=1}^{\tilde{T}}(X_i - \mu_i)\right)\left(\sum_{i=1}^{\tilde{T}}\mu_i\right) + \mathbb{E}\left(\sum_{i=1}^{\tilde{T}}\mu_i\right)^2 \\ &= \mathbb{E}\sum_{i=1}^{\tilde{T}}\sigma_i^2 + \mathbb{E}\left(\sum_{i=1}^{\tilde{T}}\mu_i\right)^2,\end{aligned}$$

where we establish the last equation by conditioning $\tilde{T}$, and (5) holds due to (3) and (4).

Lemma

$$\mathbb{E}\left(\sum_{i=1}^{T}\mu_i\right)^2 = \mathbb{E}S_T^2 + \mathbb{E}\left(\sum_{i=1}^{T}(X_i-\mu_i)\right)^2 - 2\mathbb{E}\left(S_T\sum_{i=1}^{T}(X_i-\mu_i)\right)$$
$$\leq \left(\sqrt{\mathbb{E}S_T^2} + \sqrt{\mathbb{E}\sum_{i=1}^{T}\sigma_i^2}\right)^2, \tag{6}$$

$$\mathbb{E}S_T^2 = \mathbb{E}\left(\sum_{i=1}^{T}\mu_i\right)^2 + \mathbb{E}\left(\sum_{i=1}^{T}(X_i-\mu_i)\right)^2 - 2\mathbb{E}\left(\sum_{i=1}^{T}\mu_i\sum_{i=1}^{T}(X_i-\mu_i)\right)$$
$$\leq \left(\sqrt{\mathbb{E}\left(\sum_{i=1}^{T}\mu_i\right)^2} + \sqrt{\mathbb{E}\sum_{i=1}^{T}\sigma_i^2}\right)^2. \tag{7}$$

Both first equations are obtained directly from $\sum_{i=1}^{T}\mu_i = S_T - \sum_{i=1}^{T}(X_i-\mu_i)$, and both second inequalities are due to the Cauchy-Schwarz inequality and completing the square.

From the inequality (7) and equation (5), we have that

$$
\begin{aligned}
\mathbb{E} S_T^2 &\le \left(\sqrt{\mathbb{E}\left(\sum_{i=1}^{T} \mu_i\right)^2} + \sqrt{\mathbb{E} \sum_{i=1}^{T} \sigma_i^2} \right)^2 \\
&\le 2\left[\mathbb{E}\left(\sum_{i=1}^{T} \mu_i\right)^2 + \mathbb{E} \sum_{i=1}^{T} \sigma_i^2 \right] = 2\mathbb{E} S_{\tilde{T}}^2.
\end{aligned}
$$

Note that $S_{\tilde{T}}$ and $\tilde{S}_T$ have the same distributions, the following inequality is induced.

Theorem (de la Peña & Govindarajulu [4])

$$0 \le \mathbb{E} S_T^2 \le 2\mathbb{E} \tilde{S}_T^2. \tag{8}$$

In addition, since almost surely $\sum_{i=1}^{T} \mathbb{E} X_i^2 \geq \sum_{i=1}^{T} \sigma_i^2$,

$$\begin{aligned}\mathbb{E} \sum_{i=1}^{T} X_i^2 &= \sum_{i=1}^{\infty} \mathbb{E}(X_i^2 I_{T \geq i}) = \sum_{i=1}^{\infty} [\mathbb{E}(X_i^2) \mathbb{E} I_{T \geq i}] \\ &= \mathbb{E}[(\sum_{i=1}^{\infty} \mathbb{E}(X_i^2) I_{T \geq i}] = \mathbb{E} \sum_{i=1}^{T} \mathbb{E} X_i^2 .\end{aligned}$$

When all X_i's are non-negative, we have $\mathbb{E} S_T^2 \geq \mathbb{E} \sum_{i=1}^{T} X_i^2 \geq \mathbb{E} \sum_{i=1}^{T} \sigma_i^2$. Then from (6) we have

$$\mathbb{E} \left(\sum_{i=1}^{T} \mu_i \right)^2 \leq \left(\sqrt{\mathbb{E} S_T^2} + \sqrt{\mathbb{E} \sum_{i=1}^{T} \sigma_i^2} \right)^2 \leq 4 \mathbb{E} S_T^2 . \tag{9}$$

Therefore, we have

$$\mathbb{E}S_{\tilde{T}}^2 = \mathbb{E}\left(\sum_{i=1}^{T}\mu_i\right)^2 + \mathbb{E}\sum_{i=1}^{T}\sigma_i^2 \leq 5\mathbb{E}S_T^2,$$

which leads to the following theorem.

Theorem (de la Peña & Govindarajulu [4])

With the same assumption above, we further suppose that for all $i = 1, \dots, n$, $X_i \geq 0$ *almost surely, then*

$$\mathbb{E}S_T^2 \geq \frac{1}{5}\mathbb{E}S_{\tilde{T}}^2. \tag{10}$$

We further note that the bounds $0 \le \mathbb{E}S_T^2 \le 2\mathbb{E}\tilde{S}_T^2$ is sharp, from the following example, provided by Aryeh Dvoretzky.
Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with

$$Y_1 = \begin{cases} 1, \ w.p.\dfrac{1}{n} \\ \dfrac{-1}{n-1}, \ w.p.\dfrac{n-1}{n} \end{cases}$$

Then $\mathbb{E}Y_1 = 0$ and $\mathbb{E}Y_1^2 = (n-1)^{-1}$. Let the stopping time

$$T_n = \begin{cases} 1, \text{ if } Y_1 < 0 \\ k_n, \text{ if } Y_1 > 0 \end{cases},$$

for some k_n such that $k_n/n \to 0$ and $k_n^2/n \to \infty$ when $n \to \infty$.
In this case, marginally, when $n \to \infty$, $\mathbb{E}T_n = 1 - n^{-1} + \frac{k_n}{n} \to 1$, and $\mathbb{E}T_n^2 = 1 - n^{-1} + \frac{k_n^2}{n} \sim \frac{k_n^2}{n}$.

Setting some constant $a \in \mathbb{R}$ and we then have

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^{T_n}(a+Y_i)\right)^2 &= a^2\mathbb{E}T_n^2 + 2a\mathbb{E}[T_n\sum_{i=1}^{T_n}Y_i] + \mathbb{E}[\sum_{i=1}^{T_n}Y_i]^2 \\ &\sim a^2\frac{k_n^2}{n} + 2a\frac{k_n}{n} + \frac{1}{n}.\end{aligned}$$

When we let $a = \frac{1}{k_n}$, $\mathbb{E}\left(\sum_{i=1}^{T_n}(a+Y_i)\right)^2 \sim \frac{4}{n}$. And when $a = \frac{-1}{k_n}$, $\mathbb{E}\left(\sum_{i=1}^{T_n}(a+Y_i)\right)^2 = o(n^{-1})$. By comparison, for the i.i.d. copy $\tilde{T}_n$ of T_n, we also have when $a = \pm\frac{1}{k_n}$

$$\mathbb{E}\left(\sum_{i=1}^{\tilde{T}_n}(a+Y_i)\right)^2 = \mathbb{E}\tilde{T}_n\mathbb{E}Y_1^2 + a^2\mathbb{E}\tilde{T}_n^2 \sim \frac{1}{n} + \frac{a^2k_n^2}{n} \sim \frac{2}{n}.$$

Hence, both the upper bound and the lower bound are sharp.

Application

Consider the hitting time $T_r := \inf\{n : S_n^2 \geq r\}$ for some nonnegative r, and the function $a : \mathbb{N}_0 \to \mathbb{R}_0^+$ induced by S_n such that

$$a(n) := \mathbb{E}\left[\max_{0\leq j\leq n} S_j^2\right], \ \ \forall n \in \mathbb{N}_0.$$

Then we can lower bound the expectation of the random variable $a(T_r)$, via the following procedure

$$\begin{aligned} r &\leq \mathbb{E}[S_{T_r}^2] \leq 2\mathbb{E}[\tilde{S}_{T_r}^2] \leq 2\mathbb{E}\left[\max_{0\leq j\leq T_r} \tilde{S}_j^2\right] \\ &\iff \mathbb{E}[a(T_r)] \geq r/2. \end{aligned}$$

Remark: This result can be extended to the case for all nonnegative, measurable process X_t with $a(t) = \mathbb{E} \sup_{0 \le s \le t} X_s$ and $T_r := \inf\{t : X_t \ge r\}$, such that $r/2 \le \mathbb{E}[a(T_r)]$ (see Brown, de la Peña & Sit [1]).
If $a(t)$ is assumed to be concave, we obtain that

$$a^{-1}(r/2) \le \mathbb{E}[T_r]. \tag{11}$$

If $a(\cdot)$ is continuous and strictly increasing, there is a sharp inequality for any Cadlag stochastic process, X_t with $X_0 = 0$ and $g(\cdot)$ non-decreasing (see Brown, de la Peña, Klass & Sit [2])

$$\mathbb{E}g(T_r) \ge \int_0^1 g\left(a^{-1}(r\alpha)\right) d\alpha. \tag{12}$$

Extension

Hitczenko [3] extended the inequality to p-th moment.

Theorem

With the same assumptions above, we further assume that for all $i = 1, 2 \ldots, X_i \geq 0$ *almost surely, then for all* $1 \leq p < \infty$,

$$\mathbb{E} S_T^p \leq 2^{p-1} \mathbb{E} \tilde{S}_T^p. \tag{13}$$

Remark 1: This bound is proved to be sharp.
Remark 2: This bound is established through a more general result in tangent decoupling.

References I

[1] Mark Brown, Victor de la Peña, and Tony Sit. "From boundary crossing of non-random functions to boundary crossing of stochastic processes". In: *Probability in the Engineering and Informational Sciences* 29.3 (2015), pp. 345–359.

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References II

[5] Abraham Wald. "On cumulative sums of random variables". In: *The Annals of Mathematical Statistics* 15.3 (1944), pp. 283–296.