Sample path large deviations for Lévy processes and random walks with regularly varying increments

March 20, 2025

Topics Preview

1. Setting and Overview
   - Centered and scaled Lévy process
   - $\mathbb{M}$-Convergence
2. One-sided large deviations
   - Limiting theorems
   - Application 1: Crossing high levels with moderate jumps
3. Two-sided large deviations
   - Limiting theorems
   - Application 2: A two-sided barrier crossing problem
   - Application 3: Identifying the optimal number of jumps for sets of the form $A = \{\xi : l \leq \xi \leq u\}$
   - Limit theorem: Non-unique argument minimums
   - Application 4: Multiple optima
4. Implications
5. Application to Reinsurance Model
6. Application to Rare Event Simulation

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Lévy process

X is a Lévy process with a Lévy measure ν with Itô representation

$$X_n(s) = nsa + B(ns) + \int_{|x|\leq 1} x[N([0, ns] \times dx) - ns\nu(dx)] + \int_{|x|>1} xN([0, ns] \times dx).$$

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- B: a Brownian motion
- N: a Poisson random measure with mean measure Leb $\times\, \nu$ on $[0, n] \times (0, \infty)$

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- B: a Brownian motion
- N: a Poisson random measure with mean measure Leb $\times\, \nu$ on $[0,n]\times(0,\infty)$
- $\nu[x,\infty)$ regularly varying with index $-\alpha$
- $\nu(-\infty,-x]$ is regularly varying with index $-\beta$

Centered and scaled Lévy process

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Example: $S_k, k\geq 0$ a mean-zero random walk. $\mathbb{P}(S_1\geq x) = L_+(x)x^{-\alpha}$, $\mathbb{P}(S_1\leq -x) = L_-(x)x^{-\beta}$. We let

$$X(t) = S_{N(t)}, \quad \bar{X}_n(t) = X(nt)/n \quad t\geq 0$$

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Questions: For "general" $A \subseteq \mathbb{D}$ (the Skorokhod space). Do we have

- $\epsilon_n \mathbb{P}(\bar{X}_n \in A) \to C(A)$ in some sense, for some $C(\cdot)$ and ϵ_n?
- LDP for $\mathbb{P}(\bar{X}_n \in A)$ with some rate function $I(\cdot)$ and speed b_n?
- Full LDP for $\bar{X}_n$?

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- Full LDP for $\bar{X}_n$? No

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- $\mathcal{C}_{\mathbb{S}\setminus C} \subseteq b\mathcal{C}(\mathbb{S} \setminus C, \mathbb{R}_0^+)$ is the set functions f with $supp(f)$ bounded away from C.
- A sequence $\mu_n \in \mathbb{M}(\mathbb{S} \setminus C)$ converges to $\mu \in \mathbb{M}(\mathbb{S} \setminus C)$ if $\mu_n(f) \to \mu(f)$ for each $f \in \mathcal{C}_{\mathbb{S}\setminus C}$. =Weak convergence when $C = \emptyset$.

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- $C_j \in \mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$, $j \geq 0$, a measure concentrated on $\mathbb{D}_j$ s.t.

$$C_j(\cdot) \triangleq \mathbb{E}\left[\nu_\alpha^j \left(y \in (0, \infty)^j : \sum_{i=1}^{j} y_i \mathbf{1}_{[U_i, 1]} \in \cdot \right)\right],$$

- $\nu_\gamma(x, \infty) \triangleq x^{-\gamma}$, $\gamma > 1$.
- ν_γ^j the restriction (to $\mathbb{R}_+^{j\downarrow}$) of the j-fold product measure of ν_γ.
- $U_i, i \geq 1$ i.i.d. $Uniform[0, 1]$.

$\mathbb{M}$-convergence of $(n\nu[n,\infty))^{-j}\mathbb{P}(\bar{X}_n \in \cdot)$

Theorem 3.1. *For each* $j \geq 0$,

$$(n\nu[n,\infty))^{-j}\mathbb{P}(\bar{X}_n \in \cdot) \to C_j(\cdot), \tag{1}$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j})$, *as* $n \to \infty$.

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Proof Sketch:

1. $\bar{X}_n$ is asymptotically equivalent to $\hat{J}_n^{\leq j}$, a process obtained by just keeping its j biggest jumps. (Prop. 5.1 & 5.2.)
2. $\mathbb{M}$-convergences of jump times and jump sizes (independent) leads to $\mathbb{M}$-convergence of product measures. (Lem 5.3)
3. $\mathbb{M}$-convergence of the push-back measure (path space) is preserved. (Cor 2.2)

Large Deviation Theorem for $\bar{X}_n$

Theorem 3.2. *If $\mathcal{J}(A) < \infty$, and $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$ is bounded away from $\mathbb{D}_{<\mathcal{J}(A)}$ for some $\delta > 0$, then*

$$C_{\mathcal{J}(A)}(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \leq \limsup_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \leq C_{\mathcal{J}(A)}(\bar{A}). \quad (2)$$

- $\mathcal{J}(A) =$ min #jumps for step functions in $A \triangleq \inf_{\xi \in \mathbb{D}_s^\uparrow \cap A} \mathcal{D}_+(\xi)$.
- $\mathbb{D}_s^\uparrow \subseteq \mathbb{D}$ set of non-decreasing step functions vanishing at 0.
- $\mathcal{D}_+(\xi)$: # of upward jumps of an element $\xi \in \mathbb{D}$.

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$$\mathbb{P}(\bar{X}_n \in A) = \Omega\left((n\nu[n,\infty))^{\mathcal{J}(A)}\right) = \Omega\left(n^{(1-\alpha)\mathcal{J}(A)}\right)$$

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Proof Sketch:

- $\mathcal{J}(A^o) > \mathcal{J}(A) \Rightarrow A^\circ \cap \mathbb{D}_{\leq \mathcal{J}(A)} = \emptyset$.
- $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$ is b.a.f. $\mathbb{D}_{< \mathcal{J}(A)} \Rightarrow \mathcal{J}(A) = \mathcal{J}(\bar{A})$.
- $\mathcal{J}(A^\circ) = \mathcal{J}(A) = \mathcal{J}(\bar{A}) = j$:

$$C_{\mathcal{J}(A)}(A^\circ) = C_{\mathcal{J}(A^\circ)}(A^\circ) \leq \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A^\circ)}{(n\nu[n,\infty])^{\mathcal{J}(A^\circ)}} \leq \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty])^{\mathcal{J}(A)}}.$$

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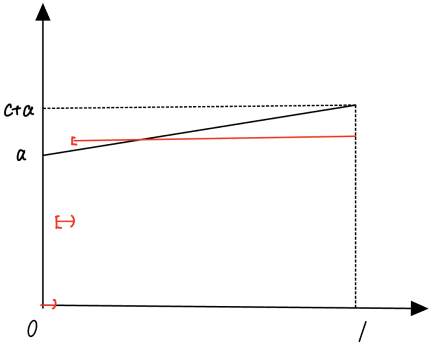
Assumptions:

- a is not a multiple of b, and $j \triangleq \lceil a/b \rceil$.
- $\bar{X}_n$ is spectrally positive.

$A \triangleq \{\zeta \in \mathbb{D} : \sup_{t\in[0,1]}[\zeta(t) - ct] \geq a; \sup_{t\in[0,1]}[\zeta(t) - \zeta(t-)] \leq b\}$.

What is $\mathbb{P}(\bar{X}_n \in A)$?



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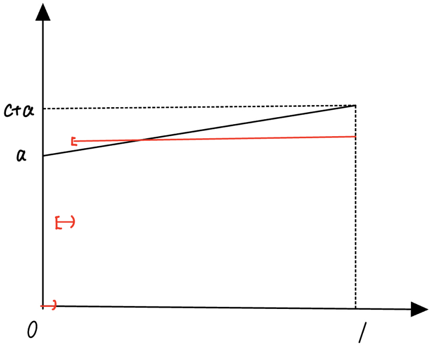

Answer: $\mathbb{P}\left(\bar{X}_n \in A\right) \sim C_j(A)(n\nu[n,\infty])^j$, regularly varying with index $-(\alpha-1)\lceil a/b \rceil$.

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Fact:

1. $\mathcal{J}(A) = j$.
2. $A_\delta \cap \mathbb{D}_j$ is b.a.f. the closed set $\mathbb{D}_{\leq j-1}$ for some $\delta > 0$.
3. $C_j(A^\circ) = C_j(A) = C_j(\bar{A})$.

Application 1: Crossing High Levels with Moderate Jumps

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With the set

$$S = \{(x, u) \in \hat{S}_j : \sum_{i=1}^{j} x_i > a + c \max_{i=1,\ldots,j} u_i, \max_{i=1,\ldots,j} x_i < b\},$$

where $\hat{S}_j := \{(x, u) \in \mathbb{R}_+^{j\downarrow} \times [0, 1]^j : 0, 1, u_1, \ldots, u_j$ are all distinct$\}$, we have

$$C_j(A) = \mathbb{E}[\nu_\alpha^j\{(0, \infty)^j : \sum_{i=1}^{j} x_i \mathbf{1}_{[U_i,1]} \in A\}] = \int_S \prod_{i=1}^{j}[\alpha x_i^{-\alpha-1} dx_i du_i] > 0.$$

$$\Rightarrow \mathbb{P}\left(\bar{X}_n \in A\right) \sim C_j(A)(n\nu[n, \infty])^j. \tag{3}$$

which is regularly varying with index $-(\alpha - 1)\lceil a/b \rceil$.

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- $C_{j,k} \in \mathbb{M}(\mathbb{D}\setminus\mathbb{D}_{<j,k})$, a measure concentrated on $\mathbb{D}_{j,k}$

$$C_{j,k}(\cdot) \triangleq \mathbb{E}\left[\nu_\alpha^j \times \nu_\beta^k \left\{(x,y)\in(0,\infty)^j \times (0,\infty)^k : \sum_{i=1}^{j} x_i \mathbf{1}_{[U_i,1]} - \sum_{i=1}^{k} y_i \mathbf{1}_{[V_i,1]} \in \cdot \right\}\right],$$

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- $\nu_\gamma(x,\infty) \triangleq x^{-\gamma}$, $\gamma > 1$.
- $\mathbb{D}_{<j,k} = \bigcup_{(l,m)\in I_{<j,k}} \mathbb{D}_{l,m}$
- $I_{<j,k} = \{(l,m)\in\mathbb{Z}_+^2 \setminus \{(j,k)\} : (\alpha-1)l + (\beta-1)m \le (\alpha-1)j + (\beta-1)k\}.$
- U_i, V_i, $i \ge 1$ i.i.d. $Uniform[0,1]$.

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in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$ as $n \to \infty$.

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Theorem 3.3. *For each* $(j,k) \in \mathbb{Z}_+^2$,

$$(n\nu[n,\infty])^{-j}(n\nu(-\infty,-n])^{-k}\mathbb{P}(\bar{X}_n \in \cdot) \to C_{j,k}(\cdot) \tag{4}$$

in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$ as $n \to \infty$.

Proof Sketch:

- Decomposition: $\bar{X}_n^{(-)} = \int_{\mathbb{R}-} xN([0,ns] \times dx)$, $\bar{X}_n^{(+)} = \bar{X}_n - \bar{X}_n^{(-)}$.
- Thm 5.1: $\frac{\mathbb{P}\{(\bar{X}_n^{(+)},\bar{X}_n^{(-)})\in\cdot\}}{(n\nu[n,\infty])^j(n\nu(-\infty,-n])^k} \to C_j^+ \times C_k^-(\cdot)$ in $\mathbb{M}(\mathbb{D}^2 \setminus \mathbb{D}_{<(j,k)})$.
- Lem 2.6: $h(\xi,\zeta) = \xi - \zeta$ continuous at $(\xi,\zeta) \in \mathbb{D}^2$ s.t. $(\xi(t) - \xi(t-))(\zeta(t) - \zeta(t-)) = 0$.
  $h^{-1}(A)$ b.a.f. $\mathbb{D}_{<(j,k)} := \bigcup_{(l,m)\in I_{<j,k}} \mathbb{D}_l \times \mathbb{D}_m$ if A b.a.f. $\mathbb{D}_{<j,k}$.
- Lem 2.4: $\frac{\mathbb{P}\{h(\bar{X}_n^{(+)},\bar{X}_n^{(-)})\in\cdot\}}{(n\nu[n,\infty])^j(n\nu(-\infty,-n])^k} \to (C_j^+ \times C_k^-) \circ h^{-1}(\cdot) = C_{i,j}(\cdot)$ in $\mathbb{M}(\mathbb{D} \setminus \mathbb{D}_{<j,k})$.

Let $\mathcal{I}(j,k) \triangleq (\alpha-1)j + (\beta-1)k$, and consider a pair of integers $(\mathcal{J}(A), \mathcal{K}(A))$ such that

$$(\mathcal{J}(A), \mathcal{K}(A)) \in \underset{(j,k)\in\mathbb{Z}_+^2, \mathbb{D}_{j,k}\cap A\neq\emptyset}{\arg\min} \mathcal{I}(j,k). \tag{5}$$

Theorem 3.4. *If the argmin in* (5) *is nonempty and A is b.a.f. $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$, then the argmin is unique and*

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty])^{\mathcal{J}(A)}(n\nu(-\infty,-n])^{\mathcal{K}(A)}} \geq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ), \tag{6}$$

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$$\begin{aligned}\mathbb{P}(\bar{X}_n \in A) &= \Omega\left((n\nu[n,\infty))^{\mathcal{J}(A)}(n\nu(-\infty,-n])^{\mathcal{K}(A)}\right)\\ &= \Omega\left(n^{(1-\alpha)\mathcal{J}(A)(1-\beta)\mathcal{K}(A)}\right)\end{aligned}$$

A Two-sided Barrier Crossing Problem

- What's the probability that $\bar{X}_n$ hits below b on $[0, 1]$ and ends up above a?
- $A \triangleq \{\xi \in \mathbb{D} : \inf_{0 \leq t \leq 1} \xi(t) \leq b, \xi(1) \geq a\}$.

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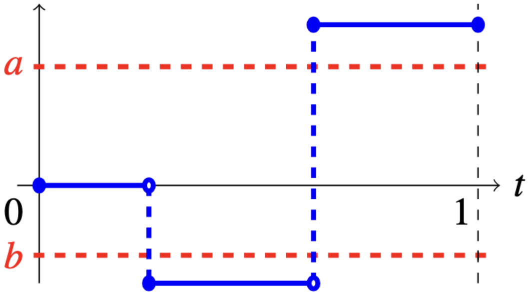

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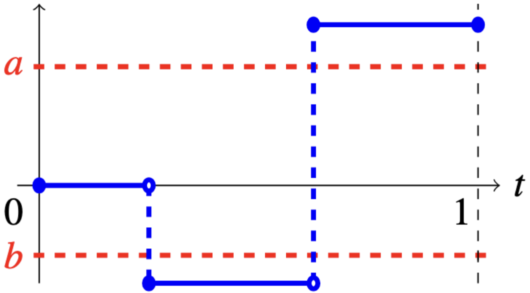
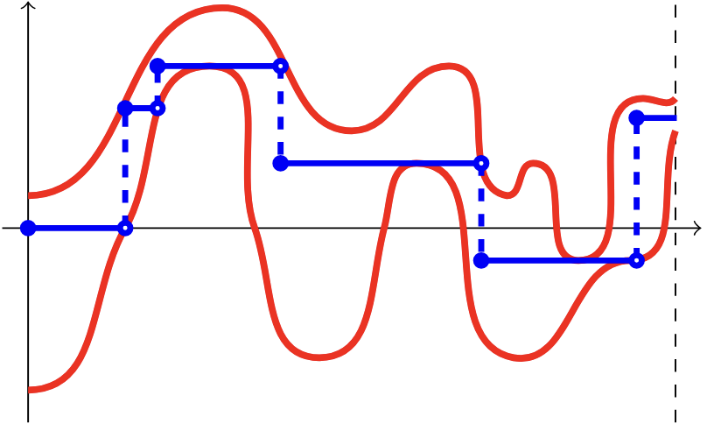


- $\mathbb{P}(\bar{X}_n \in A) \sim n\nu[n,\infty) n\nu(-\infty,-n] C_{1,1}(A)$, regularly varying with index $2-\alpha-\beta$.

Optimal Number of Jumps with Restrictions



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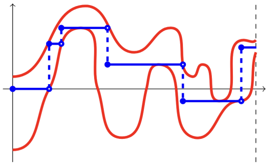


- $A_t \triangleq \{x : l(t) \leq x \leq u(t)\}, \quad A^*_{s,t} \triangleq \bigcap_{s \leq r \leq t} A_r.$
- $t_1 \triangleq 1 \wedge \inf\{t > 0 : 0 \notin A_t\}.$
- $t_{n+1} \triangleq 1 \wedge \inf\{t > t_n : A_{t_n,t} = \emptyset\} \quad \text{for } n \geq 2.$
- $n^* = \inf\{n \geq 1 : t_n = 1\}.$

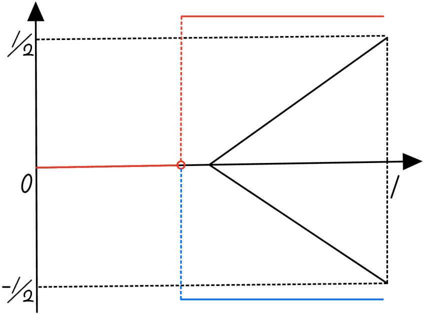
Properties:

1. Either $l(t_1) = 0$ or $u(t_1) = 0$.
2. For every $n = 1, \ldots, n^* - 2$, $\sup_{t \in [t_n, t_{n+1}]} l(t) = \inf_{t \in [t_n, t_{n+1}]} u(t)$.
3. $H_{fin} \triangleq [\sup_{t \in [t_{n^*-1}, t_{n^*}]} l(t), \inf_{t \in [t_{n^*-1}, t_{n^*}]} u(t)] \neq \emptyset.$

Non-unique argument minimums

The limiting behavior may not be dominated by a single $\mathbb{D}_{l,m}$ (i.e., A is not b.a.f. $\mathbb{D}_{\mathcal{J}(A),\mathcal{K}(A)}$)

Example: $\alpha = \beta$ and $A := \{\xi \in \mathbb{D} : |\xi(t)| \geq t - \frac{1}{2}\}$.



Non-unique argument minimums

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- $\mathbb{I}_{=j,k} \triangleq \{(l,m) : (\alpha-1)l + (\beta-1)m = (\alpha-1)j + (\beta-1)k\}$.
  "Contour line" in $\mathbb{Z}^+ \times \mathbb{Z}^+$.
- $\mathbb{I}_{\leq j,k} \triangleq \{(l,m) : (\alpha-1)l + (\beta-1)m < (\alpha-1)j + (\beta-1)k\}$.
- $\mathbb{D}_{=j,k} \triangleq \bigcup_{(l,m)\in\mathbb{I}_{=j,k}} \mathbb{D}_{l,m}$, $\mathbb{D}_{\leq j,k} \triangleq \bigcup_{(l,m)\in\mathbb{I}_{\leq j,k}} \mathbb{D}_{l,m}$.
- $L_+(n) = n^\alpha \nu[n,\infty)$, $L_-(n) = n^\beta \nu(-\infty,-n]$.

Multiple Dominant Configurations

Theorem 3.5. *Let* $(\mathcal{J}(A), \mathcal{K}(A))$ *contain a pair of integers. If* $\exists \delta > 0$, $A_\delta \cap \mathbb{D}_{=\mathcal{J}(A),\mathcal{K}(A)}$ *is b.a.f.* $\mathbb{D}_{\leq \mathcal{J}(A),\mathcal{K}(A)}$*, then* $\forall \epsilon > 0$, $\exists N$ *s.t.* $\forall n \geq N$

$$\mathbb{P}(\bar{X}_n \in A) \geq \frac{\sum_{(l,m)} (C_{l,m}(A^\circ) - \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}},$$

$$\mathbb{P}(\bar{X}_n \in A) \leq \frac{\sum_{(l,m)} (C_{l,m}(\bar{A}) + \epsilon) L_+^l(n) L_-^m(n)}{n^{(\alpha-1)\mathcal{J}(A)+(\beta-1)\mathcal{K}(A)}},$$

where the summations are over $(l, m) \in \mathbb{I}_{=\mathcal{J}(A),\mathcal{K}(A)}$.

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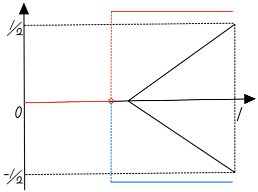
Proof Sketch: Do partition s.t. $A \cap (\mathbb{D}_{l,m})_\rho$ disjoint and b.a.f. $\mathbb{D}_{\leq l,m}$, then apply Theorem 3.4.

Back to Multiple Optima Example

- $\alpha = \beta$, $A := \{\xi \in \mathbb{D} : |\xi(t)| \geq t - \frac{1}{2}\}$.
- $\arg\min_{(j,k)\in\mathbb{Z}_+^2, \mathbb{D}_{j,k}\cap A\neq\emptyset} \mathcal{I}(j,k) = \{(1,0),(0,1)\}$.
- $\mathbb{D}_{\leq 1,0} = \mathbb{D}_{\leq 0,1} = \mathbb{D}_{0,0}$.
- $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n \geq N$,

$$\mathbb{P}(\bar{X}_n \in A) \geq \frac{(C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) - \epsilon)L_+(n) + (C_{0,1}(A^\circ \cap \mathbb{D}_{0,1}) - \epsilon)L_-(n)}{n^{\alpha-1}},$$

$$\mathbb{P}(\bar{X}_n \in A) \leq \frac{(C_{1,0}(\bar{A} \cap \mathbb{D}_{1,0}) + \epsilon)L_+(n) + (C_{0,1}(\bar{A} \cap \mathbb{D}_{0,1}) + \epsilon)L_-(n)}{n^{\alpha-1}}.$$



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- $C_{1,0}(A^\circ \cap \mathbb{D}_{1,0}) = C_{1,0}(\bar{A} \cap \mathbb{D}_{1,0}) = (1/2)^{1-\alpha}$.
- $C_{0,1}(A^\circ \cap \mathbb{D}_{0,1}) = C_{0,1}(\bar{A} \cap \mathbb{D}_{0,1}) = (1/2)^{1-\alpha}$.
-

$$\begin{aligned} ((1/2)^{1-\alpha} - \epsilon)(L_+(n) + L_-(n))n^{1-\alpha} &\leq \mathbb{P}(\bar{X}_n \in A) \\ &\leq ((1/2)^{1-\alpha} + \epsilon)(L_+(n) + L_-(n))n^{1-\alpha} \\ \Rightarrow \lim_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(L_+(n) + L_-(n))n^{1-\alpha}} &= \left(\frac{1}{2}\right)^{1-\alpha}. \end{aligned}$$

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6. Application to Rare Event Simulation

Random Walks

- $S_k, k \geq 0$ a mean-zero random walk.
- $\bar{S}_n = \{\bar{S}_n(t) := S_{[nt]}/n : t \in [0,1]\}$.
- $\mathbb{P}(S_1 \geq x) = L_+(x)x^{-\alpha}$, $\mathbb{P}(S_1 \leq -x) = L_-(x)x^{-\beta}$.
- $X(t) = S_{N(t)}, \quad \bar{X}_n(t) = X(nt)/n \quad t \geq 0.$

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Theorem 4.1. *Let A b.a.f. $\mathbb{D}_{<\mathcal{J}(A),\mathcal{K}(A)}$.*

$$\liminf_{n\to\infty} \frac{\mathbb{P}(\bar{S}_n \in A)}{(n\mathbb{P}(S_1 \geq n))^{\mathcal{J}(A)}(n\mathbb{P}(S_1 \leq -n))^{\mathcal{K}(A)}} \geq C_{\mathcal{J}(A),\mathcal{K}(A)}(A^\circ), \tag{8}$$

$$\limsup_{n\to\infty} \frac{\mathbb{P}(\bar{S}_n \in A)}{(n\mathbb{P}(S_1 \geq n))^{\mathcal{J}(A)}(n\mathbb{P}(S_1 \leq -n))^{\mathcal{K}(A)}} \leq C_{\mathcal{J}(A),\mathcal{K}(A)}(\bar{A}). \tag{9}$$

Conditional Limit Theorem

Corollary 4.2. *Suppose that $B \subseteq \mathbb{D}$ satisfies the conditions in Theorem 3.4 and that*

$$C_{\mathcal{J}(B),\mathcal{K}(B)}(B^{\circ}) = C_{\mathcal{J}(B),\mathcal{K}(B)}(B) = C_{\mathcal{J}(B),\mathcal{K}(B)}(\bar{B}) > 0.$$

Let $\bar{X}_n^{|B}$ have the conditional law of $\bar{X}_n$ given $\bar{X}_n \in B$, then there exists a process $\bar{X}_\infty^{|B}$ such that

$$\bar{X}_n^{|B} \Rightarrow \bar{X}_\infty^{|B},$$

in $\mathbb{D}$. Moreover, if $\mathbb{P}^{|B}(\cdot)$ is the law of $\bar{X}_\infty^{|B}$, then

$$\mathbb{P}^{|B}(\bar{X}_\infty^{|B} \in \cdot) := \frac{C_{\mathcal{J}(B),\mathcal{K}(B)}(\cdot \cap B)}{C_{\mathcal{J}(B),\mathcal{K}(B)}(B)}.$$

Weak Large Deviation Principle

Theorem 4.2. *$\bar{X}_n$ satisfies the weak LDP with rate function I and speed* $\log n$*, that is,*

$$-\inf_{x\in G} I(x) \le \liminf_{n\to\infty} \frac{\log \mathbb{P}(\bar{X}_n \in G)}{\log n}, \quad \forall G \text{ open}, \tag{10}$$

$$\limsup_{n\to\infty} \frac{\log \mathbb{P}(\bar{X}_n \in K)}{\log n} \le -\inf_{x\in K} I(x), \quad \forall K \text{ compact}. \tag{11}$$

$$I(\xi) \triangleq \begin{cases} (\alpha-1)\mathcal{D}_+(\xi) + (\beta-1)\mathcal{D}_-(\xi) & \text{if } \xi \text{ is a step function} \\ & \text{ and } \xi(0)=0; \\ \infty & \text{otherwise,} \end{cases} \tag{12}$$

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But the strong LDP does not hold.

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Cramér-Lundberg Model

$$Y(t) = u + p_D t - S_{N_t} + R(t)$$

- u: Initial capital.
- p_D: The premium after reinsurance has been purchased.
- $S_{N_t} = \sum_{i=1}^{N_t} X_i$: The aggregate claim amount at time t.
- N_t: a Poisson process with rate λ.
- $R(t)$: The reinsured amount at time t (depending on the policy).
- X_i: i.i.d. positive claim with a **regularly varying** tail.

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$$Y(nt) \overset{u=an}{=} an + p_D nt - (S_{N_{nt}} - \lambda nt\mathbb{E}X) - \lambda nt\mathbb{E}X + R(nt)$$

$$\overset{c=p_D-\lambda\mathbb{E}X}{\Rightarrow} \quad \bar{Y}_n(t) \triangleq \frac{1}{n}Y(nt) = a + ct - \bar{S}_n(t) + \frac{1}{n}N(nt).$$

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Question: Ruin probability before time n:

$$\begin{aligned}\psi(an, n) &= \mathbb{P}\left(\inf_{0\le t\le n} Y(t) \le 0\right) \\ &= \mathbb{P}\left(\inf_{0\le t\le 1} \bar{Y}_n(t) \le 0\right) \\ &= \mathbb{P}\left(\sup_{0\le t\le 1} \bar{S}_n(t) - ct - \frac{1}{n}R(nt) \ge a\right)\end{aligned}$$

A rare event when $a + c > 0$!

Reinsurance policies: LCR

LCR (largest claim reinsurance) contract ($R(\cdot) = L_r(\cdot)$):

- Cover the largest r claims ($r \geq 1$).

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- For every $\xi \in \mathbb{D}$ and $m \in \mathbb{N}$, we define, for $t \in (0,1]$,

$$\mathfrak{I}_\xi^m(t) = \sup_{\substack{(s_1,\ldots,s_m)\in[0,t]^m \\ s_i \neq s_j, \forall i \neq j}} \sum_{i=1}^m (\xi(s_i) - \xi(s_i^-)) = \max_{\substack{(s_1,\ldots,s_m)\in[0,t]^m \\ s_i \neq s_j, \forall i \neq j}} \sum_{i=1}^m (\xi(s_i) - \xi(s_i^-))$$

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- $\psi(an, n) = \mathbb{P}(\phi_r(\bar{S}_n(t)) \geq a) = \mathbb{P}(\bar{S}_n(t) \in A^r)$
- $\phi_r(\xi) = \sup_{t\in[0,1]} \left\{ \xi(t) - ct - \mathfrak{I}_\xi^r(t) \right\}$.
- $A^r := A_{c,a}^r = \phi_r^{-1}([a, \infty))$.

Reinsurance policies: ECOMOR

ECOMOR (excédent du coût moyen relatif) contract ($R(\cdot) = E_r(\cdot)$):

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ECOMOR (excédent du coût moyen relatif) contract ($R(\cdot) = E_r(\cdot)$):

- Covers the excess of the r largest claims over the $(r+1)$-th largest claim
- $\frac{1}{n}R(nt) = \frac{1}{n}E_r(nt) = \frac{1}{n}\left((r+1)L_r(nt) - rL_{r+1}(nt)\right)$
- $\psi(an, n) = \mathbb{P}(\varphi_r(\bar{S}_n(t)) \geq a) = \mathbb{P}(\bar{S}_n(t) \in \mathcal{A}^r)$.
- $\varphi_r(\xi) = \sup_{t\in[0,1]} \left\{\xi(t) - ct - (r+1)\mathfrak{I}_\xi^r(t) + r\mathfrak{I}_\xi^{r+1}(t)\right\}$.
- $\mathcal{A}^r := \mathcal{A}^r_{c,a} = \varphi_r^{-1}([a,\infty))$.

Recall: Spectrally positive Lévy process

$$X_n(s) = nsa + B(ns) + \int_{|x|\le 1} x[N([0,ns]\times dx) - ns\nu(dx) + \int_{|x|>1} xN([0,ns]\times dx).$$

- $\bar{X}_n(s) \triangleq \frac{1}{n}X_n(s) - sa - \mu_1^+\nu_1^+ s, \quad s\in[0,1].$
- $C_j \in \mathbb{M}(\mathbb{D}\setminus\mathbb{D}_{<j})$, $j\ge 0$, a measure concentrated on $\mathbb{D}_j$ s.t.

$$C_j(\cdot) \triangleq \mathbb{E}\left[\nu_\alpha^j\left(y\in(0,\infty)^j : \sum_{i=1}^j y_i \mathbf{1}_{[U_i,1]} \in \cdot\right)\right],$$

- $\nu_\gamma(x,\infty) \triangleq x^{-\gamma}$, $\gamma > 1$.
- ν_γ^j the restriction (to $\mathbb{R}_+^{j\downarrow}$) of the j-fold product measure of ν_γ.
- U_i, $i\ge 1$ i.i.d. *Uniform*$[0,1]$.

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Theorem 3.2. *If $\mathcal{J}(A)<\infty$, and $A_\delta \cap \mathbb{D}_{\le \mathcal{J}(A)}$ is bounded away from $\mathbb{D}_{<\mathcal{J}(A)}$ for some $\delta>0$, then*

$$C_{\mathcal{J}(A)}(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n\in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le \limsup_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n\in A)}{(n\nu[n,\infty))^{\mathcal{J}(A)}} \le C_{\mathcal{J}(A)}(\bar{A}). \quad (13)$$

Theorem 3.2 for $A = A^r$ or $\mathcal{A}^r$

Theorem 3.2. *If $\mathcal{J}(A) < \infty$, and $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$ is bounded away from $\mathbb{D}_{<\mathcal{J}(A)}$ for some $\delta > 0$, then*

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Checklist:

1. $\mathcal{J}(A) =?$.
2. Is $A_\delta \cap \mathbb{D}_{\leq \mathcal{J}(A)}$ b.a.f. $\mathbb{D}_{<\mathcal{J}(A)}$ for some δ?
3. $C_{\mathcal{J}(A)}(\bar{A}) \stackrel{?}{=} C_{\mathcal{J}(A)}(A^\circ) =?$

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Checklist:

1. $\mathcal{J}(A) =?$.
   Lemma 4: $\mathcal{J}(A) = r + 1$ for both.
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   Lemma 5: Yes for both, which can be proved via Lipschitz continuity of ϕ_r and φ_r (Lemmas 2 & 3).
3. $C_{\mathcal{J}(A)}(\bar{A}) \stackrel{?}{=} C_{\mathcal{J}(A)}(A^\circ) =?$

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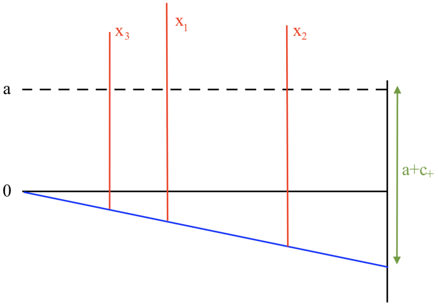
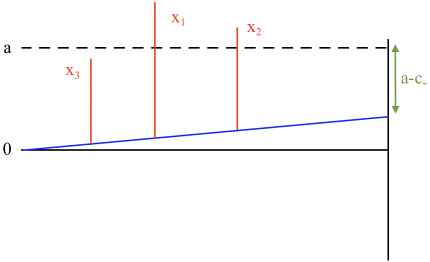
   Lemma 6: Yes for both.

Ruin Probability: LCR

$$A^r \cap \mathbb{D}_{j+1} = \{\sum_{i=1}^{j+1} x_i \mathbb{I}_{[u_i,1]} : x_1 \geq \ldots \geq x_{r+1} \geq a', 0, u_1, \ldots, u_{r+1}, 1 \text{ distinct}\}$$

- $a' = a + c_+ \max\{u_1, \ldots, u_{r+1}\} - c_-$.
- $c_+ = \max\{c, 0\}$, $c_- = \max\{-c, 0\}$.





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$$\mathcal{C}^L_{r+1} = \frac{1}{(r+1)!} \times \begin{cases} a^{-(r+1)\alpha} \cdot {}_2F_1[r+1, (r+1)\alpha; r+2; -c/a], & c > 0, \\ (a+c)^{-(r+1)\alpha}, & c < 0; \end{cases}$$

$$\psi(na, n) \sim \mathcal{C}^L_{r+1}(\lambda L(n))^{r+1} n^{-(r+1)(\alpha-1)}, \quad n \to \infty$$

Ruin Probability: ECOMOR

$$\mathcal{A}^r \cap \mathbb{D}_{j+1} = \{\sum_{i=1}^{j+1} x_i \mathbb{I}_{[u_i,1]} : x_1 \geq \ldots \geq x_{r+1} \geq a'', 0, u_1, \ldots, u_{r+1}, 1 \text{ distinct}\}$$

- $a'' = \frac{1}{r+1}(a + c_+ \max\{u_1, \ldots, u_{r+1}\} - c_-)$.
- $c_+ = \max\{c, 0\}$, $c_- = \max\{-c, 0\}$.

$$\mathcal{C}^E_{r+1} = \frac{(r+1)^{(r+1)\alpha}}{(r+1)!} \times \begin{cases} a^{-(r+1)\alpha} \cdot {}_2F_1[r+1, (r+1)\alpha; r+2; -c/a], & c > 0, \\ (a+c)^{-(r+1)\alpha}, & c < 0. \end{cases}$$

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Topics Preview

1. Setting and Overview
   - Centered and scaled Lévy process
   - $\mathbb{M}$-Convergence
2. One-sided large deviations
   - Limiting theorems
   - Application 1: Crossing high levels with moderate jumps
3. Two-sided large deviations
   - Limiting theorems
   - Application 2: A two-sided barrier crossing problem
   - Application 3: Identifying the optimal number of jumps for sets of the form $A = \{\xi : l \leq \xi \leq u\}$
   - Limit theorem: Non-unique argument minimums
   - Application 4: Multiple optima
4. Implications
5. Application to Reinsurance Model
6. Application to Rare Event Simulation

Recap: Spectrally positive process

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1D Compensated compound Poisson process

$$X_n(s) = nsa + B(ns) + \int_{|x|\le 1} x[N([0,ns]\times dx) - ns\nu(dx) + \int_{|x|>1} xN([0,ns]\times dx).$$

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$$X_n(s) = \sum_{i=0}^{N_t} W(k) - \lambda t \mathbb{E} W(1).$$

- $\{N(t)\}_{t\geq 0} \sim PP(\lambda)$.
- $\mathbb{P}(W(1) > x)$ is regularly varying of index $-\beta < -1$.
- $\bar{X}_n(t) = \frac{X_n(t)}{n}, \quad t \in [0,1]$.

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Theorem 3.2. *If $l^* < \infty$, and A is bounded away from $\mathbb{D}_{l^*}$, then*

$$C_{l^*}(A^\circ) \le \liminf_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{l^*}} \le \limsup_{n\to\infty} \frac{\mathbb{P}(\bar{X}_n \in A)}{(n\nu[n,\infty))^{l^*}} \le C_{l^*}(\bar{A}). \qquad (15)$$

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We want to simulate $\mathbb{P}(\bar{X}_n \in A)$, where A is b.a.f. zero function and $C_{l^*}(A^\circ) > 0$.

Importance Sampling I: Importance Measure

1. A set where A happens with higher probability:
$B_n^\gamma \triangleq \{\bar{X}_n \in B^\gamma\}$: $\gamma > 0$, $B^\gamma \triangleq \{\xi \mid \#\{t \mid \xi(t) - \xi(t^-) > \gamma\} \geq l^*\}$

Importance Sampling I: Importance Measure

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2. $\mathbb{P}(B_n^\gamma)$ has a closed form.

$$\mathbb{P}(B_n^\gamma) = 1 - \exp\left(-\lambda n \mathbb{P}(W(1) > n\gamma)\right) \sum_{j=0}^{l^*-1} \frac{(\lambda n)^j}{j!} \mathbb{P}(W(1) > n\gamma)^j.$$

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3. Conditional distribution given B_n^γ, $\mathbb{Q}_\gamma(\cdot) \triangleq \mathbb{P}(\cdot \mid B_n^\gamma)$:

$$\frac{d\mathbb{Q}_\gamma}{d\mathbb{P}} = \mathbb{P}(B_n^\gamma)^{-1}\mathbb{I}_{B_n^\gamma}.$$

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$$\mathbb{Q}_{\gamma,w}(\cdot) \triangleq w\mathbb{P}(\cdot) + (1-w)\mathbb{Q}_\gamma(\cdot).$$

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5. The estimator for $\mathbb{P}(A_n)$ is:

$$Z_n = \frac{d\mathbb{P}}{d\mathbb{Q}_{\gamma,w}}\mathbb{I}_{A_n} = \frac{\mathbb{I}_{A_n}}{w + \frac{1-w}{\mathbb{P}(B_n^\gamma)}\mathbb{I}_{B_n^\gamma}}.$$

Importance Sampling II: Rejection Sampling

How to sample $\bar{X}_n$ under $\mathbb{Q}_\gamma$?

- $\mathbb{Q}_\gamma(\bar{X}_n \in \cdot) = \sum_{m=l^*}^{\infty} h_m \mathbb{P}(\bar{X}_n \in \cdot \mid B_n^\gamma, N(n) = m)$, where $h_m = \frac{\mathbb{P}(B_n^\gamma, N(n)=m)}{\mathbb{P}(B_n^\gamma)}$.

Importance Sampling II: Rejection Sampling

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- When we have some m:
  1. Sample $\{b_k\}_{k \leq l^*}$ uniformly from $\mathcal{C}(\{1, ..., m\}, l^*)$;
  2. Sample each $W(b_k), k \leq l^*$, conditional on $W(b_k) > n\gamma$;
  3. Sample $W(m'), m' \leq m, m' \notin \{b_k\}_{k \leq l^*}$, under the nominal measure.

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Proposal density $f_{\text{proposal};m}(w_1, \ldots, w_m)$:

$$\frac{1}{\binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j) \sum_{(b_1,\ldots,b_{l^*}) \in \mathcal{C}(\{1,\ldots,m\},l^*)} \mathbb{I}_{\{W(b_k) > n\gamma, \forall k \le l^*\}}.$$

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BUT Target density $f_{\text{target};m}(w_1, \ldots, w_m)$:

$$\frac{1}{\mathbb{P}(B_n^\gamma \mid N(n) = m)} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j)\mathbb{I}_{B_n^\gamma}(w_1, \ldots, w_m).$$

Importance Sampling II: Rejection Sampling (Cont.)

$$\frac{1}{\binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \leq w_j) \sum_{(b_1,\ldots,b_{l^*})\in\mathcal{C}(\{1,\ldots,m\},l^*)} \mathbb{I}_{\{W(b_k)>n\gamma,\forall k\leq l^*\}}.$$

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$$\frac{1}{\binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j) \sum_{(b_1,\ldots,b_{l^*})\in\mathcal{C}(\{1,\ldots,m\},l^*)} \mathbb{I}_{\{W(b_k)>n\gamma,\forall k\le l^*\}}.$$

$$\frac{1}{\mathbb{P}(B_n^\gamma \mid N(n) = m)} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j)\mathbb{I}_{B_n^\gamma}(w_1,\ldots,w_m).$$

$$\frac{f_{\text{target};m}(w_1,\ldots,w_m)}{f_{\text{proposal};m}(w_1,\ldots,w_m)} \le M_m(n) = \binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}\mathbb{P}(B_n^\gamma \mid N(n) = m)^{-1}.$$

Importance Sampling II: Rejection Sampling (Cont.)

$$\frac{1}{\binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j) \sum_{(b_1,\ldots,b_{l^*})\in\mathcal{C}(\{1,\ldots,m\},l^*)} \mathbb{I}_{\{W(b_k)>n\gamma,\forall k\le l^*\}}.$$

$$\frac{1}{\mathbb{P}(B_n^\gamma \mid N(n) = m)} \prod_{j=1}^{m} \frac{d}{dw_j}\mathbb{P}(W(j) \le w_j)\mathbb{I}_{B_n^\gamma}(w_1,\ldots,w_m).$$

$$\frac{f_{\text{target};m}(w_1,\ldots,w_m)}{f_{\text{proposal};m}(w_1,\ldots,w_m)} \le M_m(n) = \binom{m}{l^*}\mathbb{P}(W(1) > n\gamma)^{l^*}\mathbb{P}(B_n^\gamma \mid N(n) = m)^{-1}.$$

Acceptance probability: $a(W(1),\ldots,W(m)) = \binom{\#\{i|W(i)>n\gamma\}}{l^*}^{-1}$.

Importance Sampling III: Algorithm

Algorithm 1 (Generating the Sample Path of $\bar{X}_n$ Under $\mathbb{Q}_\gamma$)

sample $m \sim h_m$ $\quad \triangleright\ m = m'$ with probability $h_{m'} = \mathbb{P}(N(n) = m' \mid B_n^\gamma)$
$R \leftarrow$ **true**
while $R =$ **true do**
  sample $\{b_k\}_{k \le l^*} \sim \text{unif}\,(\mathscr{C}(\{1,\dots,m\},k))$ $\quad \triangleright$ uniform distribution on $\mathscr{C}(\{1,\dots,m\},k)$
  for $i \in \{b_k\}_{k \le l^*}$ **do**
    sample $W(i) \sim W(1) \mid W(1) > n\gamma$
  for $i \notin \{b_k\}_{k \le l^*}$ **do**
    sample $W(i) \sim W(1)$
  $c \leftarrow \#\{j \in \{1,\dots,m\} \mid W(j) > n\gamma\}$; $a \leftarrow \binom{c}{l^*}^{-1}$; sample $u \sim \text{uniform}[0,1]$; $R \leftarrow$ **true**
  if $u < a$ **then**
    $R \leftarrow$ **false**
  return $\bar{X}_n$

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  for $i \in \{b_k\}_{k \le l^*}$ **do**
    sample $W(i) \sim W(1) \mid W(1) > n\gamma$
  for $i \notin \{b_k\}_{k \le l^*}$ **do**
    sample $W(i) \sim W(1)$
  $c \leftarrow \#\{j \in \{1, \ldots, m\} \mid W(j) > n\gamma\}$; $a \leftarrow \binom{c}{l^*}^{-1}$; sample $u \sim \text{uniform}[0,1]$; $R \leftarrow$ **true**
  if $u < a$ **then**
    $R \leftarrow$ **false**
  return $\bar{X}_n$

- Importance distribution: $\mathbb{Q}_{\gamma,w}(\cdot) \triangleq w\mathbb{P}(\cdot) + (1-w)\mathbb{Q}_\gamma(\cdot)$.
- The estimator for $\mathbb{P}(A_n)$ is: $Z_n = \frac{d\mathbb{P}}{d\mathbb{Q}_{\gamma,w}}\mathbb{I}_{A_n} = \frac{\mathbb{I}_{A_n}}{w + \frac{1-w}{\mathbb{P}(B_n^\gamma)}\mathbb{I}_{B_n^\gamma}}$.

Importance Sampling III: Algorithm

Algorithm 2 (Efficient Sampling of $\mathbb{P}(\bar{X}_n \in A)$)

1: sample $u \sim \text{uniform}[0,1]$
2: sample $\bar{X}_n \sim \mathbb{P}(\bar{X}_n \in \cdot \,|\bar{X}_n \in B^\gamma)$
3: **if** $u < w$ **then**
4: sample $\bar{X}_n \sim \mathbb{P}(\bar{X}_n \in \cdot\,)$
5: **if** $\bar{X}_n \in A$ **then**
6: $L \leftarrow [w + (1-w)\mathbb{I}_{B_n^\gamma} / \mathbb{P}(\mathrm{B}_\mathrm{n}^\gamma)]^{-1}$
7: **else**
8: $L \leftarrow 0$
return L

- Importance distribution: $\mathbb{Q}_{\gamma,w}(\cdot) \triangleq w\mathbb{P}(\cdot) + (1-w)\mathbb{Q}_\gamma(\cdot)$.
- The estimator for $\mathbb{P}(A_n)$ is: $Z_n = \frac{d\mathbb{P}}{d\mathbb{Q}_{\gamma,w}}\mathbb{I}_{A_n} = \frac{\mathbb{I}_{A_n}}{w + \frac{1-w}{\mathbb{P}(B_n^\gamma)}\mathbb{I}_{B_n^\gamma}}$.

Theoretical Guarantee

Proposition 1. Let $T_{\text{alg1}}(n)$ denote the expected running time of Algorithm 1. If $W(1)$ is regularly varying with index $-\beta < -1$, then $T_{\text{alg1}}(n) = \sum_{m \geq l^*} h_m(n) M_m(n)$ is uniformly bounded as $n \to \infty$.

Proof Sketch:

1. $T_{\text{alg1}}(n) = \sum_{l \geq l^*} h_l M_l = \frac{n^{l^*}(\lambda \mathbb{P}(W(1) > n\gamma))^{l^*}}{\mathbb{P}(B_n^\gamma)}$.
2. B_n^γ is b.a.f. $\mathbb{D}_{<l^*}$ and $l^* = \min\{l \in \mathbb{Z}_+ | \mathbb{D}_l \cap B_n^\gamma\}$.
3. LDP lower bound:

$$\limsup_{n\to\infty} T_{\text{alg1}}(n) \leq \limsup_{n\to\infty} n^{l^*}(\lambda \mathbb{P}(W(1) > n\gamma))^{l^*} \mathbb{P}(B_n^\gamma)^{-1}$$
$$\leq C_{l^*}((B^\gamma)^\circ)^{-1} < \infty.$$

Theoretical Guarantee

Theorem 1. There exists a $\gamma > 0$ such that the estimator Z_n is strongly efficient for estimating $\mathbb{P}(A_n)$.

Proof Sketch:

$$\mathbb{E}^{Q,w}[Z_n^2] = \mathbb{E}[Z_n] = \mathbb{E}[Z_n \mathbb{I}_{B_n^\gamma}] + \mathbb{E}[Z_n \mathbb{I}_{(B_n^\gamma)^c}]$$
$$\leq \frac{1}{1-w}\mathbb{P}(A_n)\mathbb{P}(B_n^\gamma) + \frac{1}{w}\mathbb{P}(A_n \cap (B_n^\gamma)^c).$$

- B^γ captures dominant paths contributing to A, ensuring $\mathbb{P}(B^\gamma) = \Theta(\mathbb{P}(A))$.
- Residual paths in $A \cap (B^\gamma)^c$ are rarer and negligible.

Theorem 1. There exists a $\gamma > 0$ such that the estimator Z_n is strongly efficient for estimating $\mathbb{P}(A_n)$.

Proof Sketch:

- Both A and B^γ are b.a.f. $\mathbb{D}_{<l^*}$. Also from the LDP upper and lower bound,

$$\mathbb{P}(\bar{X}_n \in A) = \Theta\left((n\mathbb{P}(W(1) \geq n))^{l^*}\right),$$

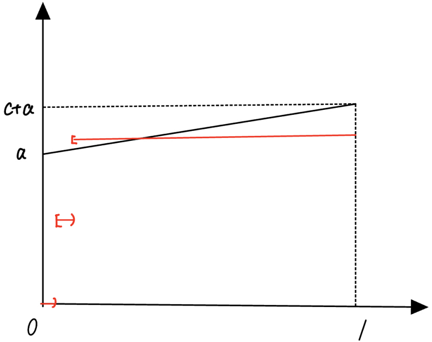
  and similarly for B^γ. Hence, $\mathbb{P}(B^\gamma) = \Theta(\mathbb{P}(A))$.
- By choosing γ small enough, we have $A_n \cap (B_n^\gamma)^c$ is b.a.f. D_{2l^*+1}, and thus $\mathbb{P}(A_n \cap (B_n^\gamma)^c) = o(\mathbb{P}^2(A_n)) = o\left((n\mathbb{P}(W(1) \geq n))^{2l^*}\right)$.
  1. $d(A, \mathbb{D}_{<l^*}) \geq r \Rightarrow A \in \{\xi : d(\xi, D_{<l^*}) \geq r\}$.
  2. $(B_n^\gamma)^c = \{\xi : \#\{t : \xi(t) - \xi(t-) \geq \gamma\} \leq l^* - 1\}$.
  3. $(B_n^\gamma)^c \cap A \cap D_{l^*+m} \neq \emptyset \Rightarrow \exists \xi = \sum_{i=1}^{l^*+m} c_i \mathbb{I}_{[t_i,1]}$ s.t. $c_i \leq \gamma$, $i \geq l^*$,

$$\sum_{i=l^*}^{l^*+m} c_i \geq d\left(\sum_{i=1}^{l^*+m} c_i, \sum_{i=1}^{l^*-1} c_i\right) \geq r.$$

  4. Let $\gamma < r/(l^* + 1)$.

Example 1: Finite-time Ruin Probability: Setting

- Centered random walk $S_k = \sum_{i=1}^{j} Y_i$, $k \geq 0$, $\bar{S}_n(t) = S_{\lfloor nt \rfloor}/n$



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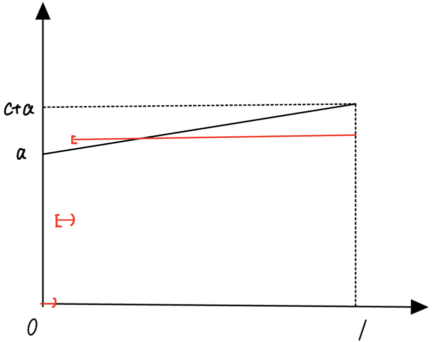

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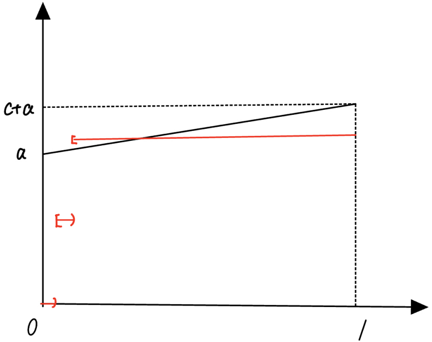
- $\mathbb{P}(Y_1 > x)$ regularly varying $-\beta$.
- $A_n \triangleq \{\max_{0 \leq k \leq n} S_k \geq nb, \max_{0 \leq k \leq n} S_k - ck \geq na\}$, $a/b \notin \mathbb{Z}$, $c > 0$.



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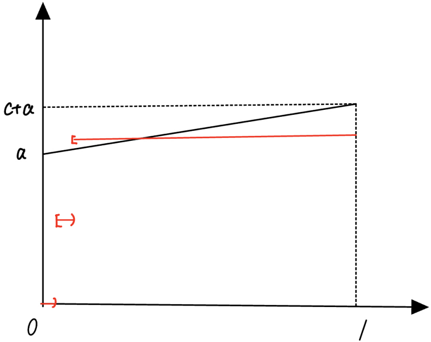
- $A \triangleq \{\xi \in \mathbb{D} : \sup_{t\in[0,1]}[\xi(t) - ct] \geq a; \sup_{t\in[0,1]}[\xi(t) - \xi(t^-)] \leq b\}$.



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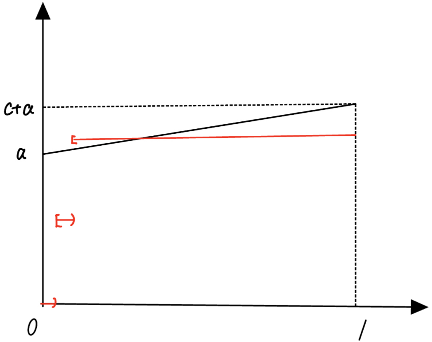

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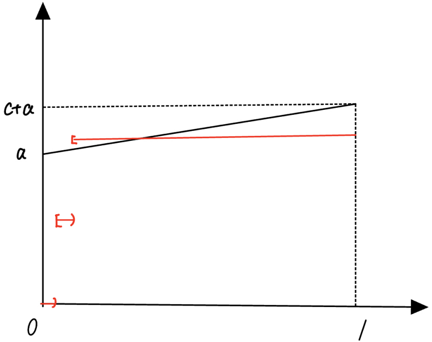
- $\mathbb{P}(A_n) = \mathbb{P}(\bar{S}_n \in A)$.
- $l^* = \lceil a/b \rceil$, $\mathbb{P}(A_n) = \Theta(n^{l^*}\mathbb{P}(S_1 \geq n)^{l^*})$.



Example 1: Finite-time Ruin Probability: B^γ

- $B^\gamma = \{\xi \in \mathbb{D} \mid \#\{t \mid \xi(t) - \xi(t^-) > \gamma\} \geq l^*\}$.
- $B^\gamma_n = \{\bar{S}_n \in B^\gamma\} = \#\{k \in \{1, \dots, n\} \mid Y_k > \gamma\} \geq l^*\}$.

$$\mathbb{P}(B^\gamma_n) = \sum_{i=l^*}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{l^*-1} \binom{n}{i} p^i (1-p)^{n-i}.$$

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- $B^\gamma = \{\xi \in \mathbb{D} \mid \#\{t \mid \xi(t) - \xi(t^-) > \gamma\} \geq l^*\}$.
- $B_n^\gamma = \{\bar{S}_n \in B^\gamma\} = \#\{k \in \{1, \ldots, n\} \mid Y_k > \gamma\} \geq l^*\}$.

$$\mathbb{P}(B_n^\gamma) = \sum_{i=l^*}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{l^*-1} \binom{n}{i} p^i (1-p)^{n-i}.$$

- γ needs to satisfy $\mathbb{P}(A_n \cap (B_n^\gamma)^c) = o(\mathbb{P}(A_n)^2)$.

Example 1: Finite-time Ruin Probability: B^γ

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$$\mathbb{P}(B_n^\gamma) = \sum_{i=l^*}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{l^*-1} \binom{n}{i} p^i (1-p)^{n-i}.$$

- γ needs to satisfy $\mathbb{P}(A_n \cap (B_n^\gamma)^c) = o(\mathbb{P}(A_n)^2)$.
- Thus $(a - (l^* - 1)b)/\gamma \notin \mathbb{Z}_+$ and

$$\left\lceil \frac{a - (l^* - 1)b}{\gamma} \right\rceil > l^* + 1.$$

Example 1: Finite-time Ruin Probability: Numerical Results

- $Y \sim Parteo(1, \beta)$.
- $c = 0.05$, $\omega = 0.05$.
- level of precision (ratio between the radius of the 95% confidence interval and the estimated value).

EstPr	$n = 80$		$n = 100$		$n = 200$	
Pr	$\beta = 1.5$	$\beta = 2.0$	$\beta = 1.5$	$\beta = 2.0$	$\beta = 1.5$	$\beta = 2.0$
$a = 2$, $b = 1.2$	1.171×10^{-3}	3.904×10^{-5}	1.043×10^{-3}	2.361×10^{-5}	6.316×10^{-4}	5.167×10^{-6}
$(l^* = 2)$	2.053×10^{-2}	3.133×10^{-2}	2.057×10^{-2}	3.376×10^{-2}	2.130×10^{-2}	3.975×10^{-2}
$a = 4$, $b = 1.2$	5.099×10^{-7}	3.778×10^{-10}	3.860×10^{-7}	1.592×10^{-10}	1.326×10^{-7}	8.911×10^{-12}
$(l^* = 4)$	1.799×10^{-2}	2.278×10^{-2}	1.761×10^{-2}	2.366×10^{-2}	1.717×10^{-2}	2.780×10^{-2}
$a = 2$, $b = 0.3$	1.635×10^{-10}	1.147×10^{-12}	1.795×10^{-10}	3.983×10^{-13}	1.202×10^{-10}	6.775×10^{-15}
$(l^* = 7)$	6.441×10^{-2}	1.662×10^{-2}	5.456×10^{-2}	1.635×10^{-2}	3.535×10^{-2}	1.826×10^{-2}

d-Dim Compensated compound Poisson process

$\bar{X}_n(t) = X(nt)/n,\ t \in [0,1]$.

- $X \triangleq (X^{(1)}, \dots, X^{(d)})$.
- $X^{(i)}(t) = \sum_{k=1}^{N^{(i)}(t)} W^{(i)}(k) - \lambda_i t \mathbb{E} W^{(i)}(1)$.
  - $\{N^{(i)}(t)\} \sim PP(\lambda_i)$.
  - $\mathbb{P}(X^{(i)}(1) > x)$ regularly varying of index $-\beta_i < -1$.
- $\mathbb{P}(\bar{X}_n \in A)$ depend heavily on $(l_1^*, \dots, l_d^*)$, where

$$(l_1^*, \dots, l_d^*) = \arg \min_{(l_1,\dots,l_d)\in\mathbb{Z}_+^d, \prod_{i=1}^d \mathbb{D}_{l_i} \cap A \neq \emptyset} \mathcal{G}(l_1, \dots, l_d) = \sum_{i=1}^d (\beta_i - 1) l_i.$$

Choice of B^γ

Let $\gamma \in \mathbb{R}^d$ with $\gamma_i > 0$ for all $i \in \{1, \dots, d\}$, and define

$$B^\gamma \triangleq \bigcup_{(l_1,\dots,l_d)\in J_{(l_1^*,\dots,l_d^*)}} B^{\gamma,l},$$

where

$$B^{\gamma,l} \triangleq \{(\xi^{(1)},\dots,\xi^{(d)}) \in \mathbb{D}^d \mid \#\{t \mid \xi^{(i)}(t) - \xi^{(i)}(t^-) > \gamma_i\} \geq l_i, \forall i\}.$$

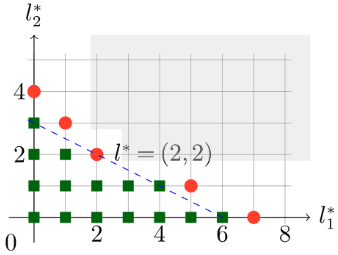


Figure: Example when $l^* = (2,2)$, $(\beta_1 - 1)/(\beta_2 - 1) = 2$.

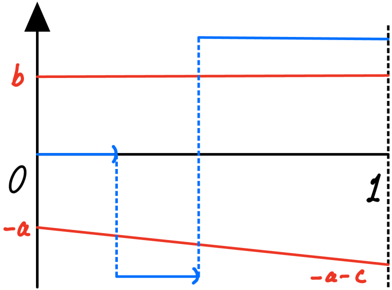
Algorithm: Sampling under $\mathbb{Q}(\cdot) = \mathbb{P}(\cdot | B_n^\gamma)$

Algorithm 3 (Generating the Sample Path of $\bar{X}_n^{(1)}, \ldots, \bar{X}_n^{(d)}$ Under $\mathbb{Q}_\gamma$)

sample $(m_1, \ldots, m_{d-1}) \sim h_{1; m_1, \ldots, m_{d-1}}$
for $i = 1$ to d **do**
  sample $q_i \sim h_{2; q_i}$; $R \leftarrow$ **true**
  while $R =$ **true do**
    sample $\{b_k\}_{k \leq \check{l}(m_i; i)} \sim \mathrm{unif}(\mathscr{C}(\{1, \ldots, q_i\}, \check{l}(m_i; i)))$
    for $j \in \{b_k\}_{k \leq \check{l}(m_i; i)}$ **do**
      sample $W^{(i)}(j) \sim W^{(i)}(1) \mid W^{(i)}(1) > n\gamma_i$
    for $j \notin \{b_k\}_{k \leq \check{l}(m_i; i)}$ **do**
      sample $W^{(i)}(j) \sim W^{(i)}(1)$
    $c \leftarrow \#\{j \in \{1, \ldots, q_i\} \mid W^{(i)}(j) > n\gamma_i\}$; $a \leftarrow 0$
    if $c < \hat{l}(m_i; i)$ **then**
      $a \leftarrow \binom{c}{\check{l}(m_i; i)}^{-1}$
    sample $u \sim \mathrm{uniform}[0, 1]$; $R \leftarrow$ **true**
    if $u < a$ **then**
      $R \leftarrow$ **false**
  return $\bar{X}_n^{(1)}, \ldots, \bar{X}_n^{(d)}$

Example 2: Barrier Option Pricing: Setting

- $S_k = \sum_{i=1}^{k} Y_k,\ k \geq 0$, a centered random walk.
- $\mathbb{P}(Y_1 \leq x)$ regularly varying with index $-\alpha$; $\mathbb{P}(Y_1 \geq x)$ regularly varying with index $-\beta$.
- $A \triangleq \{\xi \in \mathbb{D} : \xi(1) \geq b, \inf_{0 \leq t \leq 1}[\xi(t) + ct] \leq -a\}$.
- $A_n \triangleq \{S_n \geq bn, \min_{0 \leq k \leq n} S_k + ck \leq -an\}$.
- $\mathbb{P}(A_n)$ is caused by two large jumps.



Example 2: Barrier Option Pricing: Choice of B_n^γ

- $(l_-^*, l_+^*) = (1, 1)$.
- $J_{1,1} = \{(1, 1), (l, 0), (0, m)\}$
  - $l = \min\{l' \in \mathbb{Z}_+ \mid (l' - 1)(\beta - 1) > (\alpha - 1)\}$.
  - $m = \min\{m' \in \mathbb{Z}_+ \mid (m' - 1)(\alpha - 1) > (\beta - 1)\}$.

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  - $m = \min\{m' \in \mathbb{Z}_+ \mid (m' - 1)(\alpha - 1) > (\beta - 1)\}$.
- A is b.a.f. both $\mathbb{D}_{i,0}$ and $\mathbb{D}_{0,i}$

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- A is b.a.f. both $\mathbb{D}_{i,0}$ and $\mathbb{D}_{0,i}$
- It is sufficient to consider $\tilde{J}_{1,1} = \{(1,1)\}$.

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- $(l_-^*, l_+^*) = (1,1)$.
- $J_{1,1} = \{(1,1),(l,0),(0,m)\}$
  - $l = \min\{l' \in \mathbb{Z}_+ \mid (l'-1)(\beta-1) > (\alpha-1)\}$.
  - $m = \min\{m' \in \mathbb{Z}_+ \mid (m'-1)(\alpha-1) > (\beta-1)\}$.
- A is b.a.f. both $\mathbb{D}_{i,0}$ and $\mathbb{D}_{0,i}$
- It is sufficient to consider $\tilde{J}_{1,1} = \{(1,1)\}$.
- Modified
  $B^\gamma = \{\xi \in \mathbb{D} \mid \exists t_1 < t_2 : \xi(t_1^-) - \xi(t_1) > \gamma_-, \xi(t_2) - \xi(t_2^-) > \gamma_+\}$.

Example 2: Barrier Option Pricing: Choice of B_n^γ

- $(l_-^*, l_+^*) = (1, 1)$.
- $J_{1,1} = \{(1,1), (l, 0), (0, m)\}$
  - $l = \min\{l' \in \mathbb{Z}_+ \mid (l'-1)(\beta - 1) > (\alpha - 1)\}$.
  - $m = \min\{m' \in \mathbb{Z}_+ \mid (m'-1)(\alpha - 1) > (\beta - 1)\}$.
- A is b.a.f. both $\mathbb{D}_{i,0}$ and $\mathbb{D}_{0,i}$
- It is sufficient to consider $\tilde{J}_{1,1} = \{(1,1)\}$.
- Modified
  $B^\gamma = \{\xi \in \mathbb{D} \mid \exists t_1 < t_2 : \xi(t_1^-) - \xi(t_1) > \gamma_-, \xi(t_2) - \xi(t_2^-) > \gamma_+\}$.
- $B_n^\gamma = \{\exists i < j : Y_i < -\gamma_- n, Y_j > \gamma_+ n\}$.

Example 2: Barrier Option Pricing: Choice of B_n^γ

- $(I_-^*, I_+^*) = (1,1)$.
- $J_{1,1} = \{(1,1),(l,0),(0,m)\}$
    - $l = \min\{l' \in \mathbb{Z}_+ \mid (l'-1)(\beta-1) > (\alpha-1)\}$.
    - $m = \min\{m' \in \mathbb{Z}_+ \mid (m'-1)(\alpha-1) > (\beta-1)\}$.
- A is b.a.f. both $\mathbb{D}_{i,0}$ and $\mathbb{D}_{0,i}$
- It is sufficient to consider $\tilde{J}_{1,1} = \{(1,1)\}$.
- Modified
  $B^\gamma = \{\xi \in \mathbb{D} \mid \exists t_1 < t_2 : \xi(t_1^-) - \xi(t_1) > \gamma_-, \xi(t_2) - \xi(t_2^-) > \gamma_+\}$.
- $B_n^\gamma = \{\exists i < j : Y_i < -\gamma_- n, Y_j > \gamma_+ n\}$. Need that $\mathbb{P}(A_n \cap (B_n^\gamma)^c) = o(\mathbb{P}(A_n)^2)$: choose γ_- and γ_+ such that
    1. $((a+b)/\gamma_+, a/\gamma_-) \notin \mathbb{Z}_+^2$.
    2. $\min\left\{(\alpha-1) + \left\lfloor \frac{a+b}{\gamma_+} \right\rfloor (\beta-1), \left\lfloor \frac{a}{\gamma_-} \right\rfloor (\alpha-1) + (\beta-1)\right\} > 2(\alpha+\beta-2)$.

Example 2: Barrier Option Pricing: Numerical Result

- $Y_1 = Y_1' - \mathbb{E}Y_1'$
- $f_{Y'} = \frac{1}{3}\left(\frac{1}{y}\right)^{\beta}\mathbb{I}_{(1,\infty)}(y) + \frac{1}{3}\left(-\frac{1}{y}\right)^{\alpha}\mathbb{I}_{(-\infty,-1)}(y) + \frac{1}{6}\mathbb{I}_{[-1,1]}(y).$
- Estimate $\mathbb{P}(S_n \geq bn, \min_{0\leq k\leq n} S_k \leq -an)$, $a = 2$ and $b = 1.5$.
- $w = 0.05$

Table 2. Estimated rare-event probability and level of precision for the application as described in Section 5.

Est Pr	$n = 250$	$n = 500$	$n = 750$	$n = 1,000$	$n = 1,250$	$n = 1,500$
$\alpha = 2, \beta = 1.5$	3.913×10^{-7} 0.043	1.370×10^{-7} 0.043	6.992×10^{-8} 0.044	4.539×10^{-8} 0.044	3.305×10^{-8} 0.044	2.471×10^{-8} 0.044
$\alpha = 1.8, \beta = 1.7$	3.322×10^{-7} 0.037	1.154×10^{-7} 0.037	6.040×10^{-8} 0.038	3.840×10^{-8} 0.038	2.870×10^{-8} 0.038	2.225×10^{-8} 0.037
$\alpha = 2.3, \beta = 2$	1.923×10^{-9} 0.053	4.004×10^{-10} 0.053	1.491×10^{-10} 0.054	7.601×10^{-11} 0.054	4.632×10^{-11} 0.054	3.072×10^{-11} 0.054
$\alpha = 2.7, \beta = 1.8$	6.838×10^{-10} 0.068	1.121×10^{-10} 0.070	4.092×10^{-11} 0.070	2.079×10^{-11} 0.069	1.105×10^{-11} 0.071	6.896×10^{-12} 0.071