Last time:

Intro & Background. & Monge-Kantorovich

To day:

- Background:
  - Problem setting
  - Functional analysis complement.
    * Legendre transform
    * Fenchel duality (w. proof)
    * Hahn-Banach theorem
- General Duality proof: only for compactness & C_b.
  * Fenchel duality & Riesz-Markov Theorem
  * Approximation
- Special cases: distance cost function
  * Kantorovich-Rubinstein theorem
  * Connection to total variation distance.

✡ Problem Setting:

▷ (X, μ), (Y, ν) 2 probability spaces, $c: X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$ measurable.

Kantorovich mass transportation problem:

$$\underset{\pi \in \Pi(\mu,\nu)}{\arg\min} \int_{X \times Y} c(x,y)\, d\pi(x,y)$$

- rmk 1: $\Pi(\mu,\nu)$ is nonempty & convex.
- rmk 2: From now we only consider Borel prob. sp.:

  $(X, \mathcal{B}_X, \mu)$ where $\mathcal{B}_X := \sigma\left(\{O : O \text{ open in } X\}\right)$
- rmk 3: $\pi \in \Pi(\mu,\nu) \iff \forall (\varphi, \psi) \in L^1(\mu) \times L^1(\nu) = L^\infty(\mu) \times L^\infty(\nu)$

$$(*) \quad \int_{X \times Y} [\varphi(x) + \psi(y)]\, d\pi = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu =: J(\varphi, \psi)$$

  - rmk 3.1:

    If X, Y are Polish | separable complete metric sp, which the theorem concerns.

    $(*)$ holds for $(\varphi, \psi) \in C_b(X) \times C_b(Y)$ is sufficient.
  - rmk 3.2:

    If X, Y are Polish & locally compact | $\forall$ point has a cpt nbhd.

    $(*)$ holds for $(\varphi, \psi) \in C_0(X) \times C_0(Y)$ is sufficient.

✡ General Duality:

- Thm: [Kantorovich duality]
  - (X,μ), (Y,ν) Polish. Cost function $c: X\times Y \to \overline{\mathbb{R}}_+$ lower semi-continuous | $\forall x \in X$, $c(x) \le \liminf_{y\to x} c(y)$
  - $\forall \pi \in \mathcal{P}(X\times Y)$, $(\varphi,\psi) \in L^1(\mu)\times L^1(\nu)$, define

$$I[\pi] = \int_{X\times Y} c(x,y)\, d\pi(x,y) \qquad J(\varphi,\psi) = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu.$$

  - Define $\Pi(\mu,\nu)$ the set of all Borel coupling of μ,ν.
  - Define $\Phi_c = \{(\varphi,\psi): X\times Y \to \overline{\mathbb{R}}$ meas : $\varphi(x)+\psi(y) \le c(x,y)$ for $d\mu$-almost all $x\in X$, $d\nu$-almost all $y \in Y\} \cap \left(L^1(\mu)\times L^1(\nu)\right)$

  Then

$$\inf_{\pi\in\Pi(\mu,\nu)} I[\pi] = \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi)$$

- rmk 4: The infimum, may infinite, can be attained.
- rmk 5: The supreme does not change if $\Phi_c = \Phi_c \cap \left(C_b(X)\times C_b(Y)\right)$.

▷ Before proof:

- Legendre (- Fenchel) Transform

  E a normed vector space. $\Theta : E \to \mathbb{R} \cup \{+\infty\}$ convex.

  Then the Legendre transform is the function $\Theta^* : E^* \to \mathbb{R}$ by

$$\Theta^*(z^*) = \sup_{z \in E} \left[\langle z^*, z \rangle - \Theta(z) \right].$$

- rmk 6:

6.1.1. Definition. *Let X be a normed space. The space $X^* := \mathcal{L}(X, \mathbb{R})$ (or $X^* := \mathcal{L}(X, \mathbb{C})$ in the complex case) of all continuous linear functionals on the space X is called the dual (or topological dual) to the space X. The space X' of all linear functions on X is called the algebraic dual.*

- Fenchel (- Rockafellar) Duality

  - E (separable) normed vector sp.
  - $\Theta, \Xi : E \to \mathbb{R} \cup \{+\infty\}$. Θ^*, Ξ^* Legendre trans of Θ and Ξ.
  - Assume $\exists\ x_0 \in E$ s.t. $\Theta(z_0), \Xi(z_0) < +\infty$. Θ cts at z_0.

Then

$$\inf_E [\Theta + \Xi] = \max_{z^* \in E^*} \left[-\Theta^*(-z^*) - \Xi^*(z^*) \right]$$

$$\max_{z^* \in E^*} = \sup_{z^* \in E^*}$$

Proof: WTS:

$$RHS = \sup_{z^*\in E^*}\left[-\Theta^*(-z^*) - \Xi^*(z^*)\right]$$

$$= \sup_{z^*\in E^*}\left\{-\sup_{x\in E}\left[\langle -z^*, x\rangle - \Theta(x)\right] - \sup_{y\in E}\left[\langle z^*, y\rangle - \Xi(y)\right]\right\}$$

$$= \sup_{z^*\in E^*}\left\{\inf_{x\in E}\left[\langle z^*, x\rangle + \Theta(x)\right] + \inf_{y\in E}\left[\langle z^*, -y\rangle + \Xi(y)\right]\right\}$$

$$= \sup_{z^*\in E^*}\ \inf_{x,y\in E}\left\{\left[\langle z^*, x-y\rangle + \Theta(x) + \Xi(y)\right]\right\} \qquad (1)$$

$$= LHS = \inf_{x\in E}\left[\Theta(x) + \Xi(x)\right] \qquad (2)$$

① $(1) \leq (2)$

$x = y$

$$\Rightarrow (1) \leq \sup_{z^*}\ \inf_{x}\left[\Theta(x) + \Xi(x)\right] = \inf_{x}\left[\Theta(x) + \Xi(x)\right] = (2)$$

$\Theta(z_0)$, $\Xi(z_0)$ finite $\Rightarrow m := \inf_{x}\left[\Theta(x) + \Xi(x)\right] \leq \Theta(z_0) + \Xi(z_0) \leq +\infty$

② $(1) \geq (2)$

Consider $C := \{(x, \lambda) \in E\times\mathbb{R} : \lambda > \Theta(x)\}$ ← hypergraph of convex fct

and $C' := \{(y, \mu) \in E\times\mathbb{R} : \mu \leq m - \Xi(y)\}$ ← subgraph of concave fct

- both convex.

- $\forall x$, $\Theta(x) + \Xi(x) \geq m \Rightarrow \Theta(x) \geq m - \Xi(x)$

  $\Rightarrow C \cap C' = \phi$ (disjoint)

- $(z_0, \Theta(z_0)+1) \in C \Rightarrow \mathrm{Int}(C) \neq \phi$

By Hahn - Banach separation theorem :

Theorem[19] — Let A and B be non-empty convex subsets of a real locally convex topological vector space X. If $\operatorname{Int} A \neq \varnothing$ and $B \cap \operatorname{Int} A = \varnothing$ then there exists a continuous linear functional f on X such that $\sup f(A) \leq \inf f(B)$ and $f(a) < \inf f(B)$ for all $a \in \operatorname{Int} A$ (such an f is necessarily non-zero).

$\exists$ a non-trivial linear form $\ell \in (E \times \mathbb{R})^*$ s.t.

$$\inf \langle \ell, c \rangle = \inf_{c \in \operatorname{Int}(C)} \langle \ell, c \rangle \geqslant \sup_{c' \in C'} \langle \ell, c' \rangle . \qquad (*)$$

Write ℓ as $(w^*, \alpha) \neq \vec{0}$, $\forall (x, \lambda) \in C$ and $(y, \mu) \in C'$,

$(*) \Leftrightarrow$ $\langle w^*, x \rangle + \alpha \lambda \geqslant \langle w^*, y \rangle + \alpha \mu$, where $\alpha > 0$ (Think about subsp. $\vec{0} \times \mathbb{R}$)

Normalize w^* s.t. $z^* = \dfrac{w^*}{\alpha}$ and we have

$$(*) \Leftrightarrow \langle z^*, x \rangle + \lambda \geqslant \langle z^*, y \rangle + \mu$$

$$\Big\downarrow \lambda > \Theta(x) \qquad \qquad \Big\downarrow \mu \leq m - \Xi(y)$$

$$\langle z^*, x \rangle + \Theta(x) \geqslant \langle z^*, y \rangle + m - \Xi(y)$$

$$\Rightarrow \langle z^*, x - y \rangle + \Theta(x) + \Xi(y) \geqslant m = LHS$$

$$\Rightarrow \inf_{x,y} [\quad -\|- \quad] \geqslant LHS = (2)$$

$$\Rightarrow \quad (1) = \sup_{z^*} \inf_{x,y} [-\|-] \geqslant LHS = (2)$$

$\therefore$ (1) = (2) ■

▷ Proof of Kantorovich Duality (Using Fenchel's Duality)

- We mainly consider step 1:
  $\mathcal{X}, \mathcal{Y}$ both compact, $c \in (C_b(\mathcal{X}\times\mathcal{Y}) =: E, \|\cdot\|_\infty)$

Recall Fenchel Duality

(i) E (separable) normed vector sp. (Satisfied!)

(ii) $\Theta, \Xi : E \to \mathbb{R}\cup\{+\infty\}$. Θ^*, Ξ^* Legendre trans of Θ and Ξ.

(iii) Assume $\exists\ z_0 \in E$ s.t. $\Theta(z_0), \Xi(z_0) < +\infty$. Θ cts at z_0.

Then $\inf_E [\Theta + \Xi] = \max_{z^*\in E^*} [-\Theta^*(-z^*) - \Xi^*(z^*)]$
$= \sup_{z^*\in E^*}$

- Two convex functions:

$$\Theta : u \in C_b(\mathcal{X},\mathcal{Y}) \longmapsto \begin{cases} 0, & \text{if } u(x,y) \geq -c(x,y) \\ +\infty, & \text{o.w.} \end{cases}$$

$$\Xi : u \in C_b(\mathcal{X},\mathcal{Y}) \longmapsto \begin{cases} \int_{\mathcal{X}} \varphi\, d\mu + \int_{\mathcal{Y}} \psi\, d\nu, & \text{if } u(x,y) = \varphi(x) + \psi(y) \ (\cdots \text{ a.s.}) \\ +\infty & \text{, o.w.} \end{cases}$$

(ii) satisfied.

- $z_0(x,y) \equiv 1 \geq 0 \Rightarrow \Theta(z_0) = 0 < \infty$ and Θ cts at $z_0 \Rightarrow$ (iii) Satisfied.
  $z_0 \equiv \underset{\frac{1}{2}}{\underset{\parallel}{\varphi}} + \underset{\frac{1}{2}}{\underset{\parallel}{\psi}}$

Hence: $\inf_E [\Theta + \Xi] = \sup_{z^*\in E^*} [-\Theta^*(-z^*) - \Xi^*(z^*)]$

$$= \inf_E \left\{ \left(\int_{\mathcal{X}} \varphi\, d\mu + \int_{\mathcal{Y}} \psi\, d\nu\right) \mathbb{1}_{\{\varphi(x)+\psi(y) = u(x,y) \geq -c(x,y)\}} + (+\infty)\, \mathbb{1}_{\{\cdots c\}} \right\}$$

$$= \inf_E \left\{ \int_{\mathcal{X}} \varphi\, d\mu + \int_{\mathcal{Y}} d\nu : \varphi(x) + \psi(y) \geq -c(x,y) \right\}$$

$$= -\sup_{\Phi_c} J(\varphi, \psi)$$

Now it remains to show $\sup_{z^* \in E^*} \left[-\Theta^*(-z^*) - \Xi^*(z^*) \right] = -\inf_{\Pi(\mu,\nu)} I(\pi)$.

$$\boxed{\Theta^*(z^*) = \sup_{z \in E} \left[\langle z^*, z \rangle - \Theta(z) \right].}$$

Question: What is z^*?
How to characterize z^*?

Riesz-Markov Theorem:

Theorem. Let X be a locally compact Hausdorff space. For any continuous linear functional ψ on $C_0(X)$, there is a unique *regular* countably additive complex Borel measure μ on X such that

$$\forall f \in C_0(X): \qquad \psi(f) = \int_X f(x)\, d\mu(x).$$

The norm of ψ as a linear functional is the total variation of μ, that is

$$\|\psi\| = |\mu|(X).$$

Finally, ψ is positive if and only if the measure μ is non-negative.

Here, we adopt Radon measure | Inner regular
locally finite

So, $\forall u^*$, $\exists$ unique Radon measure π s.t.

$$\langle u^*, u \rangle = u^*(u) = \int_{X \times Y} u(x,y)\, d\pi(x,y)$$

Okay. now we have

$$\Theta : u \in C_b(X,Y) \longmapsto \begin{cases} 0 & \text{if } u(x,y) \geq -c(x,y) \\ +\infty & \text{o.w.} \end{cases}$$

$$\Rightarrow \Theta^*(-\pi) = \sup_{u \in C_b(X \times Y)} \Big\{ -\int u(x,y)\, d\pi - \underbrace{\Theta(u)}_{\text{not be infinite}} \Big\}$$

$$= \sup_{u \in C_b(X \times Y)} \Big\{ \int -u(x,y)\, d\pi : -u(x,y) \leq c(x,y) \Big\}$$

$$= \begin{cases} \int c\, d\pi, & \pi \text{ nonnegative measure} \\ +\infty, & \text{o.w.} \end{cases}$$

$$\Xi : u \in C_b(X,Y) \longmapsto \begin{cases} \int_X \varphi\, d\mu + \int_Y \psi\, d\nu, & \text{if } u(x,y) = \varphi(x) + \psi(y) \ (\cdots \text{a.s.}) \\ +\infty & \text{, o.w.} \end{cases}$$

$$\Rightarrow \Xi^*(\pi) = \sup_u \Big\{ \int u(x,y)\, d\pi - \Xi(u) \Big\}$$

$$= \sup_u \Big\{ \int [\varphi(x) + \psi(y)]\, d\pi - \Big(\int_X \varphi\, d\mu + \int_Y \psi\, d\nu\Big) \Big\}$$

$$= \begin{cases} 0, & \text{if } \int_{X \times Y} [\varphi(x) + \psi(y)]\, d\pi = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu \\ +\infty, & \text{o.w.} \end{cases}$$

$$\Updownarrow$$

$$(\varphi, \psi) \in \Phi_c$$

$$\sup_{z^* \in E^*} \left[-\Theta^*(-z^*) - \Xi^*(z^*) \right] = \sup_\pi \left[-\int c\, d\pi\ \mathbb{1}_{\{\pi \text{ nonnegative measure}\}} - \infty\, \mathbb{1}_{\{\ldots c\}} \right]$$

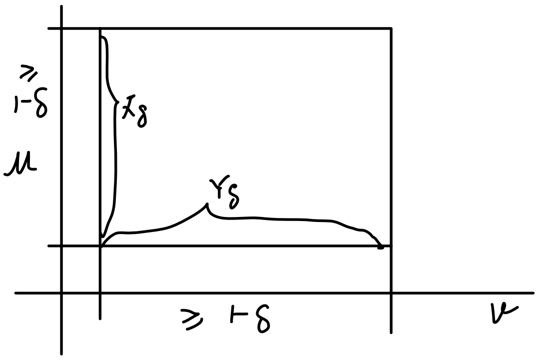
$$\int [\varphi(x) + \psi(y)]\, d\pi = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu$$

$$= -\inf_{\pi \in \Pi(\mu,\nu)} \int c\, d\pi = -\inf_{\pi \in \Pi(\mu,\nu)} I(\pi)$$

Q.E.D.

- Step 2: $\begin{cases} c \in C_b \\ \text{No assumption for } X, Y \text{ (May not compact)} \end{cases}$

Tightness $\Pi(\mu, \nu) \to$ compact subset $\to$ approximation (page 28 – 31)



$\delta \to 0$

- Step 3: No assumption for X, Y

No assumption for c (May be L^1 but not C_b)

$c = \sup c_n$, c_n non-decreasing seq of nonneg, unif cts cost fcts.

$I_n[\pi] = \int_{X\times Y} c_n \, d\pi$, $\sup_{(\varphi,\psi)\in\Phi_{c_n}} J(\varphi,\psi)$.

step 2: $\inf_{\pi\in\Pi(\mu,\nu)} I_n[\pi] = \sup_{(\varphi,\psi)\in\Phi_{c_n}} J(\varphi,\psi)$

$\begin{cases} \sup_{(\varphi,\psi)\in C_n} J(\varphi,\psi) \le \sup_{(\varphi,\psi)\in\Phi_c} J(\varphi,\psi) \\ \underbrace{\inf_{(\varphi,\psi)\in\Phi_{c_n}} I[\pi] = \sup_n \inf_{\pi\in\Phi_c} I_n[\pi]}_{} \end{cases}$

$\inf_{\Pi(\mu,\nu)} I[\pi] \le \sup_{\Phi_c} J(\varphi,\psi)$

[$\Pi(\mu,\nu)$ tight & Prokhorov's Theorem
MCT]

- Step 4: Infimum can be attained

✡ Distance cost function

- $c(x,y)$ may not be a metric.
- When the cost function is a metric $d(x,y)$, then there is more structure in the Kantorovich Duality principle
- Kantorovich – Rubinstein Theorem
  - $X = Y$ be a Polish space
  - $d: X \times X \to \mathbb{R}$ a lower semi-continuous metric on X.

  Let $T_d(\mu,\nu) = \inf\limits_{\pi \in \Pi(\mu,\nu)} \int_{X\times X} d(x,y)\, d\pi$ the cost of OT.

  Let $Lip(X)$ denote the space of all Lipschitz functions on X.

  Define $\|\varphi\|_{Lip} = \sup\limits_{x \neq y} \dfrac{|\varphi(x) - \varphi(y)|}{d(x,y)}$.

  Then

$$T_d(\mu,\nu) = \sup\left\{ \int_X \varphi \, d(\mu-\nu) ;\ \varphi \in L^1(|\mu-\nu|),\ \|\varphi\|_{Lip} \le 1 \right\}$$

$$|\varphi(x) - \varphi(y)| \le d(x,y)$$

- rmk 7: Let $P_1(X)$ be the space of those probability measures μ such that $\int d(x_0, x)\, d\mu(x) < +\infty$ for some (and thus any) x_0, and let $M_1(X)$ be the vector space generated by $P_1(X)$. On $M_1(X)$ we can define the norm

$$(1.20) \qquad \|\sigma\|_{KR} = \sup\left\{ \int_X \varphi \, d\sigma; \quad \varphi \in L^1(d|\sigma|), \quad \|\varphi\|_{\mathrm{Lip}} \le 1 \right\}.$$

Then the Kantorovich-Rubinstein theorem states that $\mathcal{T}_d(\mu,\nu) = \|\mu - \nu\|_{KR}$ for all probability measures μ, ν in $P_1(X)$.

When d is bounded, $\mathcal{T}_d$ and $\|\cdot\|_{KR}$ are well-defined on $P(X)$ and $M(X)$ respectively; then $\mathcal{T}_d$ is sometimes called the "bounded Lipschitz distance", and $\|\cdot\|_{KR}$ the "bounded Lipschitz norm".

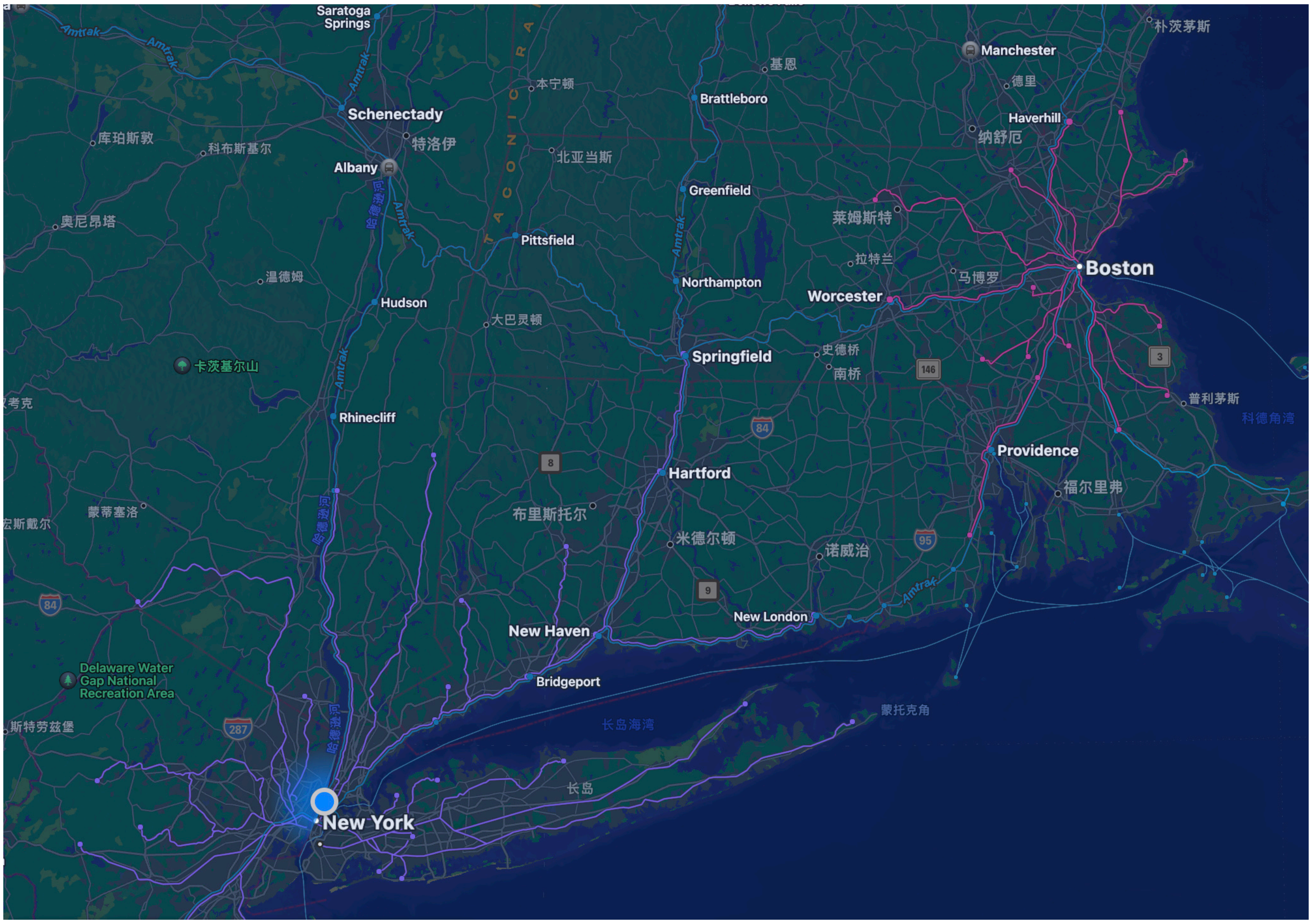
- Invariance of K-R distance
  - $X = Y$ Polish, d semi-cts metric
  - μ, ν, σ Borel non-negative finite measures on X.

  Then $T_d(\mu,\nu) = T_d(\mu+\sigma, \nu+\sigma)$

▷ Example: Total variation distance.

- X polish sp.
- cost function $c(x,y) = \mathbb{1}_{x \neq y}$
- $\mu, \nu \in P(X)$

$$\inf_{\pi \in \Pi(\mu,\nu)} \pi(\{x \neq y\}) = \sup_{0 \le f \le 1} \int_X f \, d(\mu - \nu)$$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