Lab09-Approximation Algorithm

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∗ If there is any problem, please contact TA Jiahao Fan.
∗ Name:Bowen Zhang   Student ID:517021910797   Email: 372799293@qq.com

1. **Metric k-center:** Let $G = (V, E)$ be an complete undirected graph with nonnegative edge costs satisfying the triangle inequality, and k be a positive integer. For any set $S \subseteq V$ and vertex $v \in V$, define $cost(v, S)$ to be the cost of the cheapest edge from v to a vertex in S ($cost(v, S) = 0$ if $v \in S$). The problem is to find a set $S \subseteq V$, with $|S| = k$, so as to minimize $\max_v\{cost(v, S)\}$.

   (a) Design a greedy approximation algorithm (in the form of pseudo code) with approximation ratio 2 for this problem.

   (Basic idea: start with an arbitrary center, and in each round, add the 'farthest' vertex to the center set until there are totaly k centers)

Algorithm 1: Greedy approximation algorithm

Input: An complete undirected graph $G = (V, E)$ with non-negative edge costs satisfying the triangle inquality;a positive integer k;
Output: a set $S \subseteq V$;

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S = {v_1} ;
for i = 2 → k do
    forall v_j ∈ V do
        cost(v_j, S) = min_{v∈S} cost(v_j, v);
    v_j = max_v{cost(v, S)} ;
    S = S ∪ {v_j} ;
```

   (b) Prove that your greedy algorithm achieves an approximation ratio of 2 for the metric k-center problem. (Hint: prove by contradiction and use the triangle inequality.)

Proof. Suppose that we select $v_1, v_2, \ldots, v_k$ in turn by Greedy algorithm. $S_i = \{v_1, v_2, \ldots, v_i\}$ and $S = \{v_1, v_2, \ldots, v_k\}$.We can easily induce that $max_v\{cost(v, S_i)\} \geq max_v\{cost(v, S_{i+1})\}$.

Lemma: $max_v\{cost(v, S)\}$ is less than any pair in S.

We prove it by mathematical induction:

It's obviously satisfied when $k = 1, 2$.

If $k = n$,$n \geq 2$ satisfied,that is,$max_v\{cost(v, S_n)\}$ is less than any pair in S_n.$S_{n+1} = \{v_{n+1}\} \cup S_n$,$cost(v_{n+1}, S_n)$= $max_v\{cost(v, S_n)\}$.

As we have $max_v\{cost(v, S_{n+1})\} \leq max_v\{cost(v, S_n)\} = cost(v_{n+1}, S_n)$, and $cost(v_{n+1}, S_n)$ is the minimum in any pair in S_{n+1}.

Thus,$max_v\{cost(v, S_{n+1})\}$ is less than any pair in S_{n+1}.

And then we proof the greedy algorithm achieves an approximation ratio of 2 for the problem by contradiction:

We suppose that the optimal cost is OPT and $max_v\{cost(v, S) \geq 2OPT$.

We assume $v_{k+1} = arg\{max_v\{cost(v, S)\}$,and we can easily induce that any pair in $S_{k+1} = \{v_{k+1}\} \cup S$ is more than $2OPT$.As $|S_{k+1}| = k + 1$,there exist two vertex v_p, v_q in S_{k+1} belong to the same center in optimal solution,and we denote it as v_o.

Thus,we will have $cost(v_p, v_o) \leq OPT$,$cost(v_q, v_o) \leq OPT$,$cost(v_p, v_q) \geq 2OPT$,and it's contradict with triangle inequality. □

2. Let $G = (V, E)$ be a complete undirected graph with nonnegative edge costs satisfying the triangle inequality, and its vertices are partitioned into two sets, R and S. The goal is to find a minimum cost tree in G that contains R and any subset of S. Obviously, a minimum spanning tree (MST) on R is a feasible solution. Prove that finding an MST on R achieves an approximation ratio of 2 for this problem.

Proof. Lemma: According to the triangle inequality,if we want to travel from V_1 to V_2 with minimum cost,then travel directly from V_1 to V_2 will be the optimal solution.

Suppose the optimal cost is OPT,and the corresponding tree is T^*.We construct a new graph T' based on T^*,T' will double each edge of T^* with same weight.Figure 1 is an example:

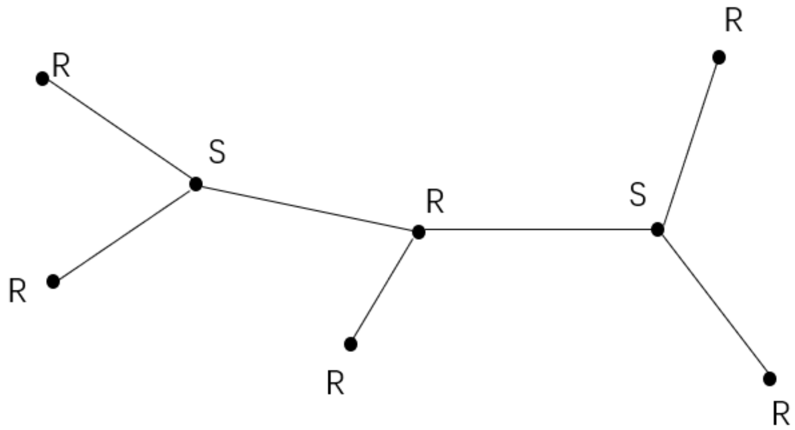

图 1: T^*

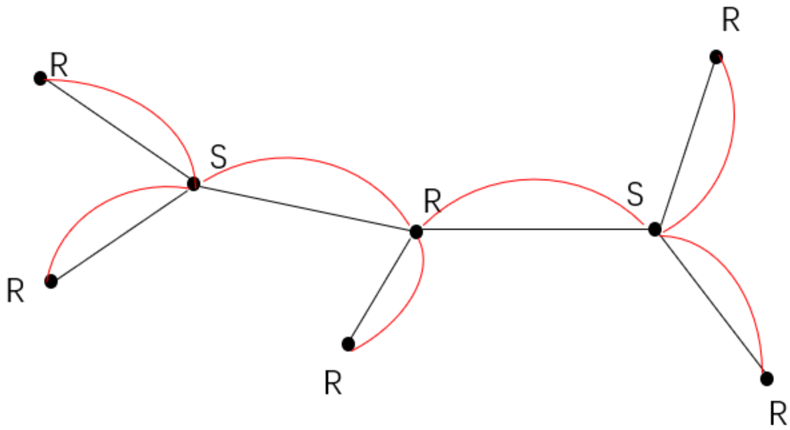

图 2: $T^{'}$

The total cost of $T^{'}$ is $2OPT$.As the degree of each vertex is even,T' has an **Euler Circuit**.Starting with a random vetex,travelling along with the Euler Circuit,we sorted the vertex in R by the sequential order that travelled in the Euler Circuit.Suppose the order is $R_1, R_2, \dots, R_n$.We construct the a tree $T = (V, E)$ with $V = \{R_1, R_2, \dots, R_n\}$ and $E = \{< R_1, R_2 >, < R_2, R_3 >, \dots, < R_{n-1}, R_n >\}$.According to the **Lemma**,the cost of $< R_k, R_{k+1} >$ will be less than the cost traveling from R_k to R_{k+1} in Euler Circuit. Thus the total cost of T will be less than $T^{'}$,that is the cost of T is less than $2OPT$.As T is a spanning tree on R,which means the cost of MST is less than T.Thus,the cost of MST is less than $2OPT$,which means that finding an MST on R achieves an approximation ratio of 2 for this problem.

□

3. **Minimum Weighted Vertex Cover:** Consider the weighted version of the Minimum Vertex Cover problem in which a non-negative weight c_i is associated with each vertex v_i and we look for a vertex cover having minimum total weight.

   (a) Given a weighted graph $G = (V, E)$ with a non-negative weight c_i associated with each vertex v_i, please formulate the Minimum Weighted Vertex Cover problem as an integer linear program.

   Solution.

$$\min \quad \sum_{j=1}^{|V|} c_j x_j$$

$$\begin{aligned} s.t. \qquad & x_i + x_j \geq 1 \quad < i, j > \in E \\ & x_i = 0, 1 \quad i = 1, 2, \dots, |V| \end{aligned}$$

□

   (b) Prove that the following algorithm finds a feasible solution of the Minimum Weighted Vertex Cover problem with value $m_{LP}(G)$ such that $m_{LP}(G)/m^*(G) \leq 2$.

Algorithm 2: Rounding Weighted Vertex Cover

Input: Graph $G = (V, E)$ with non-negative vertex weights;
Output: Vertex cover V' of G;

Let ILP_{VC} be the integer linear programming formulation of the problem;
Let LP_{VC} be the problem obtained from ILP_{VC} by LP-relaxation;
Let $x^*(G)$ be the optimal solution for LP_{VC};
$V' \leftarrow \{v_i \mid x_i^*(G) \geq 0.5\}$;
return V';

Proof. Let $x^*(G)$ be the optimal solution for ILP_{VC},$x(G)$ be the optimal solution for LP_{VC},U^* be the vertex cover solved by ILP_{VC} and U be the vertex cover solved by LP_{VC}.

   i. Firstly,we need to prove that V' is a feasible solution
   For any $< i, j > \in E$,we have $x_i + x_j \geq 1$.Thus, there is at least one of x_i,x_j will be greater than 0.5,that is the element of V'.So V' is a feasible solution.

   ii. As $x_i \geq \frac{1}{2}$ for $v_i \in U$,$x_i^* = 1$ for $v_i \in U^*$,we have

$$\sum_{v_i \in U} \frac{1}{2} c_i \leq \sum_{v_i \in U} c_i x_i$$

$$\sum_{v_i \in U^*} c_i = \sum_{v_i \in U^*} c_i x_i^*$$

As $\sum_{v_i \in U^*} c_i x_i^*$ is the optimal solution of ILP_{VC},$\sum_{v_i \in U} c_i x_i$ is the optimal solution of LP_{VC},and LP_{VC} is a relaxation of ILP_{VC},thus we have

$$\sum_{v_i \in U} c_i x_i \leq \sum_{v_i \in U^*} c_i x_i^*$$

Thus

$$\sum_{v_i \in U} \frac{1}{2} c_i \leq \sum_{v_i \in U^*} c_i$$

That is

$$\sum_{v_i \in U} c_i / \sum_{v_i \in U^*} c_i \leq 2$$

That is

$$m_{LP}(G)/m^*(G) \leq 2$$

□

4. Give the corresponding $(I, sol, m, goal)$ for **Metric k-center** and **Minimum Weighted Vertex Cover** respectively.

   Metric k-center:

   $I = \{(G,k)|G = (V,E)$ is an complete undirected graph with nonnegative edge costs satisfying the triangle inequality,k is a positive integer $\}$

   $sol((G,k)) = \{S|S \subseteq V$ with $|S| = k\}$

   $m((G,k),S) = \max_v\{cost(v,S)\}$

   goal=min

   Minimum Weighted Vertex Cover:

   $I = \{G = (V,E)|G$ is a graph with a non-negative weight c_i associated with each vetex $v_i\}$

   $sol(G) = \{U \subseteq V|\forall(v_i,v_j) \in E[v_i \in U \vee v_j \in U]\}$

   $m(G,U) = \sum_{v_i \in U} c_i$

   goal=min

Remark: You need to include your .pdf and .tex files in your uploaded .zip file.