Sorting Network*

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Outline

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Introduction

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- (now) comparison-network
  $\rightarrow$ n comparison operations can be performed simultaneously.

Comparison Network **VS** RAM's

- Comparison network can only perform comparisons.
  (Cannot deal with Counting Sort etc.)
- Comparison network runs parallel operations.

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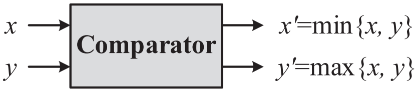
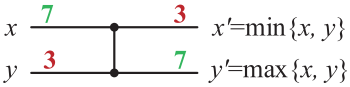
comparator: device with two inputs x and y, and two outputs x' and y', that performs the following function:



A comparator



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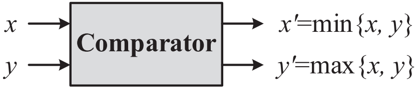

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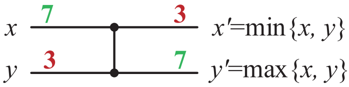

Logical representation

Each comparator operates in $O(1)$ time.

wire: transmits a value from place to place.

- ▷ Connect the output of one comparator to the input of another;
- ▷ The network input wires or output wires.

Objective of Comparison Network

Assume a **comparison network** contains n input wires $\langle a_1, a_2, \cdots, a_n \rangle$, through which the values to be sorted enter the network, and n output wires $\langle b_1, b_2, \cdots, b_n \rangle$, which produce the results computed by the network.

Goal: Draw a comparison network on n inputs as a collection of n horizontal lines with comparators stretched vertically.

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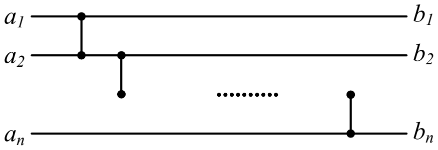
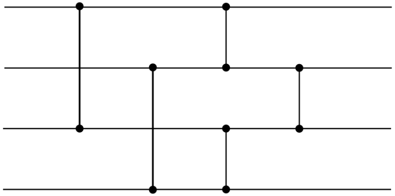


An Example



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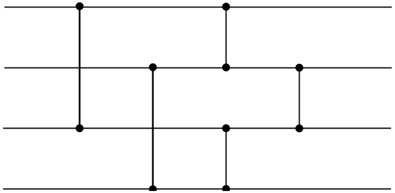

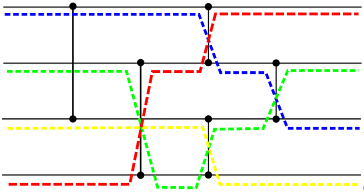

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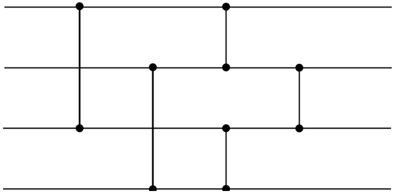
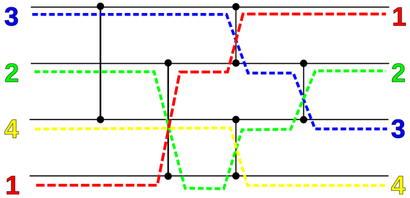
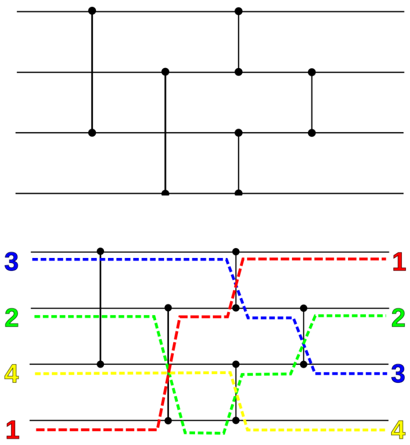




Data move from left to right.

An Example



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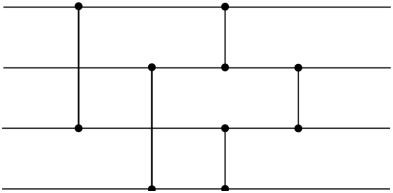

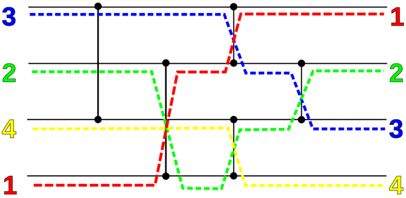

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Interconnections must be acyclic.

If a comparator has two input wires with **depths** d_x and d_y, then its output wire have depth $\max\{d_x, d_y\} + 1$. (Initially is 0)

Outline

Sorting Network

A sorting network is a comparison network for which the output sequence is monotonically increasing ($b_1 \leq b_2 \leq \cdots \leq b_n$) for **every** input sequence.

Note: We are discussing a family of comparison networks according to the input size.

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Zero-One Principle

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Domain Conversion Lemma: If a comparison network transforms the input sequence $\mathbf{a} = \langle a_1, a_2, \cdots, a_n \rangle$ into the output sequence $\mathbf{b} = \langle b_1, b_2, \cdots, b_n \rangle$, then for any monotonically increasing function f, the network transforms the input sequence $f(\mathbf{a}) = \langle f(a_1), f(a_2), \cdots, f(a_n) \rangle$ into the output sequence $f(\mathbf{b}) = \langle f(b_1), f(b_2), \cdots, f(b_n) \rangle$.

Proof (by Induction)

Basis Step: Consider a comparator whose input values are x and y. The upper output is $\min\{x, y\}$ while the lower output is $\max\{x, y\}$.

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Since f is monotonically increasing, $x \leq y$ implies $f(x) \leq f(y)$. Thus we have

$$\min\{f(x), f(y)\} = f(\min\{x, y\}),$$

$$\max\{f(x), f(y)\} = f(\max\{x, y\}),$$

which completes the proof of the claim as the base case.

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Since the output wires are included in this statement, proving it will prove the lemma.

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Induction: A wire at depth $d \geq 1$ is the output of a comparator at depth d, and the input wires to this comparator are at a depth strictly less than d. By inductive hypothesis, if the input wires carry values a_i and a_j with input sequence $\mathbf{a}$, then they carry $f(a_i)$ and $f(a_j)$ with input sequence $f(\mathbf{a})$.

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By previous claim, the output wires of this comparator then carry $f(\min\{a_i, a_j\})$ and $f(\max\{a_i, a_j\})$. Since the carry $\min\{a_i, a_j\}$ and $\max\{a_i, a_j\}$ when the input sequence is $\mathbf{a}$, the lemma is proved.

An Example

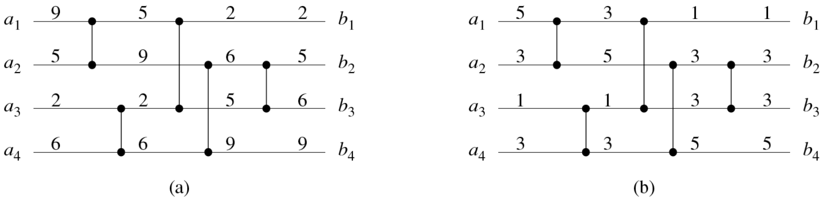

Figure 27.5 **(a)** The sorting network from Figure 27.2 with input sequence $\langle 9, 5, 2, 6\rangle$. **(b)** The same sorting network with the monotonically increasing function $f(x) = \lceil x/2 \rceil$ applied to the inputs. Each wire in this network has the value of f applied to the value on the corresponding wire in (a).

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1. Basic Concepts
   - Comparison Network
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2. Zero-One Principle
   - Domain Conversion Lemma
   - Zero-One Principle
3. Construction of a Sorting Network
   - Bitonic Sorter
   - Merger
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Theorem: If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

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Proof: (Contradiction) Suppose there exists a sequence of arbitrary numbers that the network does not correctly sort. That is, there exists an input sequence $\langle a_1, a_2, \cdots, a_n \rangle$ containing elements a_i and a_j, such that $a_i < a_j$, but the network places a_j before a_i in the output sequence.

Proof (Continued)

Define a monotonically increasing function f as

$$f(x) = \begin{cases} 0 & \text{if } x \leq a_i, \\ 1 & \text{if } x > a_i. \end{cases}$$

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However, since $f(a_j) = 1$ and $f(a_i) = 0$, the network fails to sort the zero-one sequence $\langle f(a_1), f(a_2), \cdots, f(a_n) \rangle$ correctly.

A contradiction!

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Step 3: Construct a Sorter $\Rightarrow$ to sort an arbitrary sequence.

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Examples: $\langle 1, 4, 6, 8, 3, 2 \rangle$, $\langle 6, 9, 4, 2, 3, 5 \rangle$, $\langle 9, 8, 3, 2, 4, 6 \rangle$

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Zero-one bitonic sequence have the form $0^i 1^j 0^k$ or the form $1^i 0^j 1^k$.

A monotonically increasing or monotonically decreasing sequence is also bitonic.

Half-Cleaner

A half-cleaner is a comparison network of depth 1, in which input line i is compared with line $i + \dfrac{n}{2}$ for $i = 1, 2, \cdots, \dfrac{n}{2}$ (assume n is even).

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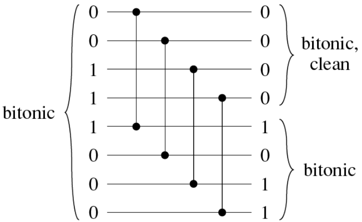

Figure 27.7 The comparison network HALF-CLEANER[8]. Two different sample zero-one input and output values are shown. The input is assumed to be bitonic. A half-cleaner ensures that every output element of the top half is at least as small as every output element of the bottom half. Moreover, both halves are bitonic, and at least one half is clean.

Half-Cleaner Lemma

Lemma: If the input to a half-cleaner is a bitonic sequence of 0's and 1's, then the output satisfies the following properties:

- both the top half and the bottom half are bitonic;
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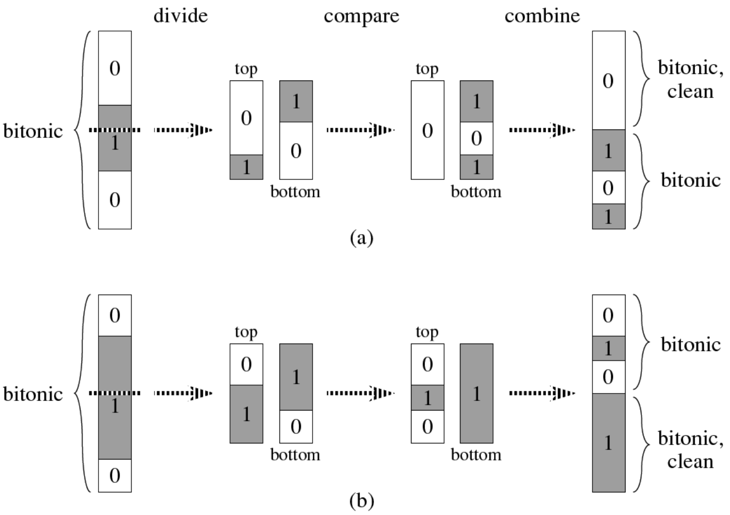
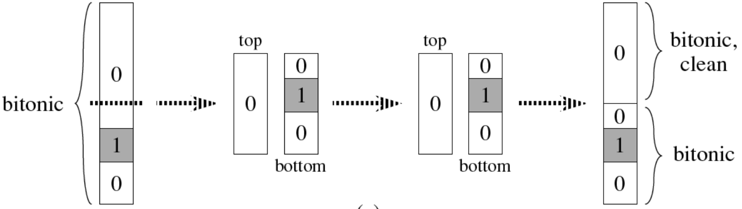
Proof: HALF-CLEANER[n] compares inputs i and $i + n/2$ for $i = 1, 2, \cdots, n/2$. Without loss of generality, suppose the input is of the form $00\cdots011\cdots100\cdots0$ (the situation of $11\cdots100\cdots011\cdots1$ are symmetric).

Note: There are three possible cases depending upon the block of consecutive 0's and 1's in which the midpoint $n/2$ falls, and one of these cases is further split into two cases.

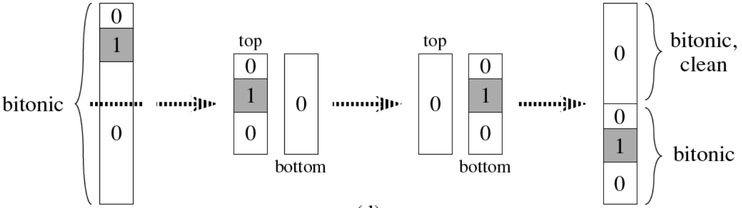
Case Analysis



Case Analysis (Cont.)



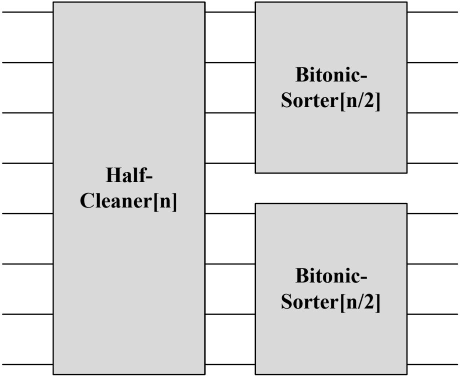
(c)



(d)

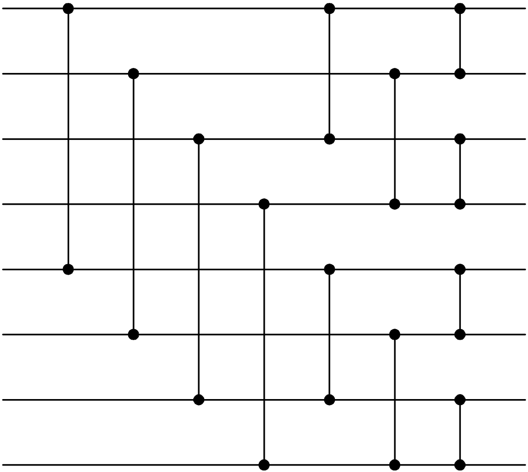
The Bitonic Sorter

By recursively combing half-cleaners, we can build a bitonic sorter, which is a network that sorts bitonic sequences.



Bitonic-Sorter[n]

An Example of $n = 8$



Depth $D(n)$

The depth $D(n)$ of BITONIC-SORTER$[n]$ is given by the recurrence

$$D(n) = \begin{cases} 0 & \text{if } n = 1; \\ D(n/2) + 1 & \text{if } n = 2^k \text{ and } k \geq 1, \end{cases}$$

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Thus, a zero-one bitonic sequence can be sorted by BITONIC-SORTER$[n]$, which has a depth of $\log n$.

By zero-one principle, any bitonic sequence of arbitrary numbers can be sorted by this network.

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For instance:

$$\begin{aligned} X &= 00000111; \\ Y &= 00001111; \\ Y^R &= 11110000; \\ X \circ Y^R &= 0000011111110000. \end{aligned}$$

MERGER$[n]$

Given two sorted sequences $\langle a_1, a_2, \cdots, a_{n/2} \rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \cdots, a_n \rangle$, we want the effect of bitonically sorting the sequence $\langle a_1, a_2, \cdots, a_{n/2}, a_n, a_{n-1}, \cdots, a_{n/2+1} \rangle$.

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Since the first half-cleaner of BITONIC-SORTER[n] compares inputs i with $n/2 + i$, for $i = 1, 2, \cdots, n/2$, we make the first stage of the merging network compare inputs i and $n - i + 1$.

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Since the first half-cleaner of BITONIC-SORTER$[n]$ compares inputs i with $n/2 + i$, for $i = 1, 2, \cdots, n/2$, we make the first stage of the merging network compare inputs i and $n - i + 1$.

The order of the outputs from the bottom of the first stage of MERGER$[n]$ are reversed compared with the order of outputs from an ordinary half-cleaner.

Comparison between MERGER[n] and HALF-CLEANER[n]

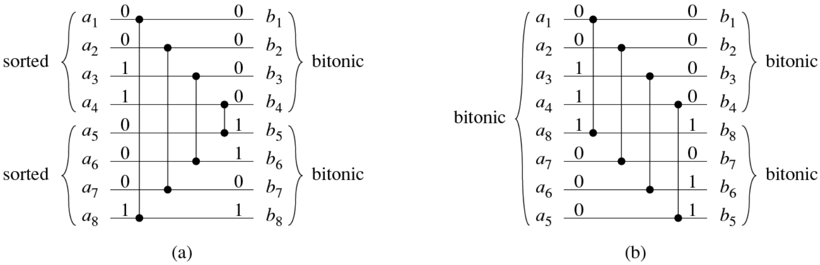
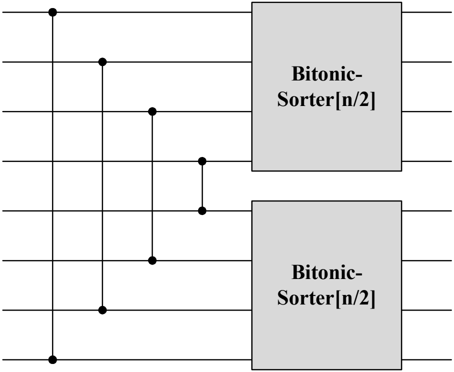


Figure 27.10 Comparing the first stage of MERGER[n] with HALF-CLEANER[n], for $n = 8$. **(a)** The first stage of MERGER[n] transforms the two monotonic input sequences $\langle a_1, a_2, \ldots, a_{n/2}\rangle$ and $\langle a_{n/2+1}, a_{n/2+2}, \ldots, a_n\rangle$ into two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_{n/2+1}, b_{n/2+2}, \ldots, b_n\rangle$. **(b)** The equivalent operation for HALF-CLEANER[n]. The bitonic input sequence $\langle a_1, a_2, \ldots, a_{n/2-1}, a_{n/2}, a_n, a_{n-1}, \ldots, a_{n/2+2}, a_{n/2+1}\rangle$ is transformed into the two bitonic sequences $\langle b_1, b_2, \ldots, b_{n/2}\rangle$ and $\langle b_n, b_{n-1}, \ldots, b_{n/2+1}\rangle$.

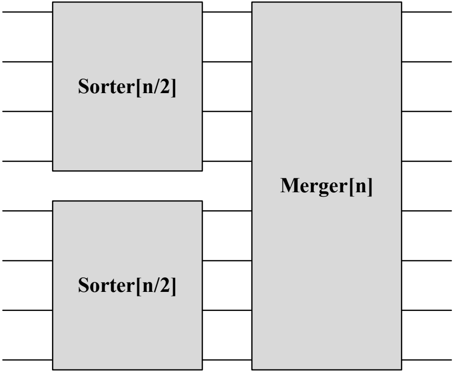
MERGER[n]



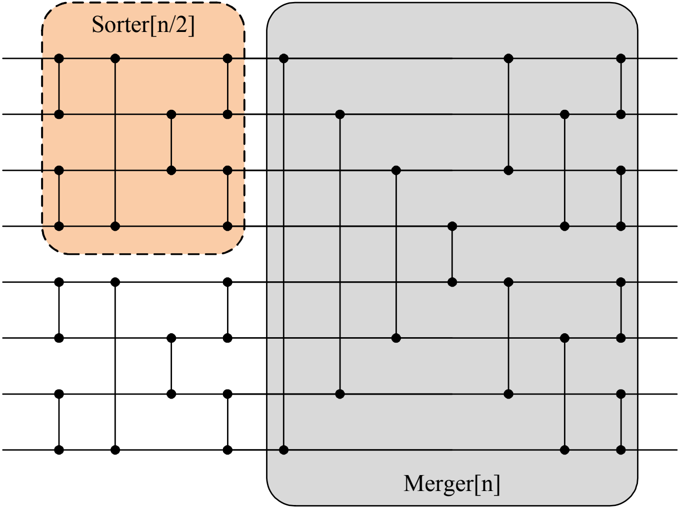
Outline

Step 3: Construct a Sorter

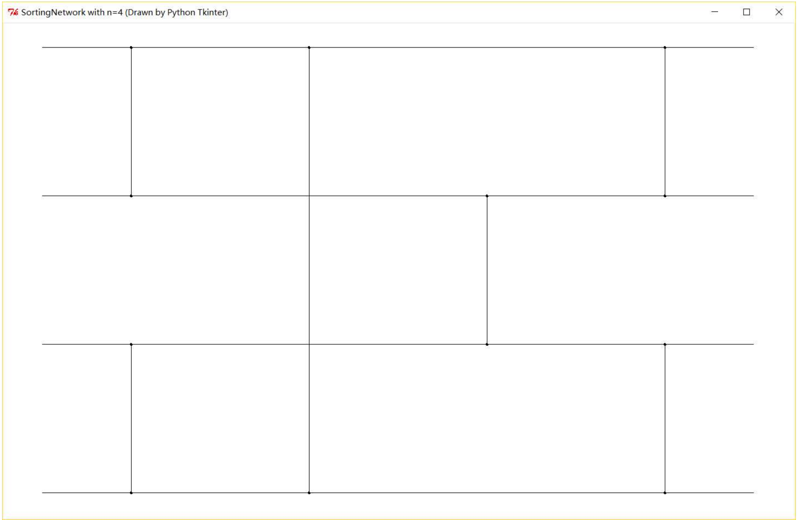
The sorting network SORTER[n] are composed by two copies of SORTER[$n/2$] and one MERGER[n] recursively.



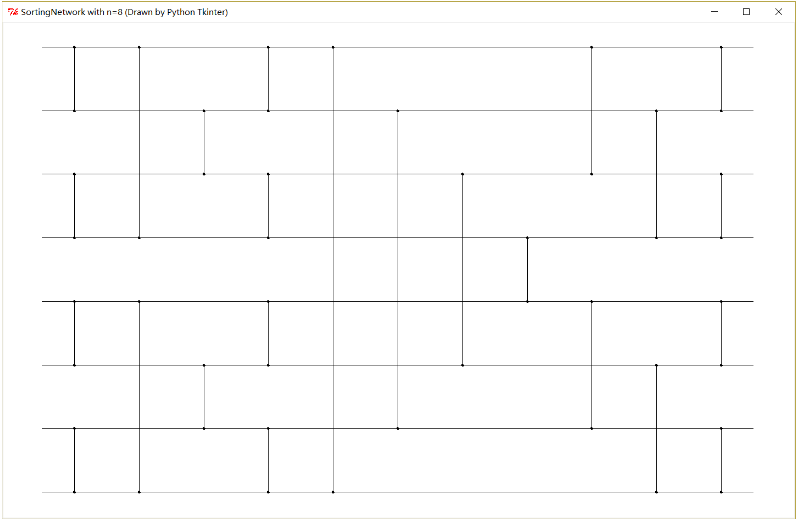
An Example with $n = 8$



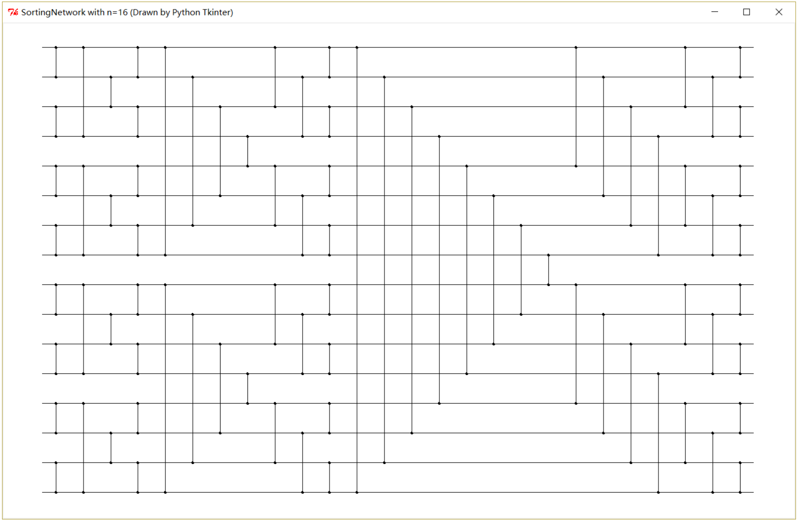
Growing Trend (Drawn by Python with $n = 4$)



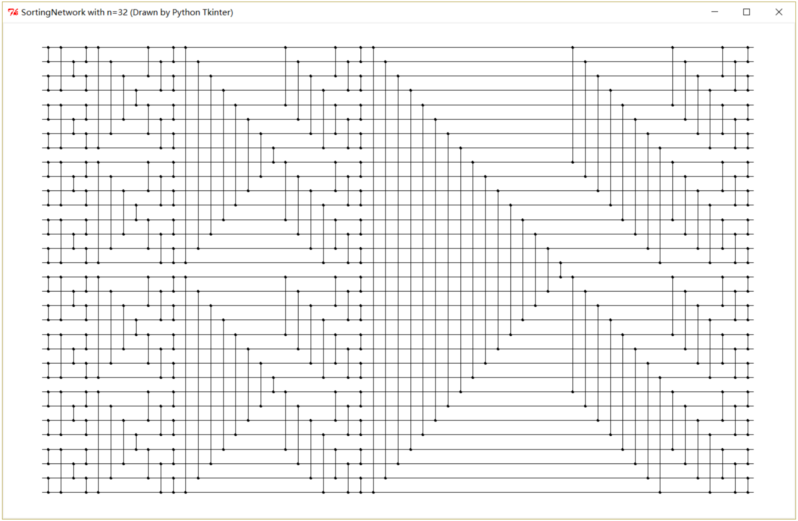
Growing Trend (Drawn by Python with $n = 8$)



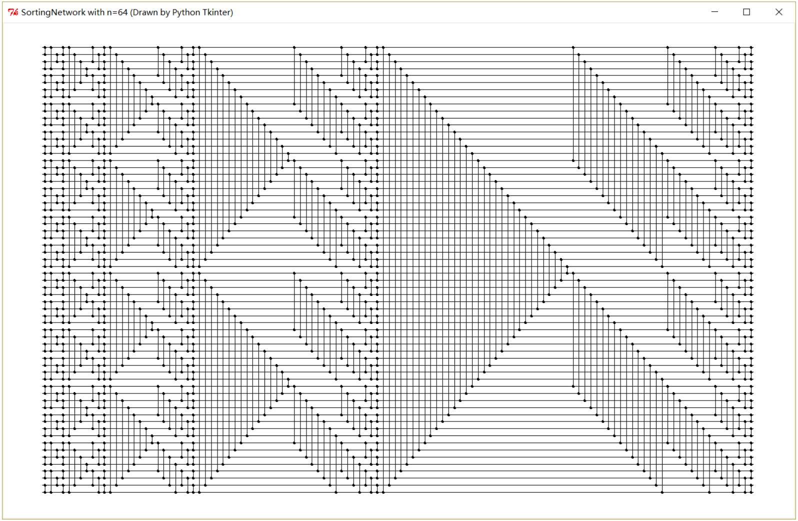
Growing Trend (Drawn by Python with $n = 16$)



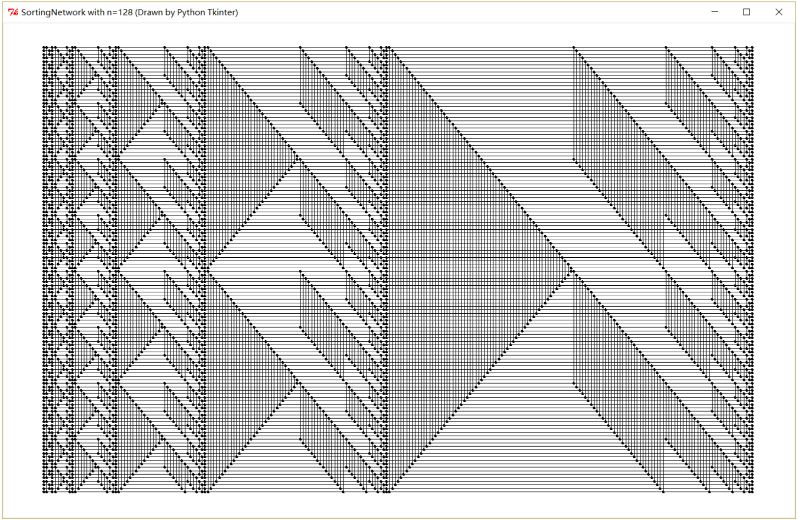
Growing Trend (Drawn by Python with $n = 32$)



Growing Trend (Drawn by Python with $n = 64$)



Growing Trend (Drawn by Python with $n = 128$)



Performance Analysis

The depth $D(n)$ of SORTER$[n]$ is the depth $D(n/2)$ of SORTER$[n/2]$ plus the depth $\ln n$ of MERGER$[n]$.

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$$D(n) = \begin{cases} 0 & \text{if } n = 1; \\ D(n/2) + O(\log n) & \text{if } n \geq 1. \end{cases}$$

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By recurrence computation, the solution is $D(n) = O(\log^2 n)$. Thus we can sort n numbers in parallel in $O(\log^2 n)$ time.