Amortized Analysis

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Outline

1. Amortized Analysis
   - Definition
   - Types
2. Three Methods
   - Aggregate Analysis
   - Accounting Method
   - Potential Function Method
3. Dynamic Tables
   - Description
   - Supporting TABLEINSERT Only
   - Supporting TABLEINSERT and TABLEDELETE

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Basic Concepts

Motivation: given a **sequence** of operations, majority of them are cheap, but some rare might be expensive; thus a standard worst-case analysis might be overly pessimistic.

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Example: serving coffee in a bar

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Amortized analysis: average over operations, e.g., TABLEINSERTION algorithm performs well on "average" over all operations even if some operations use a lot of time.

- Probability is not involved;
- Guarantees the average performance of each operation in the worst case.

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Aggregate Analysis: determine an upper bound $T(n)$ on the total cost of a sequence of n operations, and the average cost per operation is then $T(n)/n$.

Accounting Method: determine an amortized cost of each operation, different cost for different operations. Store “prepaid credit" for overcharge at early stage and pay for operations later in the sequence.

Potential Method: determine costs for operations, and maintain credit as the “potential energy” as a whole instead of associating the credit within individual objects.

Examples

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Stack Operations: Push and pop elements from an empty stack;

Binary Counter: Count a series of numbers by binary flip flops;

Dynamic Table: A continuous storage array that could change size dynamically.

Outline

First Method: Aggregate Analysis

In aggregate analysis, we compute the worst time $T(n)$ in total for a sequence of n operations. In the worst case, the average cost, or *amortized cost*, per operation is therefore $T(n)/n$.

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- Cost $T(n)/n$ applies to each operation (There may be several types of operations)
- The other two methods may assign different amortized costs to different types of operation.

Example: Stack with Multipop Operations

There are two fundamental stack operations, each takes $O(1)$ time:

PUSH(S, x): push object x onto stack S.

POP(S): pop the top of stack S and returns the popped object.

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POP(S): pop the top of stack S and returns the popped object.

Assign cost for each operation as **1**.

Time Complexity: The total cost of a sequence of n PUSH and POP operations is n, and the actual running time for n operations is $\Theta(n)$.

Multipop Operation

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ALGORITHM 3: MULTIPOP(S, k)

```
while S is not empty and k > 0 do
    POP (S);
    k ← k − 1;
```

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ALGORITHM 4: MULTIPOP(S, k)

while S *is not empty* **and** $k > 0$ **do**
    POP (S);
    $k \leftarrow k - 1$;

The total cost of MULTIPOP is $\min\{|S|, k\}$.

A Sequence of Operations

Consider a sequence of n POP, PUSH, and MULTIPOP operations on an initially empty stack.

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ALGORITHM 6: Stack with MULTIPOP

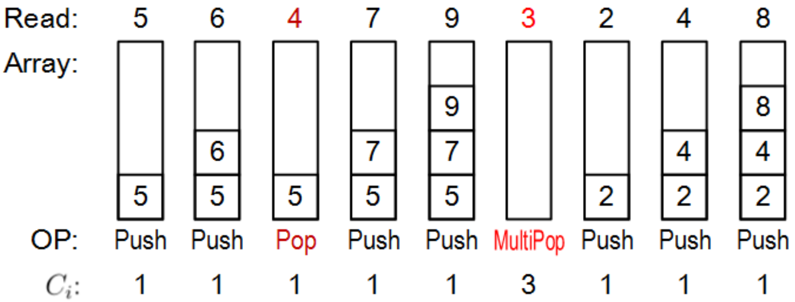
Input : An array $A[1..n]$ of n elements and an integer k.
Output: Stack S.

```
for i = 1 to n do
    if A[i] ≥ A[i − 1] then
        PUSH(S, A[i]);
    else if A[i] ≤ A[i − 1] − k then
        MULTIPOP(S, k);
    else
        POP(S);
```

An Example Scenario

Read:	5	6	4	7	9	3	2	4	8
Array:	5	6 5	5	7 5	9 7 5		2	4 2	8 4 2
OP:	Push	Push	Pop	Push	Push	MultiPop	Push	Push	Push
C_i:	1	1	1	1	1	3	1	1	1

An Example Scenario



Cursory analysis: MULTIPOP(S, k) may take $O(n)$ time; thus,

$$T(n) = \sum_{i=1}^{n} C_i \leq n^2.$$

Cursory Analysis versus Tighter Analysis

In a sequence of operations, some operations may be cheap, but some operations may be expensive, say MULTIPOP(S, k).

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Objective: For each operation we hope to assign an amortized cost $\widehat{C}_i$ to bound the actual total cost.

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Objective: For each operation we hope to assign an amortized cost $\widehat{C}_i$ to bound the actual total cost.

For any sequence of n operations, we have

$$T(n) = \sum_{i=1}^{n} C_i \leq \sum_{i=1}^{n} \widehat{C}_i.$$

Here, C_i denotes the actual cost of step i.

Tighter Analysis: Aggregate Technique

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Key observation: $\#Pop \leq \#Push$; Thus, we have:

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\begin{aligned}
T(n) &= \sum_{i=1}^{n} C_i \\
&= \#Push + \#Pop \\
&\leq 2 \times \#Push \\
&\leq 2n
\end{aligned}
$$

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$$\begin{aligned} T(n) &= \sum_{i=1}^{n} C_i \\ &= \#Push + \#Pop \\ &\leq 2 \times \#Push \\ &\leq 2n \end{aligned}$$

Conclusion: on average, the MULTIPOP(S, k) step takes only $O(1)$ time rather than $O(k)$ time.

Another Example: Incrementing a Binary Counter

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$$x = \sum_{i=0}^{k-1} A[i] \cdot 2^i.$$

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$$x = \sum_{i=0}^{k-1} A[i] \cdot 2^i.$$

Initially, $x = 0$, $A[i] = 0$ for $i = 0, \cdots, k-1$.

An Example Scenario

Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Cost	Total Cost
0	0	0	0	0	0	0	0	0	**0**	**0**

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Counter Value	$A[7]$	$A[6]$	$A[5]$	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	Cost	Total Cost
0	0	0	0	0	0	0	0	0	**0**	**0**
1	0	0	0	0	0	0	0	1	**1**	**1**

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0	0	0	0	0	0	0	0	0	**0**	**0**
1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**

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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**

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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**
4	0	0	0	0	0	1	0	0	**3**	**7**

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3	0	0	0	0	0	0	1	1	**1**	**4**
4	0	0	0	0	0	1	0	0	**3**	**7**
5	0	0	0	0	0	1	0	1	**1**	**8**

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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**
4	0	0	0	0	0	1	0	0	**3**	**7**
5	0	0	0	0	0	1	0	1	**1**	**8**
6	0	0	0	0	0	1	1	0	**2**	**10**

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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**
4	0	0	0	0	0	1	0	0	**3**	**7**
5	0	0	0	0	0	1	0	1	**1**	**8**
6	0	0	0	0	0	1	1	0	**2**	**10**
7	0	0	0	0	0	1	1	1	**1**	**11**

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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**
4	0	0	0	0	0	1	0	0	**3**	**7**
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6	0	0	0	0	0	1	1	0	**2**	**10**
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3	0	0	0	0	0	0	1	1	**1**	**4**
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9	0	0	0	0	1	0	0	1	**1**	**16**
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1	0	0	0	0	0	0	0	1	**1**	**1**
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1	0	0	0	0	0	0	0	1	**1**	**1**
2	0	0	0	0	0	0	1	0	**2**	**3**
3	0	0	0	0	0	0	1	1	**1**	**4**
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8	0	0	0	0	1	0	0	0	**4**	**15**
9	0	0	0	0	1	0	0	1	**1**	**16**
10	0	0	0	0	1	0	1	0	**2**	**18**
11	0	0	0	0	1	0	1	1	**1**	**19**
12	0	0	0	0	1	1	0	0	**3**	**22**

Pseudo Code for Binary Counter

Procedure INCREMENT is used to add 1 (modulo 2^k) to the value in the counter.

ALGORITHM 7: INCREMENT(A)

$i \leftarrow 0$;
while $i \leq k-1$ **and** $A[i] = 1$ **do**
　$A[i] \leftarrow 0$; $i \leftarrow i+1$
if $i \leq k-1$ **then**
　$A[i] \leftarrow 1$;

Objective

Consider a sequence of n operations that counts upward from 0:

ALGORITHM 8: BinaryCounter

for $i = 1$ *to* n **do**
    Increment(A);

Question: $T(n) \leq ?$

Objective

Consider a sequence of n operations that counts upward from 0:

ALGORITHM 9: BINARYCOUNTER

---

for $i = 1$ *to* n **do**
    INCREMENT(A);

Question: $T(n) \leq ?$

Cursory analysis: $T(n) \leq kn$ since an increment step might change all k bits.

Tighter Analysis: Aggregate Technique

Basic operations: `flip(1→0)`, `flip(0→1)`

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During a sequence of n INCREMENT operations:

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$A[0]$ flips each time INCREMENT is called $\leftarrow n$ times;

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$A[0]$ flips each time INCREMENT is called $\leftarrow n$ times;

$A[1]$ flips every other time $\leftarrow \lfloor n/2 \rfloor$ times;

$\ldots \; \ldots \; \ldots$

$A[i]$ flips $\lfloor n/2^i \rfloor$ times.

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$T(n) = \sum_{i=1}^{n} C_i$$

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$\begin{aligned} T(n) &= \sum_{i=1}^{n} C_i \\ &= 1+2+1+3+1+2+1+4+\cdots \qquad \text{(add by row)} \end{aligned}$$

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n} C_i \\
&= 1+2+1+3+1+2+1+4+\cdots && \text{(add by row)} \\
&= \#\mathit{flip}(A[0]) + \#\mathit{flip}(A[1]) + \cdots + \#\mathit{flip}(A[k]) && \text{(add by column)}
\end{aligned}
$$

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$
\begin{aligned}
T(n) &= \sum_{i=1}^{n} C_i \\
&= 1+2+1+3+1+2+1+4+\cdots && \text{(add by row)} \\
&= \#\mathit{flip}(A[0]) + \#\mathit{flip}(A[1]) + \cdots + \#\mathit{flip}(A[k]) && \text{(add by column)} \\
&= n + \frac{n}{2} + \frac{n}{4} + \cdots
\end{aligned}
$$

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$
\begin{aligned}
T(n) = \sum_{i=1}^{n} C_i & \\
&= 1+2+1+3+1+2+1+4+\cdots && \text{(add by row)} \\
&= \#\mathit{flip}(A[0]) + \#\mathit{flip}(A[1]) + \cdots + \#\mathit{flip}(A[k]) && \text{(add by column)} \\
&= n + \frac{n}{2} + \frac{n}{4} + \cdots \\
&\leq 2n
\end{aligned}
$$

Tighter Analysis: Aggregate Technique (Cont.)

Thus,

$$
\begin{aligned}
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&= 1+2+1+3+1+2+1+4+\cdots && \text{(add by row)} \\
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&= n + \frac{n}{2} + \frac{n}{4} + \cdots \\
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Amortized cost of each operation: $O(n)/n = O(1)$.

Outline

Accounting Method

Basic idea: for each operation OP with actual cost C_{OP}, an amortized cost $\widehat{C_{OP}}$ is assigned such that for any sequence of n operations,

$$T(n) = \sum_{i=1}^{n} C_i \le \sum_{i=1}^{n} \widehat{C}_i.$$

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The requirement that $\sum_{i=1}^{n} C_i \leq \sum_{i=1}^{n} \widehat{C}_i$ is essentially credit never goes negative.

Example 1: Stack with MULTIPOP Operation

Example: For stack with MULTIPOP, assign amortized cost as:

Operation	Real Cost C_{op}	Amortized Cost $\widehat{C_{op}}$
PUSH	1	2
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Here $n = n_1 + n_2 + n_3$.

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Note: when there are more than one type of operations, each type of operation might be assigned with different amortized cost.

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If "average" cost $<$ actual cost: credit will be used to pay actual cost.

Constraint: $\sum_{i=1}^{n} C_i \leq \sum_{i=1}^{n} \widehat{C}_i$ for arbitrary n operations, i.e. you have enough credit in your account.

An Example Scenario

Read:	5
Array:	5
OP:	Push
C_i:	1
$\widehat{C}_i$:	2
Credit:	1

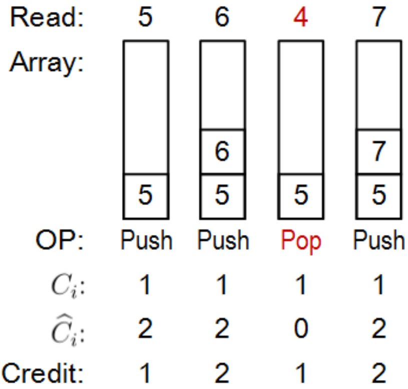
An Example Scenario

Read:	5	6
Array:	5	6 5
OP:	Push	Push
C_i:	1	1
$\widehat{C}_i$:	2	2
Credit:	1	2

An Example Scenario

Read:	5	6	4
Array:	5	6 5	5
OP:	Push	Push	Pop
C_i:	1	1	1
$\widehat{C}_i$:	2	2	0
Credit:	1	2	1

An Example Scenario



An Example Scenario

Read:	5	6	4	7	9
Array:	5	5, 6	5	5, 7	5, 7, 9
OP:	Push	Push	Pop	Push	Push
C_i:	1	1	1	1	1
$\widehat{C}_i$:	2	2	0	2	2
Credit:	1	2	1	2	3

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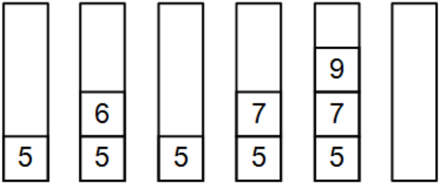
Read:	5	6	4	7	9	3
Array:	5	5, 6	5	5, 7	5, 7, 9	



OP:	Push	Push	Pop	Push	Push	MultiPop
C_i:	1	1	1	1	1	3
$\widehat{C}_i$:	2	2	0	2	2	0
Credit:	1	2	1	2	3	0

An Example Scenario

Read:	5	6	4	7	9	3	2
Array:	5	5, 6	5	5, 7	5, 7, 9		2
OP:	Push	Push	Pop	Push	Push	MultiPop	Push
C_i:	1	1	1	1	1	3	1
$\widehat{C}_i$:	2	2	0	2	2	0	2
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Read:	5	6	4	7	9	3	2	4
Array:	5	6 5	5	7 5	9 7 5		2	4 2
OP:	Push	Push	Pop	Push	Push	MultiPop	Push	Push
C_i:	1	1	1	1	1	3	1	1
$\widehat{C}_i$:	2	2	0	2	2	0	2	2
Credit:	1	2	1	2	3	0	1	2

An Example Scenario

Read:	5	6	4	7	9	3	2	4	8
Array:	5	5, 6	5	5, 7	5, 7, 9		2	2, 4	2, 4, 8
OP:	Push	Push	Pop	Push	Push	MultiPop	Push	Push	Push
C_i:	1	1	1	1	1	3	1	1	1
$\widehat{C}_i$:	2	2	0	2	2	0	2	2	2
Credit:	1	2	1	2	3	0	1	2	3

Example 2: Incrementing Binary Counter

Set amortized cost as follows:

OP	Real Cost C_{OP}	Amortized Cost $\widehat{C_{OP}}$
`flip(0→1)`	1	2
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Key observation: $\#\mathit{flip}(0 \to 1) \geq \#\mathit{flip}(1 \to 0)$

$$\begin{aligned} T(n) &= \sum_{i=1}^{n} C_i \\ &= \#\mathit{flip}(0 \to 1) + \#\mathit{flip}(1 \to 0) \\ &\leq 2\#\mathit{flip}(0 \to 1) \\ &\leq 2n \end{aligned}$$

Outline

Potential Technique: "Physicist's View"

Basic idea: sometimes it is not easy to set $\widehat{C_{op}}$ for each operation OP directly.

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Amortized Cost Setting: $\widehat{C_i} = C_i + \Phi(S_i) - \Phi(S_{i-1})$.

Potential Technique: “Physicist’s View” (Cont.)

Then we have

$$\begin{aligned}\sum_{i=1}^{n} \widehat{C}_i &= \sum_{i=1}^{n} (C_i + \Phi(S_i) - \Phi(S_{i-1})) \\ &= \sum_{i=1}^{n} C_i + \Phi(S_n) - \Phi(S_0)\end{aligned}$$

Potential Technique: "Physicist's View" (Cont.)

Then we have

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Requirement: To guarantee $\sum_{i=1}^{n} C_i \le \sum_{i=1}^{n} \widehat{C}_i$, it suffices to assure

$\Phi(S_n) \ge \Phi(S_0)$.

Stack Example: Potential Changes

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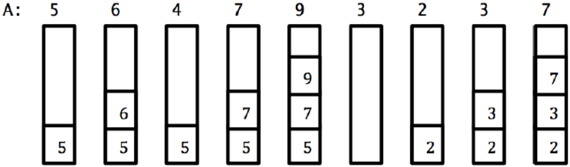
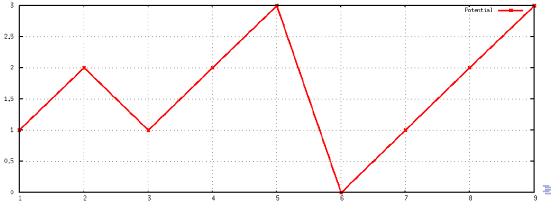
Correctness: $\Phi(S_i) \geq 0 = \Phi(S_0)$ for any i;

An Example Scenario

States of Stack S:



Polyline of Potential Function $\Phi(S_i)$:



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\textsc{MultiPop}: & \Phi(S_i) - \Phi(S_{i-1}) = -\#Pop \\
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\end{array}$$

Thus, starting from an empty stack, any sequence of n_1 PUSH, n_2 POP, and n_3 MULTIPOP operations takes at most

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Binary Counter

Definition: Set potential function as $\Phi(S) = \#1$ in counter

Counter Value	A[7]	A[6]	A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	Cost	Total Cost
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1	1	1
2	0	0	0	0	0	0	1	0	2	3
3	0	0	0	0	0	0	1	1	1	4
4	0	0	0	0	0	1	0	0	3	7
5	0	0	0	0	0	1	0	1	1	8
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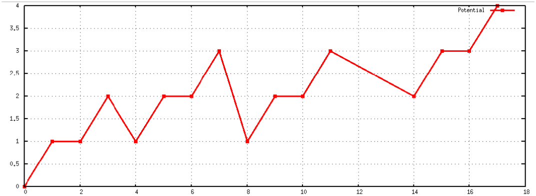

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Thus we have

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In other words, starting from 00....0, a sequence of n INCREMENT operations takes at most $2n$ time.

Outline

1. Amortized Analysis
   - Definition
   - Types
2. Three Methods
   - Aggregate Analysis
   - Accounting Method
   - Potential Function Method
3. Dynamic Tables
   - Description
   - Supporting TableInsert Only
   - Supporting TableInsert and TableDelete

A Practical Problem

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- `push_back`: to add a new object onto the tail;
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Recall that `vector` uses a **contiguous memory area** to store objects.

Question: How to design an efficient **memory-allocation strategy** for `vector`?

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DYNAMIC CONTRACTION: Similarly, if many objects have been removed from a table, it is worthwhile to reallocate the table with a smaller size.

We will show a **memory allocation strategy** such that the amortized cost of insertion and deletion is $O(1)$, even if the actual cost of an operation is large when it triggers an expansion or contraction.

Outline

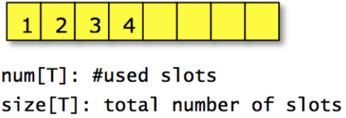
Table Expansion Operation

ALGORITHM 10: TABLE_INSERT(T, i)

if $size[T] = 0$ **then**
    allocate a table with 1 slot;
    $size[T] = 1$;
if $num[T] = size[T]$ **then**
    allocate a new table with $2 \times size[T]$ slots; //**double size**
    $size[T] = 2 \times size[T]$;
    copy all items into the new table;
    free the original table;
insert the new item i into T;
$num[T] \leftarrow num[T] + 1$;

An Example

An Example Dynamic Table T:



Example: TABLEINSERT

Consider a sequence of operations starting with an empty table:

ALGORITHM 11: TABLE_INSERT

```
Table T;
for i = 1 to n do
    TABLE_INSERT(T, i);
```

TABLEINSERT(1)

INSERT(1) $\boxed{1}$ $C_1=1$

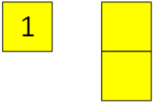
TABLEINSERT(2)

INSERT(1)	1	$C_1=1$
INSERT(2)	*overflow*	

TABLEINSERT(2)

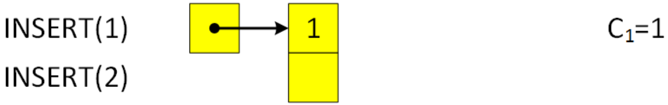
INSERT(1)

INSERT(2)

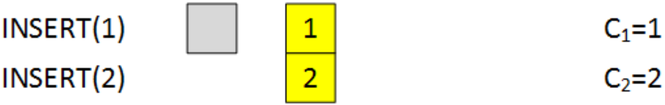


C_1=1

TABLEINSERT(2)



TABLEINSERT(2)



TABLEINSERT(3)

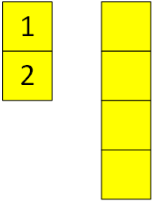
INSERT(1)	1	$C_1=1$
INSERT(2)	2	$C_1=2$
INSERT(3)	*overflow*	

TABLEINSERT(3)

INSERT(1)

INSERT(2)

INSERT(3)



$C_1=1$

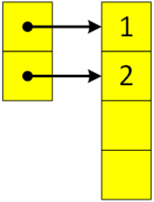
$C_1=2$

TABLEINSERT(3)

INSERT(1)

INSERT(2)

INSERT(3)



$C_1=1$

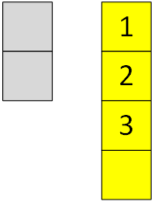
$C_2=2$

TABLEINSERT(3)

INSERT(1)

INSERT(2)

INSERT(3)



$C_1=1$

$C_2=2$

$C_3=3$

TABLEINSERT(4)

INSERT(1)	1	$C_1=1$
INSERT(2)	2	$C_2=2$
INSERT(3)	3	$C_3=3$
INSERT(4)	4	$C_4=1$

TABLEINSERT(5)

INSERT(1)	1	$C_1=1$
INSERT(2)	2	$C_2=2$
INSERT(3)	3	$C_3=3$
INSERT(4)	4	$C_4=1$
INSERT(5)	*overflow*	

TABLEINSERT(5)

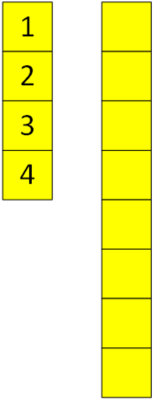
INSERT(1)

INSERT(2)

INSERT(3)

INSERT(4)

INSERT(5)



$C_1=1$

$C_2=2$

$C_3=3$

$C_4=1$

TABLEINSERT(5)

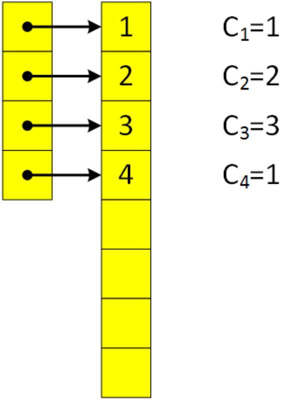
INSERT(1)

INSERT(2)

INSERT(3)

INSERT(4)

INSERT(5)



TABLEINSERT(5)

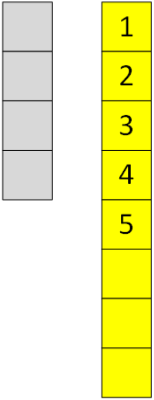
INSERT(1)

INSERT(2)

INSERT(3)

INSERT(4)

INSERT(5)



$C_1=1$

$C_2=2$

$C_3=3$

$C_4=1$

$C_5=5$

TABLEINSERT(6)

INSERT(1)	1	$C_1=1$
INSERT(2)	2	$C_2=2$
INSERT(3)	3	$C_3=3$
INSERT(4)	4	$C_4=1$
INSERT(5)	5	$C_5=5$
INSERT(6)	6	$C_6=1$

TABLEINSERT(7)

INSERT(1)	1	$C_1=1$
INSERT(2)	2	$C_2=2$
INSERT(3)	3	$C_3=3$
INSERT(4)	4	$C_4=1$
INSERT(5)	5	$C_5=5$
INSERT(6)	6	$C_6=1$
INSERT(7)	7	$C_7=1$

TABLEINSERT(8)

INSERT(1)	1	C_1=1
INSERT(2)	2	C_2=2
INSERT(3)	3	C_3=3
INSERT(4)	4	C_4=1
INSERT(5)	5	C_5=5
INSERT(6)	6	C_6=1
INSERT(7)	7	C_7=1
INSERT(8)	8	C_8=1

Cursory analysis: $O(n^2)$

Consider a sequence of operations starting with an empty table. If we define the cost in terms of elementary insertions or deletions, what is the actual cost C_i of the ith operation?

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If n operations are performed, the worst-case cost of an operation will be $O(n)$.

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If n operations are performed, the worst-case cost of an operation will be $O(n)$.

Thus, the total running time for a total of n operations is $O(n^2)$. **Not tight!**

Tighter Analysis 1: Aggregate Method

Key Observation: **Table expansions are rare**.

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i	1	2	3	4	5	6	7	8	9	10
$Size_i$	1	2	4	4	8	8	8	8	16	16
C_i	1	2	3	1	5	1	1	1	9	1

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We can decompose C_i as follows:

i	1	2	3	4	5	6	7	8	9	10
$Size_i$	1	2	4	4	8	8	8	8	16	16
C_i	1	1	1	1	1	1	1	1	1	1
		1	2		4				8	

Total cost of n operations

The total cost of n operations is:

$$\sum_{i=1}^{n} C_i = 1+2+3+1+5+1+1+1+9+1+\ldots$$

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$$

Thus the amortized cost of an operation is 3.

In other words, the average cost of each TABLEINSERT operation is $O(n)/n = O(1)$.

Tighter Analysis 2: Accounting Technique

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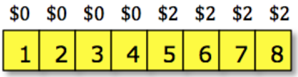
Any amount not immediately consumed is stored in a "bank" for use for subsequent operations.

Thus for the i-th insertion, the \$3 is used as follows:

- \$1 pays for the insertion **itself**;
- \$2 is stored for **later table doubling**, including \$1 for copying one of the recent $\frac{i}{2}$ items, and \$1 for copying one of the old $\frac{i}{2}$ items.

Tighter Analysis 2: Accounting Technique

Original:



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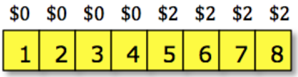
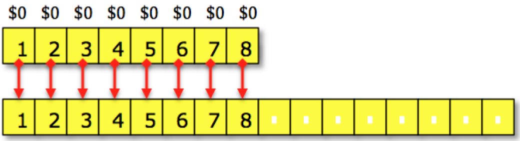


Expansion:



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$Size_i$	1	2	4	4	8	8	8	8	16	16
C_i	1	1	1	1	1	1	1	1	1	1
$\widehat{C}_i$	3	3	3	3	3	3	3	3	3	3
Credit	2	3	3	5	3	5	7	9	3	5

Tighter Analysis 3: Potential Function Technique

Motivation: sometimes it is not easy to find an appropriate amortized cost **directly**. An alternative way is to use a **potential function** as a bridge.

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Basic idea: the **bank account** can be viewed as potential function of the dynamic set. More specifically, we prefer a potential function $\Phi : \{T\} \to R$ with the following properties:

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Basic idea: the **bank account** can be viewed as potential function of the dynamic set. More specifically, we prefer a potential function $\Phi : \{T\} \rightarrow R$ with the following properties:

- $\Phi(T) = 0$ immediately **after** an expansion;
- $\Phi(T) = size[T]$ immediately **before** an expansion; thus, the next expansion can be paid for by the potential.

A Possible Function

A possibility: $\Phi(T) = 2 \times num[T] - size[T]$

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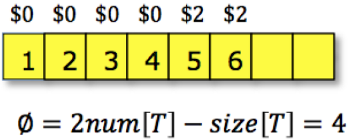

$\Phi(T) = 2 \times num[T] - size[T]$: An Example

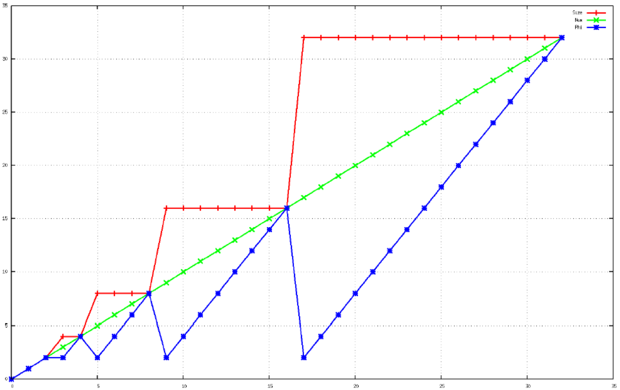

Figure: The effect of a sequence of n TABLEINSERT on $size_i$ (red), num_i (green), and Φ_i (blue).

Correctness of $\Phi(T) = 2 \times num[T] - size[T]$

Correctness: Initially $\Phi_0 = 0$, and it is easy to verify that $\Phi_i \geq \Phi_0$ since the table is always at least half full.

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Thus $\sum_{i=1}^{n} \widehat{C}_i = \sum_{i=1}^{n} C_i + \Phi_n - \Phi_0$ is really an upper bound of the actual cost $\sum_{i=1}^{n} C_i$.

Calculate $\widehat{C}_i$ with respect to Φ

Case 1: the i-th insertion does not trigger an expansion

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$$\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + 2 \\
&= 3
\end{aligned}$$

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```
1. Insert(1)
2. Insert(2)
3. Insert(3)
4. Insert(4)
```


```
C1: 1
C2: 2
C3: 3
C4: 1
```

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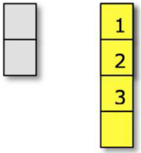

C1: 1
C2: 2
C3: 3

Conclusion

Starting with an empty table, a sequence of n TABLEINSERT operations cost $O(n)$ time in the worst case.

Outline

TableDelete Operation

To implement TableDelete operation, it is simple to remove the specified item from the table, followed by a Contraction operation when the **load factor** (denoted as $\alpha(T) = \frac{num[T]}{size[T]}$) is small, so that the wasted space is not exorbitant.

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Specifically, when the number of the items in the table drops too low, we allocate a new, smaller space, copy the items from the old table to the new one, and finally free the original table.

We would like the following two properties:

1. The load factor is bounded below by a constant;
2. The amortized cost of a table operation is bounded above by a constant.

Trial 1

Trial 1: load factor $\alpha(T)$ never drops below 1/2

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A natural strategy is:

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The strategy guarantees that load factor $\alpha(T)$ never drops below 1/2.

However, the amortized cost of an operation might be quite large.

An Example of Large Amortized Cost

Consider a sequence of $n = 16$ operations:

- The first 8 operations: `I, I, I, ....`
- The second 8 operations: `I, D, D, I, I, D, D, I, I, ...`

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Note:

- After the 8-th `I`, we have $num_{16} = size_{16} = 16$.
- The 9-th `I` leads to a table expansion;
- The following two `D` lead to a table contraction;
- The following two `I` lead to a table expansion, and so on.

An Example of Large Amortized Cost

After 8 Insertions

1	2	3	4	5	6	7	8

Insert(9) causes an expansion

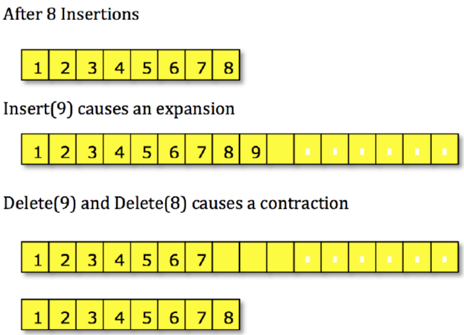
1	2	3	4	5	6	7	8	9		.	.	.	.	.	.

Delete(9) and Delete(8) causes a contraction

1	2	3	4	5	6	7				.	.	.	.	.	.

1	2	3	4	5	6	7	8

An Example of Large Amortized Cost



The expansion/contraction takes $O(n)$ time, and there are n of them.

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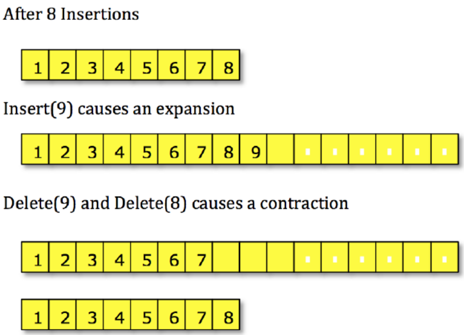

The expansion/contraction takes $O(n)$ time, and there are n of them.

Thus the total cost of n operations are $O(n^2)$, and the amortized cost of an operation is $O(n)$.

Trial 2

Trial 2: load factor $\alpha(T)$ never drops below 1/4

Trial 2

Trial 2: load factor $\alpha(T)$ never drops below 1/4

Another strategy is:

- To double the table size when inserting an item into a full table;
- To halve the table size when deletion causes $\alpha(T) < \frac{1}{4}$.

Trial 2

Trial 2: load factor $\alpha(T)$ never drops below 1/4

Another strategy is:

- To double the table size when inserting an item into a full table;
- To halve the table size when deletion causes $\alpha(T) < \frac{1}{4}$.

The strategy guarantees that load factor $\alpha(T)$ never drops below 1/4.

Amortized Analysis

We start by defining a potential function $\Phi(T)$ that is 0 immediately after an expansion or contraction, and builds as $\alpha(T)$ increases to 1 or decreases to $\frac{1}{4}$.

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$$\Phi(T) = \begin{cases} 2 \times num[T] - size[T] & \texttt{if} \;\; \alpha(T) \geq \frac{1}{2} \\ \frac{1}{2} size[T] - num[T] & \texttt{if} \;\; \alpha(T) < \frac{1}{2} \end{cases}$$

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Correctness: the potential is 0 for an empty table, and $\Phi(T)$ never goes negative. Thus, the total amortized cost of a sequence of n operations with respect to Φ is an upper bound of the actual cost.

Amortized Cost of TABLEINSERT

Case 1: $\alpha_{i-1} \geq \frac{1}{2}$ and no expansion

Amortized Cost of TABLEINSERT

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The amortized cost is:

$$\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (2(num_{i-1} + 1) - size_i) - (2num_{i-1} - size_i) \\
&= 3
\end{aligned}$$

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\end{aligned}
$$

1. Insert(1)
2. Insert(2)
3. Insert(3)
4. Insert(4)

C1: 1
C2: 2
C3: 3
C4: 1

Amortized Cost of TABLEINSERT

Case 2: $\alpha_{i-1} \geq \frac{1}{2}$ and an expansion was triggered

Amortized Cost of TABLEINSERT

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\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= num_i + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= num_{i-1} + 1 + (2(num_{i-1} + 1) - 2size_{i-1}) - (2num_{i-1} - size_{i-1}) \\
&= 3 + num_{i-1} - size_{i-1} \\
&= 3
\end{aligned}
$$

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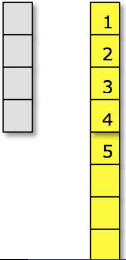
&= 3 + num_{i-1} - size_{i-1} \\
&= 3
\end{aligned}
$$

```
1. Insert(1)
2. Insert(2)
3. Insert(3)
4. Insert(4)
5. Insert(5)
```




```
C1: 1
C2: 2
C3: 3
C4: 1
C5: 5
```

Amortized Cost of TABLEINSERT

Case 3: $\alpha_{i-1} < \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

Amortized Cost of TableInsert

Case 3: $\alpha_{i-1} < \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

The amortized cost is:

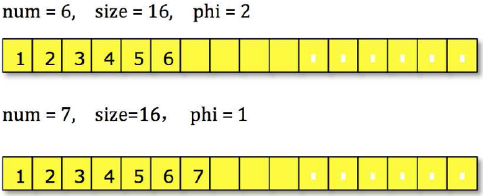
$$\begin{aligned}\widehat{C_i} &= C_i + \Phi_i - \Phi_{i-1} \\ &= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\ &= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_i - (num_i - 1)) \\ &= 0\end{aligned}$$

Amortized Cost of TABLEINSERT

Case 3: $\alpha_{i-1} < \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

The amortized cost is:

$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_i - (num_i - 1)) \\
&= 0
\end{aligned}
$$



Amortized Cost of TABLEINSERT

Case 4: $\alpha_{i-1} < \frac{1}{2}$ but $\alpha_i \geq \frac{1}{2}$

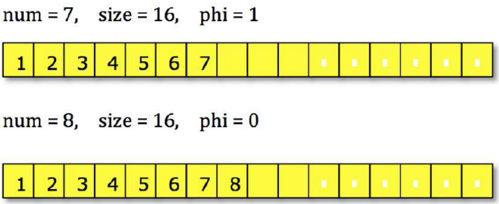
Amortized Cost of TABLEINSERT

Case 4: $\alpha_{i-1} < \frac{1}{2}$ but $\alpha_i \geq \frac{1}{2}$

The amortized cost is:

$$\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2num_i - size_i) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 1 + (2(num_{i-1} + 1) - size_{i-1}) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 3num_{i-1} - \frac{3}{2}size_{i-1} + 3 \\
&= 3\alpha_{i-1}size_{i-1} - \frac{3}{2}size_{i-1} + 3 \\
&< \frac{3}{2}size_{i-1} - \frac{3}{2}size_{i-1} + 3 \\
&= 3
\end{aligned}$$

Amortized Cost of TABLEINSERT



Amortized Cost of TABLEDELETE

Case 1: $\alpha_{i-1} < \frac{1}{2}$ and no contraction

Amortized Cost of TableDelete

Case 1: $\alpha_{i-1} < \frac{1}{2}$ and no contraction

The amortized cost is:

$$\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 1 + (\frac{1}{2}size_{i-1} - (num_{i-1} - 1)) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 2
\end{aligned}$$

Amortized Cost of TABLEDELETE

Case 1: $\alpha_{i-1} < \frac{1}{2}$ and no contraction

The amortized cost is:

$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (\frac{1}{2}size_i - num_i) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 1 + (\frac{1}{2}size_{i-1} - (num_{i-1} - 1)) - (\frac{1}{2}size_{i-1} - num_{i-1}) \\
&= 2
\end{aligned}
$$

num = 7, size = 16, phi = 1

1	2	3	4	5	6	7									

num = 6, size = 16, phi = 2

1	2	3	4	5	6										

Amortized Cost of TABLEDELETE

Case 2: $\alpha_{i-1} < \frac{1}{2}$ and a contraction was triggered

Amortized Cost of TABLEDELETE

Case 2: $\alpha_{i-1} < \frac{1}{2}$ and a contraction was triggered

The amortized cost is:

$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= num_i + 1 + (\tfrac{1}{2}size_i - num_i) - (\tfrac{1}{2}size_{i-1} - num_{i-1}) \\
&= num_{i-1} + (\tfrac{1}{4}size_{i-1} - (num_{i-1} - 1)) - (\tfrac{1}{2}size_{i-1} - num_{i-1}) \\
&= 1 + num_{i-1} - \tfrac{1}{4}size_{i-1} \\
&= 1
\end{aligned}
$$

Amortized Cost of TABLEDELETE

Case 2: $\alpha_{i-1} < \frac{1}{2}$ and a contraction was triggered

The amortized cost is:

$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= num_i + 1 + \left(\tfrac{1}{2}size_i - num_i\right) - \left(\tfrac{1}{2}size_{i-1} - num_{i-1}\right) \\
&= num_{i-1} + \left(\tfrac{1}{4}size_{i-1} - (num_{i-1} - 1)\right) - \left(\tfrac{1}{2}size_{i-1} - num_{i-1}\right) \\
&= 1 + num_{i-1} - \tfrac{1}{4}size_{i-1} \\
&= 1
\end{aligned}
$$

num=4, size=16, phi=4

1	2	3	4												

num=3, size=8, phi=1

1	2	3					

Amortized Cost of TABLEDELETE

Case 3: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i \geq \frac{1}{2}$

Amortized Cost of TABLEDELETE

Case 3: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i \geq \frac{1}{2}$

The amortized cost is:

$$\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (2(num_{i-1} - 1) - size_{i-1}) - (2num_{i-1} - size_{i-1}) \\
&= -1
\end{aligned}$$

Amortized Cost of TABLEDELETE

Case 3: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i \geq \frac{1}{2}$

The amortized cost is:

$$\begin{aligned}
\widehat{C_i} &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (2(num_{i-1} - 1) - size_{i-1}) - (2num_{i-1} - size_{i-1}) \\
&= -1
\end{aligned}$$

num = 10, size = 16, phi = 4

1	2	3	4	5	6	7	8	9	10						

num = 9, size = 16, phi = 2

1	2	3	4	5	6	7	8	9							

Amortized Cost of TABLEDELETE

Case 4: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

Amortized Cost of TABLEDELETE

Case 4: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

The amortized cost is:

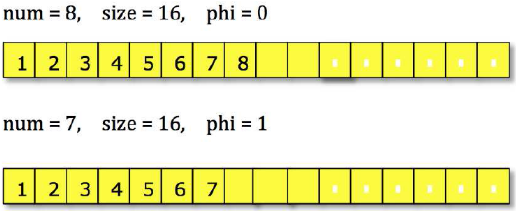
$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (\tfrac{1}{2}size_i - num_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (\tfrac{1}{2}size_{i-1} - (num_{i-1} - 1)) - (2num_{i-1} - size_{i-1}) \\
&= 2 + \tfrac{3}{2}size_{i-1} - 3num_{i-1} \\
&\leq 2
\end{aligned}
$$

Amortized Cost of TABLEDELETE

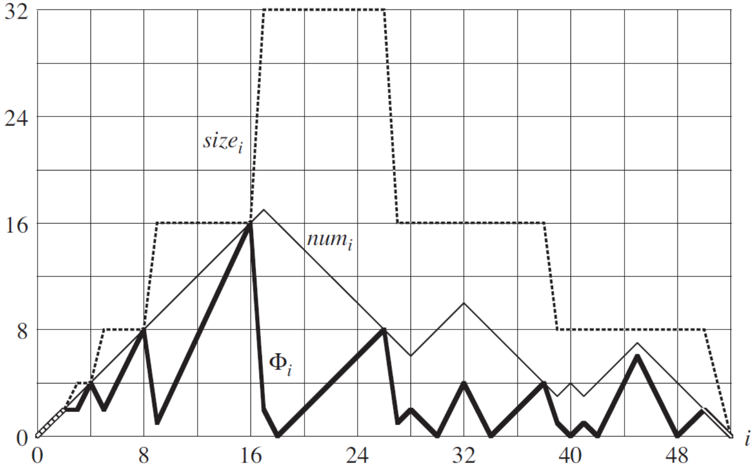
Case 4: $\alpha_{i-1} \geq \frac{1}{2}$ and $\alpha_i < \frac{1}{2}$

The amortized cost is:

$$
\begin{aligned}
\widehat{C}_i &= C_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (\tfrac{1}{2} size_i - num_i) - (2num_{i-1} - size_{i-1}) \\
&= 1 + (\tfrac{1}{2} size_{i-1} - (num_{i-1} - 1)) - (2num_{i-1} - size_{i-1}) \\
&= 2 + \tfrac{3}{2} size_{i-1} - 3num_{i-1} \\
&\leq 2
\end{aligned}
$$



An Example Polyline of Φ_i



Conclusion

In summary, since the amortized cost of each operation is bounded above by a constant, the actual cost of any sequence of n TABLEINSERT and TABLEDELETE operations on a dynamic table is $O(n)$ if starting with an empty table.

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Amortized costs can provide a clean abstraction of data-structure performance.

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Any of the analysis methods can be used when an amortized analysis is called for, but each method has some situations where it is arguably the simplest.

Different schemes may work for assigning amortized costs in the accounting method, or potentials in the potential method, sometimes yielding radically different bounds.