Divide and Conquer*

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Outline

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Divide-and-Conquer Strategy

The divide-and-conquer strategy solves a problem P by:

(1) Breaking P into smaller subproblems of the same type.

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The key works lay in three different places:

(1) How to partition problem into subproblems.

(2) At the very tail end of the recursion, how to solve the smallest subproblems outright.

(3) How to glue together the partial answers.

Outline

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1777 - 1855

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$$1 + 2 + \cdots + 100 = \frac{100 \cdot (1 + 100)}{2} = 5050.$$

Multiplication for Complex Numbers

Gauss once noticed that although the product of two complex numbers

$$(a+bi)(c+di) = ac - bd + (bc+ad)i$$

seems to involve four real-number multiplications, it can in fact be done with just three: ac, bd, and $(a+b)(c+d)$, since

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In our big-O way of thinking, reducing the number of multiplications from four to three seems wasted ingenuity. However, this modest improvement becomes *very significant when applied recursively*.

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The additions take linear time, as do the multiplications by powers of 2 (merely left-shifts). The significant operations are the four $n/2$**-bit multiplications**; these we can handle by *four recursive calls*.

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Optimization: By **Gauss**'s trick, three multiplications, $x_L y_L$, $x_R y_R$, and $(x_L + x_R)(y_L + y_R)$, suffice, as

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R.$$

A Divide-and-Conquer Algorithm for Integer Multiplication

Algorithm 1: MULTIPLY(x,y)

Input: Positive integers x and y, in binary.
Output: Their product xy.

1 $n = \max(\text{size of } x, \text{size of } y)$ rounded as a power of 2 ;
2 **if** $n = 1$ **then**
3 return xy;
4 $x_L, x_R =$ leftmost $n/2$, rightmost $n/2$ bits of x;
5 $y_L, y_R =$ leftmost $n/2$, rightmost $n/2$ bits of y;
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7 $P_2 = \text{MULTIPLY}(x_R, y_R)$;
8 $P_3 = \text{MULTIPLY}(x_L + x_R, y_L + y_R)$;
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New recurrence relation: $T(n)=3T(n/2)+O(n)$ $\rightarrow$ How well?

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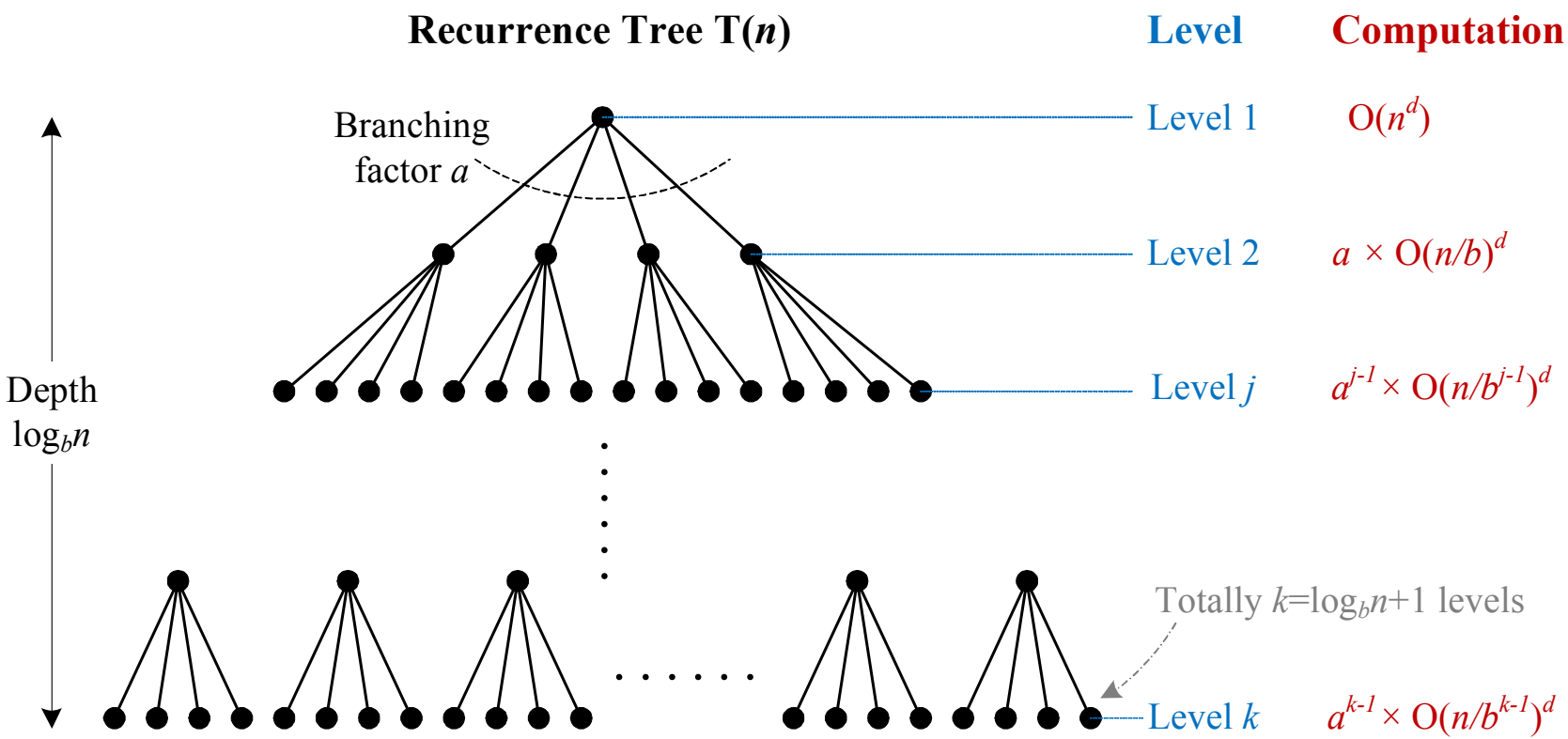
$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a. \end{cases}$$

Proof of Master Theorem: $T(n) = aT(\lceil n/b \rceil) + O(n^d)$



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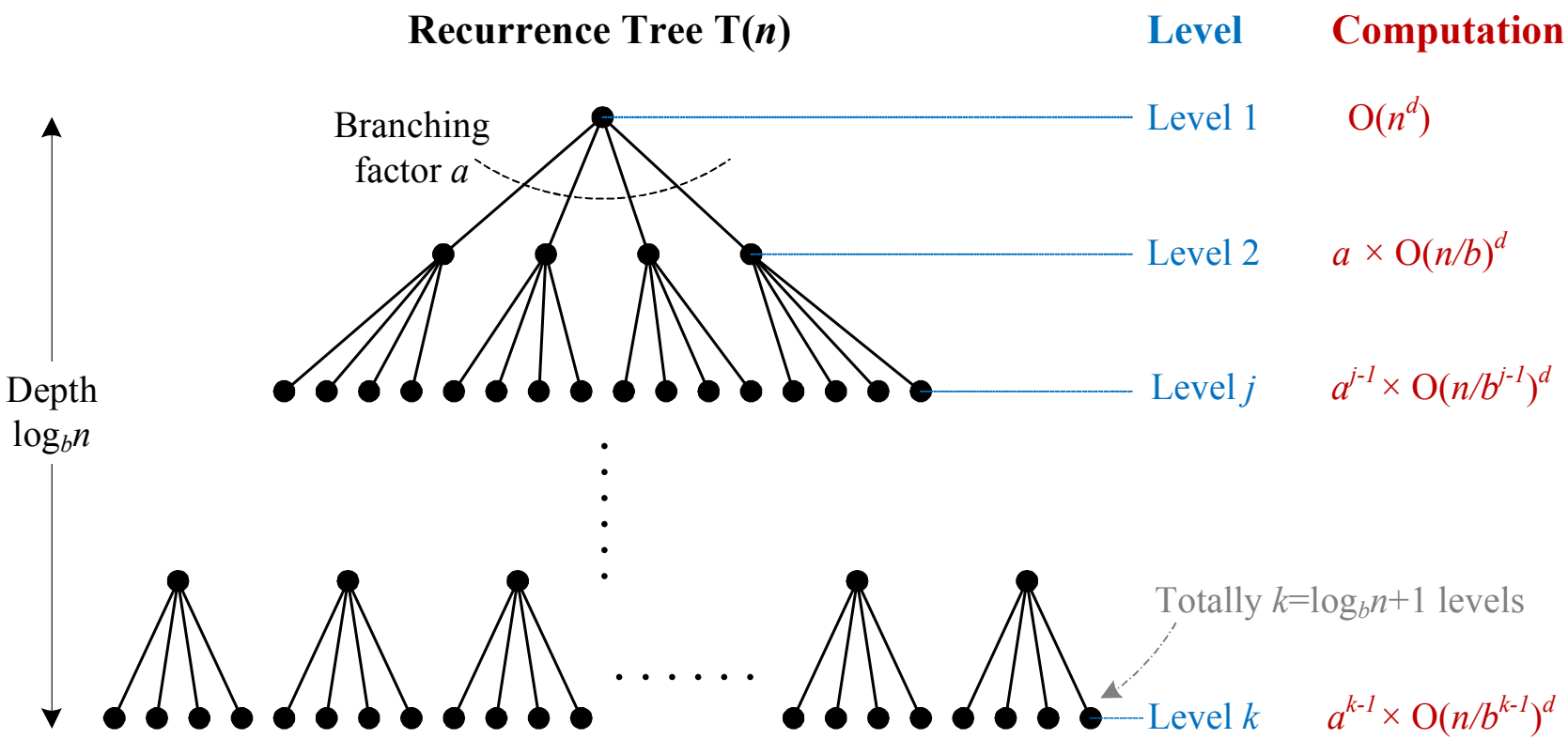

Complexity of $T(n)$ = Sum up all computations at each level.

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The total work done at the j-th level is

$$a^{j-1} \times O\left(\frac{n}{b^{j-1}}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^{j-1}.$$

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$$\sum_{j=1}^{\log_b n+1}\left(a^{j-1}\times O\left(\frac{n}{b^{j-1}}\right)^d\right)=\sum_{j=0}^{\log_b n}\left(O(n^d)\times\left(\frac{a}{b^d}\right)^j\right)=O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j.$$

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(1) $\frac{a}{b^d}<1\Rightarrow d>\log_b a$:

$$O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j\le O(n^d)\frac{1}{1-\frac{a}{b^d}}=O(n^d).$$

(Sum of GS: $S_n=\sum_{j=1}^{n}a_1q^{j-1}=a_1\frac{1-q^n}{1-q}\le a_1\frac{1}{1-q}$ if $q<1$)

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(2) $\frac{a}{b^d} = 1 \Rightarrow d = \log_b a$:

$$O(n^d) \sum_{j=0}^{\log_b n} \left(\frac{a}{b^d}\right)^j = O(n^d)(\log_b n + 1) = O(n^d \log_b n) = O(n^d \log n).$$

($\log_b n = \frac{\log n}{\log b} = \frac{1}{\log b} \log n = O(\log n)$ by changing the base)

Proof of Master Theorem

(3) $\frac{a}{b^d} > 1 \Rightarrow d < \log_b a$: (reverse the GS in decreasing order)

$$
\begin{aligned}
O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j &= O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^{\log_b n}\cdot\left(\frac{b^d}{a}\right)^j \\
&= O(n^d)\sum_{j=0}^{\log_b n}\frac{a^{\log_b n}}{(b^{\log_b n})^d}\cdot\left(\frac{b^d}{a}\right)^j \\
&\le O(n^d)\frac{n^{\log_b a}}{n^d}\cdot\frac{1}{1-\frac{b^d}{a}} \\
&= O(n^{\log_b a})
\end{aligned}
$$

$\left(a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}\right)$

Time Complexity of Multiplication

Original recurrence relation: $T(n) = 4T(n/2) + O(n)$

$a = 4, b = 2, d = 1, d < \log_b a.$

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Optimized recurrence relation: $T(n) = 3T(n/2) + O(n)$

$a = 3, b = 2, d = 1, d < \log_b a.$

$\Rightarrow$ **Time complexity:** $O(n^{\log_2 3}) \approx O(n^{1.59})$.

Outline

Binary Search

Algorithm 2: BinarySearch

Input: An array $A[1..n]$ of n elements sorted in nondecreasing order and an element x.

Output: j if $x = A[j]$, $1 \leq j \leq n$, and 0 otherwise.

```
1  low ← 1; high ← n; j ← 0;
2  while low ≤ high and j = 0 do
3      mid ← ⌊(low + high)/2⌋;
4      if x = A[mid] then
5          j ← mid  break;
6      else if x < A[mid] then
7          high ← mid − 1;
8      else
9          low ← mid + 1;
10 return j;
```

Time Complexity

To find a key x in $A[1, \cdots, n]$ in sorted order, we first compare x with $A[n/2]$, and depending on the result we recurse either on the first half of the array $A[1, \cdots, n/2 - 1]$, or on the second half $A[n/2, \cdots, n]$.

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The recurrence function is

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + O(1),$$

By Master Theorem, $a = 1$, $b = 2$, $d = 0$, and thus the running time should be $O(\log n)$.

Outline

Merging Two Sorted Lists

Algorithm 3: Merge

Input: $A[1..m]$, p, q and r with $1 \leq p \leq q < r \leq m$.
Output: $A[p..r]$ (merging two subarrays $A[p..q]$ and $A[q+1..r]$).

```
1  s ← p; t ← q + 1; k ← p;
2  while s ≤ q and t ≤ r do
3  |   if A[s] ≤ A[t] then
4  |   |   B[k] ← A[s]; s ← s + 1; (B[p..r] is an auxiliary array)
5  |   else B[k] ← A[t]; t ← t + 1;
6  |   k ← k + 1;
7  if s = q + 1 then
8  |   B[k..r] ← A[t..r];
9  else B[k..r] ← A[s..q];
10 return A[p..r] ← B[p..r];
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E.g.

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7	11	13	45	57

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If the two array sizes are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, the number of comparisons is between $\lfloor n/2 \rfloor$ and $n - 1$.

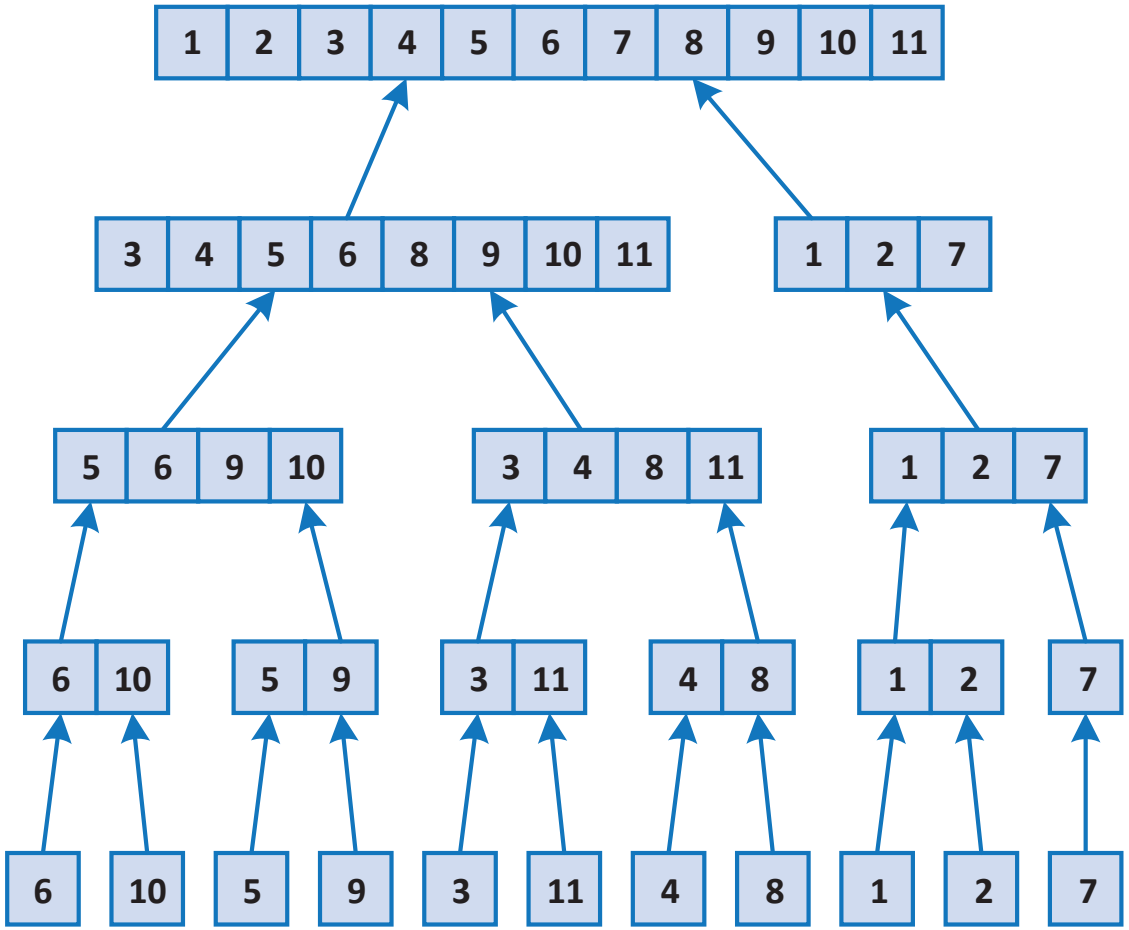
Bottom-Up MergeSort Algorithm

Algorithm 4: MergeSort

Input: An array $A[1..n]$ of n elements.
Output: $A[1..n]$ sorted in nondecreasing order.

```
1 t ← 1;
2 while t < n do
3     s ← t; t ← 2s; i ← 0;
4     while i + t ≤ n do
5         Merge(A, i + 1, i + s, i + t);
6         i ← i + t;
7     if i + s < n then
8         Merge(A, i + 1, i + s, n);
9 return A[1..n];
```

An Example


1 2 3 4 5 6 7 8 9 10 11
3 4 5 6 8 9 10 11
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4 8
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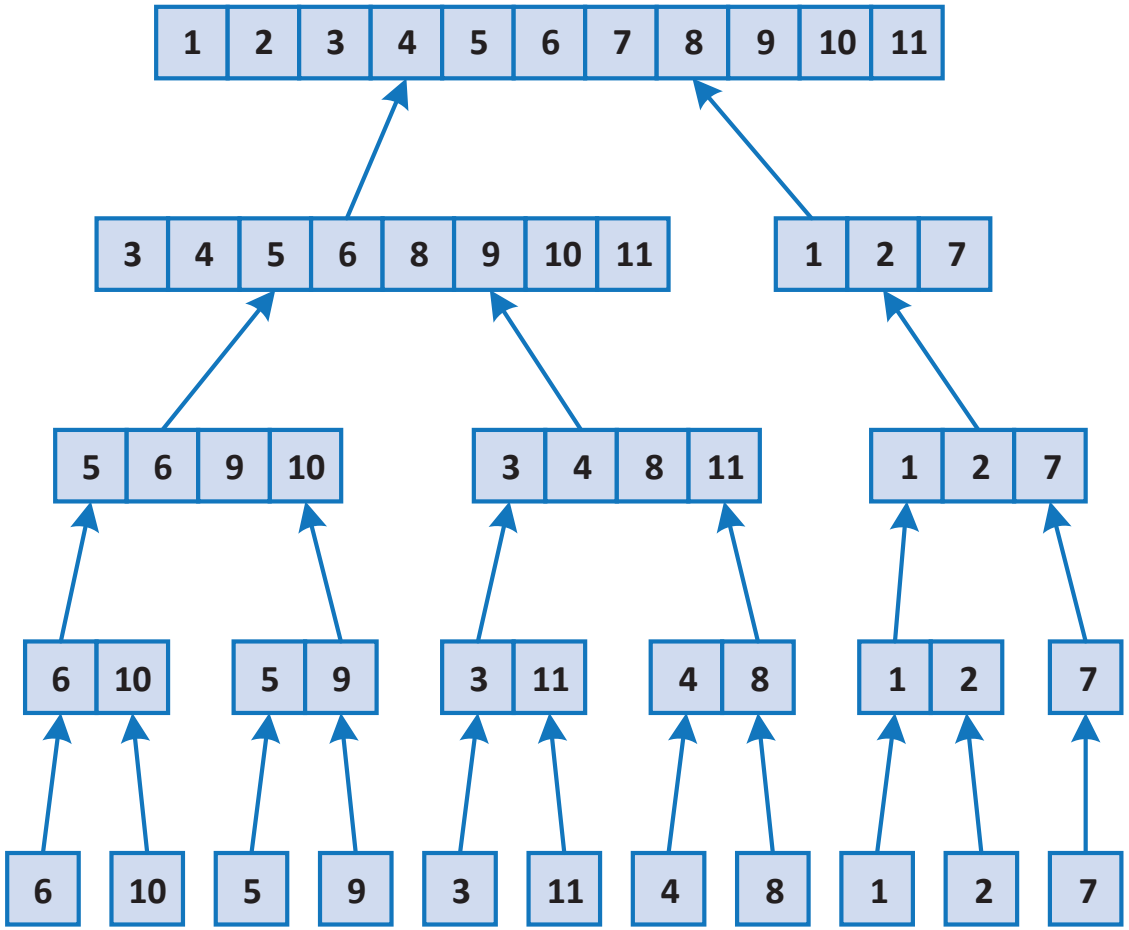


Time Complexity:

Recurrence:

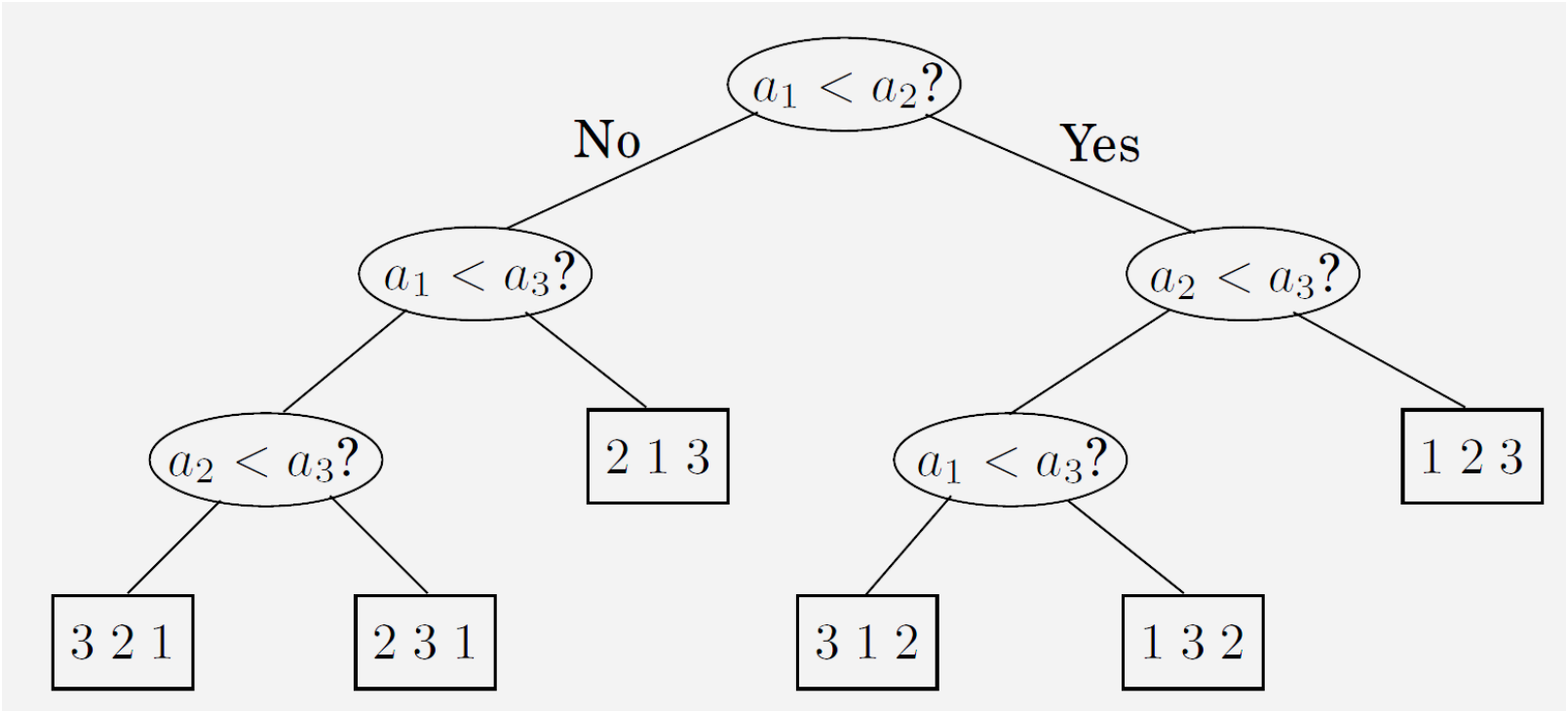
$$T(n) = 2T(n/2)+O(n);$$

By Master Theorem

$$T(n) = O(n \log n).$$

An $n \log n$ Lower Bound for Sorting

An example sorting permutation tree for $\{a_1, a_2, a_3\}$:



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This is a binary tree with $n!$ leaves. Thus, the depth of our tree – and the complexity of our algorithm – must be at least

$$\log(n!) \approx \log\left(\sqrt{\pi\,(2n + 1/3)} \cdot n^n \cdot e^{-n}\right) = \Omega(n \log n),$$

where we use Stirling's formula.

Outline

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That is, Z_{ij} is the dot product of the ith row of X with the jth column of Y.

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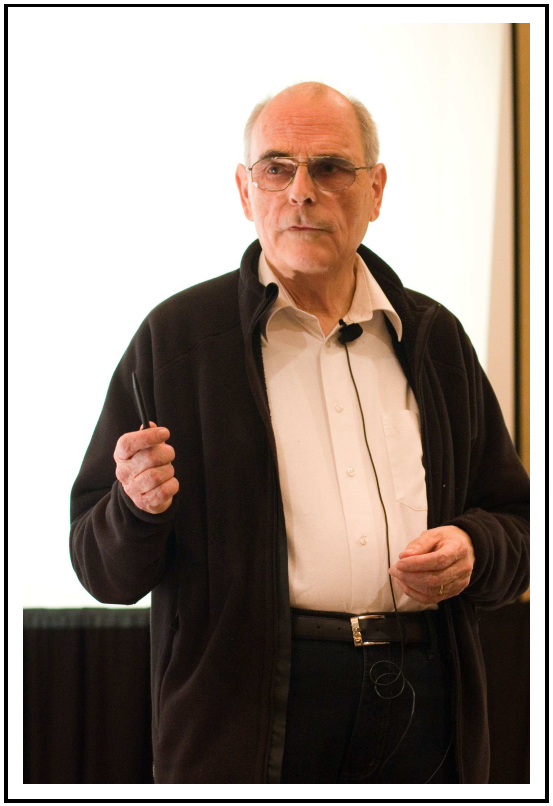
The preceding formula implies an $O(n^3)$ algorithm for matrix multiplication.

Volker Strassen



Volker Strassen (1936 –)

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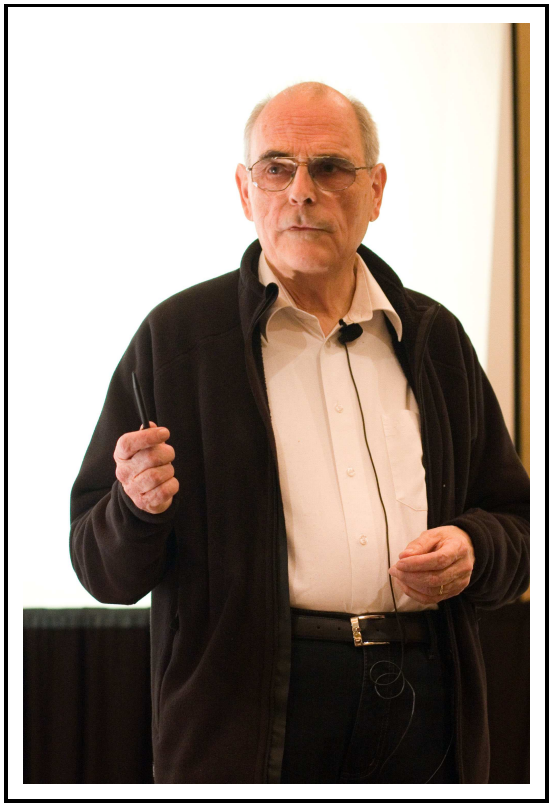

Volker Strassen (1936 –)

In 1969, the German mathematician **Volker Strassen** announced a surprising $O(n^{2.81})$ algorithm.

Divide and conquer

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The recurrence is

$$T(n) = 8T(n/2) + O(n^2)$$

with solution $O(n^3)$.

Strassen's trick

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$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

where

$$\begin{aligned} P_1 &= A(F-H) & P_5 &= (A+D)(E+H) \\ P_2 &= (A+B)H & P_6 &= (B-D)(G+H) \\ P_3 &= (C+D)E & P_7 &= (A-C)(E+F) \\ P_4 &= D(G-E) \end{aligned}$$

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The recurrence is

$$T(n) = 7T(n/2) + O(n^2)$$

with solution $O(n^{\log_2 7}) \approx O(n^{2.81})$.