

Linear Programming *

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* Special thanks is given to Mr. Xinyu Wu from EE2014@SJTU for producing this lecture.

Outline

1 Introduction

- An Introductory Example
- Standard Form of LP
- Other Programmings

2 Duality

- Primal and Dual Form
- Duality Theorem

3 Simplex Method

- Brief Overview
- Introductory Example
- Summarization and Further Topics

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An Example : Profit Maximization

Suppose that a factory produces two products **1** and **2**, using resources **A**, **B**, and **C**, under following settings :

- We have 200 units of **A**, 300 units of **B**, and 400 units of **C**.
- Making a unit of product **1** requires a unit of **A** and **C**.
- Making a unit of product **2** requires a unit of **B** and **C**.
- The price for product **1** and **2** are respectively 1 and 6.

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The factory aims to achieve the **maximum** profit, so how many units of product **1** and **2** the factory should produce ?

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According to the settings, we have

- 200 units **A**, one for each product **1** : $x_1 \leq 200$.
- 300 units **B**, one for each product **2** : $x_2 \leq 300$.
- 400 units **C**, one for both **1** and **2** : $x_1 + x_2 \leq 400$.
- Nonnegative Production : $x_1, x_2 \geq 0$.

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- Nonnegative Production : $x_1, x_2 \geq 0$.

Maximizing Profits : $\max f(x_1, x_2) = x_1 + 6x_2$.

Formulation of Linear Programming

$$\begin{array}{ll}\max & f(x_1, x_2) = x_1 + 6x_2 \\s.t. & x_1 + x_2 \leq 400, \\& x_1 \leq 200, \\& x_2 \leq 300, \\& x_1, x_2 \geq 0.\end{array}$$

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Linear Programming (LP) :

Both objective function and constraints are linear.

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Standard Form of LP

Given

- n real numbers c_1, c_2, \dots, c_n ;
- m real numbers b_1, b_2, \dots, b_m ;
- $m \times n$ real numbers $\{a_{ij}\}_{i=1,2,\dots,m; j=1,2,\dots,n}$.

We wish to find n real numbers x_1, x_2, \dots, x_n such that

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m \\ & x_j \geq 0, \quad j = 1, 2, \dots, n \end{aligned}$$

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Example :

$$\begin{aligned} \min \quad & -2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 7, \\ & x_1 - 2x_2 \leq 4, \\ & x_1 \geq 0. \end{aligned}$$

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However, we can equivalently transform different LP variants to the standard form.

Transformation to Standard Form

1. From min to max :

$$\min \sum_{j=1}^n c_j x_j \quad \Rightarrow \quad \max \quad - \sum_{j=1}^n c_j x_j$$

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Transformation to Standard Form

2. Equality Constraint :

$$\sum_{j=1}^n a_{ij}x_j = b_i \quad \Rightarrow \quad \begin{cases} \sum_{j=1}^n a_{ij}x_j \leq b_i \\ \sum_{j=1}^n a_{ij}x_j \geq b_i \end{cases}$$

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3. Inequality Constraint with \geq :

$$\sum_{j=1}^n a_{ij}x_j \geq b_i \quad \Rightarrow \quad -\sum_{j=1}^n a_{ij}x_j \leq -b_i$$

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4. Variables without Constraints :

x_2 is without constraints. \Rightarrow Introducing x_2^+ and x_2^-
 $x_2 = x_2^+ - x_2^-, x_2^+, x_2^- \geq 0.$

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$$\begin{array}{ll} \max & 2x_1 - 3x_2^+ + 3x_2^- \\ \text{s.t.} & x_1 + x_2^+ - x_2^- \leq 7, \\ & -x_1 - x_2^+ + x_2^- \leq -7, \\ & x_1 - 2x_2^+ + 2x_2^- \leq 4, \\ & x_1, x_2^+, x_2^- \geq 0. \end{array}$$

Standard Form

Slack Form of LP

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Key : Introducing **slack variables** : s_i

$$\sum_{j=1}^n a_{ij}x_j \leq b_i \quad \Rightarrow \quad \sum_{j=1}^n a_{ij}x_j + s_i = b_i, s_i \geq 0$$

Slack Form of LP

Example : Profit Maximization

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x_3, x_4, x_5 are slack variables.

Slack Form of LP

We can use a tuple $(N, B, \mathbf{A}, \mathbf{b}, \mathbf{c}, v)$ to represent a slack form.

- N : Nonbasic Variable Set. $\{x_1, x_2\}$
- B : Basic Variable Set. $\{x_3, x_4, x_5\}$
- $\mathbf{A}, \mathbf{b}, \mathbf{c}$: Constant Terms and Coefficients.
- v : Optional Constant Term in Objective Function.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{b} = [400 \quad 200 \quad 300]^T$$

$$\mathbf{c} = [1 \quad 6]^T$$

Matrix-Vector Form of LP

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If we create

- an $m \times n$ matrix $\mathbf{A} = (a_{ij})_{m \times n}$;
- an m vector $\mathbf{b} = (b_1, b_2, \dots, b_m)^T$;
- an n vector $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$;
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then we can equivalently transform the standard form as

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m \\ & x_j \geq 0, \quad j = 1, 2, \dots, n \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max \quad & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

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General Form of Programming

The **general form** of a programming is

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_i(\mathbf{x}) = 0. \quad i = m + 1, m + 2, \dots, n \end{aligned}$$

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Linear Programming should satisfy that $f(\mathbf{x})$, $\{g_i(\mathbf{x})\}$, $\{h_i(\mathbf{x})\}$ are all linear functions.

Classification of Programming

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- $\left\{ \begin{array}{l} \text{Programming **with** constraints ; (We only study this.)} \\ \text{Programming **without** constraints.} \end{array} \right.$
- $\left\{ \begin{array}{l} \text{Linear programming ;} \\ \text{Nonlinear programming. (including quadratic)} \end{array} \right.$

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- { Programming **with** constraints ; (We only study this.)
Programming **without** constraints.
- { **Linear** programming ;
Nonlinear programming. (including **quadratic**)
- { **Single-objective** programming ; (We only study this.)
Multiple-objective programming.
- ...

Integer Linear Programming (ILP)

ILP is an LP problem with an additional constraint that variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ must take on integral values.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m \\ & x_j \in \mathbb{Z}. \quad j = 1, 2, \dots, n. \end{aligned}$$

Examples : the amount of products, people, data packets,...

Note : ILP is an **NP** problem, which no efficient algorithms can solve *directly*.

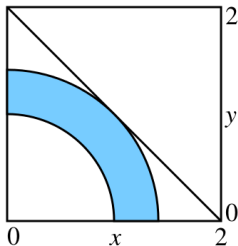
Nonlinear Programming

Nonlinear Programming : At least one of f_i , g_i , and h_i is nonlinear.

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \dots, m \\ & h_i(\mathbf{x}) = 0. \quad i = m + 1, m + 2, \dots, n. \end{aligned}$$

Example :

$$\begin{aligned} \max \quad & x + y \\ \text{s.t.} \quad & x^2 + y^2 \geq 1, \\ & x^2 + y^2 \leq 2, \\ & x, y \geq 0. \end{aligned}$$



Quadratic Programming

Quadratic Programming (QP) is a special case of nonlinear programming in the form of

$$\begin{aligned} \max \quad & \frac{1}{2} \mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A} \mathbf{x} \leq \mathbf{b} \end{aligned}$$

where

- \mathbf{Q} : an $n \times n$ real symmetric matrix ;
- \mathbf{c} : an n vector ;
- \mathbf{A} : an $m \times n$ real matrix ;
- \mathbf{b} : an m vector.

Brief History of LP

- First formal application to problems in economics by Leonid Kantorovich in the 1930s, however the work was ignored ;
- Rediscovered by Tjalling Koopmans in the 1940s, along with applications to economics ;
- First algorithm (**Simplex Algorithm**) to solve linear programs by George Dantzig in 1947 ;
- Kantorovich and Koopmans receive Nobel Prize for economics in 1975 ; Dantzig, however, was ignored.
- LP was first shown to be solvable in polynomial time via **Ellipsoid Method** by Leonid Khachiyan in 1979, but a larger breakthrough came in 1984 when Narendra Karmarkar introduced a novel **Interior-Point Method** to solve LP.

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Duality

Recall the **Profit Maximization** problem :

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$$x_1 \leq 200,$$

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$$x_1, x_2 \geq 0.$$

Try to find the optimum.

(x_1, x_2)	$f(x_1, x_2)$
(100, 200)	1300
(200, 200)	1400
(100, 300)	1900

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However, we do NOT know whether (100, 300) is exactly the optimal solution.

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(x_1, x_2)	$f(x_1, x_2)$
(100, 200)	1300
(200, 200)	1400
(100, 300)	1900

However, we do NOT know whether (100, 300) is exactly the optimal solution.

Duality enables us to prove that a solution is indeed optimal.

Intuitive Way to Show Optimality

Whether $(100, 300)$ is optimal ? (The objective value is 1900)

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Multiplier	Constraint
y_1	$x_1 + x_2 \leq 400$
y_2	$x_1 \leq 200$
y_3	$x_2 \leq 300$

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When $(y_1, y_2, y_3) = (0, 1, 6)$, $x_1 + 6x_2 \leq 2000$;

Intuitive Way to Show Optimality

Whether $(100, 300)$ is optimal ? (The objective value is **1900**)

Upper bounding $x_1 + 6x_2$ by **linearly combining** constraints.

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y_2	$x_1 \leq 200$
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When $(y_1, y_2, y_3) = (0, 1, 6)$, $x_1 + 6x_2 \leq 2000$;

When $(y_1, y_2, y_3) = (1, 0, 5)$, $x_1 + 6x_2 \leq \mathbf{1900}$! **Optimal** !

Intuitive Way to Show Optimality

Whether $(100, 300)$ is optimal ? (The objective value is **1900**)

Upper bounding $x_1 + 6x_2$ by **linearly combining** constraints.

$$\max \quad x_1 + 6x_2$$

$$s.t. \quad x_1 + x_2 \leq 400,$$

$$x_1 \leq 200,$$

$$x_2 \leq 300,$$

$$x_1, x_2 \geq 0.$$

Multiplier	Constraint
y_1	$x_1 + x_2 \leq 400$
y_2	$x_1 \leq 200$
y_3	$x_2 \leq 300$

When $(y_1, y_2, y_3) = (0, 1, 6)$, $x_1 + 6x_2 \leq 2000$;

When $(y_1, y_2, y_3) = (1, 0, 5)$, $x_1 + 6x_2 \leq \mathbf{1900 ! Optimal !}$

However, it is only a coincidence...

Rational Way to Show Optimality

$(y_1, y_2, y_3) = (1, 0, 5)$ serves as a certificate of the optimality, but how can we find it rationally ?

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Multiply horizontally and add vertically, we obtain

$$(y_1 + y_2)x_1 + (y_1 + y_3)x_2 \leq 400y_1 + 200y_2 + 300y_3.$$

$y_1, y_2, y_3 \geq 0$ to ensure no flipping from \leq to \geq in constraints.

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We want the left-hand side to look like our objective function $x_1 + 6x_2$, so that the right-hand side becomes an upper bound on the objective function.

$$x_1 + 6x_2 \leq 400y_1 + 200y_2 + 300y_3 \text{ if } \begin{cases} y_1, y_2, y_3 \geq 0 \\ y_1 + y_2 \geq 1 \\ y_1 + y_3 \geq 6 \end{cases}$$

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We should minimize $400y_1 + 200y_2 + 300y_3$ to get the tightest upper bound of $x_1 + 6x_2$. **A new LP problem !**

Dual LP

The new LP problem :

$$\begin{array}{ll}\min & 400y_1 + 200y_2 + 300y_3 \\s.t. & y_1 + y_2 \geq 1, \\ & y_1 + y_3 \geq 6, \\ & y_1, y_2, y_3 \geq 0.\end{array}$$

We call it the **dual form** of the original LP problem.

Primal and Dual Form

Correspondingly, we call the original LP problem as **primal form**.

$$\begin{array}{ll} \max & x_1 + 6x_2 \\ \text{s.t.} & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & 400y_1 + 200y_2 + 300y_3 \\ \text{s.t.} & y_1 + y_2 \geq 1, \\ & y_1 + y_3 \geq 6, \\ & y_1, y_2, y_3 \geq 0. \end{array}$$

Primal Form

Dual Form

Primal and Dual Form

Generally we have the primal and dual form of LP as

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ s.t. & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i \\ & x_j \geq 0, \quad \forall j \end{array} \quad \Rightarrow \quad \begin{array}{ll} \min & \sum_{i=1}^m b_i y_i \\ s.t. & \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j \\ & y_i \geq 0, \quad \forall i \end{array}$$

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It is obvious that the dual of dual form is the primal form.

Matrix-Vector Form

More generally, we can write primal and dual form in matrices and vectors.

$$\max \quad \mathbf{c}^T \mathbf{x}$$

$$s.t. \quad \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}.$$

Primal Form

\Rightarrow

$$\min \quad \mathbf{y}^T \mathbf{b}$$

$$s.t. \quad \mathbf{y}^T \mathbf{A} \geq \mathbf{c}^T, \\ \mathbf{y} \geq \mathbf{0}.$$

Dual Form

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- Other Programmings

2 Duality

- Primal and Dual Form
- **Duality Theorem**

3 Simplex Method

- Brief Overview
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Observations of Duality

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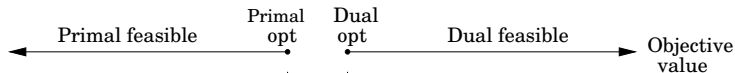
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If we find a pair of primal and dual feasible objective values that are equal, then they must both be optimal.

One such pair, making the objective value both 1900, is :

Primal : $(x_1, x_2) = (100, 300)$; **Dual** : $(y_1, y_2, y_3) = (1, 0, 5)$.

Duality Theorem



Theorem (Weak Duality Theorem)

Let x be any *feasible* solution to the primal LP, and let y be any *feasible* solution to its dual LP. Then $\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i y_i$.

Theorem (Strong Duality Theorem)

x and y are *optimal* solutions to primal and dual LPs respectively if and only if $\sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$.

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Proof.

$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \leq \sum_{i=1}^m b_i y_i.$$

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“ \Leftarrow ” : By **Weak Duality Theorem**, $\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i y_i$. The primal LP is a maximization problem and the dual LP is a minimization problem. Thus, if feasible solutions x and y have the same objective value, neither can be improved.

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“ \Rightarrow ” : It involves using Simplex Method, which contains complex mathematical derivations. We omit here.

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Recall : Profit Maximization–2D Case

Products : **1**, **2**;

Resources : **A**, **B**, **C**.

- 200 **A** ; 300 **B** ; 400 **C**.
- Product **1** = 1**A** + 1**C**.
- Product **2** = 1**B** + 1**C**.
- Price(**1**)=1, Price(**2**)=6.

Goal : Maximum profit.

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Optimal Solution :

$$(x_1, x_2) = (100, 300)$$

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Where is the optimum in
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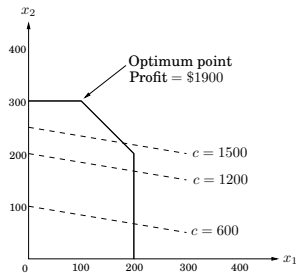
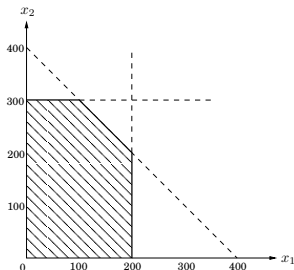
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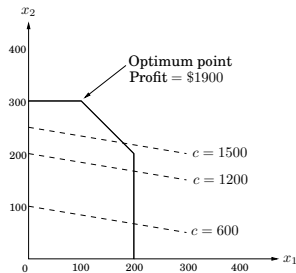
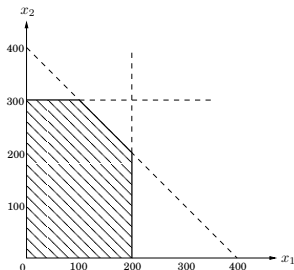
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On the Vertex !

3D Case

Problem :

$$\begin{aligned} \max \quad & x_1 + 6x_2 + 13x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 400, \\ & x_2 + 3x_3 \leq 600, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Where is the optimum ?

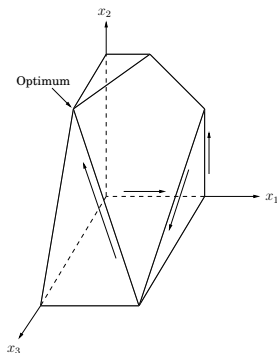
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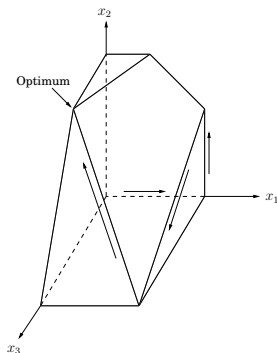
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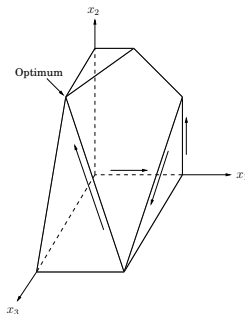
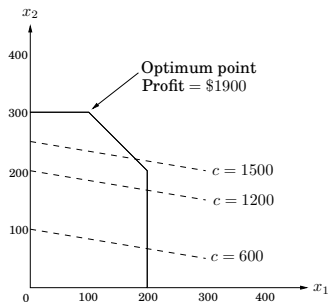
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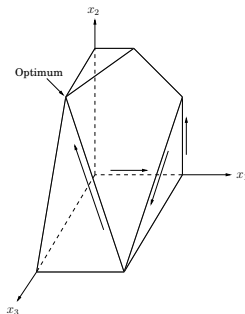
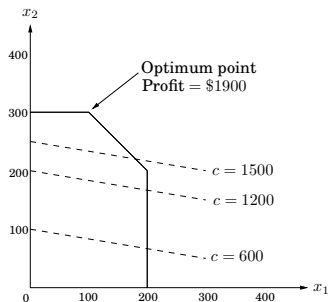
How to Find Optimum ?

Based on 2D and 3D situation, how can we find optimum of LP ?



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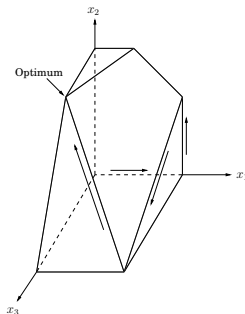
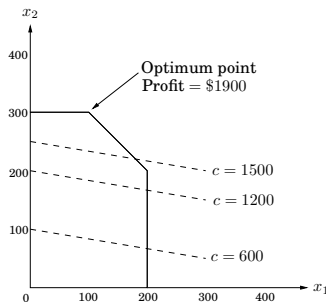
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- Move on the boundary ;
- Find the vertex with largest (smallest) objective value.

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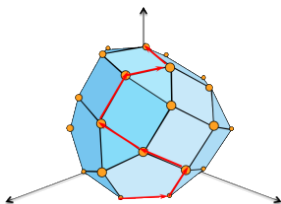
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Simplex Algorithm : For LP with arbitrary n variables.

General Description of Simplex Algorithm

Definitions :

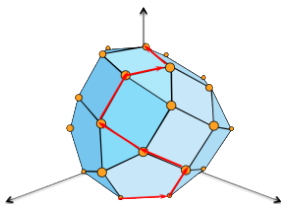
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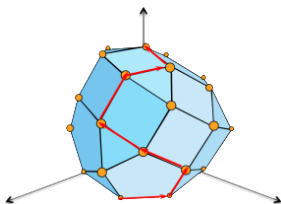
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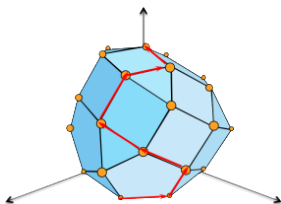
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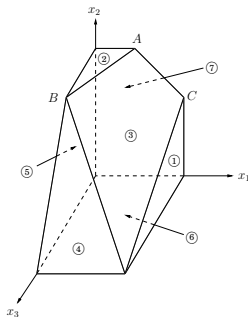
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- Two vertices are **neighbors** if they are adjacent on the polyhedron.

Observation : The optimal solution of LP exists on the **vertex** of the feasible region.

General Description of Simplex Algorithm

Example :

- A **polyhedron** defined by 7 inequalities (thus 7 **hyperplanes**).
- **Vertices** : A, B, C, \dots
- **Neighbors** : $\{A, B\}, \{A, C\}, \dots$



$$\begin{array}{ll}\max & x_1 + 6x_2 + 13x_3 \\s.t. & x_1 \leq 200, \quad \textcircled{1} \\& x_2 \leq 300, \quad \textcircled{2} \\& x_1 + x_2 + x_3 \leq 400, \quad \textcircled{3} \\& x_2 + 3x_3 \leq 600, \quad \textcircled{4} \\& x_1 \geq 0, \quad \textcircled{5} \\& x_2 \geq 0, \quad \textcircled{6} \\& x_3 \geq 0. \quad \textcircled{7}\end{array}$$

General Description of Simplex Algorithm

On each iteration, the Simplex Algorithm will do :

- Check whether the current vertex is optimal (if so, halt).
- Determine where to move next. → The one contributes to the increase of objective function.

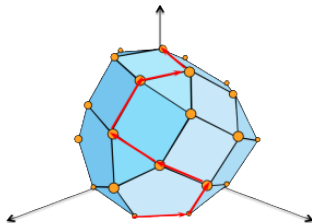
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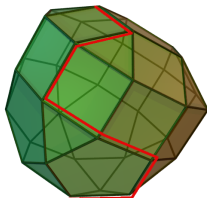
Initial Idea :

- Start at a vertex.
- Compare objective value with the neighbors.
- Move to neighbor that improves objective function, and repeat step 2.
- If no improving neighbor, stop.

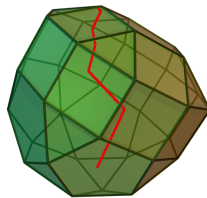


Brief Overview of Algorithms for LP

- **Simplex Algorithm**
 - ▷ Efficient in practice, but exponential in worst case.
- Interior Point Algorithms
 - ▷ Ellipsoid Algorithm $O(n^4 L)^\dagger$
 - ▷ Karmarkar's Algorithm $O(n^{3.5} L)$
 - ▷ Path-Following Method (Barrier Function Method)



Simplex (Boundary)



Interior Point (Inside)

$^\dagger L$ is bit length.

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- 3 **Simplex Method**
 - Brief Overview
 - **Introductory Example**
 - Summarization and Further Topics

An Introductory Example

How to perform simplex algorithm on a specific LP problem ?

Example : Profit Maximization

$$\begin{aligned} \max \quad & f(x_1, x_2) = x_1 + 6x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{aligned}$$

An Introductory Example

Step 1 : Converting LP into slack form

$$\begin{array}{ll} \max & x_1 + 6x_2 \\ \text{s.t.} & x_1 + x_2 \leq 400, \\ & x_1 \leq 200, \\ & x_2 \leq 300, \\ & x_1, x_2 \geq 0. \end{array} \quad \Rightarrow$$

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x_3, x_4, x_5 are slack variables.

An Introductory Example

Step 2 : Obtaining Basic Solution

$$\max \quad x_1 + 6x_2$$

$$s.t. \quad 400 - x_1 - x_2 = x_3,$$

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Define :

- **nonbasic variables** : x_1, x_2
(in the objective)
- **basic variables** : x_3, x_4, x_5
(others in the constraints)

An Introductory Example

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There are infinite number of feasible solutions for x_1, \dots, x_5 . The **basic solution** is obtained by setting all nonbasic variables to be 0.

An Introductory Example

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There are infinite number of feasible solutions for x_1, \dots, x_5 . The **basic solution** is obtained by setting all nonbasic variables to be 0.

In this example, the basic solution is

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_5) = (0, 0, 400, 200, 300)$$

An Introductory Example

Step 3 : Selecting Nonbasic Variable

Our goal, in each iteration, is to reformulate the linear program so that the basic solution gives a greater value of objective function.

An Introductory Example

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Currently for the basic solution $\bar{x} = (0, 0, 400, 200, 300)$, we have

$$f(\bar{x}_1, \bar{x}_2) = \bar{x}_1 + 6\bar{x}_2 = 0.$$

How to enhance $f(\bar{x}_1, \bar{x}_2)$?

An Introductory Example

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$$f(\bar{x}_1, \bar{x}_2) = \bar{x}_1 + 6\bar{x}_2 = 0.$$

How to enhance $f(\bar{x}_1, \bar{x}_2)$?

- Select a **nonbasic** x_e with **positive** coefficient in $f(x_1, x_2)$;
- Increase the value of x_e without violating constraints.

An Introductory Example

Step 3 : Selecting Nonbasic Variable

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An Introductory Example

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- Choose the nonbasic variable x_2 .

⇓

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\Downarrow

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○ Choose the nonbasic variable x_2 .

○ When $x_2 \uparrow$, $x_3 \downarrow$ and $x_5 \downarrow$.

However x_3 and x_5 should be nonnegative.

▷ $x_3 \leq 0$ when $x_2 \geq 400$;

▷ $x_5 \leq 0$ when $x_2 \geq 300$.

An Introductory Example

Step 3 : Selecting Nonbasic Variable

$$\begin{array}{ll}\max & x_1 + 6x_2 \\ \text{s.t.} & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - x_2 = x_5, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0.\end{array}$$

$$\begin{array}{ll}\max & x_1 + 6x_2 \\ \text{s.t.} & 400 - x_1 - x_2 = x_3, \\ & 200 - x_1 = x_4, \\ & 300 - \textcolor{red}{x_5} = \textcolor{red}{x_2}, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0.\end{array}$$

- Choose the nonbasic variable x_2 .

- When $x_2 \uparrow$, $x_3 \downarrow$ and $x_5 \downarrow$.

However x_3 and x_5 should be nonnegative.

- $x_3 \leq 0$ when $x_2 \geq 400$;
- $x_5 \leq 0$ when $x_2 \geq 300$.

- $300 - x_2 = x_5$ is the tightest constraint for x_2 .

We transform it into

$$300 - x_5 = x_2.$$

An Introductory Example

Step 4 : Pivoting

In **pivoting**, we **exchange** a nonbasic and a basic variable.

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$$\bar{x} = \{0, 0, 400, 200, 300\} \quad \Rightarrow \quad \bar{x} = \{0, 300, 100, 200, 0\}.$$

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$$\bar{x} = \{0, 0, 400, 200, 300\} \quad \Rightarrow \quad \bar{x} = \{0, 300, 100, 200, 0\}.$$

- The nonbasic variables become x_1 and x_5 .
- In next round we select a new nonbasic variable with positive coefficient. (x_1 in this example).

An Introductory Example

Step 5 : Repeat Step 2 to Step 4

- Repeat pivoting until all coefficients in the objective are **negative**.

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$$\begin{aligned} \max \quad & \boxed{1900 - x_3 - 5x_5} \\ \text{s.t.} \quad & 100 - x_3 + x_5 = x_1, \\ & 200 - x_1 = x_4, \\ & 300 - x_5 = x_2, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

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$$\bar{x} = \{0, 300, 100, 200, 0\} \quad \Rightarrow \quad \bar{x} = \{100, 300, 0, 100, 0\}.$$

An Introductory Example

Step 5 : Repeat Step 2 to Step 4

- We can prove that when all coefficients are negative, the basic solution \bar{x} is an optimal solution.

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- The maximum of the objective function is 1900, when $x_3 = x_5 = 0$;

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- The maximum of the objective function is 1900, when $x_3 = x_5 = 0$;
- $\bar{x} = \{100, 300, 0, 100, 0\}$ is the optimal solution ;

The optimal solution for original problem $f(x_1, x_2)$ is :

$$(x_1^*, x_2^*) = (100, 300)$$

Outline

- 1 Introduction
 - An Introductory Example
 - Standard Form of LP
 - Other Programmings
- 2 Duality
 - Primal and Dual Form
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- 3 Simplex Method
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 - Summarization and Further Topics

Summarization and Further Topics

Summarization :

- Simplex Algorithm searches the optimal **vertex** on the **boundary of feasible region** ;
- Simplex Algorithm iteratively **exchanges the nonbasic and basic variables** until the objective function cannot be further enhanced.

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- Simplex Algorithm searches the optimal **vertex** on the **boundary of feasible region** ;
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Further Topics :

- How to implement Simplex Algorithm ?
 - ▷ How to do the **pivoting** ?
 - ▷ How to find an initial **basic solution** ?
- Can Simplex Algorithm always find the optimal solution ?
- What will happen when the feasible region is unbounded ?

Please refer to Chapter 29.3 and 29.5 in “Introduction to Algorithms” (CLRS) for details.

Tools for Solving LP

Common Softwares and Toolboxes for solving LP :



MATLAB : Toolboxes



Mathematica : Toolboxes



Lingo : Large-scale LP (ILP)



Cplex : Large-scale ILP (Mixed ILP)



Gurobi : Similar to but better than Cplex synthetically.



Yalmip : Toolbox used by Matlab.