Lab03-Greedy Strategy

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1. Suppose there is a street with length n, described by an array $A[1...n]$ where $A[i] = 1$ means that there is a house at position i and $A[i] = 0$ means position i is vacant.

   According to some law, every house must be protected by fire hydrant. If a fire hydrant is placed at position i, then all houses at postion $i-1, i, i+1$ will be considered protected. Note that hydrants can be placed at the same place with a house.

   Using what you learnt in class, please design an algorithm that computes the minimum number of hydrants needed to protect all houses. You need to write pseudo code, analyze the time complexity, and prove its correctness.

Proof. Idea: The greedy strategy starts from the left side. When the position is vacant, just check next position. When there is a house at the position, put hydrant at next position to cover houses as many as possible and jump out of the scope of the hydrant and continue to check.

Algorithm:

Algorithm 1: Hydrant

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input  : An array A[1...n] of n elements.
output: The minimum number of hydrants num
i ← 1; num ← 0;
while i ≤ n do
    if A[i] == 1 then
        num ← num + 1;
        i ← i + 3;
    else
        i ← i + 1;
```

Complexity: Since we only have one **While** loop iterating no more than n times and the operations inside loop is $O(1)$, the time complexity of the algorithm is $O(n)$.

Correctness Proof:

Suppose the greedy algorithm(Ag) isn't optimal.

Define A* to be the optimal schedule which have the longest same arrangement with Ag from the beginning. Assume A* and Ag have different arrangement in position i for the first time. Thus, there are two cases:

Case 1: Ag place a fire hydrant at position i, but A* not.

According to the Ag, placing a fire hydrant at position i means that there is a house at position $i-1$ which can't be protected by previous fire hydrants. So is it in A*. Not placing a fire hydrant at position i will lead to an unprotected house in A*. Thus, this case won't happen.

Case 2: A* place a fire hydrant at position i, but Ag not.
According to the Ag, not placing a fire hydrant at position i means that positions before i have been protected by previous fire hydrants. Thus, A* can move this fire hydrant to where Ag place next fire hydrant to get a new arrangement, say S. It's obvious that S is also optimal. But S has longer same arrangement with Ag from the beginning than A*, which is inconsistent with our assumption.

Thus, the greedy algorithm is optimal. □

2. (a) Given a set A containing n real numbers, and you are allowed to choose k numbers from A. The bigger the sum of the chosen numbers is, the better. What is your algorithm to choose? Prove its correctness using **Matroid**.

   Remark: This is a very easy problem. Denote $\mathbf{C}$ be the collection of all subsets of A that contains no more than k elements. Try to prove $(A, \mathbf{C})$ is a matroid.

   Proof. Idea: The greedy strategy first sorts numbers in A and then chooses the largest number of the remaining numbers every time. That is, it chooses the top k largest numbers.

   Matroid Proof:
   Denote $\mathbf{C}$ be the collection of all subsets of A that contains no more than k elements.
   Hereditary: Assume $D \subset B$ and $B \in \mathbf{C}$. B contains no more than k elements and $|D| < |B|$. Thus, D contains no more than k elements. Since $\mathbf{C}$ is the collection of all subsets of A that contains no more than k elements, we have $D \in \mathbf{C}$.
   Exchange Property: Assume $B, D \in \mathbf{C}$ and $|B| > |D|$. For any $x \in B/D$, we have $|D \cup \{x\}| \leq |B|$. That is, $D \cup \{x\}$ contains no more than k elements. Thus $D \cup \{x\} \in \mathbf{C}$.

   $(A, \mathbf{C})$ is a matroid and our algorithm is a Greedy-MAX algorithm. As we have proved in the slides, Greedy-MAX algorithm performs the optimal solution. Thus, our algorithm is optimal.

   □

   (b) Consider that $B_1, B_2 ... B_n$ are n disjoint sets, and let d_i be integers with $0 \leq d_i \leq |B_i|$. Define $\mathbf{C}$ is a collection of set $X \subseteq \cup_{i=1}^{n} B_i$, where X has such property:

   $$\forall i \in \{1, 2, 3 ... n\}, |X \cap B_i| \leq d_i$$

   Prove that $(\cup_{i=1}^{n} B_i, \mathbf{C})$ is a matroid.
   Remark: You may easily find that the matroid in (a) is a special case of matroid in (b).

   Proof. Hereditary: Assume that $D \subset A$ and $A \in \mathbf{C}$. Thus, $|A| > |D|$ and A has all the elements D has. Since $A \subseteq \cup_{i=1}^{n} B_i$ and $\forall i \in \{1, 2, 3 ... n\}, |A \cap B_i| \leq d_i$, D also has $D \subseteq \cup_{i=1}^{n} B_i$ and $\forall i \in \{1, 2, 3 ... n\}, |D \cap B_i| \leq d_i$. Thus we have $D \in \mathbf{C}$.

   Exchange Property: Assume $A, D \in \mathbf{C}$ and $|A| > |D|$. Since $A, D \subseteq \cup_{i=1}^{n} B_i$, for any $x \in A/D$, it's obvious that $x \in \cup_{i=1}^{n} B_i$. Thus we have $D \cup \{x\} \subseteq \cup_{i=1}^{n} B_i$.

   We only need to consider those B_i which intersect with A and D for empty intersect set always meets the condition. Suppose $\mathbf{M} = \{B_i | \forall i \in \{1, 2, 3 ... n\}, A \cap B_i \neq \emptyset\}$ and $\mathbf{N} = \{B_i | \forall i \in \{1, 2, 3 ... n\}, D \cap B_i \neq \emptyset\}$. There are two cases:

Case 1: $\mathbf{M} \not\subset \mathbf{N}$
For any $B_j \in \mathbf{M}/\mathbf{N}$, we can choose a $x \in B_j$. Since $x \in A \cap B_j$, we have $d_j \geq 1$, and then $D \cap B_j = 1 \leq d_j$. For all $B_i \in \mathbf{N}$, it's obvious that $(D \cup \{x\}) \cap B_i = D \cap B_i$, and then $|(D \cup \{x\}) \cap B_i| \leq d_i$. Thus, we have $D \cup \{x\} \in \mathbf{C}$
Case 2: $\mathbf{M} \subseteq \mathbf{N}$
If for all $B_i \in \mathbf{N}$, $|D \cap B_i| = d_i$, we have $|D| = \sum_{B_i \in \mathbf{N}} d_i$. Since $|A| > |D|$ and $A \in \mathbf{C}$, we have $|A| > \sum_{B_i \in \mathbf{N}} d_i$ and $|A| \leq \sum_{B_i \in \mathbf{M}} d_i$, which contradicts with $\mathbf{M} \subseteq \mathbf{N}$. Thus, there must exist a $B_k \in \mathbf{N}$, $|D \cap B_k| < |A \cap B_k| \leq d_k$.
Let $x \in (A \cap B_k)/(D \cap B_k)$. For B_k, we have $|(D \cup \{x\}) \cap B_k| \leq |A \cap B_k| \leq d_k$. For all $B_i \in \mathbf{N}$ and $B_i \neq B_k$, $|(D \cup \{x\}) \cap B_i| = |D \cap B_i| \leq d_i$. Thus, $D \cup \{x\} \in \mathbf{C}$

Therefore, $(\cup_{i=1}^{n} B_i, \mathbf{C})$ is a matroid. $\square$

Remark: You need to include your .pdf and .tex files in your uploaded .rar or .zip file.