# Mathematical Foundations of Computer Science

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## Group: navigator



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# 3 Tossing Coins

Let us toss a coin infinitely often, producing a bit string $x = x_1x_2x_3\ldots$. For a finite bit string $z \in \{0,1\}^*$, let $T_z$ denote the number of tosses until $z$ appears the first time. For example, if we toss 0010110, then $T_{10} = 4$ and $T_{110} = 7$.

**Exercise 3.1.** Consider an unbiased coin, i.e., 0 and 1 come up with probability $p = 1/2$ each.

1. Give an explicit[1] formula for $\Pr[T_{10} = n]$.
2. Give an explicit formula for $\Pr[T_{11} = n]$.
3. Compute $\mathbb{E}[T_{11}]$ and $\mathbb{E}[T_{10}]$. You can choose any of the three proof methods above (but two of them won't be fun; in particular, using Point 1 and Point 2 of this exercise will not help you so much).

**Solution.**

1. Consider a string that does not contain any "10" in first $n$ tosses. Let us denote the string as $B_n$, and analyse the form it should take.

    - If $B_n$ starts with a '0', then the next bit of the string can be either a '0' or '1'. In other words, the following $n-1$ bits can be any specific element in $B_{n-1}$.
    - If $B_n$ starts with a '1', then the next bit can only be '1' as $B_n$ does not contain "10". Like wise, the second, third, ... bit can only be '1' as well. The only possible form for $B_n$ starting with '1' is a string that only contains '1'.

    From the two cases we can conclude that

    $$B_n = \{\{0,1\}^n | 0x, x \in B_{n-1} \text{ or } 111\cdots1\},$$

    and derive that

    $$|B_n| = |B_{n-1}| + 1.$$

[1] By *explicit* in this context I mean something not containing a recurrence; not containing $\sum$ or $\prod$. You may, however, use stuff we have used before, like $\binom{n}{k}$, Fibonacci numbers $F_n$, Catalan numbers $C_n$.

We have $|B_1| = 2$ and $|B_2| = 3$, from which we can easily get that $|B_n| = n+1$.

Back to our problem, the event $T_{10} = n$ refers to such string $S_n$ that

$$S_n = \{\{0,1\}^n | y10, y \in B_{n-2}\},$$

for which we have

$$|S_n| = |B_{n-2}| = n-1.$$

Therefore we have

$$\Pr[T_{10} = n] = \frac{n-1}{2^n}.$$

2. The event $T_{11} = n$ refers to such string $S_n$ that

$$S_n = \{\{0,1\}^n | z011, z \in A_{n-3}\},$$

for which we have

$$|S_n| = |A_{n-3}| = F_{n-1}.$$

Therefore we have

$$Pr[T_{11} = n] = \frac{F_{n-1}}{2^n}.$$

3. Since you have just implied the first two points won't be fun, let us just try with the automaton, which is shown in Figure 1. For every transmitting line, the corresponding probability is $p = 1/2$ as this is an unbiased coin.

For solving $\mathbb{E}[T_{10}]$, see Figure 1(a), from which we can derive

$$\begin{aligned}
\mathbb{E}[T_{10}] = \mathbb{E}[a] &= 1 + \frac{1}{2}\mathbb{E}[a] + \frac{1}{2}\mathbb{E}[b], \\
\mathbb{E}[b] &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2}\mathbb{E}[c], \\
\mathbb{E}[c] &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2}\mathbb{E}[c],
\end{aligned}$$

where state $a$ can been seen as the initial state, and $\mathbb{E}[a]$ means the expectation of number of tosses from state $a$ till the pattern "10" appears. State $d$ is the final state, therefore having $\mathbb{E}[d] = 0$. By solving these equations, we get $\mathbb{E}[T_{10}] = 4$.

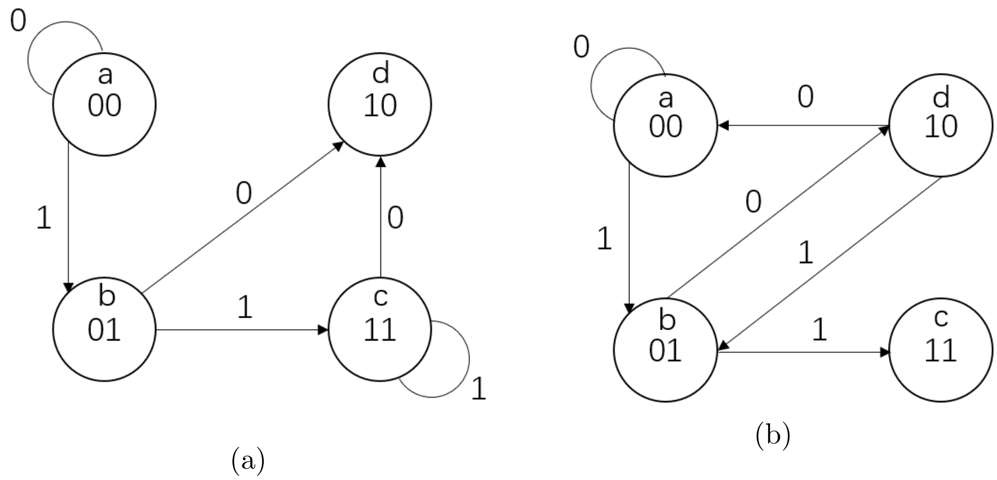



Figure 1: Automatons for two different finishing states

For solving $\mathbb{E}[T_{11}]$, see Figure 1(b), from which we can derive

$$\begin{aligned}
\mathbb{E}[T_{11}] = \mathbb{E}[a] &= 1 + \frac{1}{2}\mathbb{E}[a] + \frac{1}{2}\mathbb{E}[b], \\
\mathbb{E}[b] &= 1 + \frac{1}{2} \cdot 0 + \frac{1}{2}\mathbb{E}[d], \\
\mathbb{E}[d] &= 1 + \frac{1}{2}\mathbb{E}[a] + \frac{1}{2}\mathbb{E}[b],
\end{aligned}$$

where state $a$ can been seen as the initial state, and $\mathbb{E}[a]$ means the expectation of number of tosses from state $a$ till the pattern "11" appears. State $c$ is the final state, therefore having $\mathbb{E}[c] = 0$. By solving these equations, we get $\mathbb{E}[T_{11}] = 6$.

$\square$

You might have noticed that $\mathbb{E}[T_{10}] < \mathbb{E}[T_{11}]$, i.e., 10 appears earlier than 11, on average.

**Exercise 3.2.** Let $\mathcal{E}$ be the event that 10 appears earlier than 11. What is $\Pr[\mathcal{E}]$?

**Solution.** The 4 possible status and there conversion relationship are shown as the following picture. To reach the status *11* or *10*, we should first reach status *01*. From the picture we can know that the possibility from status *01*

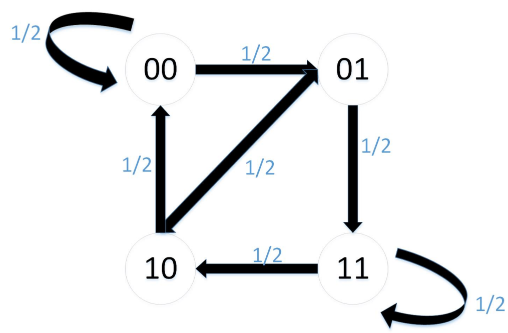



Figure 2: Recurrence Tree

to status *11* and *10* are equal, which means that the $Pr[10\, appears\, earlier] = Pr[11\, appears\, earlier] = \frac{1}{2}$. □

**Exercise 3.3.** Toss two coins repeatedly, producing two sequences $x_1x_2\dots$ and $y_1y_2\dots$. We stop once we see 10 in the first sequence or 11 in the second sequence. Formally, we toss the two coins

$$T := \min(T_{10}(x), T_{11}(y))$$

times.

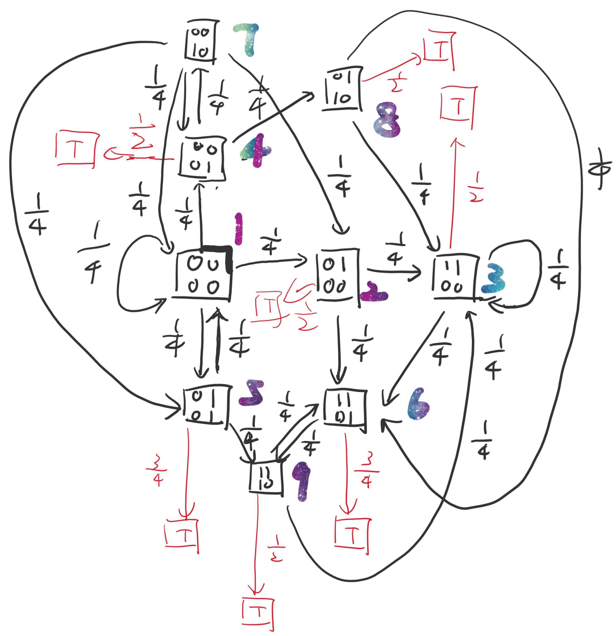
1. What is $\mathbb{E}[T]$?

2. What are the probabilities that 10 appears in $x$ (a) before 11 appears in $y$, (b) at the same time as 11, (c) later than 11?

**Hint.** I think the only way to solve this exercise without going crazy is by applying proof method 3 from above, drawing a small automaton; or in this case, not so extremely small anymore.

**Solution.**

1. Draw an automaton as shown. Every state is $2 \times 2$, of which the two rows respectively denote the last two bits of the sequences $x_1x_2\dots$ and $y_1y_2\dots$. The state T denotes either of the two sequences reaches termination condition, i.e. 10 appears in $x$ sequence or 11 appears in $y$ sequence. Name these states from $s_1, s_2, \cdots, s_9$.



Then we can create a $9 \times 9$ Markov matrix $A^{(9)}$ to denote this automaton.

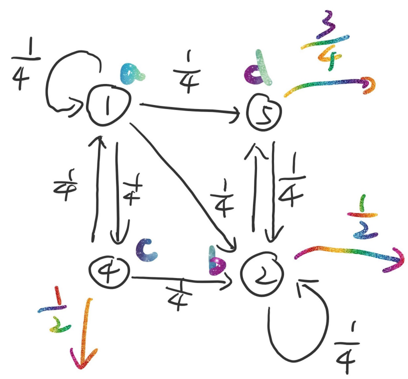
$$A^{(9)} = \frac{1}{4} \times \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \!\!\begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1 & 1 & & 1 & 1 & & & & \\ & & 1 & & & 1 & & & \\ & & 1 & & & 1 & & & \\ & & & & & & 1 & 1 & \\ & & & & & & & & 1 \\ & & & & & & & & 1 \\ 1 & 1 & & 1 & 1 & & & & \\ & & 1 & & & 1 & & & \\ & & 1 & & & 1 & & & \end{pmatrix} \end{array}$$

$$a_{ij} = \begin{cases} 0 & P_r(s_i \to s_j) = 0 \\ 1 & P_r(s_i \to s_j) = \frac{1}{4} \end{cases}$$

Observation: some of the rows are just the same e.g. $r_1 \& r_7$, thus they can be merged. Then we have a merged matrix $B^{(4)}$.

$$B = \frac{1}{4} \times \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ 1 & 1 & & \\ & 1 & & \end{pmatrix} \end{array}$$

According to matrix $B$, we can create a simplified automaton as shown.




Let $e_i$ be $\mathbb{E}(s_i \to T)$, then we have the equation set:

$$\begin{cases} e_1 & = 1 + \frac{1}{4}e_1 + \frac{1}{4}e_2 + \frac{1}{4}e_4 + \frac{1}{4}e_5 \\ e_2 & = 1 + \frac{1}{4}e_2 + \frac{1}{4}e_5 \\ e_4 & = 1 + \frac{1}{4}e_1 + \frac{1}{4}e_2 \\ e_5 & = 1 + \frac{1}{4}e_2 \end{cases}$$

which can be expressed in the form of matrix:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_4 \\ e_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4} \times \begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & & 1 \\ 1 & 1 & & \\ & 1 & & \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_4 \\ e_5 \end{pmatrix}$$

The solution:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_4 \\ e_5 \end{pmatrix} = \frac{1}{121} \times \begin{pmatrix} 384 \\ 220 \\ 272 \\ 176 \end{pmatrix}$$

Finally, let us calculate $\mathbb{E}(T)$. Toss coin $x$ and coin $y$ for the first time, and you will get four situations: $(x_1 = 0, y_1 = 0), (0,1), (1,0), (1,1)$. These 4 sitations are equivalent to $s_1, s_2, s_4, s_5$ because what really matters in one state is the very last bit.

$$\mathbb{E}(T) = 1 + \frac{e_1 + e_2 + e_4 + e_5}{4} = \frac{384}{121}$$

2. The automaton will finally end up in 7 states, which can be divided into three categories.

   (a) First Termination States: 1011, where 10 appears in $x$ (a) at the same time as 11 appears in $y$

   (b) Second Termination States: $1000, 1001, 1010$, where 10 appears in $x$ (a) before 11 appears in $y$

   (c) Third Termination States: $0011, 0111, 1111$, where 10 appears in $x$ (a) later than 11 appears in $y$

We denote these seven termination states $\{1011, 1000, 1001, 1010, 0011, 0111, 1111\}$ as $\{T_1, T_2, \cdots, T_7\}$. Also, we denote nine intermediate states $\{0000, 0100, 1100, 0001$ $, 0101, 1101, 0010, 0110, 1110\}$ as $\{M_1, M_2, \cdots, M_9\}$. Consider the following probabilities:

- The probability of: at $t^{th}$ step we are at intermediate state $M_i$, which we denote as $PM_i^t$
- The probability of: after $t$ steps or before the $t^{th}$ step we terminate at termination state $T_i$, which we denote as $PT_i^t$

Define a vector $y^t$ to represent the overall probability at $t^{th}$ step:

$$y^t = \begin{bmatrix} PT_1^t \\ \cdots \\ PT_7^t \\ PM_1^t \\ \cdots \\ PM_9^t \end{bmatrix}$$

At $t^{th}$ step, the system can either be at a intermediate state, or terminates at or even before the $t^{th}$ step. Therefore, the sum of each element

in $y^t$ is always 1.
Define transfer matrix for $y^t$ as $T$:

$$y^{t+1} = T \cdot y^t$$

And we the structure of $T$ is:

$$T = \begin{bmatrix} I & L_1 \\ O & L_2 \end{bmatrix}$$

$I$ is a $7 \times 7$ identity matrix, $O$ is a $9 \times 7$ null matrix.
If there is no termination and keep tossing coins permanently, we can form a vector $y'^t$. Each element in it means the probability of the corresponding state in $t^{th}$ step. We then create the $16 \times 16$ transfer matrix $T'$ for $y'^t$:

$$y'^{t+1} = T' \cdot y'^t$$

The sum of each column in $T'$ is 1. Then we take the last nine columns of $T'$ to form $\begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$.
The following figure shows the structure of $T$: Therefore, we get the the recurrence formula:

$$y^n = T^n \cdot y^0$$

$y^0$ is a one-hot vector, the $8^{th}$ element of which (corresponding to 0000) is 1. We use python to compute the result when n is approximate to infinity and we get:

$$\lim_{n \to +\infty} y^n = \begin{bmatrix} 17/121 \\ 24/121 \\ 24/121 \\ 17/121 \\ 11/121 \\ 11/121 \\ 17/121 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

That is to say the system will finally terminate, the probability of end up in First Termination States is 17/121, Second Termination States is 65/121, Third Termination States is 39/121.

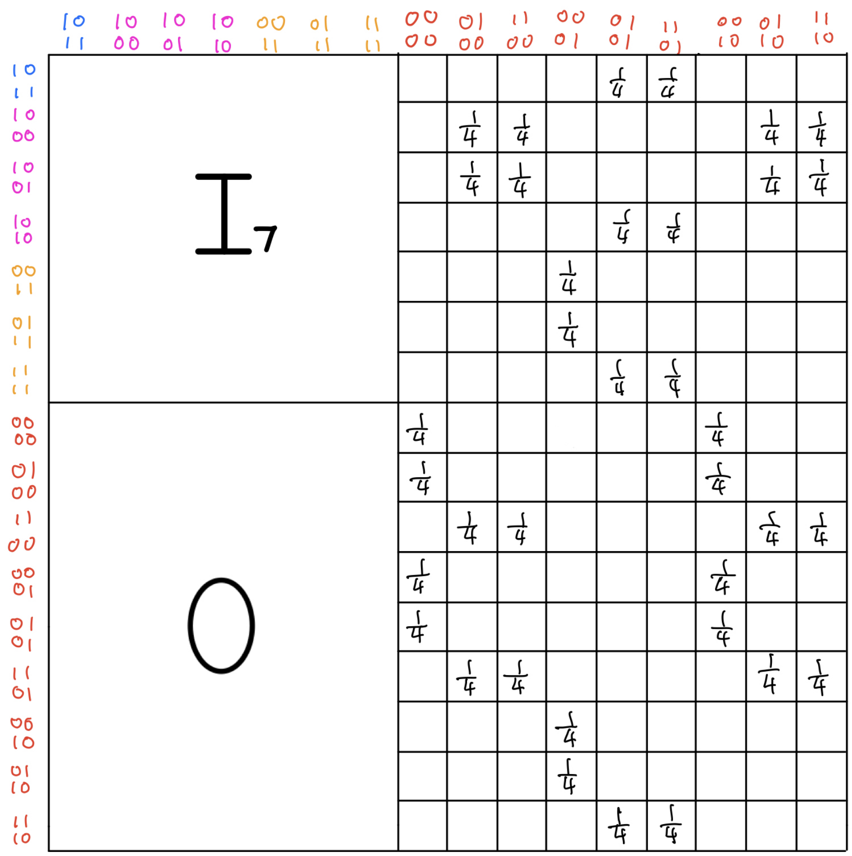



Figure 3: Structure of Matrix T

$\square$

**Exercise 3.4.** Let us toss a biased coin, that is 1 comes up with some probability $p \in [0, 1]$. Let $T$ be the number of tosses until the first 1 appears.

1. Give an explicit formula for $\mathbb{E}[T^2]$ in terms of $p$.
2. Give an explicit formula for $\mathbb{E}\left[\frac{1}{T}\right]$ in terms of $p$.

**Solution.** Define $x = 1 - p$.

1.

$$\begin{aligned}
\mathbb{E}[T^2] &= \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p \\
&= p\sum_{n=1}^{\infty} n^2 x^{n-1} \\
&= p(\sum_{n=1}^{\infty} n(n-1)x^{n-1} + \sum_{n=1}^{\infty} nx^{n-1}) \\
&= px\sum_{n=1}^{\infty}(x^n)'') + p\sum_{n=1}^{\infty}(x^n)' \\
&= px(\sum_{n=1}^{\infty} x^n)'' + p(\sum_{n=1}^{\infty}(x^n))' \\
&= px(\frac{2}{(1-x)^3}) + p(\frac{1}{p^2}) \\
&= \frac{2-p}{p^2}
\end{aligned} \tag{1}$$

2.

$$\begin{aligned}
\mathbb{E}[\frac{1}{T}] &= \sum_{n=1}^{\infty} \frac{1}{n}(1-p)^{n-1}p \\
&= p\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \\
&= \frac{p}{1-p}(\sum_{n=1}^{\infty} \frac{x^{n}}{n}) \\
&= \frac{p}{1-p}(-ln(1-x)) \\
&= \frac{plnp}{p-1}
\end{aligned} \tag{2}$$

$\square$