Mathematical Foundations of Computer Science

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9 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 9.1. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$. Here, ϵ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Solution. As we know, there are infinite prime numbers. Given $k \in \mathbb{N}$, we can reflect it as the k^{th} smallest prime number, which we denote as p_k. For example:

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5$$

We define the following reflection from $x \in \mathbb{N}$ to $\mathbb{N}^*$.

$$\begin{aligned} x &:= x + 1 \\ \text{if } x &= 1, \text{ then } x \Rightarrow \epsilon \\ x &= p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r} \text{ , where } c_r > 0 \text{ and } c_1, c_2, \cdots, c_{r-1} \geq 0 \\ \text{let } y &= (c_1, c_2, \cdots, c_{r-1}, c_r - 1) \in \mathbb{N}^r \\ x &\Rightarrow y \end{aligned}$$

We subtract 1 from c_r in case the last dimension of y is 0. It can help distinguish (x_1, x_2) and $(x_1, x_2, 0)$. For example:

$$83 \Rightarrow 84 \Rightarrow 2^2 3^1 5^0 7^1 \Rightarrow (2, 1, 0, 0) \in \mathbb{N}^4$$

$$(2, 1, 0) \Rightarrow 2^2 3^1 5^1 \Rightarrow 60 \Rightarrow 59$$

It's easy to see that this conversion is bijection between $\mathbb{N}$ and $\mathbb{N}^*$.

Exercise 9.2. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0, 1\}^{\mathbb{N}}$ and thus show that $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Proof. We first prove this lemma: *For sets A, B, C and D, if $A \cong C$, $B \cong D$, then $A \times B \cong C \times D$.*

Since $A \cong C$, $B \cong D$, there exists bijection $f : A \to C$ and $g : B \to D$. Let $h : A \times B \to C \times D$, $h(< x, y >) =< f(x), g(y) >$, then it's easy to see that h is a bijection. Therefore $A \times B \cong C \times D$.

Then we show that $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$. Suppose $x \in \{0, 1\}^{\mathbb{N}} = (x_0 x_1 x_2 x_3 \cdots)$, let $f : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$, $f(x) = (x_0 x_2 x_4 \cdots, x_1 x_3 x_5 \cdots)$.

1. f is an injection. $\forall x, y \in \{0,1\}^{\mathbb{N}_0}$ with $x \neq y$, $\exists i \in \mathbb{N}$ s.t. $x_i \neq y_i$. Without loss of generality, suppose i is an odd number, then $f(x) = (x_0x_2x_4\cdots, x_1x_3x_5\cdots x_i\cdots) \neq (y_0y_2y_4\cdots, y_1y_3y_5\cdots y_i\cdots) = f(y)$.

2. f is a surjection. $\forall (x,y) \in \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} = (x_0x_1x_2\cdots, y_0y_1y_2\cdots)$, $\exists z = (x_0y_0x_1y_1x_2y_2\cdots) \in \{0,1\}^{\mathbb{N}}$ s.t. $f(z) = (x,y)$.

Therefore f is a bijection, and $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Recall the fact that $R \cong \{0,1\}^{\mathbb{N}}$, according to the lemma, we have $R \times R \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Then according to the transitivity of $\cong$ relation, we have $R \cong \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \cong R \times R$. $\square$

Exercise 9.3. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \ldots)$ of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$.

Proof. Remember the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}} \cong 2^{\mathbb{N}}$. That is to say, each real number is a function $f : \mathbb{N} \to \{0,1\}$, and $\mathbb{R}$ is the set of all functions from $\mathbb{N} \to \{0,1\}$. $\mathbb{R}^{\mathbb{N}}$ is therefroe a sequence of such functions.

Let function g be that $g : \mathbb{N} \times \mathbb{N} \to \{0,1\}$, $\mathcal{G} : g(m,n) = f_n(m)$, where $f_n : \mathbb{N} \to \{0,1\}$ is a function as well as a real number, and the subscript n means that it is the n-th element in the real number sequence. Let us denote the set of functions from $\mathbb{N} \times \mathbb{N} \to \{0,1\}$ as A, then $\mathcal{G}$ is a bijection between A and $\mathbb{R}^{\mathbb{N}}$.

1. For two different functions g_1, g_2, $\exists (m,n) \in \mathbb{N} \times \mathbb{N}$ s.t. $g_1(m,n) \neq g_2(m,n)$, accordingly we have $f_n(m)_1 \neq f_n(m)_2$, which means different real numbers as the n-th element in the sequence, leading to different sequences. Therefore $\mathcal{G}$ is injective.

2. For any sequence in $\mathbb{R}^{\mathbb{N}}$, we can construct a function $g : \mathbb{N} \times \mathbb{N} \to \{0,1\}$, by following $g(m,n) = f_n(m)$. That is to say, $\mathcal{G}$ is surjective.

Also recall that $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, thus there is a bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then there is a bijection from the set of functions $f : \mathbb{N} \to \{0,1\}$ to A, the set of functions $g : \mathbb{N} \times \mathbb{N} \to \{0,1\}$, the former of which is just $\mathbb{R}$ and the latter of which $\cong R^{\mathbb{N}}$. Therefore $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. $\square$

Exercise 9.4. Let $\mathcal{F}$ be the set of all *continuous* functions $f : \mathbb{R} \to \mathbb{R}$. Show that $\mathcal{F} \cong \mathbb{R}$.

Proof. The cardinality of $\mathcal{F}$ is at least $|\mathbb{R}|$ because every real number corresponds to a constant function.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $x \in \mathbb{R}$. Then there is a Cauchy sequence of rational numbers such that $\lim_{n\to\infty} q_n = x$. Continuity of f means that $\lim_{n\to\infty} f(q_n) = f(\lim_{n\to\infty} q_n) = f(x)$.
This implies that the values of f at rational numbers already determine f. In other words, the mapping Φ: $\mathcal{F} \to \mathbb{R}^{\mathbf{Q}}$ is an injection, which means $|\mathcal{F}| \leq |\mathbb{R}^{\mathbf{Q}}|$. Notice that $|\mathbb{R}^{\mathbf{Q}}| = |2^{\mathbf{N}^{\mathbf{N}}}| = |2^{\mathbf{N}\times\mathbf{N}}| = |2^{\mathbf{N}}| = |\mathbb{R}|$. Hence, the cardinality of $\mathcal{F}$ is at most $|\mathbb{R}|$.
By the Schröder-Bernstein Theorem, we have that $\mathcal{F} \cong \mathbb{R}$. □

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 9.5. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Proof. Assume $x \in \mathbb{N}$, we create a sequence $00\cdots011\cdots1$, in which there are x 1's. In this way, $1 \to 2 \to 3\cdots$ is corresponding to $00\cdots001 \to 00\cdots011 \to 00\cdots0111 \to \cdots$ and this makes a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. □

Exercise 9.6. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

Proof. Given $x \in \mathbb{N}$, we create a sequence $00\cdots010\cdots0$, in which 1 is in the last x^{th} digit of this sequence. In this way, $1 \to 2 \to 3 \to \cdots$ is corresponding to $00\cdots001 \to 00\cdots010 \to 00\cdots0100 \to \cdots$ and this makes a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$. □

Exercise 9.7. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Proof. For the number 0, we corresponding create a sequence $100\cdots00$.

Now we exclude 0 from the following steps. Given $a, b \in \mathbb{N}$ and a, b are mutually prime and $a < b$, we consider four situations: (1)$+\frac{a}{b}$ (2) $-\frac{a}{b}$ (3) $+\frac{b}{a}$ (4) $-\frac{b}{a}$.

First we create a sequence in which the $(a+1)^{th}$ and $(b+1)^{th}$ digits are 1.

Second, we consider the following four digits after the latter 1 of the two. For case (1), the four digits are 1000. For case (2), the four digits are 0100. For case (3), the four digits are 0010. For case (4), the four digits are 0001.

Third, set all other digits to 0.

For example, given $a = 3$, $b = 5$.

$\frac{3}{5} \to 0001011000\cdots$

$-\frac{3}{5} \to 0001010100\cdots$

$\frac{5}{3} \to 00010100100\cdots$

$-\frac{5}{3} \to 000101000100\cdots$

In this way, we can construct an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. $\square$

****Exercise 9.8.** *Find an uncountable chain in* $\{0,1\}^{\mathbb{N}}$*. That is, a chain* A *with* $A \cong \mathbb{R}$*.*

****Exercise 9.9.** *Find a set* $X \subseteq 2^{\mathbb{N}}$ *(that is,* X *is a set of subsets of* $\mathbb{N}$*) such that (1) every* $x \in X$ *is an infinite subset of* $\mathbb{N}$*, (2)* $x \cap y$ *is finite whenever* $x, y \in X$ *are distinct, (3)* X *is uncountable.*