上 海 交 通 大 学 试 卷（A卷）

（2017 至 2018 学年 第二学期）

班级号________________________ 学号______________ 姓名 ____

课程名称 __Mathematical Foundations of Computer Science (CS499)_________ 成绩 ____

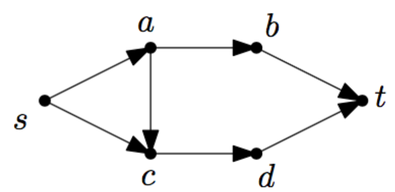
Each problem is worth 20 points. 100 points will give you full grade. Please note: (1) Justify your answers. (2) The problems are in random order, so Problem 1 is not necessarily the easiest. So check first before wasting time on the hardest problem. Also, all problems have some easy sub-problems.

Problem 1

We consider network flows (G, s, t, c) where $G = (V, E)$ is a directed graph. As usual, $c :\to \mathbf{R}^+$ denotes the capacity of an edge. By convention we set $c(u, v) = 0$ whenever $(u, v) \notin E$.

We write $\text{mincut}(G)$ to denote the capacity of a minimum s–t-cut in G. Recall that a cut is a set $S \subseteq V$ containing s but not t. By $\text{MinCuts}(G)$ we denote the set of all minimum cuts of G. So $\text{mincut}(G)$ is a real number but $\text{MinCuts}(G)$ is a set of subsets of V. Note that the set $\text{MinCuts}(G)$ is partially ordered by $\subseteq$.

1. Look at the network below. All edges have capacity 1.



   This network has several minimum cuts. Find all minimum cuts and draw the Hasse diagram of the corresponding partial ordering. **Hint.** To get you started, $\{s\}$ is a minimum cut and so is $\{s, c\}$. So your Hasse diagram will contain the arrow $\{s\} \to \{s, c\}$.

2. Consider a flow network (G, s, t, c). Let S_1, S_2 be two (not necessarily minimum) cuts. That is, S_1, S_2 are subsets of V containing s but not t. Recall that $\text{cap}(S) := \sum_{u \in S} \sum_{v \in V \setminus S} c(u, v)$. Prove that

$$\text{cap}(S_1) + \text{cap}(S_2) \geq \text{cap}(S_1 \cap S_2) + \text{cap}(S_1 \cup S_2) \ . \qquad (1)$$

3. Suppose S_1 and S_2 are minimum cuts. Show that $S_1 \cap S_2$ and $S_1 \cup S_2$ are minimum cuts, too.

4. Show that the partial ordering of minimum cuts has a minimum. That is, there is a minimum cut $S_{\min}$ such that $S_{\min} \subseteq S$ for every minimum cut S.

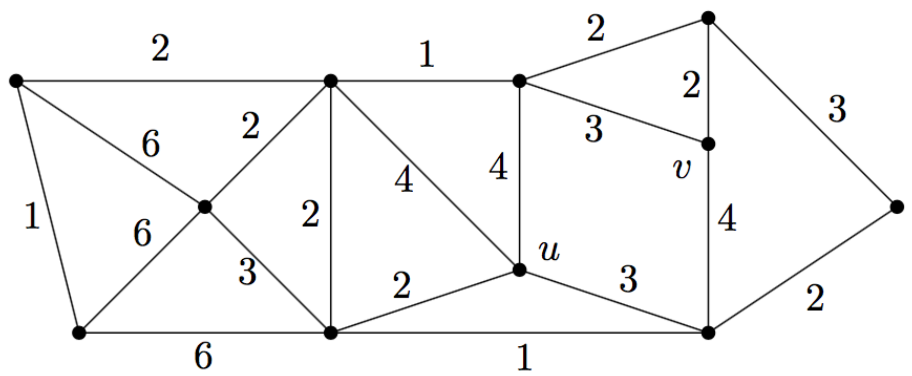
5. For a maximum flow f of the network, denote

   $$S_f := \{v \in V \mid \text{there exists a path from } s \text{ to } v \text{ in the residual network}\}.$$

   We have shown in the video lecture that S_f is a minimum cut. Show that $S_f = S_{\min}$.

Problem 2

In this problem, we consider minimum spanning trees (MST).



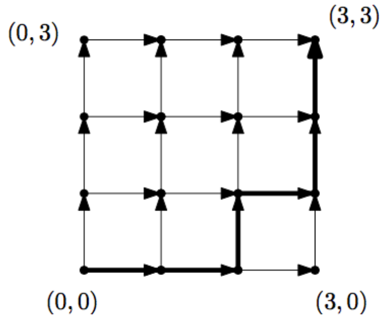
1. Draw a minimum spanning tree of this graph.

2. Suppose we add an edge between u and v and give it weight c. Call the resulting graph G^c. What is the largest c such that $\{u, v\}$ is contained in some minimum spanning tree of G^c?

3. Let G be some connected graph with edge weights (not the one above). Let $e \in \binom{V}{2} \setminus E$. We add e to G, set its cost to some $c > 0$, and call this new graph G^c. Show that there is some c^* such that (1) if $c < c^*$ then every MST of G^c contains e, (2) if $c > c^*$ then no MST of G^c contains e, (3) if $c = c^*$ then some MST of G contains e but some other MST does not. **Remark.** First think about how to find / define c^*. Then show that (1), (2), (3) hold. **Hint.** You can use everything from Homework 8 without proving it.

Problem 3

Consider the grid $\{0,1,\ldots,n\} \times \{0,1,\ldots,n\}$. We view it as a directed graph where (i,j) is connected to $(i+1,j)$ and $(i,j+1)$. Here is an illustration of the case $n=3$, together with a path from $(0,0)$ to (n,n).



1. How many paths from $(0,0)$ to (n,n) are there? Note that paths have to respect the direction of the edges. Give a closed formula!

2. Suppose $0 \leq i,j \leq n$ are given. How many paths are there that go from $(0,0)$ to (n,n) and pass through (i,j)?

3. Find a closed formula for the summation

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}\binom{2n-i-j}{n-j}.$$

Problem 4

We define a partial order P_n on $\binom{[n]}{2}$: given two elements $A, B \in \binom{[n]}{2}$ we write $A \leq B$ if $\max(A) \leq \max(B)$ and $\min(A) \geq \min(B)$. For example, $\{5,6\} \leq \{5,7\} \leq \{4,7\} \leq \{3,8\}$.

1. Draw the Hasse diagram of P_5.

2. What are the minimal and maximal elements of P_n? Does P_n have a minimum? A maximum?

3. What is the largest antichain in P_{100}? Justify your answer! **Hint.** You may use Dilworth's Theorem.

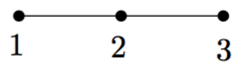
Problem 5

Consider trees on vertex set $V_n = \{1, \dots, n\}$. We say a tree T is 3-regular if every vertex has degree 3 or 1.

1. How many 3-regular trees are there on the set V_{99}?
2. Characterize 3-regular trees in terms of their Prüfer code. State and prove something like "A tree on V_n is 3-regular if and only if its Prüfer code..."
3. Give and justify an explicit formula for the number of 3-regular trees on V_n. Your formula may contain $a!$, $\binom{b}{a}$, etc., but not $\sum$ or $\prod$.

Problem 6

For a graph G, let $a(G)$ be the number of vertex covers of G. Let $b(G)$ be the number of minimal vertex covers of G. Recall that a vertex cover C is *minimal* if there is no $C' \subsetneq C$ such that C' is a vertex cover. Let P_n denote the path with n vertices (and thus $n-1$ edges). This is P_3:



So $a(P_3) = 5$ since $\{2\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1,2,3\}$ are all vertex covers. However, $b(P_3) = 2$ since only $\{2\}$ and $\{1,3\}$ are minimal vertex covers.

1. Find a recursive formula for $a(P_n)$.
2. Find a recursive formula for $b(P_n)$.
3. Let $G = (V, E)$ be a graph and $e \in \binom{V}{2} \setminus E$. Denote by $G + e$ the graph created by adding edge e. Show that $a(G) \geq a(G+e)$.
4. Give an example where $b(G) < b(G+e)$, i.e., adding an edge increases the number of minimal vertex covers.
5. Give an example where $b(G) > b(G+e)$.

题号	1	2	3	4	5	6				
得分										
批阅人(流水阅卷教师签名处)										

我承诺，我将严格遵守考试纪律。

承诺人：____________