Group: navigator

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5 The Graph Score Theorem

- Homework assignment published on Thursday, 2019-03-27.
- Submit questions and first solution by Wednesday, 2019-04-03, by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Monday, 2019-04-08.
- Submit your final solution by Sunday, 2019-04-14 to me and the two TAs.

Exercise 5.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Proof.

1. Assume that we have a graph $G = (V, E)$. Define d_i be the degree of the ith vertex. Then $score(G) = (d_1, d_2, \cdots, d_n)$.

2. **Theorem**

   Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ with $d_1 \leq \cdots \leq d_n$. Define **d'** by

   $$d' := \begin{cases} d_i - 1 & \text{for } i = n - d_n, \cdots, n-1 \\ d_i & \text{for } i = 1, \cdots, n - d_n - 1 \end{cases}$$

   Then there exists a graph with score $\mathbf{d}$ if and only if there exists a graph with score **d'**.

3. **Graph Score Algorithm:**

   find-graph$(d_1, d_2, \cdots, d_n)$

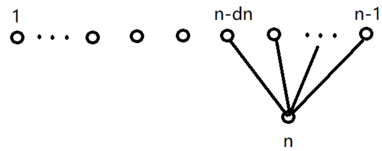
   sort$(d_1, d_2, \cdots, d_n)$

   $$d' := \begin{cases} d_i - 1 & \text{for } i = n - d_n, \cdots, n-1 \\ d_i & \text{for } i = 1, \cdots, n - d_n - 1 \end{cases}$$

   G':=find-graph$(d'_1, d'_2, \cdots, d'_{n-1})$

   if G'=null return null

   else G:=

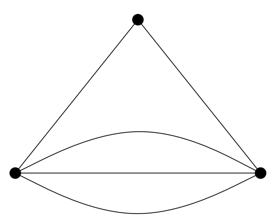


4. To prove the above theorem, we see that we can easily find a graph with **d** from **d'**. That is to create a point n and connect it to all points from $n - d_n$ to $n - 1$. But it is not equally easy to prove the other side. □

5.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 5.2. State a score theorem for multigraphs. That is, something like

Theorem 5.3 (Multigraph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$*. There is a multigraph with this score if and only if* <fill in some simple criterion here>.

Remark. This is actually simpler than for graphs.

Exercise 5.4. Prove your theorem.

Proof. Theorem

Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ with $d_1 \leq \cdots \leq d_n$. Define **d'** by

$$d' := \begin{cases} \begin{cases} d_i - 1 & \text{for } i = n - d_n, \cdots, n-1 \\ d_i & \text{for } i = 1, \cdots, n - d_n - 1 \end{cases} \text{if } n \geq d_n \\ \\ \begin{cases} d_i - 1 & \text{for } i = 1, \cdots, n-1 \\ d_i - (n-1) & \text{for } i = n \end{cases} \text{if } n < d_n \end{cases}$$

Then there exists a graph with score **d** if and only if there exists a graph with score **d'**.

Proof:

If $n \geq d_n$:

d' $\Rightarrow$ d: It is easy to find d based on d'. We just need to create a point n and connect it to all points from $n - d_n$ to $n - 1$.

d $\Rightarrow$ d':

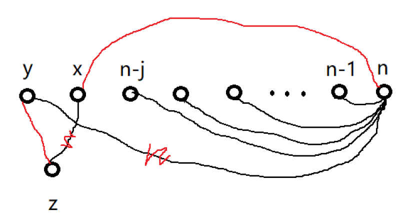
Claim1: If $\exists$ G such that $score(G) = (d_1, d_2, \cdots, d_n)$, then $\exists$ G such that $score(G) = (d_1, d_2, \cdots, d_n)$ and in which exists a point n that connects to all points from $n - d_n$ to $n - 1$.

With this claim, $G - n =: G'$, $score(G') = d'$.

Now we prove the claim. Define j(G) be the largest number j such that vertex n has an edge to vertex $n-1, n-2, \cdots, n-j$.

Claim2: If G maximizes j(G) then $j(G) = d_n$.

proof: Suppose not, then $j(G) < d_n$.



As the figure above (the black lines), we obtain that: $\{x, n\} \notin \mathbb{E}$, $deg(n) = d_n$, $\{n, y\} \in \mathbb{E}$, $deg(y) \leq deg(x)$. Hence, $\exists$ vertex z such that $\{x, z\} \in \mathbb{E}$ and $\{y, z\} \notin \mathbb{E}$. We can replace $\{n, y\}$ and $\{x, z\}$ by $\{n, x\}$ and$\{y, z\}$.

Now we get a new graph H and we have $score(G) = score(H)$ but $j(H) > j(G)$. This contradicts with G maximizes j(G). So we have proved **Claim2** which can easily lead to **Claim1** and we finally prove $d \Rightarrow d'$.

If $n < d_n$**:**

d' ⇒ d: It is also easy to find d based on d'. We just need to connect vertex n with all other vertices.

d ⇒ d':

Claim3: If ∃ G such that $score(G) = (d_1, d_2, \cdots, d_n)$, then ∃ G such that $score(G) = (d_1, d_2, \cdots, d_n)$ and in which exists a vertex n that connects to all vertices from 1 to $n-1$.

With this claim,

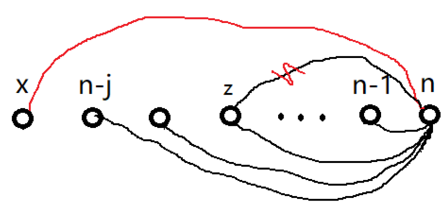
$$G' := G - \sum_{k=1}^{n-1} \{k, n\}$$

and we have $score(G') = d'$.

Now we prove the claim. Define j(G) be the largest number j such that vertex n has an edge to vertex $n-1, n-2, \cdots, n-j$.

Claim4: If G maximizes j(G) then $j(G) = n - 1$.

proof: Suppose not, then $j(G) < n - 1$.



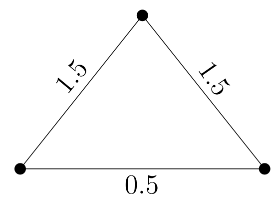
As the figure above (the black lines), we obtain that: $x < j$ so $\{x, n\} \notin \mathbb{E}$. Since $d_n > n$, there must exist an vertex z such that 2 edges are between vertex z and vertex n. We can replace one of them by an edge between vertex n and vertex x. Hence, we obtain a new graph H with a bigger j(H). Continue the operation, we may finally connect all other vertices with vertex n. This means $j(G) = n - 1$ and leads to an contradiction.

Now we have proved **Claim4** and we can easily obtain **Claim3** and finally prove d ⇒ d'.

□

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u

is $\mathrm{wdeg}(u) = \sum_{\{u,v\}\in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 5.5. State a score theorem for weighted graphs. That is, state something like

Theorem 5.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. *There is a weighted graph with this score if and only if* `<fill in some simple criterion here>`.

Remark. This is actually even simpler.

Exercise 5.7. Prove your theorem.

Proof. Theorem Let $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ with $d_1 \leq \cdots \leq d_n$. Define **d'** by

$$d' = d_i - x_i, \text{ for } i = 1, 2, \cdots, n-1; \sum_{i=1}^{n-1} x_i = d_n, \; x_i \in \mathbb{R}^+$$

Then there exists a graph with score $\mathbf{d}$ if and only if there exists a graph with score **d'**.

Proof.

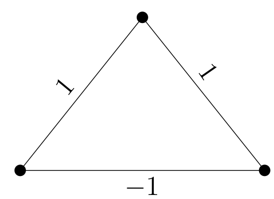
$\rightarrow$:

If there exists a graph with score $\mathbf{d}$, then the vertice whose degree is d_n must have some edges with weight of x_i connected to other vertices. Delete this vertice and related edges, the remains is obviously a graph with score **d'**.

$\leftarrow$:

If there exists a graph with score **d'**, we can add vertice with degree of d_n by creating several edges connected to vertices $\{v_1, v_2, \cdots, v_{n-1}\}$ whose weight is x_i. Then we get a weighted graph with score $\mathbf{d}$. $\square$

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 5.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like *if we view the edge between vertex i, j with weight x_{ij} as unknown numbers, then we can write down equations like*

$$\sum_{j=1,2,\cdots,i-1,i+1,\cdots,n} x_{ij} = a_i,$$

where a_i is the degree of i-th vertex in the score. If and only if we can find a solution for these equations, then there exists a graph with real edge weights with this score.

To make this criteria explicit, let us use the aid of linear algebra. Say, $\mathbf{x} = (x_{12}, x_{13}, \cdots, x_{1n}, x_{23}, x_{24}, \cdots, x_{n-1,n})^T$ *is an* $\binom{n}{2} \times 1$ *vector with no repeating elements.* $\beta = (a_1, \ldots, a_n)^T$ *is an* $n \times 1$ *vector.* $\mathbf{A}$ *is an* $n \times \binom{n}{2}$ *matrix whose elements are coefficients in the equations mentioned above, so that the equation in matrix form*

$$\mathbf{A}\mathbf{x} = \beta \tag{1}$$

holds. Let $\widetilde{\mathbf{A}}$ *denote the argumented matrix of* $\mathbf{A}$*. We know that the equation group* (1) *has a solution if and only if* $\mathrm{r}(\mathbf{A})=\mathrm{r}(\widetilde{\mathbf{A}})$*. But we can further claim that*

Lemma 5.9. *For any $n > 2$, equation group* (1) *satisfies that* $\mathrm{r}(\mathbf{A})=\mathrm{r}(\widetilde{\mathbf{A}})=n$*.*

Proof. The argumented matrix $\widetilde{\mathbf{A}}$ has the form

$$\widetilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & 0 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 & a_n \end{bmatrix},$$

when $n > 2$, we can always change it into the following form via elementary row operations:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & -1 & \cdots & -1 & 1 & \cdots & 1 & 0 & \cdots & 0 & a_2 - a_1 \\ 0 & 0 & -1 & \cdots & -1 & 2 & \cdots & 1 & 1 & \cdots & 0 & a_3 + a_2 - a_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 2 & \cdots & 2 & 2 & \cdots & 2 & \sum_{i=2}^{n} a_i - a_1 \end{bmatrix},$$

from which we can easily see that r($\mathbf{A}$)=r($\widetilde{\mathbf{A}}$)=n. □

With this we can write the theorem formally as follows:

Theorem 5.10 (Score Theorem for Graphs with Real Edge Weights)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}^n$*. There is a graph with real edge weights with this score if and only if* r($\mathbf{A}$)=r($\widetilde{\mathbf{A}}$)*. Specially, if* $n > 2$*, there is a graph with real edge weights with this score.*

Exercise 5.11. Prove your theorem.

Proof. Sufficiency. If r($\mathbf{A}$)=r($\widetilde{\mathbf{A}}$), equation group (1) has a solution in $\mathbb{R}^n$. Specially if $n > 2$,then according to Lemma 5.9, the same statement holds. That means, we can find a complete graph with real edge weights satisfying this score.

Necessity. If there is a graph with real edge weights with this score, we can always expand the graph to a complete graph, assigning weight of 0 to the edges that previously do not exist. Then we get a solution for the equation group (1), which means r($\mathbf{A}$)=r($\widetilde{\mathbf{A}}$). □

Exercise 5.12. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

Solution.

- (5,1,7,0,2,1,9,1,0,6,2,3)

  1. This is not a graph score, as there are odd number of odd degrees in the score, which violates the Handshaking lemma.
  2. This is not a multigraph score, as the Handshaking lemma still holds for multigraph, and there are odd number of odd degrees in the score, which violates the Handshaking lemma.
  3. This is a weighted graph score. See the diagram of graph in Figure 1.
  4. This is a score of graph with real edge weights, as the answer for (3) is yes.

- (5,1,7,0,2,1,9,1,0,8,8,2)

  First, this is not a graph score. According to the graph score theorem, this is a graph score iff 406010000771 is a graph score. However, $d_1 0 = 7$ but there are only five other vertices whose degree are greater than 0. Hence, this is not a graph score.

  Second, this is a multigraph score. The graph is shown below.

  Since this is a multigraph score, it is also a weighted graph score and the allowing negative edge weighted graph score. These two graph are the same as the following figure shows.

- (5,1,7,0,2,1,9,1,0,7,2,4)

  1. This is neither a graph nor a multigraph because the sum of degree is odd, which go against the Handshaking Lemma.
  2. This is a weighted graph. The figure of graph is shown below.

- (5,1,7,0,2,1,9,1,0,5,4,0)

  1. This is neither a graph nor a multigraph because the sum of degree is odd, which go against the Handshaking Lemma.
  2. This is a weighted graph. The figure of graph is shown below.

□

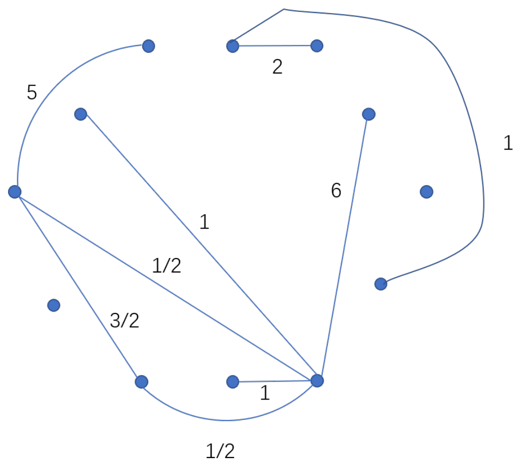


Figure 1: weighted graph: 517021910623

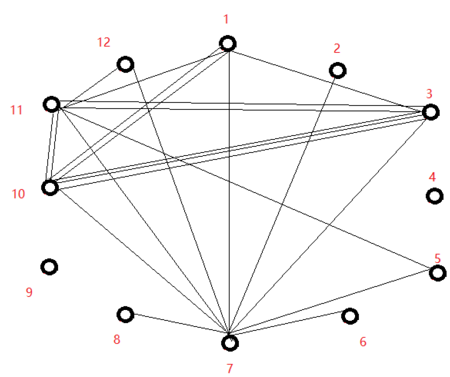


Figure 2: Multigraph: 517021910882

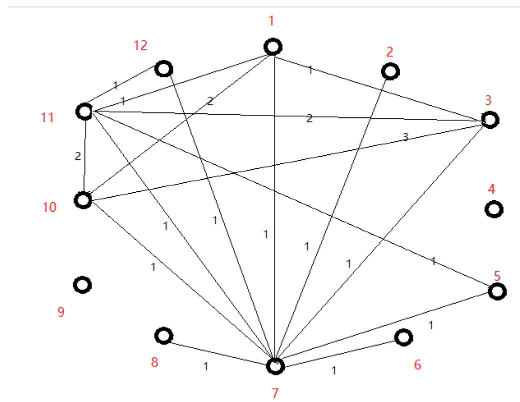


Figure 3: Weighted graph: 517021910882

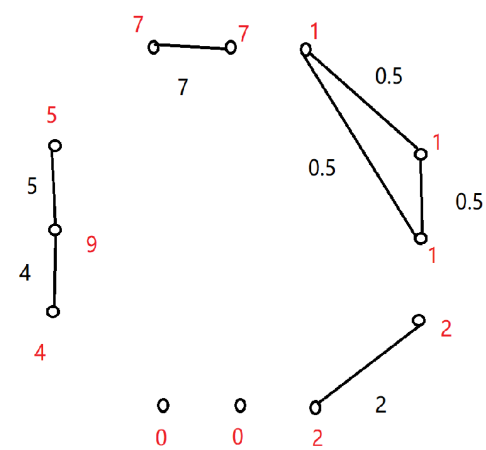


Figure 4: Weighted graph: 517021910724

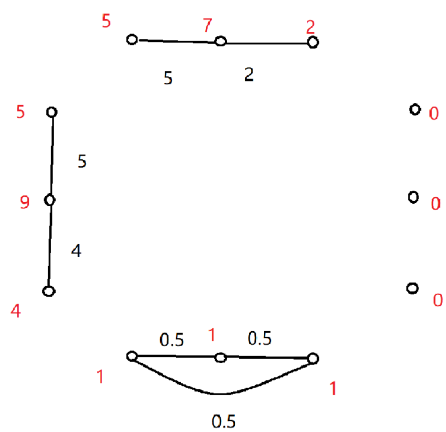


Figure 5: Weighted graph: 517021910540