Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Spring 2019

- Homework assignment published on Thursday, 2019-03-07
- Submit questions or first draft solutions by Sunday, 2019-03-10, by email to the TA and to me (dominik.scheder@gmail.com)
- We will discuss some problems on Monday, 2019-03-11.
- You will receive feedback by Wednesday, 2019-03-13.
- Revise your solution and hand in your final submission by Sunday, 2019-03-17.

2 Fibonacci Numbers and Other Recurrences

2.1 Identities among the Fibonacci Numbers

Exercise 2.1. Prove the following identity:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1 \ .$$

Actually, prove it twice:

1. Give an inductive proof.
2. Give a *combinatorial* argument. Remember that $F_{n+2} = |A_n|$, where $A_n := \{x \in \{0,1\}^n \mid x \text{ does not contain } 11\}$. Find a way to partition A_n into sets such that the identity above becomes obvious.

Solution.

1. *We first give the inductive proof.*

   Basis Step: *For $n = 0$, we have*

$$\sum_{i=1}^{0} F_i = 0 = F_2 - 1.$$

   Induction Hypothesis: *Assume that the statement is true for some $k \geq 0$. Then we have $F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$.*

   Proof of Induction Step: *Remember that we have $F_{k+2} + F_{k+1} = F_{k+3}$ derived from the definition of the Fibonacci number. For $k + 1$, we can derive that*

$$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1,$$

   which means that the statement holds true for $k + 1$.

2. *We can see that the set A_n can be divided as*

$$A_n = \{x \in \{0,1\}^n \mid 0y, y \in A_{n-1} \textit{ or } 10z, z \in A_{n-2}\}.$$

   And $A_{n-1}, \cdots, A_2$ can further be divided recursively. Let us view the sequence from left to right, so $|A_{n-2}|$ equals to the number of such sequences in A_n that "1" first appears at the very beginning. And $|A_{n-3}|$ equals to the number of such sequences in A_n that "1" first appears at the second place. This pattern continues until we have that $|A_0|$ equals to the number of such sequences in A_n that "1" first appears at the last but one place.

   In this case we have two sequences left in A_n, one sequence is where "1" first appears at the last place, the other is where there is only "0"s. That means

$$\sum_{i=0}^{n-2} |A_i| = |A_n| - 2 = F_{n+2} - 2.$$

   We also have that

$$\sum_{i=1}^{n} F_i = F_1 + \sum_{i=2}^{n} F_i = F_1 + \sum_{i=0}^{n-2} |A_i|,$$

where F_1 equals 1*. Therefore*

$$\sum_{i=1}^{n} F_i = 1 + F_{n+2} - 2 = F_{n+2} - 1.$$

Exercise 2.2. Prove the following identity:

$$F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2} \ .$$

Again, give two proofs, one using induction on n and one using a combinatorial argument involving the sets A_i.

Solution.

1. *Inductive proof.*

   Basis Step: *For $n = 0$, we have*

   $$\sum_{i=0}^{0} F_{2i+1} = 1 = F_2.$$

   Induction Hypothesis: *Assume that the statement is true for some $k \geq 0$. Then we have $F_1 + F_3 + \cdots + F_{2k+1} = F_{2k+2}$.*

   Proof of Induction Step: *Remember that we have $F_{k+2} + F_{k+1} = F_{k+3}$ derived from the definition of the Fibonacci number. For $k + 1$, we can derive that*

   $$\sum_{i=0}^{k+1} F_{2i+1} = \sum_{i=0}^{k} F_{2i+1} + F_{2k+3} = F_{2k+2} + F_{2k+3} = F_{2k+4},$$

   which means that the statement holds true for $k + 1$.

2. *Similar with Exercise 2.1, which divides A_n according to the number of continuous 0. In Exercise 2.2, we divides A_n according to the number of continuous "10".*

   $$A_n = \{x \in \{0,1\}^n \mid 0y, y \in A_{n-1} \ \textit{or} \ 10z, z \in A_{n-2}\}.$$

   In last exercise, the subset " $0y, y \in A_{n-1}$ " is further divided to smaller subsets, while the subset " $10z, z \in A_{n-2}$ " is kept. For formula in

Exercise 2.2, we will keep " $0y, y \in A_{n-1}$*" and divide "* $10z, z \in A_{n-2}$*" to smaller subsets recursively.*

To be specific, " $10z, z \in A_{n-2}$*" can be divided as "* $100y, y \in A_{n-3}$*" and "* $1010z, z \in A_{n-4}$*". We keep "* $100y, y \in A_{n-3}$*" and further divide "* $1010z, z \in A_{n-4}$*", and so on. Finally,* A_n *can be divided to subsets "* $0y, y \in A_{n-1}$*", "* $100y, y \in A_{n-3}$*", "* $10100y, y \in A_{n-5}$*",* $\cdots$.

Therefore, $|A_{2n-1}|$ *equals to the number of such sequences in* A_{2n} *that no "10" appears at the very beginning. And* $|A_{2n-3}|$ *equals to the number of such sequences in* A_{2n} *that one "10" first appears at the second place. This pattern continues until we have that* $|A_1|$ *equals to the number of such sequences in* A_2n *that n-1 continuous "10" appears at the beginning.*

In this case we have one sequence left in A_2n *when the recurrence comes to the end, which is "*$1010\cdots101$*". That means:*

$$\sum_{i=0}^{n-2} |A_{2i+1}| + 1 = \sum_{i=0}^{n-2} |A_{2i+1}| + F_1 = |A_{2n}|.$$

Therefore

$$\sum_{i=0}^{n} F_{2i+1} = F_{2n+2}.$$

2.2 General Linear Recurrences

For $a_1, \ldots, a_k \geq 0$ we can consider the recursively defined numbers:

$$\begin{array}{lll} F_n = & f_n & \text{if } n < k \\ F_n = & a_1F_{n-1} + a_2F_{n-2} + \cdots + a_kF_{n-k} & \text{if } n \geq k\ . \end{array}$$

The values $f_0, \ldots, f_{k-1}$ are the "start values" of the recurrence. For example, if we set $k = 1$, $f_0 = 1$, and $a_1 = 2$ then $F_n = 2^n$; setting $k = 2$, $f_0 = 0$, $f_1 = 1$, and $a_2 = a_1 = 1$ yields the Fibonacci numbers. As with the Fibonacci numbers, we can write the recursion in matrix-vector form:

$$\begin{pmatrix} F_n \\ F_{n-1} \\ \vdots \\ F_{n-k+1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{n-1} \\ F_{n-2} \\ \vdots \\ F_{n-k} \end{pmatrix}$$

Let us denote the matrix by A.

Exercise 2.3. Show that λ is an eigenvalue of A if and only if

$$\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-2}\lambda^2 + a_{k-1}\lambda + a_k \ . \tag{1}$$

For an eigenvalue λ, show what the corresponding eigenvector is. **Hint.** You can do this by computing $\det(A-\lambda I)$. But there is a simpler way by thinking directly in terms of what eigenvectors are.

Proof.

We assume that λ is an eigenvalue and the corresponding eigenvector is $\boldsymbol{x}$. With the assumption, we can have and only have $A\boldsymbol{x} = \lambda\boldsymbol{x}$, in which $\boldsymbol{x} = (x_1, x_2, x_3, \cdots, x_n)$. Then we have $(A - \lambda E)\boldsymbol{x} = \mathbf{0}$.

$$\begin{pmatrix} a_1-\lambda & a_2 & a_3 & \cdots & a_{k-2} & a_{k-1} & a_k \\ 1 & -\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{cases} (a_1-\lambda)x_1 + a_2x_2 + \cdots + a_kx_k = 0 \\ x_1 - \lambda x_2 = 0 \\ x_2 - \lambda x_3 = 0 \\ \qquad \vdots \\ x_{k-1} - \lambda x_k = 0 \end{cases}$$

Since $x_1, x_2, \cdots, x_k \neq 0$, we have

$$\lambda = \frac{x_1}{x_2} = \cdots = \frac{x_{k-1}}{x_k}$$

So we can obtain

$$\begin{cases} x_1 = \lambda^{k-1} x_k \\ x_2 = \lambda^{k-2} x_k \\ \quad\vdots \\ x_{k-1} = \lambda^1 x_k \end{cases}$$

Then we have

$$[(a_1 - \lambda)\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_k]x_k = 0$$

For $x_k \neq 0$, we obtain

$$(a_1 - \lambda)\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_k = 0$$

That is exactly

$$\lambda^k = a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-2}\lambda^2 + a_{k-1}\lambda + a_k \ . \tag{2}$$

The corresponding eigenvector is

$$\boldsymbol{x} = \begin{pmatrix} \lambda_{k-1} \\ \lambda_{k-2} \\ \lambda_{k-3} \\ \vdots \\ \lambda_0 \end{pmatrix}$$

□

Exercise 2.4. Recall that $a_1, \ldots, a_k \geq 0$. Assume further that $a_1 + \cdots + a_k > 1$. Show that among the solutions to (2), there is exactly one solution λ_1 with $\lambda_1 > 0$, and this λ_1 is actually greater than 1.

Proof.

1. Prove that there exist one eigenvalue λ_1 with $\lambda_1 > 0$.

   Construct function:

   $$f(\lambda) = \lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_{k-2}\lambda^2 - a_{k-1}\lambda - a_k.$$

Substitute 1 into $f(\lambda)$, then we have $f(1) = 1 - a_1 - \cdots - a_k < 0$. Substitute $a_1 + \cdots + a_k$ into $f(\lambda)$,

$$\begin{aligned}
f(\sum_{i=1}^{k} a_i) &= (\sum_{i=1}^{k} a_i)^k - a_1(\sum_{i=1}^{k} a_i)^{k-1} - \cdots - a_{k-1}\sum_{i=1}^{k} a_i - a_k \\
&= (a_1 + \cdots + a_k)(\sum_{i=1}^{k} a_i)^{k-1} - a_1(\sum_{i=1}^{k} a_i)^{k-1} - \cdots - a_{k-1}\sum_{i=1}^{k} a_i - a_k \\
&= a_2(\sum_{i=1}^{k} a_i)^{k-2}(\sum_{i=1}^{k} a_i - 1) + a_3(\sum_{i=1}^{k} a_i)^{k-3}((\sum_{i=1}^{k} a_i)^2 - 1) \\
&+ \cdots + a_k((\sum_{i=1}^{k} a_i)^{k-1} - 1) > 0.
\end{aligned}$$

Since $f(\lambda)$ is obviously continuous on $[1, \sum_{i=1}^{k} a_i]$, according to the zero point theorem, we have that there exists one solution λ_1 to $f(\lambda) = 0$ with $\lambda_1 > 1$.

2. Prove that λ_1 is the unique positive root.

   Assume that there exists at least another positive root named λ_2. We have $p\lambda_1 = \lambda_2$, with $p = \frac{\lambda_2}{\lambda_1} > 0$.

   (a) $p > 1$, then we have:

   $$\begin{aligned}
   f(\lambda_2) = f(p\lambda_1) &= (p\lambda_1)^k - a_1(p\lambda_1)^{k-1} - \cdots - a_k \\
   &= \lambda_1^k p^k - a_1\lambda_1^{k-1}p^{k-1} - \cdots - a_k \\
   &< (\lambda_1^k - a_1\lambda_1^{k-1} - \cdots - a_k)p^k = 0
   \end{aligned}$$

   Therefore $f(\lambda_2) < 0$, which contradicts the assumption that λ_2 is one solution.

   (b) $0 < p < 1$, then we have:

   $$\begin{aligned}
   f(\lambda_2) = f(p\lambda_1) &= (p\lambda_1)^k - a_1(p\lambda_1)^{k-1} - \cdots - a_k \\
   &= \lambda_1^k p^k - a_1\lambda_1^{k-1}p^{k-1} - \cdots - a_k \\
   &> (\lambda_1^k - a_1\lambda_1^{k-1} - \cdots - a_k)p^k = 0
   \end{aligned}$$

   Therefore $f(\lambda_2) > 0$, which contradicts the assumption that λ_2 is one solution.

We have derived a contradiction, which allows us to conclude that our original assumption is false.

$\square$