Mathematical Foundations of Computer Science

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Spring 2019

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4 Random Walks

4.1 Random Walks on $\{0, \ldots, k\}$

Consider a random walk starting at j. Let p_j be the probability that this walk reaches k before it reaches 0. I showed in the lecture that $p_j = j/k$. Let me give you an alternative proof.

Proof. For $t \in \mathbb{N}_0$ let X_t denote the position after t steps. So $X_0 = j$, $X_1 \in \{j-1, j+1\}$, and so on. Note that if $X_t = l$ then $\mathbb{E}[X_{t+1}] = \frac{l-1}{2} + \frac{l+1}{2} = l$. This implies that in fact $\mathbb{E}[X_t] = j$ for all $t \in \mathbb{N}_0$. Thus, it must also hold for $t = T$, where T is the time it takes until the walk reaches 0 or k. Therefore, $\mathbb{E}[X_T] = j$.

Now let's compute $\mathbb{E}[X_T]$ in a different way: by definition of T, we will be at 0 or at k. The latter happens with probability p_j, the former with probability $1 - p_j$. Thus,

$$\mathbb{E}[X_T] = k \cdot p_j + 0 \cdot (1 - p_j) \ .$$

Therefore, $j = \mathbb{E}[X_T] = k \cdot p_j$ and $p_j = j/k$. $\square$

Exercise 4.1. Explain why this proof is wrong.

Solution.

$$\mathbb{E}[X_T] = k \cdot p_j + 0 \cdot (1 - p_j) \ .$$

This equation is wrong. T is the time it takes until walk reaches 0 or k, but it doesn't promise that it definitely reaches 0 or k after T steps. Starting from j, There can be other terminal positions.

4.2 Random Walks on $\mathbb{Z}$

Consider the biased random walk on $\mathbb{Z}$. We start at some point $j \geq 1$. In every step, we go one step left with probability p and one step right with probability $1 - p$. In the lecture we have seen three things: (1) if $p < 1/2$ then with some positive probability, this walk never reaches 0; (2) if $p \geq 1/2$ then the walk reaches 1 with probability 1; (3) if $p = 1/2$ then although the walk reaches 0 with probability 1, the expected number of steps until it does so is infinite. Let T denote the number of steps until we reach 0 when the walk starts at 1; if the walk never reaches 0, we define $T = \infty$.

Exercise 4.2. Suppose $p > 1/2$.

1. Show that $\mathbb{E}[T]$ is finite. That is, the sum $\sum_{n=1}^{\infty} n \Pr[T = n]$ converges.
2. Compute $\mathbb{E}[T]$. **Hint.** Recall that

$$\mathbb{E}[T] = \sum_{n=0}^{\infty} (2n+1)\, C_n\, p^{n+1}\, (1-p)^n \ .$$

   Apply some tricks from the previous lectures to evaluate this sum.

Solution. *1. We give the explicit form of* $\mathbb{E}[T]$ *as*

$$\mathbb{E}[T] = \sum_{n=0}^{\infty} (2n+1)\, C_n\, p^{n+1}\, (1-p)^n,$$

where we have

$$C_n = \frac{\binom{2n}{n}}{n+1} = \frac{(2n)!}{(n+1)!\, n!} \leq 4^n.$$

We can prove it by induction, but for short we shall leave it as what we already know. Besides, when $p > 1/2$*, we have* $p(1-p) < 1/4$*. Now consider the infinite series* A

$$A = \sum_{n=0}^{\infty} (2n+1)\, 4^n\, p^{n+1}\, (1-p)^n \geq \mathbb{E}[T],$$

if it converges, then $\mathbb{E}[T]$ *converges. We have*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n+3}{2n+1} \cdot 4 \cdot p(1-p) = 4p(1-p) < 1.$$

So by the ratio test, the series converges when $p > 1/2$*. Therefore,* $\mathbb{E}[T]$ *also converges.*

2. We find the hint is somewhat misleading in this sub-problem, as it implies that we had better use calculus to solve this problem. However, it is simpler to compute $\mathbb{E}[T]$ *if we consider it as what expectation is. Say, we are at 1 and go with one move. With probability* p *we are at 0 and have no movement eversince, and with probability* $1-p$ *we are at 2 and need to first go back to 1, then go to 0. That is,*

$$\mathbb{E}[T] = 1 + p \cdot \mathbb{E}[0 \rightsquigarrow 0] + (1-p) \cdot (\mathbb{E}[2 \rightsquigarrow 1] + \mathbb{E}[T]).$$

Note that to reach 1 from 2 is identical to to reach 0 from 1. Therfore $\mathbb{E}[2 \rightsquigarrow 1] = \mathbb{E}[T]$*, and we can give*

$$\mathbb{E}[T] = 1 + 2(1-p)\mathbb{E}[T].$$

We solve it and get $\mathbb{E}[T] = \frac{1}{2p-1}$.

Exercise 4.3. Suppose $p < 1/2$. Let T be the number of steps until we reach 0. On the one hand, we can compute $\mathbb{E}[T]$ as follows:

$$\begin{aligned}\mathbb{E}[T] &= \sum_{n=0}^{\infty}(2n+1)C_n p^{n+1}(1-p)^n \\ &= p\sum_{n=0}^{\infty}(2n+1)C_n(p\,(1-p))^n \\ &< \sum_{n=0}^{\infty}(2n+1)\,4^n\,(p\,(1-p))^n < \infty\ ,\end{aligned}$$

since $C_n \leq 4^n$ and $p\,(1-p) < 1/4$. On the other hand, $\mathbb{E}[T] = \infty$ for the unbiased random walk, so for the p-biased random walk with $p < 1/2$, it should take even longer to reach 0, right? Explain what went wrong here.

Solution. *For one thing, the inequalities above mean anything but* $\mathbb{E}[T] \neq \infty$*. For another, there is some fixed positive probability that random walk goes infinite steps right and never goes left. Taking that case into consideration,* $\mathbb{E}[T]$ *is obviously* ∞*.*

If $p = 1/2$ then $\mathbb{E}[T]$ is infinite. We have shown this already last week in two different ways. However, what about $\mathbb{E}\left[\frac{1}{T}\right]$? This should exist. After all,

$$\sum_{n=1}^{\infty}\frac{1}{n}\Pr[T=n] \leq \sum_{n=1}^{\infty}\Pr[T=n] = 1\ ,$$

so the sum is bounded, each term is non-negative, and so the series must converge. I think computing $\mathbb{E}\left[\frac{1}{T}\right]$ is quite nasty. So let me give you a simpler problem:

Exercise 4.4. Compute $\mathbb{E}\left[\frac{1}{T+1}\right]$.

Solution.

$$
\begin{aligned}
\mathbb{E}[\frac{1}{T+1}] &= \sum_{n=0}^{\infty} \frac{1}{2n+2} C_n p^{n+1}(1-p)^n \\
&= \frac{1}{2(1-p)} \sum_{n=0}^{\infty} \frac{1}{n+1} C_n (p(1-p))^{n+1} \\
&= \frac{1}{2(1-p)} \int_0^x \sum_{n=0}^{\infty} C_n x^n \, dx \\
&= \frac{1}{2(1-p)} \int_0^{p(1-p)} \frac{1}{p} \sum_{n=0}^{\infty} C_n p^{n+1}(1-p)^n \, d(p(1-p)) \\
&= \frac{1}{2(1-p)} \int_0^{p(1-p)} \frac{1}{1-p} \, d(p(1-p)) \\
&= \frac{1}{2(1-p)} \int_0^{p(1-p)} \frac{1-2p}{1-p} \, dp \\
&= p + \frac{ln(1-p+p^2)}{2(1-p)}
\end{aligned}
$$

4.3 Monster Wars and Robot Wars

Alice and Bob are the captains of a team of monsters. Each monster has some strength $a > 0$. If two monsters of strengths a and b fight, the first monster wins with probability $\frac{a}{a+b}$ and the second with probability $\frac{b}{a+b}$. The winning monster devours the loser and its strength increases to $a + b$. The losing monster gets eaten (obviously) and dies.

Alice heads a team of m monsters of strengths $a_1, \ldots, a_m$. Bob's monsters have strengths $b_1, \ldots, b_n$. In each round, Alice and Bob each choose a monster to be sent into the arena. There, the two monsters then fight, the winning monster devours the loser, and returns to its team (with increased strength). After at most $n + m - 1$ rounds, one team has died completely. The other team is declared winner.

Exercise 4.5. Show that the probability of Alice's team winning does not depend on the order in which Alice and Bob send their monsters into the arena. Compute the exact winning probability, it terms of $a_1, \ldots, a_m, b_1, \ldots, b_n$.

Solution. Recall that in a random walking problem, $p_j = j/k$ (the probability that a random walk starting at j and reaches k before it reaches 0). Therefore, we can treat monster competition as a random walking process.

To build a random walking model, we make following definitions:

Notation	Definition
A_t	The set of each monster's strength in Alice's team after round t.
B_t	The set of each monster's strength in Bob's team after round t.
$p(X)$	The position of monster strength set X, i.e, $p(X) = \sum_{x_i \in X} x_i$

The initial position of Alice's team is $p(A_0) = \sum_{i=1}^{m} a_i$, Bob's team is $p(B_0) = \sum_{i=1}^{n} b_i$. Also, a team is declared winner when the other team's position reaches 0.

Consider that after round t, the position of Alice's team is $p(A_t)$, the position of Bob's team is $p(B_t)$. Suppose monster a_i and b_j are fighting in round $t+1$:

$$Pr\Big(p(A_{t+1}) = p(A_t) + b_j\Big) = a_i/(a_i + b_j)$$

$$Pr\Big(p(A_{t+1}) = p(A_t) - a_i\Big) = b_j/(a_i + b_j)$$

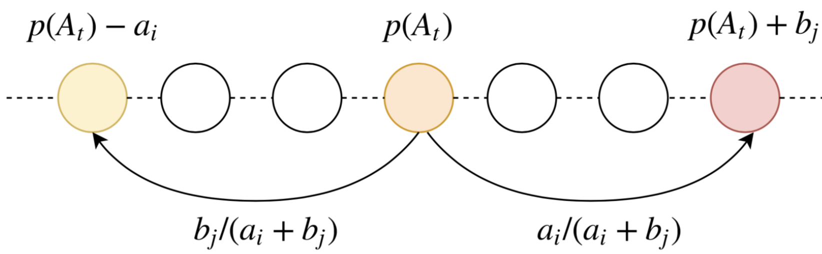

Figure 1: Position Transition from Round t to Round $t+1$

Recall that in a random walking process, let p_j be the probability that a walk starts at j and reaches k before it reaches 0, $p_j = j/k$ as showed in class.

Also, a fight between monsters with strength a_i and b_j can be treated as: a_i monsters with strength 1 are fighting with b_i monsters of strength 1 until there is no monster in one side (Without loss of generalization, we assume a_i and b_j are positive integers).

Therefore, we can treat the monster fighting as a random walk. Correspondingly, this walk model starts at $p(A_0) = \sum_{i=1}^m a_i$ and either end up at $p(A_{n+m-1}) = 0$ or $p(A_{n+m-1}) = \sum_{i=1}^m a_i + \sum_{i=1}^n b_i$.

According to the random walking formula, the probability of Alice's team winning the game is $\sum_{i=1}^m a_i/(\sum_{i=1}^m a_i + \sum_{i=1}^n b_i)$, Bob's team winning the game is $\sum_{i=1}^n b_i/(\sum_{i=1}^m a_i + \sum_{i=1}^n b_i)$.

Now let us repeat this setting with a slight change: it's not monsters but robots, and again strength-a-robot beats strength-b-robot with probability $\frac{a}{a+b}$. The only difference: the losing robot explodes, and the strength of the winning robot does not change.

***Exercise 4.6.** Show that the probability of Alice's team winning does not depend on the order in which Alice and Bob send their monsters into the arena. **Warning.** This is much more difficult.

Solution.

Observation: Every robot war with fixed robot order can be represented as a state-transition diagram. As shown in Figure 2, the state in i^{th} row and j^{th} column means "robot a_i fights with robot b_j".

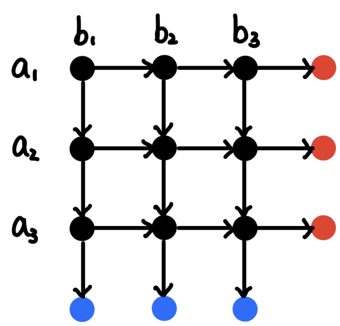

Figure 2: An example of robot war state-transition diagram

We define $P_{i,j}$ as the probability of this state occurring in the game. Correspondingly:

$$P_{i+1,j+1} = P_{i+1,j} \cdot \frac{a_{i+1}}{a_{i+1}+b_j} + P_{i,j+1} \cdot \frac{b_{j+1}}{a_i + b_{j+1}}$$

Suppose number of robots in team A is m, team B is n. Then:

$$\begin{cases} Pr(\text{team A win the game}) = \sum_{i=1}^{n} P_{m+1,i} = \text{Sum(blue vertices in the figure)} \\ Pr(\text{team B win the game}) = \sum_{i=1}^{m} P_{i,n+1} = \text{Sum(red vertices in the figure)} \end{cases}$$

Lemma 1: If there are only two robots in team A, the probability of A losing the game won't change after switching the order of those two robots.
Define P_{ij} as the probability before switching, P'_{ij} the probability after switching. A stronger proposition is that $P_{3,i} = P'_{3,i}$ for all $i \in 1, 2, \cdots, n$,

Proof Lemma 1: The corresponding state-transition diagram is shown in Figure 3.

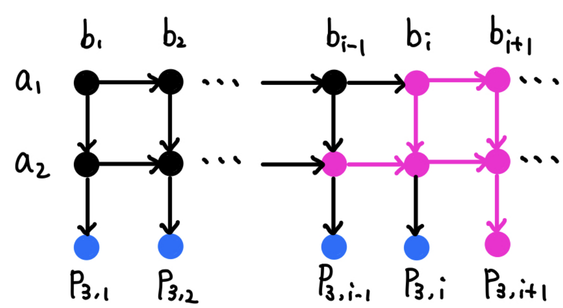

Figure 3: State-transition diagram of a special case

It's obvious that $P_{3,1} = P'_{3,1}$. Also, $P_{3,2} = P'_{3,2}$ is very easy to prove.
We assume, for any $k \leq i, (i \geq 2)$:

$$P_{3,k} = P'_{3,k}$$

Then, according to the relation formula of vertices given above, we get:

$$\begin{aligned}
P_{3,i-1} &= P_{2,i-1} \cdot \frac{b_{i-1}}{a_2 + b_{i-1}} \\
P_{3,i} &= P_{2,i} \cdot \frac{b_i}{a_2 + b_i} \\
P_{3,i+1} &= P_{2,i+1} \cdot \frac{b_{i+1}}{a_2 + b_{i+1}} \\
P_{2,i} &= P_{2,i-1} \cdot \frac{a_2}{a_2 + b_{i-1}} + P_{1,i} \cdot \frac{b_i}{a_1 + b_i} \\
P_{2,i+1} &= P_{2,i} \cdot \frac{a_2}{a_2 + b_i} + P_{1,i+1} \cdot \frac{b_{i+1}}{a_1 + b_{i+1}} \\
P_{1,i+1} &= P_{1,i} \cdot \frac{a_1}{a_1 + b_i}
\end{aligned}$$

The unknown variables in these 6 equations corresponds to the 6 pink vertices in Figure 3. By solving them, we get:

$$\begin{aligned}
P_{3,i+1} &= \frac{a_1 a_2 M + (a_1 + a_2) N}{b_{i-1} b_i^2 (a_1 + b_{i+1})(a_2 + b_{i+1})} \\
M &= b_{i+1}(P_{3,i}(b_{i+1} b_{i-1} + b_{i-1} b_i) - P_{3,i-1} b_i b_{i+1}) \\
N &= b_{i-1} b_i b_{i+1}^2 P_{3,i}
\end{aligned}$$

It's easy to see a_1 and a_2 are symmetrical for $P_{3,i+1}$. Also, because $P_{3,i-1} = P'_{3,i-1}$ and $P_{3,i} = P'_{3,i}$. Therefore:

$$P_{3,i+1} = \frac{a_1 a_2 M + (a_1 + a_2) N}{b_{i-1} b_i^2 (a_1 + b_{i+1})(a_2 + b_{i+1})} = \frac{a_2 a_1 M' + (a_2 + a_1) N'}{b_{i-1} b_i^2 (a_2 + b_{i+1})(a_1 + b_{i+1})} = P'_{3,i+1}$$

Then, by mathematical induction, we successfully prove lemma 1.

Lemma 2: If there are two or more than two robots in team A, the probability of A losing the game won't change after switching the order of two arbitrary neighboring robots.

Proof Lemma 1: The corresponding state-transition diagram is shown in Figure 4.

We zoom in and construct a new diagram shown as Figure 5.

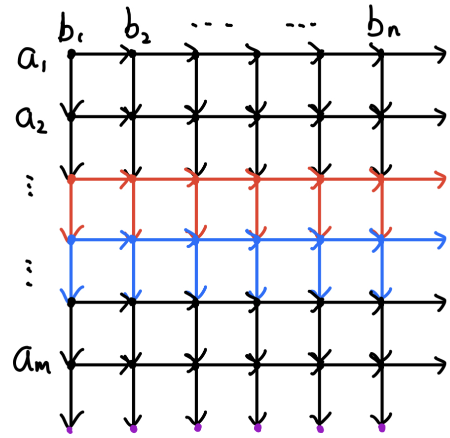

Figure 4: State-transition diagram of general case

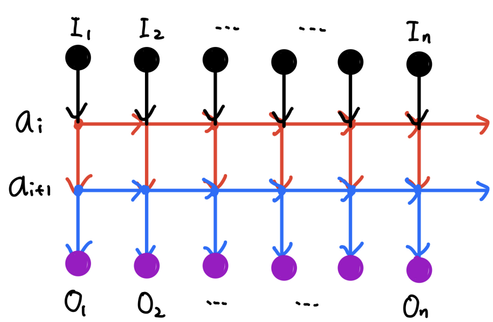

Figure 5: Constructed state-transition diagram

I_k is the input corresponding to $P_{i-1,k}$(when $i = 1$, then $I_1 = 1$, $I_k = 0$ for $k \in 2, 3, \cdots, n$). O_k is the output.
To prove lemma 2, we only have to prove: given inputs, outputs won't change after switching.
Suppose $I_0 = 1$ while other inputs are zero, we get outputs $(O_1^{<1>}, O_2^{<1>}, \cdots, O_n^{<1>})$.
Similarly, suppose $I_j = 1$ while other inputs are zero, we get outputs $(O_1^{<j>}, O_2^{<j>}, \cdots, O_n^{<j>})$.
Actually $O_k^{<j>} = 0$ when $k < j$. It's shown in Figure 6

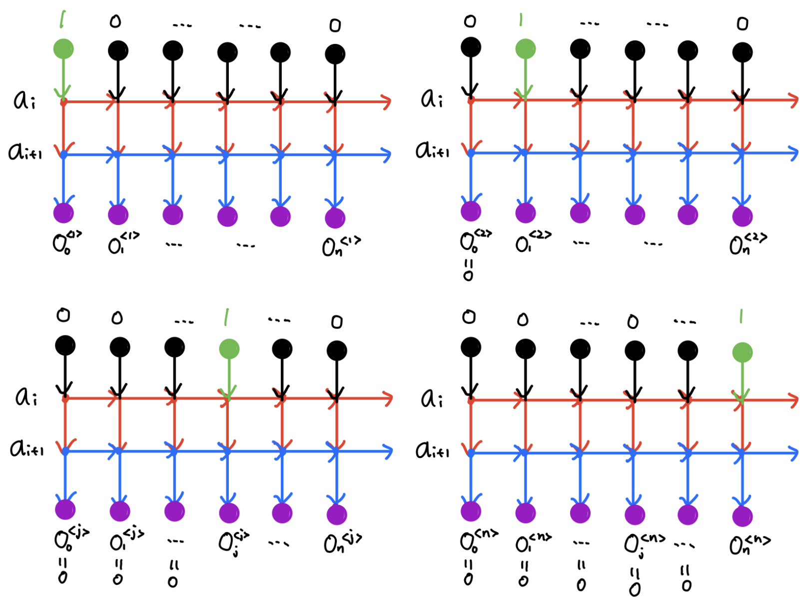

Figure 6: Divided state-transition diagram

It's easy to get that:

$$O_k = \sum_{j=1}^{n} I_j \cdot O_k^{<j>}$$

According to our lemma 1, $O^{<j>}$ won't change when after switching. Therefore, O won't change after switching. Then we successfully prove lemma 2.

Proof Exercise 4.6: As a truth, given a sequence $(1, 2, \cdots, n)$, we can get any permutation of this sequence by constantly switching two neighboring

elements.
According to Lemma 2, the probability of winning stays unchanged after switching two neighboring robots of team A. It's also true that probability of winning stays unchanged after switching two neighboring robots of team B.
We can get any sequence of team A and team B after constantly switching. Therefore, the probability of Alice's team winning does not depend on the order in which Alice and Bob send their robots into the arena