Problem Solving II: Homework #1

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Problem 1

4.1 Ask the oracle. Suppose that we have access to an oracle, as in Problem 1.10, who can answer the decision problem SAT. Show that, by asking her a polynomial number of questions, we can find a satisfying assignment if one exists. Hint: first describe how setting a variable gives a smaller SAT formula on the remaining variables. Show that this is true of SUBSETSUM as well. The property is called self-reducibility, and we will meet it again in Chapter 9 and 13.

solution:
首先给出 SAT 问题的变量：$x_1, x_2, \ldots, x_n$，合取范式 $C = C_1 \cap C_2 \cap \cdots \cap C_m$ ，其中 $C_1, C_2, \ldots, C_m$ 都是析取范式。
首先我们问上帝，当 $x_1 = 1$ 时，C 能够满足吗? 如果上帝的答案是 yes，那么设 $x_1 = 1$，否则设 $x_1 = 0$。一直这样问下去，直到 n。
因此问题数目是 n 次。
对于 SUBSETSUM 问题，首先给出判定 SUBSETSUM 问题的输入和输出：

> SUBSETSUM
> 输入：A set $S = \{x_1, \ldots, x_l\}$ of positive integers and an integer t
> 输出：Does there exist a subset $A \subset \{1, \ldots, l\}$ s.t. $\sum_{i \in A} x_i = t$?

下面给出在给定判定算法的情况下解决 SUBSETSUM 问题的伪代码。

输入: set $S = \{x_1, \ldots, x_l\}$ of positive integers and an integer t
输出: a subset $A \subset \{1, \ldots, l\} s.t. \sum_{i \in A} x_i = t$

function SubsetSUM(S, m)
  for j=1,j<l,j++ **do**
    $T = S - x_i$
    boolean B=Ask the oracle if T is satisfied.
    if B=True **then**
      S=T
    end if
  end for
end function

算法 1: Polynomial time algorithm to find Subsetsum when given identify algorithm

实质就是问上帝子问题能不能满足。

Problem 2

4.12 Same-sum subsets. Suppose that S is a list of 10 distinct integers, each ranging from 0 to 100. Show that there are two distinct disjointed sublists $A, B \subset S$ with the same total. (Note that unlike in INTEGER PARTITIONING, we don't demand that $A \cup B = S$.). that doesn't make it easy to find one!

solution: 这个问题可以用概率方法来解。S 是 10 个不同整数的集合，因此 S 一共有 $2^{10} - 1 = 1023$ 个不同的子集，十个不同整数的和最大为 955（$955 = 100 + 99 + \cdots + 91$），因而如果 A 是 S 的子集，那么 A 中元素的和最大也为 955（最小为 0）。因此根据容斥原理（1023>955），存在两个不同的非空

Problem 2 continued on next page...

子集 A $B \in S$ 使得 A，B 的和一样。
如果 $A \cap B \neq \emptyset$，令 $C = A \cap B$，$A' = A - C$，$B' = B - C$。我们有：

$$\begin{aligned} sum(A') &= sum(A - C) \\ &= sum(A) - sum(C) \\ &= sum(B) - sum(C) \\ &= sum(B - C) \\ &= sum(B') \end{aligned}$$

因此：对于给定的有十个整数的集合 $S \in [100]$，能够找到两个非空不相交的子集，他们有相同的和。

Problem 3

4.14. when greed works. A sequence $a_1, \dots, a_n$ of integers is called superincreasing if each elements a_i is strictly greater that the same of all previous elements. Show that SUBSET SUM can be solved in polynomial time is S is superincreasing. Hint: be greedy. What does you algorithm do in the case $S = \{1, 2, 4, \dots\}$?
solution:
直接给出贪心算法：

输入: A sequence $S = \{a_1, \dots, a_n\}$, a given number m
输出: Set $H \subset S$ satisfying $Sum(H) = m$

function SubsetSUM(S, m)
  //检查 S 是否是 Sumperincreasing Sequence
  $n = S.length()$
  for i=0;i<n-1;i++ **do**
    $sum = \sum_{j=1}^{i+1} a_j$
    if $a_{i+2} < sum$ **then**
      return Error
    end if
  end for
  //贪心求出不大于 m 的最大的值的 index
  while $m \leq a_1$ **do**
    $t = sup_k\{k : a_k \leq m\}$
    result.push(t)
    $m = m - a_k$
  end while
  return result
end function

算法 2: Greedy SUBSET SUM For Superincreasing Sequence

解释一下，上述的伪代码分为两个部分：第一个部分检验输入集合是否为 Superincreasing Sequence，如果不是，直接返回。第二个部分利用贪心算法直接求出不大于 m 的最大的 $a_k \in S$，并将 k push 到结果栈中。并更新 m，迭代。
对于集合 $S = \{1, 2, 4, \dots\}$, 输入 m，就相当于对 m 进行二进制编码，然后输出为 1 的项而已。

Problem 4

5.1. From circuits to NAESAT. Find a direct reduction from CIRCUIT SAT to NAE-3SAT. Hint: Consider the 3-SAT clause $(\bar{x}_1 \cup \bar{x}_2 \cup y)$ in our description of an AND gate. Does it ever happen that all three literals are true?

solution:

首先给出 SAT 的定义：

SAT 输入：合取范式 $\phi(x_1, \ldots, x_n)$ 输出：ϕ 可满足吗?

给出 CIRCUIT SAT 的定义：

CIRCUIT SAT 输入：A Boolean circuit C 输出：C 可满足吗?

CIRCUIT SAT 归约到 NAE-3SAT，首先将逻辑门电路写成逻辑表达式的形式，然后将逻辑表达式转化成 3SAT，然后将 3SAT 通过增加变元的方式转化成 NAE-3SAT。从而 CIRCUIT 的输入就是 NAE-3-SAT 的输入，而转化之后的 NAE-3-SAT 的一个 yes-instance 也是 CIRCUIT SAT 的一个 yes-instance。$(\bar{x}_1 \cup \bar{x}_2 \cup y)$ 即是一个主析取范式，也是一个主合取范式。所以想要用 AND gate 来描述，只需要将 $(\bar{x}_1 \cup \bar{x}_2 \cup y)$ 改写成 $(\bar{x}_1 \cup \bar{x}_2 \cup y) \cap (y \cup \bar{y})$ 即可。

Problem 5

5.3. Easy majority? Consider MAJORITY-OF-3SAT in which each clause contains three literals and at least two of them must be true for it to be satisfied. Either show that this problem is in P by reducing it to 2-SAT, or show that it is NP-complete by reducing 3-SAT to it. **solution:** MAJORITY-OF-3SAT 是 NP-complete 的，可以通过将 3-SAT 归约到该问题来证明。

只需要考虑 3-SAT 中的表达式 $x_1 \cup x_2 \cup x_3$，下面将其转换成 MAJORITY-OF-3SAT 的表达式子：

$$\begin{aligned} x_1 \cup x_2 \cup x_3 &= x_1 \cup x_2 \cup x_3 \cup x_4 \cup \bar{x}_4 \cup x_5 \cup \bar{x}_5 \\ &= x_1 \cup \bar{x}_4 \cup x_2 \cup x_3 \cup x_4 \cup \bar{x}_5 \cup x_5 \\ &= (x_1 \cup \bar{x}_4 \cup y_1) \cap (\bar{y}_1 \cup x_2 \cup y_2) \cap (\bar{y}_2 \cup x_3 \cup y_3) \cap (\bar{y}_3 \cup x_4 \cup y_4) \cap (\bar{y}_4 \cup \bar{x}_5 \cup y_5) \end{aligned}$$

因此，3-SAT 的任何输入都可以转换成 MAJORITY-OF-3SAT，从而 3-SAT 可以归约到 MAJORITY-OF-3SAT。因此 3-SAT 是 NP complete。