

Problem Solving II: Homework #1

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Problem 1

4.1 Ask the oracle. Suppose that we have access to an oracle, as in Problem 1.10, who can answer the decision problem SAT. Show that, by asking her a polynomial number of questions, we can find a satisfying assignment if one exists. Hint: first describe how setting a variable gives a smaller SAT formula on the remaining variables. Show that this is true of SUBSETSUM as well. The property is called self-reducibility, and we will meet it again in Chapter 9 and 13.

solution:

首先给出 SAT 问题的变量: x_1, x_2, \dots, x_n , 合取范式 $C = C_1 \cap C_2 \cap \dots \cap C_m$, 其中 C_1, C_2, \dots, C_m 都是析取范式。

首先我们问上帝, 当 $x_1 = 1$ 时, C 能够满足吗? 如果上帝的答案是 yes, 那么设 $x_1 = 1$, 否则设 $x_1 = 0$ 。一直这样问下去, 直到 n 。

因此问题数目是 n 次。

对于 SUBSETSUM 问题, 首先给出判定 SUBSETSUM 问题的输入和输出:

SUBSETSUM

输入: A set $S = \{x_1, \dots, x_l\}$ of positive integers and an integer t

输出: Does there exist a subset $A \subset \{1, \dots, l\}$ s.t. $\sum_{i \in A} x_i = t$?

下面给出在给定判定算法的情况下解决 SUBSETSUM 问题的伪代码。

输入: set $S = \{x_1, \dots, x_l\}$ of positive integers and an integer t

输出: a subset $A \subset \{1, \dots, l\}$ s.t. $\sum_{i \in A} x_i = t$

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1: function SUBSETSUM( $S, m$ )
2:   for  $j=1, j < l, j++$  do
3:      $T = S - x_i$ 
4:     boolean  $B = \text{Ask the oracle if } T \text{ is satisfied.}$ 
5:     if  $B = \text{True}$  then
6:        $S = T$ 
7:     end if
8:   end for
9: end function
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算法 1: Polynomial time algorithm to find Subsetsum when given identify algorithm

实质就是问上帝子问题能不能满足。

Problem 2

4.12 Same-sum subsets. Suppose that S is a list of 10 distinct integers, each ranging from 0 to 100. Show that there are two distinct disjoint sublists $A, B \subset S$ with the same total. (Note that unlike in INTEGER PARTITIONING, we don't demand that $A \cup B = S$.) that doesn't make it easy to find one!

solution: 这个问题可以用概率方法来解。 S 是 10 个不同整数的集合, 因此 S 一共有 $2^{10} - 1 = 1023$ 个不同的子集, 十个不同整数的和最大为 955 ($955 = 100 + 99 + \dots + 91$), 因而如果 A 是 S 的子集, 那么 A 中元素的和最大也为 955 (最小为 0)。因此根据容斥原理 ($1023 > 955$), 存在两个不同的非空

子集 $A, B \in S$ 使得 A, B 的和一样。

如果 $A \cap B \neq \emptyset$, 令 $C = A \cap B$, $A' = A - C$, $B' = B - C$ 。我们有:

$$\begin{aligned} \text{sum}(A') &= \text{sum}(A - C) \\ &= \text{sum}(A) - \text{sum}(C) \\ &= \text{sum}(B) - \text{sum}(C) \\ &= \text{sum}(B - C) \\ &= \text{sum}(B') \end{aligned}$$

因此: 对于给定的有十个整数的集合 $S \in [100]$, 能够找到两个非空不相交的子集, 他们有相同的和。

Problem 3

4.14. when greed works. A sequence a_1, \dots, a_n of integers is called superincreasing if each elements a_i is strictly greater than the sum of all previous elements. Show that SUBSET SUM can be solved in polynomial time if S is superincreasing. Hint: be greedy. What does your algorithm do in the case $S = \{1, 2, 4, \dots\}$?

solution:

直接给出贪心算法:

输入: A sequence $S = \{a_1, \dots, a_n\}$, a given number m

输出: Set $H \subset S$ satisfying $\text{Sum}(H) = m$

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1: function SUBSETSUM( $S, m$ )
2:   //检查 S 是否是 Superincreasing Sequence
3:    $n = S.length()$ 
4:   for  $i=0; i < n-1; i++$  do
5:      $sum = \sum_{j=1}^{i+1} a_j$ 
6:     if  $a_{i+2} < sum$  then
7:       return Error
8:     end if
9:   end for
10:  //贪心求出不大于 m 的最大的值的 index
11:  while  $m \leq a_1$  do
12:     $t = \sup_k \{k : a_k \leq m\}$ 
13:     $result.push(t)$ 
14:     $m = m - a_k$ 
15:  end while
16:  return result
17: end function

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算法 2: Greedy SUBSET SUM For Superincreasing Sequence

解释一下, 上述的伪代码分为两个部分: 第一个部分检验输入集合是否为 Superincreasing Sequence, 如果不是, 直接返回。第二个部分利用贪心算法直接求出不大于 m 的最大的 $a_k \in S$, 并将 k push 到结果栈中。并更新 m , 迭代。

对于集合 $S = \{1, 2, 4, \dots\}$, 输入 m , 就相当于对 m 进行二进制编码, 然后输出为 1 的项而已。

Problem 4

5.1. From circuits to NAE3SAT. Find a direct reduction from CIRCUIT SAT to NAE-3SAT. Hint: Consider the 3-SAT clause $(\bar{x}_1 \cup \bar{x}_2 \cup y)$ in our description of an AND gate. Does it ever happen that all three literals are true?

solution:

首先给出 SAT 的定义:

SAT

输入: 合取范式 $\phi(x_1, \dots, x_n)$

输出: ϕ 可满足吗?

给出 CIRCUIT SAT 的定义:

CIRCUIT SAT

输入: A Boolean circuit C

输出: C 可满足吗?

CIRCUIT SAT 归约到 NAE-3SAT, 首先将逻辑门电路写成逻辑表达式的形式, 然后将逻辑表达式转化成 3SAT, 然后将 3SAT 通过增加变元的方式转化成 NAE-3SAT。从而 CIRCUIT 的输入就是 NAE-3SAT 的输入, 而转化之后的 NAE-3-SAT 的一个 yes-instance 也是 CIRCUIT SAT 的一个 yes-instance。 $(\bar{x}_1 \cup \bar{x}_2 \cup y)$ 即是一个主析取范式, 也是一个主合取范式。所以想要用 AND gate 来描述, 只需要将 $(\bar{x}_1 \cup \bar{x}_2 \cup y)$ 改写成 $(\bar{x}_1 \cup \bar{x}_2 \cup y) \cap (y \cup \bar{y})$ 即可。

Problem 5

5.3. Easy majority? Consider MAJORITY-OF-3SAT in which each clause contains three literals and at least two of them must be true for it to be satisfied. Either show that this problem is in P by reducing it to 2-SAT, or show that it is NP-complete by reducing 3-SAT to it.

solution:

MAJORITY-OF-3SAT 是 P 的, 可以通过将该问题归约到 2SAT 来证明。

介于 clause 满足条件 at least two of them must be true, 因此每个 clause 中的三个变元 x_i, x_j, x_k 必须能够 $(x_i \cap x_j) \cup (x_i \cap x_k) \cup (x_k \cap x_j)$ 表达出。而:

$$(x_i \cap x_j) \cup (x_i \cap x_k) \cup (x_k \cap x_j) = (x_i \cup x_j) \cap (x_i \cup x_k) \cap (x_k \cup x_j)$$

从而 MAJORITY-OF-3SAT 可以归约到 2-SAT, 因此 MAJORITY-OF-3SAT 是 P 类的。