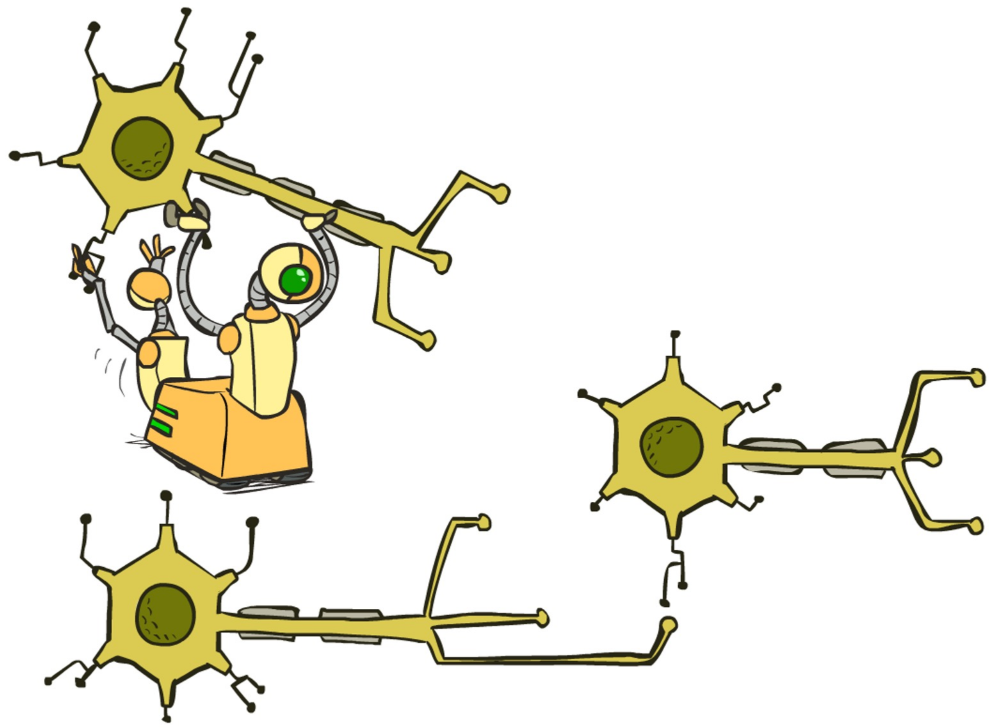
CS 188: Artificial Intelligence

Neural Networks

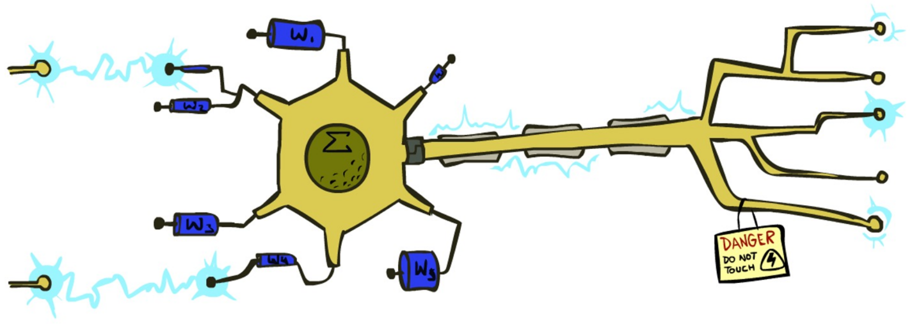


Summer 2024: Eve Fleisig & Evgeny Pobachienko

[Slides drawn from Nicholas Tomlin, Dan Klein, and Pieter Abbeel for CS188 at UC Berkeley. All CS188 materials are available at http://ai.berkeley.edu.]

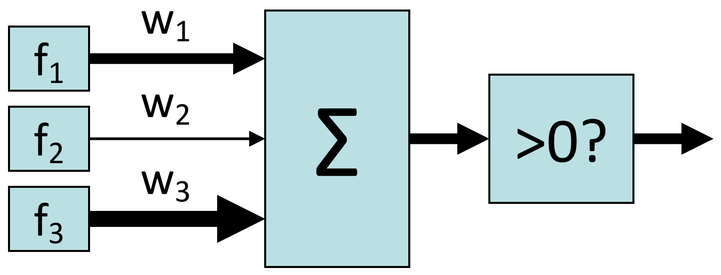
Reminder: Linear Classifiers

- Inputs are feature values
- Each feature has a weight
- Sum is the activation

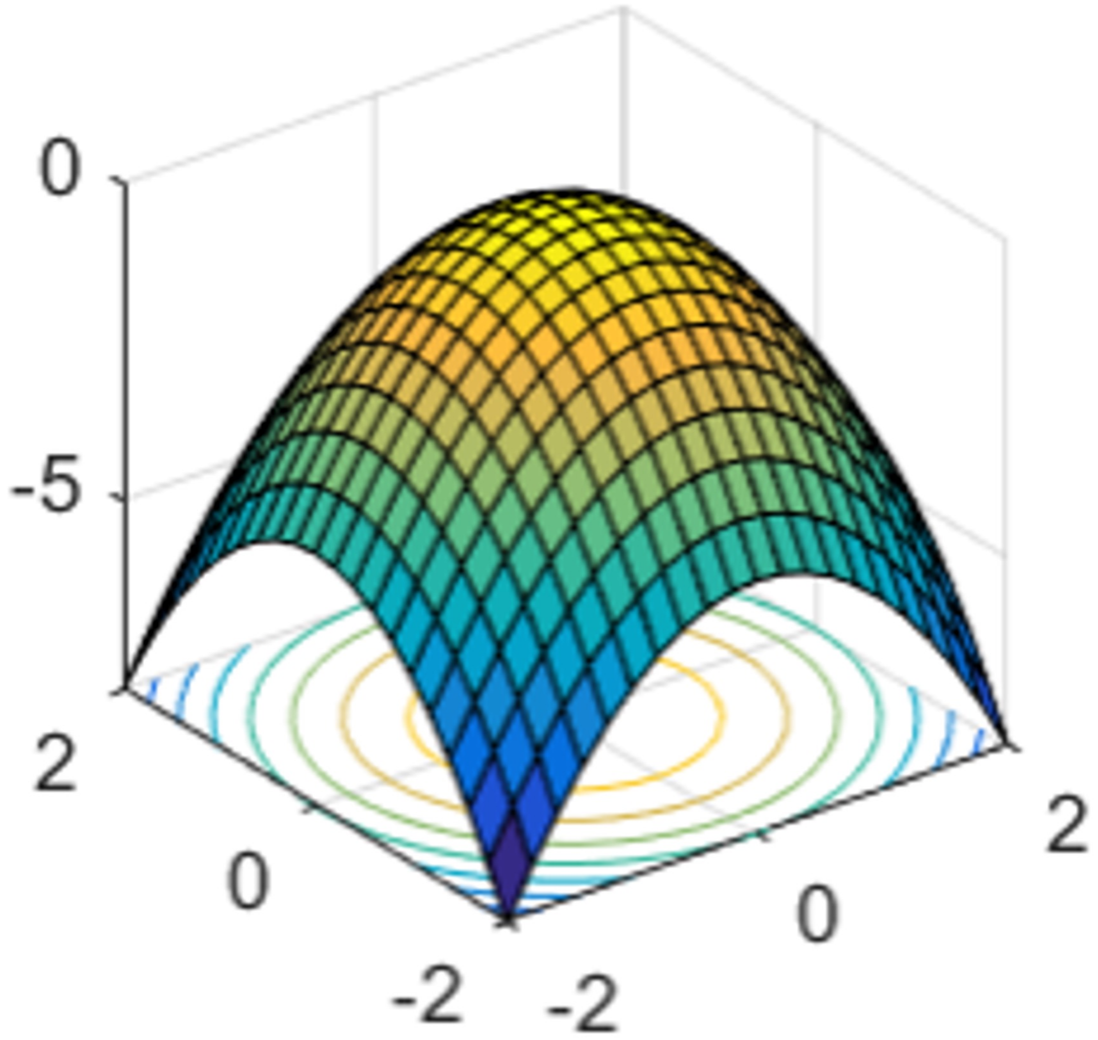


$$\text{activation}_w(x) = \sum_i w_i \cdot f_i(x) = w \cdot f(x)$$

- If the activation is:
  - Positive, output +1
  - Negative, output -1

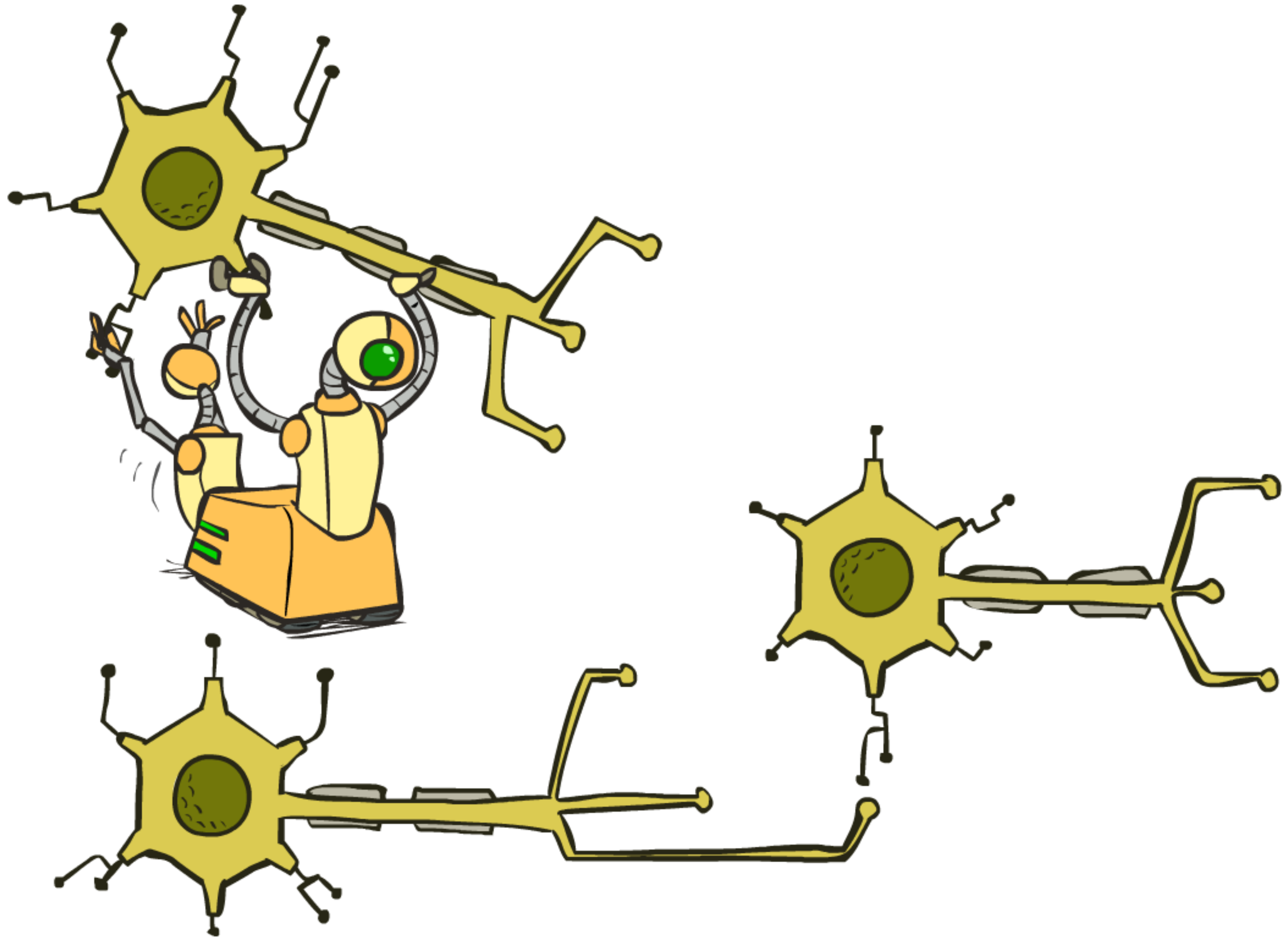


2-D Optimization



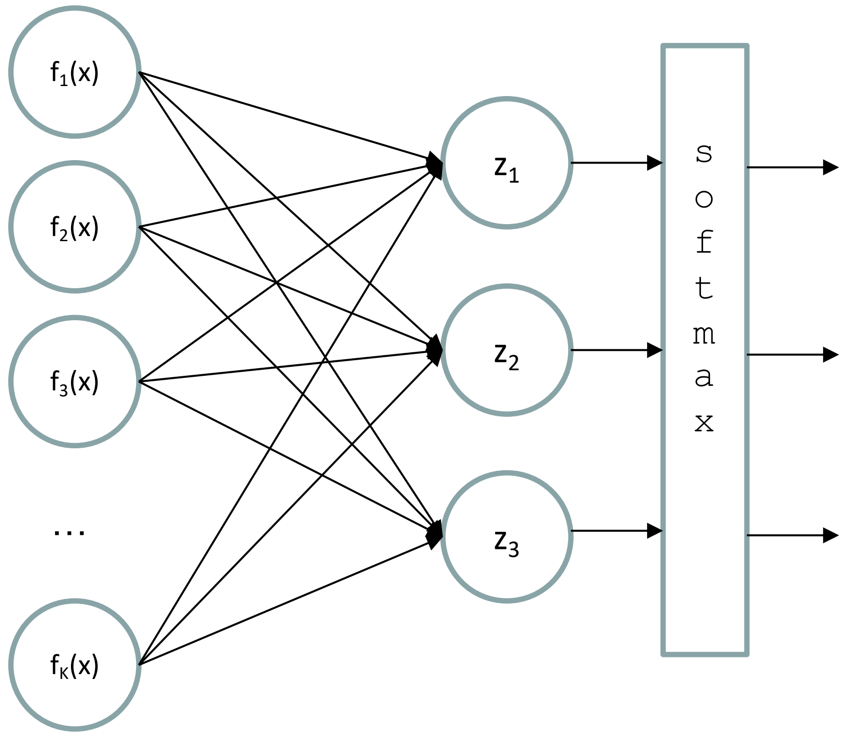
Source: offconvex.org

Neural Networks



Multi-class Logistic Regression

- = special case of neural network

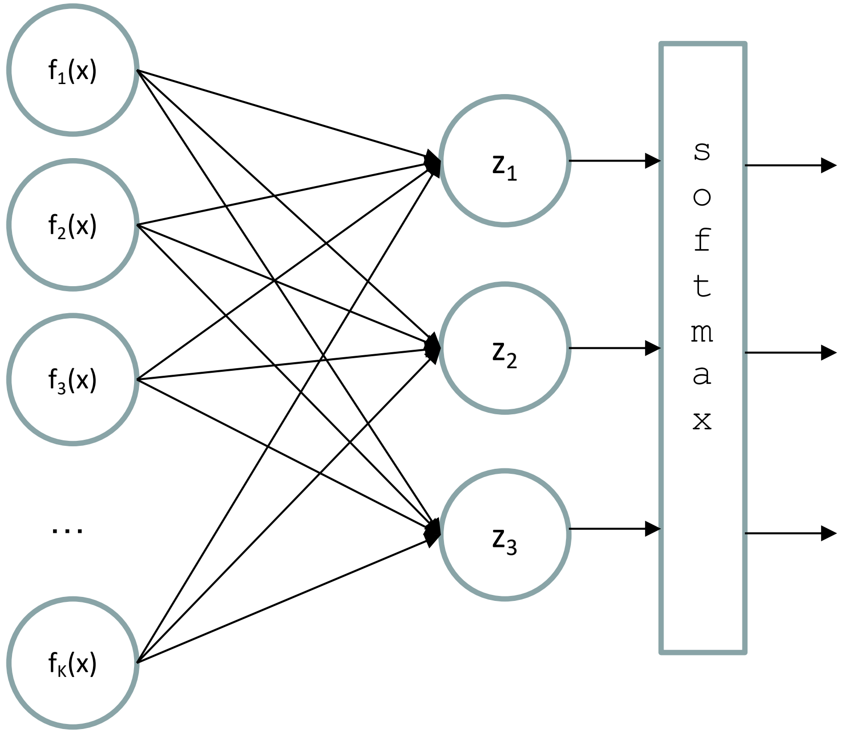


$$P(y_1|x;w) = \frac{e^{z_1}}{e^{z_1}+e^{z_2}+e^{z_3}}$$

$$P(y_2|x;w) = \frac{e^{z_2}}{e^{z_1}+e^{z_2}+e^{z_3}}$$

$$P(y_3|x;w) = \frac{e^{z_3}}{e^{z_1}+e^{z_2}+e^{z_3}}$$

Deep Neural Network = Also learn the features!



$$P(y_1|x;w) = \frac{e^{z_1}}{e^{z_1} + e^{z_2} + e^{z_3}}$$

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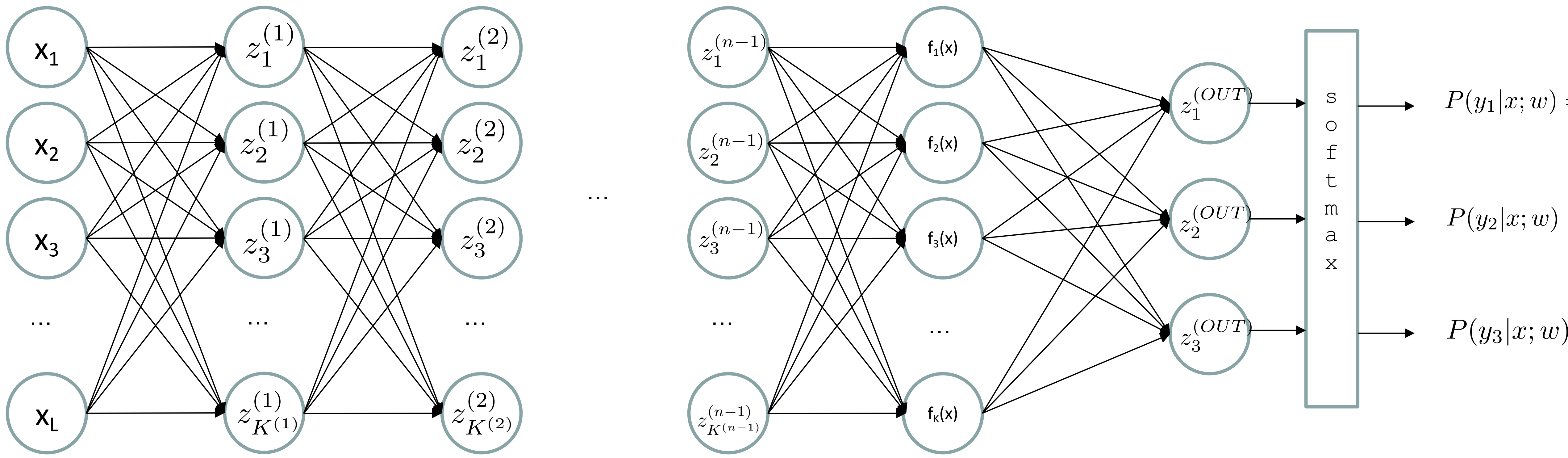
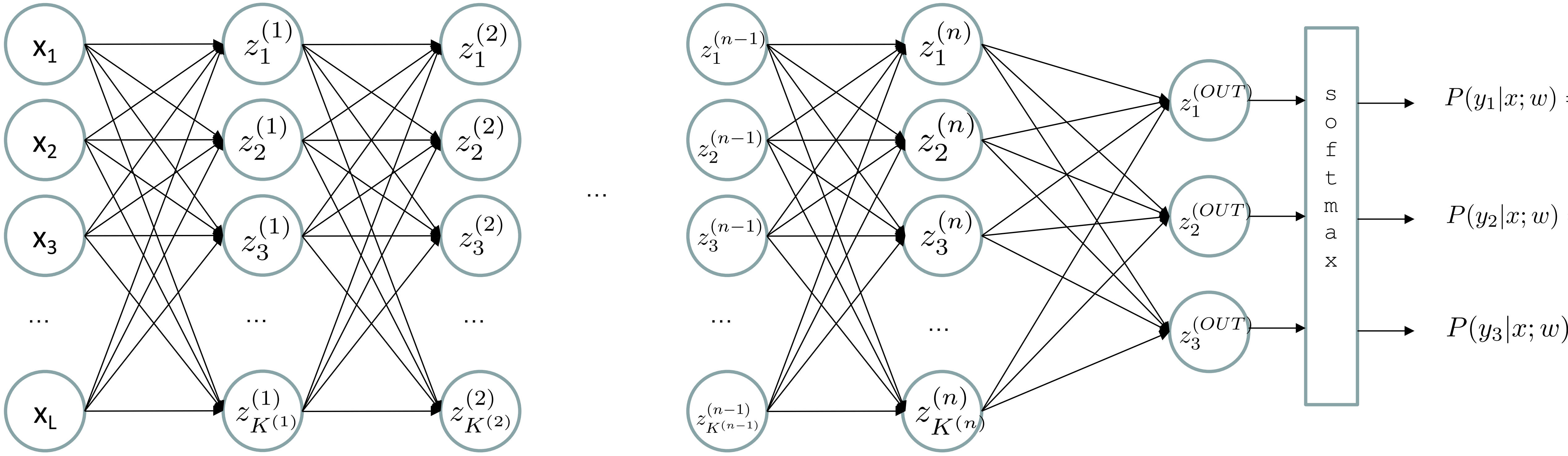


$$z_i^{(k)} = g(\sum_j W_{i,j}^{(k-1,k)} z_j^{(k-1)})$$

Deep Neural Network = Also learn the features!

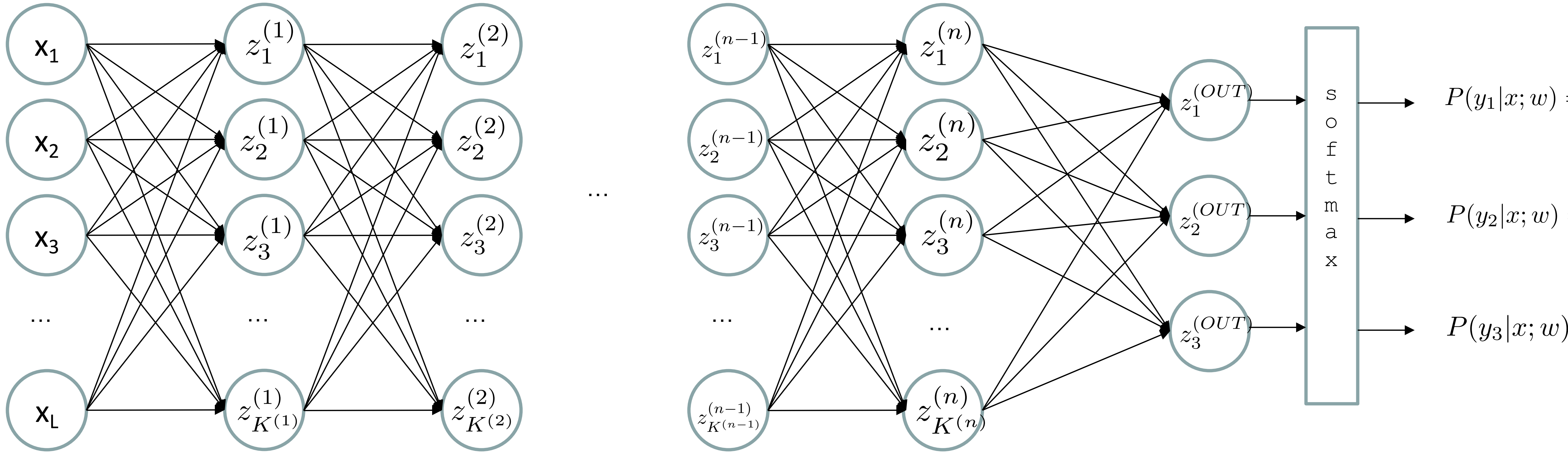


$$z_i^{(k)} = g\left(\sum_j W_{i,j}^{(k-1,k)} z_j^{(k-1)}\right)$$

Importance of Nonlinear Activation Functions

- What happens if we add more layers?
- $z_2 = W_2 (W_1 x + b_1) + b_2$
- $z_2 = W_2 (W_1 x + b_1) + b_2 = W_2 W_1 x + W_2 b_1 + b_2 = W_{new} x + b_{new}$
- No gain to adding more linear layers!
- Idea: add nonlinearities to capture more complex relationships

Deep Neural Network = Also learn the features!

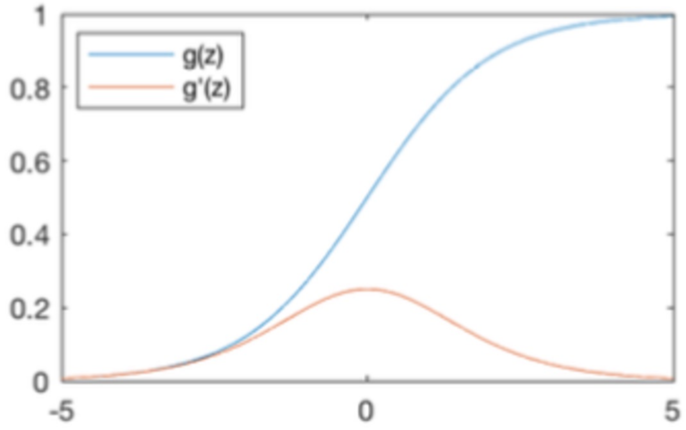


$$z_i^{(k)} = g(\sum_j W_{i,j}^{(k-1,k)} z_j^{(k-1)})$$

g = nonlinear activation function

Common Activation Functions

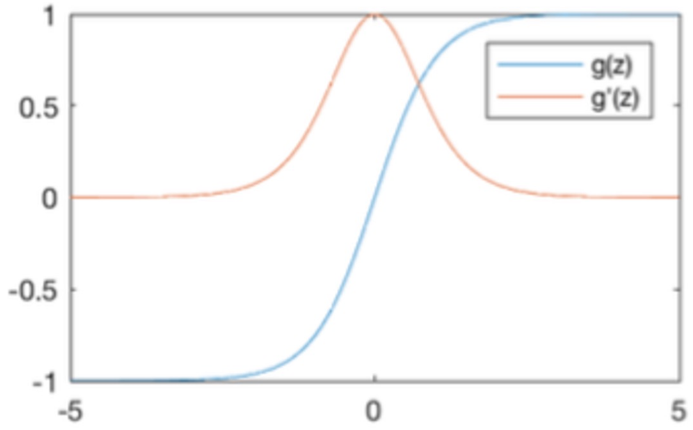
Sigmoid Function



$$g(z) = \frac{1}{1 + e^{-z}}$$

$$g'(z) = g(z)(1 - g(z))$$

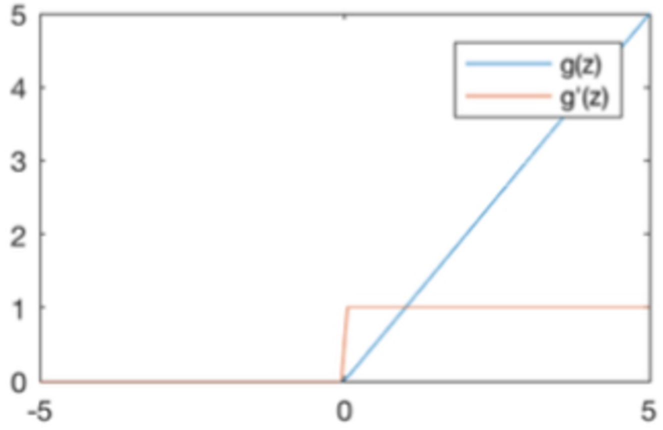
Hyperbolic Tangent



$$g(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$g'(z) = 1 - g(z)^2$$

Rectified Linear Unit (ReLU)



$$g(z) = \max(0, z)$$

$$g'(z) = \begin{cases} 1, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

[source: MIT 6.S191 introtodeeplearning.com]

Deep Neural Network: Also Learn the Features!

- Training the deep neural network is just like logistic regression:

$$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)} | x^{(i)}; w)$$

just w tends to be a much, much larger vector ☺️

⍰ just run gradient ascent

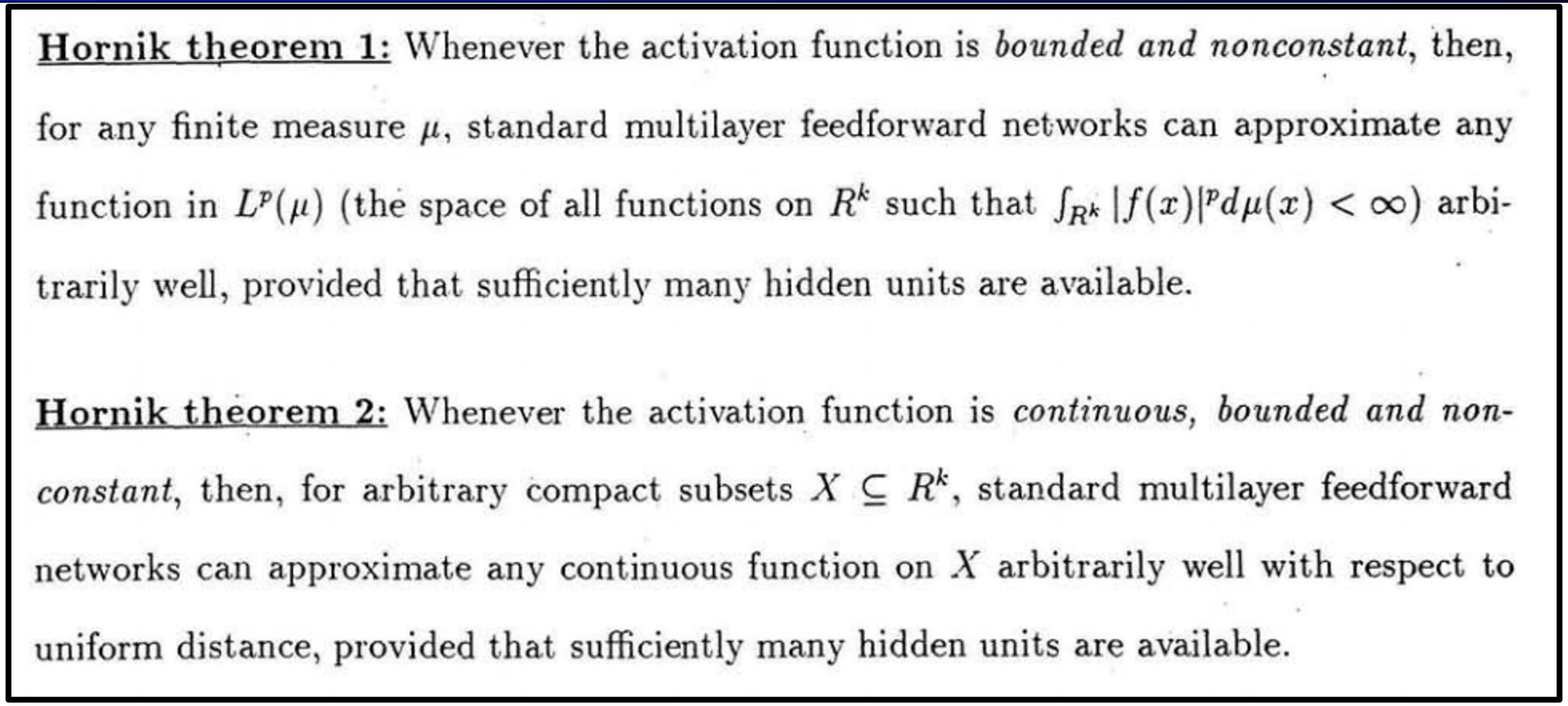
+ stop when log likelihood of hold-out data starts to decrease

Neural Networks Properties

- Theorem (Universal Function Approximators). A two-layer neural network with a sufficient number of neurons can approximate any continuous function to any desired accuracy.

- Practical considerations
  - Can be seen as learning the features
  - Large number of neurons
    - Danger for overfitting
    - (hence early stopping!)

Universal Function Approximation Theorem*



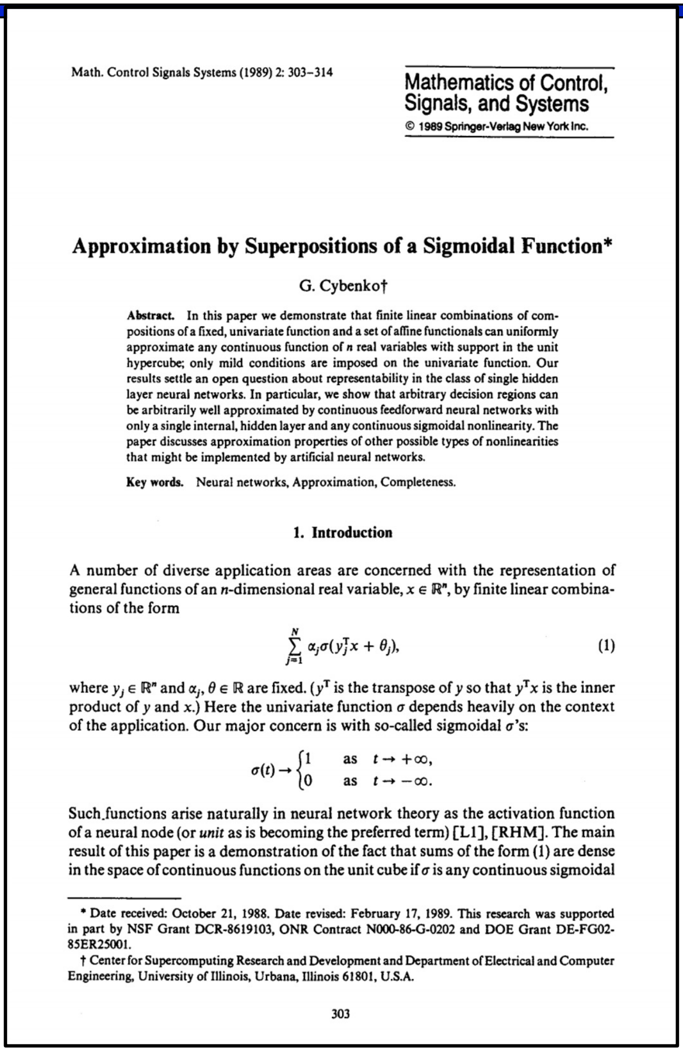
Hornik theorem 1: Whenever the activation function is *bounded and nonconstant*, then, for any finite measure μ, standard multilayer feedforward networks can approximate any function in $L^p(\mu)$ (the space of all functions on R^k such that $\int_{R^k} |f(x)|^p d\mu(x) < \infty$) arbitrarily well, provided that sufficiently many hidden units are available.

Hornik theorem 2: Whenever the activation function is *continuous, bounded and nonconstant*, then, for arbitrary compact subsets $X \subseteq R^k$, standard multilayer feedforward networks can approximate any continuous function on X arbitrarily well with respect to uniform distance, provided that sufficiently many hidden units are available.

- In words: Given any continuous function f(x), if a 2-layer neural network has enough hidden units, then there is a choice of weights that allow it to closely approximate f(x).

Cybenko (1989) "Approximations by superpositions of sigmoidal functions"
Hornik (1991) "Approximation Capabilities of Multilayer Feedforward Networks"
Leshno and Schocken (1991) "Multilayer Feedforward Networks with Non-Polynomial Activation Functions Can Approximate Any Function"

Universal Function Approximation Theorem*



Math. Control Signals Systems (1989) 2: 303–314

Mathematics of Control, Signals, and Systems
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Approximation by Superpositions of a Sigmoidal Function*

G. Cybenko†

Abstract. In this paper we demonstrate that finite linear combinations of compositions of a fixed, univariate function and a set of affine functionals can uniformly approximate any continuous function of n real variables with support in the unit hypercube; only mild conditions are imposed on the univariate function. Our results settle an open question about representability in the class of single hidden layer neural networks. In particular, we show that arbitrary decision regions can be arbitrarily well approximated by continuous feedforward neural networks with only a single internal, hidden layer and any continuous sigmoidal nonlinearity. The paper discusses approximation properties of other possible types of nonlinearities that might be implemented by artificial neural networks.

Key words. Neural networks, Approximation, Completeness.

1. Introduction

A number of diverse application areas are concerned with the representation of general functions of an n-dimensional real variable, $x \in \mathbb{R}^n$, by finite linear combinations of the form

$$\sum_{j=1}^{N} \alpha_j \sigma(y_j^{\mathrm{T}} x + \theta_j), \tag{1}$$

where $y_j \in \mathbb{R}^n$ and $\alpha_j, \theta \in \mathbb{R}$ are fixed. (y^{T} is the transpose of y so that $y^{\mathrm{T}}x$ is the inner product of y and x.) Here the univariate function σ depends heavily on the context of the application. Our major concern is with so-called sigmoidal σ's:

$$\sigma(t) \to \begin{cases} 1 & \text{as} \quad t \to +\infty, \\ 0 & \text{as} \quad t \to -\infty. \end{cases}$$

Such functions arise naturally in neural network theory as the activation function of a neural node (or *unit* as is becoming the preferred term) [L1], [RHM]. The main result of this paper is a demonstration of the fact that sums of the form (1) are dense in the space of continuous functions on the unit cube if σ is any continuous sigmoidal

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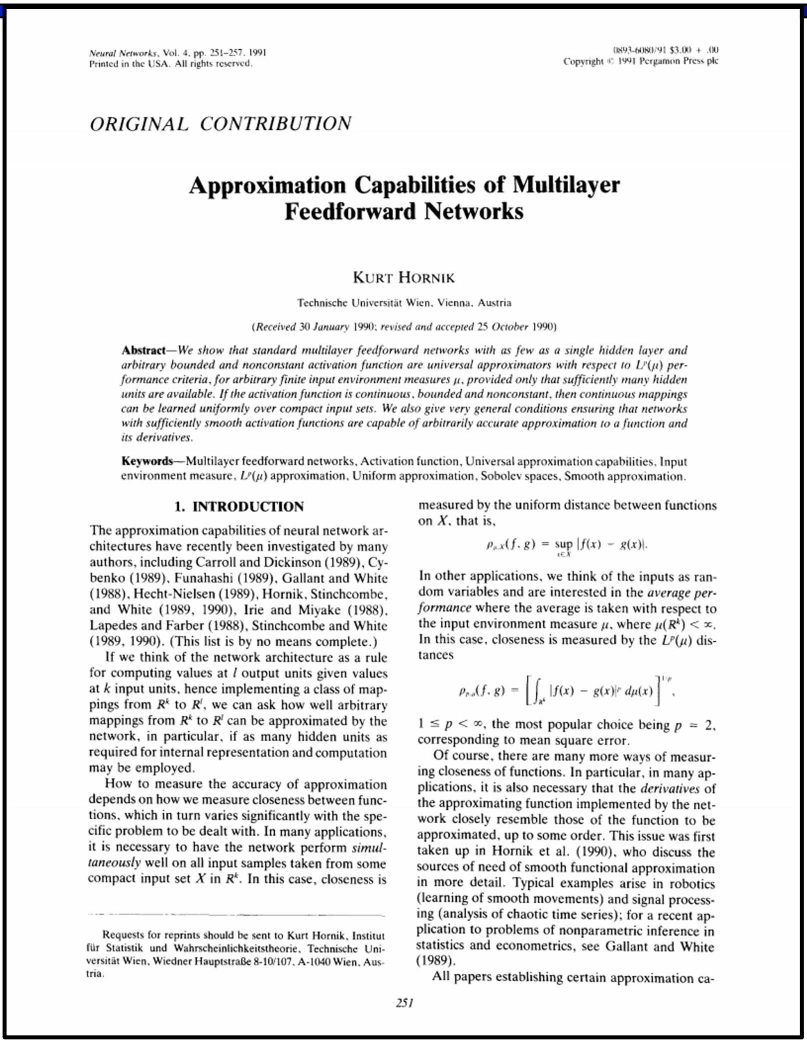

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ORIGINAL CONTRIBUTION

Approximation Capabilities of Multilayer Feedforward Networks

KURT HORNIK

Technische Universität Wien, Vienna, Austria

(*Received 30 January* 1990; *revised and accepted 25 October* 1990)

Abstract—*We show that standard multilayer feedforward networks with as few as a single hidden layer and arbitrary bounded and nonconstant activation function are universal approximators with respect to $L^p(\mu)$ performance criteria, for arbitrary finite input environment measures μ, provided only that sufficiently many hidden units are available. If the activation function is continuous, bounded and nonconstant, then continuous mappings can be learned uniformly over compact input sets. We also give very general conditions ensuring that networks with sufficiently smooth activation functions are capable of arbitrarily accurate approximation to a function and its derivatives.*

Keywords—Multilayer feedforward networks, Activation function, Universal approximation capabilities, Input environment measure, $L^p(\mu)$ approximation, Uniform approximation, Sobolev spaces, Smooth approximation.

1. INTRODUCTION

The approximation capabilities of neural network architectures have recently been investigated by many authors, including Carroll and Dickinson (1989), Cybenko (1989), Funahashi (1989), Gallant and White (1988), Hecht-Nielsen (1989), Hornik, Stinchcombe, and White (1989, 1990), Irie and Miyake (1988), Lapedes and Farber (1988), Stinchcombe and White (1989, 1990). (This list is by no means complete.)

If we think of the network architecture as a rule for computing values at l output units given values at k input units, hence implementing a class of mappings from R^k to R^l, we can ask how well arbitrary mappings from R^k to R^l can be approximated by the network, in particular, if as many hidden units as required for internal representation and computation may be employed.

How to measure the accuracy of approximation depends on how we measure closeness between functions, which in turn varies significantly with the specific problem to be dealt with. In many applications, it is necessary to have the network perform *simultaneously* well on all input samples taken from some compact input set X in R^k. In this case, closeness is measured by the uniform distance between functions on X, that is,

$$\rho_{\mu,X}(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

In other applications, we think of the inputs as random variables and are interested in the *average performance* where the average is taken with respect to the input environment measure μ, where $\mu(R^k) < \infty$. In this case, closeness is measured by the $L^p(\mu)$ distances

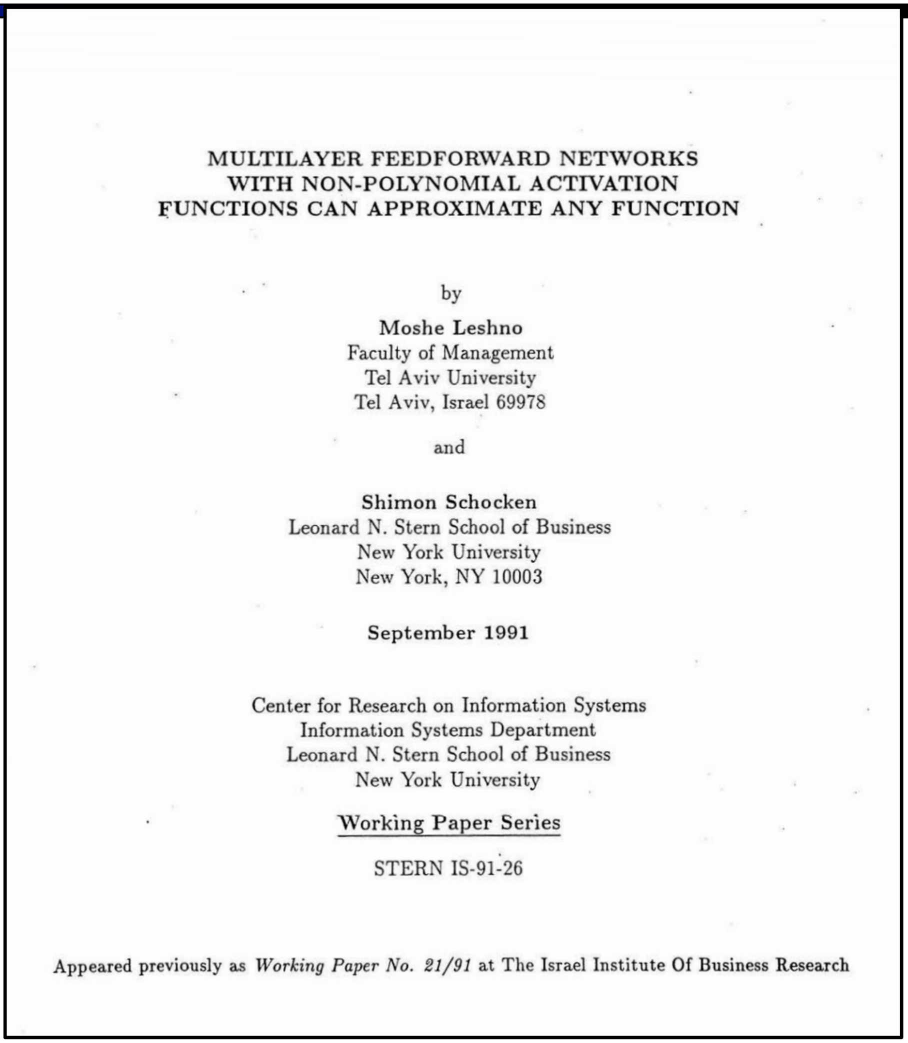
$$\rho_{p,\mu}(f, g) = \left[\int_{R^k} |f(x) - g(x)|^p \, d\mu(x) \right]^{1/p},$$

$1 \leq p < \infty$, the most popular choice being $p = 2$, corresponding to mean square error.

Of course, there are many more ways of measuring closeness of functions. In particular, in many applications, it is also necessary that the *derivatives* of the approximating function implemented by the network closely resemble those of the function to be approximated, up to some order. This issue was first taken up in Hornik et al. (1990), who discuss the sources of need of smooth functional approximation in more detail. Typical examples arise in robotics (learning of smooth movements) and signal processing (analysis of chaotic time series); for a recent application to problems of nonparametric inference in statistics and econometrics, see Gallant and White (1989).

All papers establishing certain approximation ca-

Requests for reprints should be sent to Kurt Hornik, Institut für Statistik und Wahrscheinlichkeitstheorie, Technische Universität Wien, Wiedner Hauptstraße 8-10/107, A-1040 Wien, Austria.



MULTILAYER FEEDFORWARD NETWORKS WITH NON-POLYNOMIAL ACTIVATION FUNCTIONS CAN APPROXIMATE ANY FUNCTION

by

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Fun Neural Net Demo Site

- Demo-site:
  - http://playground.tensorflow.org/

How about computing all the derivatives?

- Derivatives tables:

$$\frac{d}{dx}(a) = 0$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(au) = a\frac{du}{dx}$$

$$\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{du}{dx}$$

$$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\log_e u] = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a u] = \log_a e\frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$\frac{d}{dx}a^u = a^u \ln a\frac{du}{dx}$$

$$\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + \ln u \;\; u^v\frac{dv}{dx}$$

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u\frac{du}{dx}$$

[source: http://hyperphysics.phy-astr.gsu.edu/hbase/Math/derfunc.html

How about computing all the derivatives?

- But neural net f is never one of those?
  - No problem: CHAIN RULE:

If $$f(x) = g(h(x))$$

Then $$f'(x) = g'(h(x))h'(x)$$

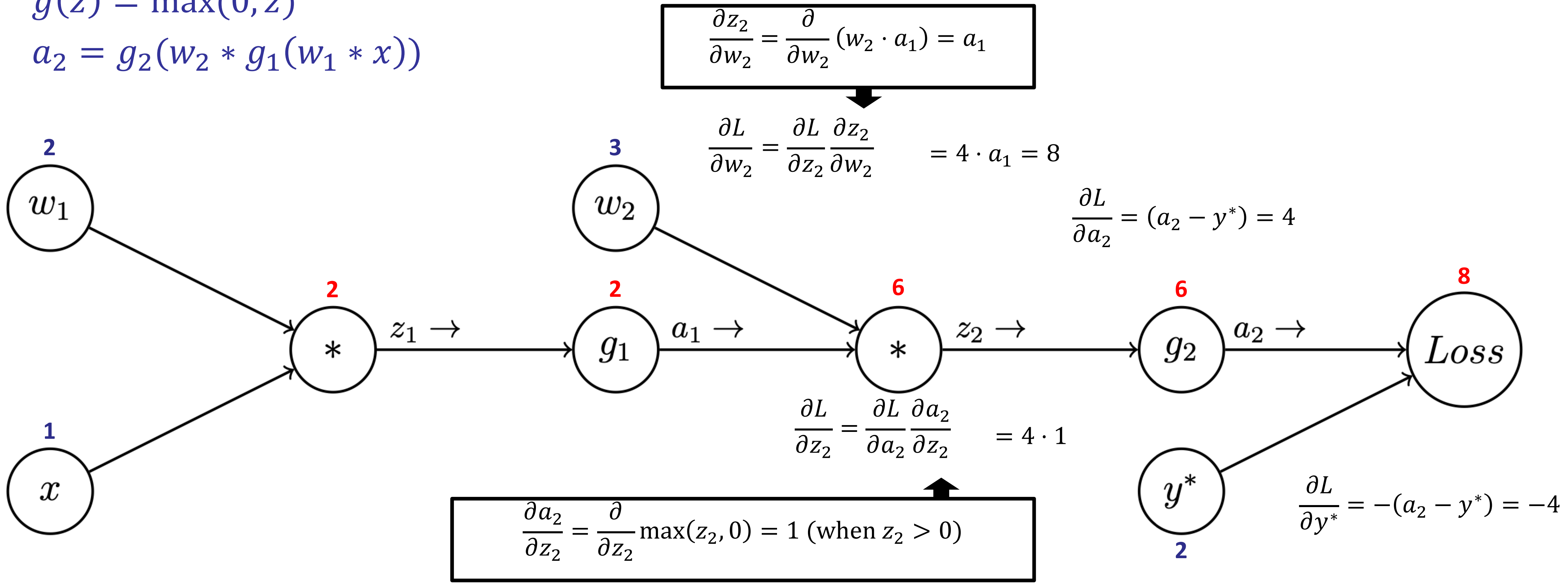
⍰ Derivatives can be computed by following well-defined procedures

Automatic Differentiation

- Automatic differentiation software
  - e.g., PyTorch, TensorFlow, Jax
  - Only need to program the function g(x,y,w)
  - Can automatically compute all derivatives w.r.t. all entries in w
  - This is typically done by caching info during forward computation pass of f, and then doing a backward pass = “backpropagation”
  - Autodiff / Backpropagation can often be done at computational cost comparable to the forward pass
- Need to know this exists
- How this is done? -- outside of scope of CS188

Example: Automatic Differentiation

- Build a computation graph and apply chain rule: $f(x) = g\big(h(x)\big)$ $f'(x) = h'(x) \cdot g'(h(x))$
- Example: neural network with quadratic loss: $L(a_2, y^*) = \frac{1}{2}(a_2 - y^*)^2$ and ReLU activations $g(z) = \max(0, z)$
- $a_2 = g_2(w_2 * g_1(w_1 * x))$



$$\boxed{\frac{\partial z_2}{\partial w_2} = \frac{\partial}{\partial w_2}(w_2 \cdot a_1) = a_1}$$

$$\frac{\partial L}{\partial w_2} = \frac{\partial L}{\partial z_2}\frac{\partial z_2}{\partial w_2} \quad = 4 \cdot a_1 = 8$$

$$\frac{\partial L}{\partial a_2} = (a_2 - y^*) = 4$$

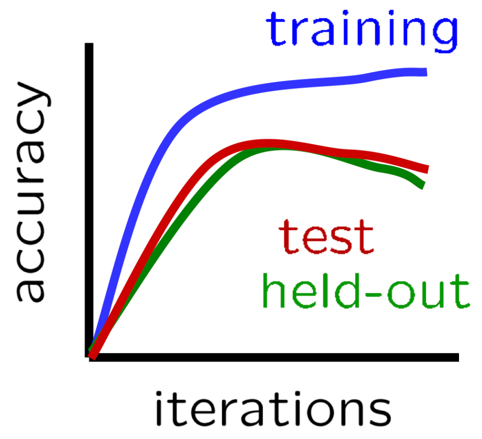
$$\frac{\partial L}{\partial z_2} = \frac{\partial L}{\partial a_2}\frac{\partial a_2}{\partial z_2} \quad = 4 \cdot 1$$

$$\boxed{\frac{\partial a_2}{\partial z_2} = \frac{\partial}{\partial z_2}\max(z_2, 0) = 1 \text{ (when } z_2 > 0)}$$

$$\frac{\partial L}{\partial y^*} = -(a_2 - y^*) = -4$$

Preventing Overfitting in Neural Networks

- Early stopping:



- Weight regularization: $\max_w \sum_i \log P\left(y^{(i)} \mid x^{(i)}; w \right) - \frac{\lambda}{2} \sum_j w_j^2$

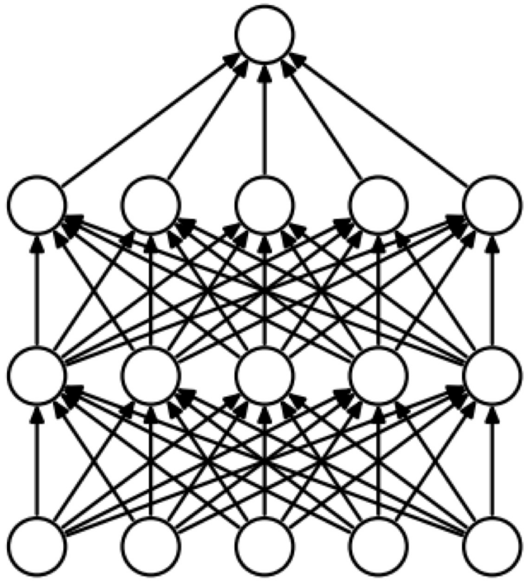
- Dropout:

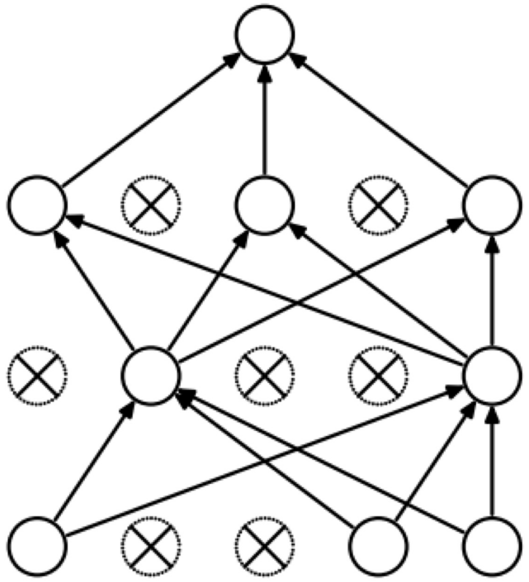
Dropout

“Damage” the network during training to increase redundancy

At each training step, with probability (1-p) set an activation to zero (i.e., drop it)

When making predictions, don’t apply dropout, but multiply weights by p (rescaling)


(a) Standard Neural Net


(b) After applying dropout.

Preventing Overfitting in Neural Networks

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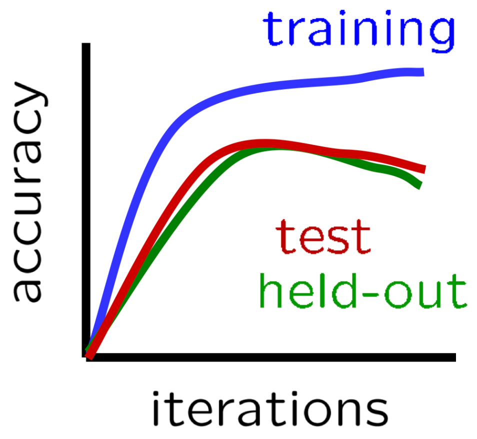

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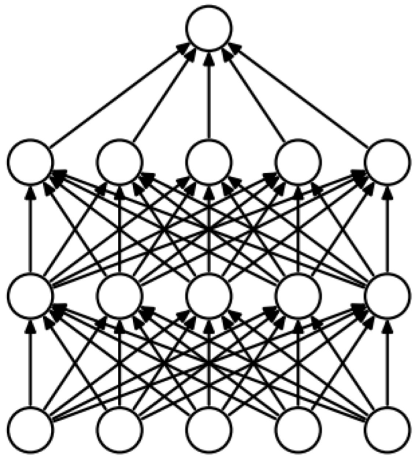
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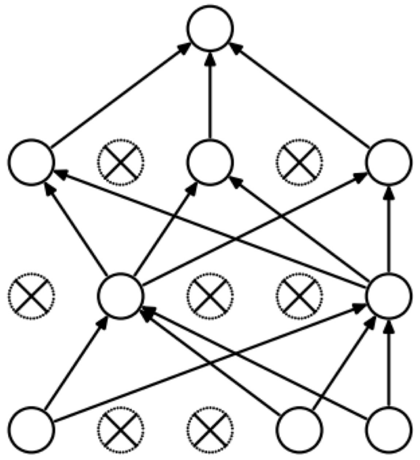
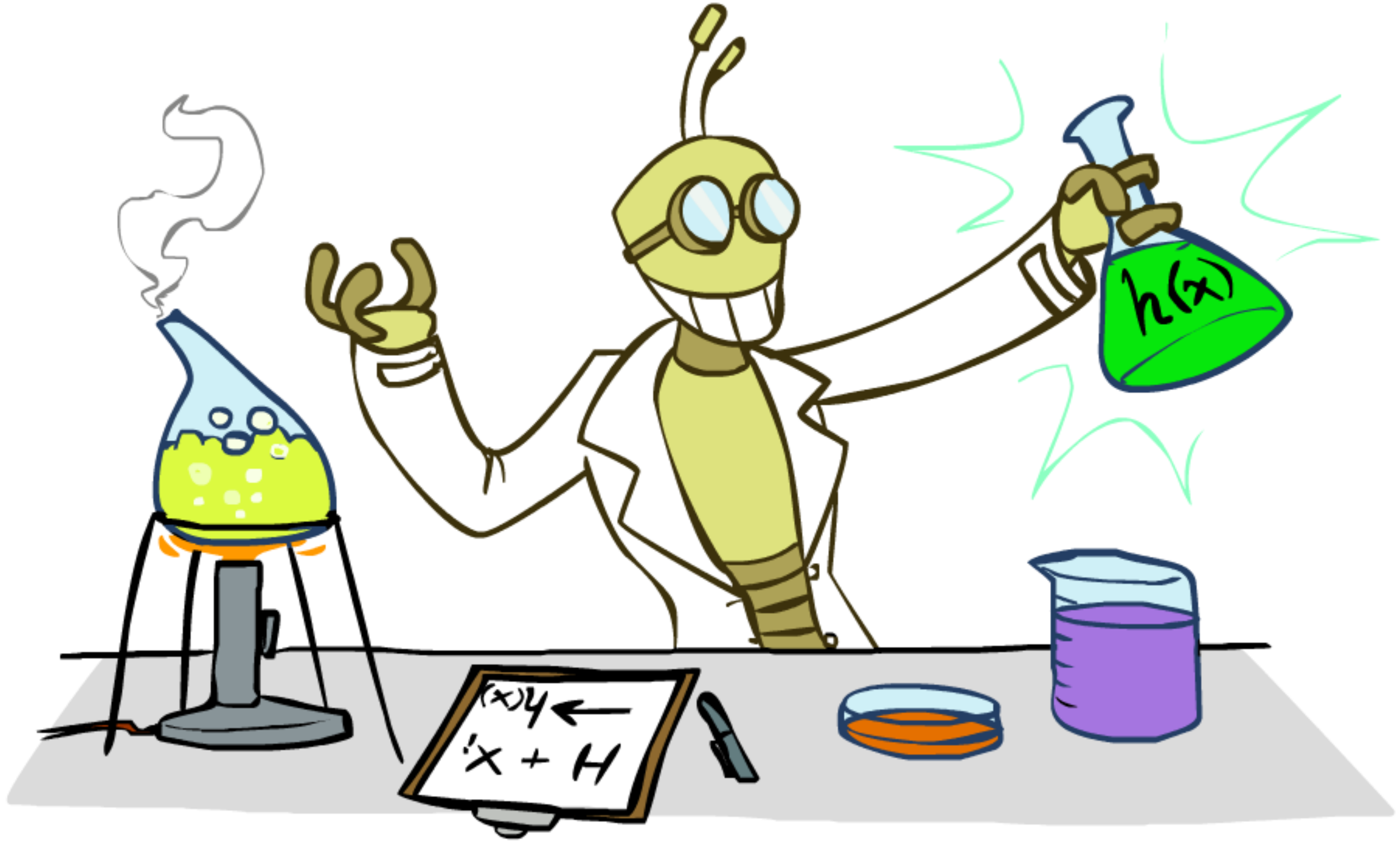

(b) After applying dropout.

Summary of Key Ideas

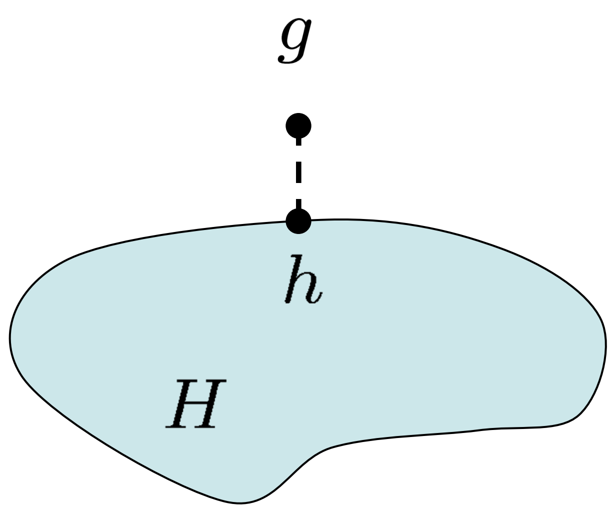
- Optimize probability of label given input $\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)}; w)$
- Continuous optimization
  - Gradient ascent:
    - Compute steepest uphill direction = gradient (= just vector of partial derivatives)
    - Take step in the gradient direction
    - Repeat (until held-out data accuracy starts to drop = “early stopping”)
- Deep neural nets
  - Last layer = still logistic regression
  - Now also many more layers before this last layer
    - = computing the features
    - → the features are learned rather than hand-designed
  - Universal function approximation theorem
    - `If` neural net is large enough
    - `Then` neural net can represent any continuous mapping from input to output with arbitrary accuracy
    - But remember: need to avoid overfitting / memorizing the training data → early stopping!
  - Automatic differentiation gives the derivatives efficiently

Inductive Learning



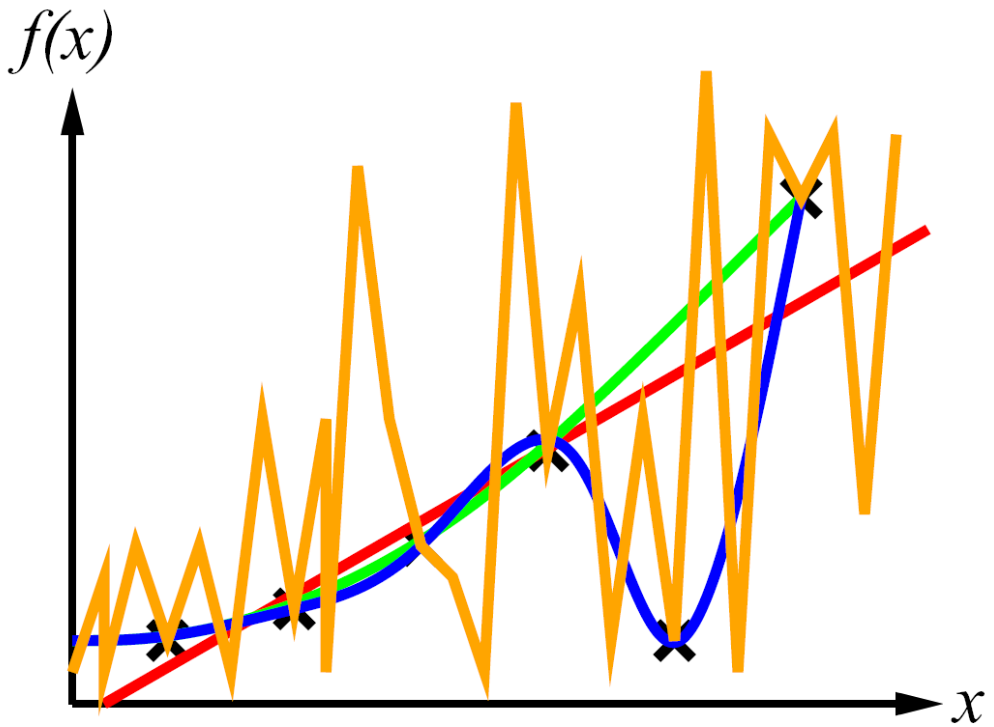
Inductive Learning (Science)

- Simplest form: learn a function from examples
  - A target function: g
  - Examples: input-output pairs $(x, g(x))$
  - E.g. x is an email and $g(x)$ is spam / ham
  - E.g. x is a house and $g(x)$ is its selling price
- Problem:
  - Given a hypothesis space H
  - Given a training set of examples x_i
  - Find a hypothesis $h(x)$ such that $h \sim g$
- Includes:
  - Classification (outputs = class labels)
  - Regression (outputs = real numbers)
- How do perceptron and naïve Bayes fit in? (H, h, g, etc.)



Inductive Learning

- Curve fitting (regression, function approximation):



- Consistency vs. simplicity
- Ockham's razor

Consistency vs. Simplicity

- Fundamental tradeoff: bias vs. variance
- Usually algorithms prefer consistency by default (why?)
- Several ways to operationalize "simplicity"
  - Reduce the hypothesis space
    - Assume more: e.g. independence assumptions, as in naïve Bayes
    - Have fewer, better features / attributes: feature selection
    - Other structural limitations (decision lists vs trees)
  - Regularization
    - Smoothing: cautious use of small counts
    - Many other generalization parameters (pruning cutoffs today)
    - Hypothesis space stays big, but harder to get to the outskirts

Decision Trees

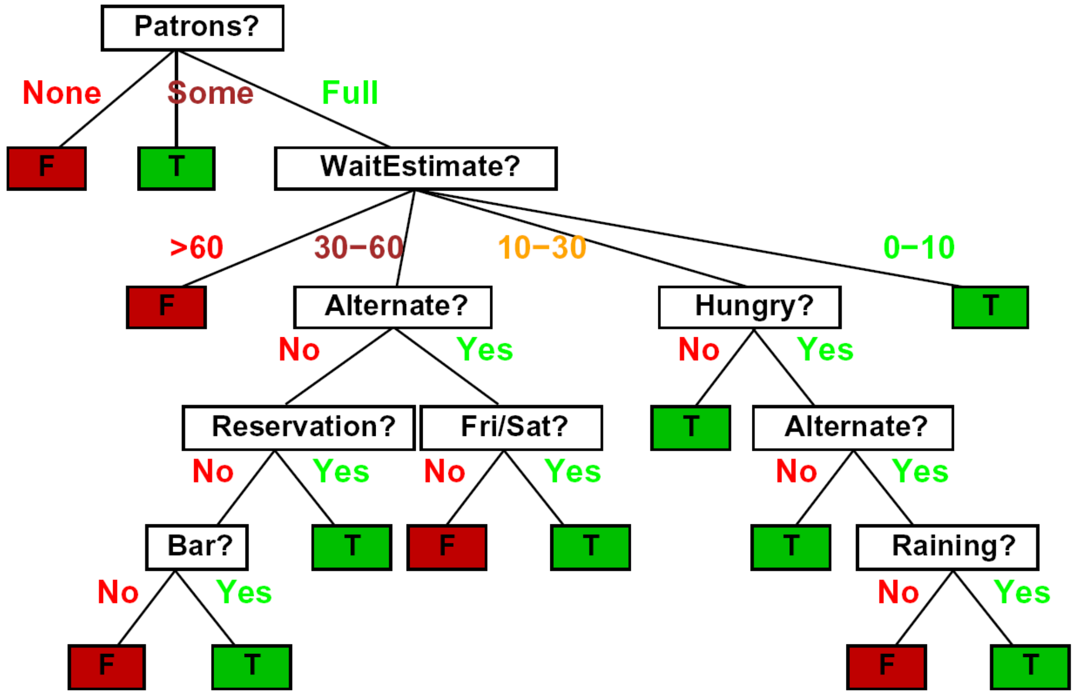
Reminder: Features

- Features, aka attributes
  - Sometimes: TYPE=French
  - Sometimes: $f_{\text{TYPE=French}}(x) = 1$

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
X_1	T	F	F	T	Some	$$$	F	T	French	0–10	T
X_2	T	F	F	T	Full	$	F	F	Thai	30–60	F
X_3	F	T	F	F	Some	$	F	F	Burger	0–10	T
X_4	T	F	T	T	Full	$	F	F	Thai	10–30	T
X_5	T	F	T	F	Full	$$$	F	T	French	>60	F
X_6	F	T	F	T	Some	$$	T	T	Italian	0–10	T
X_7	F	T	F	F	None	$	T	F	Burger	0–10	F
X_8	F	F	F	T	Some	$$	T	T	Thai	0–10	T
X_9	F	T	T	F	Full	$	T	F	Burger	>60	F
X_{10}	T	T	T	T	Full	$$$	F	T	Italian	10–30	F
X_{11}	F	F	F	F	None	$	F	F	Thai	0–10	F
X_{12}	T	T	T	T	Full	$	F	F	Burger	30–60	T

Decision Trees

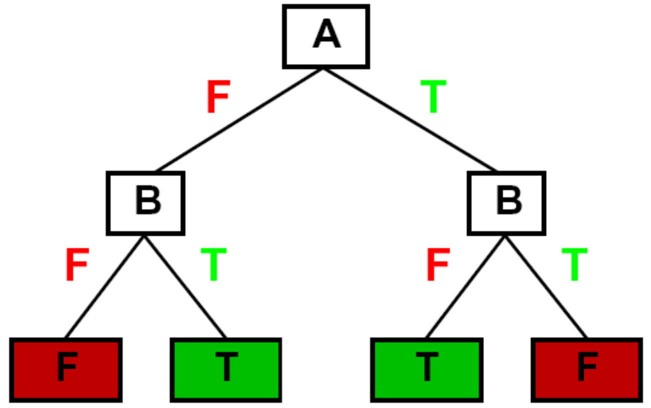
- Compact representation of a function:
  - Truth table
  - Conditional probability table
  - Regression values
- True function
  - Realizable: in H



Expressiveness of DTs

- Can express any function of the features

A	B	A xor B
F	F	F
F	T	T
T	F	T
T	T	F



$$P(C|A, B)$$

- However, we hope for compact trees

Comparison: Perceptrons

- What is the expressiveness of a perceptron over these features?

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
X_1	*T*	*F*	*F*	*T*	*Some*	*\$\$\$*	*F*	*T*	*French*	*0–10*	*T*
X_2	*T*	*F*	*F*	*T*	*Full*	*\$*	*F*	*F*	*Thai*	*30–60*	*F*

- For a perceptron, a feature's contribution is either positive or negative
  - If you want one feature's effect to depend on another, you have to add a new conjunction feature
  - E.g. adding "PATRONS=full $\wedge$ WAIT = 60" allows a perceptron to model the interaction between the two atomic features
- DTs automatically conjoin features / attributes
  - Features can have different effects in different branches of the tree!
- Difference between modeling relative evidence weighting (NB) and complex evidence interaction (DTs)
  - Though if the interactions are too complex, may not find the DT greedily

Hypothesis Spaces

- How many distinct decision trees with n Boolean attributes?

  = number of Boolean functions over n attributes

  = number of distinct truth tables with 2^n rows

  = $2^\wedge(2^n)$

  - E.g., with 6 Boolean attributes, there are 18,446,744,073,709,551,616 trees

- How many trees of depth 1 (decision stumps)?

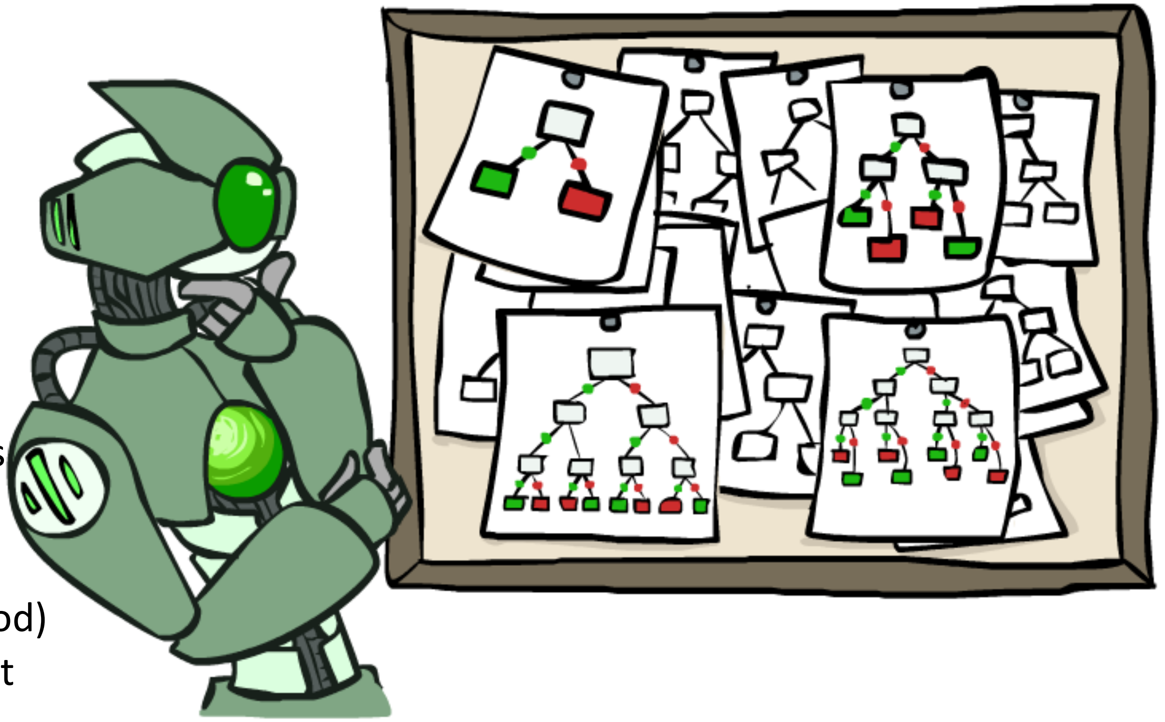
  = number of Boolean functions over 1 attribute

  = number of truth tables with 2 rows, times n

  = 4n

  - E.g. with 6 Boolean attributes, there are 24 decision stumps

- More expressive hypothesis space:
  - Increases chance that target function can be expressed (good)
  - Increases number of hypotheses consistent with training set (bad, why?)
  - Means we can get better predictions (lower bias)
  - But we may get worse predictions (higher variance)



Decision Tree Learning

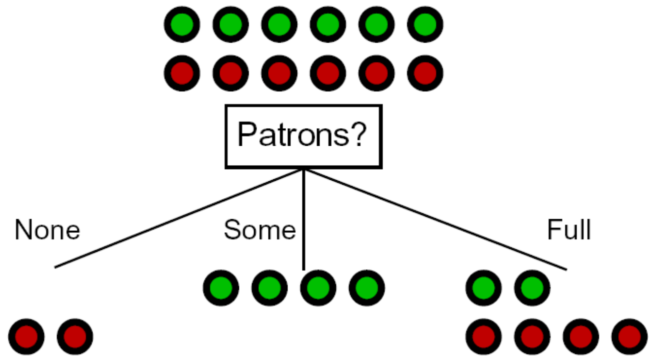
- Aim: find a small tree consistent with the training examples
- Idea: (recursively) choose “most significant” attribute as root of (sub)tree

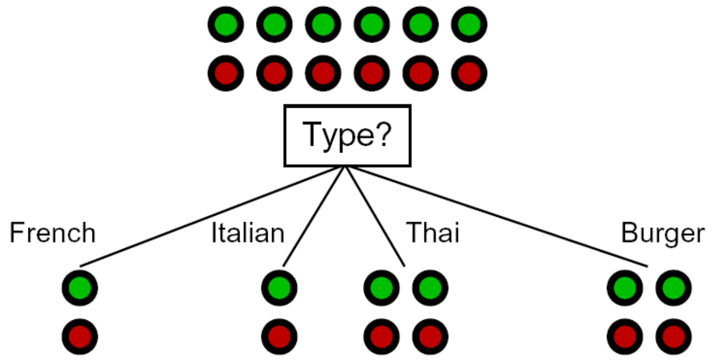
```
function DTL(examples, attributes, default) returns a decision tree

    if examples is empty then return default
    else if all examples have the same classification then return the classification
    else if attributes is empty then return MODE(examples)
    else
        best ← CHOOSE-ATTRIBUTE(attributes, examples)
        tree ← a new decision tree with root test best
        for each value $v_i$ of best do
            $examples_i$ ← {elements of examples with best = $v_i$}
            subtree ← DTL($examples_i$, attributes − best, MODE(examples))
            add a branch to tree with label $v_i$ and subtree subtree
        return tree
```

Choosing an Attribute

- Idea: a good attribute splits the examples into subsets that are (ideally) "all positive" or "all negative"





- So: we need a measure of how "good" a split is, even if the results aren't perfectly separated out

Entropy and Information

- Information answers questions
  - The more uncertain about the answer initially, the more information in the answer
  - Scale: bits
    - Answer to Boolean question with prior <1/2, 1/2>?
    - Answer to 4-way question with prior <1/4, 1/4, 1/4, 1/4>?
    - Answer to 4-way question with prior <0, 0, 0, 1>?
    - Answer to 3-way question with prior <1/2, 1/4, 1/4>?

- A probability p is typical of:
  - A uniform distribution of size 1/p
  - A code of length log 1/p

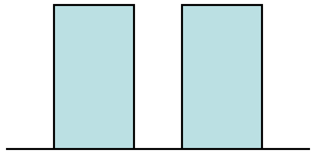
Entropy

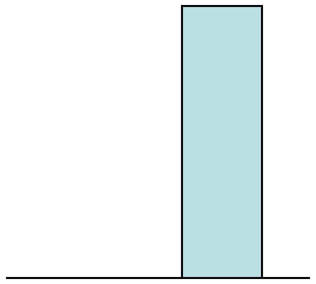
- General answer: if prior is $\langle p_1, \ldots, p_n \rangle$:
  - Information is the expected code length

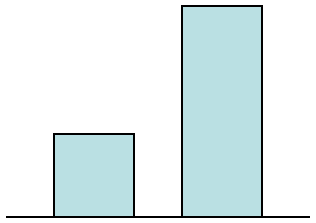
$$H(\langle p_1, \ldots, p_n \rangle) = E_p \log_2 1/p_i$$

$$= \sum_{i=1}^{n} -p_i \log_2 p_i$$

- Also called the entropy of the distribution
  - More uniform = higher entropy
  - More values = higher entropy
  - More peaked = lower entropy
  - Rare values almost "don't count"

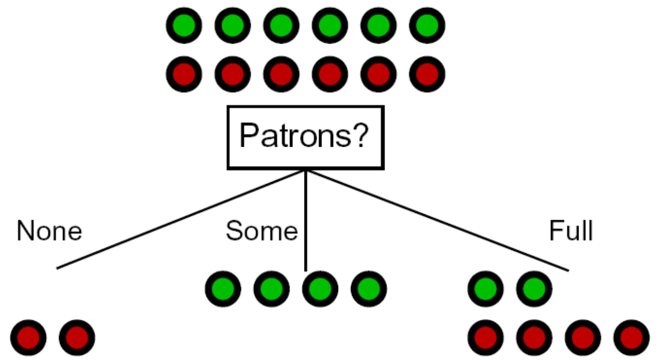

1 bit

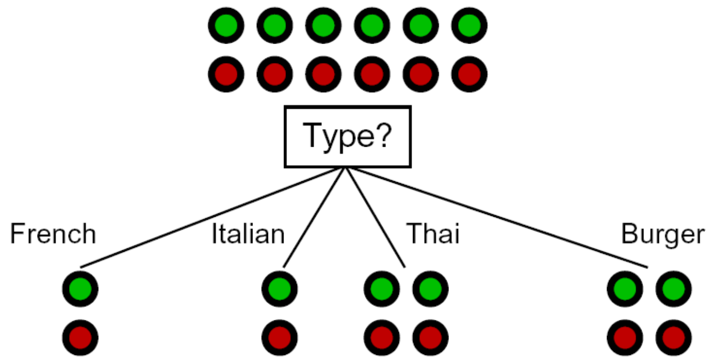

0 bits

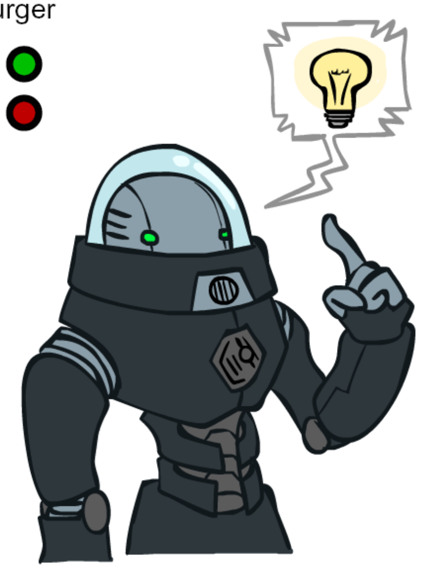

0.5 bit

Information Gain

- Back to decision trees!
- For each split, compare entropy before and after
  - Difference is the information gain
  - Problem: there's more than one distribution after split!



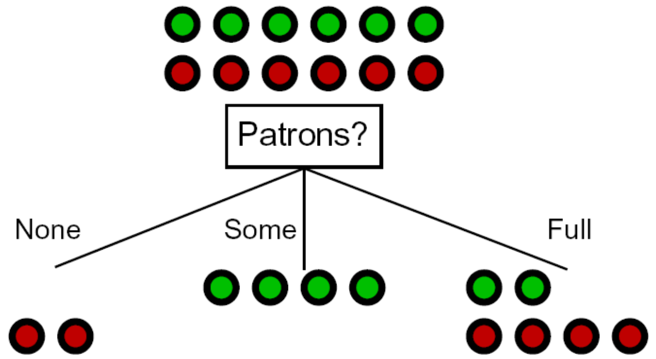




  - Solution: use expected entropy, weighted by the number of examples

Next Step: Recurse

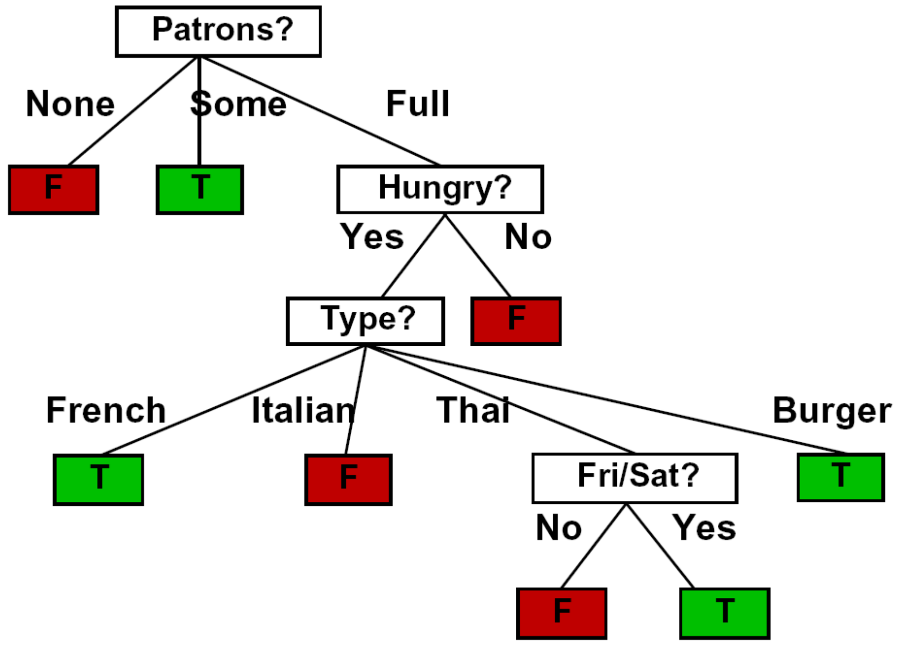
- Now we need to keep growing the tree!
- Two branches are done (why?)
- What to do under "full"?
  - See what examples are there…



Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
X_1	T	F	F	T	Some	\$\$\$	F	T	French	0–10	T
X_2	T	F	F	T	Full	\$	F	F	Thai	30–60	F
X_3	F	T	F	F	Some	\$	F	F	Burger	0–10	T
X_4	T	F	T	T	Full	\$	F	F	Thai	10–30	T
X_5	T	F	T	F	Full	\$\$\$	F	T	French	>60	F
X_6	F	T	F	T	Some	\$\$	T	T	Italian	0–10	T
X_7	F	T	F	F	None	\$	T	F	Burger	0–10	F
X_8	F	F	F	T	Some	\$\$	T	T	Thai	0–10	T
X_9	F	T	T	F	Full	\$	T	F	Burger	>60	F
X_{10}	T	T	T	T	Full	\$\$\$	F	T	Italian	10–30	F
X_{11}	F	F	F	F	None	\$	F	F	Thai	0–10	F
X_{12}	T	T	T	T	Full	\$	F	F	Burger	30–60	T

Example: Learned Tree

- Decision tree learned from these 12 examples:



- Substantially simpler than “true” tree
  - A more complex hypothesis isn't justified by data
- Also: it’s reasonable, but wrong

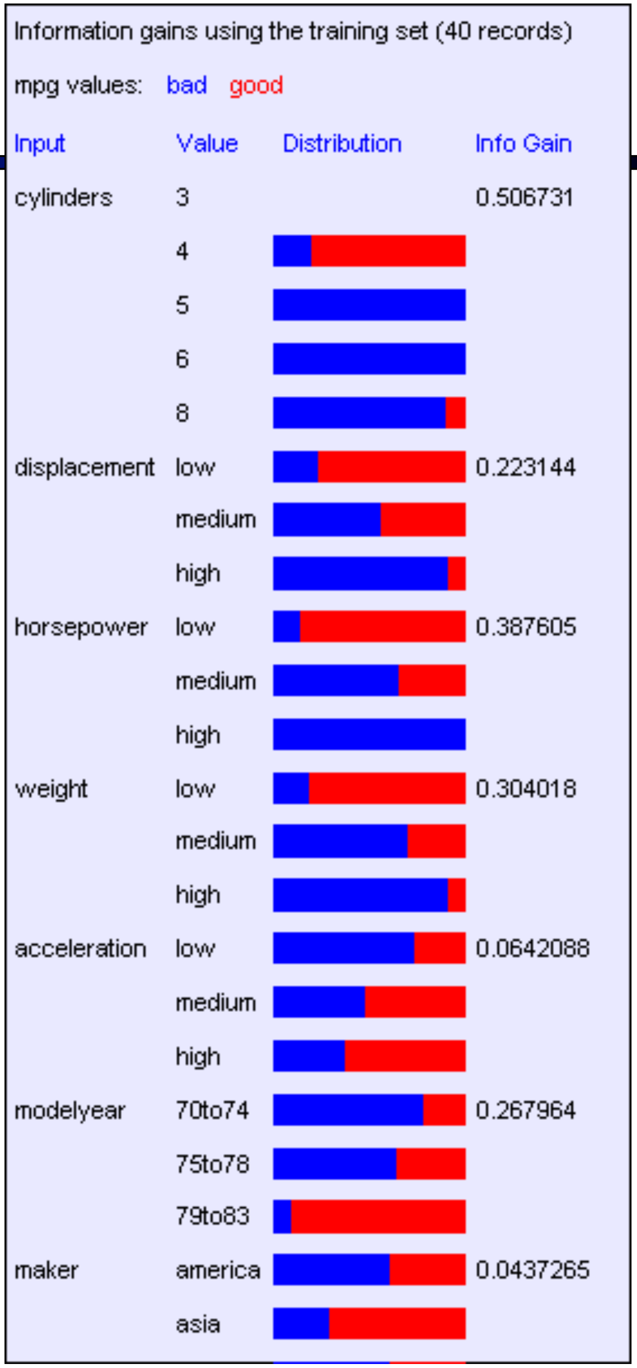
Example: Miles Per Gallon

40 Examples

mpg	cylinders	displacement	horsepower	weight	acceleration	modelyear	maker
good	4	low	low	low	high	75to78	asia
bad	6	medium	medium	medium	medium	70to74	america
bad	4	medium	medium	medium	low	75to78	europe
bad	8	high	high	high	low	70to74	america
bad	6	medium	medium	medium	medium	70to74	america
bad	4	low	medium	low	medium	70to74	asia
bad	4	low	medium	low	low	70to74	asia
bad	8	high	high	high	low	75to78	america
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
bad	8	high	high	high	low	70to74	america
good	8	high	medium	high	high	79to83	america
bad	8	high	high	high	low	75to78	america
good	4	low	low	low	low	79to83	america
bad	6	medium	medium	medium	high	75to78	america
good	4	medium	low	low	low	79to83	america
good	4	low	low	medium	high	79to83	america
bad	8	high	high	high	low	70to74	america
good	4	low	medium	low	medium	75to78	europe
bad	5	medium	medium	medium	medium	75to78	europe

Find the First Split

- Look at information gain for each attribute
- Note that each attribute is correlated with the target!
- What do we split on?

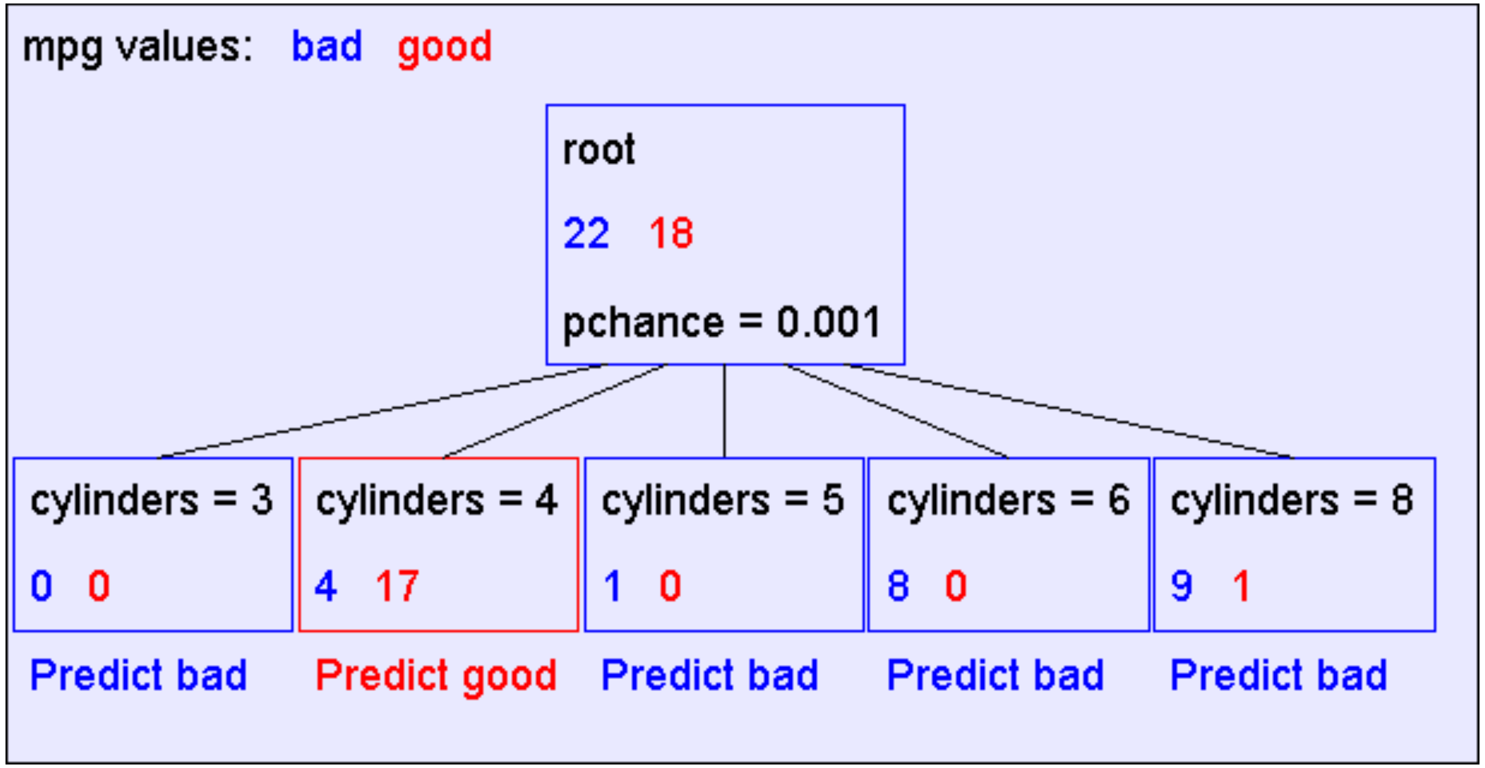


Information gains using the training set (40 records)

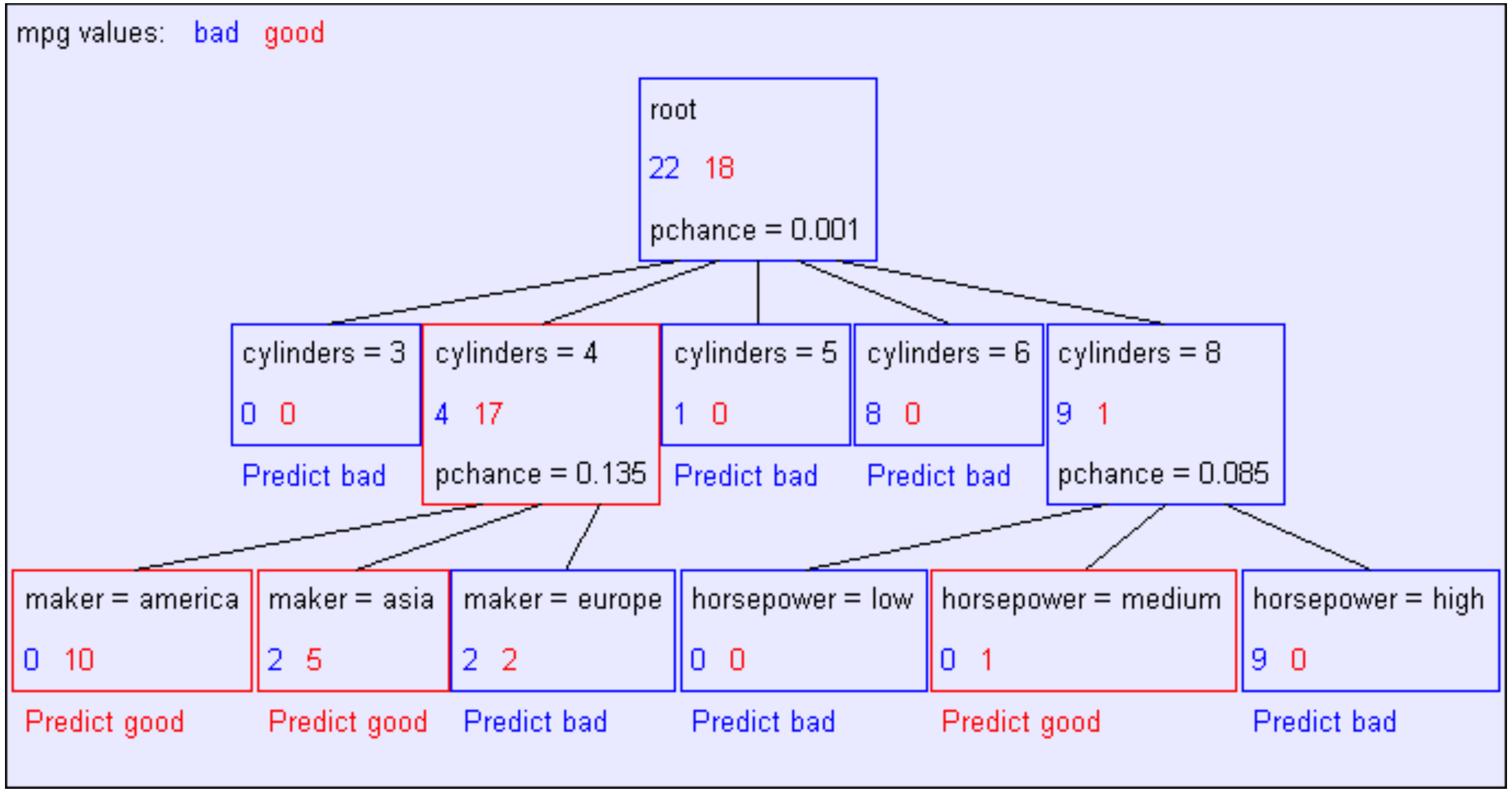
mpg values: bad good

Input	Value	Distribution	Info Gain
cylinders	3		0.506731
	4		
	5		
	6		
	8		
displacement	low		0.223144
	medium		
	high		
horsepower	low		0.387605
	medium		
	high		
weight	low		0.304018
	medium		
	high		
acceleration	low		0.0642088
	medium		
	high		
modelyear	70to74		0.267964
	75to78		
	79to83		
maker	america		0.0437265
	asia		

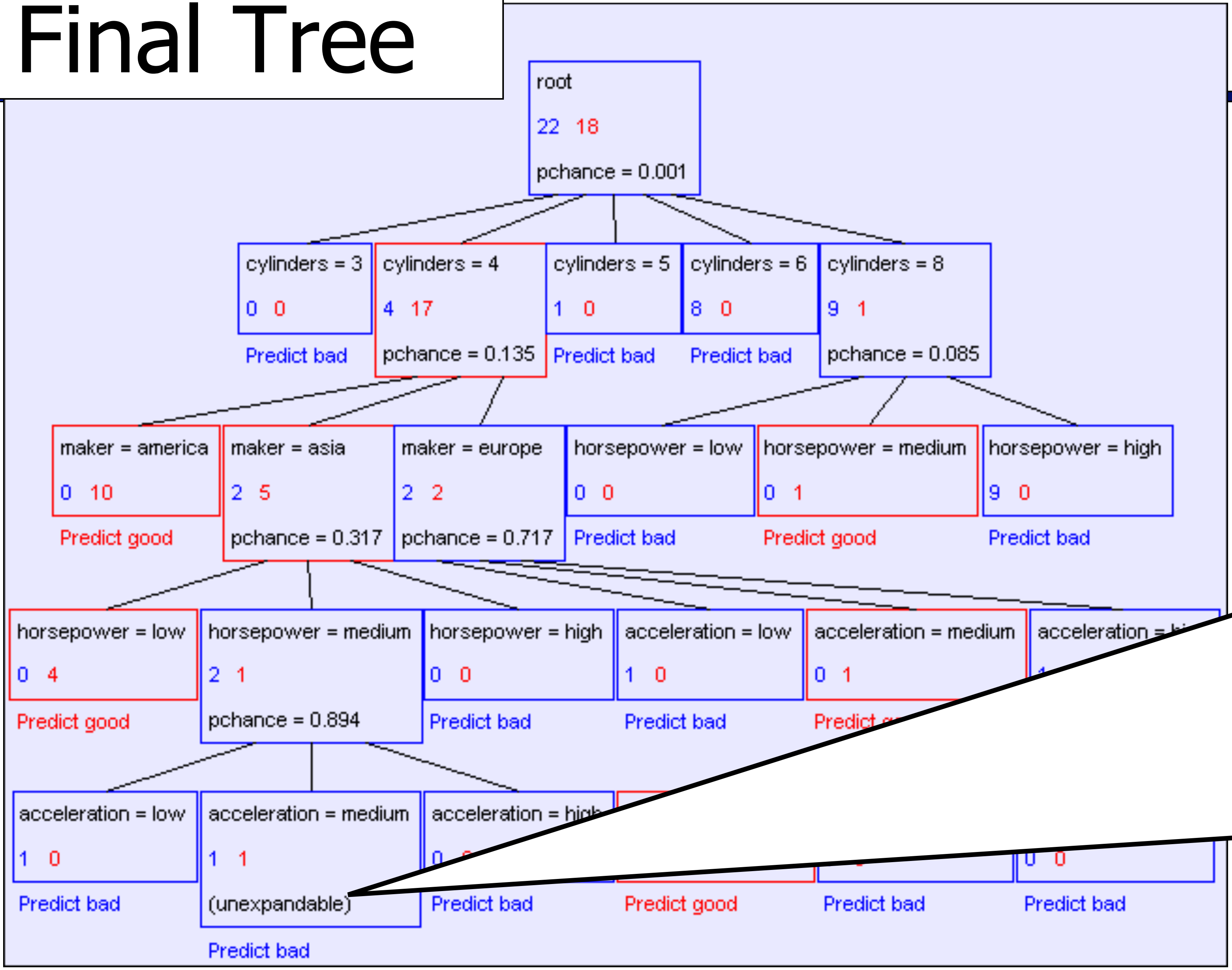
Result: Decision Stump

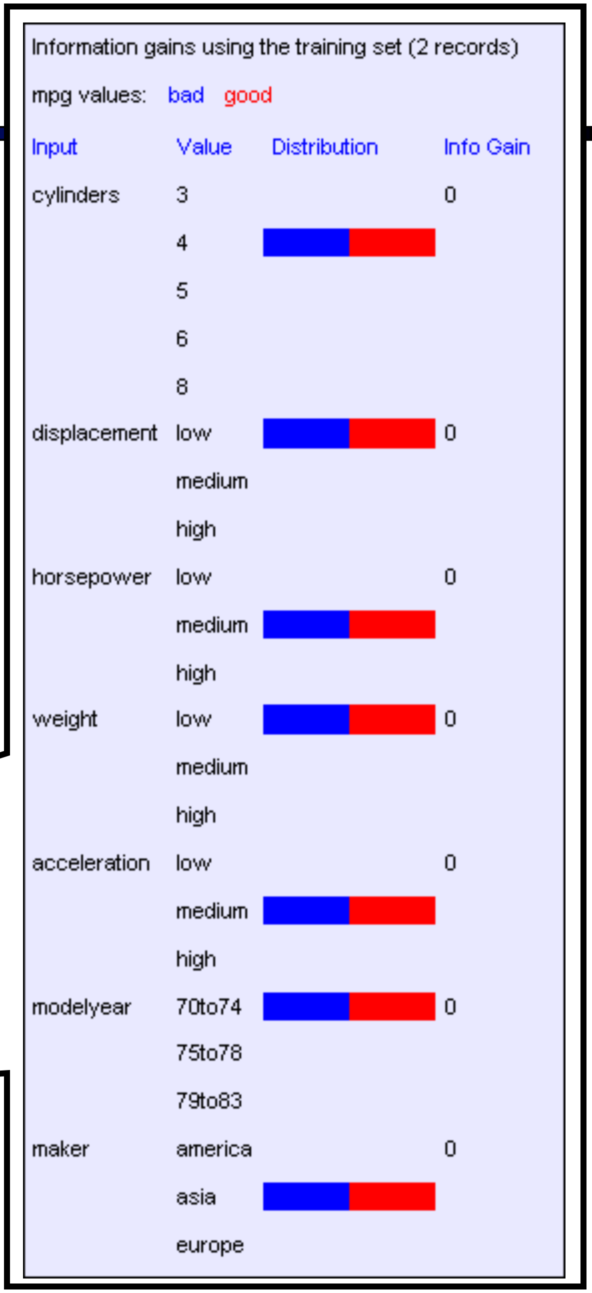


Second Level



Final Tree





Information gains using the training set (2 records)

mpg values: bad good

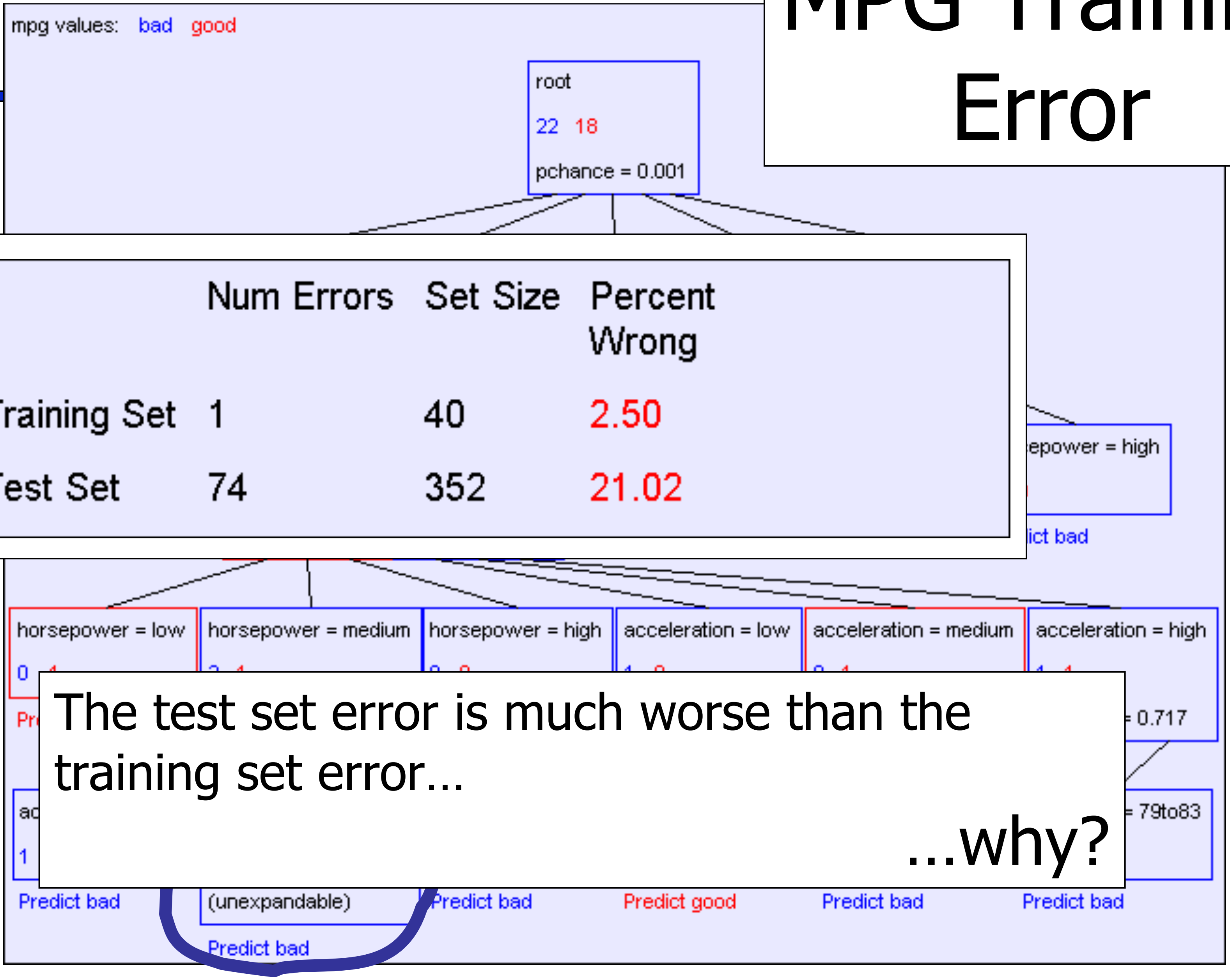
Input	Value	Distribution	Info Gain
cylinders	3		0
	4	🟦🟥	
	5		
	6		
	8		
displacement	low	🟦🟥	0
	medium		
	high		
horsepower	low		0
	medium	🟦🟥	
	high		
weight	low	🟦🟥	0
	medium		
	high		
acceleration	low		0
	medium	🟦🟥	
	high		
modelyear	70to74	🟦🟥	0
	75to78		
	79to83		
maker	america		0
	asia	🟦🟥	
	europe		

Reminder: Overfitting

- Overfitting:
  - When you stop modeling the patterns in the training data (which generalize)
  - And start modeling the noise (which doesn’t)

- We had this before:
  - Naïve Bayes: needed to smooth
  - Perceptron: early stopping

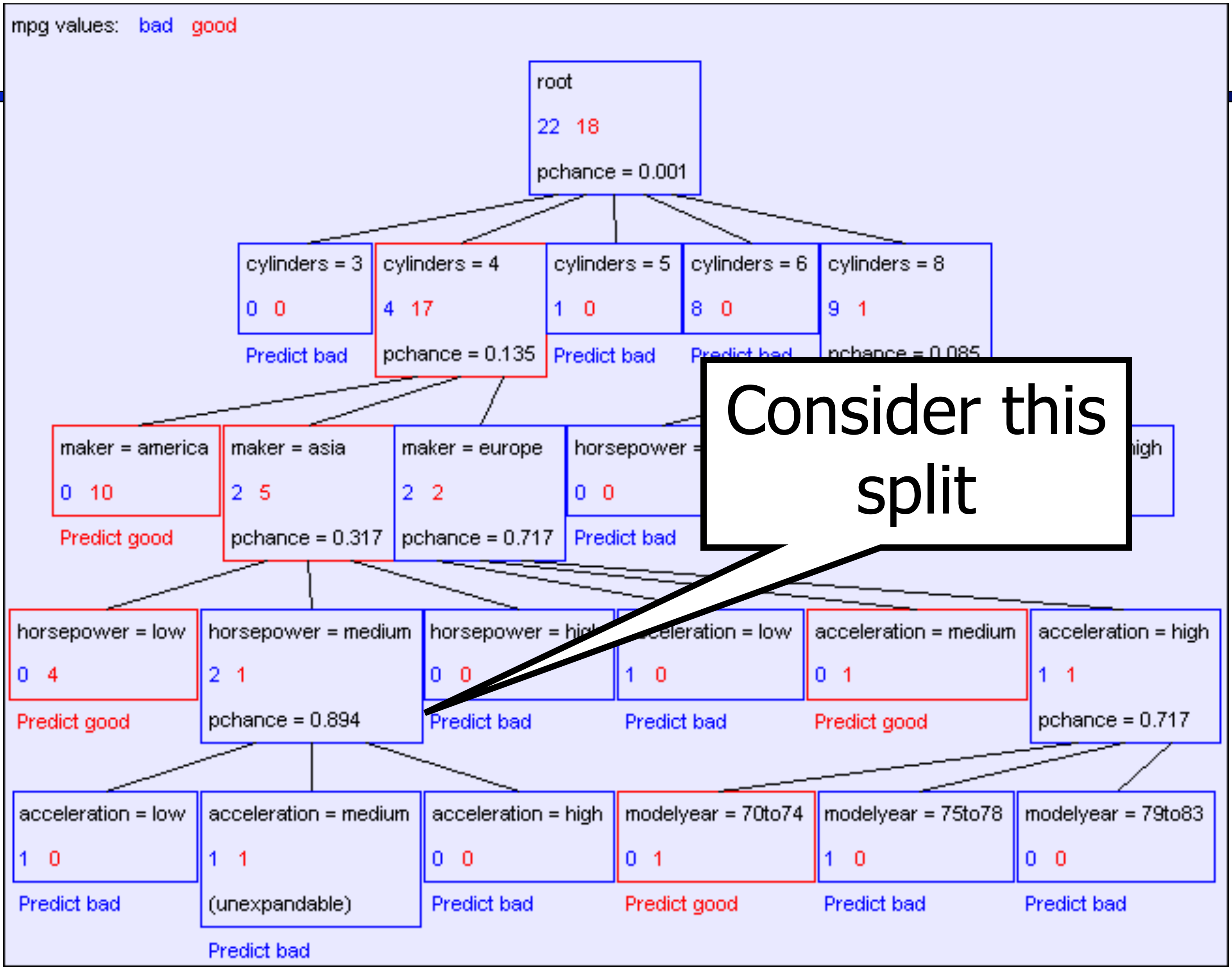
MPG Training Error



	Num Errors	Set Size	Percent Wrong
Training Set	1	40	2.50
Test Set	74	352	21.02

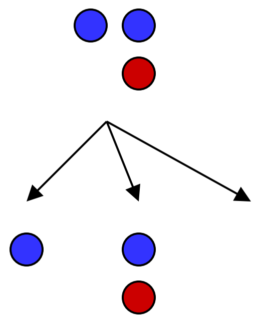
The test set error is much worse than the training set error…

…why?


mpg values: bad good
root
22 18
pchance = 0.001
cylinders = 3
0 0
Predict bad
cylinders = 4
4 17
pchance = 0.135
cylinders = 5
1 0
Predict bad
cylinders = 6
8 0
Predict bad
cylinders = 8
9 1
pchance = 0.085
maker = america
0 10
Predict good
maker = asia
2 5
pchance = 0.317
maker = europe
2 2
pchance = 0.717
horsepower =
0 0
Predict bad
high
horsepower = low
0 4
Predict good
horsepower = medium
2 1
pchance = 0.894
horsepower = high
0 0
Predict bad
acceleration = low
1 0
Predict bad
acceleration = medium
0 1
Predict good
acceleration = high
1 1
pchance = 0.717
acceleration = low
1 0
Predict bad
acceleration = medium
1 1
(unexpandable)
Predict bad
acceleration = high
0 0
Predict bad
modelyear = 70to74
0 1
Predict good
modelyear = 75to78
1 0
Predict bad
modelyear = 79to83
0 0
Predict bad
Consider this split

Significance of a Split

- Starting with:
  - Three cars with 4 cylinders, from Asia, with medium HP
  - 2 bad MPG
  - 1 good MPG



- What do we expect from a three-way split?
  - Maybe each example in its own subset?
  - Maybe just what we saw in the last slide?
- Probably shouldn't split if the counts are so small they could be due to chance
- A chi-squared test can tell us how likely it is that deviations from a perfect split are due to chance*
- Each split will have a significance value, p_{CHANCE}

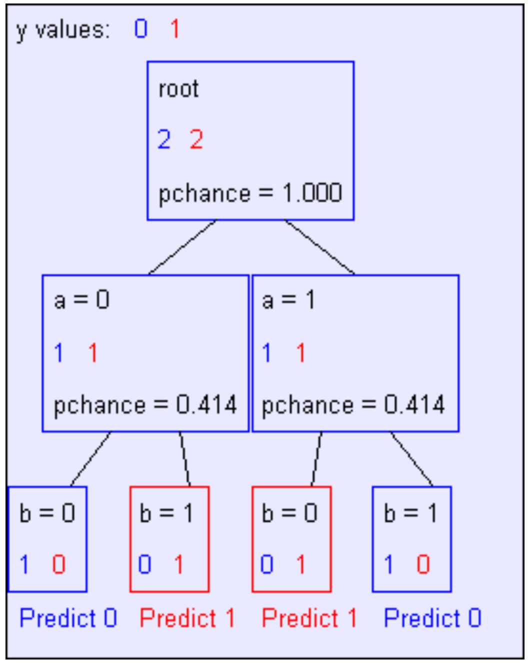
Keeping it General

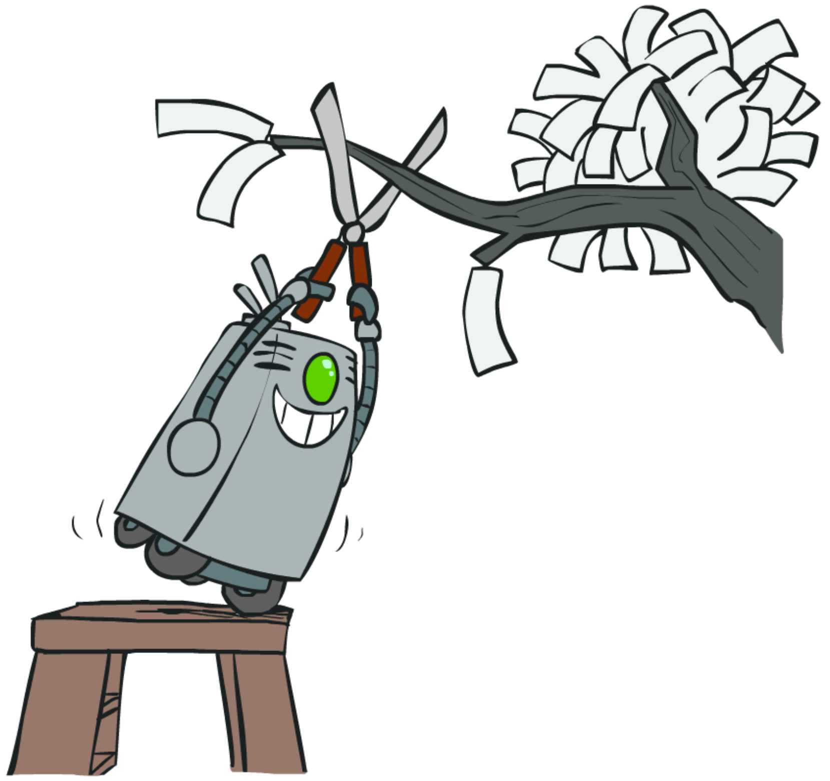
- Pruning:
  - Build the full decision tree
  - Begin at the bottom of the tree
  - Delete splits in which

    $p_{CHANCE} > MaxP_{CHANCE}$
  - Continue working upward until there are no more prunable nodes
  - Note: some chance nodes may not get pruned because they were "redeemed" later

y = a XOR b

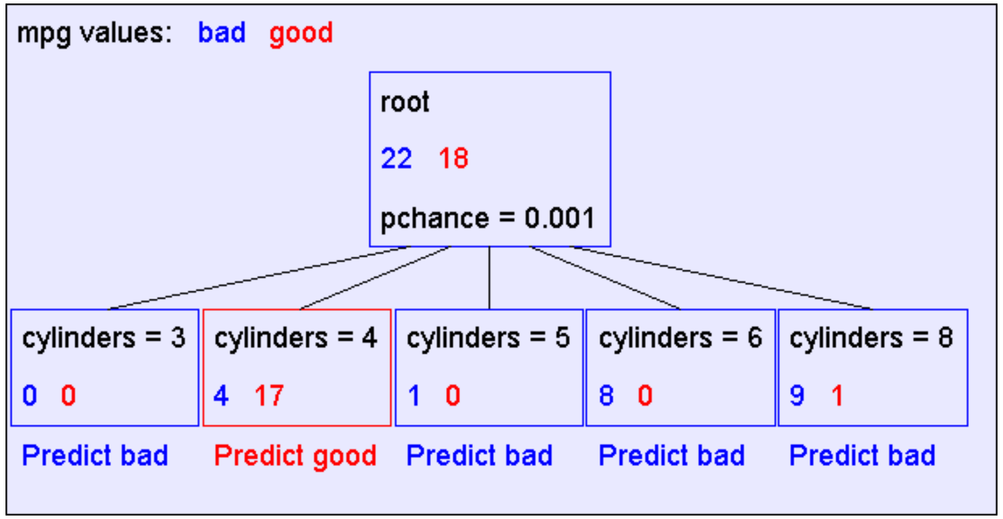
a	b	y
0	0	0
0	1	1
1	0	1
1	1	0



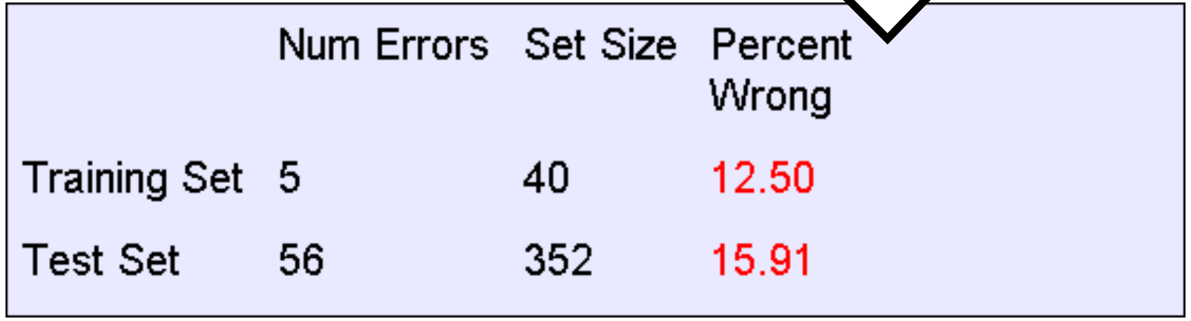


Pruning example

- With $\text{MaxP}_{\text{CHANCE}} = 0.1$:



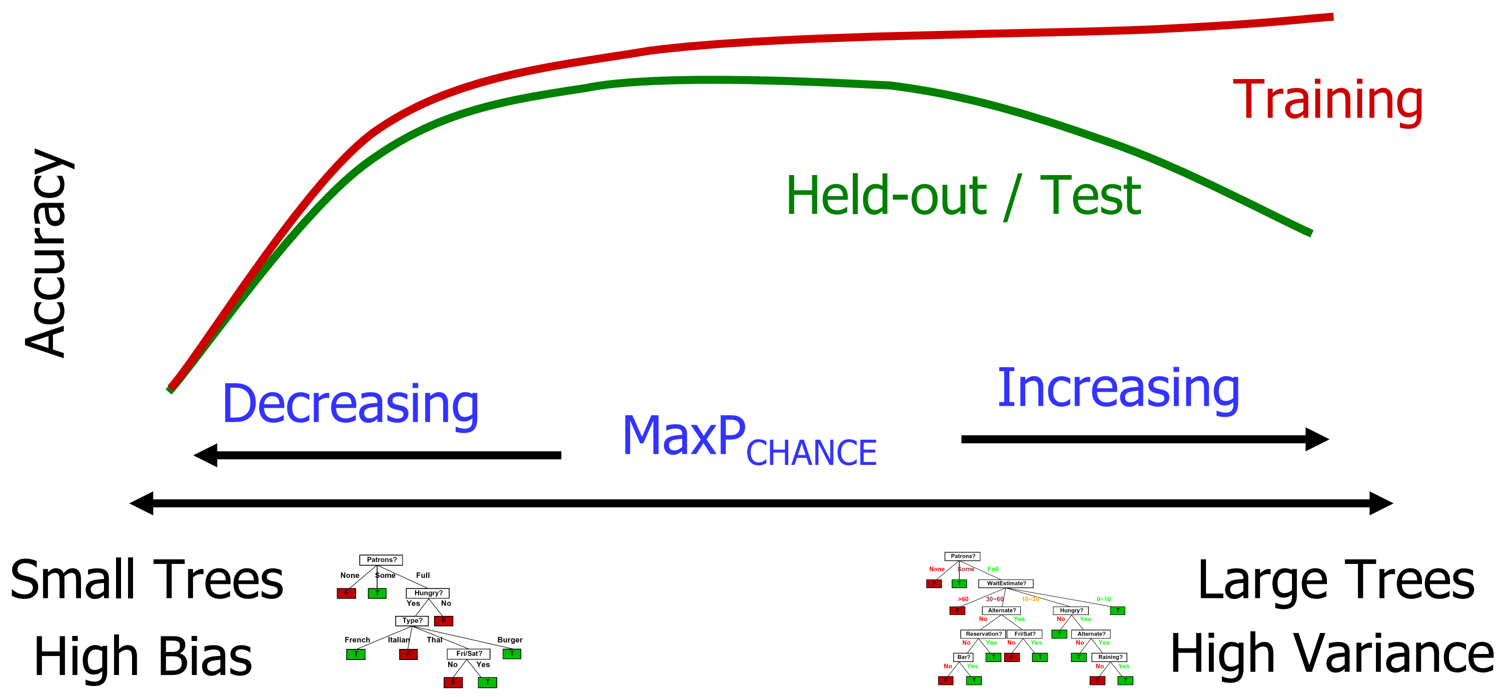
Note the improved test set accuracy compared with the unpruned tree



	Num Errors	Set Size	Percent Wrong
Training Set	5	40	12.50
Test Set	56	352	15.91

Regularization

- MaxP$_{\text{CHANCE}}$ is a regularization parameter
- Generally, set it using held-out data (as usual)



Two Ways of Controlling Overfitting

- Limit the hypothesis space
  - E.g. limit the max depth of trees
  - Easier to analyze

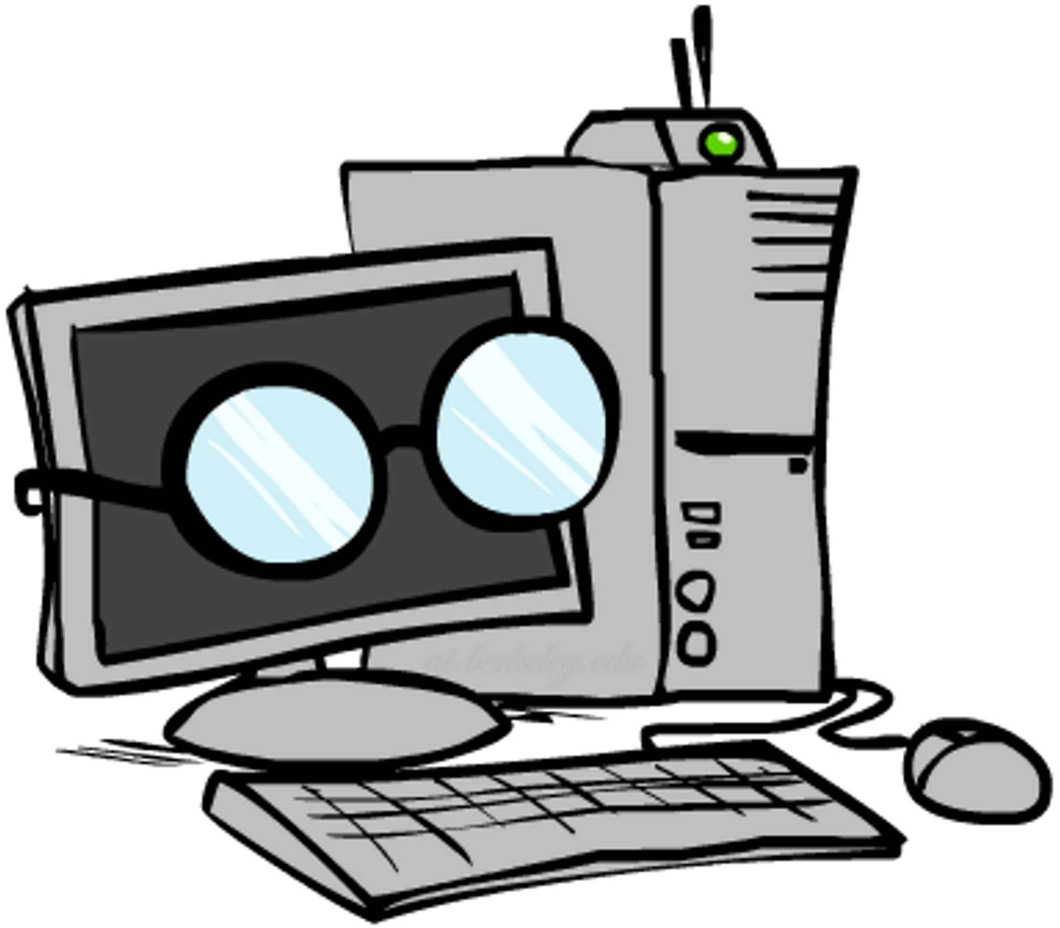
- Regularize the hypothesis selection
  - E.g. chance cutoff
  - Disprefer most of the hypotheses unless data is clear
  - Usually done in practice

Summary of Key Ideas

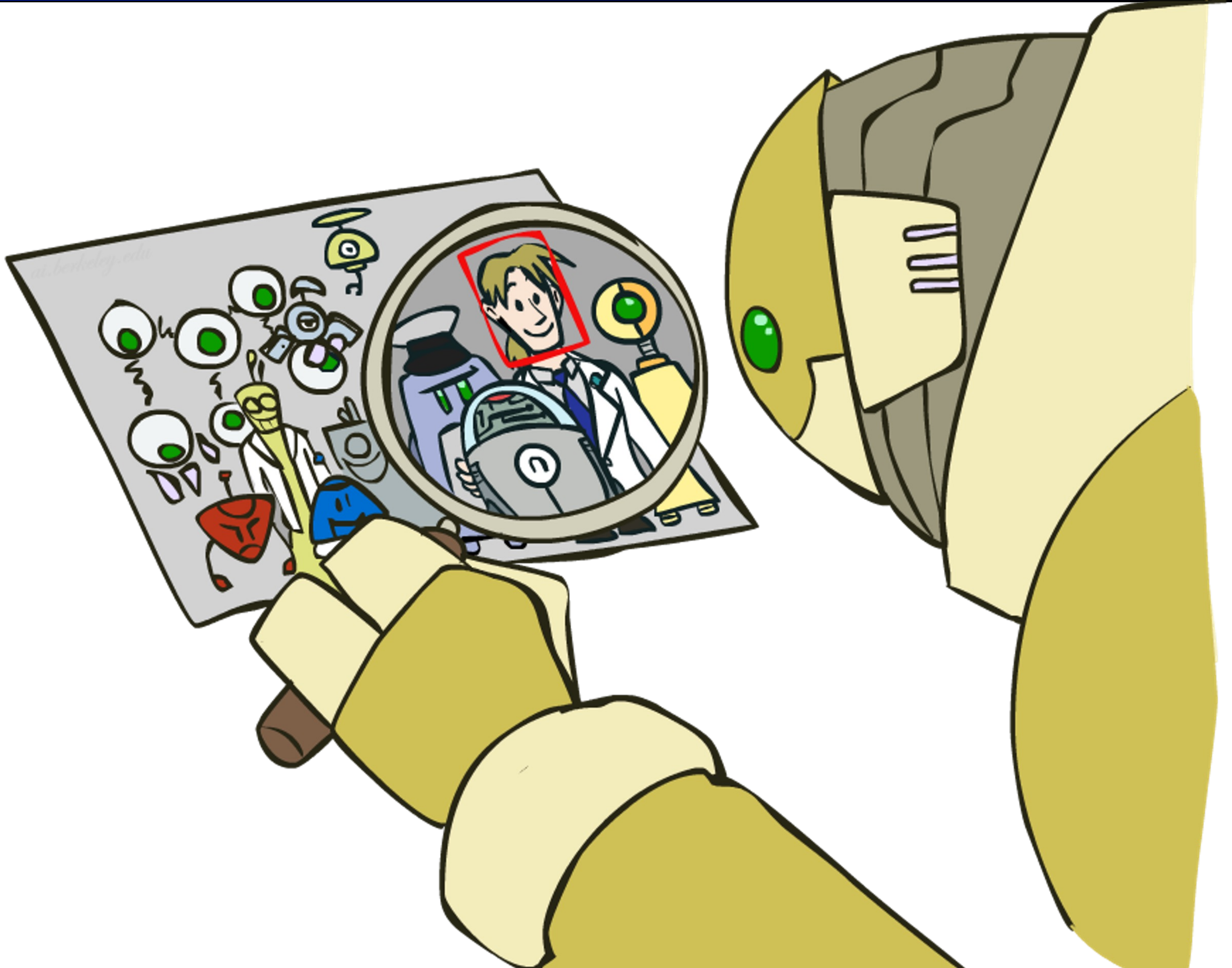
- Optimize probability of label given input $$\max_w \quad ll(w) = \max_w \quad \sum_i \log P(y^{(i)}|x^{(i)};w)$$
- Continuous optimization
  - Gradient ascent:
    - Compute steepest uphill direction = gradient (= just vector of partial derivatives)
    - Take step in the gradient direction
    - Repeat (until held-out data accuracy starts to drop = "early stopping")
- Deep neural nets
  - Last layer = still logistic regression
  - Now also many more layers before this last layer
    - = computing the features
    - ⍰ the features are learned rather than hand-designed
  - Universal function approximation theorem
    - If neural net is large enough
    - Then neural net can represent any continuous mapping from input to output with arbitrary accuracy
    - But remember: need to avoid overfitting / memorizing the training data ⍰ early stopping!
  - Automatic differentiation gives the derivatives efficiently (how? = outside of scope of 188)

How well does it work?

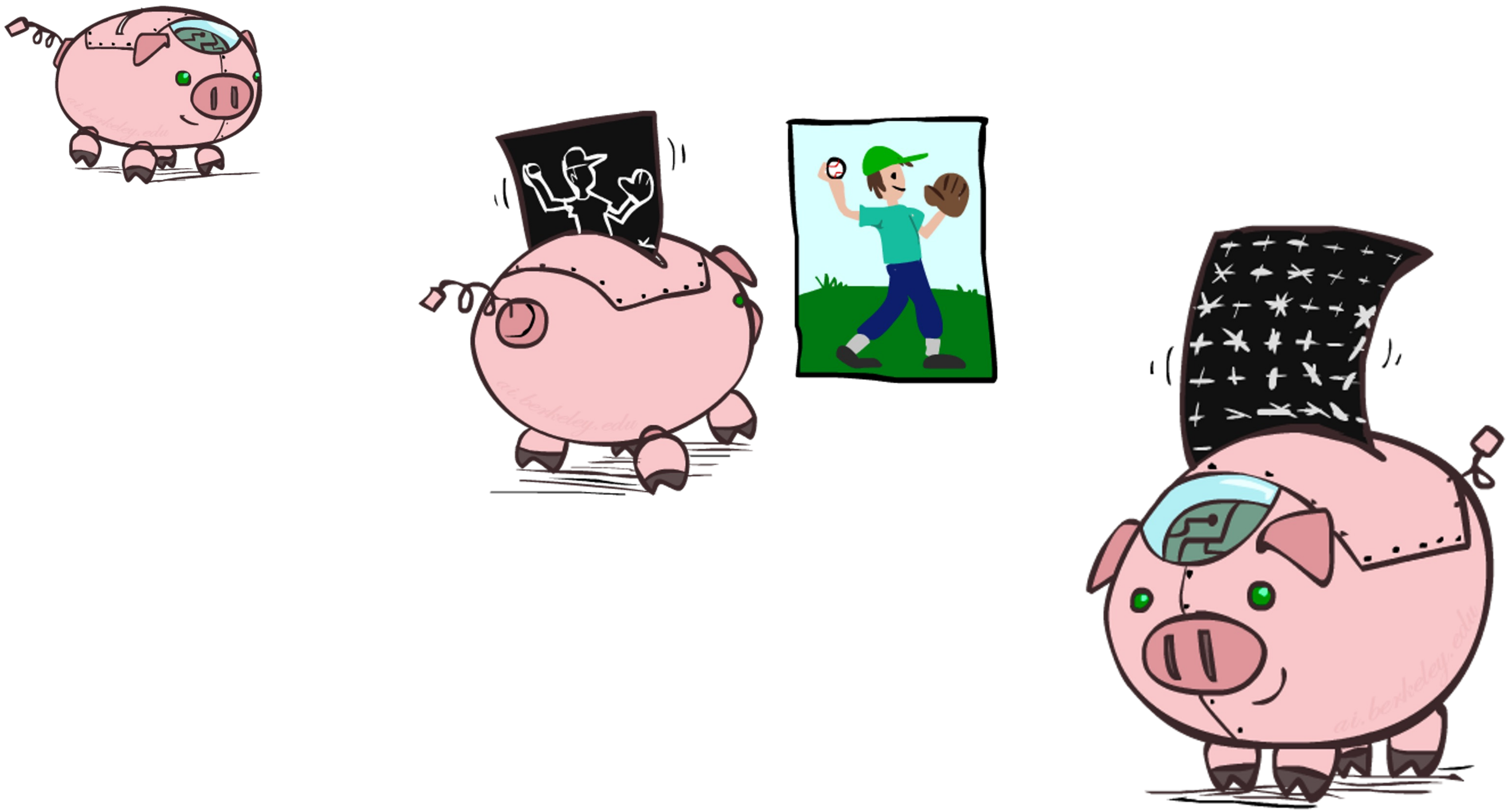
Computer Vision



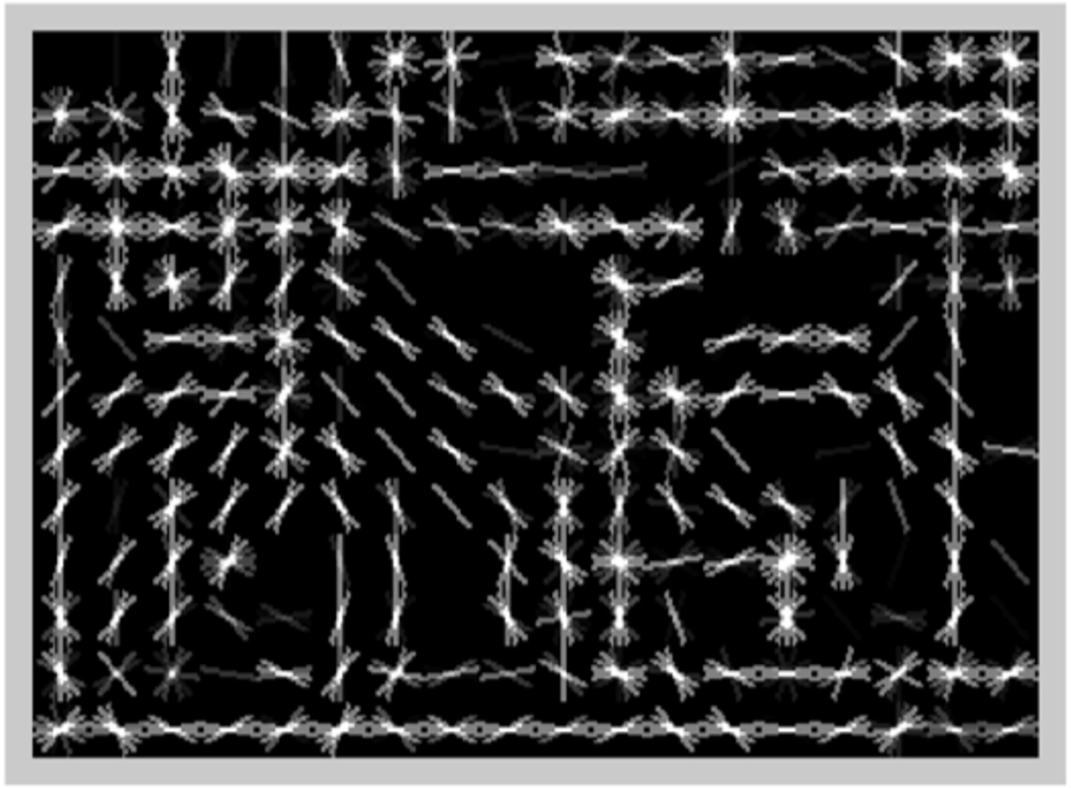
Object Detection



Manual Feature Design



Features and Generalization



[HoG: Dalal and Triggs,

Features and Generalization

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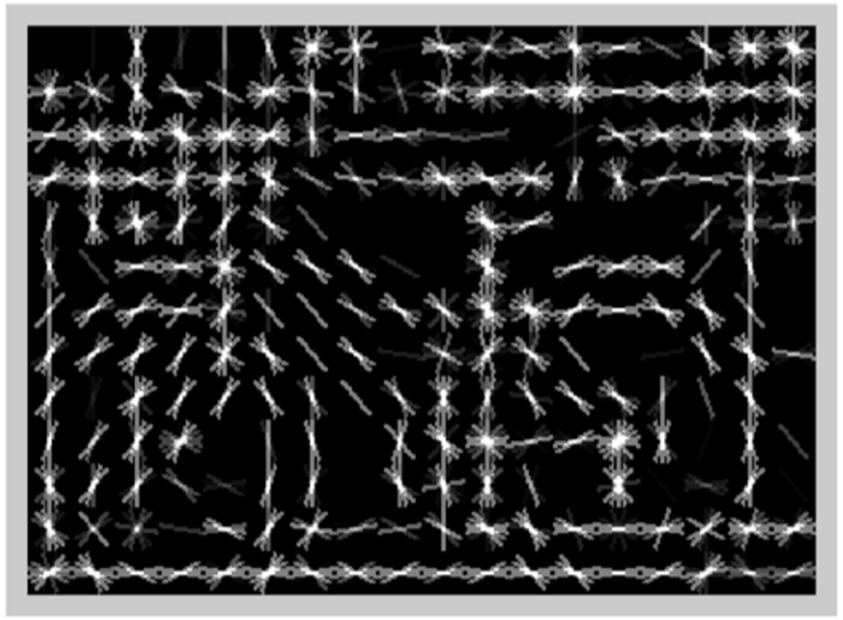
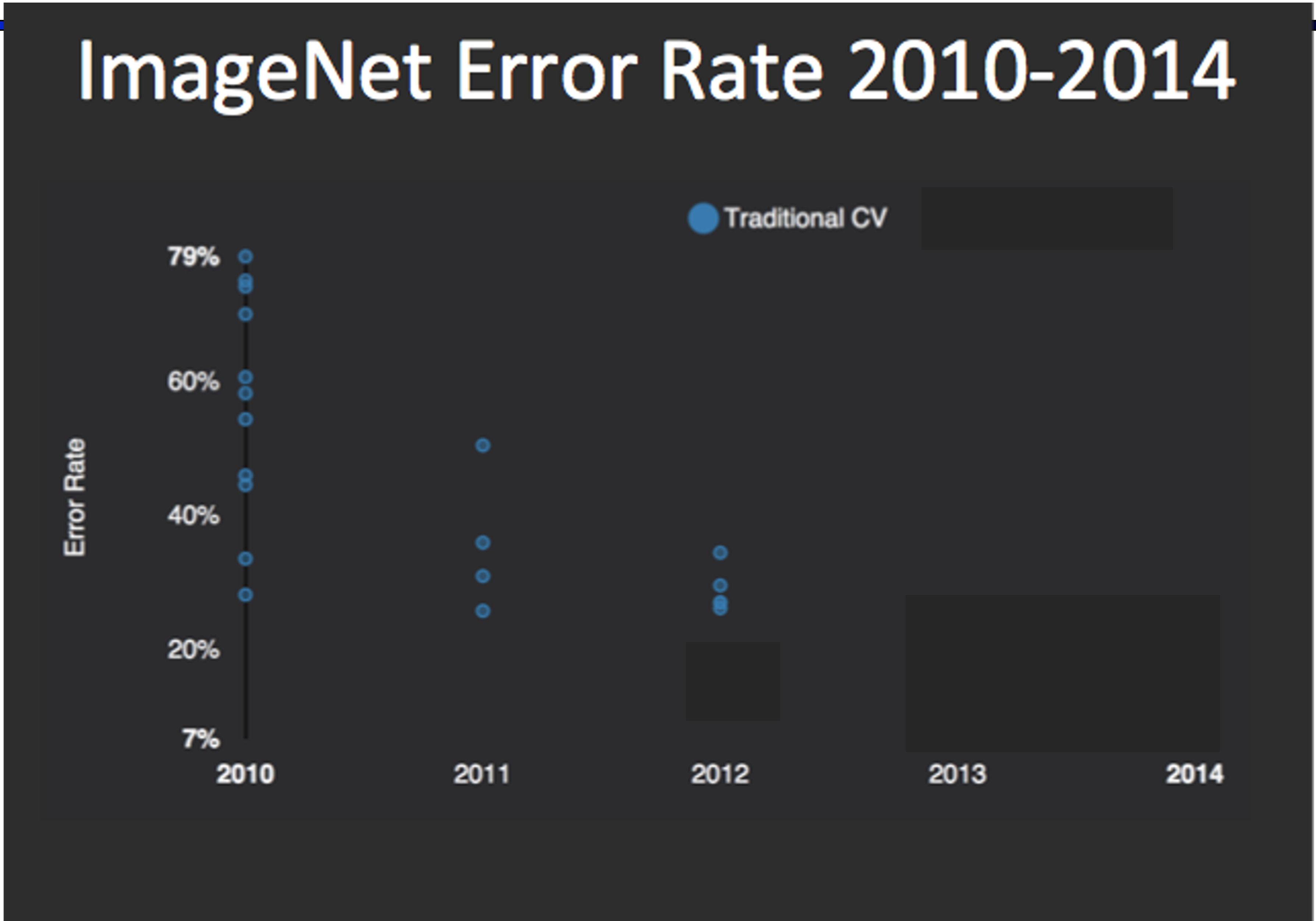


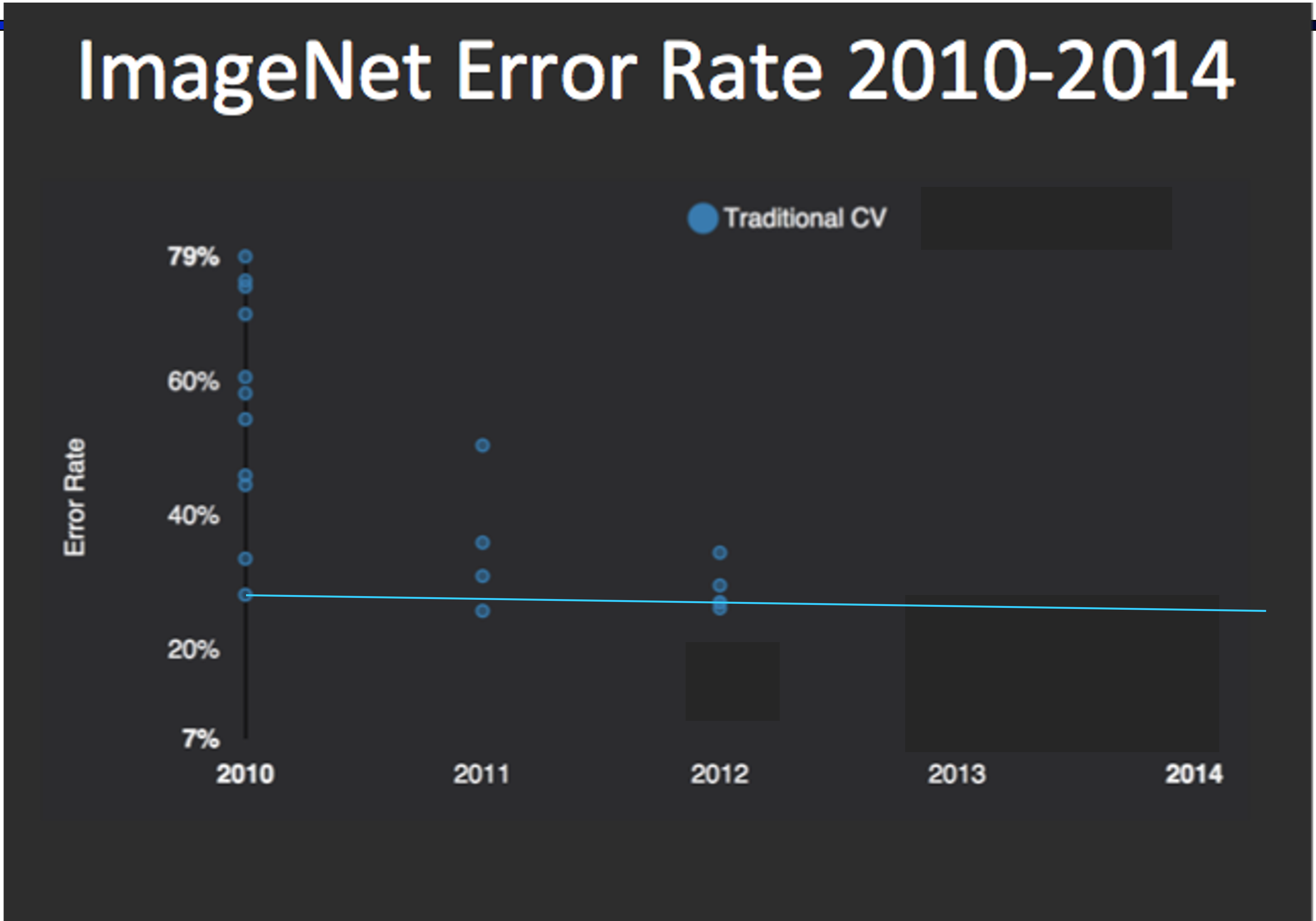
HoG

Performance



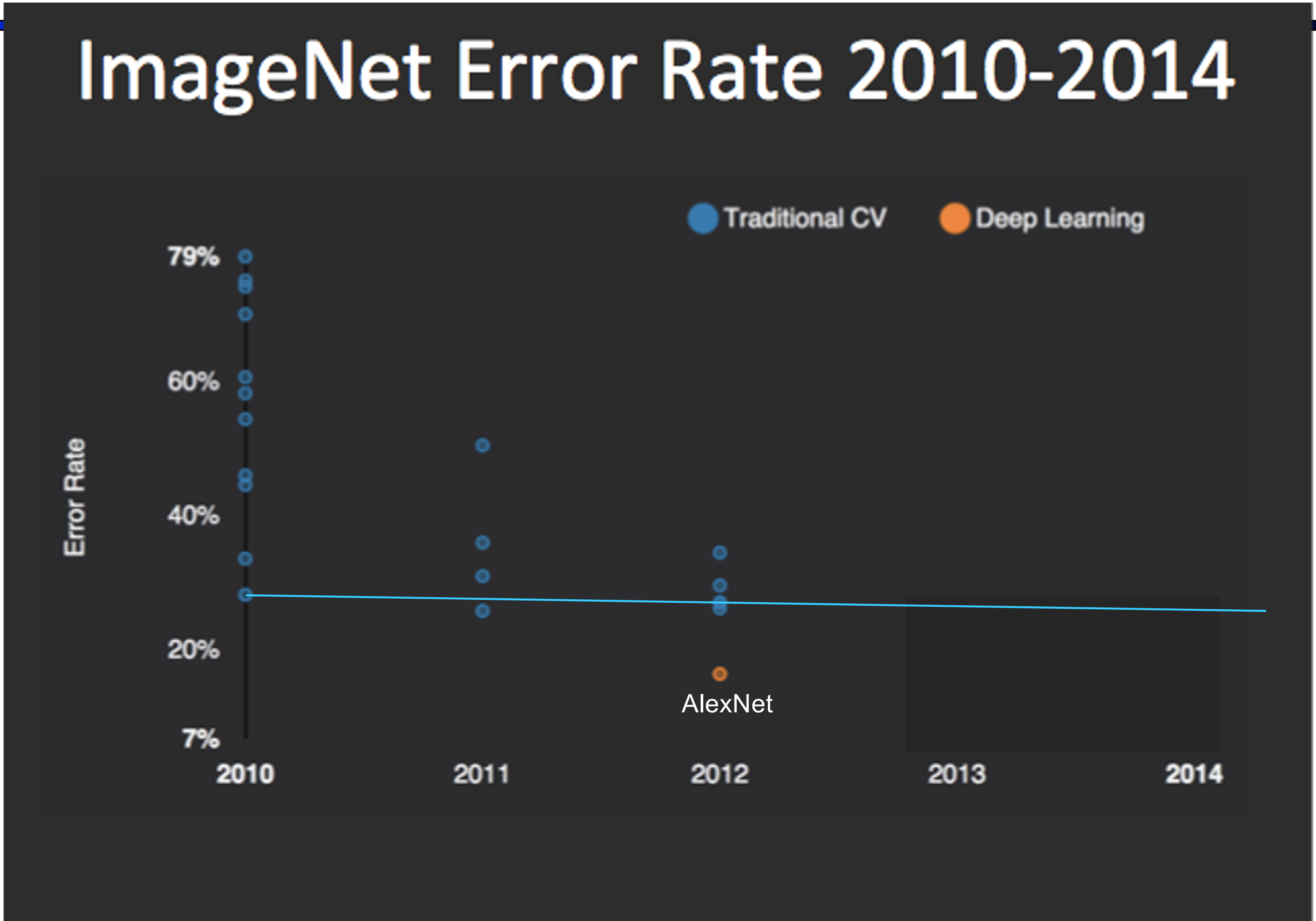
graph credit Matt Zeiler, Clarifai

Performance



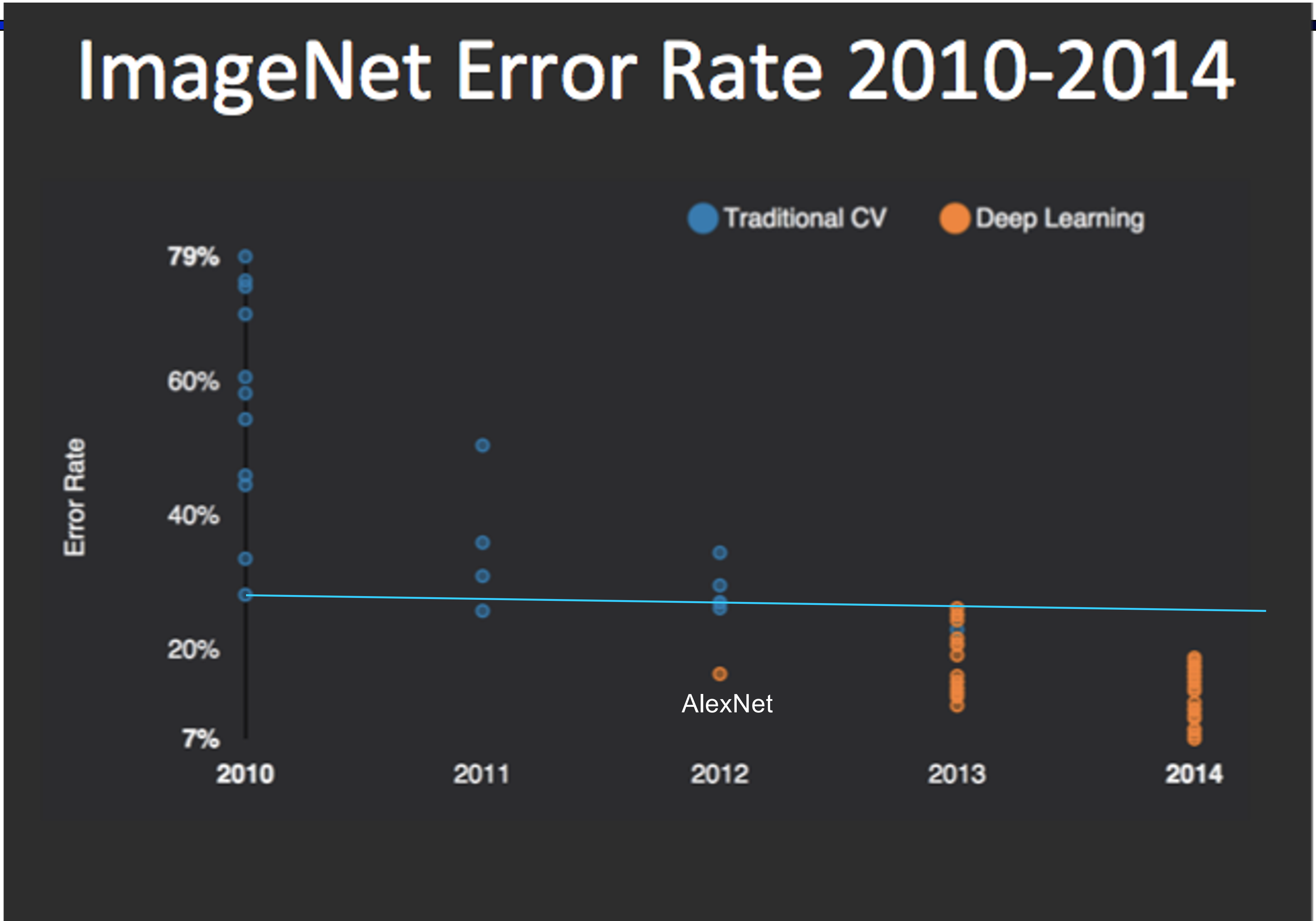
graph credit Matt Zeiler, Clarifai

Performance



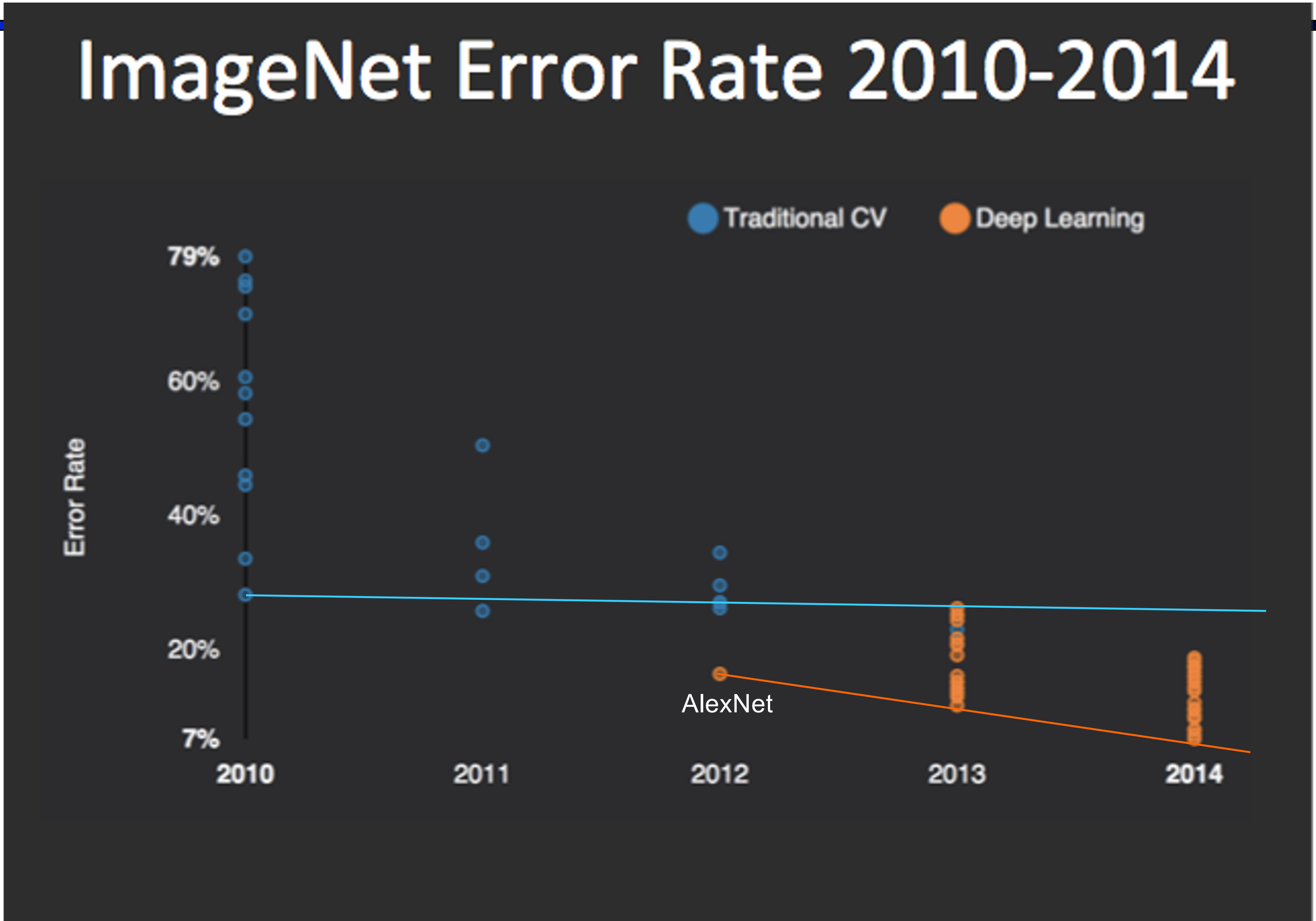
graph credit Matt Zeiler, Clarifai

Performance



graph credit Matt Zeiler, Clarifai

Performance



graph credit Matt Zeiler, Clarifai

MS COCO Image Captioning Challenge

"man in black shirt is playing guitar."

"construction worker in orange safety vest is working on road."

"two young girls are playing with lego toy."

"boy is doing backflip on wakeboard."

"girl in pink dress is jumping in air."

"black and white dog jumps over bar."

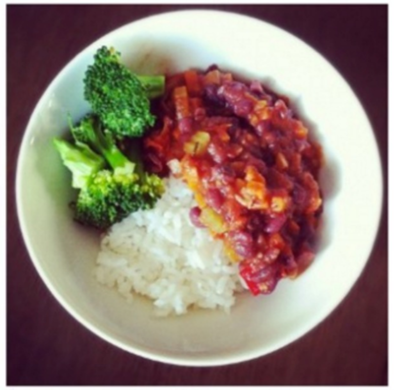
"young girl in pink shirt is swinging on swing."

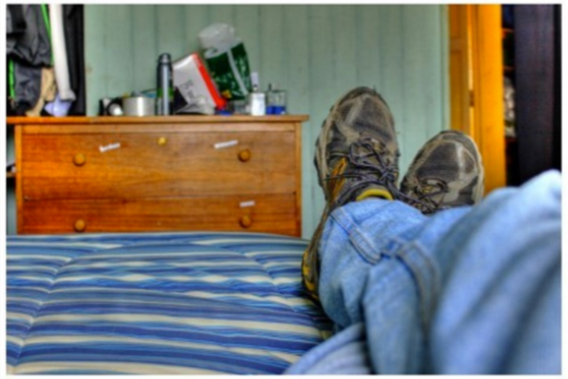
"man in blue wetsuit is surfing on wave."

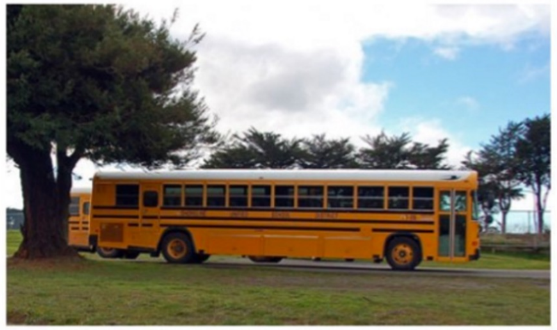
Karpathy & Fei-Fei, 2015; Donahue et al., 2015; Xu et al, 2015; many more

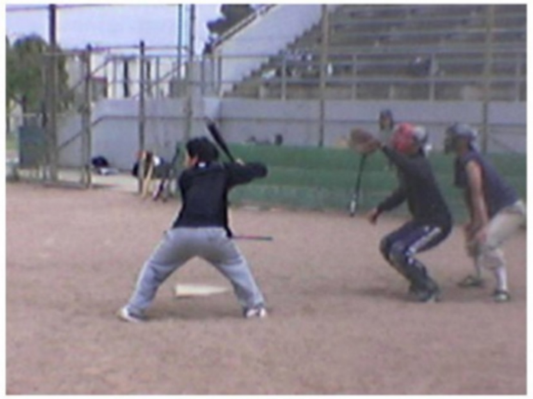
Visual QA Challenge

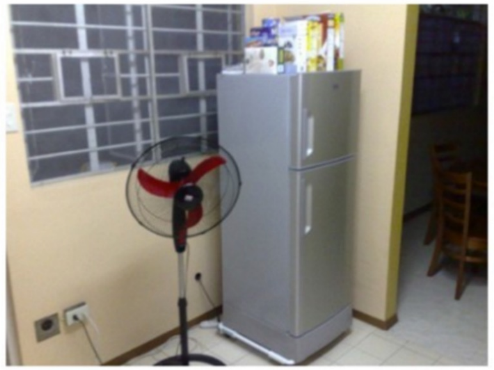
Stanislaw Antol, Aishwarya Agrawal, Jiasen Lu, Margaret Mitchell, Dhruv Batra, C. Lawrence Zitnick, Devi Parikh

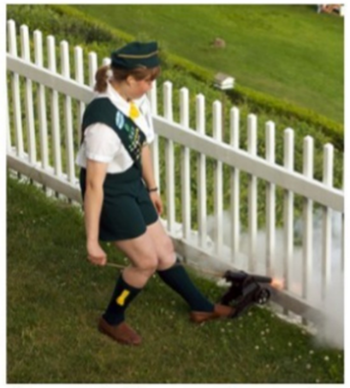

What vegetable is on the plate?
Neural Net: broccoli
Ground Truth: broccoli

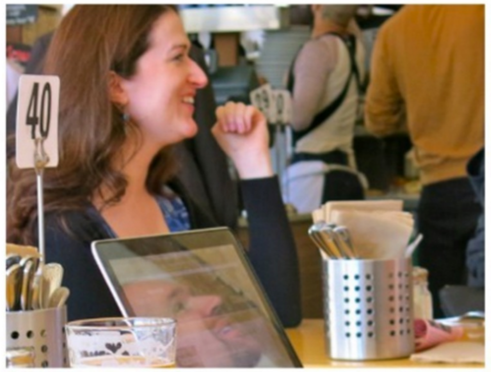

What color are the shoes on the person's feet ?
Neural Net: brown
Ground Truth: brown


How many school busses are there?
Neural Net: 2
Ground Truth: 2

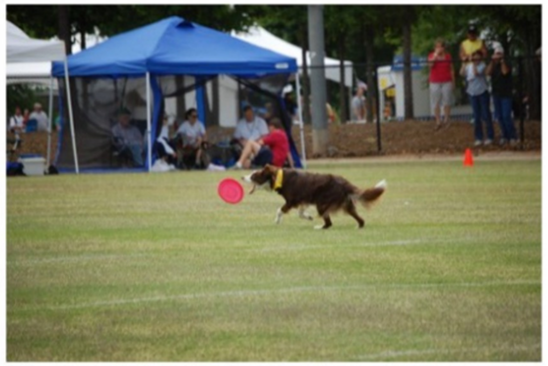

What sport is this?
Neural Net: baseball
Ground Truth: baseball


What is on top of the refrigerator?
Neural Net: magnets
Ground Truth: cereal

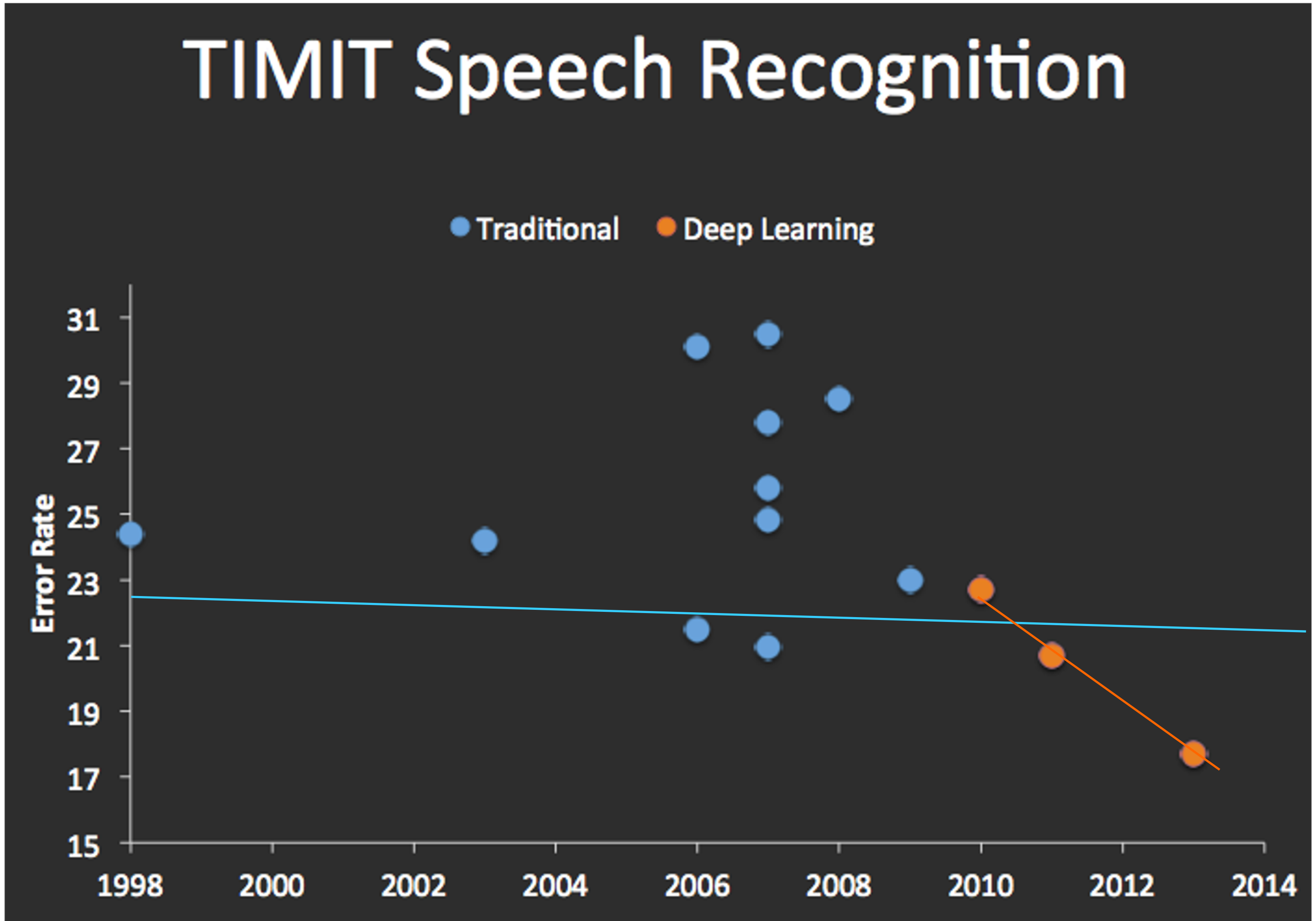

What uniform is she wearing?
Neural Net: shorts
Ground Truth: girl scout

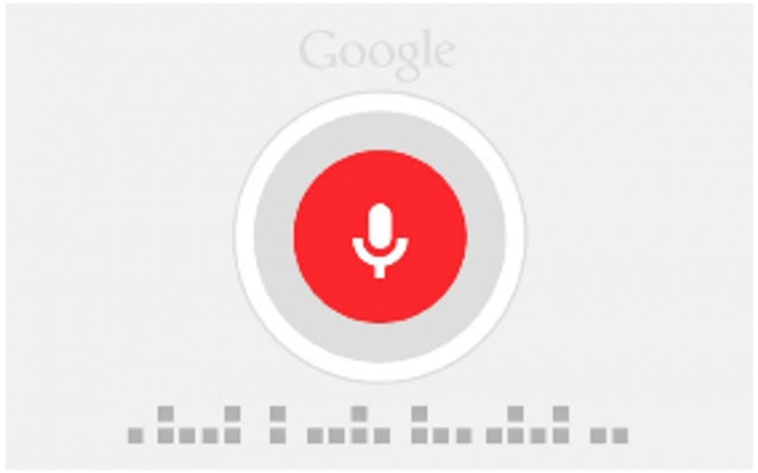


What is the table number?
Neural Net: 4
Ground Truth:40


What are people sitting under in the back?
Neural Net: bench
Ground Truth: tent

Speech Recognition

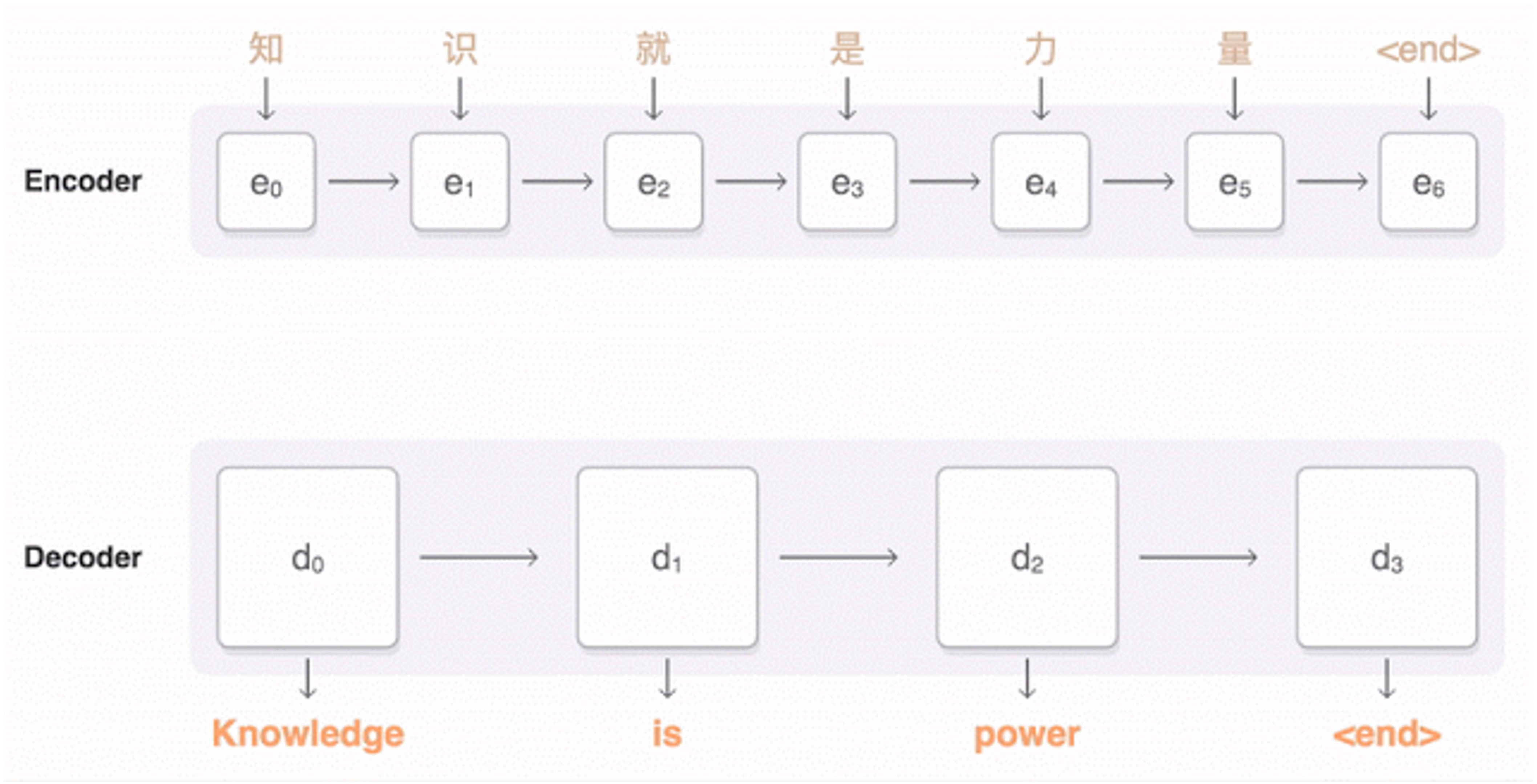




graph credit Matt Zeiler, Clarifai

Machine Translation

Google Neural Machine Translation (in production)



Next: More Neural Net Applications!