Linear Algebra

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Abstract

This is a note about Linear Algebra.

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1 The Geometric Interpretation of Systems of Equations

1.1 Two equations, two unknowns

We have the following system of equations:

$$\begin{cases} 2x - y = 0 & \text{(1a)} \\ -x + 2y = 3 & \text{(1b)} \end{cases}$$

First, let's observe the system from the perspective of a matrix. From the system,we can extract the coefficient matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{2}$$

the unknown vector:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

and the right vector is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{4}$$

so the system of equations can be written in matrix form:

$$\mathbf{Ax} = \mathbf{b} \tag{5}$$

Second, we give the **row picture**. We can find that, each row of (1) corresponds to a staright line. Draw it on the rectangular coordinate system of a plane:

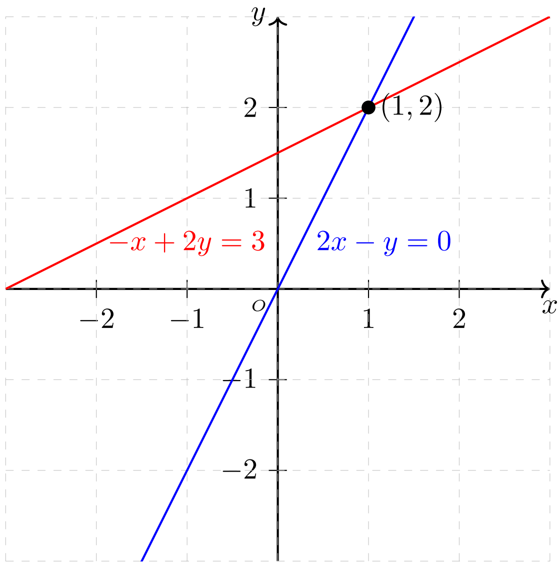

Figure 1: Two lines intersect

The intersection point $(1, 2)$ is the solution of (1).

Finally, we give the **column picture**.We can write the system in the below form:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{6}$$

We can understand the above formula as the **linear combination of column vectors**:

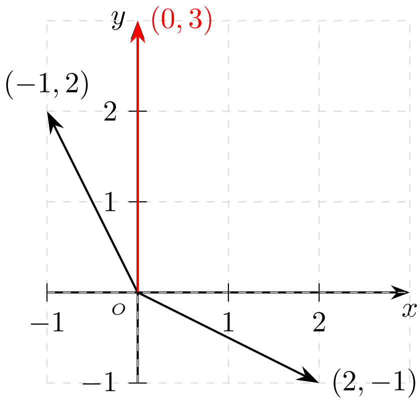

Figure 2: Column Picture

1.2 A simple summary

In the previous subsection, we use matrix form, row picture, column picture to describe the system of equations. And the most important point is the column picture.

The column picture tells us that, we can use the linear combination of two vectors (satisfied certain conditions) to obtain the third vector.

Note:If we take x and y as any combination of numbers, than the linear combination of tow vectors (satisfied certain conditions) can obtain every vectors in the plane.

1.3 Three equations,three unknowns

We have the following system of equations:

$$\begin{cases} 2x - y = 0 & \text{(7a)} \\ -x + 2y - z = 3 & \text{(7b)} \\ -3y + 4z = 4 & \text{(7c)} \end{cases}$$

First, let's give the matrix form. The coefficient matrix is:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix} \tag{8}$$

the unkown vector:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{9}$$

and the right vector is:

$$\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} \tag{10}$$

so the system can be written:

$$\mathbf{Ax} = \mathbf{b} \tag{11}$$

Second, the row picture. With the aid of the geometric knowledge, we know that, every row of (7) corresponds to a plane in 3-D space.

Finally, the column picture. Below:

$$x\begin{bmatrix}2\\-1\\3\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix} \tag{12}$$

As x, y and z are scalars, so the left-side of (12) is the linear combination of three vectors. And easily, the solution is:

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{13}$$

1.4 Extended thinking

In the previous subsection, we get the solution of (7).

So, for the fixed A (8), can I solve $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b}$? Or, in the linear combination words, the problem is: Using any combination of (x, y, z), can I probuce any vectors in a 3-D space?

For A(8), the answer is YES. Then in what case, the answer is NO?

We can give an example: Three vectors in **A** are in the same plane. In this case, we could't get any vector outside the plane.

2 Matrix Elimination