# CS 115: Maths for Computer Science

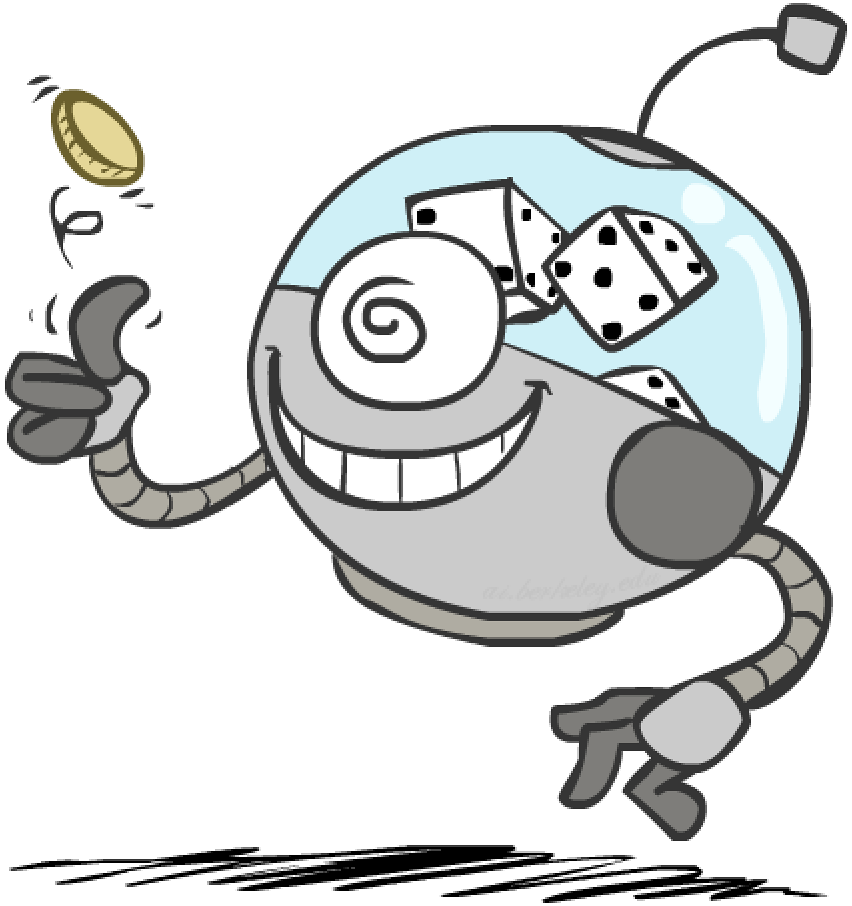
## Probability

Instructor: Ngoc-Hoang LUONG

[These slides were created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley.  All CS188 materials are available at http://ai.berkeley.edu.]

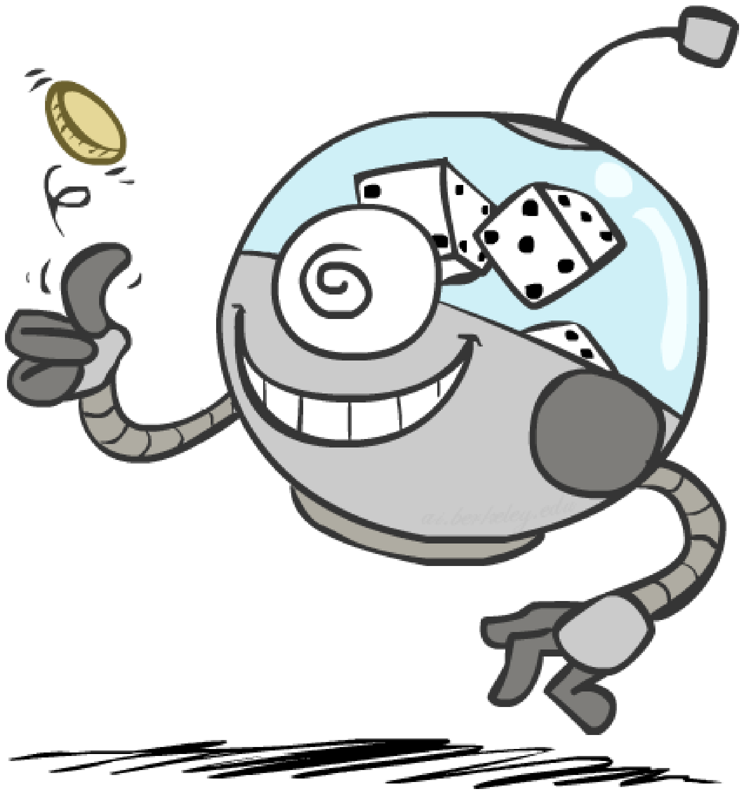
# Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes’ Rule
  - Inference
  - Independence



# Random Variables

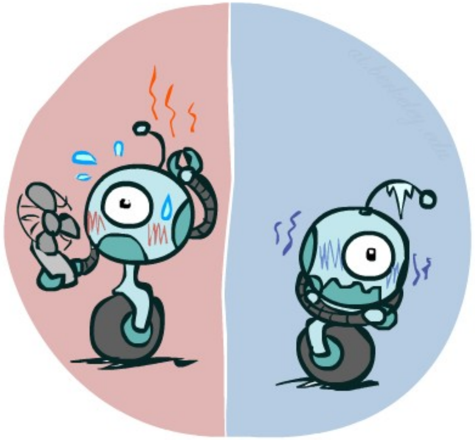
- A random variable is some aspect of the world about which we (may) have uncertainty
  - R = Is it raining?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Random variables have domains
  - R in {true, false}
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), ...}



# Probability Distributions

- Associate a probability with each value

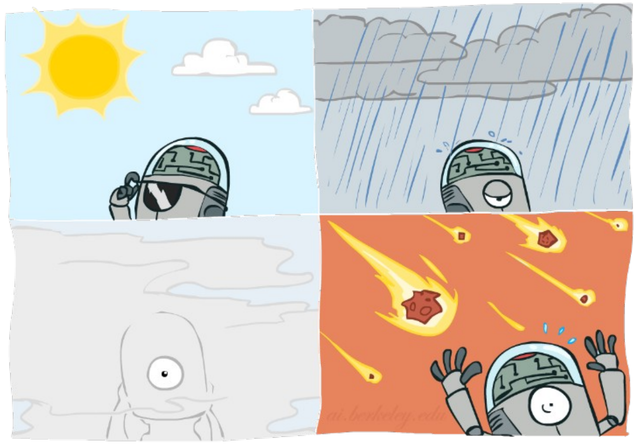
  - Temperature:



$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

  - Weather:



$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.1 |
| fog | 0.3 |
| meteor | 0.0 |

# Probability Distributions

- Unobserved random variables have distributions

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.1 |
| fog | 0.3 |
| meteor | 0.0 |

- A distribution is a TABLE of probabilities of values

- A probability (lower case value) is a single number

$P(W = rain) = 0.1$

$$\forall x \ \ P(X = x) \geq 0 \qquad \sum_x P(X = x) = 1$$

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

OK *if* all domain entries are unique

# Joint Distributions

- A *joint distribution* over a set of random variables $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or *outcome*):

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$$

$$P(x_1, x_2, \ldots x_n)$$

  - Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

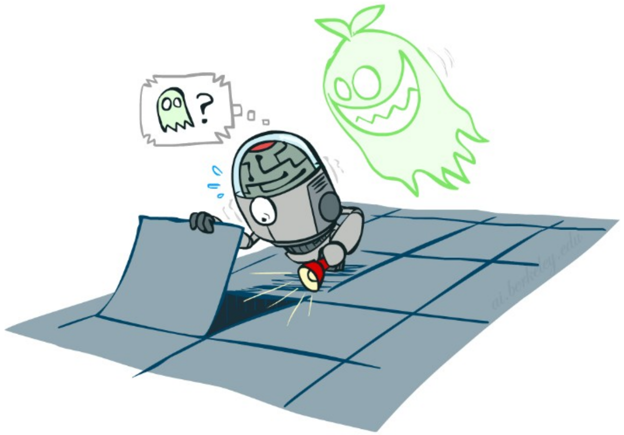
- Size of distribution if n variables with domain sizes d?
  - For all but the smallest distributions, impractical to write

# Probabilistic Models

- A probabilistic model is a joint distribution over a set of random variables
- Probabilistic models:
  - (Random) variables with domains
  - Assignments are called *outcomes*
  - Joint distributions: say whether assignments (outcomes) are likely
  - *Normalized:* sum to 1.0
  - Ideally: only certain variables directly interact
- Constraint satisfaction problems:
  - Variables with domains
  - Constraints: state whether assignments are possible
  - Ideally: only certain variables directly interact

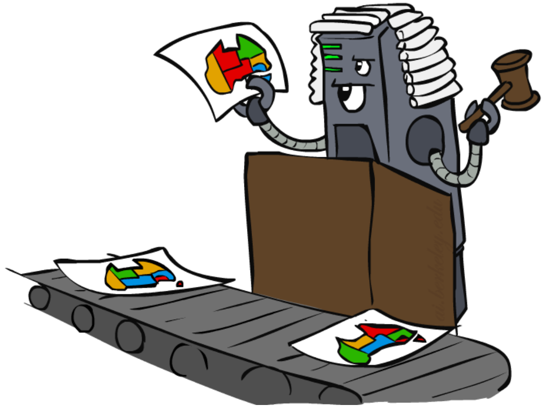
Distribution over T,W

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |



Constraint over T,W

| T | W | P |
|---|---|---|
| hot | sun | T |
| hot | rain | F |
| cold | sun | F |
| cold | rain | T |



# Events

- An *event* is a set E of outcomes

$$P(E) = \sum_{(x_1 \ldots x_n) \in E} P(x_1 \ldots x_n)$$

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny?
  - Probability that it's hot?
  - Probability that it's hot OR sunny?
- Typically, the events we care about are *partial assignments*, like P(T=hot)

$P(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

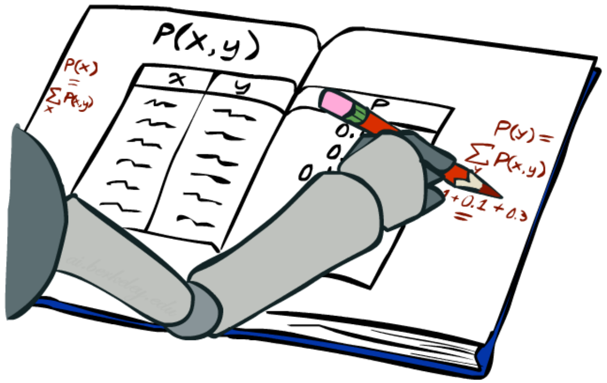
# Quiz: Events

- P(+x, +y) ?
- P(+x) ?
- P(-y OR +x) ?

$$P(X, Y)$$

| X | Y | P |
|---|---|---|
| +x | +y | 0.2 |
| +x | -y | 0.3 |
| -x | +y | 0.4 |
| -x | -y | 0.1 |

# Marginal Distributions




- Marginal distributions are sub-tables which eliminate variables
- Marginalization (summing out): Combine collapsed rows by adding

$P(T,W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$P(t) = \sum_s P(t,s)$$

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$$P(s) = \sum_t P(t,s)$$

$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.4 |

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

# Quiz: Marginal Distributions

$P(X,Y)$

| X | Y | P |
|---|---|---|
| +x | +y | 0.2 |
| +x | -y | 0.3 |
| -x | +y | 0.4 |
| -x | -y | 0.1 |

$$P(x) = \sum_y P(x,y)$$

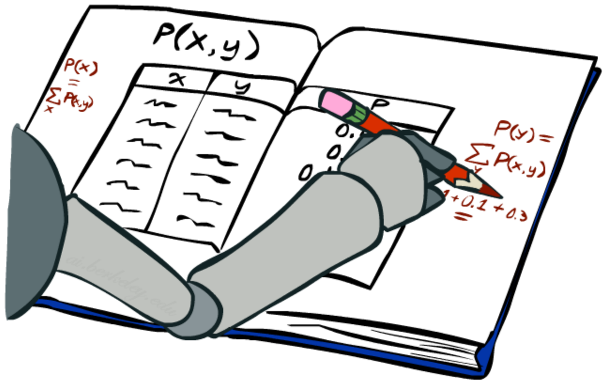
$$P(y) = \sum_x P(x,y)$$

$P(X)$

| X | P |
|---|---|
| +x | |
| -x | |

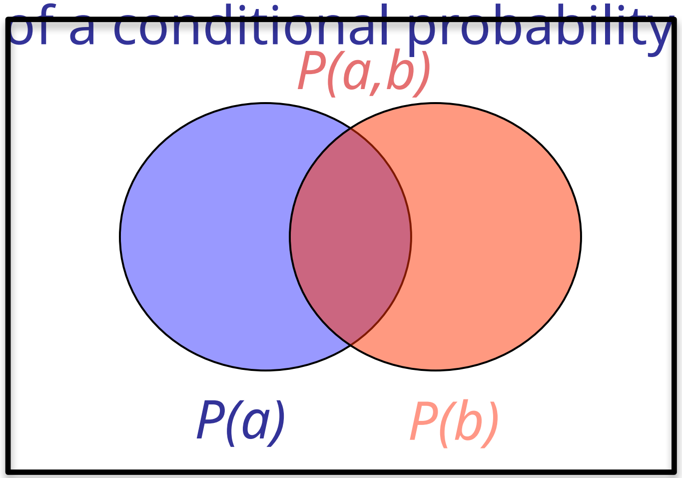
$P(Y)$

| Y | P |
|---|---|
| +y | |
| -y | |



# Conditional Probabilities

- A simple relation between joint and conditional probabilities
  - In fact, this is taken as the *definition* of a conditional probability




$$P(a|b) = \frac{P(a,b)}{P(b)}$$

$$P(T,W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$
$$= 0.2 + 0.3 \quad = 0.5$$

# Quiz: Conditional Probabilities

$$P(X,Y)$$

| X | Y | P |
|---|---|---|
| +x | +y | 0.2 |
| +x | -y | 0.3 |
| -x | +y | 0.4 |
| -x | -y | 0.1 |

- P(+x | +y) ?
- P(-x | +y) ?
- P(-y | +x) ?

# Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Conditional Distributions

$$P(W|T)$$

$$P(W|T = hot)$$

| W | P |
|---|---|
| sun | 0.8 |
| rain | 0.2 |

$$P(W|T = cold)$$

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

Joint Distribution

$$P(T, W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

# Normalization Trick

$$P(T, W)$$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

$$\longrightarrow$$

$$\begin{aligned} P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\ &= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.3}{0.2+0.3} = 0.6 \end{aligned}$$

$$P(W|T=c)$$

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

# Normalization Trick

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)}$$
$$= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.2}{0.2+0.3} = 0.4$$

$P(T,W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

**SELECT** the joint probabilities matching the evidence

$P(c,W)$

| T | W | P |
|---|---|---|
| cold | sun | 0.2 |
| cold | rain | 0.3 |

**NORMALIZE** the selection (make it sum to one)

$P(W|T=c)$

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

$$P(W=r|T=c) = \frac{P(W=r,T=c)}{P(T=c)}$$
$$= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)}$$
$$= \frac{0.3}{0.2+0.3} = 0.6$$

# Normalization Trick

$P(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

**SELECT** the joint probabilities matching the evidence

$P(c, W)$

| T | W | P |
|---|---|---|
| cold | sun | 0.2 |
| cold | rain | 0.3 |

**NORMALIZE** the selection (make it sum to one)

$P(W|T = c)$

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

- Why does this work? Sum of selection is P(evidence)! (P(T=c), here)

$$P(x_1|x_2) = \frac{P(x_1, x_2)}{P(x_2)} = \frac{P(x_1, x_2)}{\sum_{x_1} P(x_1, x_2)}$$

# Quiz: Normalization Trick

- P(X | Y=-y) ?

$P(X, Y)$

| X | Y | P |
|---|---|---|
| +x | +y | 0.2 |
| +x | -y | 0.3 |
| -x | +y | 0.4 |
| -x | -y | 0.1 |

**SELECT** the joint probabilities matching the evidence

→

**NORMALIZE** the selection (make it sum to one)

→

# To Normalize

- (Dictionary) To bring or restore $\boxed{\text{to a normal condition}}$

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

| W | P |
|---|---|
| sun | 0.2 |
| rain | 0.3 |

Normalize →

Z = 0.5

| W | P |
|---|---|
| sun | 0.4 |
| rain | 0.6 |

- Example 2

| T | W | P |
|---|---|---|
| hot | sun | 20 |
| hot | rain | 5 |
| cold | sun | 10 |
| cold | rain | 15 |

Normalize →

Z = 50

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

# Probabilistic Inference

- Probabilistic inference: compute a desired probability from other known probabilities (e.g. conditional from joint)

- We generally compute conditional probabilities
  - P(on time | no reported accidents) = 0.90
  - These represent the agent's *beliefs* given the evidence

- Probabilities change with new evidence:
  - P(on time | no accidents, 5 a.m.) = 0.95
  - P(on time | no accidents, 5 a.m., raining) = 0.80
  - Observing new evidence causes *beliefs to be updated*

# Inference by Enumeration

- General case:
  - Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
  - Query* variable: $Q$
  - Hidden variables: $H_1 \dots H_r$

$\left.\begin{matrix} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{matrix}\right\} X_1, X_2, \dots X_n$

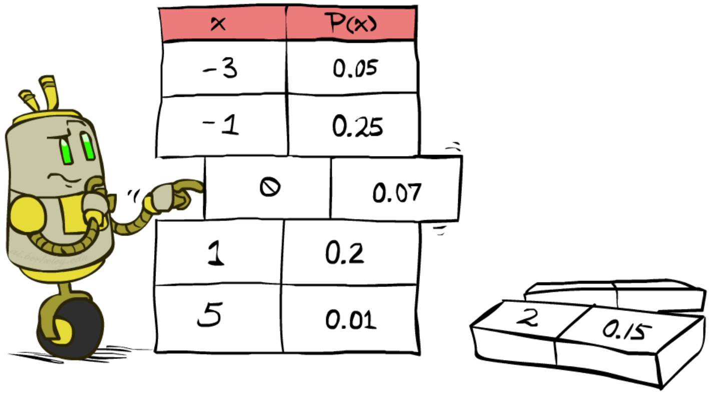
*All variables*

- We want:

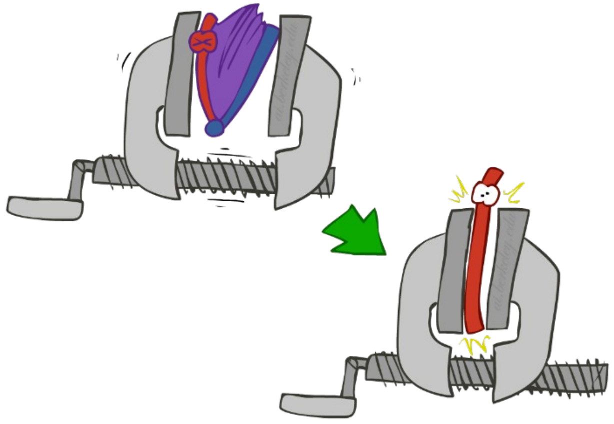
** Works fine with multiple query variables, too*

$$P(Q|e_1 \dots e_k)$$

- Step 1: Select the entries consistent with the evidence




- Step 2: Sum out H to get joint of Query and evidence



$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots X_n}$$

- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \cdots e_k)$$

$$P(Q|e_1 \cdots e_k) = \frac{1}{Z} P(Q, e_1 \cdots e_k)$$

# Inference by Enumeration

- P(W)?

- P(W | winter)?

- P(W | winter, hot)?

| S | T | W | P |
|---|---|---|---|
| summer | hot | sun | 0.30 |
| summer | hot | rain | 0.05 |
| summer | cold | sun | 0.10 |
| summer | cold | rain | 0.05 |
| winter | hot | sun | 0.10 |
| winter | hot | rain | 0.05 |
| winter | cold | sun | 0.15 |
| winter | cold | rain | 0.20 |

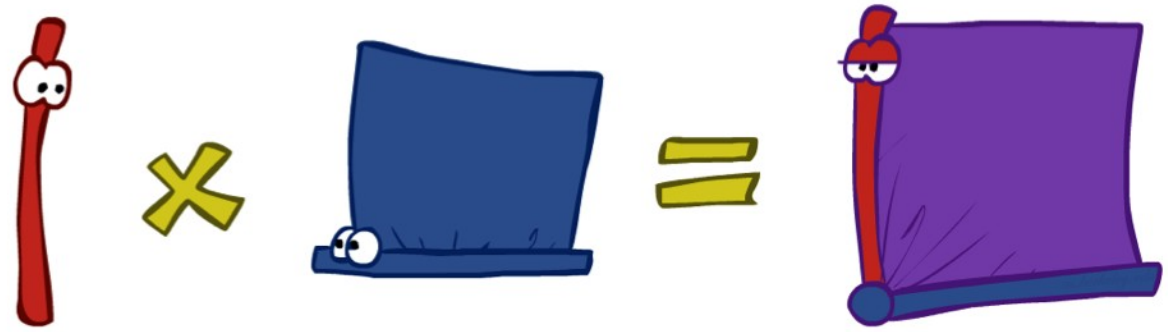
# Inference by Enumeration

- Obvious problems:
  - Worst-case time complexity $O(d^n)$
  - Space complexity $O(d^n)$ to store the joint distribution

# The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(y)P(x|y) = P(x,y) \qquad \Longleftrightarrow \qquad P(x|y) = \frac{P(x,y)}{P(y)}$$



# The Product Rule

$$P(y)P(x|y) = P(x, y)$$

- Example:

$P(W)$

| R | P |
|---|---|
| sun | 0.8 |
| rain | 0.2 |

$P(D|W)$

| D | W | P |
|---|---|---|
| wet | sun | 0.1 |
| dry | sun | 0.9 |
| wet | rain | 0.7 |
| dry | rain | 0.3 |

$P(D, W)$

| D | W | P |
|---|---|---|
| wet | sun | [illegible] |
| dry | sun | 0.72 |
| wet | rain | 0.14 |
| dry | rain | 0.06 |

# The Chain Rule

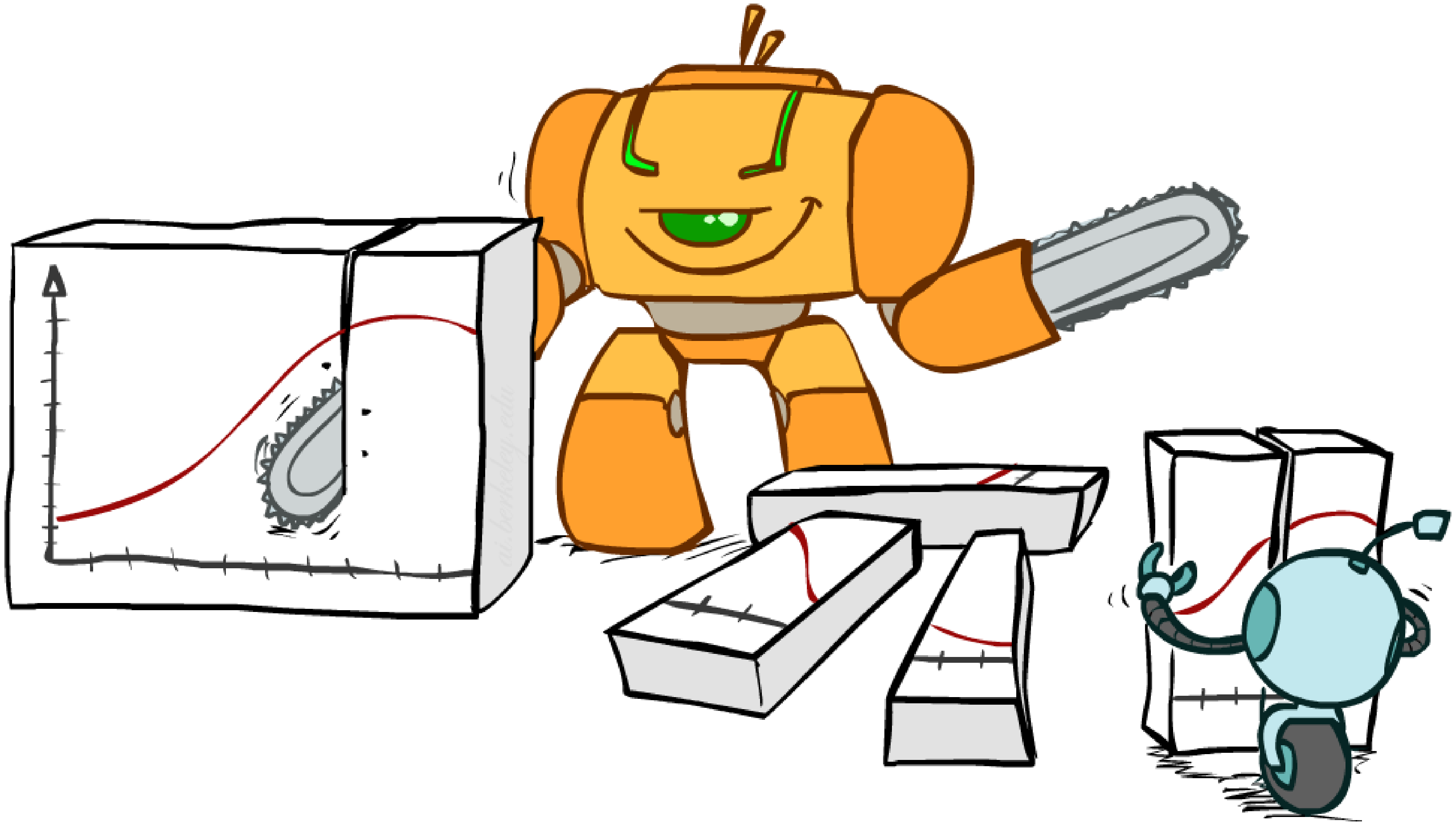
- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

- Why is this always true?

# Bayes Rule



# Bayes' Rule

- Two ways to factor a joint distribution over two variables:

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple
  - Foundation of many systems we'll see later (e.g. ASR, MT)

- In the running for most important AI equation!

That's my rule!

# Inference with Bayes' Rule

- Example: Diagnostic probability from causal probability:

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

- Example:
  - M: meningitis, S: stiff neck

$$\left.\begin{aligned} P(+m) &= 0.0001 \\ P(+s|+m) &= 0.8 \\ P(+s|-m) &= 0.01 \end{aligned}\right\} \text{Example givens}$$

$$P(+m|+s) = \frac{P(+s|+m)P(+m)}{P(+s)} = \frac{P(+s|+m)P(+m)}{P(+s|+m)P(+m) + P(+s|-m)P(-m)} = \frac{0.8 \times 0.0001}{0.8 \times 0.0001 + 0.01 \times 0.999}$$

  - Note: posterior probability of meningitis still very small
  - Note: you should still get stiff necks checked out!  Why?

# Inference with Bayes' Rule

$$P(\text{cause}|\text{effect}) = \frac{P(\text{effect}|\text{cause})P(\text{cause})}{P(\text{effect})}$$

Example:

- I am 90% confident that I'm a good singer. P(good singer) = 0.9
- If I'm a good singer, then 99% of people will like my singing. P(like | good singer) = 0.99
- If I'm a bad singer, then 10% of people will like my singing. P(like | bad singer) = 0.10
- I sing in my living room and  my roommate covers his ears.
- I need to update my beliefs to account for what I've learned.
- I need to calculate: P(good singer | roommate doesn't like my singing) = ?

# Quiz: Bayes' Rule

- Given:

$P(W)$

| R | P |
|---|---|
| sun | 0.8 |
| rain | 0.2 |

$P(D|W)$

| D | W | P |
|---|---|---|
| wet | sun | 0.1 |
| dry | sun | 0.9 |
| wet | rain | 0.7 |
| dry | rain | 0.3 |

- What is P(W | dry) ?

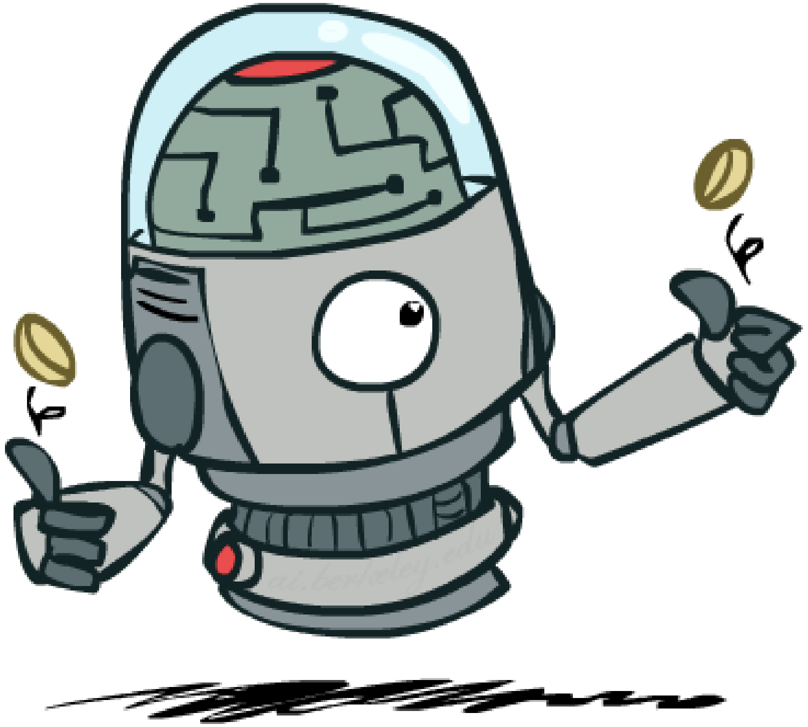
# Probabilistic Models

- Models describe how (a portion of) the world works

- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box

- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information

# Independence



# Independence

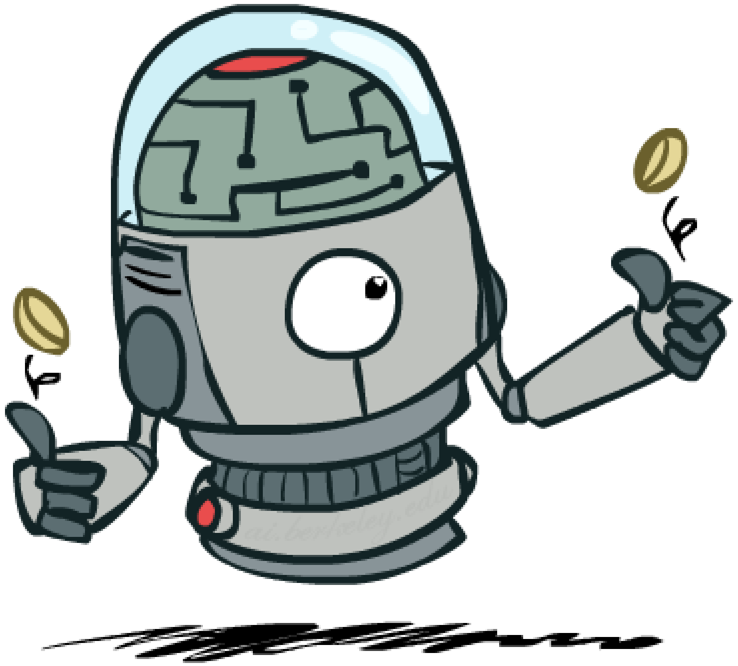
- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

  - This says that their joint distribution *factors* into a product two simpler distributions
  - Another form:

$$\forall x, y : P(x|y) = P(x)$$

  - We write: $X \perp\!\!\!\perp Y$



- Independence is a simplifying *modeling assumption*
  - *Empirical* joint distributions: at best “close” to independent
  - What could we assume for {Weather, Traffic, Cavity,

# Example: Independence?

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$P_1(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$P_2(T, W)$

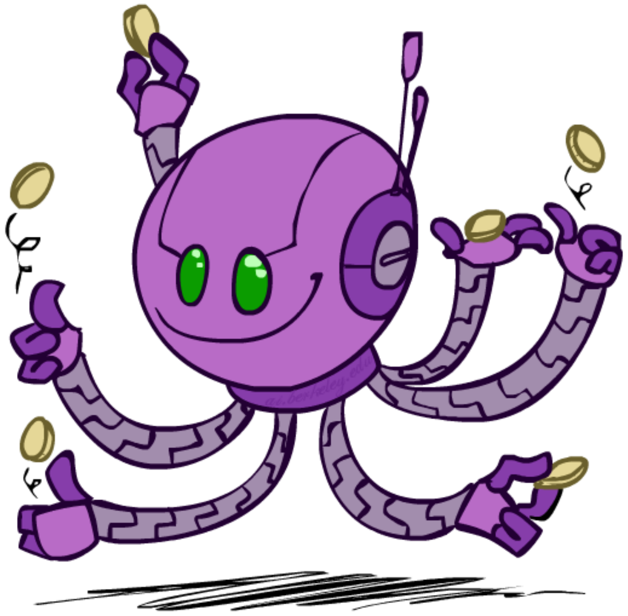
| T | W | P |
|---|---|---|
| hot | sun | 0.3 |
| hot | rain | 0.2 |
| cold | sun | 0.3 |
| cold | rain | 0.2 |

$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.4 |

# Example: Independence

- N fair, independent coin flips:



$P(X_1)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

$P(X_2)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

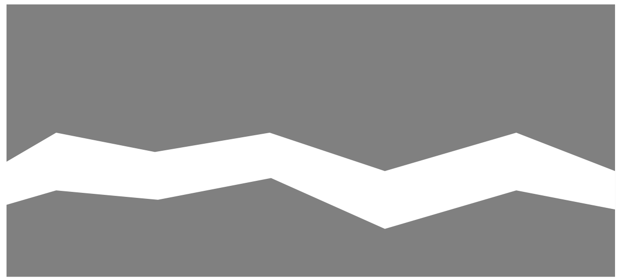
$\cdots$

$P(X_n)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

$P(X_1, X_2, \ldots X_n)$

$2^n$



# Conditional Independence

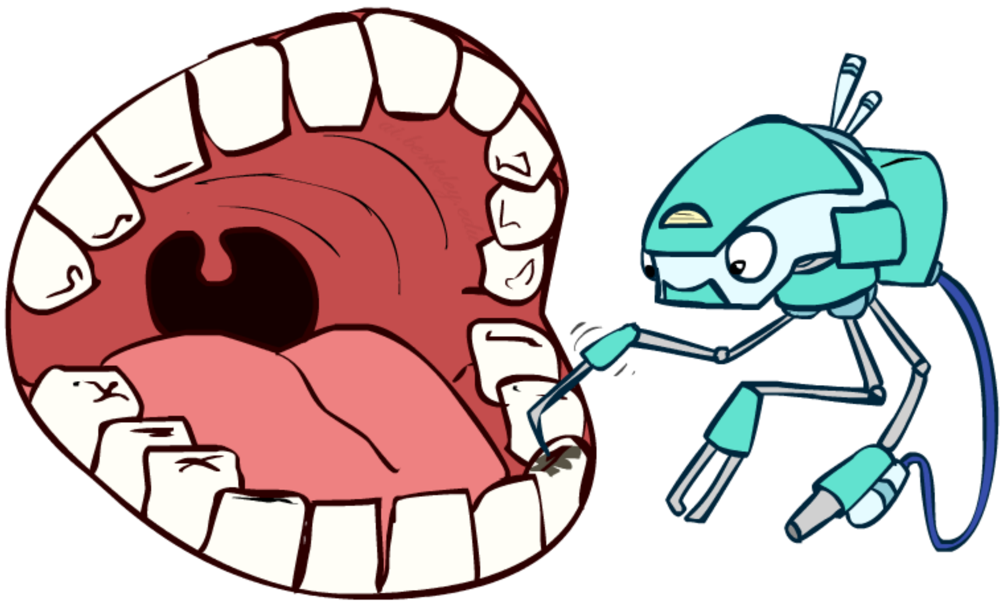
- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don’t have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)

- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily



# Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z

$$X \perp\!\!\!\perp Y | Z$$

  if and only if:

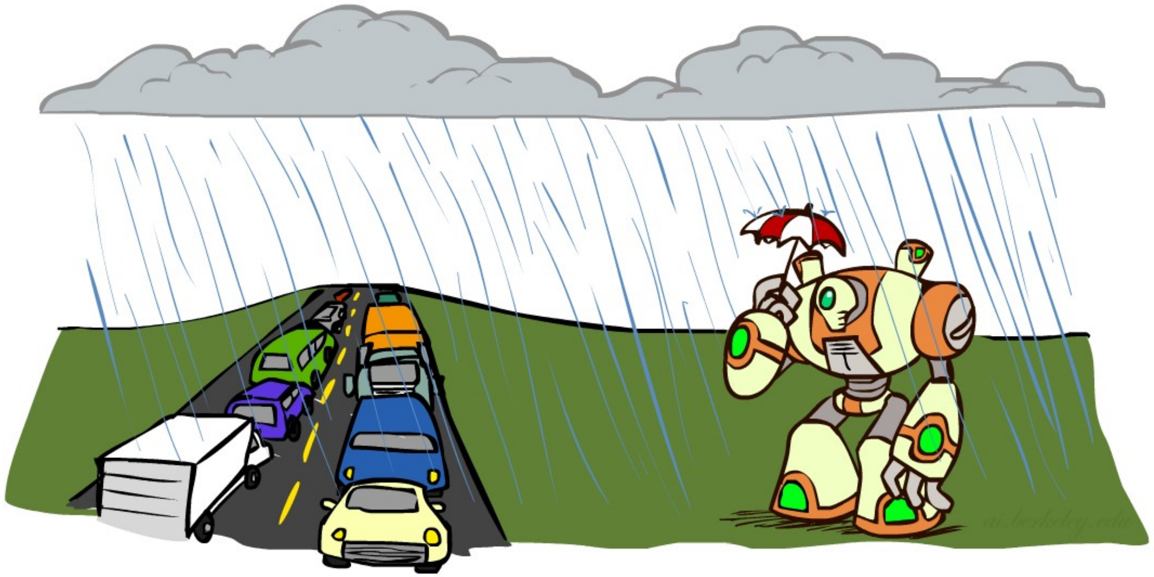
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

  or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

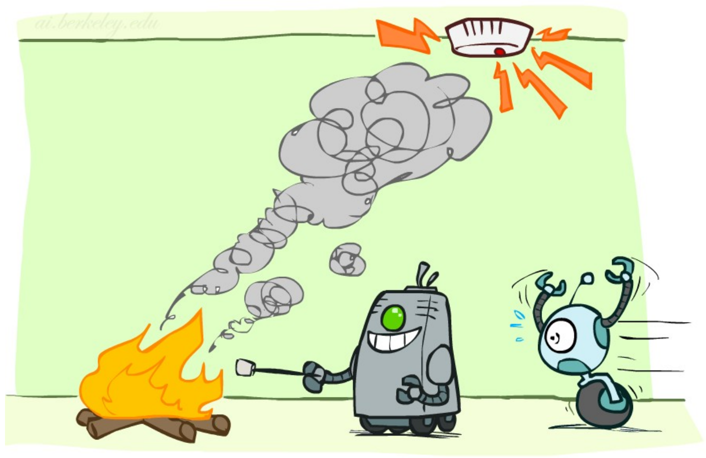
# Conditional Independence

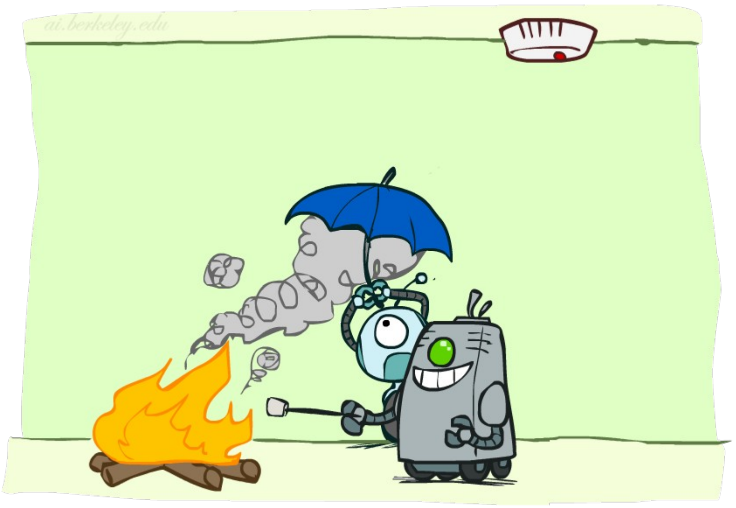
- What about this domain:
  - Traffic
  - Umbrella
  - Raining



# Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm



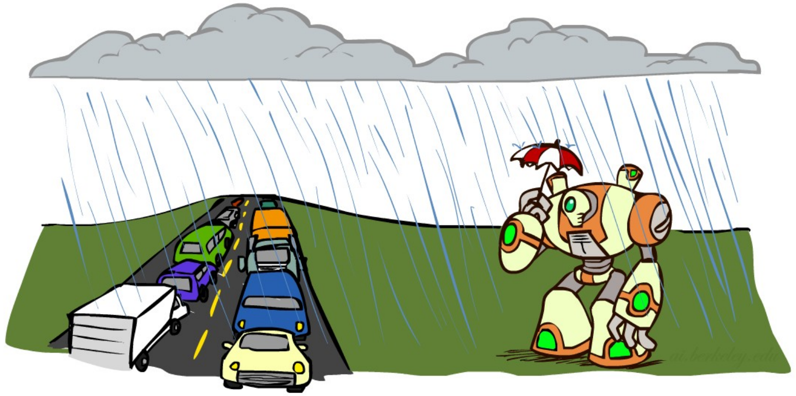


# Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$



- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes'nets / graphical models help us express conditional independence assumptions