

# Probability Theory

## Review

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The contents of this document are taken mainly from the follow sources:

- John Tsitsiklis. Massachusetts Institute of Technology. Introduction to Probability.<sup>1</sup>
- Marek Rutkowski. University of Sydney. Probability Review.<sup>2</sup>
- <https://www.probabilitycourse.com/>

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<sup>1</sup><https://ocw.mit.edu/resources/res-6-012-introduction-to-probability-spring-2018/index.htm>

<sup>2</sup>[http://www.maths.usyd.edu.au/u/UG/SM/MATH3075/r/Slides\\_1\\_Probability.pdf](http://www.maths.usyd.edu.au/u/UG/SM/MATH3075/r/Slides_1_Probability.pdf)

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# Sample Space

- List (set) of all possible states of the world,  $\Omega$ . The states are called **samples** or **elementary events**.
- List (set) of possible **outcomes**,  $\Omega$ .
- E.g., flipping a coin twice, the sample space will be a set of 4 elements:

$$\Omega = \{(heads, heads), (heads, tails), (tails, heads), (tails, tails)\}$$

- List must be:
  - Mutually exclusive
  - Collectively exhaustive
  - At the “right” granularity
- The sample space  $\Omega$  is either **countable** or **uncountable**.

A discrete sample space  $\Omega = (\omega_k)_{k \in I}$ , where the set  $I$  is countable.

## Definition (Probability)

A map  $P : \Omega \mapsto [0, 1]$  is called a **probability** on a discrete sample space  $\Omega$  if the following conditions are satisfied:

- $P(\omega_k) \geq 0$  for all  $k \in I$
- $\sum_{k \in I} P(\omega_k) = 1$

# Probability Measure

- Let  $\mathcal{F} = 2^\Omega$  be the set of all subsets of the sample space  $\Omega$ .
- $\mathcal{F}$  contains the **empty set**  $\emptyset$  and  $\Omega$ .
- Any set  $A \in \mathcal{F}$  is called an **event** (or a **random event**).
- The set  $\mathcal{F}$  is called the **event space**.
- Probability is assigned to **events**.

## Definition (Probability Measure)

A map  $P : \mathcal{F} \mapsto [0, 1]$  is called a **probability measure** on  $(\Omega, \mathcal{F})$  if

- For any sequence  $A_i \in \mathcal{F}, i = 1, 2, \dots$  of events such that  $A_i \cap A_j = \emptyset$  for all  $i \neq j$  we have

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

- $P(\Omega) = 1$

- A probability  $P : \Omega \mapsto [0, 1]$  on a discrete sample space  $\Omega$  uniquely specifies probability of all events  $A_k = \{\omega_k\}$ .
- $P(\{\omega_k\}) = P(\omega_k) = p_k$ .

## Theorem

*Let  $P : \Omega \mapsto [0, 1]$  be a probability on a discrete sample space  $\Omega$ . Then the unique probability measure on  $(\Omega, \mathcal{F})$  generated by  $P$  satisfies for all  $A \in \mathcal{F}$*

$$P(A) = \sum_{\omega_k \in A} P(\omega_k)$$



# Some properties of probability

- If  $A \subset B$ , then  $P(A) \leq P(B)$ .
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) \leq P(A) + P(B)$
- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

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## Definition (Random Variable)

A random variable  $X$  on a sample space  $\Omega$  is a real-valued function on  $\Omega$ :

$$X : \Omega \rightarrow \mathbb{R}$$

- A r.v. associates a value (a number) to every possible outcome.
- It can take discrete or continuous values.

## Notation

Random variable $X$	Numerical value $x$
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- Different r.v.'s can be defined on the same sample space.
- A function of one or several random variables is also a r.v.

# Random Variables

- E.g., flipping a coin twice. The sample space  $\Omega$  will be:

$$\Omega = \{(heads, heads), (heads, tails), (tails, heads), (tails, tails)\}$$

- If the first coin toss returns heads, you win \$10. Else, you lose \$5.
- If the second coin toss returns heads, you win \$15. Else, you lose \$12.
- We can use a random variable to model the above game:

$$X : \Omega \rightarrow \mathbb{R}$$

as

$$X(heads, heads) = 10 + 15 = 25$$

$$X(heads, tails) = 10 - 12 = -2$$

$$X(tails, heads) = -5 + 15 = 10$$

$$X(tails, tails) = -5 - 12 = -17$$

# Probability Mass Function (pmf)

Probability mass function (pmf) of a discrete random variable  $X$ .

- It is the “probability law” or “probability distribution” of  $X$ .
- If we fix some  $x$ , then “ $X = x$ ” is an event.

## Definition

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \text{ s.t. } X(\omega) = x\})$$

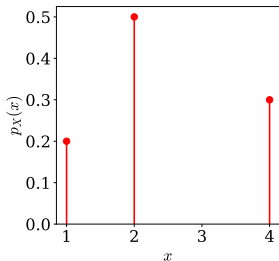
## Properties

- $p_X(x) \geq 0$
- $\sum_x p_X(x) = 1$

# Expectation

- Example: Play a game 1000 times. Random gain at each game is described by

$$X = \begin{cases} 1, & \text{with probability } 2/10 \sim 200 \\ 2, & \text{with probability } 5/10 \sim 500 \\ 4, & \text{with probability } 3/10 \sim 300 \end{cases}$$



- “Average” gain:

$$\frac{1 \cdot 200 + 2 \cdot 500 + 4 \cdot 300}{1000} = 2.4$$

- Definition:  $E[X] = \sum_x x p_X(x)$

# Expectation

- $E[X] = \sum_x xp_X(x)$
- $E(\cdot)$  is called the expectation operator.
- Average in a large number of independence experiments.
- Expectation of a r.v. can be seen as the weighted average.
- It is impossible to know the exact event to happen in the future and thus expectation is useful in making decisions when the probabilities of future outcomes are known.
- Any random variable defined on a finite set  $\Omega$  admits the expectation.
- When the set  $\Omega$  is countable but infinite, we need  $\sum_x |x|p_X(x) < \infty$  so that  $E[X]$  is well-defined.

# Expectation

## Definition

The **expectation** (**expected value** or **mean value**) of a random variable  $X$  on a discrete sample space  $\Omega$  is given by

$$E_P(X) = \mu := \sum_{k \in I} X(\omega_k) P(\omega_k) = \sum_{k \in I} x_k p_k$$

where  $P$  is a probability measure on  $\Omega$ .

## Definition

The **expectation** (**expected value** or **mean value**) of a discrete random variable  $X$  with range  $R_X = \{x_1, x_2, x_3, \dots\}$  (finite or countably infinite) is defined as

$$E(X) = \mu := \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k)$$



## Definition

$$E[X] = \sum_x xp_X(x)$$

- If  $X \geq 0$  then  $E[X] \geq 0$ .  
For all outcomes  $w : X(w) \geq 0$ .
- If  $a \leq X \leq b$  then  $a \leq E[X] \leq b$ .  
For all outcomes  $w : a \leq X(w) \leq b$ .  
$$E[X] = \sum_x xp_X(x) \geq \sum_x ap_X(x) = a \sum_x p_X(x) = a \cdot 1 = a$$
- If  $c$  is a constant,  $E[c] = c$   
$$E[c] = c \cdot p(c) = c$$

## Expected value rule, to compute $E[g(X)]$

- If  $X$  is a r.v. and  $Y = g(X)$ , then  $Y$  itself is a r.v.
- Average over  $y$ :

$$E[Y] = \sum_y y p_Y(y)$$

- Average over  $x$ :

### Theorem (Law of the unconscious statistician (LOTUS))

$$E[Y] = E[g(X)] = \sum_x g(x) p_X(x)$$

$$\begin{aligned} \sum_y \sum_{x:g(x)=y} g(x) p_X(x) &= \sum_y \sum_{x:g(x)=y} y p_X(x) = \sum_y y \sum_{x:g(x)=y} p_X(x) \\ &= \sum_y y p_Y(y) = E[Y] \end{aligned}$$

- $E[X^2] = \sum x^2 p_X(x) \neq (E[X])^2$ . **Caution:**  $E[g(X)] \neq g(E[X])$ .

# Linearity of Expectation

## Theorem

$$E[aX + b] = aE[X] + b$$

Example:  $X = \text{salary}$     $E[X] = \text{average salary}$ .

$Y = \text{new salary} = 2X + 100$     $E[Y] = E[2X + 100] = 2E[X] + 100$ .

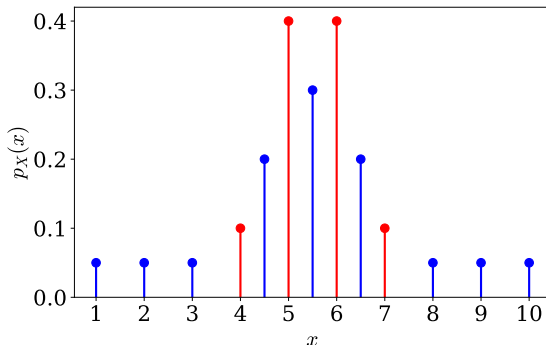
## Proof

Based on the expected value rule:  $g(x) = ax + b$ ;    $Y = g(X)$

$$\begin{aligned} E[Y] &= \sum_x g(x)p_X(x) \\ &= \sum_x (ax + b)p_X(x) \\ &= a \sum_x xp_X(x) + b \sum_x p_X(x) \\ &= aE[X] + b \end{aligned}$$

# Variance

- Variance is a measure of the spread of a random variable about its mean and also a measure of uncertainty.



- R.v.  $X$  with  $\mu = E[X]$ . Average distance from the mean?

$$E[X - \mu] = E[X] - \mu = \mu - \mu = 0$$

- Variance is a measure of the spread of a random variable about its mean and also a measure of uncertainty.
- R.v.  $X$  with  $\mu = E[X]$ . Average distance from the mean?

$$E[X - \mu] = E[X] - \mu = \mu - \mu = 0$$

- Average of the squared distance from the mean.

## Definition (Variance)

The variance of a random variable  $X$  on a discrete sample space  $\Omega$  is defined as

$$\text{Var}(X) = \sigma^2 = E_P[(X - \mu)^2],$$

where  $P$  is a probability measure on  $\Omega$ .

# Variance

- $Var(X) = \sigma^2 = E[(X - \mu)^2]$
- $g(x) = (x - \mu)^2$
- To calculate, use the expected value rule,  $E[g(X)] = \sum_x g(x)p_X(x)$

$$Var(X) = E[g(X)] = \sum_x (x - \mu)^2 p_X(x)$$

- Variance is non-negative:  $Var(X) = \sigma^2 \geq 0$ .
- $Var(X) = 0$  iff  $X$  is deterministic.

## Definition (Standard Deviation)

The **standard deviation** of a random variable  $X$  is defined as

$$SD(X) = \sigma_X = \sqrt{Var(X)}$$

# Properties of the variance

## Theorem

For a random variable  $X$  and real numbers  $a$  and  $b$ ,

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

- Notation  $\mu = E[X]$
- Let  $Y = X + b$ ,  $\gamma = E[Y] = \mu + b$ .

$$\text{Var}(Y) = E[(Y - \gamma)^2] = E[(X + b - (\mu + b))^2] = E[(X - \mu)^2] = \text{Var}(X)$$

- Let  $Y = aX$ ,  $\gamma = E[Y] = a\mu$

$$\begin{aligned}\text{Var}(Y) &= E[(aX - a\mu)^2] = E[a^2(X - \mu)^2] \\ &= a^2 E[(X - \mu)^2] = a^2 \text{Var}(X)\end{aligned}$$

## Computational formula for the variance

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\begin{aligned} Var(X) &= E[(X - \mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$



# Independence and Expectation

- In general:  $E[g(X, Y)] \neq g(E[X], E[Y])$

- Exceptions:

$$E[aX + b] = aE[X] + b \quad E[X + Y + Z] = E[X] + E[Y] + E[Z]$$

## Theorem

If  $X, Y$  are independent:  $E[X, Y] = E[X]E[Y]$ ,

$g(X)$  and  $h(Y)$  are also independent:  $E[g(X), h(Y)] = E[g(X)]E[h(Y)]$

# Independence and Variances

- Always true:  $\text{Var}(aX) = a^2\text{Var}(X)$        $\text{Var}(X + a) = \text{Var}(X)$
- In general:  $\text{Var}(X + Y) \neq \text{Var}(X) + \text{Var}(Y)$
- However

## Theorem

If  $X, Y$  are independent:  $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

## Proof.

Assume  $E[X] = E[Y] = 0$        $E[XY] = E[X]E[Y] = 0$ .

$$\begin{aligned}\text{Var}(X + Y) &= E[(X + Y)^2] = E[X^2 + 2XY + Y^2] \\ &= E[X^2] + 2E[XY] + E[Y^2] = \text{Var}(X) + \text{Var}(Y)\end{aligned}$$



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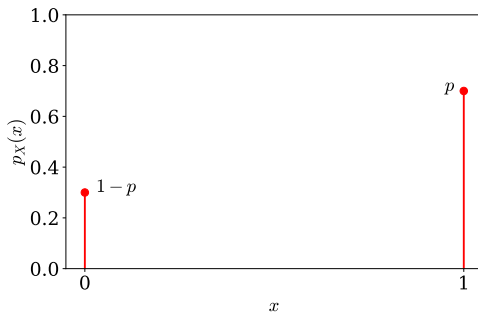
- A Bernoulli r.v.  $X$  takes two possible values, usually 0 and 1, modeling random experiments that have two possible outcomes (e.g., “success” and “failure”).
  - e.g., tossing a coin. The outcome is either Head or Tail.
  - e.g., taking an exam. The result is either Pass or Fail.
  - e.g., classifying images. An image is either Cat or Non-cat.

# Bernoulli Random Variables

## Definition

A random variable  $X$  is a Bernoulli random variable with parameter  $p \in [0, 1]$ , written as  $X \sim \text{Bernoulli}(p)$  if its PMF is given by

$$P_X(x) = \begin{cases} p, & \text{for } x = 1 \\ 1 - p, & \text{for } x = 0. \end{cases}$$



# Bernoulli & Indicator Random Variables

- A Bernoulli r.v.  $X$  with parameter  $p \in [0, 1]$  can also be described as

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

- A Bernoulli r.v. is associated with a certain event  $A$ . If event  $A$  occurs, then  $X = 1$ ; otherwise,  $X = 0$ .
- Bernoulli r.v. is also called the indicator random variable of an event.

## Definition

The indicator random variable of an event  $A$  is defined by

$$I_A = \begin{cases} 1 & \text{if the event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

The indicator r.v. for an event  $A$  has Bernoulli distribution with parameter  $p = P(I_A = 1) = P_{I_A}(1) = P(A)$ . We can write  $I_A \sim \text{Bernoulli}((P(A)))$ .

- The PMF of a Bernoulli random variable:

$$P_X(x) = \begin{cases} p, & \text{for } x = 1 \\ 1 - p, & \text{for } x = 0. \end{cases}$$

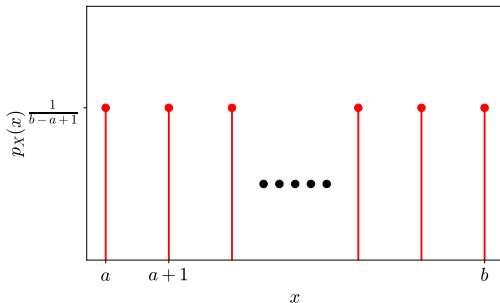
can be rewritten as:

$$P_X(x) = p^x(1 - p)^{1-x}$$

- $E[X] = p$  and  $Var(X) = p(1 - p)$

# Discrete Uniform Random Variables

- Parameters: integer  $a, b$ ;  $a \leq b$
- Experiment: Pick one of  $a, a + 1, \dots, b$  at random; all equally likely.
- Sample space:  $\{a, a + 1, \dots, b\}$
- Random variable  $X$  :  $X(\omega) = \omega$
- $b - a + 1$  possible values,  $P_X(x) = 1/(b - a + 1)$  for each value.
- Model of: complete ignorance.





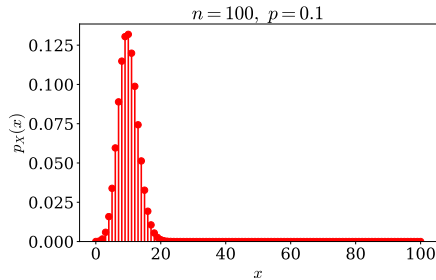
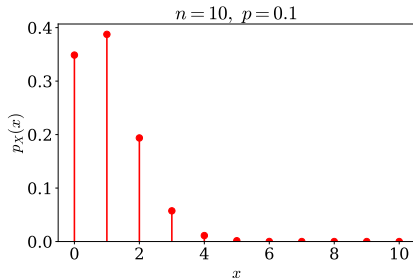
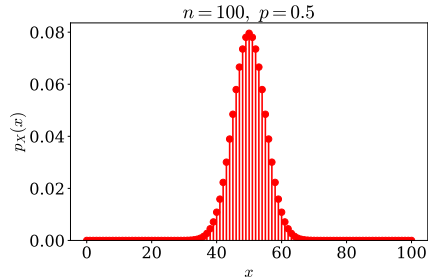
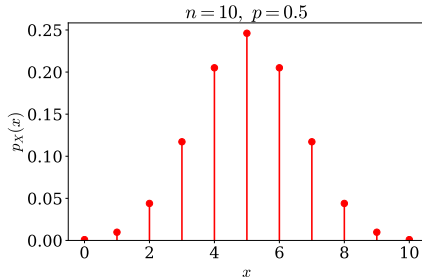
# Binomial Random Variables

- Parameters: Probability  $p \in [0, 1]$ , positive integer  $n$ .
- Experiment: e.g.,  $n$  independent tosses of a coin with  $P(\text{Head}) = p$
- Sample space: Set of sequences of  $H$  and  $T$  of length  $n$
- Random variable  $X$  : number of Heads observed.
- Model of: number of successes in a given number of independent trials.

## Examples

$$\begin{aligned}P_X(2) &= P(X = 2) \\&= P(\text{HHT}) + P(\text{HTH}) + P(\text{THH}) \\&= 3p^2(1 - p) \\&= \binom{3}{2} p^2(1 - p)\end{aligned}$$

# Binomial Random Variables



# Binomial Random Variables

- Let  $\Omega = \{0, 1, 2, \dots, n\}$  be the sample space and let  $X$  be the number of successes in  $n$  independent trials where  $p$  is the probability of success in a single Bernoulli trial.
- The probability measure  $P$  is called the binomial distribution if

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \dots, n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

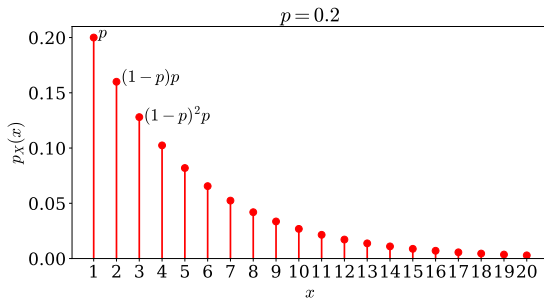
- Then

$$E[X] = np \quad \text{and} \quad \text{Var}(X) = np(1-p)$$

# Geometric Random Variables

- Parameters: Probability  $p \in (0, 1]$ .
- Experiment: infinitely many independent tosses of a coin;  
 $P(\text{Head}) = p$ .
- Sample space: Set of infinite sequences of H and T.
- Random variable  $X$  : number of tosses until the first Head.
- Model of: waiting times, number of trials until a success.

$$P_X(k) = P(X = k) = P(\underbrace{T \dots T}_{k-1} H) = (1-p)^{k-1} p$$



# Geometric Random Variables

- Let  $\Omega = \{1, 2, 3, \dots\}$  be the sample space and  $X$  be the number of independent trials to achieve the first success.
- Let  $p$  stand for the probability of a success in a single trial.
- The probability measure  $P$  is called the geometric distribution if

$$P_X(k) = (1 - p)^{k-1}p \quad \text{for } k = 1, 2, 3, \dots$$

- Then

$$E[X] = \frac{1}{p} \quad \text{and} \quad Var(X) = \frac{1-p}{p^2}$$

- $P(\text{no Heads}) \leq P(\underbrace{T \dots T}_k) = (1 - p)^k$ . As  $k \rightarrow \infty$ ,  $(1 - p)^k \rightarrow 0$

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# Continuous Random Variables

## Definition

A random variable  $X$  on the sample space  $\Omega$  is said to have a continuous distribution if there exists a real-valued function  $f$  such that

$$\begin{aligned} f(x) &\geq 0, \\ \int_{-\infty}^{\infty} f(x) \, dx &= 1, \end{aligned}$$

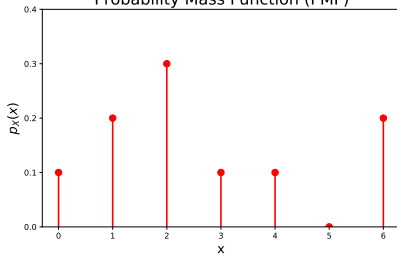
and for all real numbers  $a < b$ :

$$P(a \leq X \leq b) = \int_a^b f(x) \, dx.$$

Then  $f : \mathbb{R} \mapsto \mathbb{R}_+$  is called the **probability density function (PDF)** of a **continuous random variable**  $X$ .

# Probability Density Function (PDF)

Probability Mass Function (PMF)

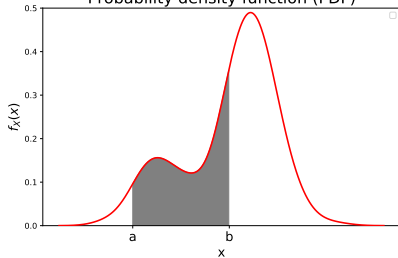


$$P(a \leq X \leq b) = \sum_{x: a \leq x \leq b} p_X(x)$$

$$p_X(x) \geq 0$$

$$\sum_x p_X(x) = 1$$

Probability density function (PDF)



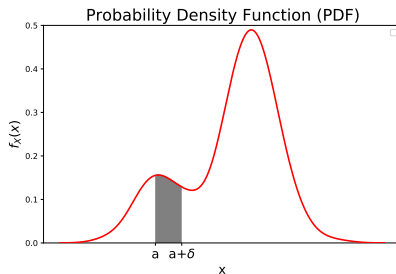
$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

$$f_X(x) \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1$$



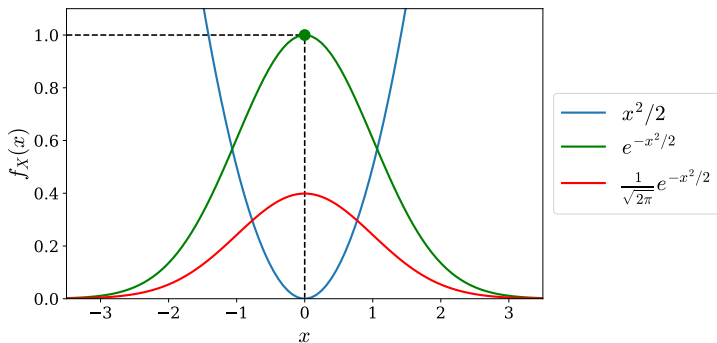
# Probability Density Function (PDF)



$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

- $\delta > 0$ , small
- $P(a \leq X \leq a + \delta) \approx f_X(a) \cdot \delta$
- $P(X = a) = 0$
- Just like, a single point has zero length.
- But, a set of lots of points has a positive length.

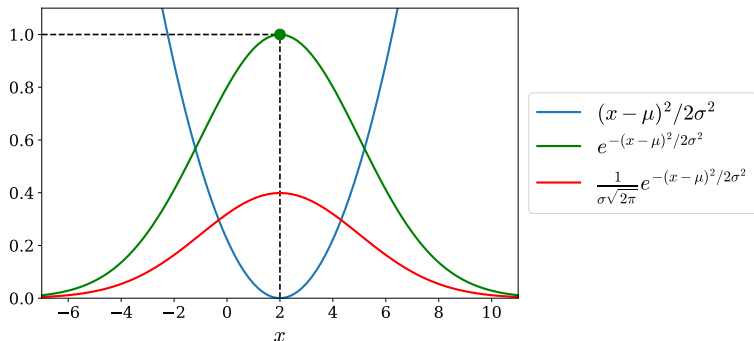
# Standard Normal (Gaussian) Random Variable $N(0, 1)$



$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

# General Normal (Gaussian) Random Variable $N(\mu, \sigma^2)$

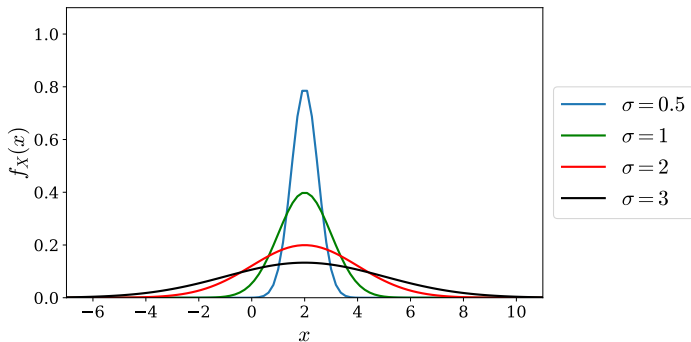


$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\mathbb{E}[X] = \mu$$

$$\text{Var}(X) = \sigma^2$$

# General Normal (Gaussian) Random Variable $N(\mu, \sigma^2)$



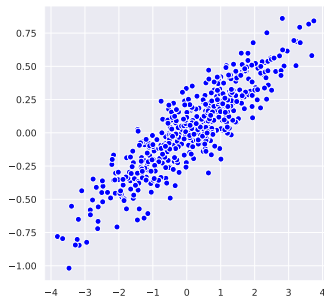
- Smaller  $\sigma$ , narrower PDF.
- Let  $Y = aX + b$        $N \sim N(\mu, \sigma^2)$
- Then,  $E[Y] = aE[X] + b$        $\text{Var}(Y) = a^2\sigma^2$  (always true)
- But also,  $Y \sim N(a\mu + b, a^2\sigma^2)$

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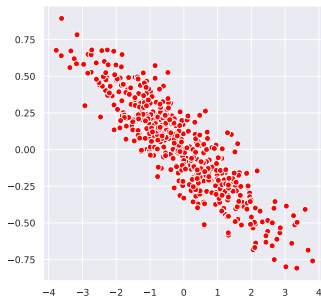
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# Covariance

- Consider zero-mean, discrete random variables  $X$  and  $Y$
- If they are independent,  $E[XY] = E[X]E[Y]=0$ .
- But if their joint PDF is as follows.



$$E[XY] > 0$$



$$E[XY] < 0$$

- Definition for general case:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

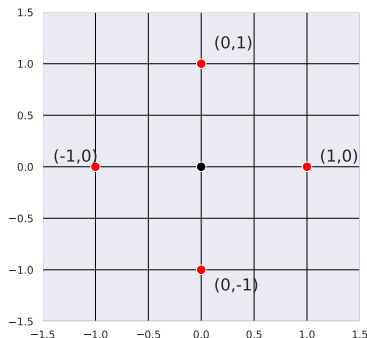
- The deviation of  $X$  from its mean value  $X - E[X]$ .
- The deviation of  $Y$  from its mean value  $Y - E[Y]$ .
- Whether these two deviations tend to have the same sign or not.
- In general, the covariance tells us whether two random variables tend to move together, both being high or both being low, on average.
- Whether they move in same direction or not.
- If the covariance is positive, it indicates that whenever the quantity  $X - E[X]$  is positive ( $X$  is above its mean), the deviation of  $Y$  from its mean will also tend to be positive  $Y - E[Y]$ .

# Covariance

- If  $X$  and  $Y$  are independent, then

$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[(X - E[X])]E[(Y - E[Y])] \\ &= 0\end{aligned}$$

- However, the inverse is not true.



- $E[X] = 0, E[Y] = 0$
- $XY = 0$
- $\text{Cov}(X, Y) = 0$
- $X = 1 \implies Y = 0$ . Knowing the value of  $X$  tells a lot about  $Y$ . Therefore,  $X$  and  $Y$  are dependent.



# Covariance properties



$$\begin{aligned}\text{Cov}(X, X) &= E[(X - E[X])^2] = \text{Var}(X) \\ &= E[X^2] - (E[X])^2\end{aligned}$$



$$\begin{aligned}\text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY - XE[Y] - E[X]Y + E[X]E[Y]] \\ &= E[XY] - E[XE[Y]] - E[E[X]Y] + E[E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

- Assume 0 means

$$\text{Cov}(aX + b, Y) = E[(aX + b)Y] = aE[XY] + bE[Y] = a\text{Cov}(X, Y)$$

# Joint distribution for multiple random variables

- The **covariance** between two random variables  $X$  and  $Y$  measures the degree to which  $X$  and  $Y$  are related

$$\text{Cov}[X, Y] \triangleq E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

- If  $\mathbf{x}$  is a  $D$ -dimensional random vector, its **covariance matrix** is defined to be the following symmetric, positive semi definite matrix

$$\begin{aligned}\text{Cov}[\mathbf{x}] &\triangleq E[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^T] \triangleq \mathbf{\Sigma} \\ &= \begin{bmatrix} V[X_1] & \text{Cov}[X_1, X_2] & \dots & \text{Cov}[X_1, X_D] \\ \text{Cov}[X_2, X_1] & V[X_2] & \dots & \text{Cov}[X_2, X_D] \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}[X_D, X_1] & \text{Cov}[X_D, X_2] & \dots & V[X_D] \end{bmatrix}\end{aligned}$$

from which we get

$$E[\mathbf{x}\mathbf{x}^T] = \mathbf{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T$$

# Multivariate Gaussian (normal) distribution

- The **multivariate normal (MVN)** density is defined

$$\mathcal{N}(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \triangleq \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp \left[ -\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}) \right]$$

where  $\boldsymbol{\mu} = E[\mathbf{y}] \in \mathbb{R}^D$ ,  $\boldsymbol{\Sigma} = \text{Cov}[\mathbf{y}]$  is the  $D \times D$  **covariance matrix**

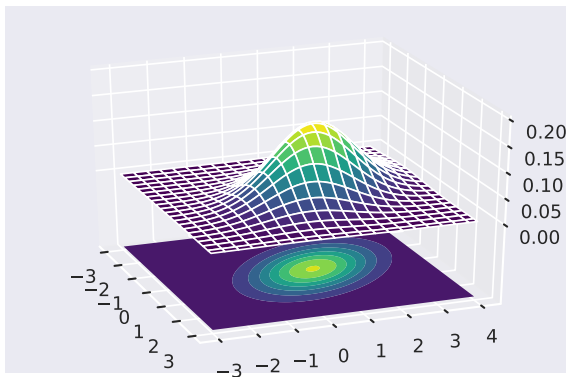
$$\begin{aligned} \text{Cov}[\mathbf{y}] &\triangleq E[(\mathbf{y} - E[\mathbf{y}])(\mathbf{y} - E[\mathbf{y}])^T] \triangleq \boldsymbol{\Sigma} \\ &= \begin{bmatrix} V[Y_1] & \text{Cov}[Y_1, Y_2] & \dots & \text{Cov}[Y_1, Y_D] \\ \text{Cov}[Y_2, Y_1] & V[Y_2] & \dots & \text{Cov}[Y_2, Y_D] \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}[Y_D, Y_1] & \text{Cov}[Y_D, Y_2] & \dots & V[Y_D] \end{bmatrix} \end{aligned}$$

- A **full covariance matrix** had  $D(D+1)/2$  parameters.

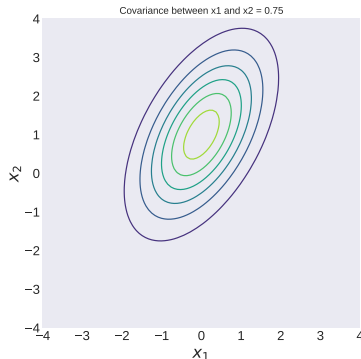
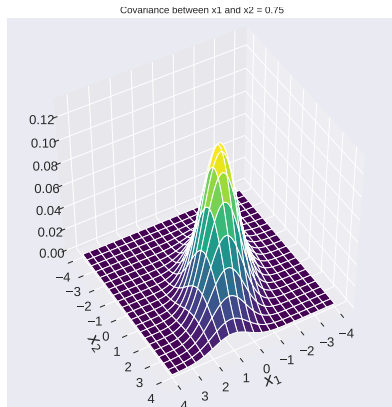
# Multivariate Gaussian (normal) distribution

$$p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \triangleq \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp \left[ -\frac{1}{2}(\mathbf{y} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu}) \right]$$

where  $\boldsymbol{\mu} = E[\mathbf{y}] \in \mathbb{R}^D$ ,  $\boldsymbol{\Sigma} = \text{Cov}[\mathbf{y}]$  is the  $D \times D$  **covariance matrix**.



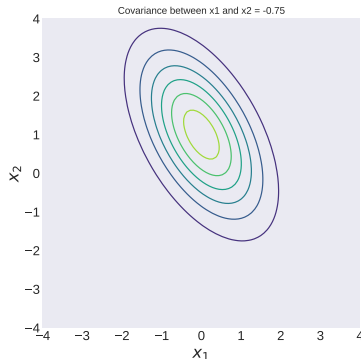
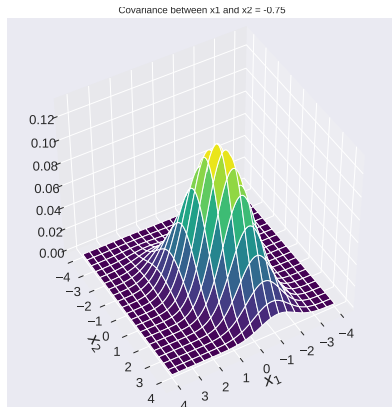
# Multivariate Gaussian - Full Covariance Matrix



$$\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 2 \end{bmatrix}$$

A **full covariance matrix** had  $D(D + 1)/2$  parameters.

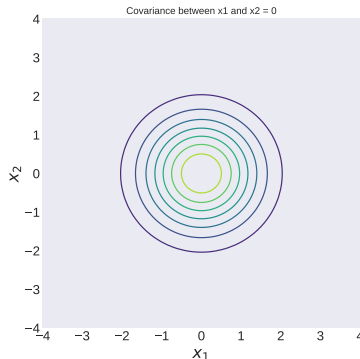
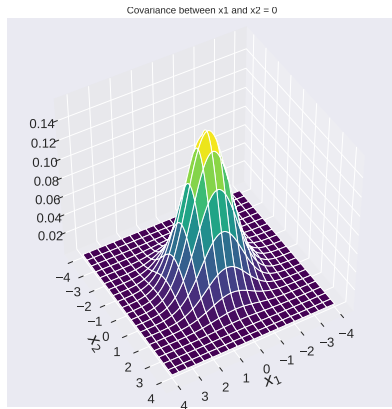
# Multivariate Gaussian - Full Covariance Matrix



$$\mu = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & -0.75 \\ -0.75 & 2 \end{bmatrix}$$

A **full covariance matrix** had  $D(D + 1)/2$  parameters.

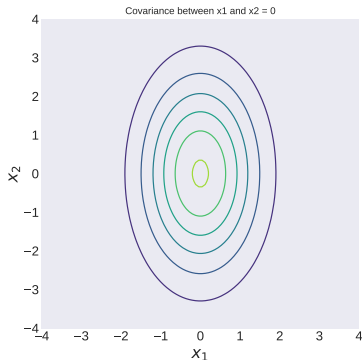
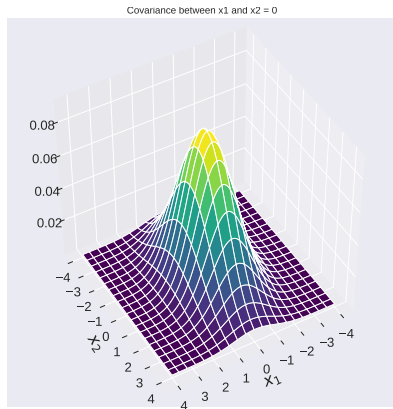
# Standard Multivariate Gaussian - Spherical Cov. Matrix



$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A spherical (isotropic) covariance matrix  $\Sigma = \sigma^2 \mathbf{I}_D$  has one free parameter  $\sigma^2$ .

# Uncorrelated Multivariate Gaussian - Diagonal Cov. Matrix



$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

A diagonal covariance matrix has  $D$  parameters.



# Bivariate Gaussian (normal) distribution

- In 2D, the MVN is known as the **bivariate Gaussian** distribution. Its PDF can be represented as  $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , where  $\mathbf{y} \in \mathbb{R}^2$ ,  $\boldsymbol{\mu} \in \mathbb{R}^2$  and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where  $\rho$  is the **correlation coefficient**

$$\text{corr}[Y_1, Y_2] \triangleq \frac{\text{Cov}[Y_1, Y_2]}{\sqrt{V[Y_1]V[Y_2]}} = \frac{\sigma_{12}^2}{\sigma_1\sigma_2}$$

# Correlation coefficient

- Dimensionless version of covariance:

$$\rho(X, Y) = \text{corr}[X, Y] \triangleq E \left[ \frac{(X - E[X])}{\sigma_X} \frac{(Y - E[Y])}{\sigma_Y} \right] = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

- $-1 \leq \rho \leq 1$ : measure of the degree of “association” between  $X$  and  $Y$ .
- Independent  $\implies \rho = 0$ : “uncorrelated” (the converse is not true).
- $\rho(X, X) = \frac{\text{Var}(X)}{\sigma_X^2} = 1$ .
- $|\rho| = 1 \iff (X - E[X]) = c(Y - E[Y])$ : linearly related, deterministic relation between the two rv.
- We have  $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y)$ . Therefore,

$$\rho(aX + b, Y) = \frac{a\text{Cov}(X, Y)}{|a|\sigma_X\sigma_Y} = \text{sign}(a) \cdot \rho(X, Y)$$

$\implies$  Changing the unit of  $X$  does not change correlation coefficient between  $X$  and  $Y$ .

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## Example

- A bag contains 3 balls, each ball is either red or blue.
- The number of blue balls  $\theta$  can be 0, 1, 2, 3.
- Choose 4 balls randomly **with replacement**.
- Random variables  $X_1, X_2, X_3, X_4$  are defined as

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th chosen ball is blue} \\ 0, & \text{if the } i\text{-th chosen ball is red} \end{cases}$$

- After doing the experiment, the following values for  $X_i$ 's are observed:  $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1$ .
- Note that  $X_i$ 's are i.i.d. (independent and identically distributed) and  $X_i \sim \text{Bernoulli}(\frac{\theta}{3})$ . For which value of  $\theta$  is the probability of the observed sample is the largest?

## Example

$$P_{X_i}(x) = \begin{cases} \frac{\theta}{3}, & \text{for } x = 1 \\ 1 - \frac{\theta}{3}, & \text{for } x = 0 \end{cases}$$

$X_i$ 's are independent, the joint PMF of  $X_1, X_2, X_3, X_4$  can be written

$$P_{X_1 X_2 X_3 X_4}(x_1, x_2, x_3, x_4) = P_{X_1}(x_1)P_{X_2}(x_2)P_{X_3}(x_3)P_{X_4}(x_4)$$

$$P_{X_1 X_2 X_3 X_4}(1, 0, 1, 1) = \frac{\theta}{3} \cdot \left(1 - \frac{\theta}{3}\right) \cdot \frac{\theta}{3} \cdot \frac{\theta}{3} = \left(\frac{\theta}{3}\right)^3 \left(1 - \frac{\theta}{3}\right)$$

$\theta$	$P_{X_1 X_2 X_3 X_4}(1, 0, 1, 1; \theta)$
0	0
1	0.0247
2	0.0988
3	0

The observed data is most likely to occur for  $\theta = 2$ .

We may choose  $\hat{\theta} = 2$  as our estimate of  $\theta$ .

# Maximum Likelihood Estimation (MLE)

## Definition

Let  $X_1, X_2, \dots, X_n$  be a random sample from a distribution with a parameter  $\theta$ .

Given that we have observed  $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$ , a maximum likelihood estimate of  $\theta$ , denoted as  $\hat{\theta}_{ML}$ , is a value of  $\theta$  that maximizes the likelihood function

$$L(x_1, x_2, \dots, x_n; \theta)$$

A maximum likelihood estimator (MLE) of the parameter  $\theta$ , denoted as  $\hat{\Theta}_{ML}$ , is a random variable  $\hat{\Theta}_{ML} = \hat{\Theta}(X_1, X_2, \dots, X_n)$  whose values  $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$  is given by  $\hat{\theta}_{ML}$ .