Markov Decision Processes

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Outline

Table of Contents

Markov Property

"The future is independent of the past given the present"

Definition

A state S_t is *Markov* if and only of

$$\mathbb{P}[S_{t+1}|S_t] = \mathbb{P}[S_{t+1}|S_1, \cdots, S_t]$$

- The state captures all relevant information from the history
- Once the state is known, the history may be thrown away
- i.e. The state is a sufficient statistic of the future

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State Transition Matrix

For a Markov state s and successor state s', the *state transition probability* is defined by

$$\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s]$$

State transition matrix $\mathcal{P}$ defines transition probabilities from all states s to all successor states s',

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \vdots \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix}$$

where each row of the matrix sums to 1.

Markov Process

A Markov process is a memoryless random process, i.e. a sequence of random states $S_1, S_2, \cdots$ with the Markov property.

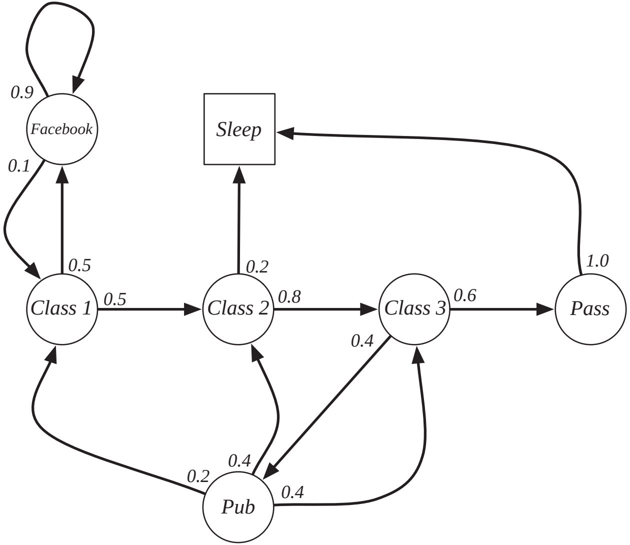
Definition

A *Markov Process (or Markov Chain)* is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

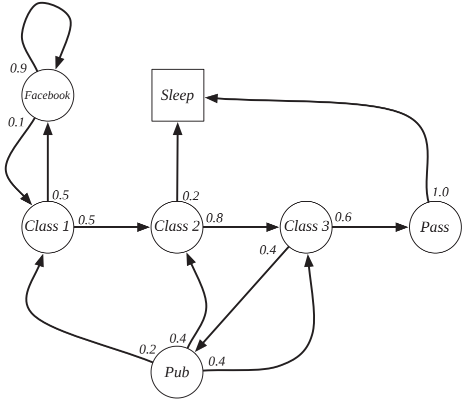
- $\mathcal{S}$ is a (finite) set of states
- $\mathcal{P}$ is a state transition probability matrix,

$$\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s]$$

Example: Student Markov Chain



Example: Student Markov Chain Episodes



Sample episodes for Student Markov Chain starting from $S_1 = C_1$

$$S_1, S_2, \cdots, S_T$$

- C1 C2 C3 Pass Sleep
- C1 FB FB C1 C2 Sleep
- C1 C2 C3 Pub C2 C3 Pass Sleep
- C1 FB FB C1 C2 C3 Pub C1 FB FB FB C1 C2 C3 Pub C2 Sleep

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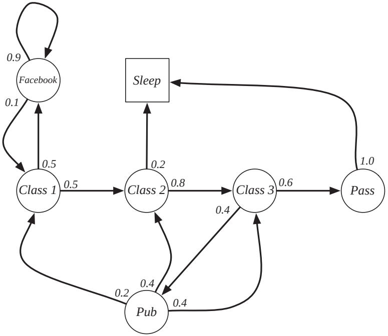
Example: Student Markov Chain Transition Matrix



$$
\mathcal{P} = \begin{array}{cc}
 & \begin{array}{ccccccc} C1 & C2 & C3 & Pass & Pub & FB & Sleep \end{array} \\
\begin{array}{c} C1 \\ C2 \\ C3 \\ Pass \\ Pub \\ FB \\ Sleep \end{array} &
\begin{bmatrix}
 & 0.5 & & & & 0.5 & \\
 & & 0.8 & & & & 0.2 \\
 & & & 0.6 & 0.4 & & \\
 & & & & & & 1.0 \\
0.2 & 0.4 & 0.4 & & & & \\
0.1 & & & & & 0.9 & \\
 & & & & & & 1
\end{bmatrix}
\end{array}
$$

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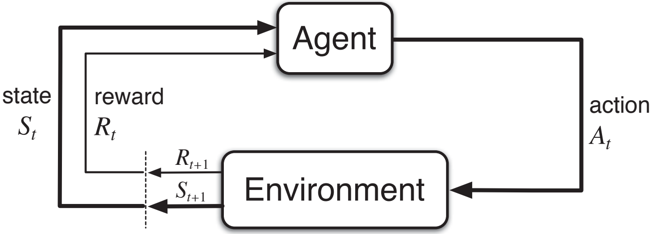
Table of Contents

Goal and Rewards



- Informally, the agent's goal is to *maximize the total amount of reward it received*.
- This means maximizing not immediate reward, but cumulative reward in the long run.
- The use of a reward signal to formalize the idea of a goal is one of the most distinctive features of reinforcement learning.

Markov Reward Process

A Markov reward process is a Markov chain with values.

Definition

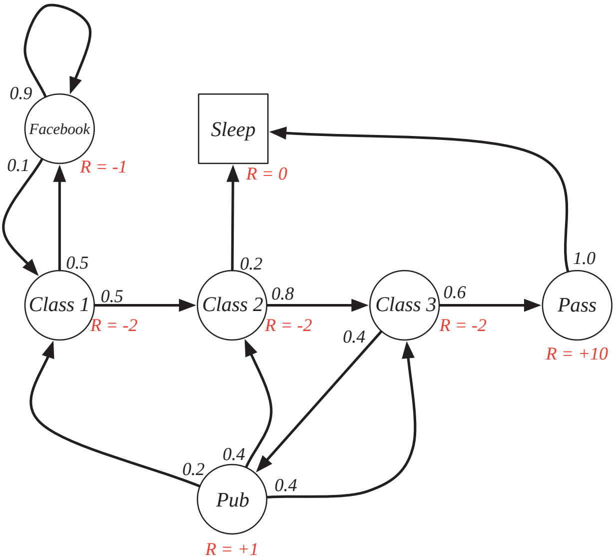
A *Markov Reward Process* is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is a state transition probability matrix,

$$\mathcal{P}_{ss'} = \mathbb{P}[S_{t+1} = s' | S_t = s]$$

- $\mathcal{R}$ is a reward function, $\mathcal{R}_s = \mathbb{E}[R_{t+1} | S_t = s]$
- γ is a discount factor, $\gamma \in [0, 1]$

Example: Student MRP



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Return

Definition

The *return* G_t is the total discounted reward from time-step t

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

- The discount $\gamma \in [0, 1]$ is the present value of future rewards
- The value of receiving reward R after $k+1$ time-steps is $\gamma^k R$
- This values immediate reward above delayed reward.
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

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Why discount?

Most Markov reward and decision processes are discounted. Why?

- Mathematically convenient to discount rewards
- Avoids infinite returns in cyclic Markov processes
    - If the task is continuous, the final time step would be $T = \infty$
- Uncertainty about the future may not be fully represented
- If the reward is financial, immediate rewards may earn more interest than delayed rewards
- Animal/human behaviour shows preference for immediate reward
- It is sometimes possible to use *undiscounted* Markov reward processes (i.e. $\gamma = 1$), e.g. if all sequences terminate, which we call *episodic task*.

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Example: Student MRP Returns

Sample returns for Student MRP: Starting from $S_1 = C1$ with $\gamma = \frac{1}{2}$

$$G_1 = R_2 + \gamma R_3 + \cdots + \gamma^{T-2} R_T$$

C1 C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 10 * \frac{1}{8}$	$=$	-2.25
C1 FB FB C1 C2 Sleep	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16}$	$=$	-3.125
C1 C2 C3 Pub C2 C3 Pass Sleep	$v_1 = -2 - 2 * \frac{1}{2} - 2 * \frac{1}{4} + 1 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.41
C1 FB FB C1 C2 C3 Pub C1 ...	$v_1 = -2 - 1 * \frac{1}{2} - 1 * \frac{1}{4} - 2 * \frac{1}{8} - 2 * \frac{1}{16} \ldots$	$=$	-3.20
FB FB FB C1 C2 C3 Pub C2 Sleep			

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Value Function

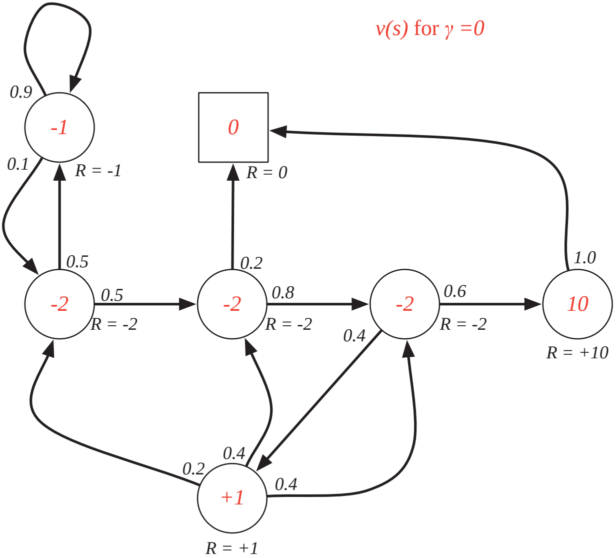
The value function $v(s)$ gives the long-term value of state s, i.e. estimate *how good* it is for the agent to be in a given state

Definition

The *state value function* $v(s)$ of an MRP is the expected return starting from state s

$$v(s) = \mathbb{E}[G_t | S_t = s]$$

Example: State-Value Function for Student MRP (1)



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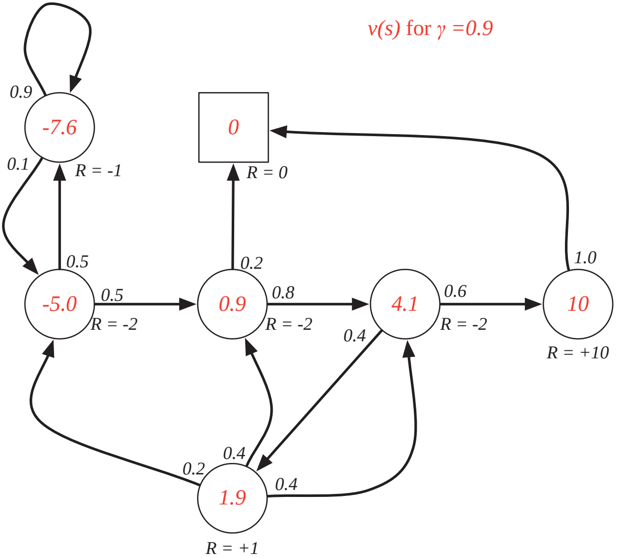
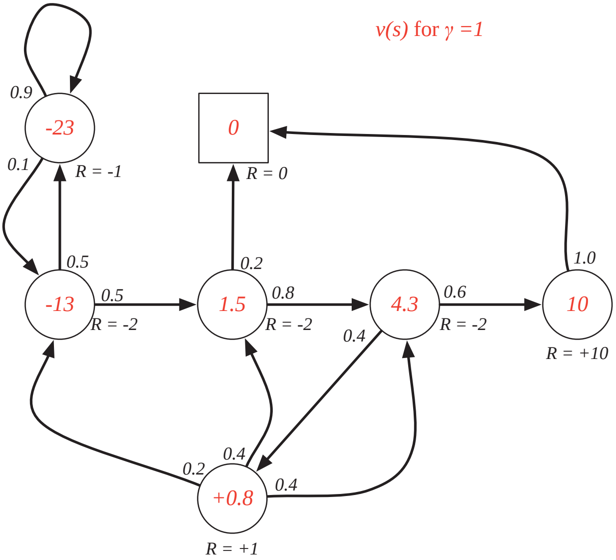
Example: State-Value Function for Student MRP (2)



Example: State-Value Function for Student MRP (3)



Bellman Equation for MRPs

The value function can be decomposed into two parts:

- immediate reward R_{t+1}
- discounted value of successor state $\gamma v(S_{t+1})$

$$
\begin{aligned}
v(s) &= \mathbb{E}[G_t | S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots | S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma (R_{t+2} + \gamma R_{t+3} + \cdots) | S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma G_{t+1} | S_t = s] \\
&= \mathbb{E}[R_{t+1} + \gamma v(S_{t+1}) | S_t = s]
\end{aligned}
$$

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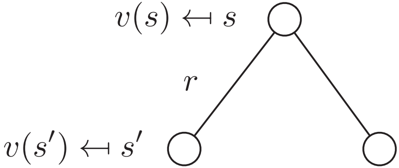
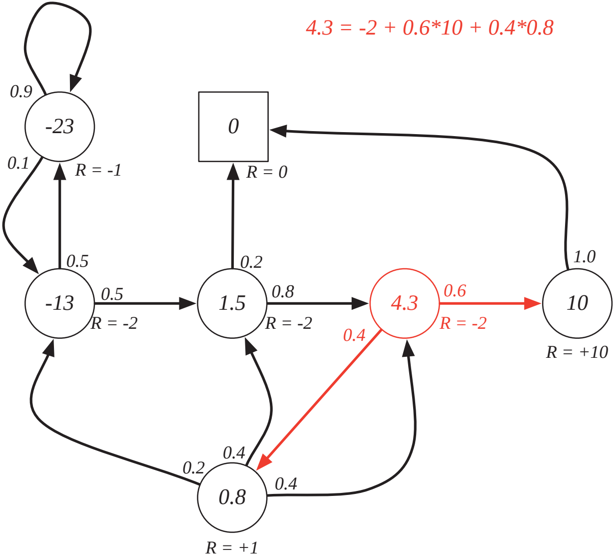
Bellman Equation for MRPs (2)

$$v(s) = \mathbb{E}[R_{t+1} + \gamma v(S_{t+1}) | S_t = s]$$



$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Example: Bellman Equation for Student MRP



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Bellman Equation in Matrix Form

The Bellman equation can be expressed concisely using matrices,

$$v = \mathcal{R} + \gamma \mathcal{P} v$$

where v is a column vector with one entry per state

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \vdots \\ \mathcal{P}_{11} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix}$$

Solving the Bellman Equation

- The Bellman equation is a linear equation
- It can be solved directly:

$$\begin{aligned} v &= \mathcal{R} + \gamma \mathcal{P} v \\ (I - \gamma \mathcal{P}) v &= \mathcal{R} \\ v &= (I - \gamma \mathcal{P})^{-1} \mathcal{R} \end{aligned}$$

- Computational complexity is $O(n^3)$ for n states
- Direct solution only possible for small MRPs
- There are many iterative methods for large MRPs, e.g.
    - Dynamic programming
    - Monte-Carlo evaluation
    - Temporal-Difference learning

Table of Contents

Markov Decison Process

A Markov decision process (MDP) is a Markov reward process with decisions. It is an *environment* in which all states are Markov.

Definition

A *Markov Decision Process* is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is a state transition probability matrix,

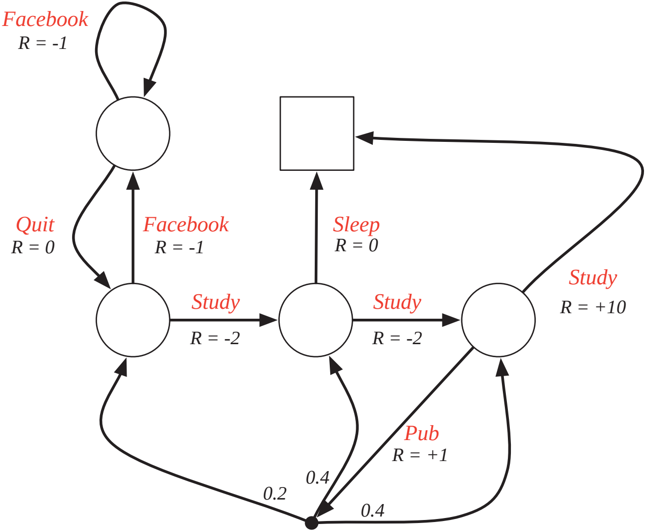
$$\mathcal{P}_{ss'}^{a} = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$$

- $\mathcal{R}$ is a reward function,

$$\mathcal{R}_{s}^{a} = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$$

- γ is a discount factor $\gamma \in [0, 1]$

Example: Student MDP



Policies (1)

Definition

A *policy* π is a distribution over actions given states,

$$\pi(a|s) = \mathbb{P}[A_t = a|S_t = s]$$

- A policy fully defined the behaviour of an agent
- MDP policies depend on the current state (not the history)
- i.e. Policies are stationary (time-independent), $A_t \sim \pi(\cdot|S_t), \forall t > 0$

Policies (2)

- Given an MDP $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and a policy π
- The state sequence $S_1, S_2, \cdots$ is a Markov process $\langle \mathcal{S}, \mathcal{P}^\pi \rangle$
- The state and reward sequence $S_1, R_2, S_2, \cdots$ is a Markov reward process $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
- where

$$\mathcal{P}^\pi_{ss'} = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{P}^a_{ss'}$$

$$\mathcal{R}^\pi_s = \sum_{a \in \mathcal{A}} \pi(a|s) \mathcal{R}^a_s$$

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Value Function Following Policy

Definition

The state-value function $v_\pi(s)$ of an MDP is the expected return starting from state s, and then *following policy* π

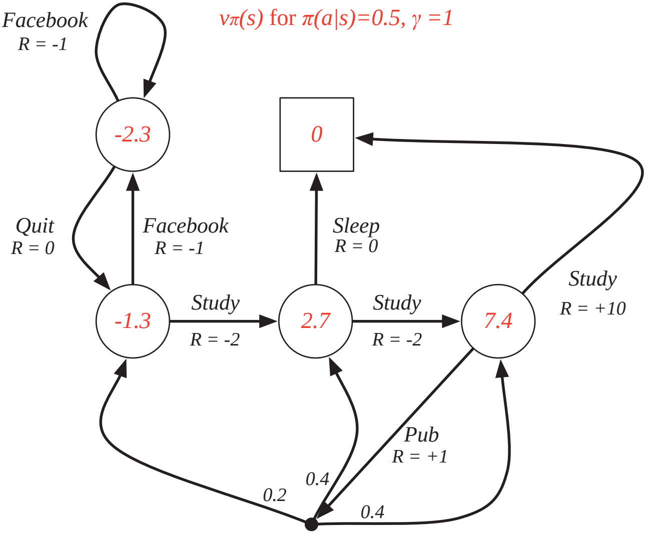
$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s]$$

Definition

The action-value function $q_\pi(s, a)$ is the expected return starting from state s, taking action a, and then *following policy* π

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$$

Example: State-Value Function for Student MDP



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Bellman Expectation Equation

The state-value function can again be decomposed into immediate reward plus discounted value of successor state,

$$v_\pi(s) = \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$$

The action-value function can similarly be decomposed,

$$q_\pi(s, a) = \mathbb{E}_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) | S_t = s, A_t = a]$$

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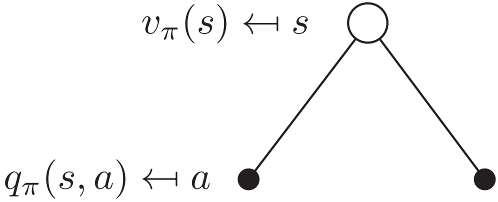
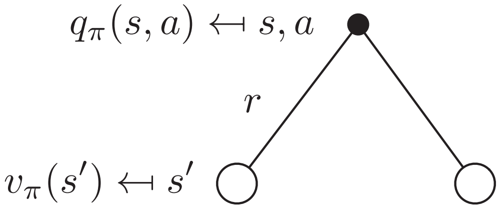
Bellman Expectation Equation for V^{π}



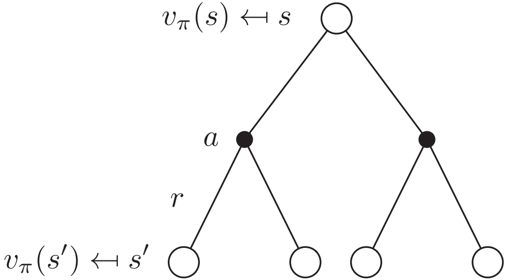
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a)$$

Bellman Expectation Equation for Q^π



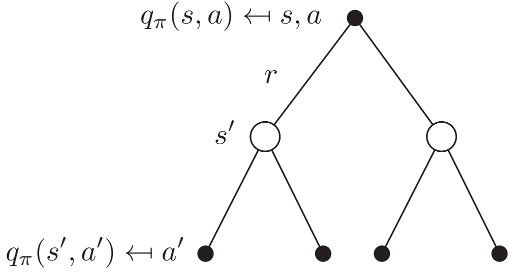
$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s')$$

Bellman Expectation Equation for v_π (2)



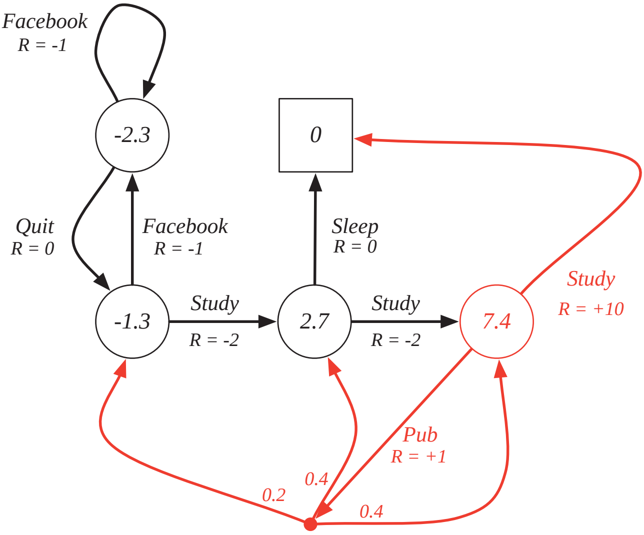
$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right)$$

Bellman Expectation Equation for q_π (2)



$$q_\pi(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \sum_{a' \in \mathcal{A}} \pi(a'|s') q_\pi(s',a')$$

Example: Bellman Expectation Equation in Student MDP



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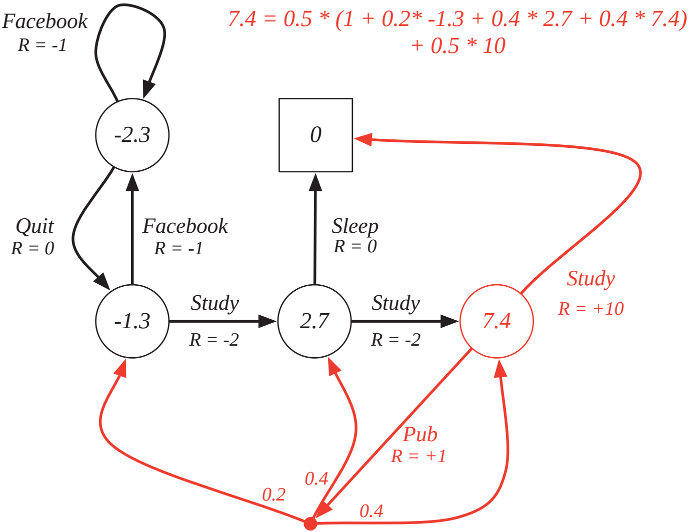
Example: Bellman Expectation Equation in Student MDP



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Bellman Expectation Equation (Matrix Form)

The Bellman Expectation equation can be expressed concisely using the induced MRP,

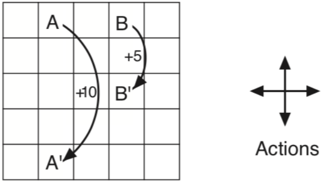
$$v_\pi = \mathcal{R}^\pi + \gamma \mathcal{P}^\pi v_\pi$$

with direct solution

$$v_\pi = (I - \gamma \mathcal{P}^\pi)^{-1} \mathcal{R}^\pi$$

Example: Bellman Expectation Equation in Gridworld

Example 3.5: Gridworld Figure 3.2 (left) shows a rectangular gridworld representation of a simple finite MDP. The cells of the grid correspond to the states of the environment. At each cell, four actions are possible: `north`, `south`, `east`, and `west`, which deterministically cause the agent to move one cell in the respective direction on the grid. Actions that would take the agent off the grid leave its location unchanged, but also result in a reward of -1. Other actions result in a reward of 0, except those that move the agent out of the special states A and B. From state A, all four actions yield a reward of $+10$ and take the agent to A′. From state B, all actions yield a reward of $+5$ and take the agent to B′.



3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

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Table of Contents

Optimal Value Function

Definition

The *optimal state-value function* $v_*(s)$ is the maximum value function over all policies

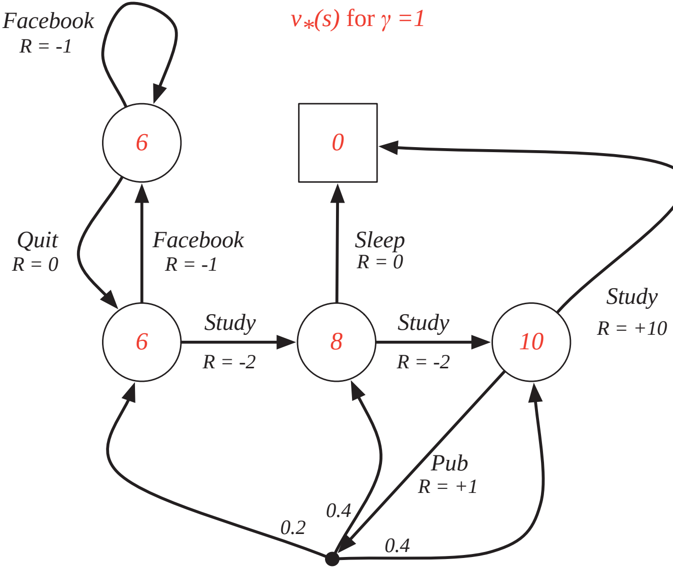
$$v_*(s) = \max_\pi v_\pi(s)$$

The *optimal action-value function* $q_*(s, a)$ is the maximum action-value function over all policies

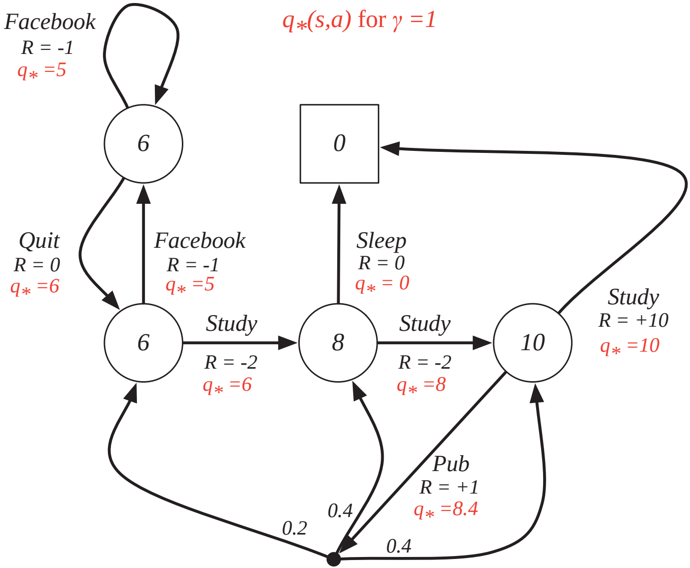
$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

- The optimal value function specifies the best possible performance in the MDP.
- An MDP is "solved" when we know the optimal value.

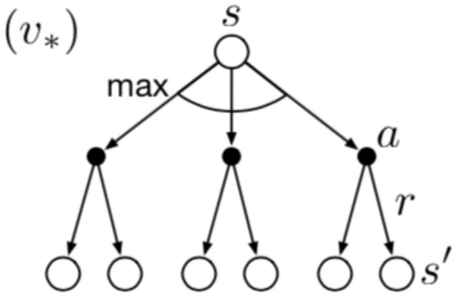
Example: Optimal Value Function for Student MDP



Example: Optimal Action-Value Function for Student MDP

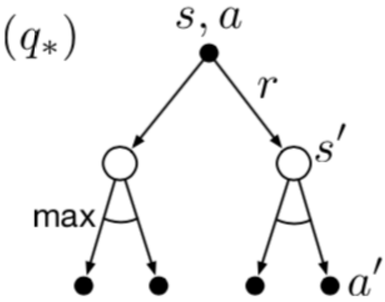


Bellman Optimality Equation for V^*



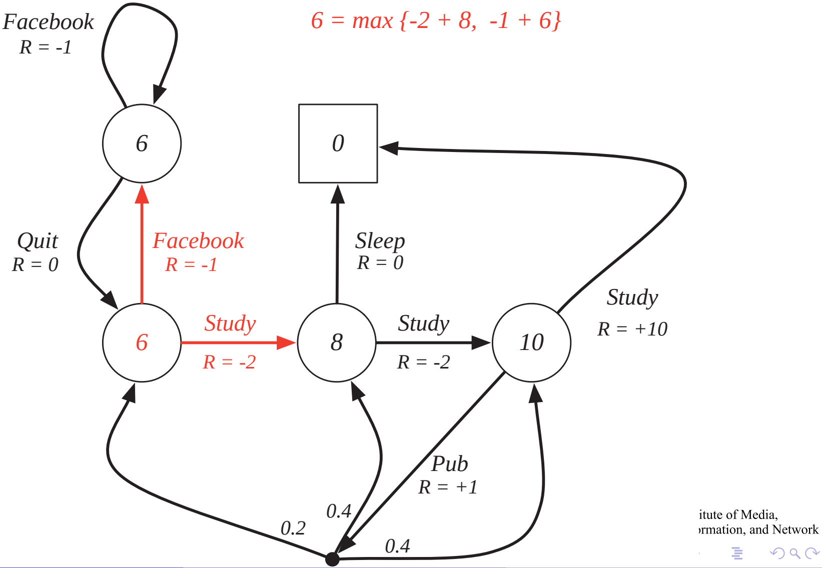
$$v_*(s) = \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

Bellman Optimality Equation for Q^*



$$q_*(s,a) = \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \max_{a'} q_*(s', a')$$

Example: Bellman Optimal Equation in Student MDP



Optimal Policy

Define a partial ordering over policies

$$\pi \geq \pi' \text{ if } v_\pi(s) \geq v_{\pi'}(s), \forall s$$

Theorem

For any Markov Decision Process

- *There exists an optimal policy π_* that is better than or equal to all other policies, $\pi_* \geq \pi, \forall \pi$*
- *All optimal policies achieve the optimal value function, $v_{\pi_*}(s) = v_*(s)$*
- *All optimal policies achieve the optimal action-value function, $q_{\pi_*}(s, a) = q_*(s, a)$*

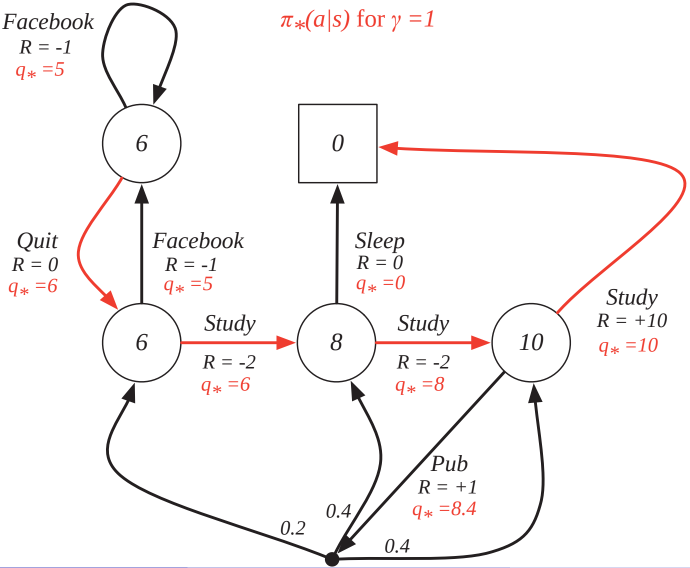
Finding an Optimal Policy

An optimal policy can be found by maximizing over $q_*(s, a)$,

$$\pi_*(a|s) = \begin{cases} 1 & \text{if } a = \underset{a \in \mathcal{A}}{\arg\max}\, q_*(s, a) \\ 0 & \text{otherwise} \end{cases}$$

- There is always a deterministic optimal policy for any MDP
- If we know $q_*(s, a)$, we immediately have the optimal policy

Example: Optimal Policy for Student MDP



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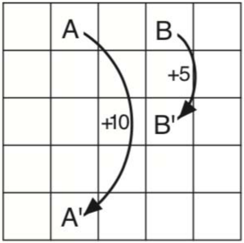
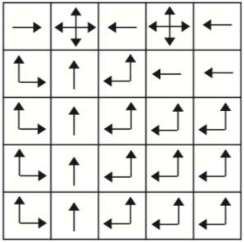
Example: Optimal Policy for Gridworld



Gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

Solving the Bellman Optimality Equation

- Bellman Optimality Equation is non-linear
- No closed form solution (in general)
- Many iterative solution methods
  - Value Iteration
  - Policy Iteration
  - Q-learning
  - Sarsa