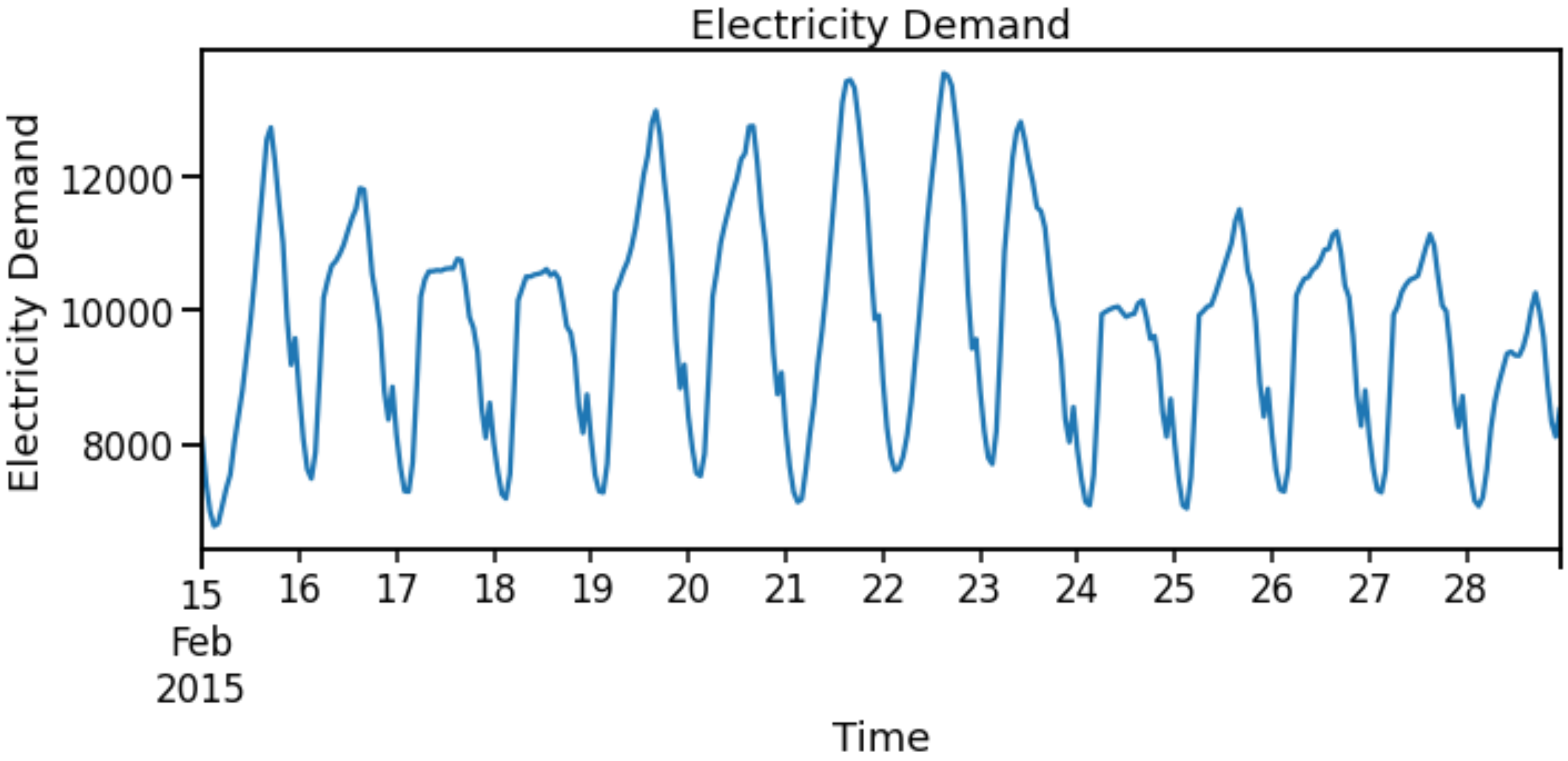
Fourier Features (part 2)

Seasonality features

How do we create Fourier features?

1. Pick a seasonal period that we want to model.

Example: Hourly data, daily seasonality → $\mathrm{T} = 24, \mathrm{f} = 1/24$



How do we create Fourier features?

2. Compute time, t. This is typically the time since the start of the time series.

Time index	Electricity Demand	t
2000-01-01 03:00:00	8000	0
2000-01-01 04:00:00	8123	1
2000-01-01 05:00:00	8340	2
2000-01-01 06:00:00	8561	3

How do we create Fourier features?

3. Pick the number of Fourier terms, N, and compute the Fourier features:

For n in 1 to N, calculate $\sin(2\pi * nft)$ & $\cos(2\pi * nft)$; $f = \frac{1}{24}$

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Time index	Electricity Demand	t	$sin\left(\frac{2\pi t}{24}\right)$	$cos\left(\frac{2\pi t}{24}\right)$
2000-01-01 03:00:00	8000	0	0.00	1.00
2000-01-01 04:00:00	8123	1	0.26	0.97
2000-01-01 05:00:00	8340	2	0.50	0.87
2000-01-01 06:00:00	8561	3	0.71	0.71

$N = 1$

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Time index	Electricity Demand	t	$sin\left(\frac{2\pi t}{24}\right)$	$cos\left(\frac{2\pi t}{24}\right)$	$sin\left(\frac{2\pi 2t}{24}\right)$	$cos\left(\frac{2\pi 2t}{24}\right)$
2000-01-01 03:00:00	8000	0	0.00	1.00	0.00	1.00
2000-01-01 04:00:00	8123	1	0.26	0.97	0.50	0.87
2000-01-01 05:00:00	8340	2	0.50	0.87	0.86	0.50
2000-01-01 06:00:00	8561	3	0.71	0.71	1.00	0.00

$N = 2$

What about multiple seasonality?

We can model additional seasonal components by adding more Fourier terms using the frequency of the additional seasonal component.

Example: We can add weekly seasonality here by adding Fourier terms using $f = \frac{1}{24*7} = \frac{1}{168}$

Time index	Electricity Demand	t	$sin\left(\frac{2\pi t}{24}\right)$	$cos\left(\frac{2\pi t}{24}\right)$	$sin\left(\frac{2\pi 2t}{24}\right)$	$cos\left(\frac{2\pi 2t}{24}\right)$
2000-01-01 03:00:00	8000	0	0.00	1.00	0.00	1.00
2000-01-01 04:00:00	8123	1	0.26	0.97	0.50	0.87
2000-01-01 05:00:00	8340	2	0.50	0.87	0.86	0.50
2000-01-01 06:00:00	8561	3	0.71	0.71	1.00	0.00

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2000-01-01 03:00:00	8000	0	0.00	1.00	0.00	1.00	0.00	1.000
2000-01-01 04:00:00	8123	1	0.26	0.97	0.50	0.87	0.04	0.999
2000-01-01 05:00:00	8340	2	0.50	0.87	0.86	0.50	0.07	0.997
2000-01-01 06:00:00	8561	3	0.71	0.71	1.00	0.00	0.11	0.993

Daily seasonality

Weekly seasonality

Fourier features: Sktime

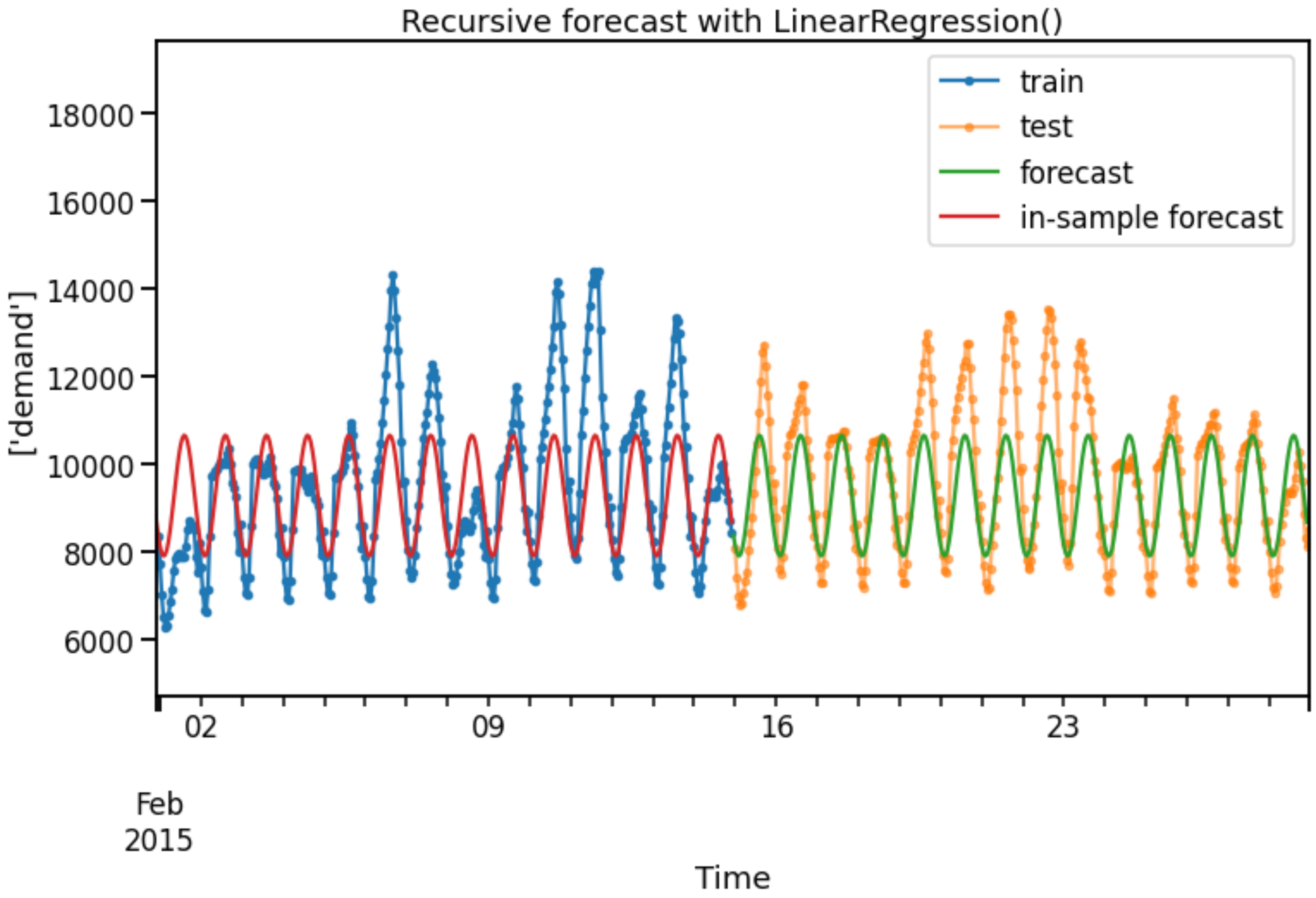
FourierFeatures

class **FourierFeatures**(*sp_list: List[float], fourier_terms_list: List[int], freq: Optional[str] = None, keep_original_columns: Optional[bool] = False*) [source]

Fourier Features for time series seasonality.

```
transformer = FourierFeatures(
                               sp_list = [24, 24*7], # list of seasonal periods
                               fourier_terms_list = [3, 3], # list of fourier terms
                               freq="H", # Frequency of the time series
                               keep_original_columns=False,
                              )
```

Example: Electricity demand



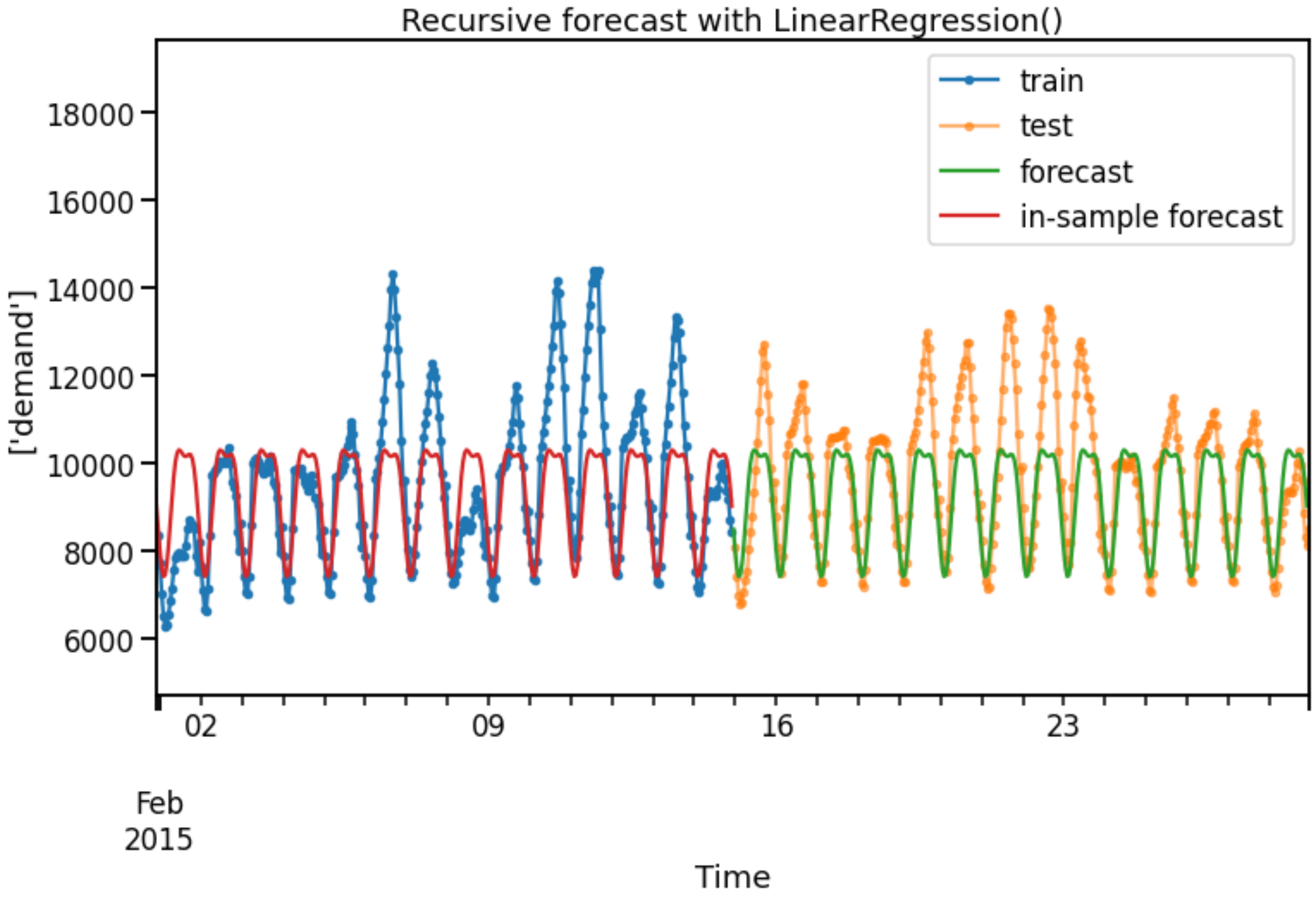
Features

- Fourier features daily seasonality (N=1)

Model

$$\hat{y}_t = \beta_0 + \beta_1 \sin\left(\frac{2\pi t}{24}\right) + \beta_2 \cos\left(\frac{2\pi t}{24}\right)$$

Example: Electricity demand



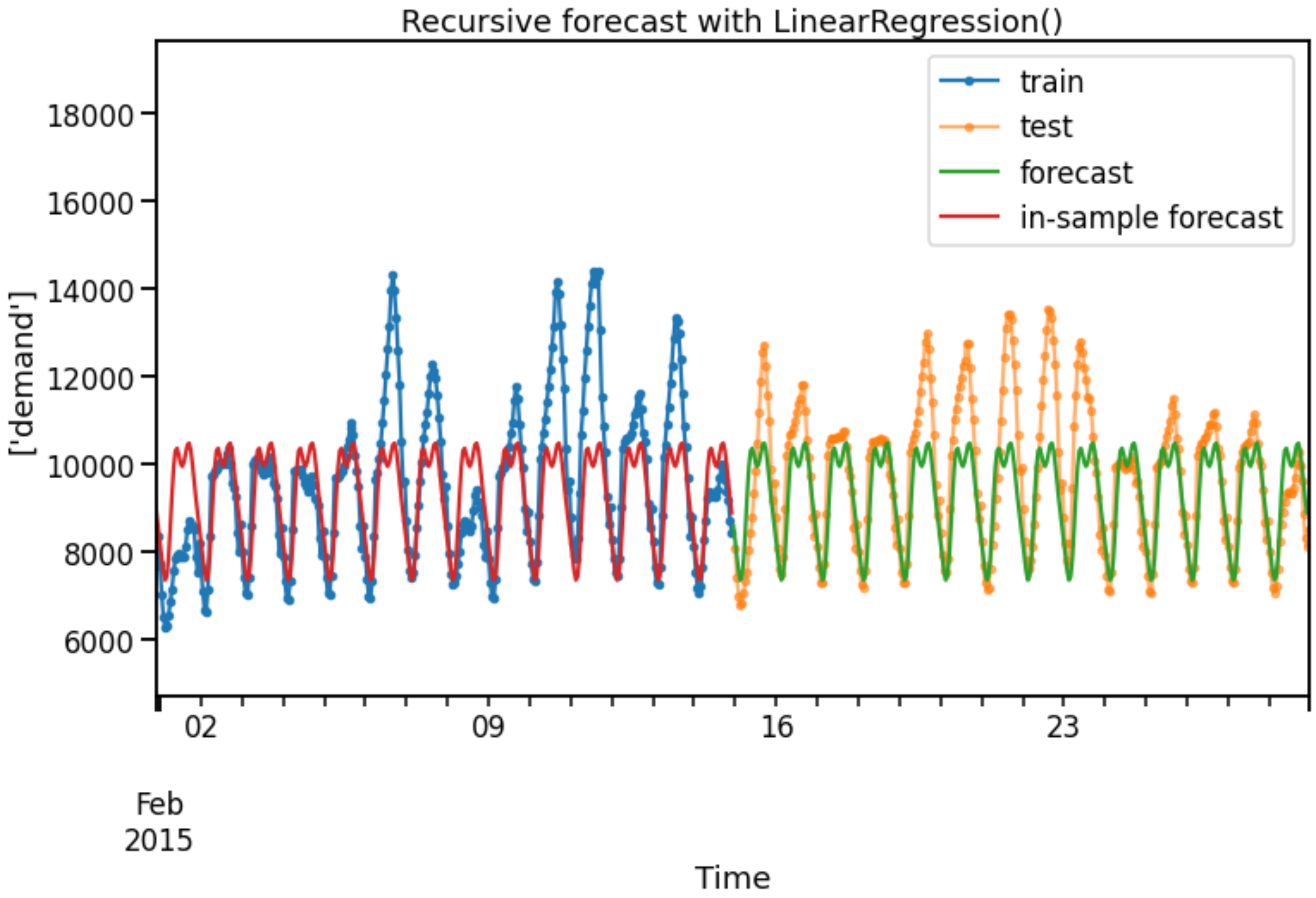
Features

- Fourier features daily seasonality (N=2)

Model

$$\hat{y}_t = \beta_0 + \beta_1 \sin\left(\frac{2\pi t}{24}\right) + \beta_2 \cos\left(\frac{2\pi t}{24}\right)$$
$$+\beta_3 \sin\left(\frac{2\pi 2t}{24}\right) + \beta_4 \cos\left(\frac{2\pi 2t}{24}\right)$$

Example: Electricity demand



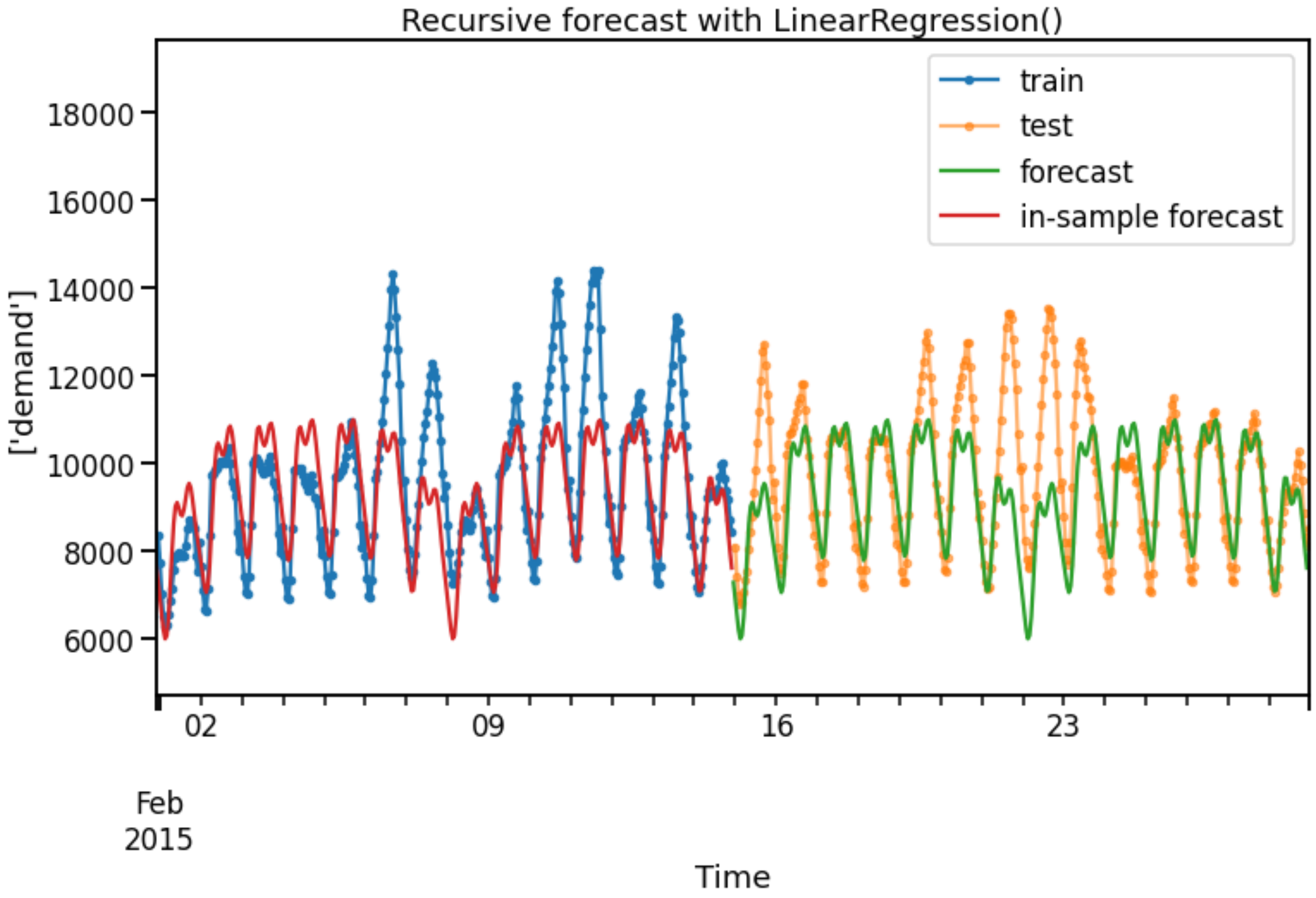
Features

- Fourier features daily seasonality (N=3)

Model

$$\hat{y}_t = \beta_0 + \beta_1 \sin\left(\frac{2\pi t}{24}\right) + \beta_2 \cos\left(\frac{2\pi t}{24}\right) + \beta_3 \sin\left(\frac{2\pi 2t}{24}\right) + \beta_4 \cos\left(\frac{2\pi 2t}{24}\right) + \beta_5 \sin\left(\frac{2\pi 3t}{24}\right) + \beta_6 \cos\left(\frac{2\pi 3t}{24}\right)$$

Example: Electricity demand



Features

- Fourier features daily seasonality (N=3)
- Fourier features weekly seasonality(N=3)

Model

$$\hat{y}_t = \beta_0 + \beta_1 \sin\left(\frac{2\pi t}{24}\right) + \beta_2 \cos\left(\frac{2\pi t}{24}\right) + \cdots$$

$$+\beta_i \sin\left(\frac{2\pi t}{168}\right) + \beta_j \cos\left(\frac{2\pi t}{168}\right) + \cdots$$

Pros and cons of Fourier features

Pros

- Makes it easier to model long period seasonality.
- Can model multiple seasonalities.
- Can control smoothness of the fit using N.
- Uses fewer features than seasonal dummies to capture seasonality.

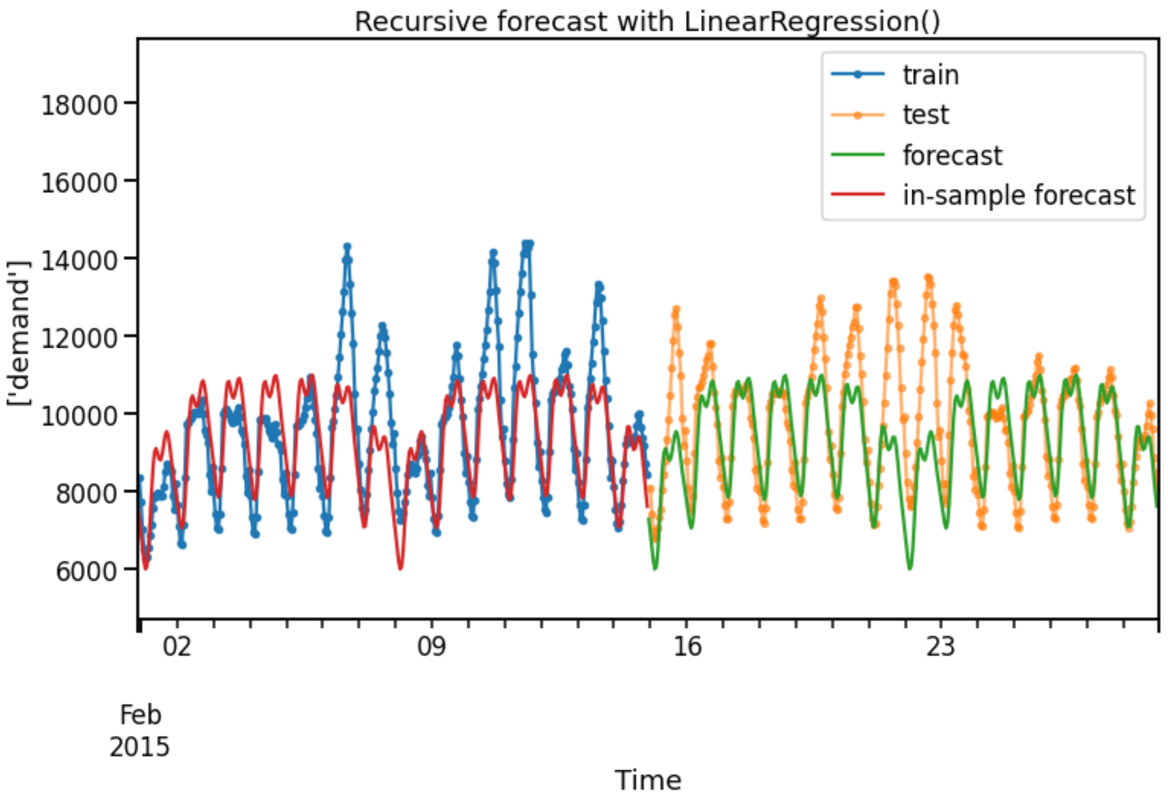
Cons

- Only useful for linear models.
- Assumes seasonality is fixed.

When are Fourier features useful?

- High frequency data with long seasonalities (e.g., hourly data with daily, monthly, and yearly seasonality).

- Can capture **short term** dynamics with **lag features** $(y_{t-1}, y_{t-2}, \dots)$ and **long term** seasonality with **Fourier features**.



Summary

Fourier series are made of sine and cosine terms and can model any periodic function.

Fourier features are the use of sine and cosine terms as features.

Fourier features allow us to model multiple long seasonalities.

Fourier features should be used with linear models.