Vectors and Vector Spaces: Foundations and Applications

Estimated reading time: 35 minutes

Learning order: Vectors: Definitions and Operations → Vector Spaces and Subspaces → Bases and Dimension → Orthogonality and the Gram-Schmidt Process

Vectors: Definitions and Operations

A vector is an object with both magnitude and direction. In mathematics, vectors are represented as ordered lists of numbers (coordinates) in a space. Vectors can be added, subtracted, and multiplied by scalars. These operations follow specific rules and are foundational for understanding higher concepts like vector spaces and orthogonality.

Key points: - A vector in $\mathbb{R}^n$ is an ordered n-tuple of real numbers. - Vector addition and scalar multiplication are defined component-wise. - The zero vector has all components zero. - The length (norm) of a vector is the square root of the sum of the squares of its components.

Formulas:

$$u + v = (u_1 + v_1, \ldots, u_n + v_n)$$

$$cu = (cu_1, \ldots, cu_n)$$

$$\|u\| = \sqrt{u_1^2 + \ldots + u_n^2}$$

$$u \cdot v = u_1 v_1 + \ldots + u_n v_n$$

Worked Example:
Add vectors $u = (2, 3)$ and $v = (1, -4)$. Find their sum and the norm of the result.

- Add components: $(2 + 1, 3 + (-4)) = (3, -1)$
- Compute norm: $\|(3, -1)\| = \sqrt{3^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$

Answer: Sum: $(3, -1)$, Norm: $\sqrt{10}$

Diagram: Vector addition in $\mathbb{R}^2$
Instructions: Draw two arrows from the origin: one for vector u, one for vector v. Draw the parallelogram formed by u and v. The diagonal from the origin represents $u + v$.

Common Pitfalls: - Confusing vector addition with multiplication. - Forgetting to add or multiply each component separately. - Misinterpreting the zero vector as a scalar zero.

Quick Quiz: - What is the zero vector in $\mathbb{R}^3$?
Ans: $(0, 0, 0)$ - How do you multiply a vector by a scalar?
Ans: Multiply each component by the scalar. - What is the norm of $(0, 4)$?
Ans: 4

Vector Spaces and Subspaces

A vector space is a set of vectors closed under addition and scalar multiplication, satisfying specific axioms (like associativity, distributivity, and existence of a zero vector). Subspaces are subsets of vector spaces that are themselves vector spaces under the same operations. Understanding these structures is crucial for advanced linear algebra topics.

Key points: - A vector space must be closed under addition and scalar multiplication. - Subspaces must contain the zero vector. - Every linear combination of vectors in a subspace remains in the subspace. - Common examples: $\mathbb{R}^n$, the set of all polynomials, solution sets to homogeneous equations.

Formulas:

$$\text{If } u, v \in V,\ c \in \mathbb{R},\ \text{then } u + v \in V,\ cu \in V$$

$$\text{Subspace test: } 0 \in W,\ u + v \in W,\ cu \in W\ \forall u, v \in W, c \in \mathbb{R}$$

Worked Example:
Is the set $W = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

- Check zero vector: $(0, 0, 0)$ is in W.
- Closed under addition: $(x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$ in W.
- Closed under scalar multiplication: $c(x, y, 0) = (cx, cy, 0)$ in W.

Answer: Yes, W is a subspace of $\mathbb{R}^3$.

Diagram: A subspace in $\mathbb{R}^3$
Instructions: Draw a 3D coordinate system. Shade a plane passing through the origin to represent a subspace.

Common Pitfalls: - Forgetting to check all subspace criteria. - Assuming any subset containing zero is a subspace. - Overlooking closure under scalar multiplication.

Quick Quiz: - Does the set of all vectors $(x, y, 1)$ form a subspace of $\mathbb{R}^3$?
Ans: No, it does not contain the zero vector. - What must every subspace contain?
Ans: The zero vector. - Is the set of all multiples of $(1, 2, 3)$ a subspace?
Ans: Yes, it is a line through the origin.

Bases and Dimension

A basis of a vector space is a set of linearly independent vectors that spans the space. The number of vectors in a basis is called the dimension of the space. Bases allow unique representation of every vector in the space as a linear combination of basis vectors. Understanding bases and dimension is essential for working with vector spaces and subspaces.

Key points: - A basis is both linearly independent and spans the space. - All bases of a finite-dimensional vector space have the same number of vectors. - Dimension is the number of vectors in any basis. - Changing the basis changes coordinates but not the underlying vector.

Formulas:

$$\text{If } \{v_1, \ldots, v_n\} \text{ is a basis for } V,\ \dim(V) = n$$

$$v = a_1 v_1 + \ldots + a_n v_n \text{ (unique representation)}$$

Worked Example:
Do the vectors $(1, 2)$ *and* $(3, 4)$ *form a basis for* $\mathbb{R}^2$*?*

- Check linear independence: Set $a(1, 2) + b(3, 4) = (0, 0)$.
- Solve: $a + 3b = 0$, $2a + 4b = 0$.
- First equation: $a = -3b$. Substitute into second: $2(-3b) + 4b = -6b + 4b = -2b = 0$, so $b = 0$, $a = 0$.
- Since only solution is $a = b = 0$, vectors are linearly independent.
- There are two vectors in $\mathbb{R}^2$, so they span $\mathbb{R}^2$.

Answer: Yes, they form a basis for $\mathbb{R}^2$.

Diagram: Basis vectors in $\mathbb{R}^2$
Instructions: Draw the x and y axes. Draw two arrows from the origin: one along the x-axis, one along the y-axis, labeled as basis vectors e_1 *and* e_2*.*

Common Pitfalls: - Confusing spanning with linear independence. - Assuming any two vectors in $\mathbb{R}^2$ form a basis. - Forgetting that basis vectors must be linearly independent.

Quick Quiz: - What is the dimension of $\mathbb{R}^3$?
Ans: 3 - Can a set of three vectors in $\mathbb{R}^2$ be a basis?
Ans: No, the maximum is two. - What does it mean for vectors to be linearly independent?
Ans: No nontrivial linear combination equals zero.

Orthogonality and the Gram-Schmidt Process

Orthogonality means vectors are perpendicular, or their dot product is zero. An orthonormal set is both orthogonal and each vector has unit length. The Gram-Schmidt process converts any linearly independent set into an orthonormal basis. This process is essential for simplifying computations and understanding projections in vector spaces.

Key points: - Orthogonal vectors have zero dot product. - Orthonormal sets are orthogonal and each vector has norm 1. - Gram-Schmidt produces an orthonormal basis from any basis. - Orthonormal bases simplify computations of coordinates and projections.

Formulas:

$$u \cdot v = 0 \implies u, v \text{ are orthogonal}$$

$$e_i = \frac{v_i - \sum_{j=1}^{i-1} \mathrm{proj}_{e_j}(v_i)}{\|v_i - \sum_{j=1}^{i-1} \mathrm{proj}_{e_j}(v_i)\|}$$

$$\mathrm{proj}_u(v) = \frac{v \cdot u}{u \cdot u} u$$

Worked Example:
Apply Gram-Schmidt to $(1, 1)$ *and* $(1, -1)$ *in* $\mathbb{R}^2$*.*

- Let $v_1 = (1, 1)$, $v_2 = (1, -1)$.
- Set $e_1 = v_1/\|v_1\| = (1, 1)/\sqrt{2} = (1/\sqrt{2}, 1/\sqrt{2})$.
- Project v_2 onto e_1: proj $= \frac{(1,-1)\cdot(1,1)}{(1,1)\cdot(1,1)}(1, 1) = \frac{0}{2}(1, 1) = (0, 0)$.
- Set $u_2 = v_2 - \mathrm{proj} = (1, -1) - (0, 0) = (1, -1)$.
- Normalize: $e_2 = (1, -1)/\sqrt{2} = (1/\sqrt{2}, -1/\sqrt{2})$.

Answer: Orthonormal basis: $(1/\sqrt{2}, 1/\sqrt{2}),\ (1/\sqrt{2}, -1/\sqrt{2})$

Diagram: Orthogonal vectors in $\mathbb{R}^2$
Instructions: Draw two arrows from the origin at a right angle (90 degrees) to each other, labeled u and v.

Common Pitfalls: - Forgetting to normalize vectors after orthogonalization. - Mixing up the order of subtraction in Gram-Schmidt. - Assuming original vectors are already orthogonal.

Quick Quiz: - What is the dot product of $(1, 0)$ and $(0, 1)$?
Ans: 0 - What does Gram-Schmidt produce?
Ans: An orthonormal basis. - Why normalize in Gram-Schmidt?
Ans: To ensure each basis vector has length 1.

Summary

This packet introduces vectors, their operations, and the structure of vector spaces. It covers subspaces, bases, and dimension, providing the foundation for understanding orthogonality. The Gram-Schmidt process is explained as a method to construct orthonormal bases, crucial for applications in linear algebra. Each section includes definitions, key properties, formulas, examples, diagrams, and quizzes to reinforce understanding.

Practice Problems

Problem 1 (Gram-Schmidt)

Let $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (0, 1, 1)$.

(a) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

(b) Write each of the original vectors as a linear combination of your orthonormal basis.

Problem 2 (Vector Spaces)

Let S and T be two subspaces of a vector space V.

(a) Definition: The sum $S + T$ contains all sums $s + t$ of a vector s in S and a vector t in T. Prove that $S + T$ is a subspace of V.

(b) Prove that

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$$

Problem 3 (Vector Spaces)

Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection of that plane with the xy plane.

Problem 4 (Gram-Schmidt)

Let $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (0, 1, 1)$.

Apply the Gram-Schmidt process to these vectors to obtain an orthonormal basis for $\mathbb{R}^3$.

Problem 5 (Vector Spaces)

Let V be the vector space of all polynomials of degree at most 3. Let W be the subspace of V consisting of all polynomials $p(x)$ such that $p(1) = 0$. Find a basis for W.

Solutions

1: Solution

(a) Let's denote the orthonormal basis vectors as $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Step 1:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1,1,0)}{\sqrt{1^2+1^2+0^2}} = \frac{(1,1,0)}{\sqrt{2}}$$

Step 2:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = \frac{1}{\sqrt{2}} \cdot \frac{(1,1,0)}{\sqrt{2}} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\mathbf{v}_2' = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = (1,0,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(\frac{1}{2}, -\frac{1}{2}, 1\right)$$

Normalize:

$$\|\mathbf{v}_2'\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2'}{\|\mathbf{v}_2'\|} = \frac{\left(\frac{1}{2}, -\frac{1}{2}, 1\right)}{\sqrt{3/2}}$$

Step 3:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) = \frac{1}{\sqrt{2}} \cdot \frac{(1,1,0)}{\sqrt{2}} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) = (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{1}{2}, \frac{1}{2}, 1\right)$$

$$\text{proj}_{\mathbf{u}_2}(\mathbf{v}_3) = \left((-\frac{1}{2}, \frac{1}{2}, 1) \cdot \mathbf{u}_2\right)\mathbf{u}_2$$

Compute $\mathbf{u}_2$ from above, then project and subtract to get $\mathbf{u}_3$, then normalize.

(b) Each original vector can be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ using the Gram-Schmidt coefficients.

2: Solution

(a) To show $S + T$ is a subspace, check closure under addition and scalar multiplication.

Let $x, y \in S + T$. Then $x = s_1 + t_1$, $y = s_2 + t_2$, with $s_1, s_2 \in S$ and $t_1, t_2 \in T$.

$$x + y = (s_1 + t_1) + (s_2 + t_2) = (s_1 + s_2) + (t_1 + t_2)$$

Since S and T are subspaces, $s_1 + s_2 \in S$ and $t_1 + t_2 \in T$, so $x + y \in S + T$.

For scalar multiplication:

$$\alpha x = \alpha(s_1 + t_1) = \alpha s_1 + \alpha t_1 \in S + T$$

Thus $S + T$ is a subspace.

(b) The formula

$$\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$$

follows from the fact that $S + T$ is the smallest subspace containing both S and T, and the intersection $S \cap T$ is counted twice in the sum $\dim S + \dim T$, so we subtract its dimension once.

3: Solution

A basis for the plane $x - 2y + 3z = 0$ can be found by solving for x in terms of y and z:

$$x = 2y - 3z$$

So any vector in the plane is of the form

$$(2y - 3z, y, z) = y(2, 1, 0) + z(-3, 0, 1)$$

Thus, $\{(2, 1, 0), (-3, 0, 1)\}$ is a basis for the plane.

For the intersection with the xy plane, set $z = 0$:

$$x - 2y = 0 \implies x = 2y$$

So the intersection is the line

$$\{(2y, y, 0)\} = y(2, 1, 0)$$

Thus, $\{(2, 1, 0)\}$ is a basis for the intersection.

4: Solution

Let's denote the orthonormal basis vectors as $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Step 1:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{(1, 0, 1)}{\sqrt{1^2 + 0^2 + 1^2}} = \frac{(1, 0, 1)}{\sqrt{2}}$$

Step 2:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \frac{1}{\sqrt{2}} \cdot \frac{(1, 0, 1)}{\sqrt{2}} = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

$$\mathbf{v}_2' = (1,1,0) - \left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(\frac{1}{2}, 1, -\frac{1}{2}\right)$$

Normalize:

$$\|\mathbf{v}_2'\| = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{3}{2}}$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2'}{\sqrt{3/2}}$$

Step 3:

$$\text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) = \frac{1}{\sqrt{2}} \cdot \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

$$\mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) = (0,1,1) - \left(\frac{1}{2}, 0, \frac{1}{2}\right) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$$

$$\text{proj}_{\mathbf{u}_2}(\mathbf{v}_3) = \left((-\frac{1}{2}, 1, \frac{1}{2}) \cdot \mathbf{u}_2\right) \mathbf{u}_2$$

Compute $\mathbf{u}_2$ from above, then project and subtract to get $\mathbf{u}_3$, then normalize.

5: Solution

A general polynomial of degree at most 3 is

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

The condition $p(1) = 0$ gives

$$a_0 + a_1 + a_2 + a_3 = 0$$

So

$$a_0 = -(a_1 + a_2 + a_3)$$

Thus,

$$p(x) = -(a_1 + a_2 + a_3) + a_1 x + a_2 x^2 + a_3 x^3 = a_1(x-1) + a_2(x^2-1) + a_3(x^3-1)$$

Therefore, a basis for W is

$$\{x-1,\ x^2-1,\ x^3-1\}$$