Quantum Physics Concepts: Core Principles and Applications

Estimated reading time: 35 minutes

Learning order: Particle-Wave Duality → Measurement Problem in Quantum Mechanics → Quantum Entanglement → Superconductors

Particle-Wave Duality

Particle-wave duality is a foundational concept in quantum physics. It states that all quantum objects, such as electrons and photons, exhibit both particle-like and wave-like properties depending on the experimental context. This duality is demonstrated by phenomena like the double-slit experiment, where particles create interference patterns typical of waves. The concept challenges classical physics, which treats particles and waves as distinct entities. The mathematical description uses the wavefunction, which encodes the probability of finding a particle at a given position.

Key points: - Quantum objects can behave as both particles and waves. - Wavefunction (ψ) describes the probability amplitude for a particle's position. - Interference patterns arise from wave-like behavior. - Measurement collapses the wavefunction to a definite outcome. - Classical physics cannot explain duality; quantum theory is required.

Formulas: - $\lambda = h/p$ - $\psi(x,t)$: Wavefunction - $P(x) = |\psi(x)|^2$

Worked Example: *Calculate the de Broglie wavelength of an electron moving at* 2.0×10^6 *m/s. (Electron mass:* 9.11×10^{-31} *kg, Planck's constant:* 6.63×10^{-34} $J \cdot s$*)*

- Calculate momentum: $p = m \times v = 9.11 \times 10^{-31}$ kg $\times 2.0 \times 10^6$ m/s $= 1.822 \times 10^{-24}$ kg $\cdot$ m/s.
- Apply de Broglie formula: $\lambda = h/p$.
- $\lambda = 6.63 \times 10^{-34}$ J $\cdot$ s $/1.822 \times 10^{-24}$ kg $\cdot$ m/s.
- $\lambda \approx 3.64 \times 10^{-10}$ m.

Answer: The de Broglie wavelength is approximately 3.64×10^{-10} meters.

Diagram: Double-slit experiment showing interference pattern.
Instructions: Draw two vertical slits on a barrier. Behind the barrier, draw a screen with alternating light and dark bands (interference pattern). Show particles (dots) approaching the slits and wavefronts spreading from each slit.

Common Pitfalls: - Assuming quantum objects are only particles or only waves. - Ignoring the probabilistic interpretation of the wavefunction. - Confusing classical and quantum interference.

Quick Quiz: - What does the wavefunction $\psi(x)$ represent?
Ans: The probability amplitude for finding a particle at position x. - What is the de Broglie wavelength formula?
Ans: $\lambda = h/p$ - What pattern is observed in the double-slit experiment with single electrons?
Ans: An interference pattern typical of waves.

Measurement Problem in Quantum Mechanics

The measurement problem addresses how and why quantum systems transition from a superposition of states to a single outcome upon measurement. Before measurement, a system is described by a wavefunction that can represent multiple possible outcomes simultaneously. Measurement appears to 'collapse' the wavefunction into one definite state, but the mechanism of this collapse is not fully understood. This problem is central to debates about the interpretation of quantum mechanics, such as the Copenhagen interpretation and many-worlds hypothesis.

Key points: - Quantum systems exist in superpositions before measurement. - Measurement causes the wavefunction to collapse to a definite state. - The process of collapse is not explained by standard quantum mechanics. - Different interpretations propose various solutions (e.g., Copenhagen, many-worlds).

Formulas: - $\psi = c_1\psi_1 + c_2\psi_2 + ...$ (superposition) - $P_i = |c_i|^2$ (probability of outcome i)

Worked Example: *An electron is in a superposition:* $\psi = (1/\sqrt{2})\psi_A + (1/\sqrt{2})\psi_B$*. What is the probability of finding it in state A?*

- Identify the coefficient for ψ_A: $c_1 = 1/\sqrt{2}$.
- Probability is $P_A = |c_1|^2$.
- $P_A = |1/\sqrt{2}|^2 = 1/2$.

Answer: The probability of finding the electron in state A is 1/2.

Diagram: Wavefunction collapse upon measurement.
Instructions: Draw a wavy line representing a superposed wavefunction approaching a detector. After the detector, show a single spike at one position, indicating collapse.

Common Pitfalls: - Thinking the system is in one state before measurement. - Assuming collapse is a physical process described by quantum equations. - Confusing probability amplitude with probability.

Quick Quiz: - What does wavefunction collapse mean?
Ans: The transition from a superposition to a single outcome upon measurement. - What is the probability of an outcome given coefficient c_i?
Ans: $P_i = |c_i|^2$ - Name one interpretation of the measurement problem.
Ans: Copenhagen interpretation.

Quantum Entanglement

Quantum entanglement is a phenomenon where the quantum states of two or more particles become linked, so that the state of one instantly determines the state of the other, regardless of distance. Entangled particles exhibit correlations that cannot be explained by classical physics. Measurement of one particle's property immediately affects the other's, a feature confirmed by experiments violating Bell's inequalities. Entanglement is fundamental to quantum information science, including quantum computing and cryptography.

Key points: - Entangled particles have correlated properties. - Measurement of one affects the state of the other instantly. - Entanglement violates classical locality assumptions. - Bell's inequalities test for non-classical correlations.

Formulas: - Bell's inequality: $|E(a,b) - E(a,b')| + |E(a',b) + E(a',b')| \leq 2$ - Example entangled state: $\psi = (1/\sqrt{2})(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

Worked Example: *Two electrons are in the singlet state:* $\psi = (1/\sqrt{2})(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$*. If electron 1 is measured spin-up, what is the spin of electron 2?*

- The singlet state means total spin is zero.
- If electron 1 is spin-up, the state collapses to $|\uparrow\downarrow\rangle$.
- Electron 2 must be spin-down.

Answer: Electron 2 will be measured as spin-down.

Diagram: Entangled particles measured at distant locations.
Instructions: Draw two particles moving apart from a common source. Show detectors at each end. Use arrows to indicate measurement outcomes that are always opposite.

Common Pitfalls: - Believing entanglement allows faster-than-light communication. - Assuming entanglement is the same as classical correlation. - Thinking measurement changes the distant particle physically.

Quick Quiz: - What is quantum entanglement?
Ans: A phenomenon where particles' states are correlated regardless of distance. - What does Bell's inequality test?
Ans: Whether quantum correlations can be explained by classical physics. - If two entangled particles are measured, are their results always independent?
Ans: No, their results are correlated.

Superconductors

Superconductors are materials that conduct electricity with zero resistance below a critical temperature. In the quantum view, electrons form Cooper pairs that move coherently without scattering. This collective quantum state leads to perfect conductivity and the expulsion of magnetic fields (Meissner effect). Superconductivity is explained by the BCS theory, which describes how electron pairing arises from attractive interactions mediated by lattice vibrations (phonons). Superconductors have important technological applications, including MRI machines and quantum computers.

Key points: - Superconductors have zero electrical resistance below a critical temperature. - Electrons form Cooper pairs, enabling frictionless flow. - The Meissner effect expels magnetic fields from the superconductor. - BCS theory explains the microscopic origin of superconductivity.

Formulas: - Critical temperature: T_c - Energy gap: $\Delta = 1.76 k_B T_c$ - London equation: $\nabla^2 B = B/\lambda_L^2$

Worked Example: *A superconductor has a critical temperature $T_c = 10$ K. Calculate the energy gap Δ at zero temperature. (Boltzmann constant $k_B = 1.38 \times 10^{-23}$ J/K)*

- Use $\Delta = 1.76 k_B T_c$.
- $\Delta = 1.76 \times 1.38 \times 10^{-23}$ J/K $\times 10$ K.
- $\Delta = 1.76 \times 1.38 \times 10^{-22}$ J.
- $\Delta \approx 2.43 \times 10^{-22}$ J.

Answer: The energy gap Δ is approximately 2.43×10^{-22} joules.

Diagram: Meissner effect in a superconductor.
Instructions: Draw a rectangular superconductor with magnetic field lines bending around it, not passing through. Indicate zero resistance with a current loop inside.

Common Pitfalls: - Assuming all materials become superconductors at low temperatures. - Confusing zero resistance with zero voltage. - Ignoring the role of Cooper pairs in superconductivity.

Quick Quiz: - What is the Meissner effect?
Ans: The expulsion of magnetic fields from a superconductor. - What forms the basis of superconductivity in the BCS theory?
Ans: Cooper pairs of electrons. - What happens to resistance in a superconductor below T_c?
Ans: It drops to zero.

Summary

This packet introduces the essential concepts of quantum physics, focusing on particle-wave duality, the measurement problem, quantum entanglement, and superconductors. Each section provides clear definitions, key formulas, diagrams, and worked examples to build a solid understanding of quantum phenomena and their applications.

Practice Problems

Problem 1

Time Dependence: Show that in one dimension

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \Psi_1^* \Psi_2 \, dx = 0$$

for any two normalizable solutions to the Schrödinger equation.

Problem 2

Prove the following relation between the uncertainty in position and the uncertainty in total energy:

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m|\langle p \rangle|}$$

Why does this not tell us much for bound state (normalizable) stationary states (in the center of mass reference frame)?

Problem 3

Use Ehrenfest's theorem, and appropriate choices for the operators that appear in the theorem, to

(a) prove classical energy conservation;

(b) prove that $\langle p \rangle = m\langle v \rangle$;

(c) prove Newton's second law: $\langle F \rangle = \frac{d\langle p \rangle}{dt}$.

Problem 4

Is it possible to measure energy of $0.75\,\hbar\omega$ for a quantum harmonic oscillator? Why? Why not? Explain.

Problem 5

Explain the connection between Planck's hypothesis of energy quanta and the energies of the quantum harmonic oscillator.

Problem 6

When an electron and a proton of the same kinetic energy encounter a potential barrier of the same height and width, which one of them will tunnel through the barrier more easily? Why?

Problem 7

Explain the difference between a box-potential and a potential of a quantum dot.

Problem 8

Can a quantum particle 'escape' from an infinite potential well like that in a box? Why? Why not?

Solutions

Problem 1

Solution:

Let Ψ_1 and Ψ_2 be two normalizable solutions to the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

Then,

$\frac{d}{dt} \int \Psi_1^* \Psi_2 \, dx = \int \left[\frac{\partial \Psi_1^*}{\partial t} \Psi_2 + \Psi_1^* \frac{\partial \Psi_2}{\partial t} \right] dx$

Using the Schrödinger equation:

$i\hbar \frac{\partial \Psi_1}{\partial t} = H\Psi_1 \implies \frac{\partial \Psi_1}{\partial t} = -\frac{i}{\hbar} H\Psi_1$

$i\hbar \frac{\partial \Psi_2}{\partial t} = H\Psi_2 \implies \frac{\partial \Psi_2}{\partial t} = -\frac{i}{\hbar} H\Psi_2$

Similarly, $\frac{\partial \Psi_1^*}{\partial t} = \left(\frac{\partial \Psi_1}{\partial t} \right)^* = \frac{i}{\hbar} H\Psi_1^*$ (since H is Hermitian).

So,

$\frac{d}{dt} \int \Psi_1^* \Psi_2 \, dx = \int \left[\frac{i}{\hbar} H\Psi_1^* \Psi_2 + \Psi_1^* \left(-\frac{i}{\hbar} H\Psi_2 \right) \right] dx$

$= \frac{i}{\hbar} \int \left[H\Psi_1^* \Psi_2 - \Psi_1^* H\Psi_2 \right] dx$

But since H is Hermitian,

$\int H\Psi_1^* \Psi_2 \, dx = \int \Psi_1^* H\Psi_2 \, dx$

Therefore,

$\frac{d}{dt} \int \Psi_1^* \Psi_2 \, dx = 0$

Problem 2

Solution:

The uncertainty relation between position and energy can be derived using the general uncertainty principle:

$\sigma_A \sigma_B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$

Let $A = x$ and $B = H$ (Hamiltonian). The commutator

$[x, H] = \left[x, \frac{p^2}{2m} + V(x) \right] = \frac{1}{2m} [x, p^2] = \frac{i\hbar}{m} p$

Therefore,

$\sigma_x \sigma_H \geq \frac{1}{2} \left| \langle \frac{i\hbar}{m} p \rangle \right| = \frac{\hbar}{2m} \left| \langle p \rangle \right|$

For bound stationary states, $\langle p \rangle = 0$ (since the expectation value of momentum in the center of mass frame is zero), so the right-hand side vanishes and the inequality does not provide a useful lower bound.

Problem 3

Solution:

(a) **Classical energy conservation:**
Ehrenfest's theorem states:

$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle[H, A]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle$

For $A = H$, $[H, H] = 0$ and $\frac{\partial H}{\partial t} = 0$, so $\frac{d\langle H\rangle}{dt} = 0$.

(b) $\langle p\rangle = m\langle v\rangle$:
$\langle p\rangle = m\frac{d\langle x\rangle}{dt}$ by Ehrenfest's theorem.

(c) **Newton's second law:**
Ehrenfest's theorem for $A = p$:

$\frac{d\langle p\rangle}{dt} = \langle-\frac{\partial V}{\partial x}\rangle = \langle F\rangle$

So $\langle F\rangle = \frac{d\langle p\rangle}{dt}$.

Problem 4

Solution:

No, it is not possible to measure energy of $0.75\,\hbar\omega$ for a quantum harmonic oscillator. The allowed energies are quantized and given by $E_n = (n + 1/2)\hbar\omega$, where $n = 0, 1, 2, ...$ Therefore, the possible energies are $0.5\,\hbar\omega$, $1.5\,\hbar\omega$, $2.5\,\hbar\omega$, etc., but not $0.75\,\hbar\omega$.

Problem 5

Solution:

Planck's hypothesis states that energy is quantized in units of $\hbar\omega$. The quantum harmonic oscillator has energy levels $E_n = (n+1/2)\hbar\omega$, which means the energy can only take discrete values separated by $\hbar\omega$. This is a direct application of Planck's idea to the vibrational energies of the oscillator.

Problem 6

Solution:

The electron will tunnel through the barrier more easily. The tunneling probability depends exponentially on the mass of the particle; a lighter particle (electron) has a higher probability to tunnel than a heavier particle (proton) for the same kinetic energy.

Problem 7

Solution:

A box-potential is an idealized potential with infinitely high walls, confining a particle strictly within a region. A quantum dot potential is more realistic, with finite walls and a confining potential that allows for some probability of tunneling outside the dot.

Problem 8

Solution:

No, a quantum particle cannot escape from an infinite potential well. The probability of finding the particle outside the well is zero because the potential is infinite and the wave function is strictly zero outside the box.

Sources

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