Campus Universitário de Castanhal

Faculdade de Computação

Comunicações Digitais

Aula 6

Professor: Diogo Acatauassú

Na aula de hoje

- Sampling and Analog-to-Digital Conversion
- Sampling Theorem
- Signal Reconstruction from Uniform Samples
- Practical Issues in Signal Sampling and Reconstruction
- The Treachery of Aliasing
- Advantages of Digital Communication

Sampling and Analog-to-Digital Conversion

- Analog signals can be digitized through sampling and quantization.

- This analog-to-digital (A/D) conversion sets the foundation of modern digital communication systems.

- In the A/D converter the sampling rate must be large enough to permit the analog signal to be reconstructed from the samples with sufficient accuracy.

- The **sampling theorem** is the basis for determining the proper sampling rate for a given signal.

Sampling Theorem

- A signal g(t) whose spectrum is band-limited to B Hz, that is,

$$G(f) = 0 \qquad \text{for } |f| > B$$

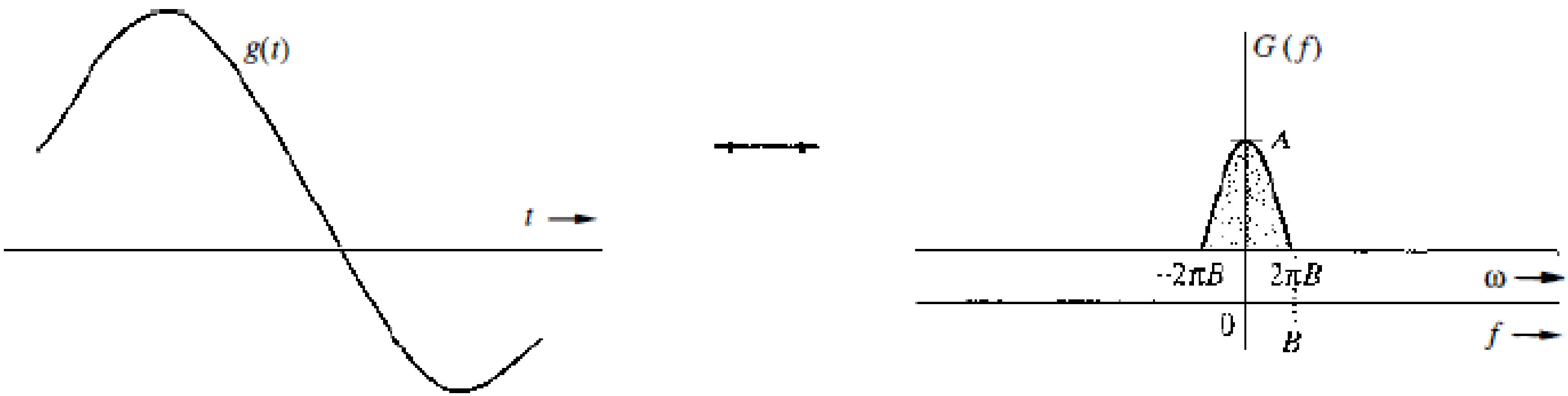
can be reconstructed exactly (without any error) from its discrete time samples taken uniformly at a rate of R samples per second.

- The condition is that R > 2*B

- In other words, the minimum sampling frequency for perfect signal recover is f_s = 2*B Hz.

Sampling Theorem

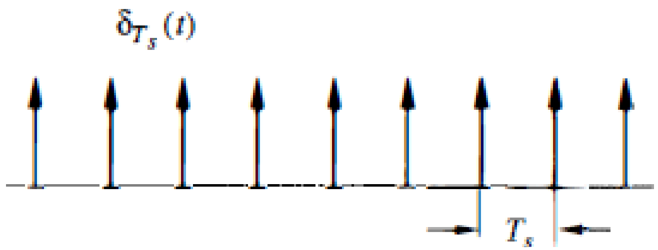
- Consider a signal g(t) whose spectrum is band-limited to B Hz.

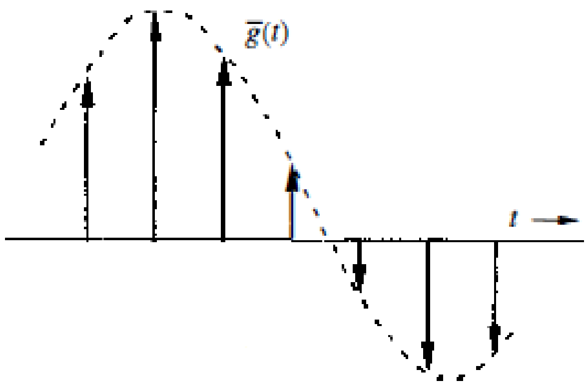


- Sampling g(t) at a rate of f_s Hz means that we take **f_s uniform samples** per second.

- This uniform sampling can be accomplished by multiplying g(t) by an impulse train $\delta_{T_s}(t)$ consisting of unit impulses repeating periodically every T_s seconds, where $T_s = 1 / f_s$

Sampling Theorem





- This results in the sampled signal ḡ(t).

- The sampled signal consists of impulses spaced every T_s seconds (the sampling interval).

- The *n*th impulse, located at t = *n* T_s , has a strength g(nT_s) which is the value of g(t) at t = nT_s.

Sampling Theorem

- Thus, the relationship between the sampled signal $\bar{g}(t)$ and the original analog signal g(t) is:

$$\overline{g}(t) = g(t)\delta_{T_s}(t) = \sum_n g(nT_s)\delta(t - nT_s)$$

- Because the impulse train δ_{Ts} (t) is a periodic signal of period T_s, it can be expressed as an exponential Fourier series, as:

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \qquad \omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

- Therefore:

$$\begin{aligned}\overline{g}(t) &= g(t)\delta_{T_s}(t) \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty} g(t)e^{jn2\pi f_s t}\end{aligned}$$

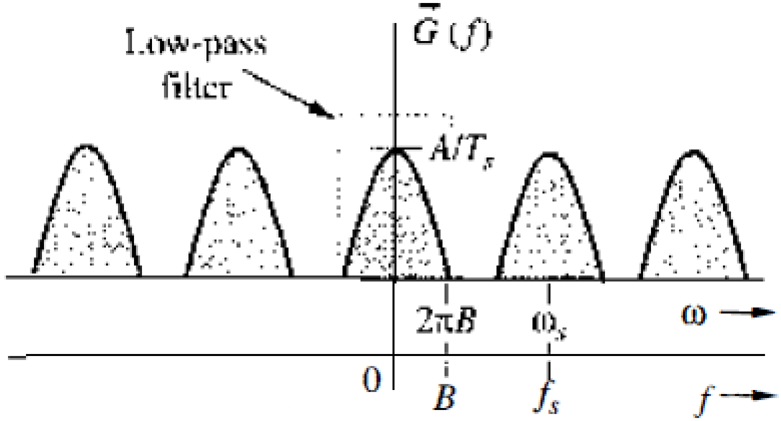
Sampling Theorem

- To find $\bar{G}(f)$, the Fourier transform of $\bar{g}(t)$, we take the Fourier transform of the summation.

- Based on the frequency-shifting property, the transform of the nth term is shifted by nf_s. Therefore:

$$\bar{G}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(f - nf_s)$$

- This means that the spectrum $\bar{G}(f)$ consists of $G(f)$, scaled by a constant $1/T_s$, repeating periodically with period $f_s = 1 / T_s$ Hz



Sampling Theorem

- Can g(t) be reconstructed from $\bar{g}(t)$ without any loss or distortion?

- If we are to reconstruct g(t) from $\bar{g}(t)$, equivalently in the frequency domain we should be able to recover G(f) from $\bar{G}(f)$.

- Perfect recovery is possible if there is no overlap among the replicas in $\bar{G}(f)$.

- This requires: $f_s > 2B$ (**Nyquist frequency**)

- Therefore: $T_s < \frac{1}{2B}$ (**Nyquist interval**)

- As long as the sampling frequency f_s is greater than twice the signal bandwidth B (in hertz), $\bar{G}(f)$ will consist of nonoverlapping repetitions of G(f).

- When this is true, g(t) can be recovered from its samples $\bar{g}(t)$ by passing $\bar{g}(t)$ through an ideal low-pass filter of bandwidth B Hz.

Signal Reconstruction from Uniform Samples

- The process of reconstructing a continuous time signal g(t) from its samples is also known as **interpolation**.

- Assuming that uniform sampling above the Nyquist rate preserves all the signal information, passing the sampled signal through an ideal low-pass filter of bandwidth B Hz will reconstruct the original message.

- The sampled signal contains a component $(1/T_s)*g(t)$, and to recover g(t) the sampled signal must be sent through an ideal low-pass filter of badwidth B Hz and gain T_s

- Transfer function of such a filter: $$H(f) = T_s \Pi\left(\frac{\omega}{4\pi B}\right) = T_s \Pi\left(\frac{f}{2B}\right)$$

Signal Reconstruction from Uniform Samples

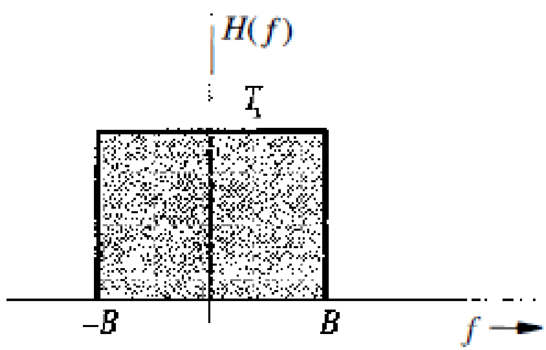
- **Ideal Reconstruction:**

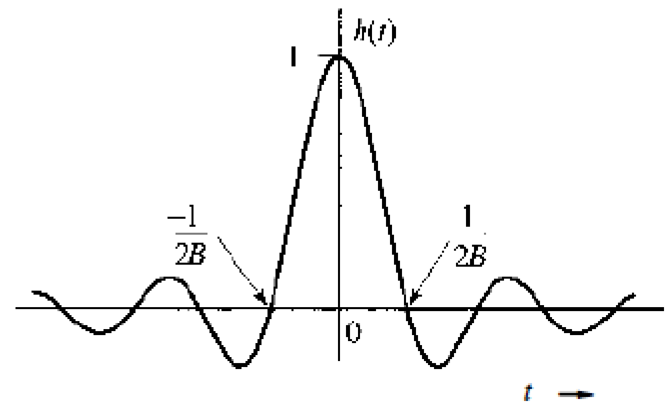
- The inverse Fourier transform of the reconstruction filter leads to:

$$h(t) = 2BT_s \operatorname{sinc}(2\pi Bt)$$

- Assuming the use of Nyquist sampling rate, that is, $2BT_s = 1$:

$$h(t) = \operatorname{sinc}(2\pi Bt)$$





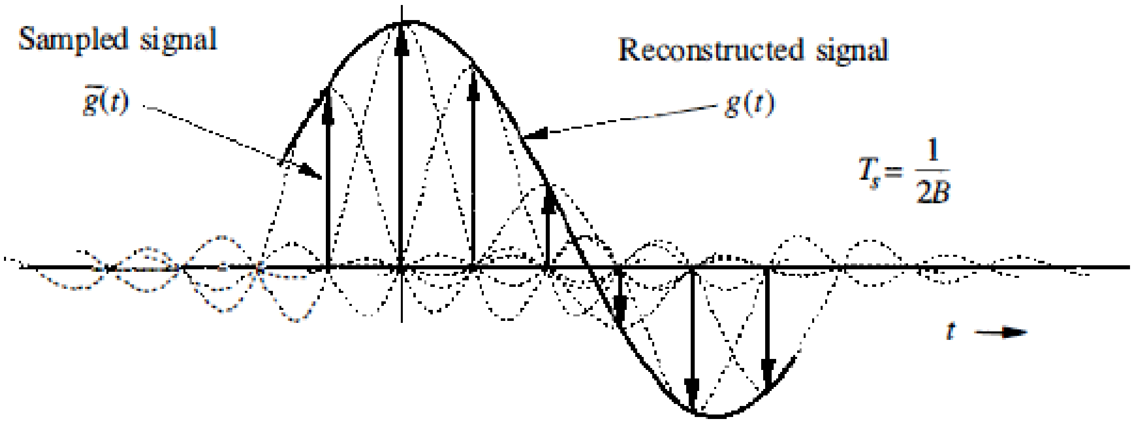
Signal Reconstruction from Uniform Samples

- **Ideal Reconstruction:**

- $h(t) = 0$ at all Nyquist sampling instants ($t = \pm n/2B$) except $t = 0$.

- When the sampled signal $\bar{g}(t)$ is applied at the input of this filter, the output is $g(t)$.

- Each sample in $\bar{g}(t)$, being an impulse, generates a sinc pulse of height equal to the strength of the sample



Signal Reconstruction from Uniform Samples

- **Ideal Reconstruction:**

- Addition of the sinc pulses generated by all the samples results in g(t).

- The *k*th sample of the input g(t) is the impulse $g(kT_s)\delta(t-kT_s)$.

- The filter output of this impulse is $g(kT_s)h(t-kT_s)$.

- Hence, the filter output to $\bar{g}(t)$, which is g(t), can now be expressed as a sum:

$$
\begin{aligned}
g(t) &= \sum_k g(kT_s)h(t-kT_s) \\
&= \sum_k g(kT_s)\ \mathrm{sinc}\ [2\pi B(t-kT_s)] \\
&= \sum_k g(kT_s)\ \mathrm{sinc}\ (2\pi Bt - k\pi)
\end{aligned}
$$

Practical Issues in Signal Sampling and Reconstruction

- If a signal is sampled at the Nyquist rate $f_s = 2B$ Hz, the spectrum $\bar{G}(f)$ consists of repetitions of $G(f)$ without any gap between successive cycles.

- To recover $g(t)$ from $\bar{g}(t)$, we need to pass the sampled signal $\bar{g}(t)$ through an ideal low-pass filter.

- However, an ideal low-pass filter is noncausal and unrealizable! ☹

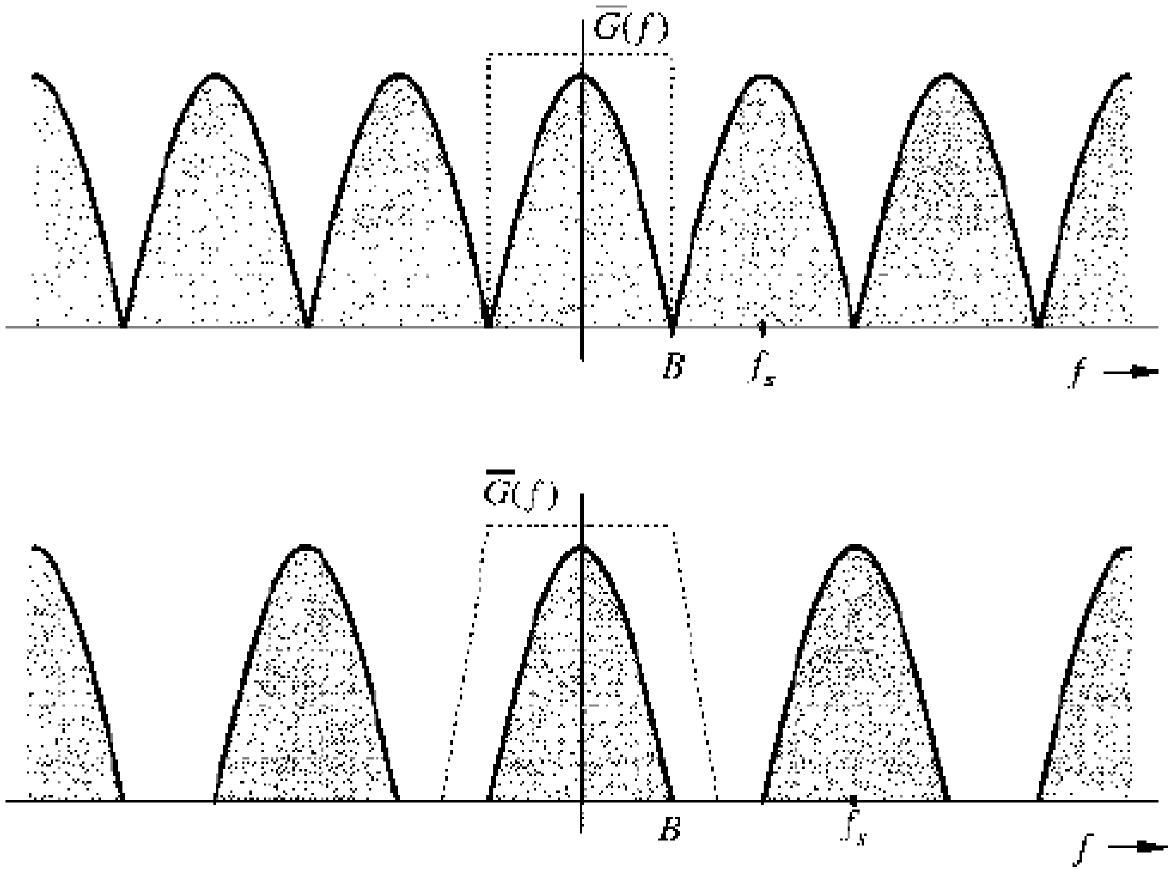
- It comes from the infinitely long nature of the sinc reconstruction pulse.

- A practical solution to this problem is to sample the signal at a rate **higher than** the Nyquist rate ($f_s > 2B$ or $\omega_s > 4\pi B$).

- This yields $\bar{G}(f)$, consisting of repetitions of $G(f)$ with a finite band gap between successive cycles.

Practical Issues in Signal Sampling and Reconstruction

- We can now recover G(f) from Ḡ(f) by using a low-pass filter with a gradual cutoff
characteristic.



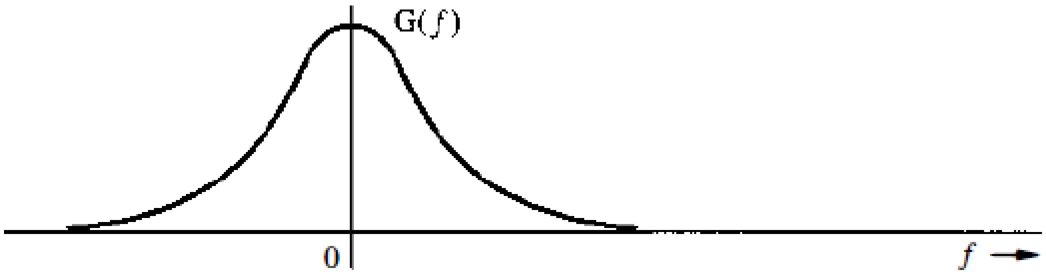
The Treachery of Aliasing

- There is another fundamental practical difficulty in reconstructing a signal from its samples.

- The sampling theorem was proved on the assumption that the signal g(t) is band-limited.

- All practical signals are **time-limited!!**

- A signal cannot be time-limited and band-limited simultaneously.



- Clearly, all practical signals, which are necessarily time-limited, are non-band-limited, they have infinite bandwidth.

The Treachery of Aliasing

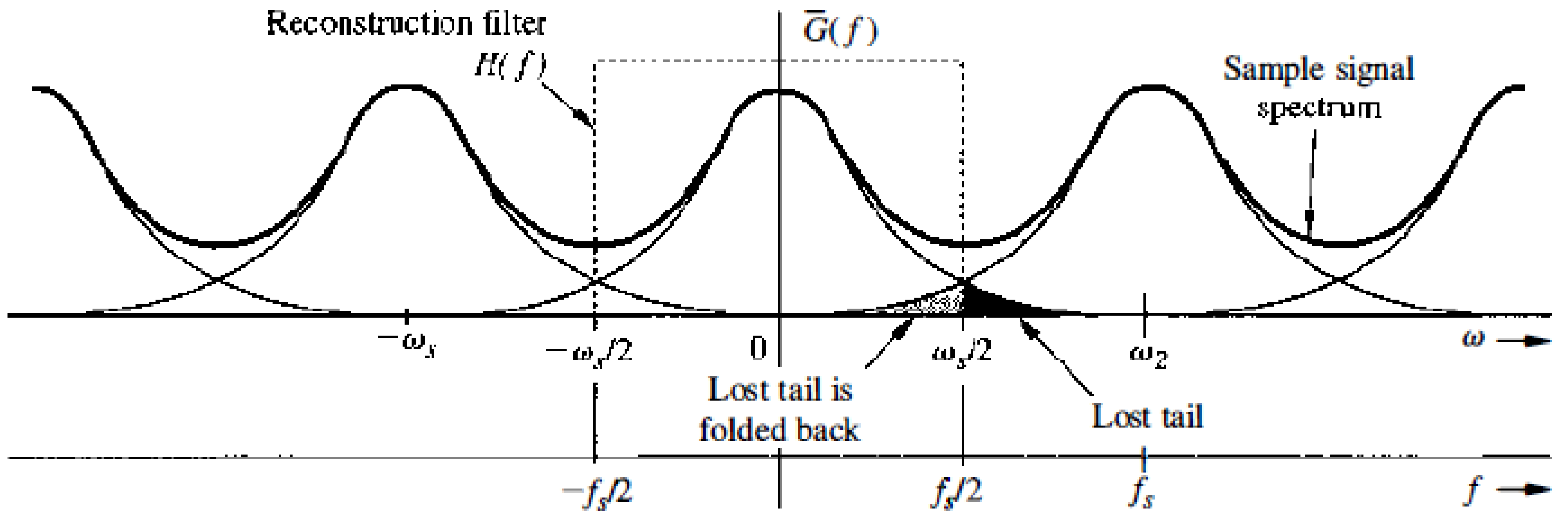
- The spectrum $\bar{G}(f)$ consists of overlapping cycles of G(f) repeating every f_s Hz (the sampling frequency).

- Because of the infinite bandwidth in this case, the spectral overlap is unavoidable, regardless of the sampling rate.

- Sampling at a higher rate reduces but does not eliminate overlapping between repeating spectral cycles.

- Because of the overlapping tails, $\bar{G}(f)$ no longer has complete information about G(f), and it is no longer possible, to recover g(t) exactly from the sampled signal $\bar{g}(t)$.

The Treachery of Aliasing

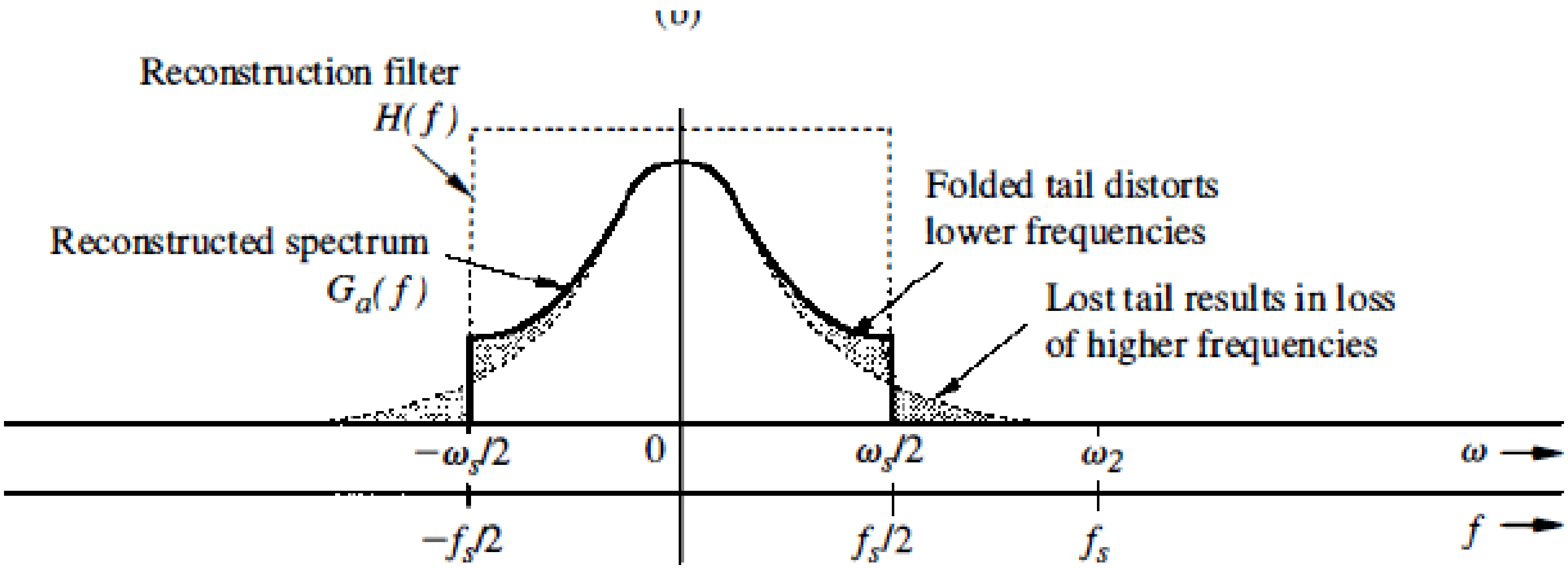


- If the sampled signal is passed through an ideal low-pass filter of cutoff frequency $f_s/2$ Hz, the output is not G(f) but $G_a(f)$.

- $G_a(f)$ is a version of G(f) distorted as a result of two separate causes:

- **The loss of the tail of G(f) beyond | f | > f_s /2 Hz.**
- **The reappearance of this tail inverted or folded back onto the spectrum.**

The Treachery of Aliasing



- The spectra cross at frequency $f_s/2 = 1 / 2T_s$ Hz, which is called the **folding frequency**.

- The spectrum may be viewed as if the lost tail is folding back onto itself at the folding frequency.

- The components of frequencies above $f_s/2$ reappear as components of frequencies below $f_s/2$.

The Treachery of Aliasing

- This tail inversion known as spectral folding or **aliasing.**

- During aliasing, not only are we losing all the components of frequencies above the folding frequency $f_s/2$ Hz, but these very components reappear (aliased) as lower frequency components.

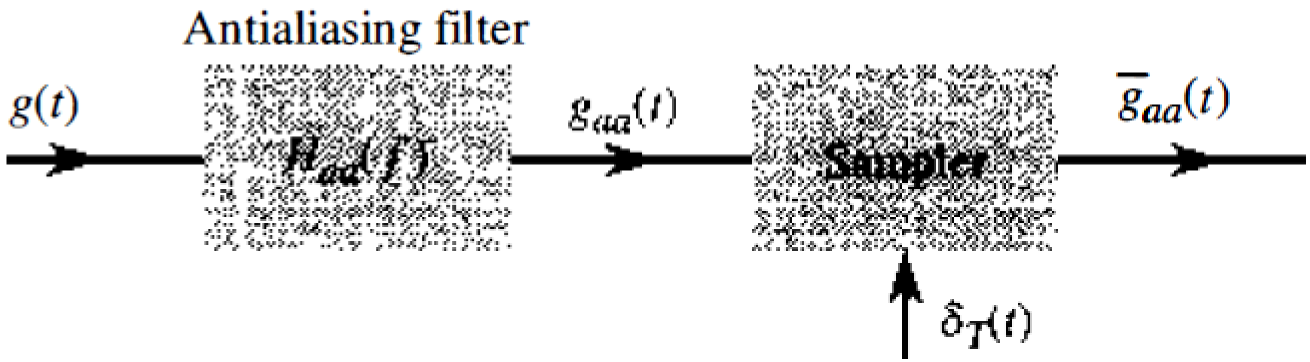
- Such aliasing destroys the integrity of the frequency components below the folding frequency $f_s/2$.

The Treachery of Aliasing

- **The Antialiasing Filter:**

- The antialiasing filter eliminate (suppress) all the frequency components beyond the folding frequency $f_s/2 = 1/2T_s$ Hz from g (t) **before sampling** g(t).

- Such suppression of higher frequencies can be accomplished by an ideal low-pass filter of cutoff $f_s/2$ Hz.



The Treachery of Aliasing

- **The Antialiasing Filter:**

- An antialiasing filter essentially band-limits the signal g(t) to $f_s/2$ Hz.

- We lose only the components beyond the folding frequency $f_s/2$ Hz.

- These suppressed components now cannot reappear, corrupting the components of frequencies below the folding frequency.

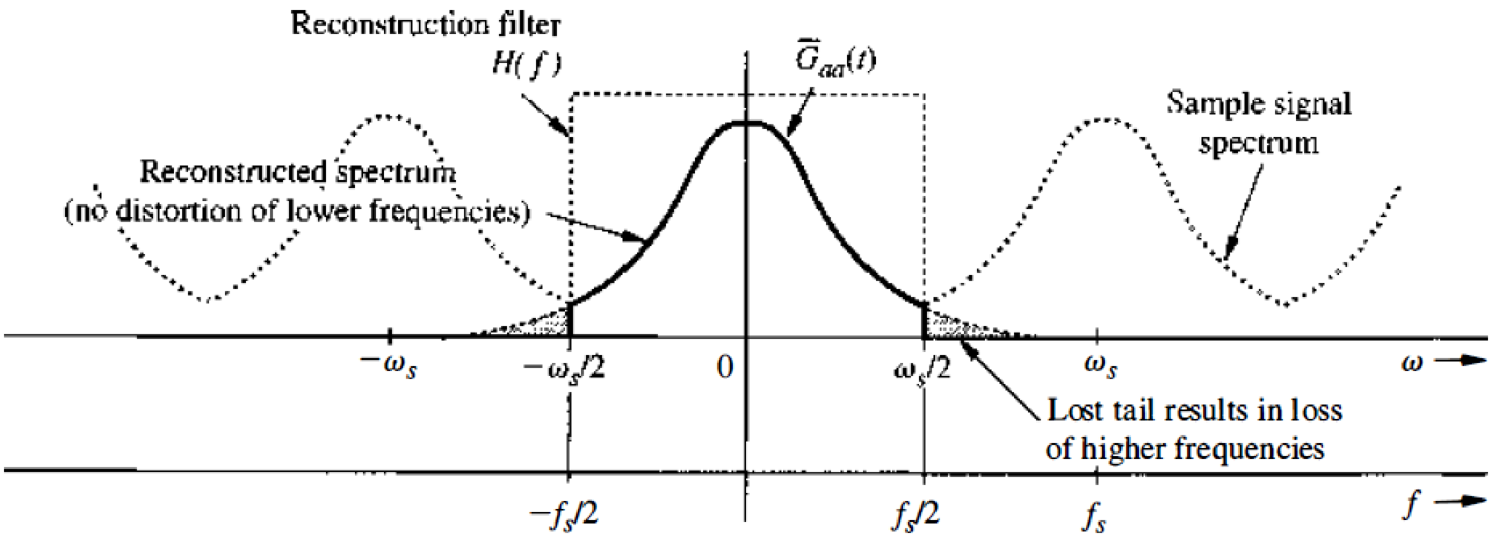
- The use of an antialiasing filter results in the reconstructed signal spectrum $G_{aa}(f) = G(f)$ for $| f | < f_s/2$.

- Although we lost the spectrum beyond $f_s/2$ Hz, the spectrum for all the frequencies below $f_s/2$ remains intact.

The Treachery of Aliasing

- **The Antialiasing Filter:**

- The effective aliasing distortion is cut in half owing to elimination of folding.



- The antialiasing filter, being an ideal filter, is unrealizable. In practice we use a steep-cutoff filter, which leaves a sharply attenuated residual spectrum beyond the folding frequency $f_s/2$.

Advantages of Digital Communication

- Digital communication, which can withstand channel noise and distortion much better than analog.

- Viability of regenerative repeaters in the former. In an analog communication system, a message signal becomes progressively weaker as it travels along the channel. Amplification enhances the signal and the noise by the same proportion.

- Digital hardware implementation is flexible.

- Digital signals can be coded to yield extremely low error rates and high fidelity.

- Reproduction with digital messages can be extremely reliable without deterioration.

- It is easier and more efficient to multiplex several digital signals.

Comments on Logarithmic Units

- Logarithmic units and logarithmic scales are very convenient when a variable has a **large dynamic range**. This is the case of frequency variables or SNRs.

- A logarithmic unit for the power ratio is the decibel (dB), defined as 10 *log10 (power ratio).

- Thus, an SNR is x dB, where: $x = 10 \log_{10} \frac{S}{N}$

- We use dB to express power gain or loss over a certain transmission medium.

- If over a certain cable the signal power is attenuated by a factor of 15, the cable gain is:

$$G = 10 \log_{10} \frac{1}{15} = -11.76 \text{ dB}$$

or the cable attenuation (loss) is 11.76 dB.

Comments on Logarithmic Units

- Although the decibel is a measure of power ratios, it is often used as a measure of power itself.

- 100 watt may be considered to be a power ratio of 100 with respect to 1 watt power, and is expressed in units of dBW:

$$P_{\mathrm{dBW}} = 10 \log_{10} 100 = 20 \text{ dBW}$$

- Thus, 100 watt power is 20 dBW.

- Similarly, power measured with respect to 1 mW power is dBm.

- For instance, 100 watt power is:

$$P_{\mathrm{dBm}} = 10 \log \frac{100 \text{ W}}{1 \text{ mW}} = 50 \text{ dBm}$$