Campus Universitário de Castanhal

Faculdade de Computação

Comunicações Digitais

Aula 4 – Parte 1

Professor: Diogo Acatauassú

Na aula de hoje

- Analysis and Transmission of Signals
- Fourier Integral
- Transforms of Some Useful Functions
- Properties of the Fourier Transform

Fourier Integral

- Spectral representation of **periodic signals** = **Fourier series**.

- Spectral representation of **aperiodic signals** = **Fourier integral**

- The Fourier series is used to represent a **periodic function** by a **discrete sum of complex exponentials**.

- The Fourier transform is then used to represent a general, **non-periodic function** by a continuous superposition **or integral of complex exponentials.**

- The Fourier transform can be viewed as **the limit of the Fourier series of a function with the period approaches to infinity**, so the limits of integration change from one period to $(-\infty,+\infty)$.

Fourier Integral

- G(f) - Direct Fourier transform of g(t) $G(f) = \mathcal{F}[g(t)]$

$$g(t) \Longleftrightarrow G(f)$$

- g(t) - Inverse Fourier transform of G(f) $g(t) = \mathcal{F}^{-1}[G(f)]$

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j\omega t}\, df$$

- For real g(t), G(f) must be complex.

Fourier Integral

- **f versus ω:**

- We use two equivalent notations of angular frequency ω and frequency f in representing signals in the frequency domain.

- **Conjugate Symmetry Property:**

- If g(t) is a real function of t, then G(f) and G(-f) are complex conjugates.

$$G(-f) = G^*(f)$$

$$|G(-f)| = |G(f)|$$

$$\theta_g(-f) = -\theta_g(f)$$

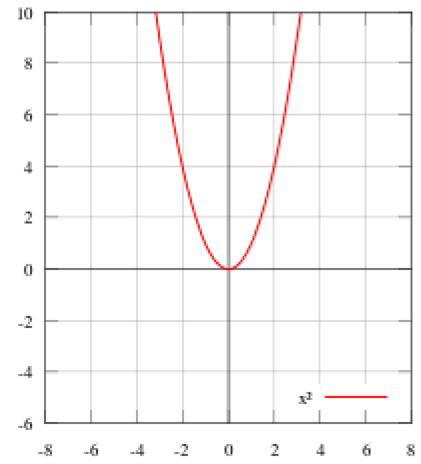
- For real g(t), the amplitude spectrum |G(f)| is an even function and the phase spectrum θ(f) is an odd function.

Fourier Integral

- Even function:

$$f(x) = f(-x)$$

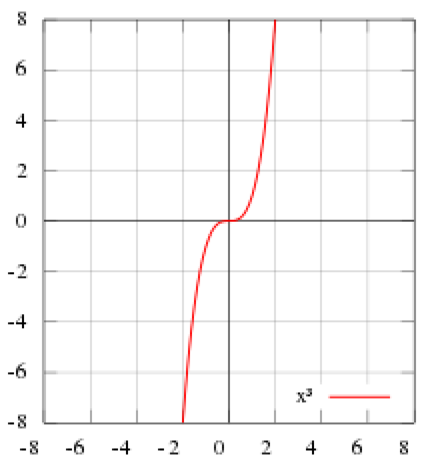
$$f(x) - f(-x) = 0.$$



- Odd function:

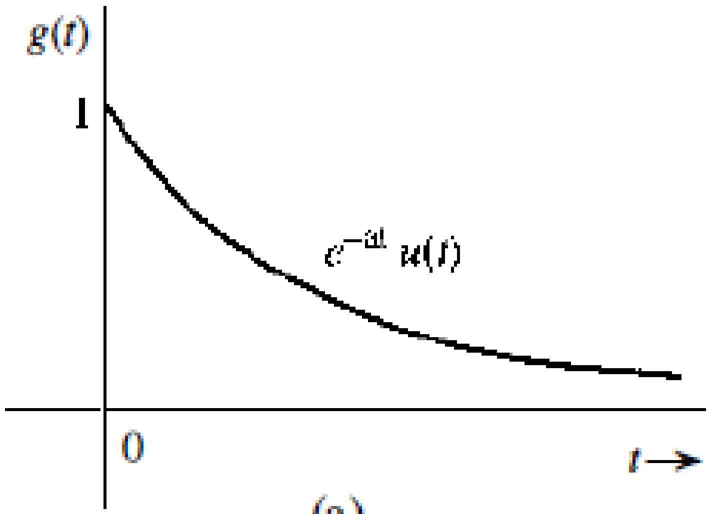
$$-f(x) = f(-x)$$

$$f(x) + f(-x) = 0.$$



Fourier Integral

- Example: Find the Fourier transform of $e^{-at}u(t)$:



(a)

Fourier Integral

- Example: Find the Fourier transform of $e^{-at}u(t)$:

$$G(f) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j2\pi ft}\,dt = \int_{0}^{\infty} e^{-(a+j2\pi f)t}\,dt = \left.\frac{-1}{a+j2\pi f}e^{-(a+j2\pi f)t}\right|_{0}^{\infty}$$

But $|e^{-j2\pi ft}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j2\pi f)t} = e^{-at}e^{-j2\pi ft} = 0$ if $a > 0$. Therefore,

$$G(f) = \frac{1}{a+j\omega} \qquad a > 0$$

$$e^{\pm j\omega_0 t} = \cos\,\omega_0 t \pm j\,\sin\,\omega_0 t$$

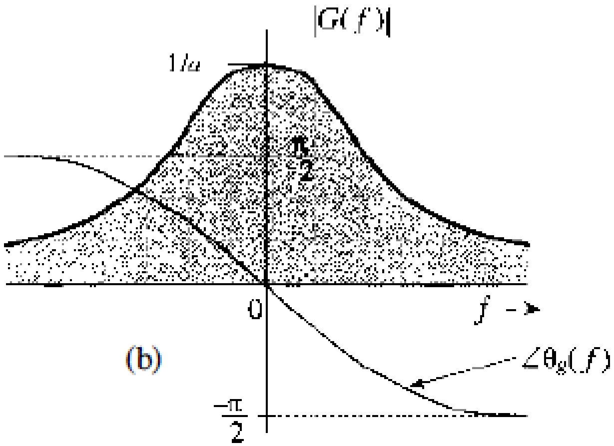
$$\begin{aligned} e^{ix} &= \cos x + i\sin x \\ \Longrightarrow \quad |e^{ix}| &= |\cos x + i\sin x| \\ &= \sqrt{(\mathrm{Re}(\cos x + i\sin x))^2 + (\mathrm{Im}(\cos x + i\sin x))^2} \\ &= \sqrt{\cos^2 x + \sin^2 x} \\ &= 1 \end{aligned}$$

Fourier Integral

- Example: Find the Fourier transform of $e^{-at}u(t)$:
- Expressing α + jω in polar form: $\sqrt{a^2+\omega^2}\,e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$

$$G(f)=\frac{1}{\sqrt{a^2+(2\pi f)^2}}e^{-j\tan^{-1}\left(\frac{2\pi f}{a}\right)}$$

- Therefore:



$$|G(f)|=\frac{1}{\sqrt{a^2+(2\pi f)^2}} \quad \text{and} \quad \theta_g(f)=-\tan^{-1}\left(\frac{2\pi f}{a}\right)$$

Fourier Integral

- **Linearity**:

$$g_1(t) \Longleftrightarrow G_1(f) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(f)$$

$$a_1 g_1(t) + a_2 g_2(t) \Longleftrightarrow a_1 G_1(f) + a_2 G_2(f)$$

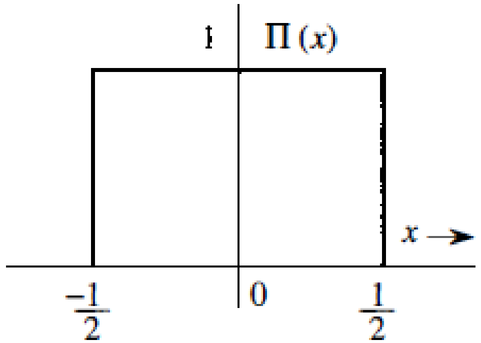
- Linear combinations of signals in the time domain correspond to linear combinations of their Fourier transforms in the frequency domain:

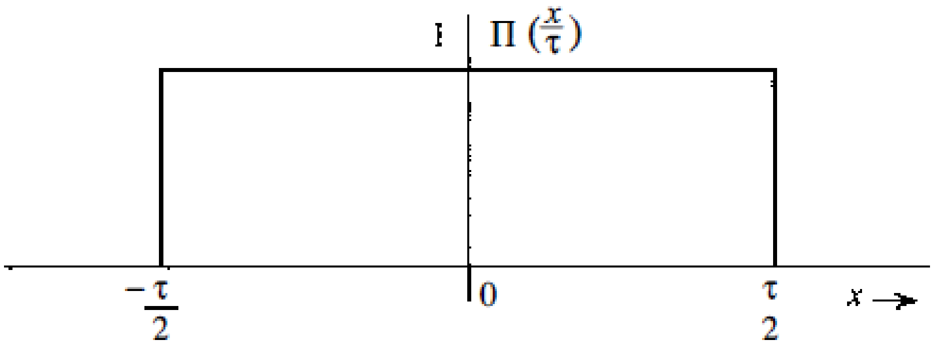
$$\sum_k a_k g_k(t) \Longleftrightarrow \sum_k a_k G_k(f)$$

Fourier Integral

- **Transforms of some useful functions**:

- Unit rectangular function:





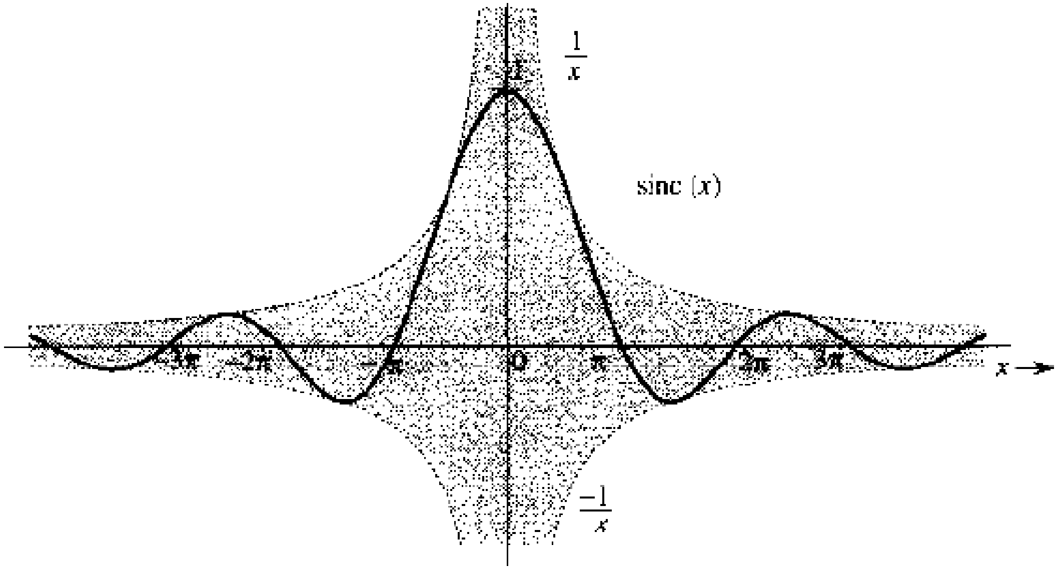
$$\Pi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0.5 & |x| = \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

Fourier Integral

- **Transforms of some useful functions**:

- sinc(x) function:

$$\text{sinc}\,(x) = \frac{\sin x}{x}$$



- sinc(x) is an even function.

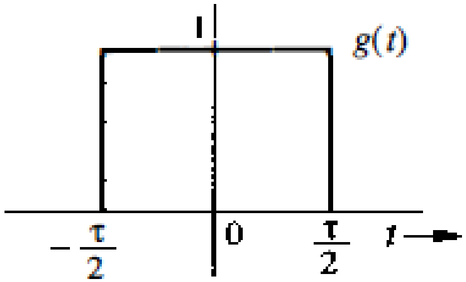
- sinc(0) = 1

- sinc(x) = 0 when sin(x) = 0 (**integer multiples of π**), except at x=0

- sinc(x) is the product of an oscillating signal sin(x) (of period 2π) and a monotonically decreasing function 1/x .

Fourier Integral

- Example: Find the Fourier transform of g(t) = Π(t/τ):



$$G(f) = \int_{-\infty}^{\infty} \Pi\left(\frac{t}{\tau}\right) e^{-j2\pi f t} dt$$

- Π(t/τ) = 1 for | t | < τ/2 and Π(t/τ) = 0 for | t | > τ/2.

$$G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi f t}\, dt$$

Fourier Integral

- Example: Find the Fourier transform of $g(t) = \Pi(t/\tau)$:

$$G(f) = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}\, dt \qquad = -\frac{1}{j2\pi f}\left(e^{-j\pi f\tau} - e^{j\pi f\tau}\right) = \frac{2\sin(\pi f\tau)}{2\pi f}$$

$$= \tau \frac{\sin(\pi f\tau)}{(\pi f\tau)} = \tau \ \text{sinc}\ (\pi f\tau)$$

$$\sin(\theta) \qquad = \qquad \frac{1}{2i}\left(e^{+i\theta} - e^{-i\theta}\right)$$

- Therefore:

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\ \text{sinc}\left(\frac{\omega\tau}{2}\right) = \tau\ \text{sinc}\ (\pi f\tau)$$

- sinc(x) = 0 when x= +- nπ

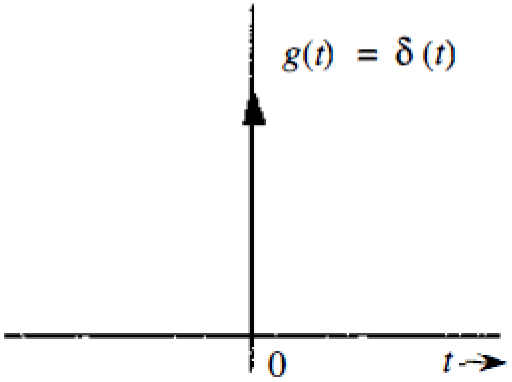
Fourier Integral

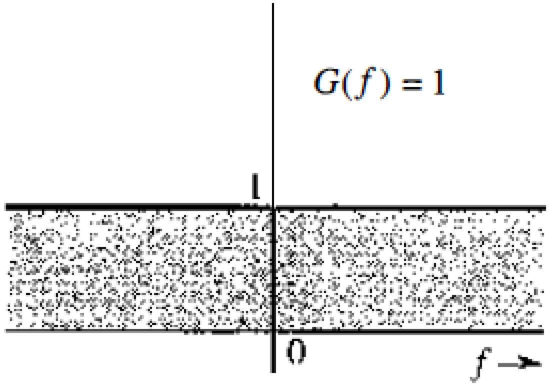
- Example: Find the Fourier transform of the impulse function δ(t):

- Sampling property of the impulse function.

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = e^{-j2\pi f \cdot 0} = 1$$

$$\delta(t) \Longleftrightarrow 1$$

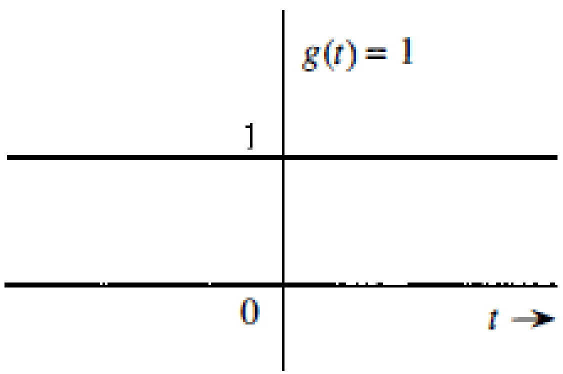


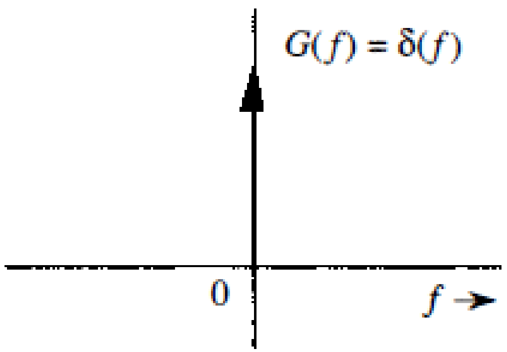


Fourier Integral

- Example: Find the Inverse Fourier transform of δ(f):

$$1 \Longleftrightarrow \delta(f)$$





- g(t) = 1 is a dc signal that has a single frequency component at f=0.

- If an impulse at f = 0 is a spectrum of a dc signal, what does an impulse at f = f_0 represent?

Fourier Integral

- Example: Find the Inverse Fourier transform of $\delta(f-f_0)$:

- Sampling property of the impulse function.

$$\mathcal{F}^{-1}[\delta(f-f_0)] = \int_{-\infty}^{\infty} \delta(f-f_0) e^{j2\pi ft}\, df = e^{j2\pi f_0 t}$$

- Therefore:

$$e^{j2\pi f_0 t} \Longleftrightarrow \delta(f-f_0)$$

- The spectrum of an everlasting exponential $e^{j2\pi f_0 t}$ is a single impuls at $f = f_0$.

$$e^{-j2\pi f_0 t} \Longleftrightarrow \delta(f+f_0)$$

Fourier Integral

- Example: Find the Inverse Fourier transform of the everlasting sinusoid $\cos(2\pi f_0 t)$

- Euler Formula:

$$\cos 2\pi f_0 t = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})$$

- So, from the last example:

$$\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[\delta(f + f_0) + \delta(f - f_0)]$$

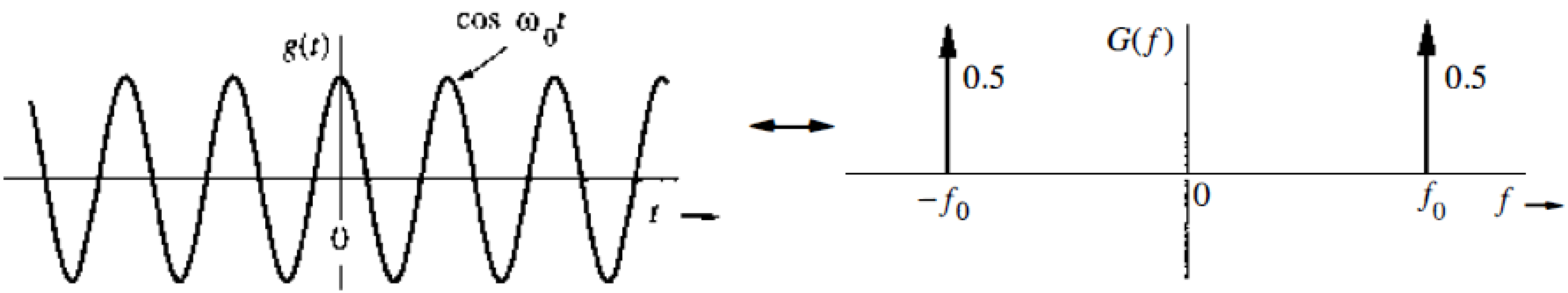
- The spectrum of $\cos(2\pi f_0 t)$ consists of two impulses at f_0 and $-f_0$ in the f-domain, or, two impulses at $\pm\omega_0 = \pm 2\pi f_0$ in the ω-domain.

Fourier Integral

- Example: Find the Inverse Fourier transform of the everlasting sinusoid $\cos(2\pi f_0 t)$

- An everlasting sinusoid $\cos(2\pi f_0 t)$ can be synthesized by two everla exponentials, $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$.

- Therefore, the Fourier spectrum consists of only two components of ω_0 and $-\omega_0$.

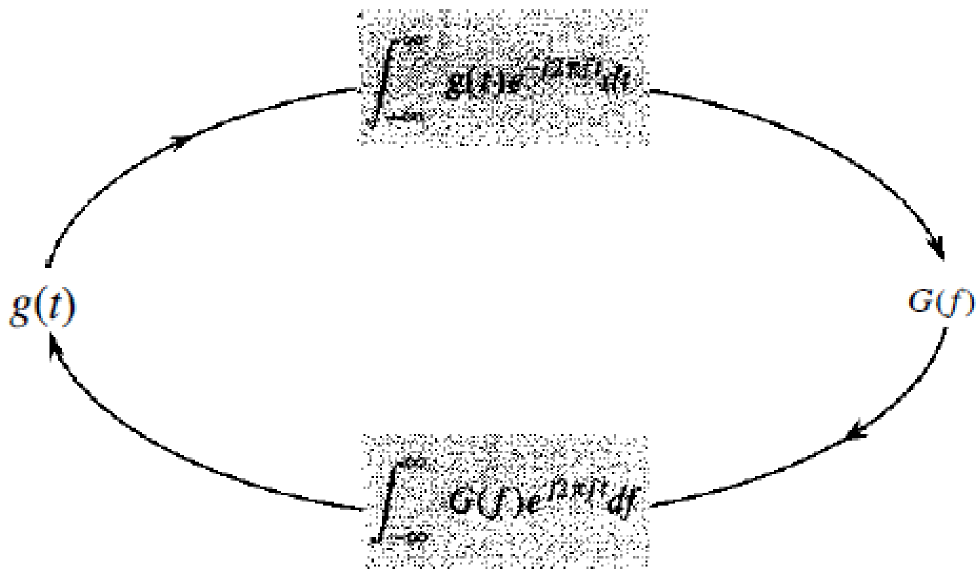


Fourier Transform Pairs

	$g(t)$	$G(f)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+(2\pi f)^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j2\pi f)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\frac{2}{j2\pi f}$	
13	$\cos 2\pi f_0 t\, u(t)$	$\frac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\frac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\frac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\frac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\frac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\frac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	$a>0$
17	$\Pi\left(\frac{t}{\tau}\right)$	$\tau \operatorname{sinc}(\pi f \tau)$	
18	$2B \operatorname{sinc}(2\pi B t)$	$\Pi\left(\frac{f}{2B}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\pi f \tau}{2}\right)$	
20	$B \operatorname{sinc}^2(\pi B t)$	$\Delta\left(\frac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	$f_0=\frac{1}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

Properties of the Fourier Transform

- **Time-Frequency Duality**:



- "A photograph can be obtained from its negative, and by using an identical procedure, the negative can be obtained from the photograph."

Properties of the Fourier Transform

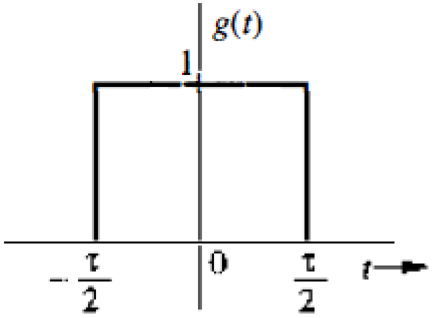
- **Duality Property**:

$$g(t) \Longleftrightarrow G(f)$$

$$G(t) \Longleftrightarrow g(-f)$$

- If the Fourier transform of g(t) is G(f) then the Fourier transform of G(t), with f replaced by t, is the g(-f) which is the original time domain signal with t replaced by -f.

- Example: Apply de duality property for g(t) = Π(t/τ)



Properties of the Fourier Transform

- **Duality Property**:

- Example: Apply the duality property for g(t) = Π(t/τ)

$$\Pi\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau \operatorname{sinc}(\pi f \tau)$$

$$\underbrace{\Pi\left(\frac{t}{\alpha}\right)}_{g(t)} \Longleftrightarrow \underbrace{\alpha \operatorname{sinc}(\pi f \alpha)}_{G(f)}$$

- G(t) is the same as G(f) with f replaced by t, and g(-f) is the same as g(t) with t replaced by –f.

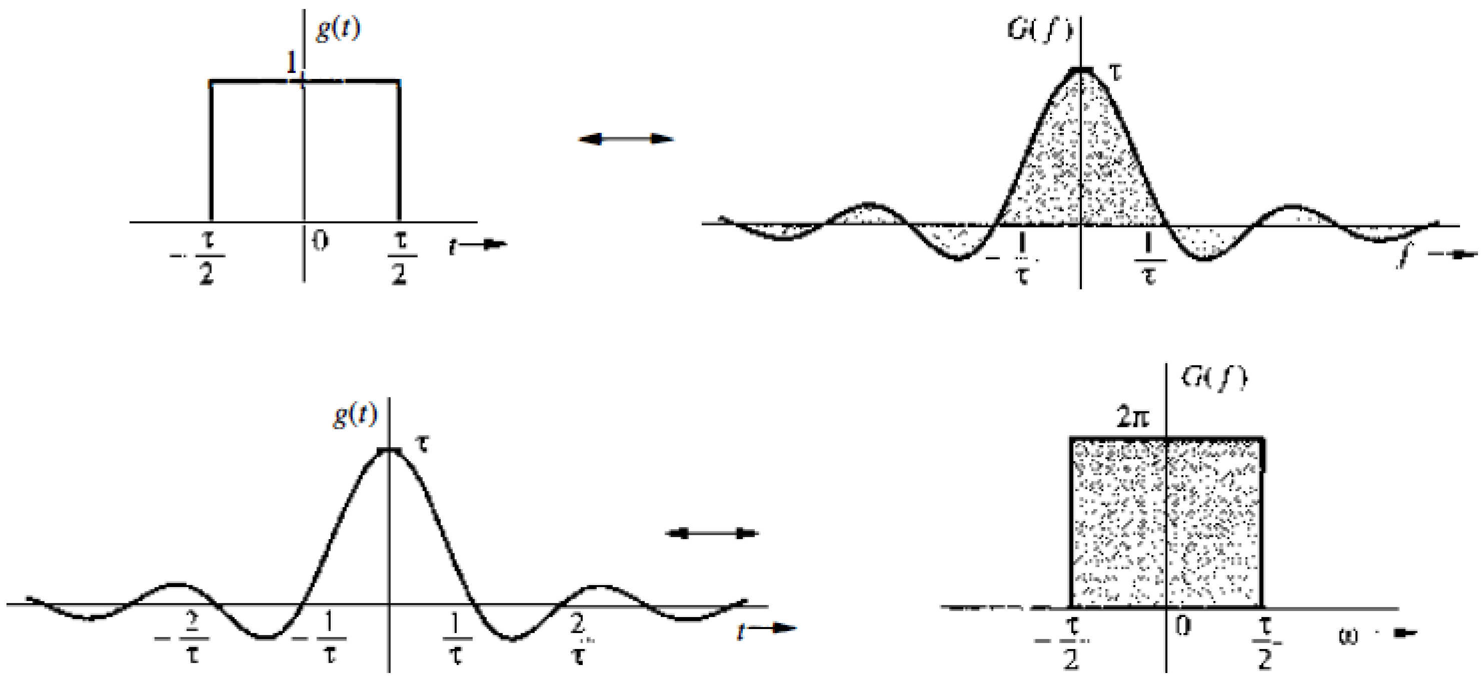
$$\underbrace{\alpha \operatorname{sinc}(\pi \alpha t)}_{G(t)} \Longleftrightarrow \underbrace{\Pi\left(-\frac{f}{\alpha}\right)}_{g(-f)} = \Pi\left(\frac{f}{\alpha}\right)$$

Properties of the Fourier Transform

- **Duality Property**:

- Π(-t) = Π(t) since Π(t) is an even function. Substituting τ = 2πα:

$$\tau \,\text{sinc}\left(\frac{\alpha t}{2}\right) \Longleftrightarrow 2\pi\, \Pi\left(\frac{2\pi f}{\tau}\right)$$



Properties of the Fourier Transform

- **Time-Scaling Property**:
- If: $g(t) \Longleftrightarrow G(f)$
- For any real constant a: $g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)$
- The function g(at) represents the function g(t) compressed in time by a factor a(|a|>1).
- The function G(f/a) represents the function G(f) expanded in frequency by the same factor a.
- For a < 0: $g(at) \Longleftrightarrow \frac{-1}{a} G\left(\frac{f}{a}\right)$
- Time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.

Properties of the Fourier Transform

- **Time-Scaling Property**:

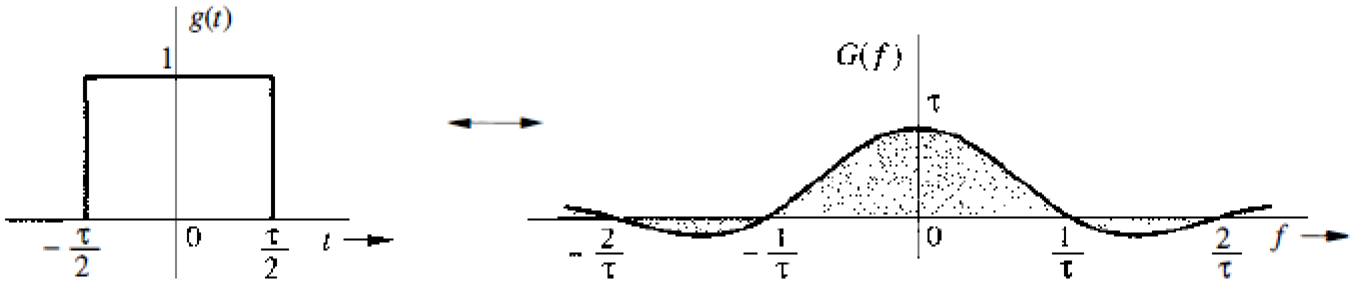
- Compression in time by a factor a means that the signal is varying more rapidly by the same factor.

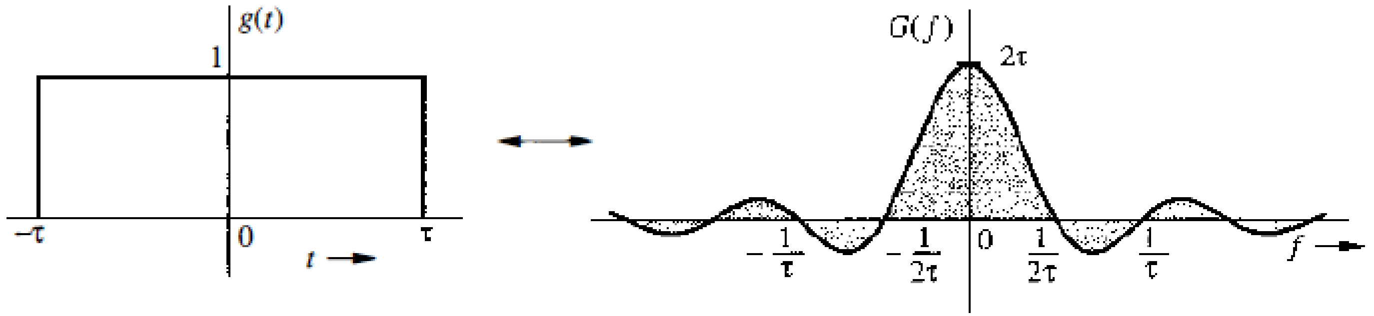
- To synthesize such a signal, the frequencies of its sinusoidal components must be increased by the factor a, implying that its frequency spectrum is expanded by the factor a.

- Similarly, a signal expanded in time varies more slowly; hence, the frequencies of its components are lowered, implying that its frequency spectrum is compressed.

Properties of the Fourier Transform

- **Time-Scaling Property**:





Properties of the Fourier Transform

- **Time-Scaling Property**:

- **Reciprocity of signal duration and its bandwidth:**

- The time-scaling property implies that if g(t) is wider, its spectrum is narrower, and vice versa.

- Doubling the signal duration halves its bandwidth, and vice versa.

- The bandwidth of a signal is inversely proportional to the signal duration or width (in seconds).

Properties of the Fourier Transform

- **Time-Shifting Property**:
- If: $g(t) \Longleftrightarrow G(f)$
- Then: $g(t - t_0) \Longleftrightarrow G(f)e^{-j2\pi f t_0}$
- *Proof*: By definition: $\mathcal{F}[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi f t}\, dt$
- Letting $t\text{-}t_0\text{=}x$:

$$\mathcal{F}[g(t-t_0)] = \int_{-\infty}^{\infty} g(x)e^{-j2\pi f(x+t_0)}\, dx$$

$$= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} g(x)e^{-j2\pi f x}\, dx = G(f)e^{-j2\pi f t_0}$$

- Delaying a signal by t_0 seconds does not change its amplitude spectrum.
- The phase spectrum, however, is changed by $-2\pi f t_0$. (linear function of f).
- Physical explanation for linear phase.

Properties of the Fourier Transform

- **Frequency-Shifting Property**:

- If: $g(t) \Longleftrightarrow G(f)$

- then: $g(t)e^{j2\pi f_0 t} \Longleftrightarrow G(f - f_0)$

- The **modulation property**.

- *Proof*:

$$\mathcal{F}[g(t)e^{j2\pi f_0 t}] = \int_{-\infty}^{\infty} g(t)e^{j2\pi f_0 t}e^{-j2\pi f t}\,dt = \int_{-\infty}^{\infty} g(t)e^{-j(2\pi f - 2\pi f_0)t}\,dt = G(f - f_0)$$

- Multiplication of a signal by a factor $e^{j2\pi f_0 t}$ shifts the spectrum

Properties of the Fourier Transform

- **Frequency-Shifting Property**:

- There is a duality between the time-shifting and the frequency-shiftir properties.

- Changing f_0 to - $f_{0:}$ $\quad g(t)e^{-j2\pi f_0 t} \Longleftrightarrow G(f + f_0)$

- Because $e^{j2\pi f_0 t}$ is not a real function that can be generated, frequen in practice is achieved by multiplying g(t) by a sinusoid:

$$g(t)\cos 2\pi f_0 t = \frac{1}{2}\left[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}\right]$$

- Then:

$$g(t)\cos 2\pi f_0 t \Longleftrightarrow \frac{1}{2}[G(f - f_0) + G(f + f_0)]$$

Properties of the Fourier Transform

- **Frequency-Shifting Property**:

- The multiplication of a signal g(t) by a sinusoid of frequency f_0 shifts the spectrum G(f) by $\pm f_0$.

- Multiplication of a sinusoid $\cos(2\pi f_0 t)$ by g(t) amounts to modulating the sinusoid amplitude.

- This type of modulation is known as **amplitude modulation.**

- The sinusoid $\cos(2\pi f_0 t)$ is called the **carrier**.

- The signal g(t) is the **modulating signal.**

- The signal $g(t)\cos(2\pi f_0 t)$ is the **modulated signal**.

Properties of the Fourier Transform

- **Frequency-Shifting Property**:

- To sketch a signal $g(t)\cos(2\pi f_0 t)$, we observe that:

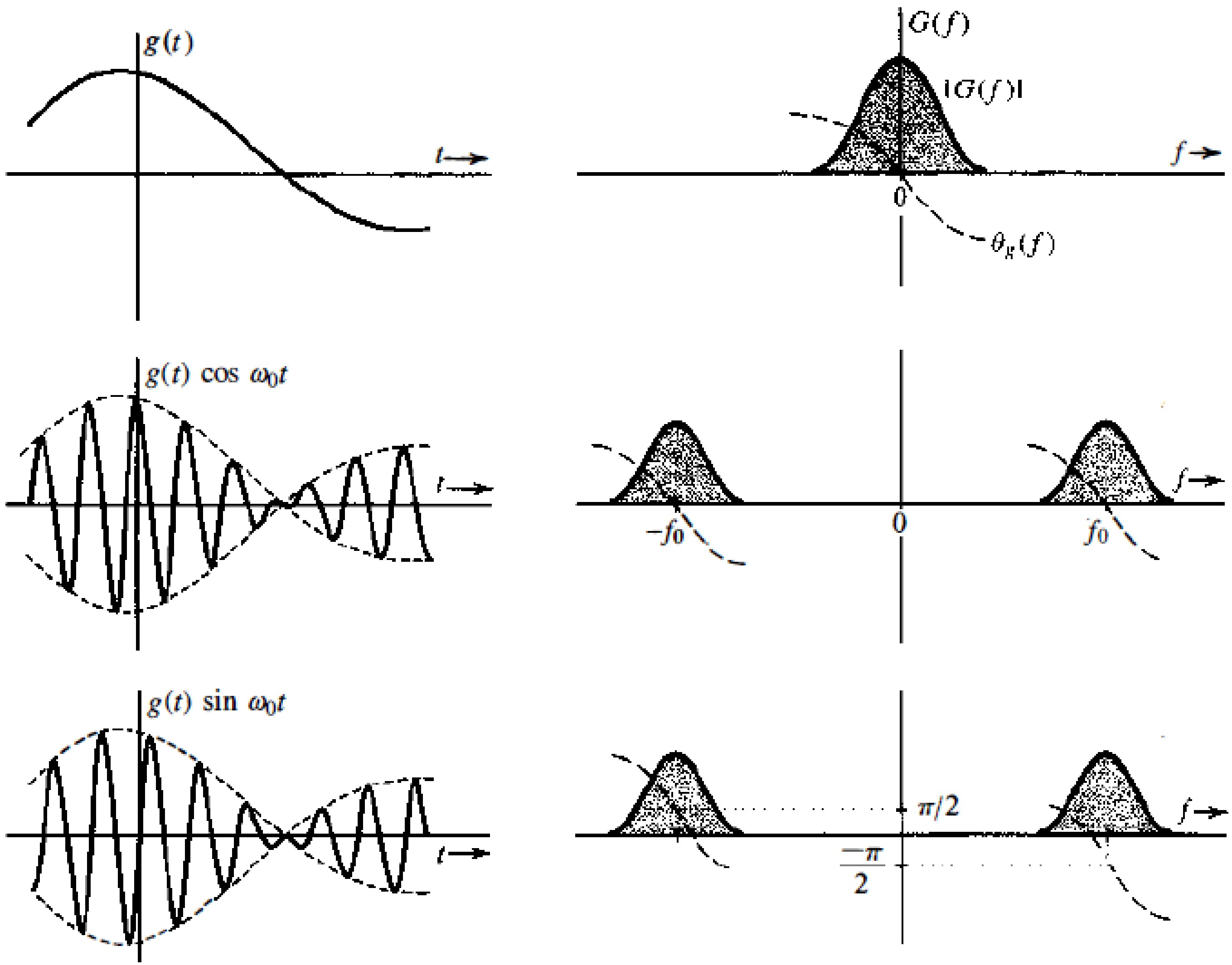
$$g(t)\cos 2\pi f_0 t = \begin{cases} g(t) & \text{when } \cos 2\pi f_0 t = 1 \\ -g(t) & \text{when } \cos 2\pi f_0 t = -1 \end{cases}$$

- $g(t)\cos(2\pi f_0 t)$ touches $g(t)$ when the sinusoid $\cos(2\pi f_0 t)$ is at its positive peaks and touches $-g(t)$ when $\cos(2\pi f_0 t)$ is at its negative peaks.

- $g(t)$ and $-g(t)$ act as **envelopes** for the signal $g(t)\cos(2\pi f_0 t)$.

- The signal $-g(t)$ is a mirror image of $g(t)$ about the horizontal axis.

Properties of the Fourier Transform



Properties of the Fourier Transform

- **Frequency-Shifting Property**:

- **Modulation** is a common application that shifts signal spectra.

- If several message signals (ex: radio signals), each occupying the same frequency band, are transmitted simultaneously over a common transmission medium, they will all interfere.

- This problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency.

- Each station transmits a modulated signal, thus shifting the signal spectrum to its allocated band, which is not occupied by any other station.

- A radio receiver can demodulate the signal. **Demodulation** consists of another spectral shift required to restore the signal to its original band.

Properties of the Fourier Transform

- **Convolution Theorem**:

- The convolution of two functions g(t) and w(t) is defined as:

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(\tau)w(t-\tau)\,d\tau$$

- If:

$$g_1(t) \Longleftrightarrow G_1(f) \qquad\qquad g_2(t) \Longleftrightarrow G_2(f)$$

- Then:

- Time convolution: $g_1(t) * g_2(t) \Longleftrightarrow G_1(f)G_2(f)$

- Frequency convolution: $g_1(t)g_2(t) \Longleftrightarrow G_1(f) * G_2(f)$

Properties of the Fourier Transform

Properties of Fourier Transform Operations

Operation	$g(t)$	$G(f)$
Superposition	$g_1(t) + g_2(t)$	$G_1(f) + G_2(f)$
Scalar multiplication	$kg(t)$	$kG(f)$
Duality	$G(t)$	$g(-f)$
Time scaling	$g(at)$	$\frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$
Time shifting	$g(t - t_0)$	$G(f)e^{-j2\pi f t_0}$
Frequency shifting	$g(t)e^{j2\pi f_0 t}$	$G(f - f_0)$
Time convolution	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
Frequency convolution	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
Time differentiation	$\frac{d^n g(t)}{dt^n}$	$(j2\pi f)^n G(f)$
Time integration	$\int_{-\infty}^{t} g(x)\, dx$	$\frac{G(f)}{j2\pi f} + \frac{1}{2} G(0)\delta(f)$