Matrix Chain Multiplication

Dynamic Programming

Matrix Chain Multiplication

Dot Product

$$\begin{bmatrix} 1 & 5 & 9 & 7 & 3 & 4 \\ 2 & 1 & 9 & 7 & 2 & 6 \\ 9 & 5 & 2 & 2 & 3 & 5 \\ 6 & 6 & 1 & 3 & 1 & 7 \end{bmatrix} \times \begin{bmatrix} 5 & 1 & 3 \\ 9 & 5 & 1 \\ 8 & 7 & 6 \\ 9 & 6 & 8 \\ 8 & 1 & 3 \\ 2 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 217 & 142 & 163 \\ 182 & 126 & 177 \\ 158 & 73 & 114 \\ 141 & 76 & 120 \end{bmatrix}$$

4x6   6x3   4x3

72 Multiplications in Total!

(4x6x3)

Matrix Chain Multiplication

$$\begin{array}{ccccccccc} A_1 & \times & A_2 & \times & A_3 & \times & A_4 & \times & A_5 \\ 4x10 & & 10x3 & & 3x12 & & 12x20 & & 20x7 \end{array}$$

4x10x3 + 4x3x12 + 4x12x20 + 4x20x7 = 1784 Multiplication Operations

Very Large Computational Times !!

Matrix Chain Multiplication

$$A_1 \times A_2 \times A_3 \times A_4 \times A_5$$

4x10	10x3	3x12	12x20	20x7

Goal: Find the optimal way to multiply these matrices to perform the fewest multiplications.

Naïve Approach: Try them all, and pick the most optimal one.

Running time: $\Omega(4^n/n^{3/2})$ - 4^n dominates! Exponential

Matrix Chain Multiplication

There is a better way! Dynamic Programming!

Step 1: Check if the problem has Optimal Substructure

If we have an optimal solution for $A_{i...j}$

Assume the solution has the following parentheses:

$$\underline{(A_{i...k})}\underline{(A_{k+1...j})}$$

If there is a better way to multiply $(A_{i...k})$, then we would have a more optimal solution.
This would be a contradiction, as we already stated that we have the optimal solution for $A_{i...j}$
Therefore this problem has optimal substructure.

Matrix Chain Multiplication

A matrix series $A_{i...j}$ can be broken up into a more efficient solution:

$$(A_{i...k})(A_{k+1...j})$$

We want to find out at which 'k' returns the fewest number of multiplications

We need to define our recursive formula:
M[i,j] is the cost of multiplying matrices from A_i to A_j

Matrix Chain Multiplication

Now we want to try out a bunch of values for ‘k’ in order to see what the best one is:

$$M[i,j] = \underbrace{M[i,k]}_{100} + \underbrace{M[k+1,j]}_{200} + \underbrace{p_{i-1}p_kp_j}_{2\text{x}3\text{x}4}$$

Since we don’t know what k is, we try this range of k:

$i \leq k < j$

$$\overset{100}{\underset{2\text{x}3}{(A_{i\ldots k})}}\overset{200}{\underset{3\text{x}4}{(A_{k+1\ldots j})}}$$

The minimum returned value is our solution!

Our Final Recursive Formula:

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1					
2	x				
3	x	x			
4	x	x	x		
5	x	x	x	x	

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} \boxed{0 \quad \text{if} \quad i=j} \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

Step 1: Fill the table for i = j

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} \boxed{0 \quad \text{if} \quad i=j} \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

Step 2: Fill the table for:

i=1, j=2

i=2, j=3

i=3, j=4

i=4, j=5

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

Matrix Chain Multiplication

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{p_0\ p_1}{\underset{4x10}{A_1}} \times \underset{p_1\ p_2}{\underset{10x3}{A_2}} \times \underset{p_2\ p_3}{\underset{3x12}{A_3}} \times \underset{p_3\ p_4}{\underset{12x20}{A_4}} \times \underset{p_4\ p_5}{\underset{20x7}{A_5}}$$

i \ j	1	2	3	4	5
1	0				
2	x	0			
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$$M[1,2] = \min_{1 \leq k < 2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$$

$$M[1,2] = \min_{1 \leq k < 2} \{0 + 0 + 4x10x3\}$$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 \quad \text{if} \quad i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{\substack{4x10 \\ p_0\ p_1}}{A_1} \times \underset{\substack{10x3 \\ p_1\ p_2}}{A_2} \times \underset{\substack{3x12 \\ p_2\ p_3}}{A_3} \times \underset{\substack{12x20 \\ p_3\ p_4}}{A_4} \times \underset{\substack{20x7 \\ p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0		
4	x	x	x	0	
5	x	x	x	x	0

$M[1,2] = \min_{1 \leq k < 2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$

$M[1,2] = \min_{1 \leq k < 2} \{0 + 0 + 4x10x3\}$

$M[1,2] = 120$

$M[2,3] = \min_{2 \leq k < 3} \{M[2,2] + M[2+1,3] + p_1p_2p_3\}$

$M[2,3] = \min_{2 \leq k < 3} \{0 + 0 + 10x3x12\}$

$M[2,3] = 360$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
$p_0\ p_1$	$p_1\ p_2$	$p_2\ p_3$	$p_3\ p_4$	$p_4\ p_5$

i \ j	1	2	3	4	5
1	0	120			
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,2] = \min_{1 \le k < 2} \{M[1,1] + M[1+1,2] + p_0p_1p_2\}$

$M[1,2] = \min_{1 \le k < 2} \{0 + 0 + 4x10x3\}$

$M[1,2] = 120$

$M[2,3] = \min_{2 \le k < 3} \{M[2,2] + M[2+1,3] + p_1p_2p_3\}$

$M[2,3] = \min_{2 \le k < 3} \{0 + 0 + 10x3x12\}$

$M[2,3] = 360$

$M[3,4] = \min_{3 \le k < 4} \{M[3,3] + M[3+1,4] + p_2p_3p_4\}$

$M[3,4] = \min_{3 \le k < 4} \{0 + 0 + 3x12x20\}$

$M[2,3] = 720$

$M[4,5] = \min_{4 \le k < 5} \{M[4,4] + M[4+1,5] + p_3p_4p_5\}$

$M[4,5] = \min_{4 \le k < 5} \{0 + 0 + 12x20x7\}$

$M[1,2] = 1680$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360		
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,3] = \min_{1 \le k < 3}$

k=1

$= M[1,1] + M[1+1,3] + p_0p_1p_3$

$= 0 + 360 + 4x10x12$

$= 840$

k=2

$= M[1,2] + M[2+1,3] + p_0p_2p_3$

$= 120 + 0 + 4x3x12$

$= 264$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
$p_0\ p_1$	$p_1\ p_2$	$p_2\ p_3$	$p_3\ p_4$	$p_4\ p_5$

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	
4	x	x	x	0	1680
5	x	x	x	x	0

$M[2,4] = \min_{2 \le k < 4}$

k=2

$= M[2,2] + M[2+1,4] + p_1p_2p_4$

$= 0 + 720 + 10x3x20$

$= 1320$

k=3

$= M[2,3] + M[3+1,4] + p_1p_3p_4$

$= 360 + 0 + 10x12x20$

$= 2760$

computer science BREAKDOWN with Karim Hamasni

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264		
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[3,5] = \min_{3 \le k < 5}$

k=3

$= M[3,3] + M[3+1,5] + p_2p_3p_5$

$= 0 + 1680 + 3x12x7$

$= 1932$

k=4

$= M[3,4] + M[4+1,5] + p_2p_4p_5$

$= 720 + 0 + 3x20x7$

$= 1140$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,4] = \min_{1 \le k < 4}$

k=1

$= M[1,1] + M[1+1,4] + p_0p_1p_4$

$= 0 + 1320 + 4x10x20$

$= 2120$

k=2

$= M[1,2] + M[2+1,4] + p_0p_2p_4$

$= 120 + 720 + 4x3x20$

$= 1080 \leftarrow$

k=3

$= M[1,3] + M[3+1,4] + p_0p_3p_4$

$= 264 + 0 + 4x12x20$

$= 1224$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$A_1 \times A_2 \times A_3 \times A_4 \times A_5$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

i \ j	1	2	3	4	5
1	0	120	264	1080	
2	x	0	360	1320	1350
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[2,5] = \min_{2 \leq k < 5}$

k=2

$= M[2,2] + M[2+1,5] + p_1p_2p_5$

$= 0 + 1140 + 10 \times 3 \times 7$

$= 1350$

k=3

$= M[2,3] + M[3+1,5] + p_1p_3p_5$

$= 360 + 1680 + 10 \times 12 \times 7$

$= 2880$

k=4

$= M[2,4] + M[4+1,5] + p_1p_4p_5$

$= 1320 + 0 + 10 \times 20 \times 7$

$= 2720$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

We want to start with i = j, then i<j starting with a spread of 1, working our way up

$$\underset{\substack{4\text{x}10 \\ p_0\ p_1}}{A_1} \times \underset{\substack{10\text{x}3 \\ p_1\ p_2}}{A_2} \times \underset{\substack{3\text{x}12 \\ p_2\ p_3}}{A_3} \times \underset{\substack{12\text{x}20 \\ p_3\ p_4}}{A_4} \times \underset{\substack{20\text{x}7 \\ p_4\ p_5}}{A_5}$$

i \ j	1	2	3	4	5
1	0	120	264	1080	1344
2	x	0	360	1320	1350
3	x	x	0	720	1140
4	x	x	x	0	1680
5	x	x	x	x	0

$M[1,5] = \min_{1 \leq k < 5}$

k=1

$= M[1,1] + M[1+1,5] + p_0p_1p_5$

$= 0 + 1350 + 4\text{x}10\text{x}7$

$= 1630$

k=2

$= M[1,2] + M[2+1,5] + p_0p_2p_5$

$= 120 + 1140 + 4\text{x}3\text{x}7$

$= 1344$

k=3

$= M[1,3] + M[3+1,5] + p_0p_3p_5$

$= 264 + 1680 + 4\text{x}12\text{x}7$

$= 2280$

k=4

$= M[1,4] + M[4+1,5] + p_0p_4p_5$

$= 1080 + 0 + 4\text{x}20\text{x}7$

$= 1640$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$\underset{\substack{4x10 \\ p_0\ p_1}}{A_1} \times \underset{\substack{10x3 \\ p_1\ p_2}}{A_2} \times \underset{\substack{3x12 \\ p_2\ p_3}}{A_3} \times \underset{\substack{12x20 \\ p_3\ p_4}}{A_4} \times \underset{\substack{20x7 \\ p_4\ p_5}}{A_5}$$

We now know that we can multiply A_1 to A_5 in as few as 1344 multiplication operations!

But where do we put our brackets?

We must focus on the selected k values

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } \; i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

$$\underset{\substack{4x10 \\ p_0 \; p_1}}{A_1} \times \underset{\substack{10x3 \\ p_1 \; p_2}}{A_2} \times \underset{\substack{3x12 \\ p_2 \; p_3}}{A_3} \times \underset{\substack{12x20 \\ p_3 \; p_4}}{A_4} \times \underset{\substack{20x7 \\ p_4 \; p_5}}{A_5}$$

k=2

$M[1,5] = M[1,2] + M[3,5] + p_0p_2p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \leq k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$(A_1 \times A_2)(A_3 \times A_4 \times A_5)$$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
p_0 p_1	p_1 p_2	p_2 p_3	p_3 p_4	p_4 p_5

k=2

$$M[1,5] = M[1,2] + M[3,5] + p_0p_2p_5$$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_kp_j\} \end{cases}$$

$$(A_1 \times A_2)((A_3 \times A_4)\ A_5)$$

A_1	A_2	A_3	A_4	A_5
4x10	10x3	3x12	12x20	20x7
$p_0\ p_1$	$p_1\ p_2$	$p_2\ p_3$	$p_3\ p_4$	$p_4\ p_5$

k=2

$M[1,5] = \underline{M[1,2]} + \underline{M[3,5]} + p_0p_2p_5$

k=4

$M[3,5] = M[3,4] + M[5,5] + p_2p_4p_5$

Matrix Chain Multiplication

$$M[i,j] = \begin{cases} 0 & \text{if } i=j \\ \min_{i \le k < j} \{M[i,j] = M[i,k] + M[k+1,j] + p_{i-1}p_k p_j\} \end{cases}$$

Check our work!

$$(A_1 \times A_2)((A_3 \times A_4)\ A_5)$$

$$\begin{array}{ccccc} 4\text{x}10 & 10\text{x}3 & 3\text{x}12 & 12\text{x}20 & 20\text{x}7 \end{array}$$

$$\left(\begin{array}{c} 120 \\ 4\text{x}3 \end{array}\right)\left(\begin{array}{cc} 720 & \\ 3\text{x}20 & 20\text{x}7 \end{array}\right)$$

$$\begin{array}{cc} & 420 \\ 4\text{x}3 & 3\text{x}7 \end{array}$$

$$\begin{array}{c} 84 \\ 4\text{x}7 \end{array}$$

$$120 + 720 + 420 + 84 = 1344$$

Matrix Chain Multiplication

Example: Given a chain of four matrices A_1, A_2, A_3 and A_4, with $p_0 = 5$, $p_1 = 4$, $p_2 = 6$, $p_3 = 2$ and $p_4 = 7$. Find $m[1, 4]$.