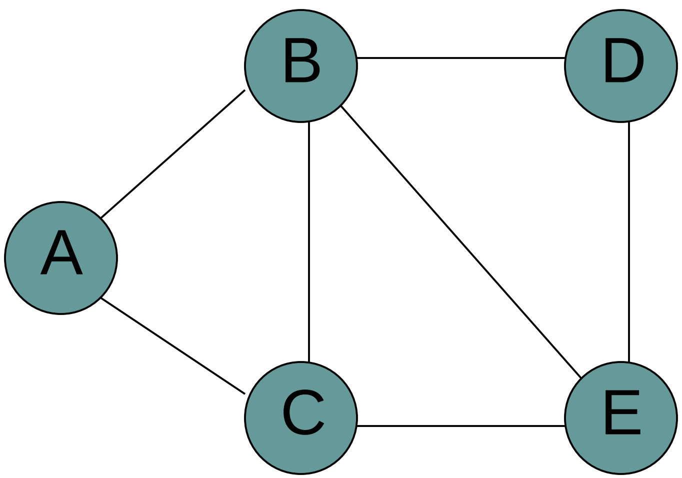
Graphs: MSTs and Shortest Paths

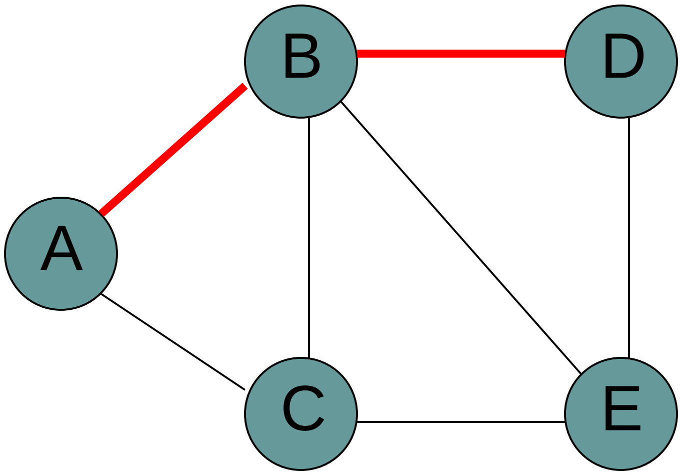
Shortest paths

- What is the shortest path from a to d?



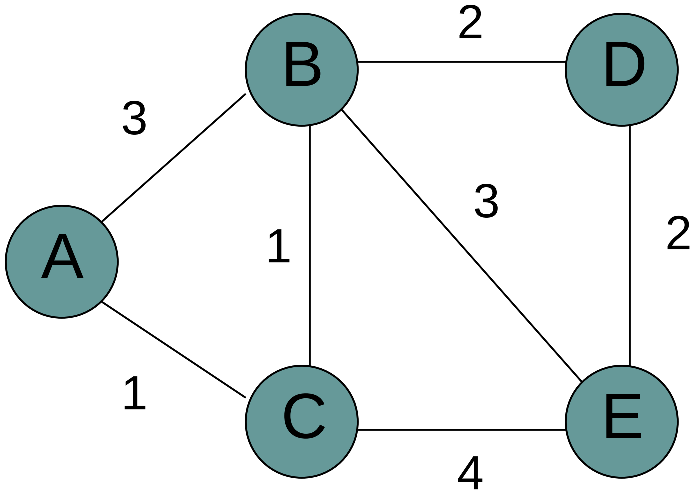
Shortest paths

- BFS



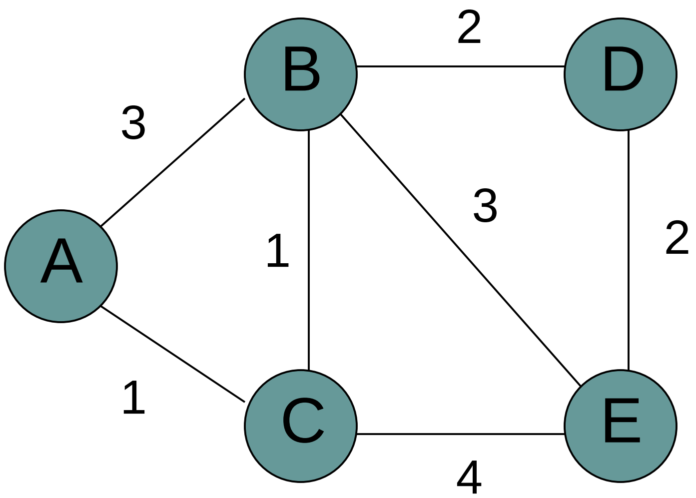
Shortest paths

- What is the shortest path from a to d?



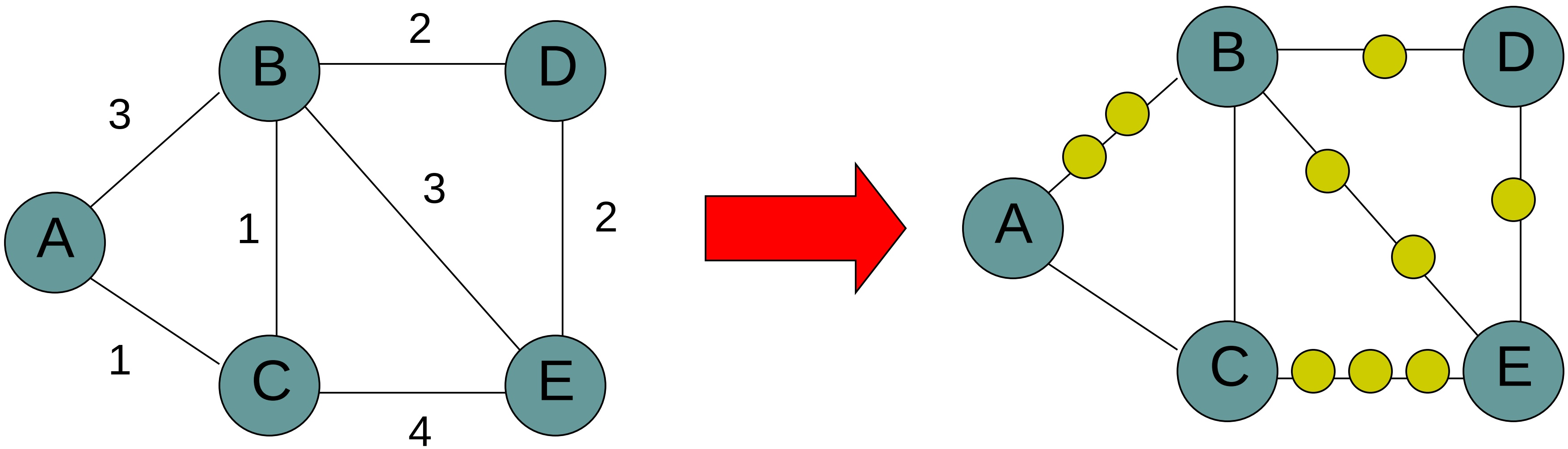
Shortest paths

- We can still use BFS



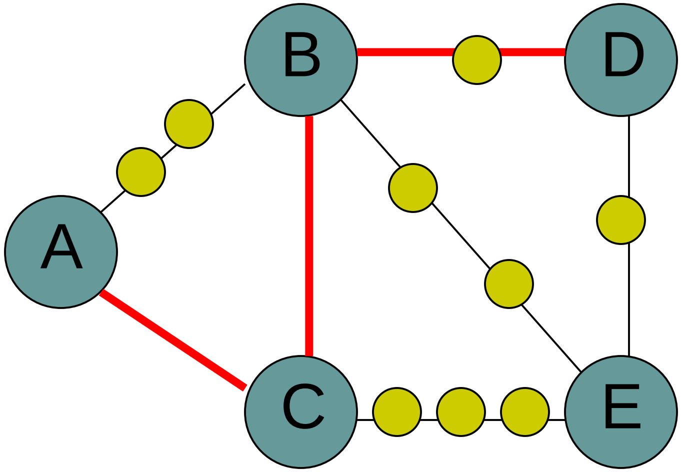
Shortest paths

- We can still use BFS



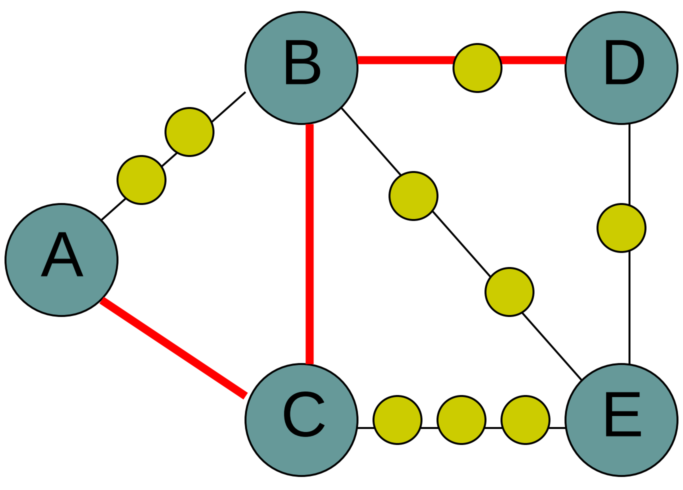
Shortest paths

- We can still use BFS



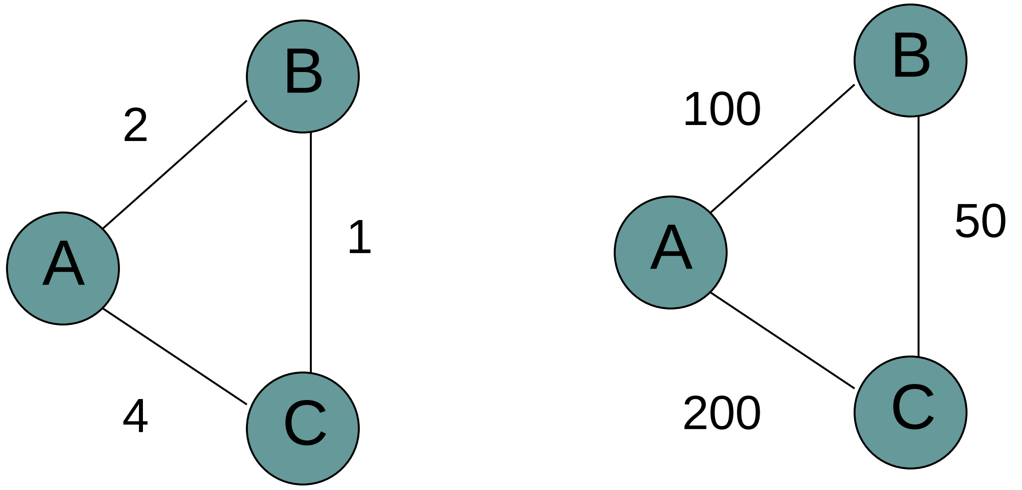
Shortest paths

- What is the problem?

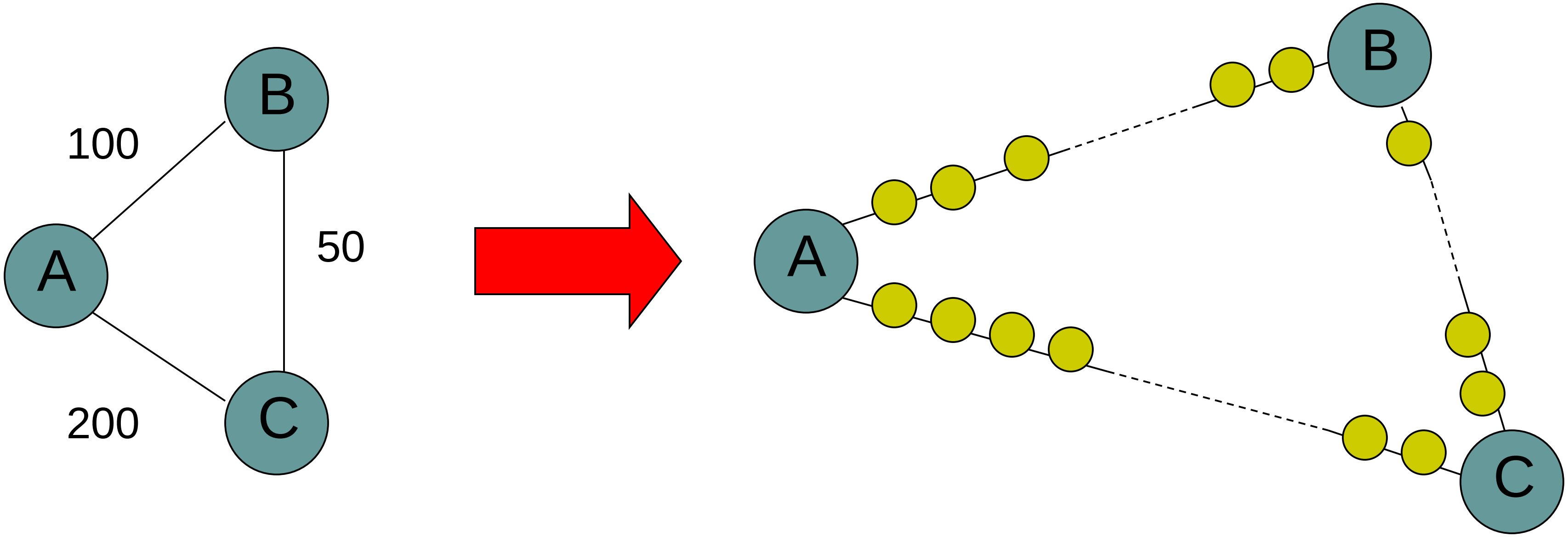


Shortest paths

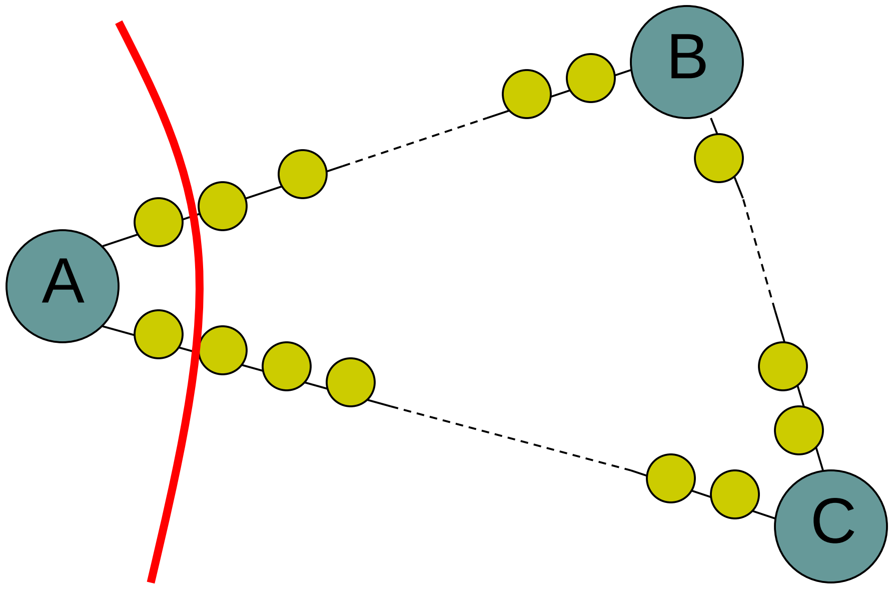
- Running time is dependent on the weights



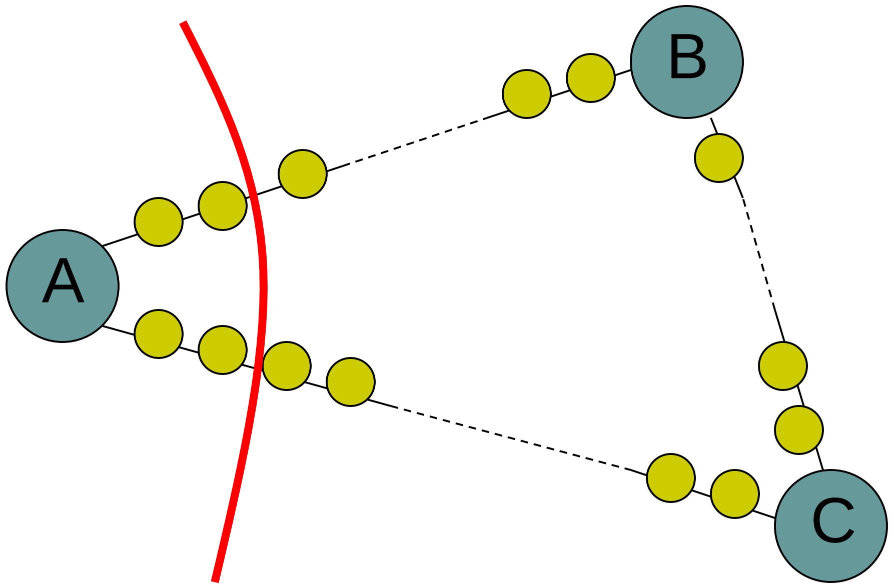
Shortest paths



Shortest paths

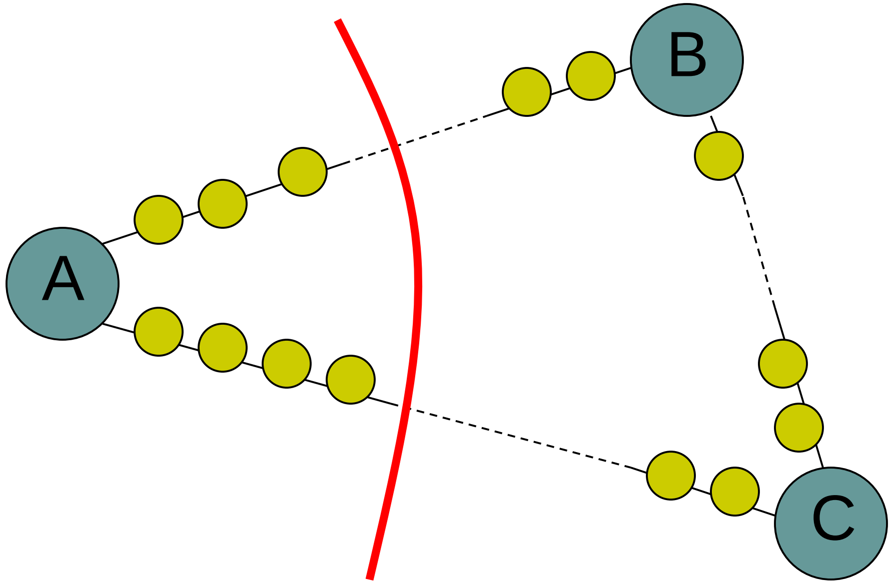


Shortest paths



Shortest paths

- Nothing will change as we expand the frontier until we've gone out 100 levels



Dijkstra’s algorithm

$\textsc{Dijkstra}(G, s)$

for all $v \in V$
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow null$
$dist[s] \leftarrow 0$
$Q \leftarrow \textsc{MakeHeap}(V)$
while $!\textsc{Empty}(Q)$
    $u \leftarrow \textsc{ExtractMin}(Q)$
    for all edges $(u, v) \in E$
        if $dist[v] > dist[u] + w(u, v)$
            $dist[v] \leftarrow dist[u] + w(u, v)$
            $\textsc{DecreaseKey}(Q, v, dist[v])$
            $prev[v] \leftarrow u$

Dijkstra's algorithm

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(U)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```

Dijkstra's algorithm

prev keeps track of the shortest path

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(U)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```

Dijkstra's algorithm

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(U)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```

Dijkstra's algorithm

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(U)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```

Dijkstra's algorithm

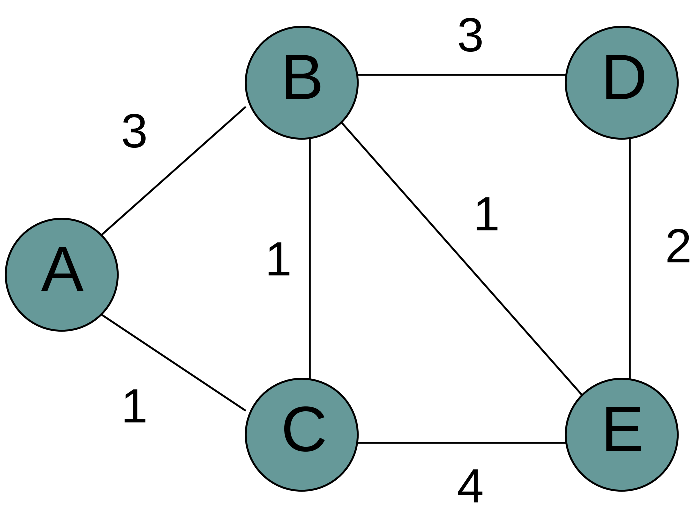
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

```
BFS(G, s)
for each v ∈ V
        dist[v] = ∞
dist[s] = 0
ENQUEUE(Q, s)
while !EMPTY(Q)
        u ← DEQUEUE(Q)
        VISIT(U)
        for each edge (u, v) ∈ E
                if dist[v] = ∞
                        ENQUEUE(Q, v)
                        dist[v] ← dist[u] + 1
```

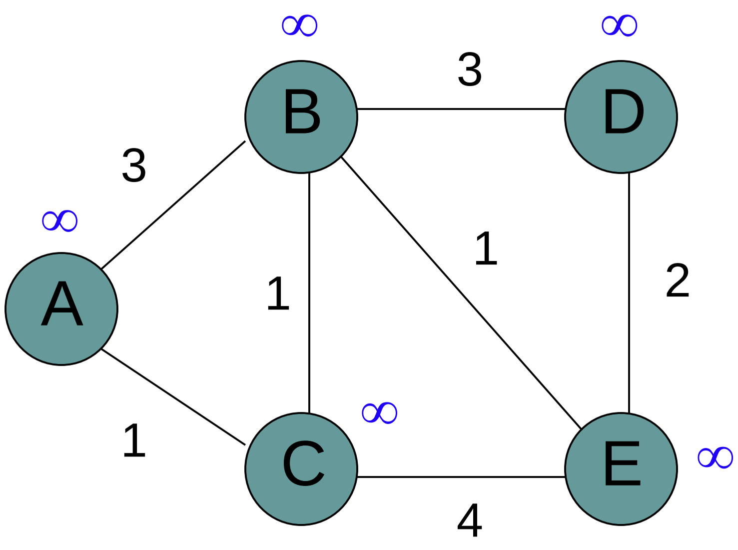
Single source shortest paths

- All of the shortest path algorithms we'll look at today are call "single source shortest paths" algorithms
- Why?

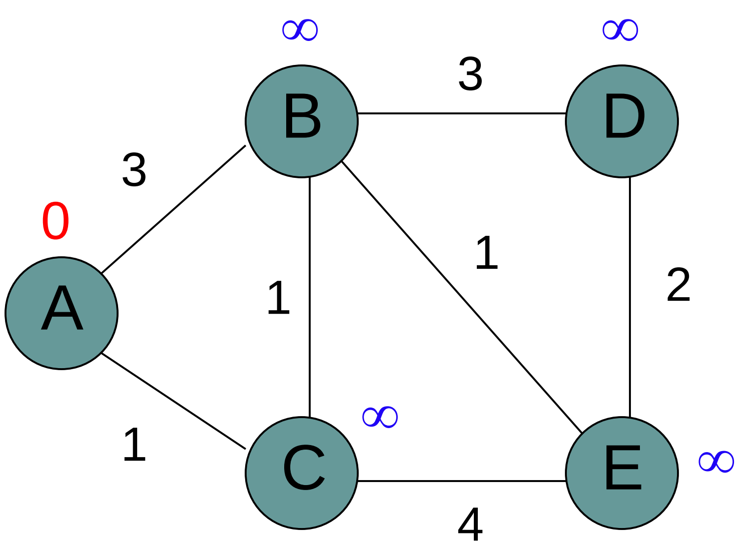
```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



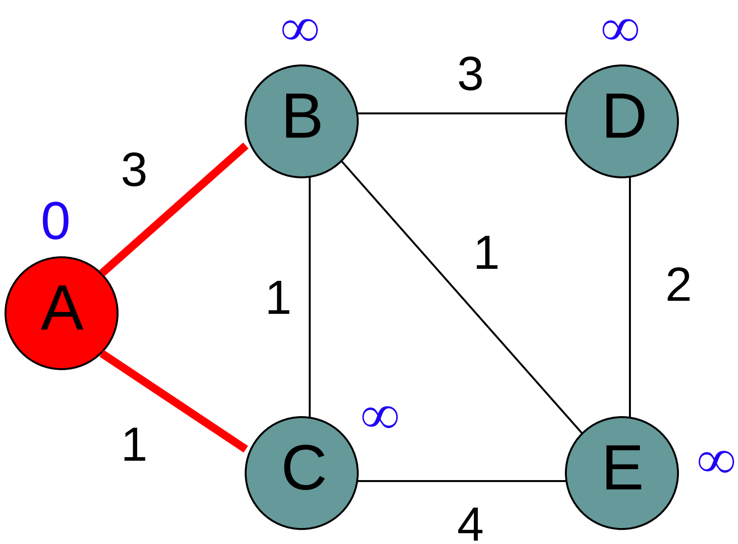
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



Heap

A	0
B	∞
C	∞
D	∞
E	∞

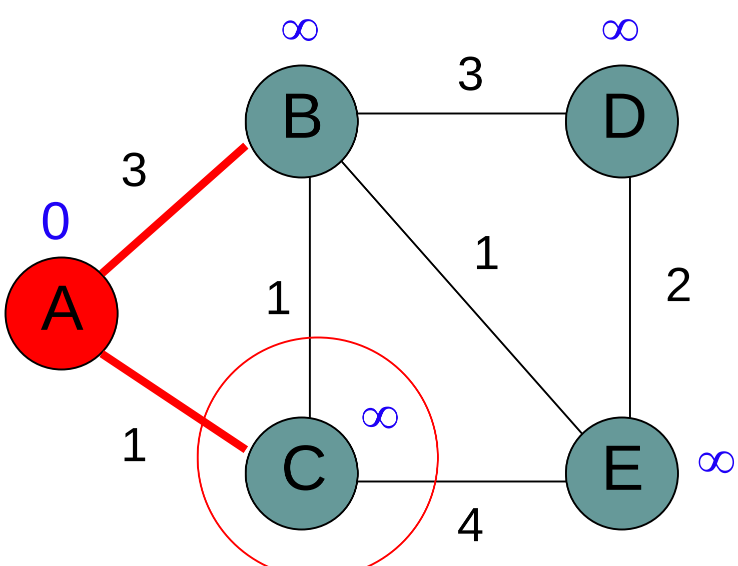
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



Heap

B ∞
C ∞
D ∞
E ∞

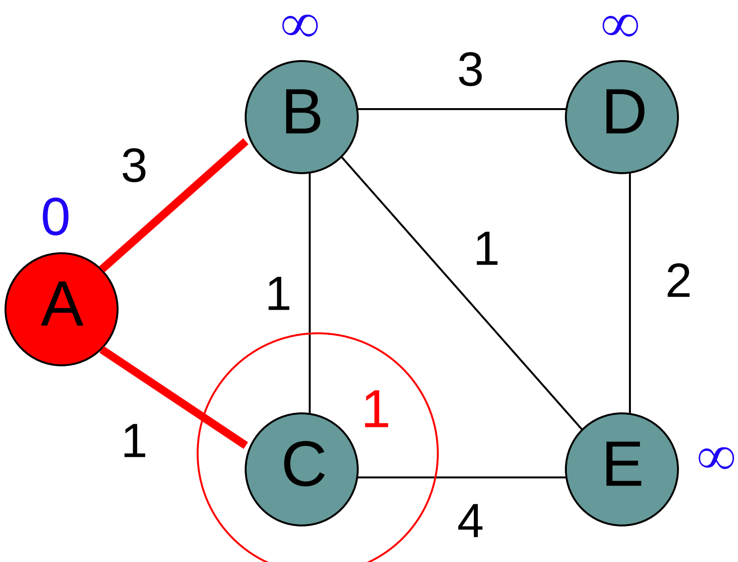
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



Heap

B	∞
C	∞
D	∞
E	∞

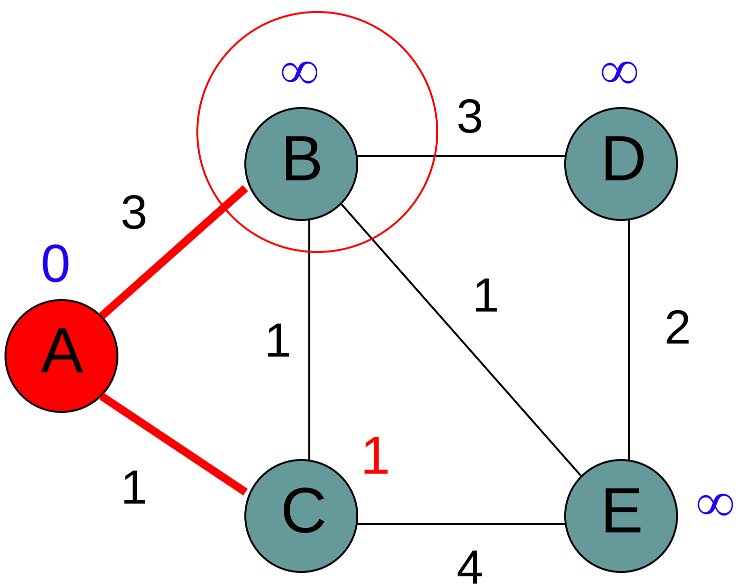
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



Heap

C 1
B ∞
D ∞
E ∞

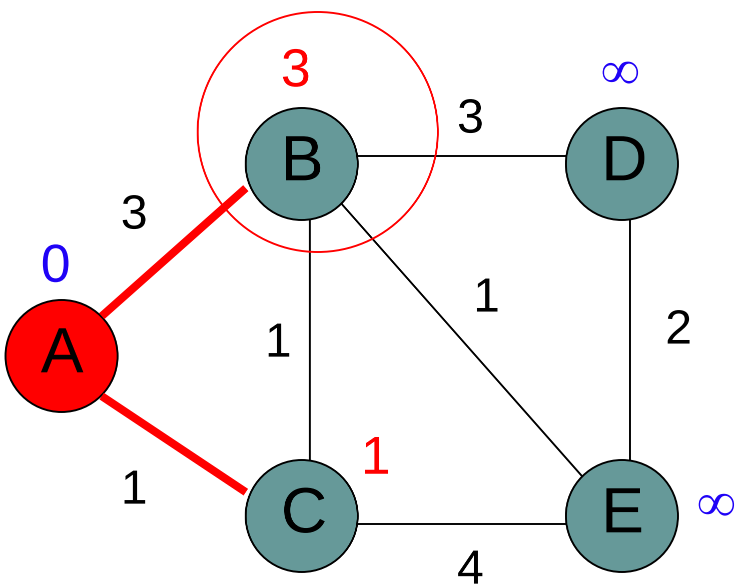
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```



Heap

C 1
B ∞
D ∞
E ∞

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

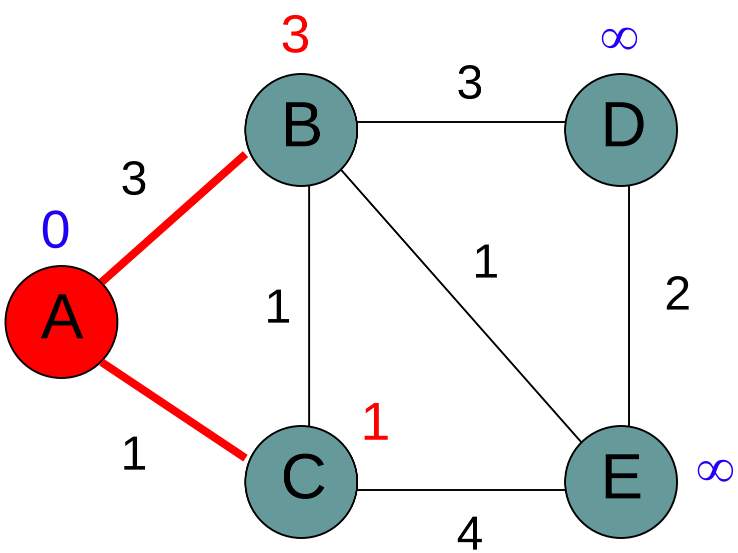


Heap

C 1
B 3
D ∞
E ∞

Dijkstra(G, s)

```
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```



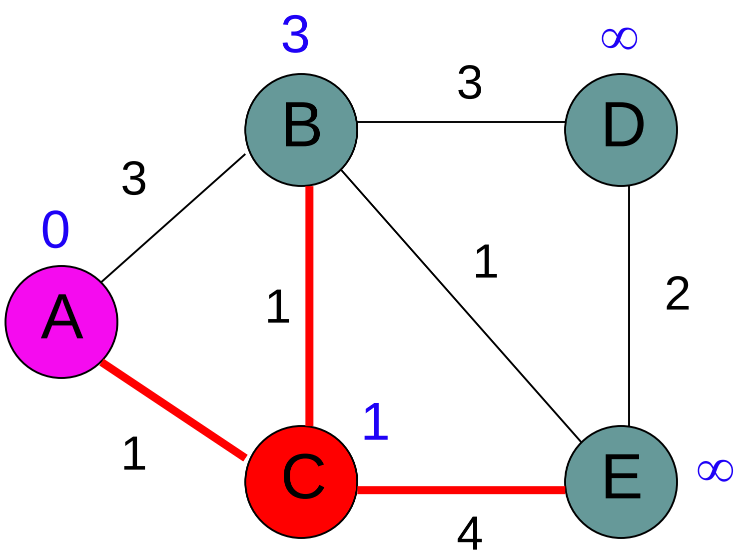
Heap

C	1
B	3
D	∞
E	∞

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

B	3
D	∞
E	∞



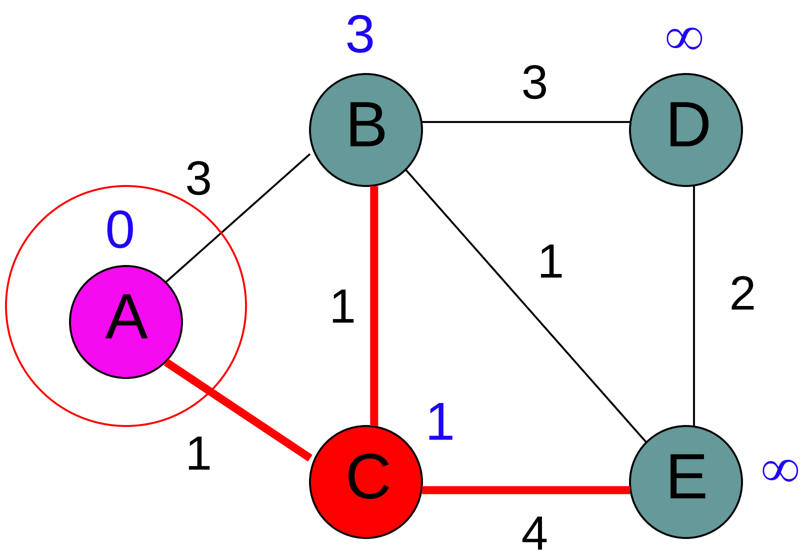
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

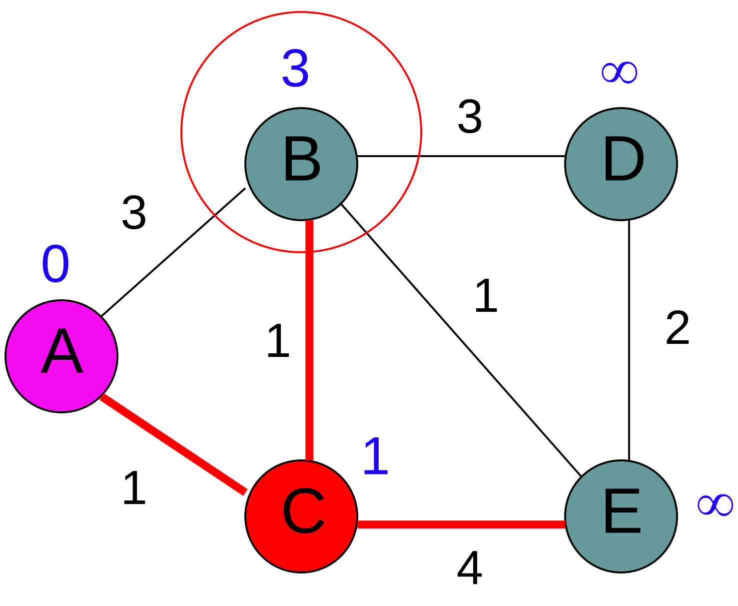
B 3

D ∞

E ∞



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

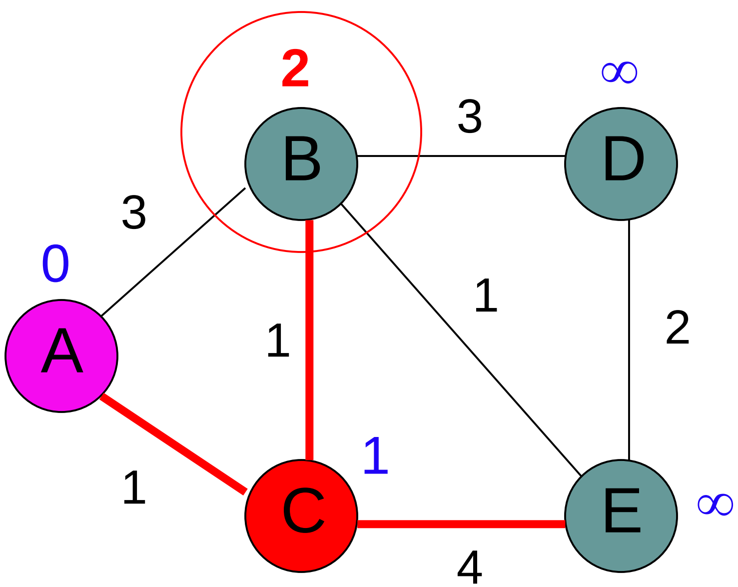


Heap	
B	3
D	∞
E	∞

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

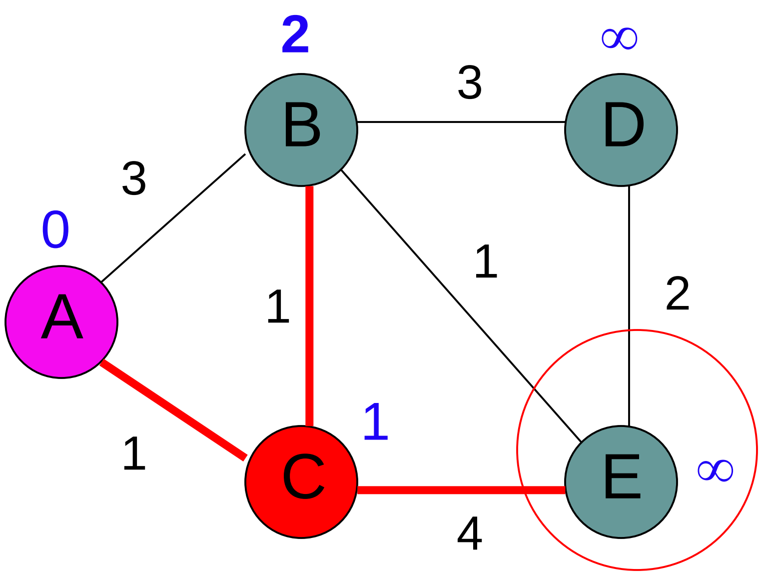
Heap

B 2
D ∞
E ∞



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

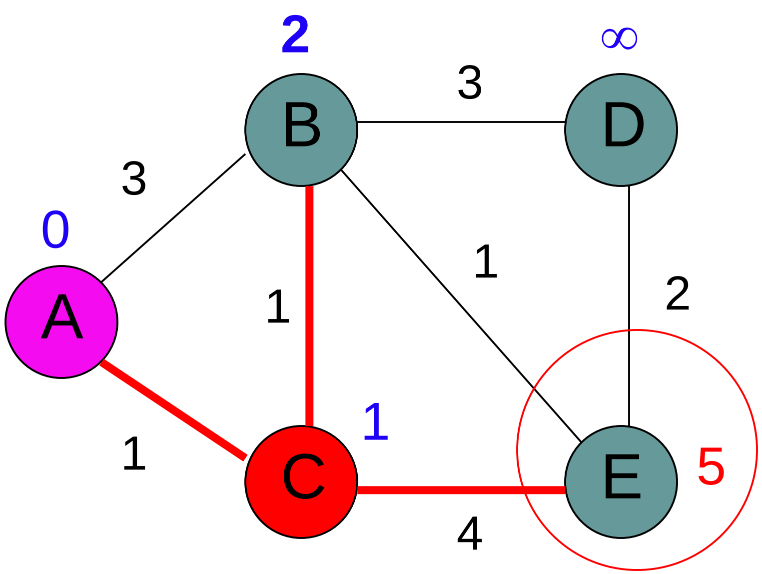
Heap	
B	2
D	∞
E	∞



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

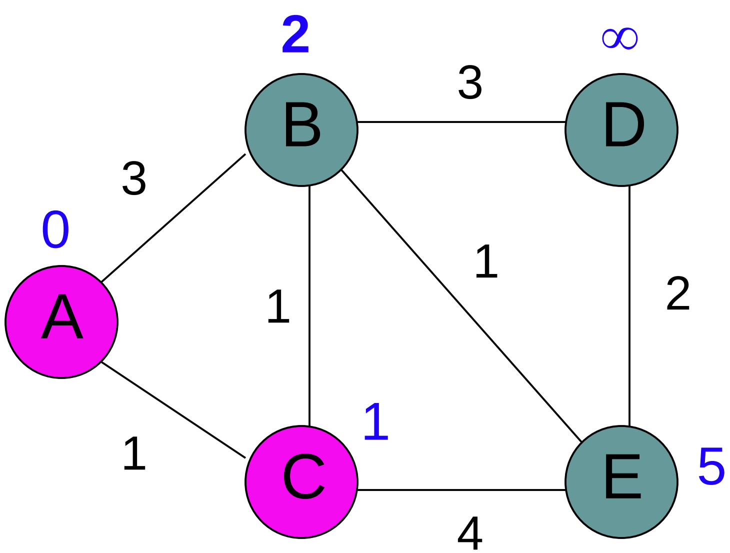
Heap

B	2
E	5
D	∞



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap	
B	2
E	5
D	∞

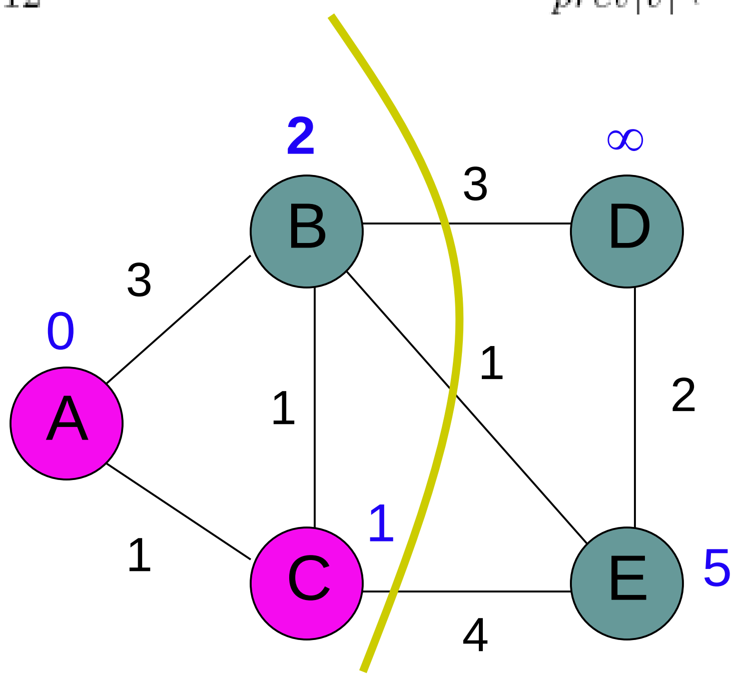


Frontier?

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

B	2
E	5
D	∞

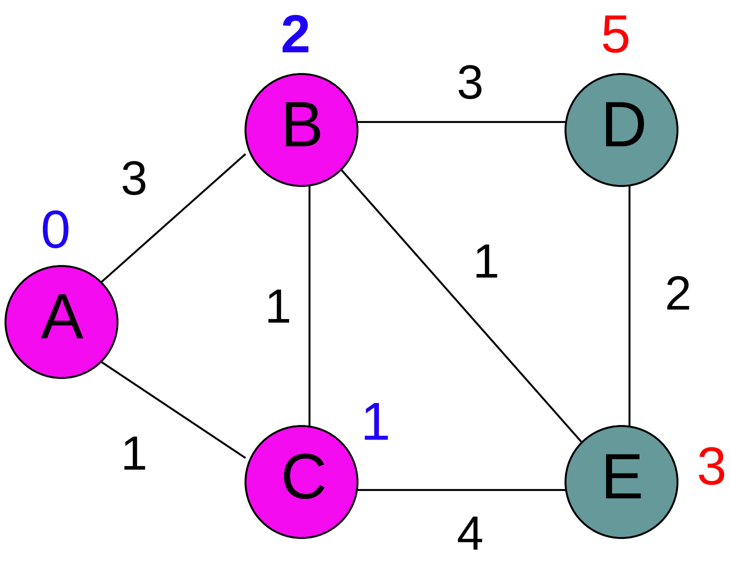


All nodes reachable from starting node within a given distance

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

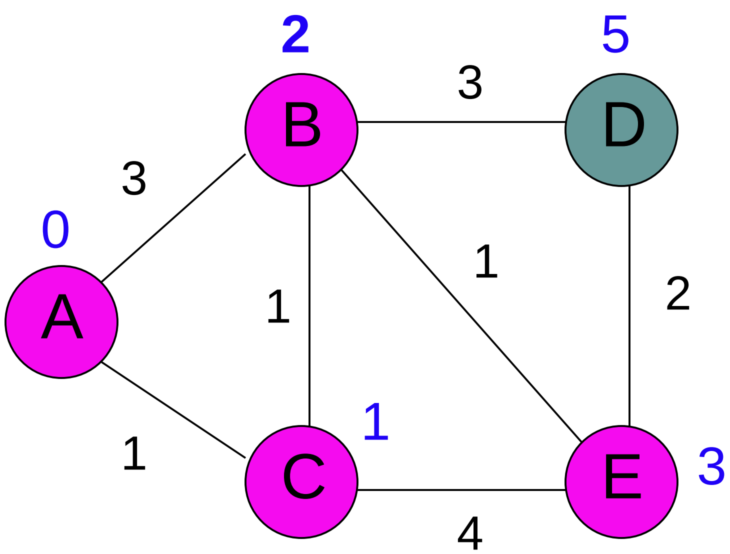
E 3
D 5



```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

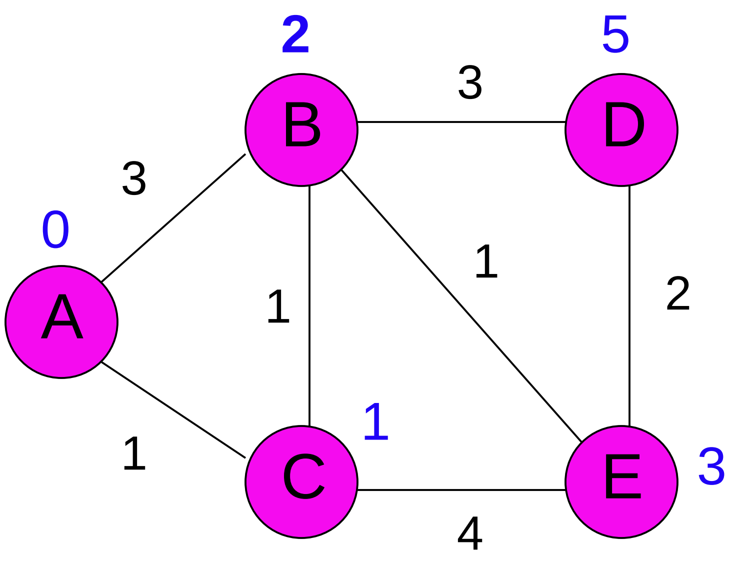
Heap

D 5



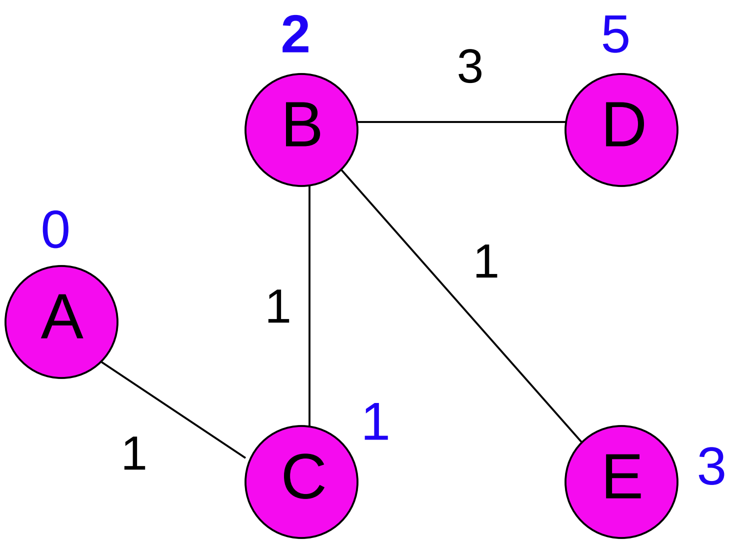
```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Heap

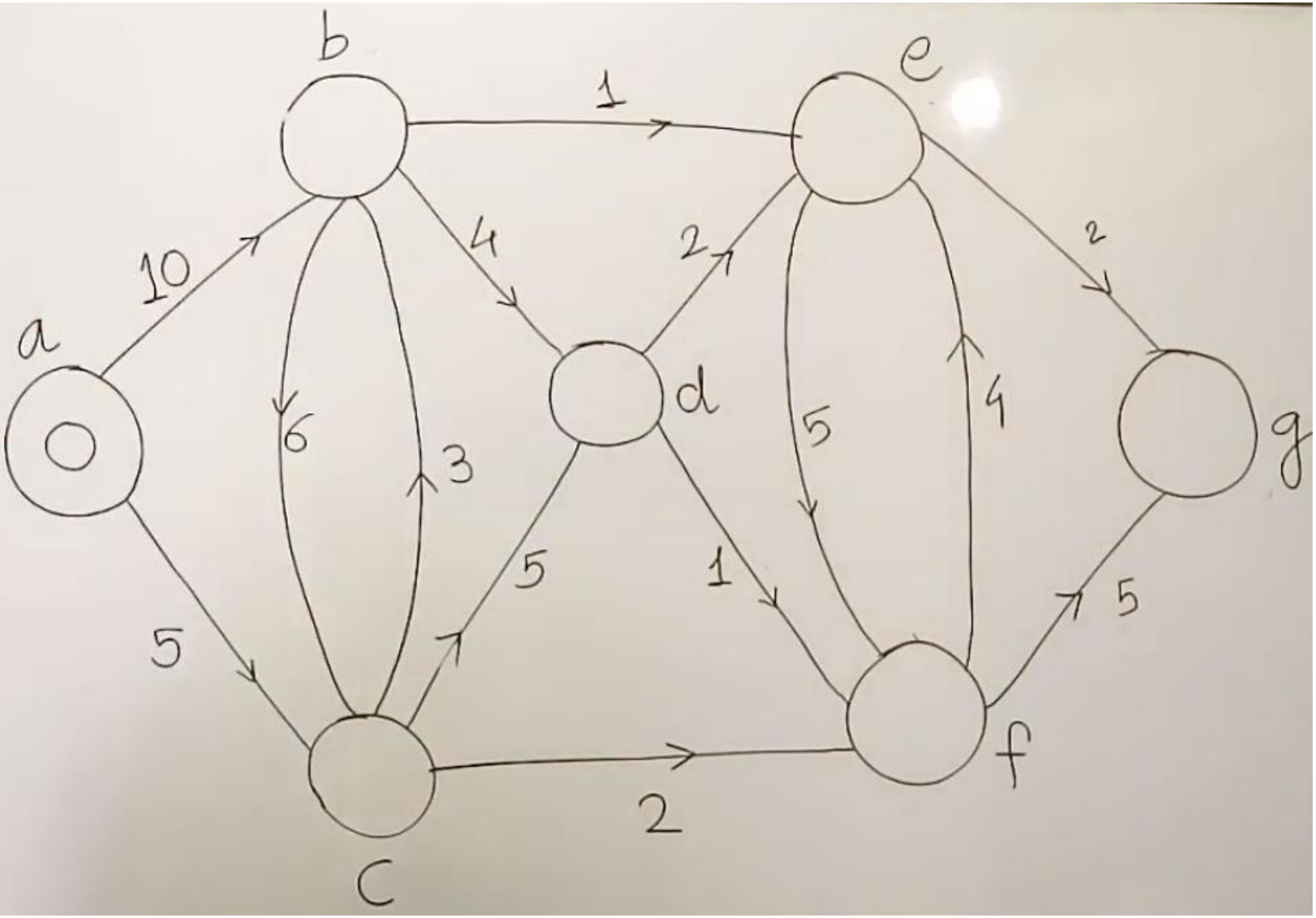


```
DIJKSTRA(G, s)
for all v ∈ V
    dist[v] ← ∞
    prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
    u ← EXTRACTMIN(Q)
    for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
            dist[v] ← dist[u] + w(u, v)
            DECREASEKEY(Q, v, dist[v])
            prev[v] ← u
```

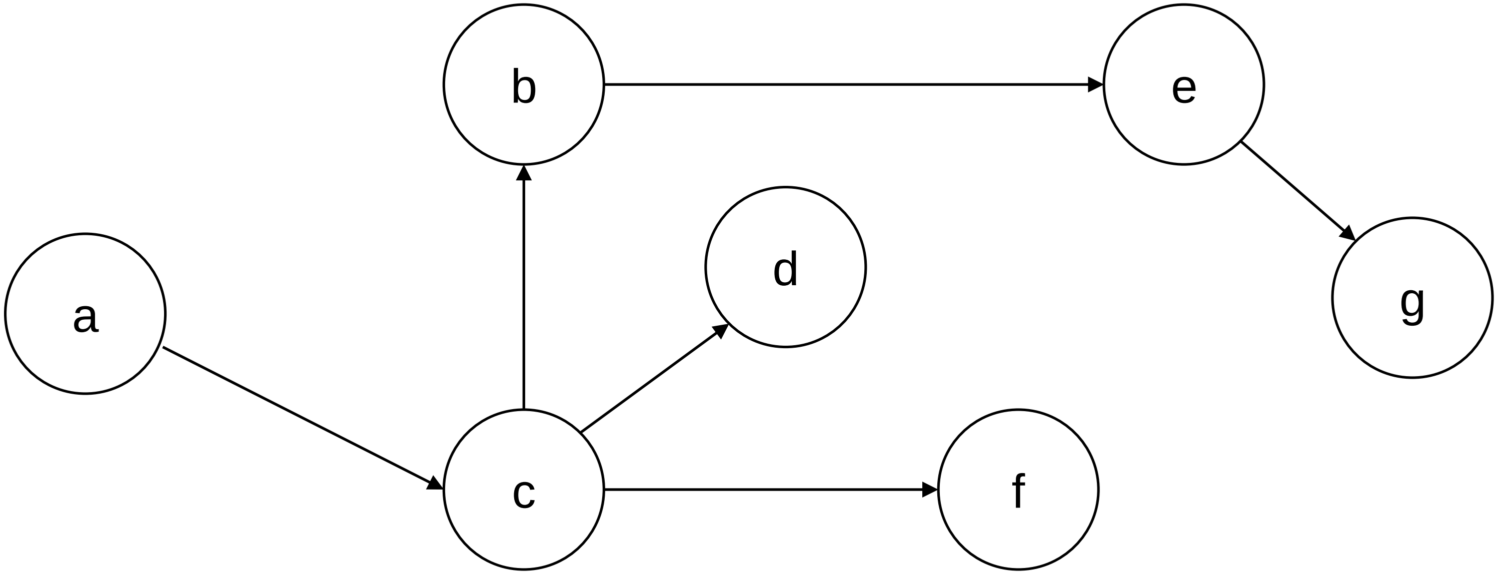
Heap



Example Problem



Example Problem



Is Dijkstra's algorithm correct?

- Invariant:

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Is Dijkstra's algorithm correct?

- Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

Is Dijkstra’s algorithm correct?

- Invariant: For every vertex removed from the heap, dist[v] is the actual shortest distance from s to v
  - The only time a vertex gets visited is when the distance from s to that vertex is smaller than the distance to any remaining vertex
  - Therefore, there cannot be any other path that hasn’t been visited already that would result in a shorter path

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

1 call to MakeHeap

Running time?

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

|V| iterations

Running time?

```
Dijkstra(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MakeHeap(V)
while !Empty(Q)
        u ← ExtractMin(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DecreaseKey(Q, v, dist[v])
                        prev[v] ← u
```

|V| calls

Running time?

```
DIJKSTRA(G, s)
  for all v ∈ V
          dist[v] ← ∞
          prev[v] ← null
  dist[s] ← 0
  Q ← MAKEHEAP(V)
  while !EMPTY(Q)
          u ← EXTRACTMIN(Q)
          for all edges (u, v) ∈ E
                  if dist[v] > dist[u] + w(u, v)
                          dist[v] ← dist[u] + w(u, v)
                          DECREASEKEY(Q, v, dist[v])
                          prev[v] ← u
```

O(|E|) calls

Running time?

- Depends on the heap implementation

	1 MakeHeap	$|V|$ ExtractMin	$|E|$ DecreaseKey	Total
Array	$O(|V|)$	$O(|V|^2)$	$O(|E|)$	$O(|V|^2)$
Bin heap	$O(|V|)$	$O(|V| \log |V|)$	$O(|E| \log |V|)$	$O((|V|+|E|) \log |V|)$ $O(|E| \log |V|)$

Running time?

- Depends on the heap implementation

	1 MakeHeap	$\lvert V\rvert$ ExtractMin	$\lvert E\rvert$ DecreaseKey	Total
Array	$O(\lvert V\rvert)$	$O(\lvert V\rvert^2)$	$O(\lvert E\rvert)$	$O(\lvert V\rvert^2)$
Bin heap	$O(\lvert V\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert)$	$O(\lvert E\rvert \log \lvert V\rvert)$	$O((\lvert V\rvert+\lvert E\rvert) \log \lvert V\rvert)$ $O(\lvert E\rvert \log \lvert V\rvert)$

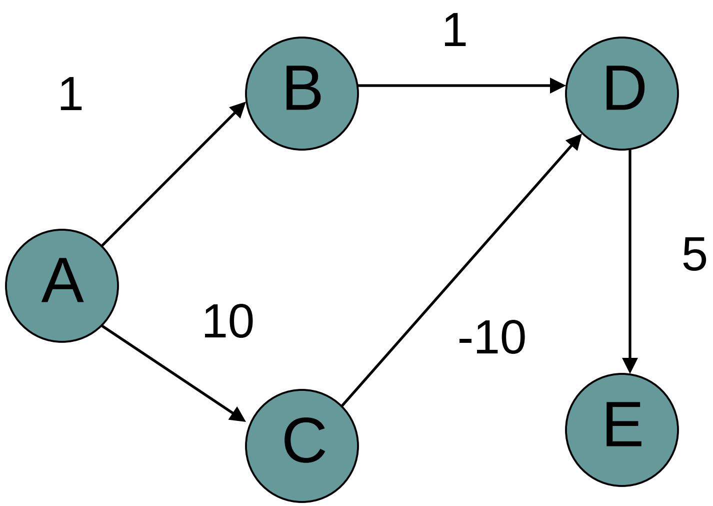
Is this an improvement? If $\lvert E\rvert < \lvert V\rvert^2 / \log \lvert V\rvert$

Running time?

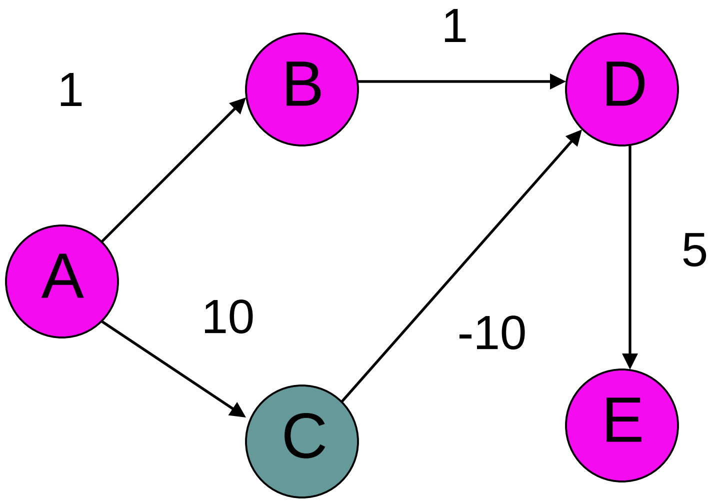
- Depends on the heap implementation

	1 MakeHeap	$\lvert V\rvert$ ExtractMin	$\lvert E\rvert$ DecreaseKey	Total
Array	$O(\lvert V\rvert)$	$O(\lvert V\rvert^2)$	$O(\lvert E\rvert)$	$O(\lvert V\rvert^2)$
Bin heap	$O(\lvert V\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert)$	$O(\lvert E\rvert \log \lvert V\rvert)$	$O((\lvert V\rvert+\lvert E\rvert) \log \lvert V\rvert)$ $O(\lvert E\rvert \log \lvert V\rvert)$
Fib heap	$O(\lvert V\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert)$	$O(\lvert E\rvert)$	$O(\lvert V\rvert \log \lvert V\rvert + \lvert E\rvert)$

What about Dijkstra's on…?

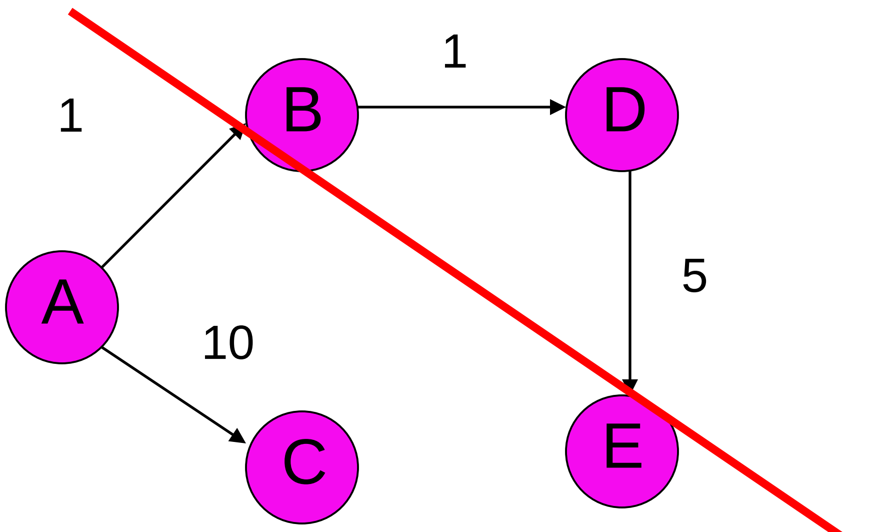


What about Dijkstra's on…?



What about Dijkstra's on…?

Dijkstra's algorithm only works for positive edge weights



Bounding the distance

- Another invariant: For each vertex v, dist[v] is an upper bound on the actual shortest distance
  - start of at ∞
  - only update the value if we find a shorter distance
- An update procedure

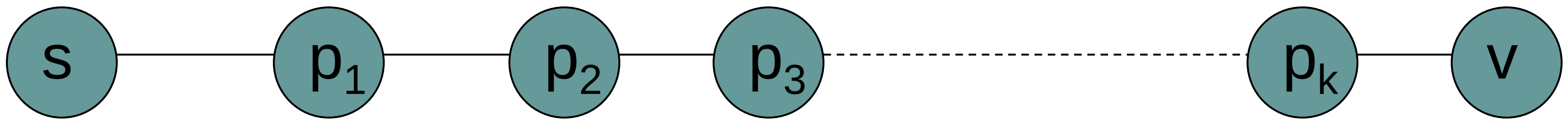
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- Can we ever go wrong applying this update rule?
  - We can apply this rule as many times as we want and will never underestimate dist[v]
- When will dist[v] be right?
  - If u is along the shortest path to v and dist[u] is correct

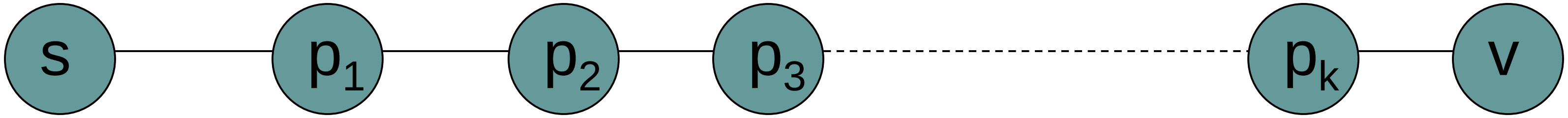
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- Consider the shortest path from s to v



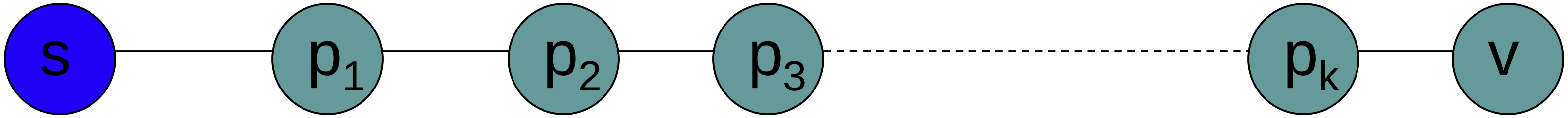
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- What happens if we update all of the vertices with the above update?



$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

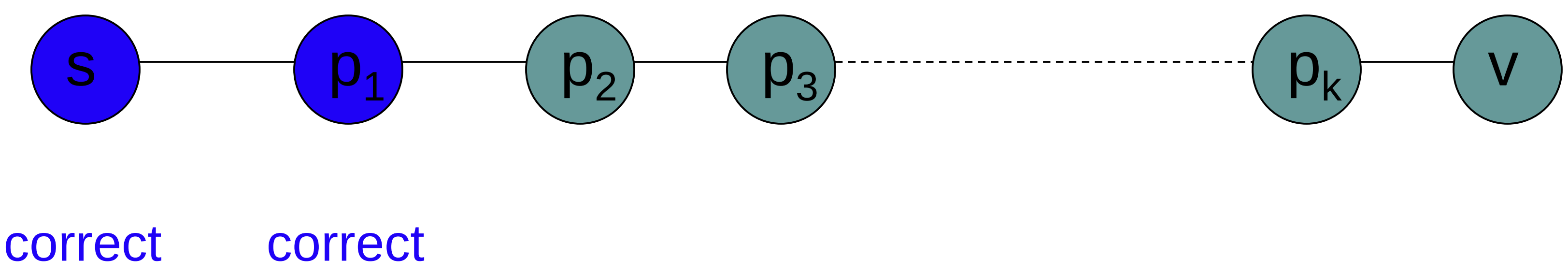
- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- What happens if we update all of the vertices with the above update?



correct

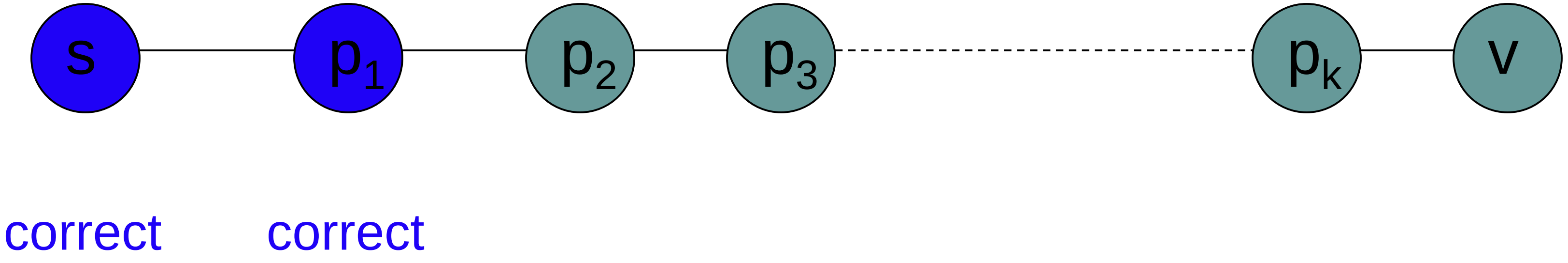
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- What happens if we update all of the vertices with the above update?



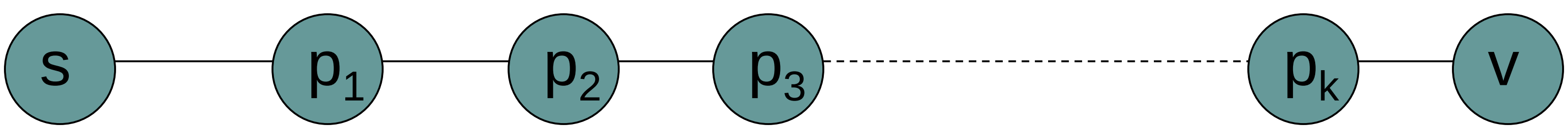
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- Does the order that we update the vertices matter?



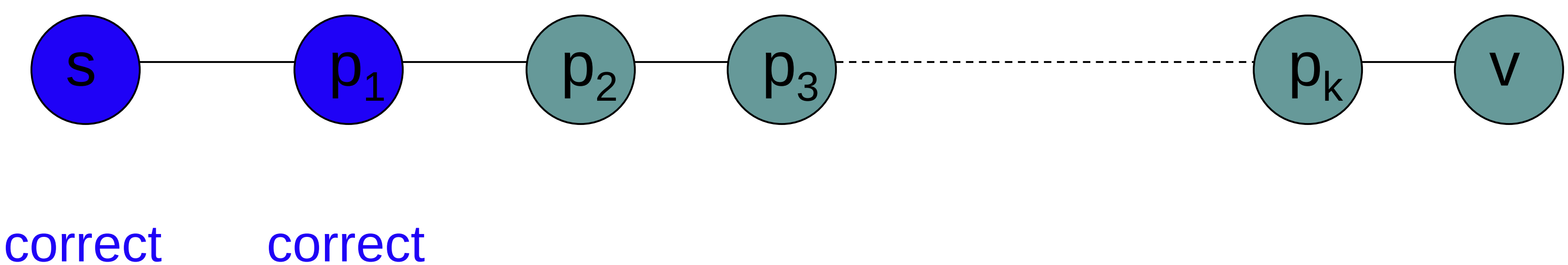
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



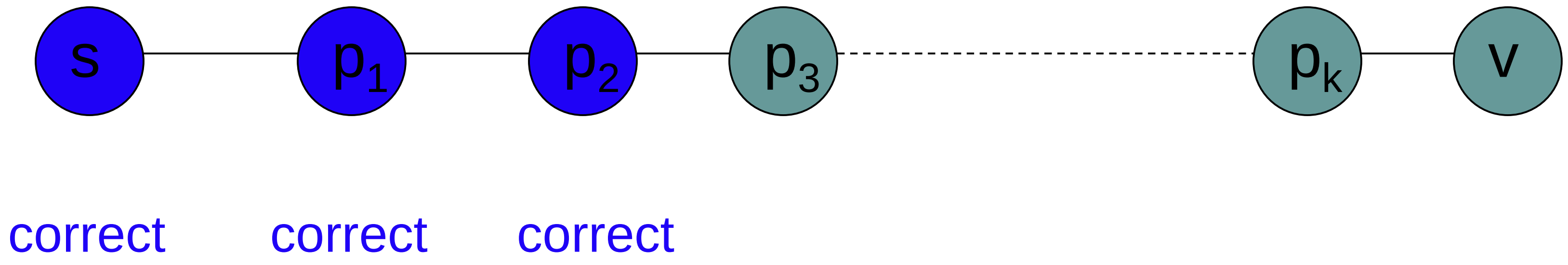
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



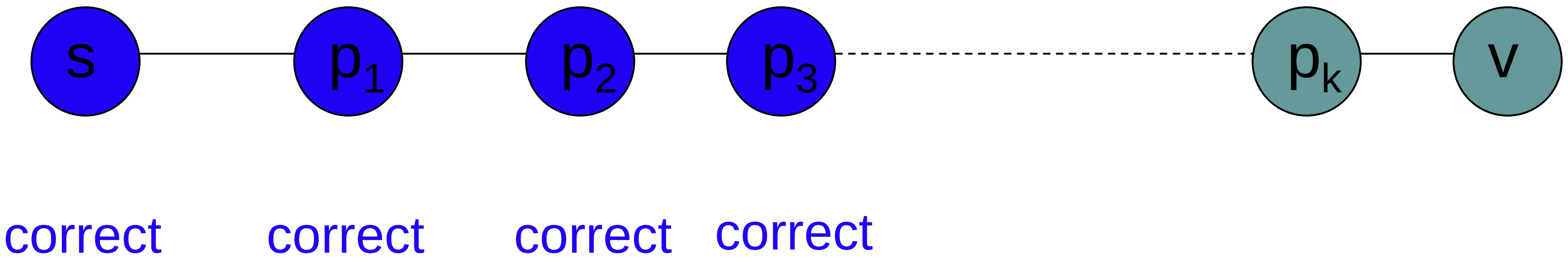
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



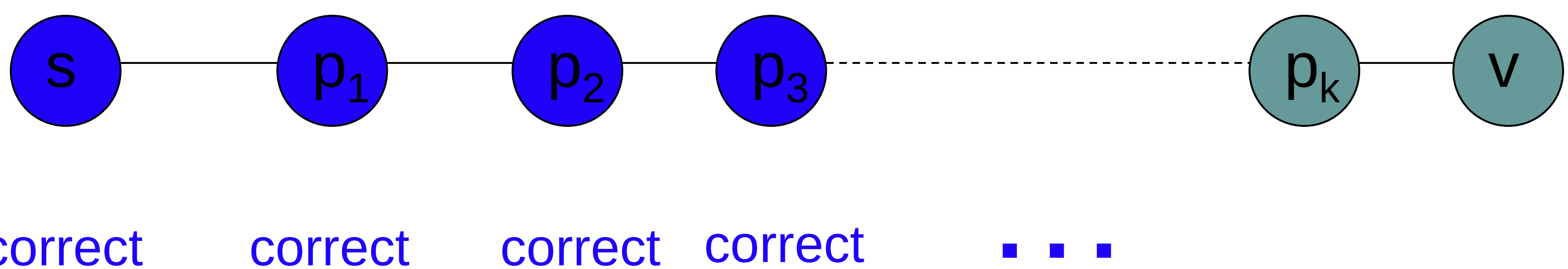
$$dist[v] = \min\{dist[v], dist[u] + w(u,v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



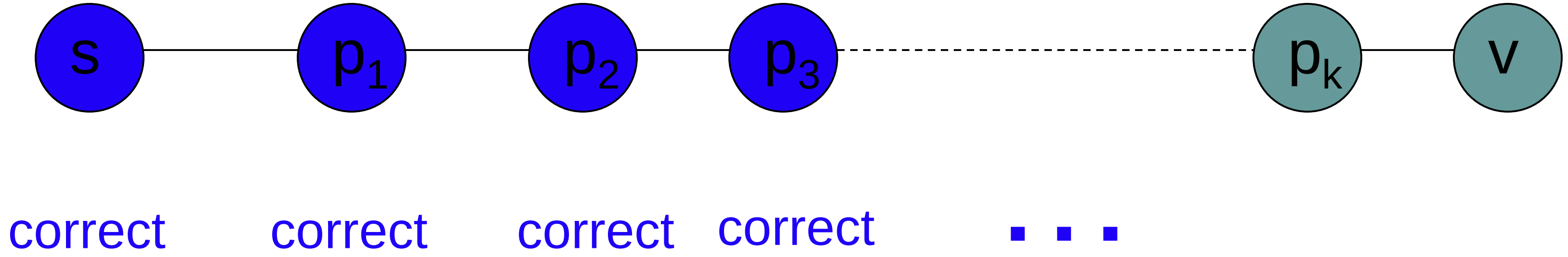
$$dist[v] = \min\{dist[v], dist[u] + w(u, v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- How many times do we have to do this for vertex p_i to have the correct shortest path from s?
  - i times



$$dist[v] = \min\{dist[v], dist[u] + w(u, v)\}$$

- dist[v] will be right if u is along the shortest path to v and dist[u] is correct
- What is the longest (vetex-wise) the path from s to any node v can be?
  - |V| - 1 edges/vertices



Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Bellman-Ford algorithm

BELLMAN-FORD(G, s)

for all $v \in V$
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow null$
$dist[s] \leftarrow 0$
for $i \leftarrow 1$ **to** $|V| - 1$
    for all edges $(u, v) \in E$
        if $dist[v] > dist[u] + w(u, v)$
            $dist[v] \leftarrow dist[u] + w(u, v)$
            $prev[v] \leftarrow u$
for all edges $(u, v) \in E$
    if $dist[v] > dist[u] + w(u, v)$
        return $false$

Initialize all the distances

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

iterate over all edges/vertices and apply update rule

Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

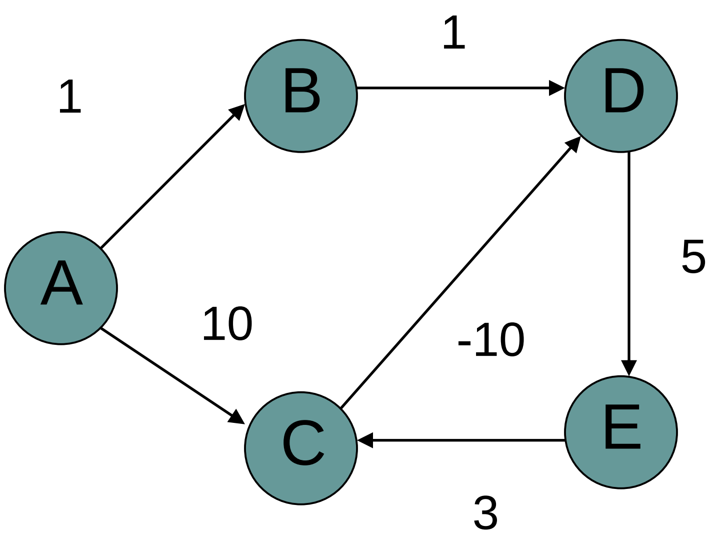
Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

check for negative cycles

Negative cycles

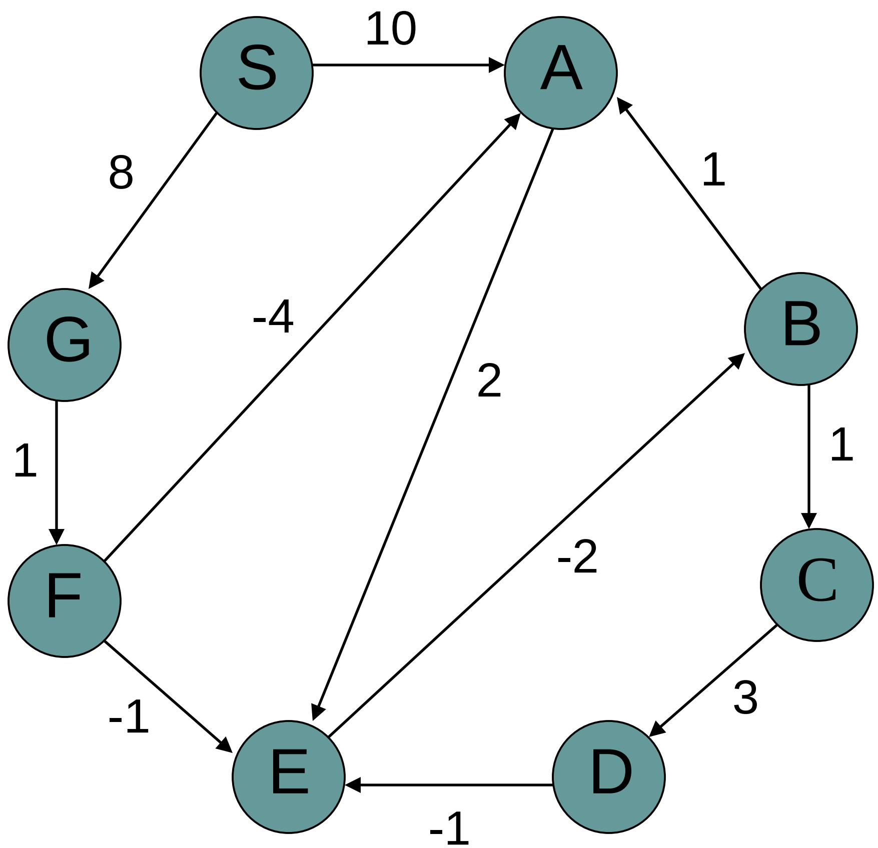
What is the shortest path from a to e?



Bellman-Ford algorithm

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

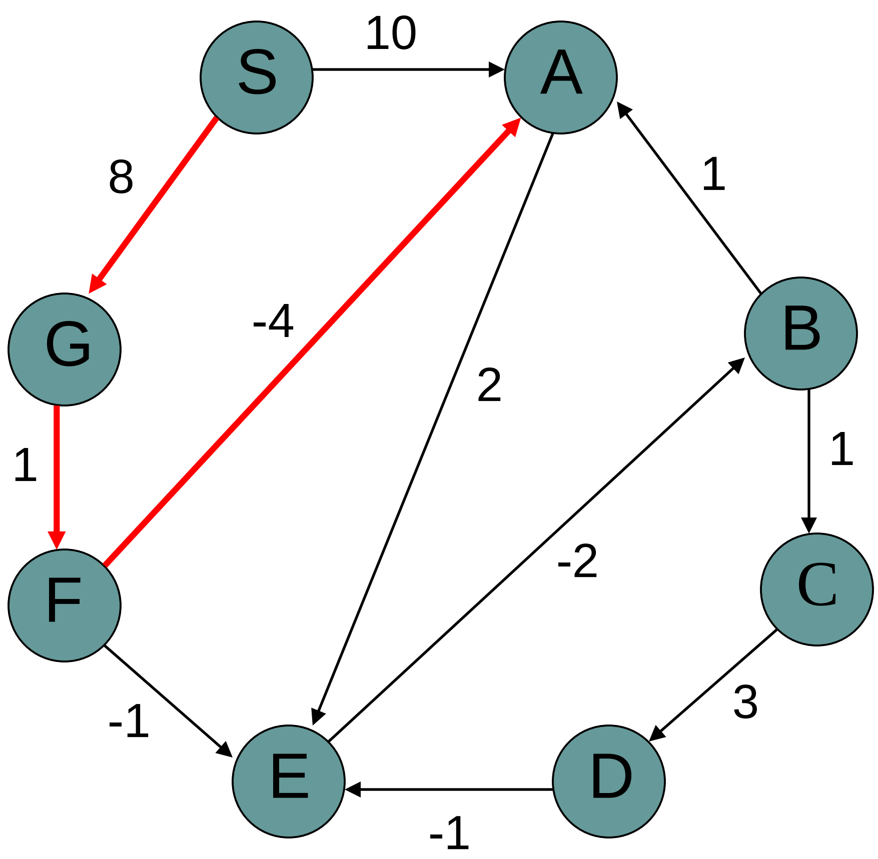
Bellman-Ford algorithm



How many edges is the shortest path from s to:

A:

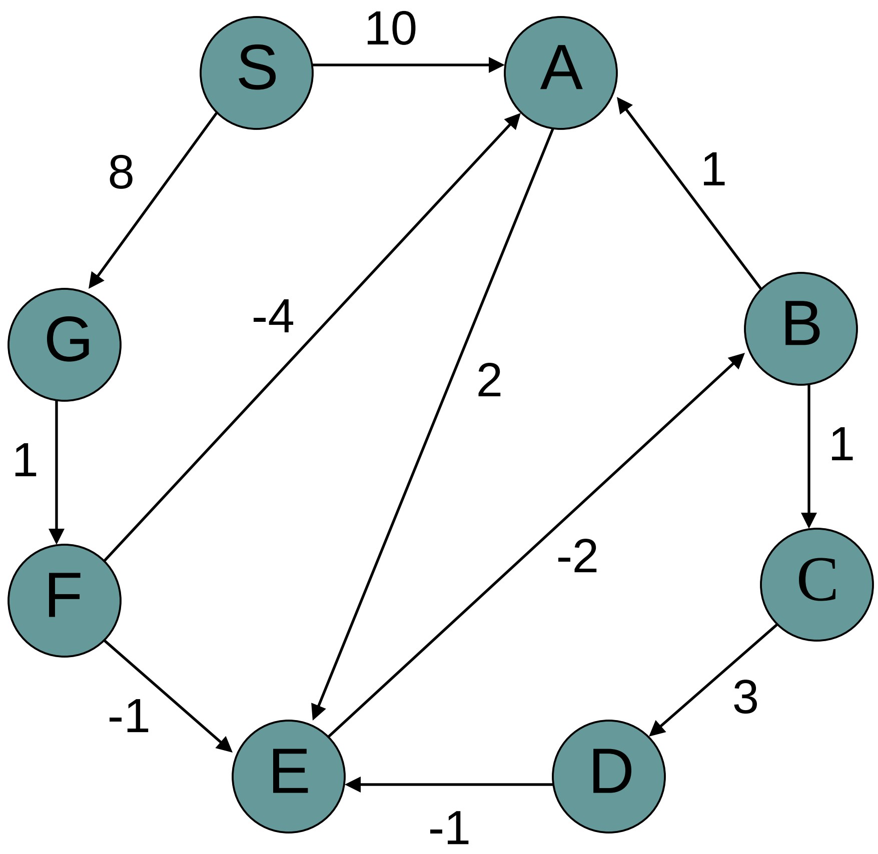
Bellman-Ford algorithm



How many edges is the shortest path from s to:

A: 3

Bellman-Ford algorithm

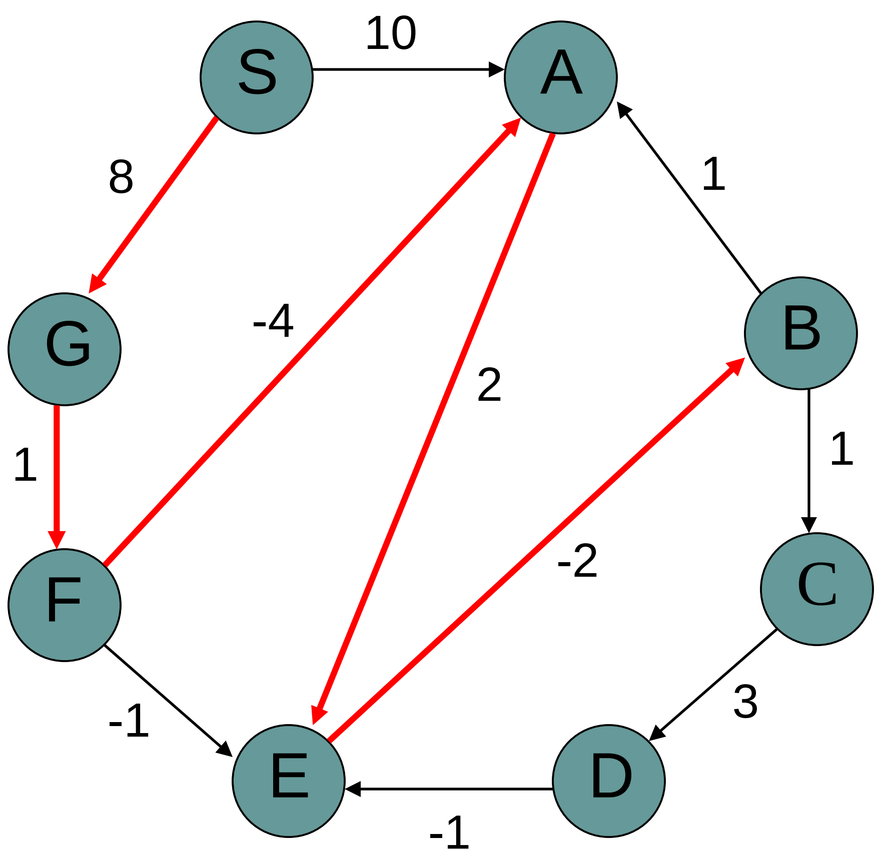


How many edges is the shortest path from s to:

A: 3

B:

Bellman-Ford algorithm

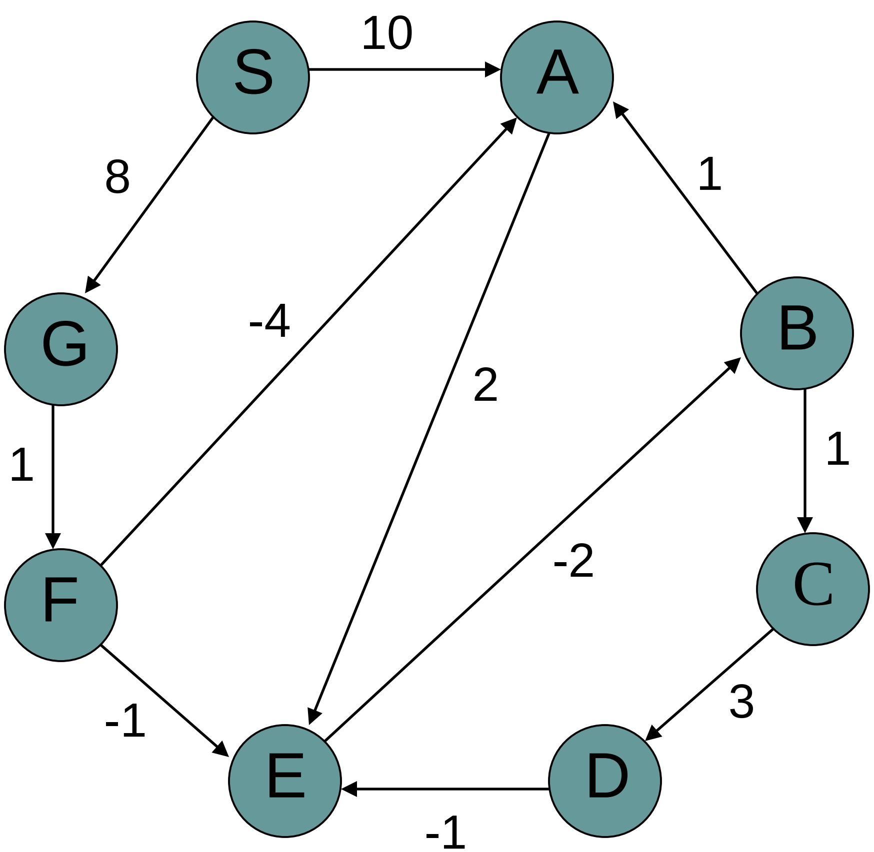


How many edges is the shortest path from s to:

A: 3

B: 5

Bellman-Ford algorithm



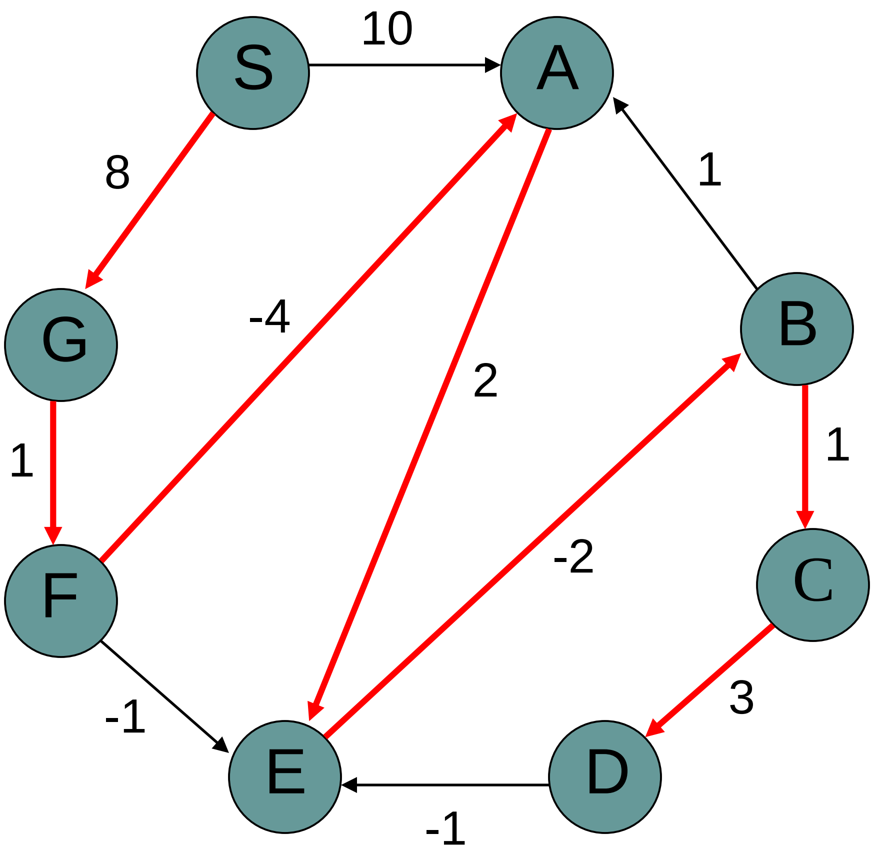
How many edges is the shortest path from s to:

A: 3

B: 5

D:

Bellman-Ford algorithm



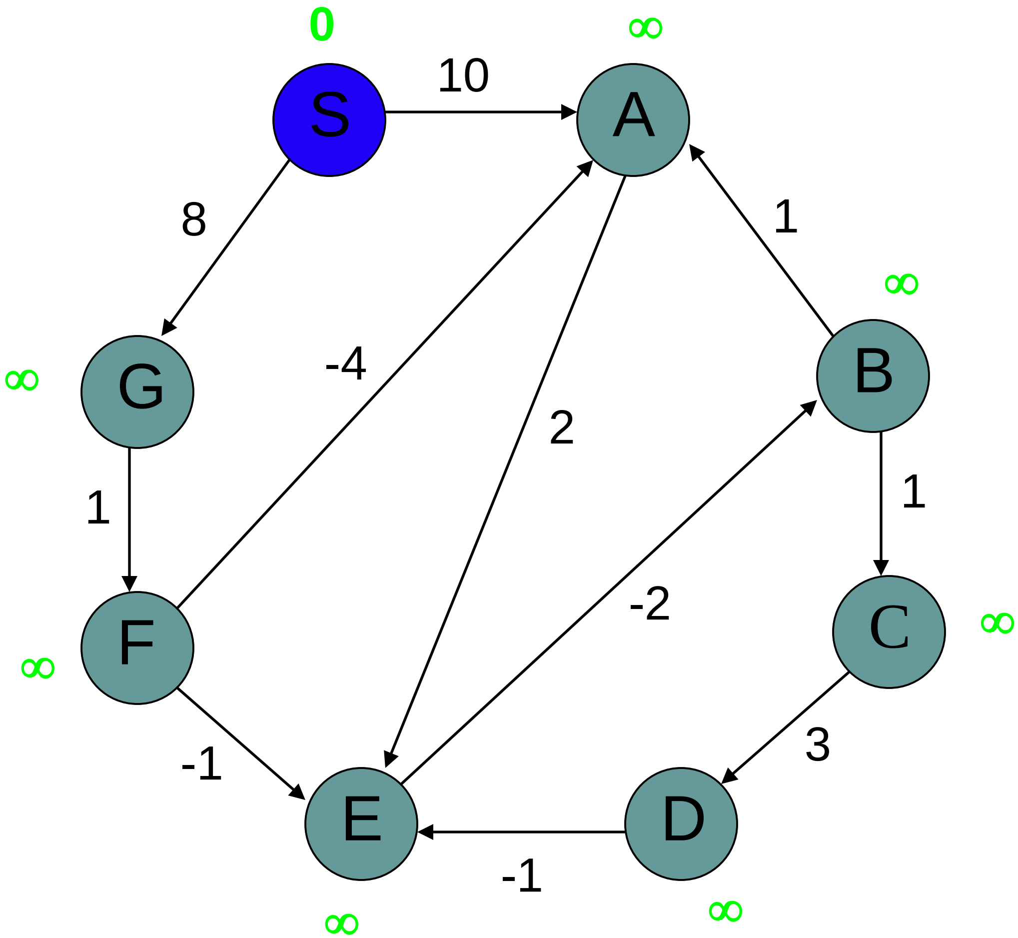
How many edges is the shortest path from s to:

A: 3

B: 5

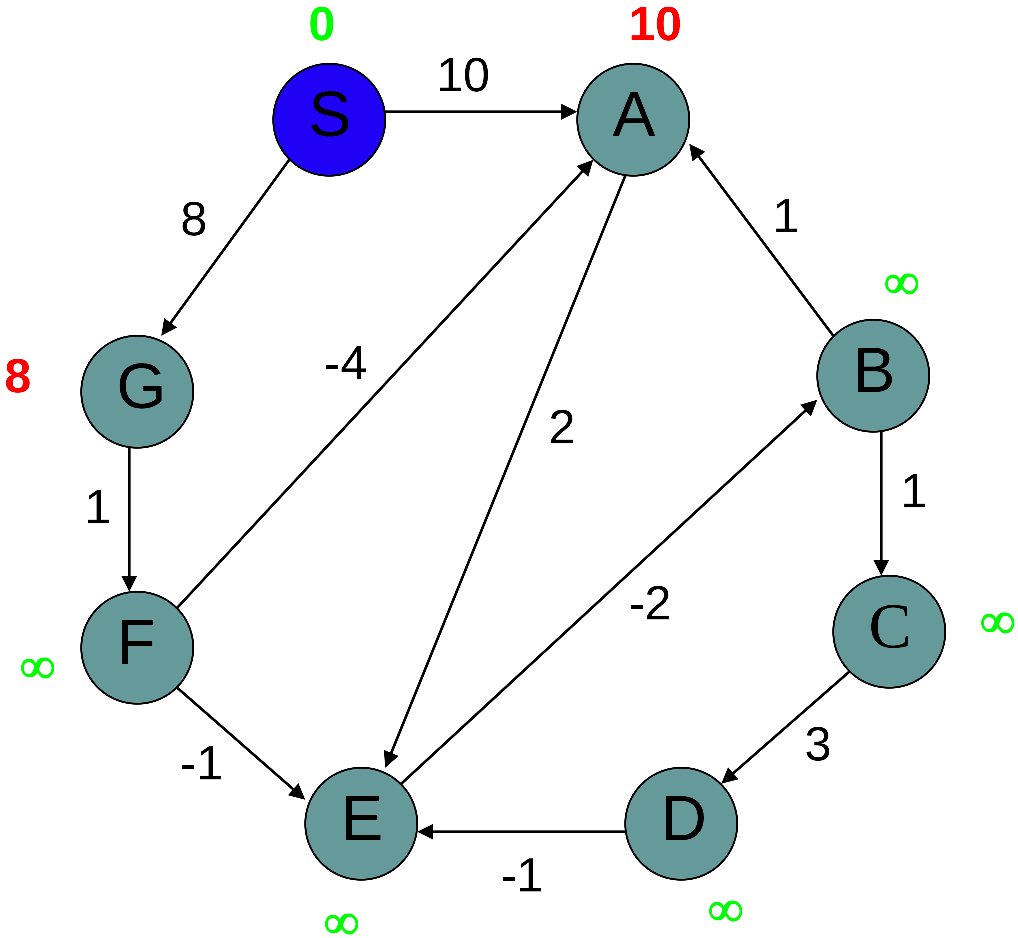
D: 7

Bellman-Ford algorithm



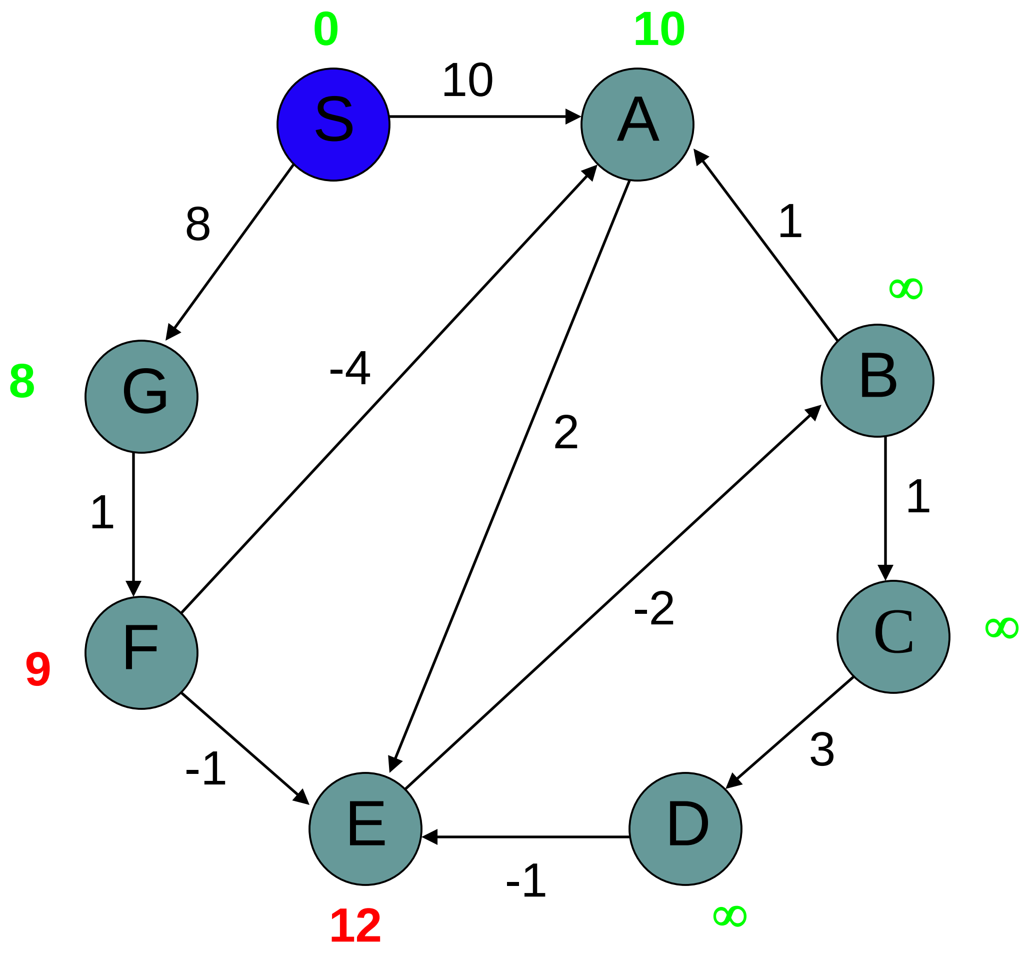
Iteration: 0

Bellman-Ford algorithm



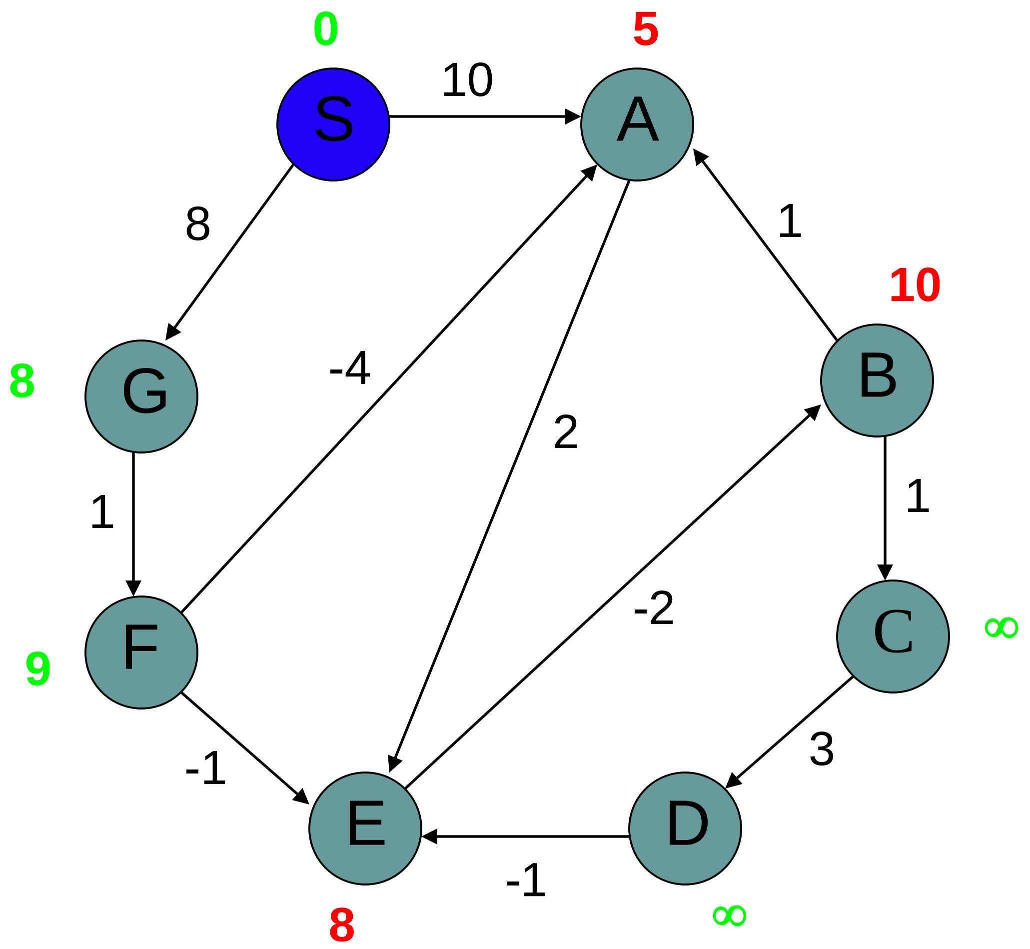
Iteration: 1

Bellman-Ford algorithm



Iteration: 2

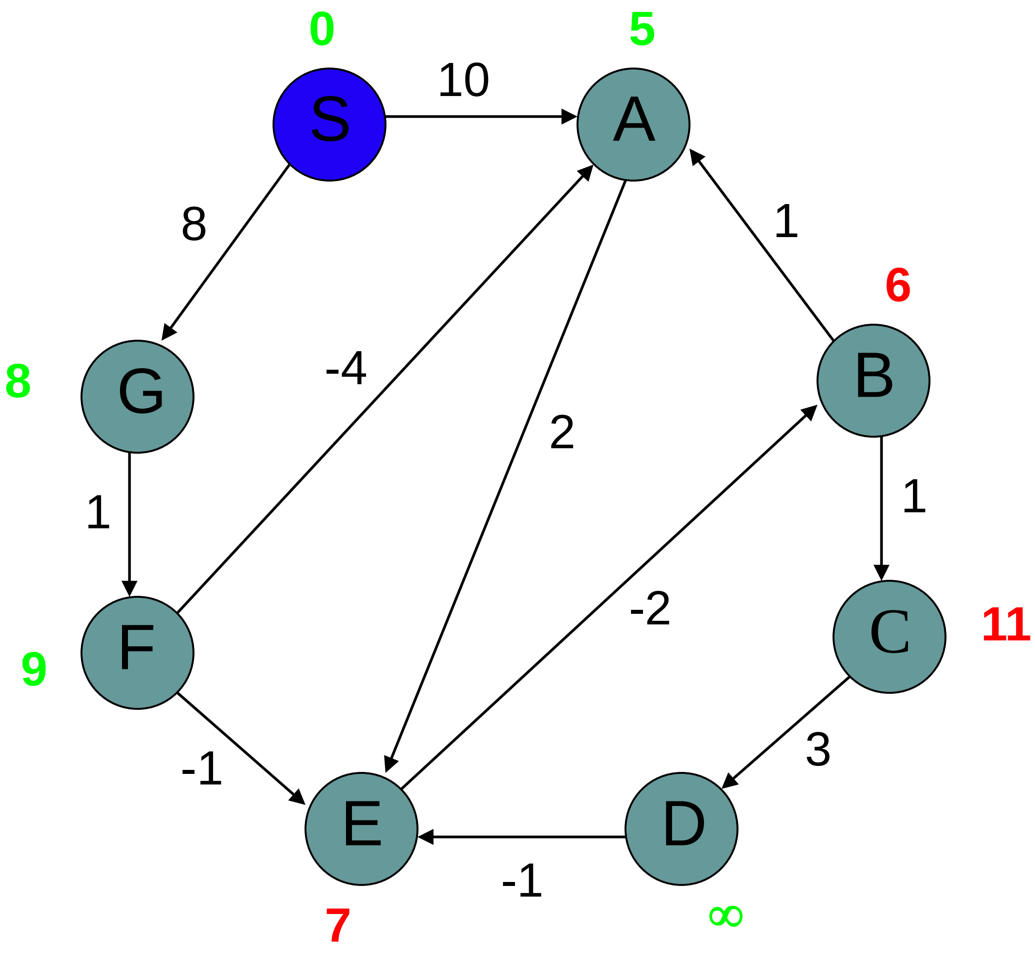
Bellman-Ford algorithm



Iteration: 3

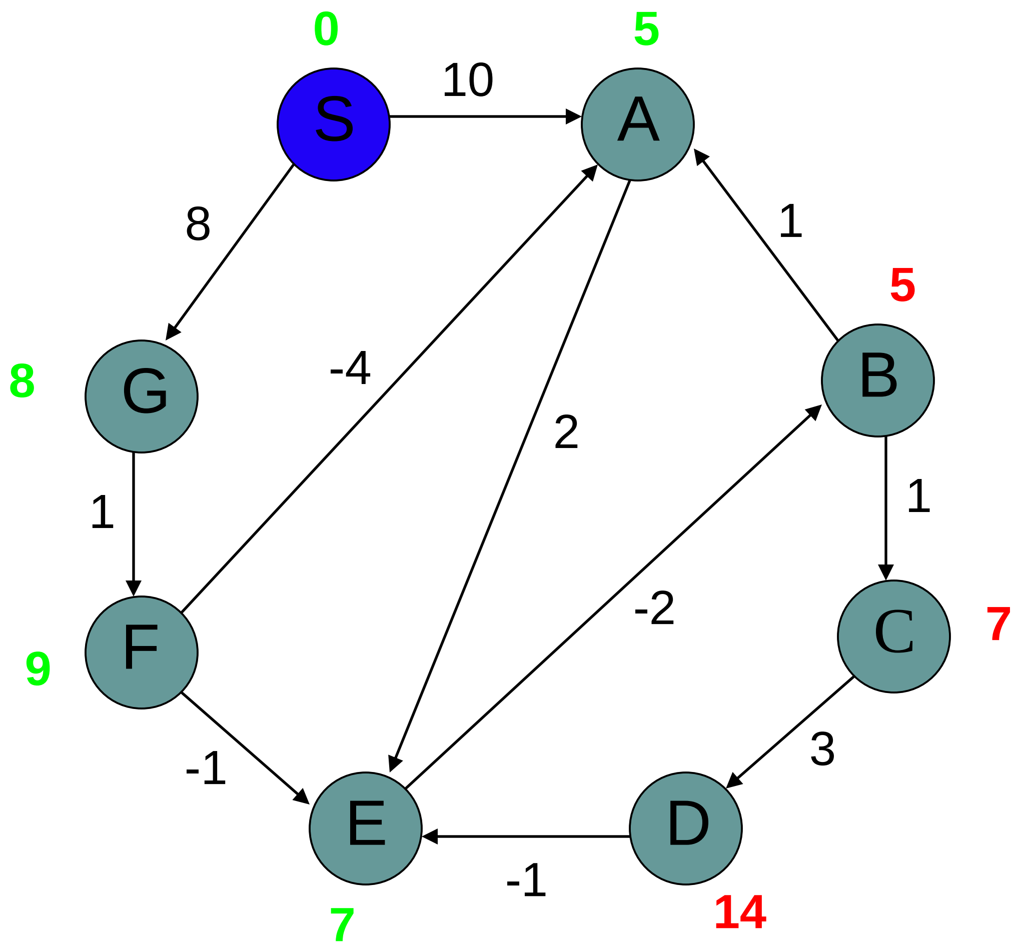
A has the correct distance and path

Bellman-Ford algorithm



Iteration: 4

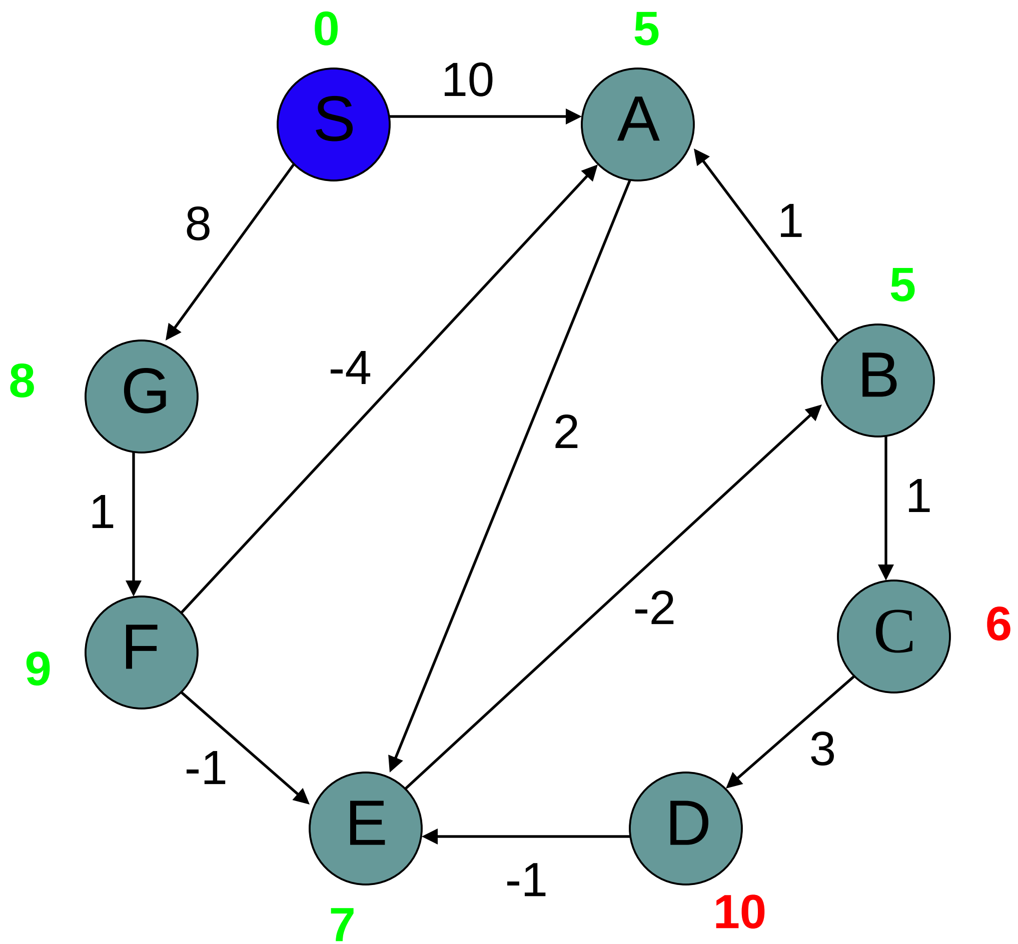
Bellman-Ford algorithm



Iteration: 5

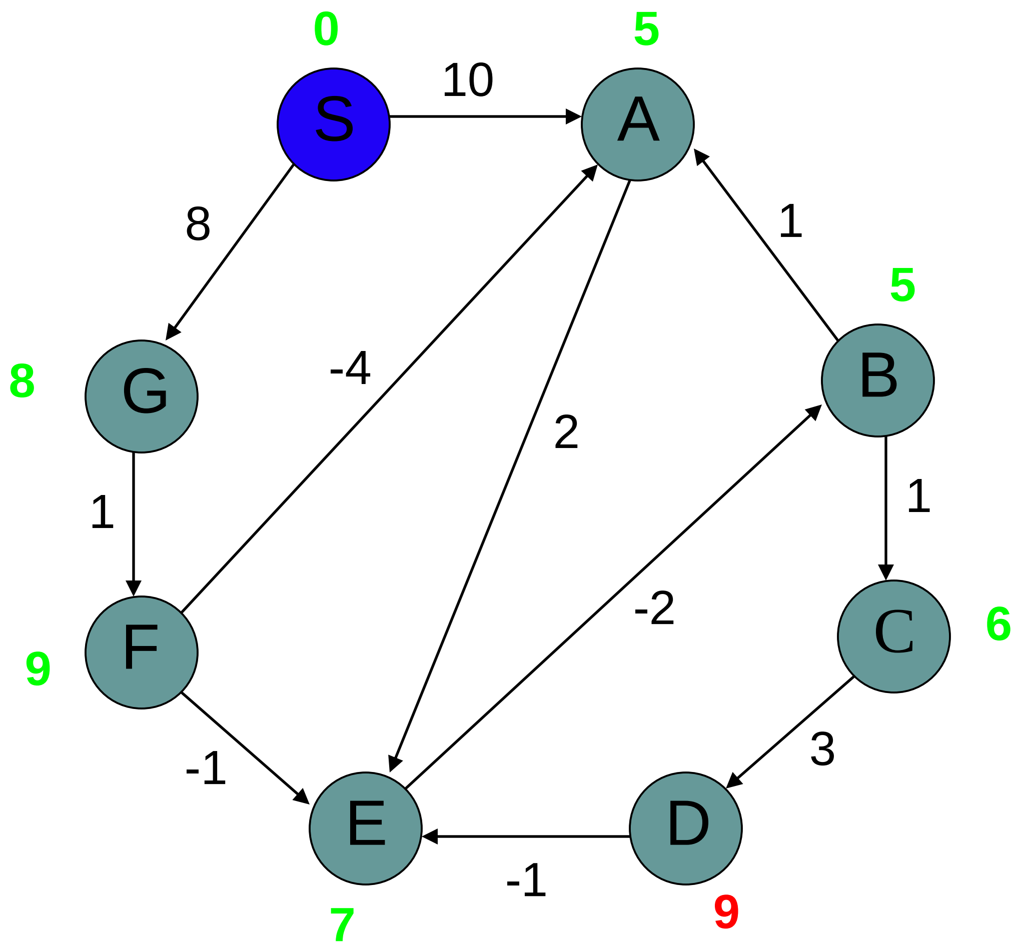
B has the correct distance and path

Bellman-Ford algorithm



Iteration: 6

Bellman-Ford algorithm



Iteration: 7

D (and all other nodes) have the correct distance and path

Correctness of Bellman-Ford

- Loop invariant:

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Correctness of Bellman-Ford

- Loop invariant: After iteration i, all vertices with shortest paths from s of length i edges or less have correct distances

```
Bellman-Ford(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Runtime of Bellman-Ford

BELLMAN-FORD(G, s)

for all $v \in V$
    $dist[v] \leftarrow \infty$
    $prev[v] \leftarrow null$
$dist[s] \leftarrow 0$
for $i \leftarrow 1$ **to** $|V| - 1$
    for all edges $(u, v) \in E$
        if $dist[v] > dist[u] + w(u, v)$
            $dist[v] \leftarrow dist[u] + w(u, v)$
            $prev[v] \leftarrow u$
for all edges $(u, v) \in E$
    if $dist[v] > dist[u] + w(u, v)$
        return $false$

O(|V| |E|)

Runtime of Bellman-Ford

```
BELLMAN-FORD(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
for i ← 1 to |V| − 1
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        prev[v] ← u
for all edges (u, v) ∈ E
        if dist[v] > dist[u] + w(u, v)
                return false
```

Can you modify the algorithm to run faster (in some circumstances)?

All pairs shortest paths

- Simple approach
  - Call Bellman-Ford |V| times
  - $O(|V|^2 |E|)$
- Floyd-Warshall – $\Theta(|V|^3)$
- Johnson's algorithm – $O(|V|^2 \log |V| + |V| |E|)$

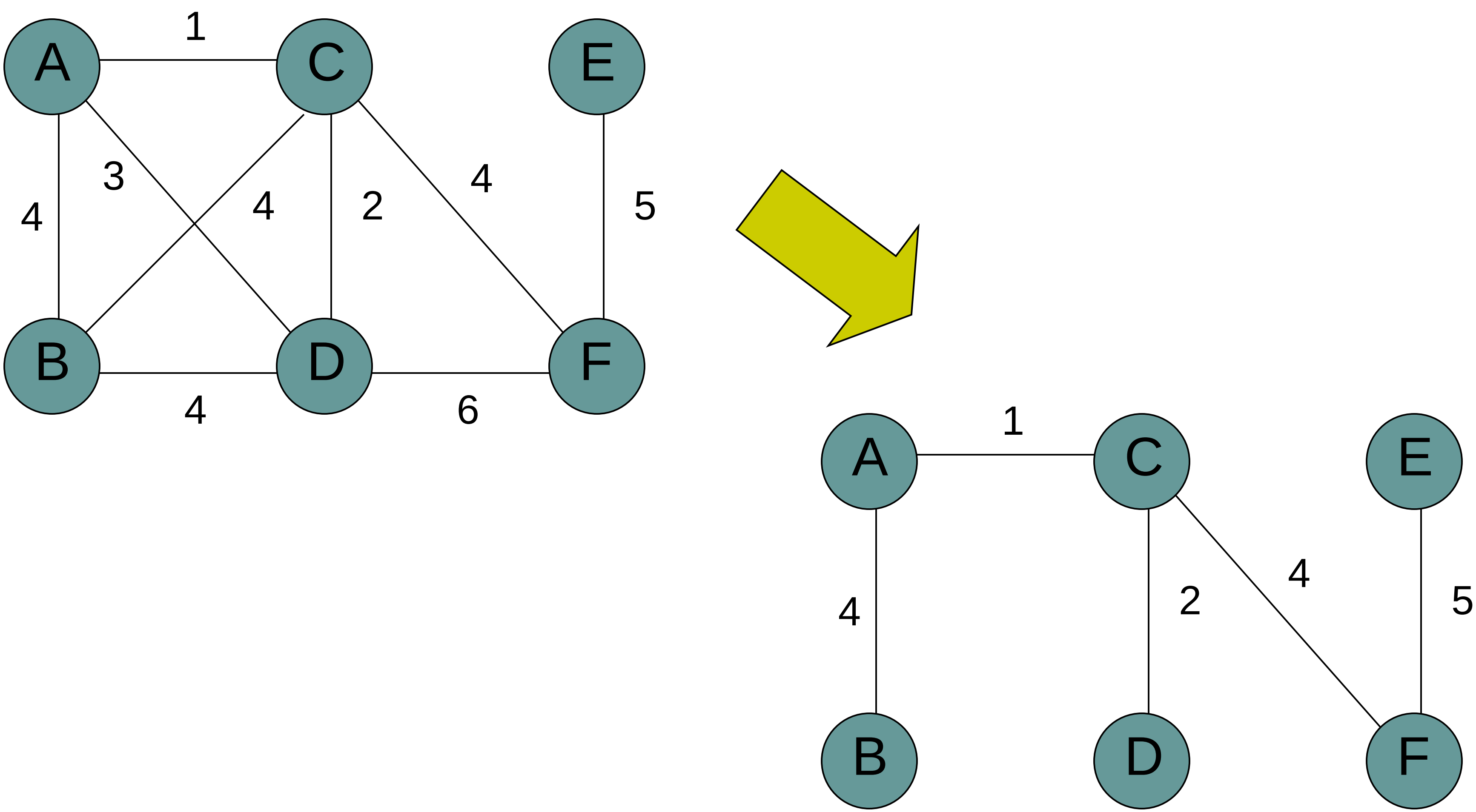
Minimum spanning trees

- What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights

- Input: An undirected, positive weight graph, G=(V,E)
- Output: A tree T=(V,E') where E' $\subseteq$ E that minimizes

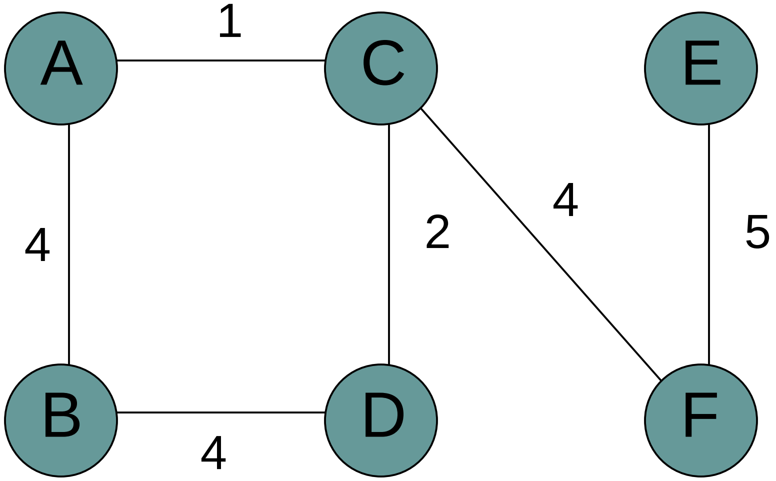
$$weight(T) = \sum_{e \in E'} w_e$$

MST example



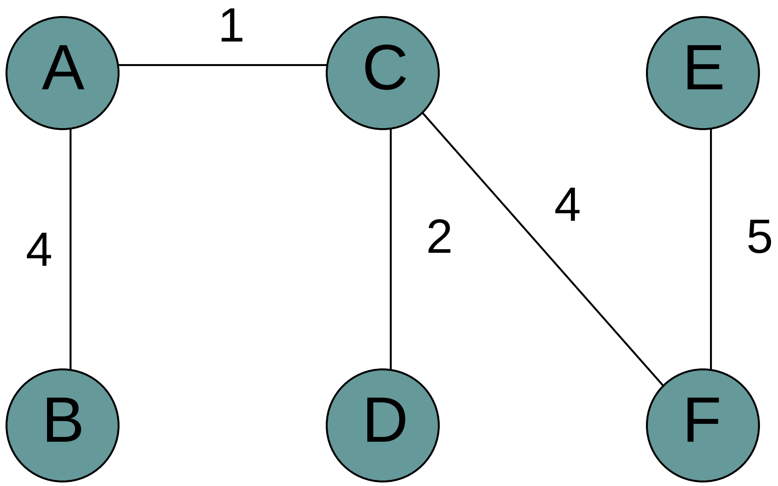
MSTs

- Can an MST have a cycle?



MSTs

- Can an MST have a cycle?

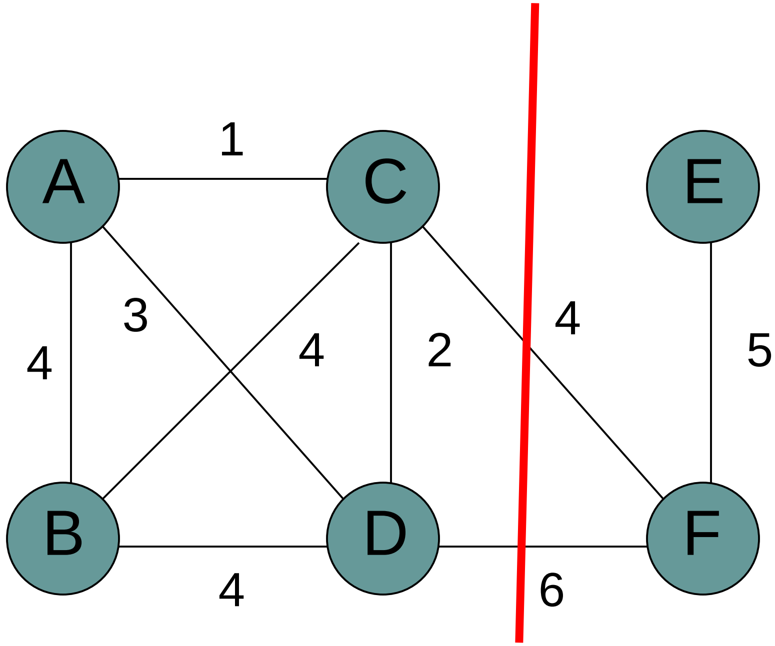


Applications?

- Connectivity
  - Networks (e.g. communications)
  - Circuit desing/wiring
- hub/spoke models (e.g. flights, transportation)
- Traveling salesman problem?

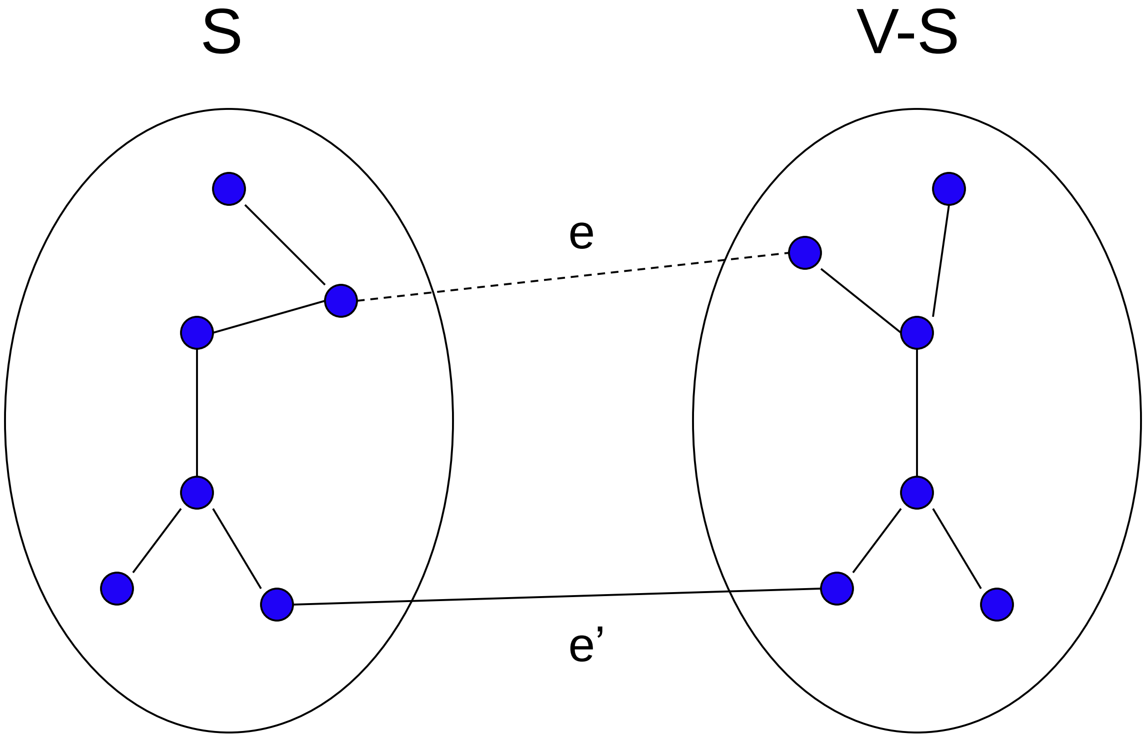
Cuts

- A cut is a partitioning of the vertices into two sets S and V-S
- An edges "crosses" the cut if it connects a vertex u$\in$V and v$\in$V-S



Minimum cut property

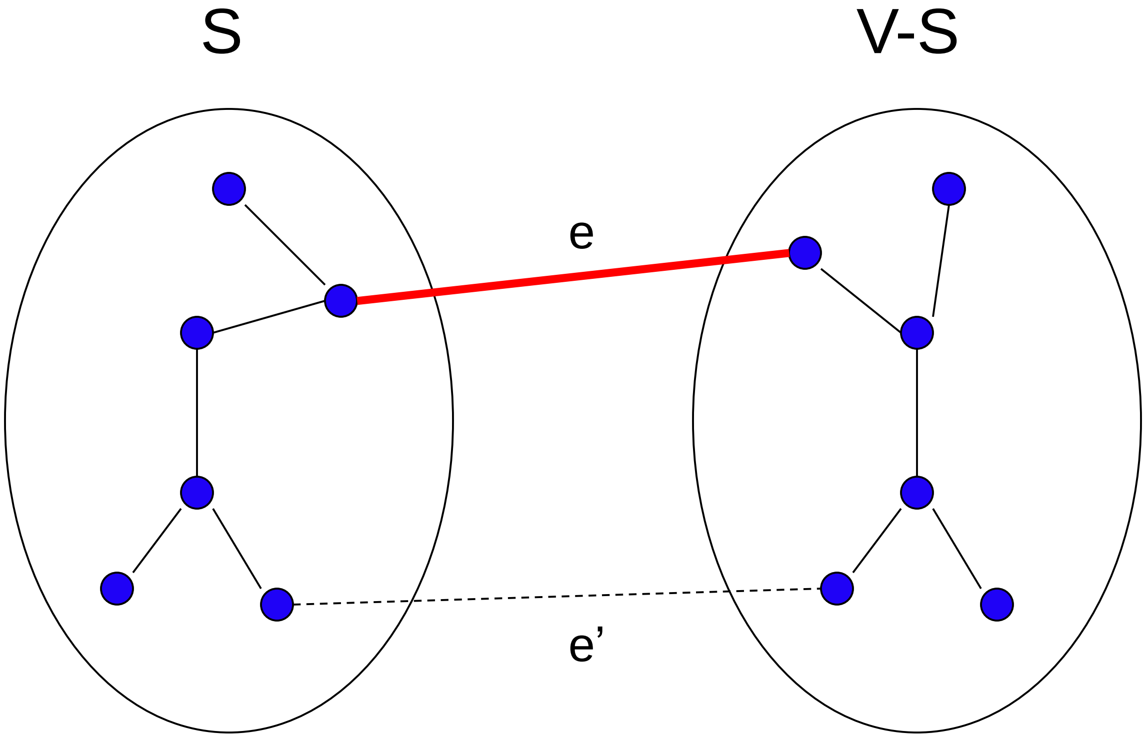
- Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.



Consider an MST with edge e’ that is not the minimum edge

Minimum cut property

- Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.



Using e instead of e', still connects the graph, but produces a tree with smaller weights

Algorithm ideas?

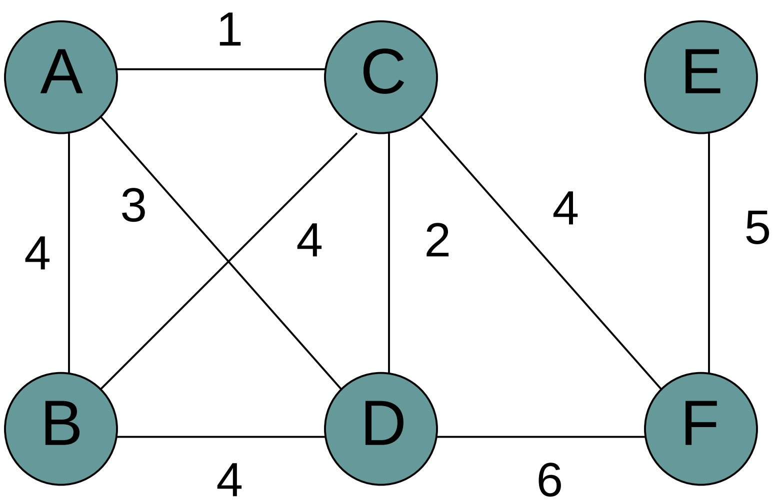
- Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.

KRUSKAL(G)

for all $v \in V$
    MAKESET(v)
$T \leftarrow \{\}$
sort the edges of E by weight
for all edges $(u, v) \in E$ in increasing order of weight
    if FIND-SET(u) $\neq$ FIND-SET(v)
        add edge to T
        UNION(FIND-SET(u),FIND-SET(v))

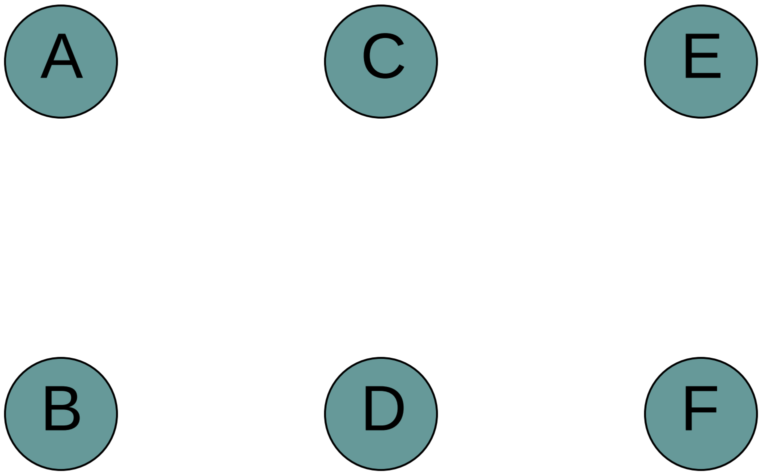
Kruskal's algorithm

Add smallest edge that connects two sets not already connected



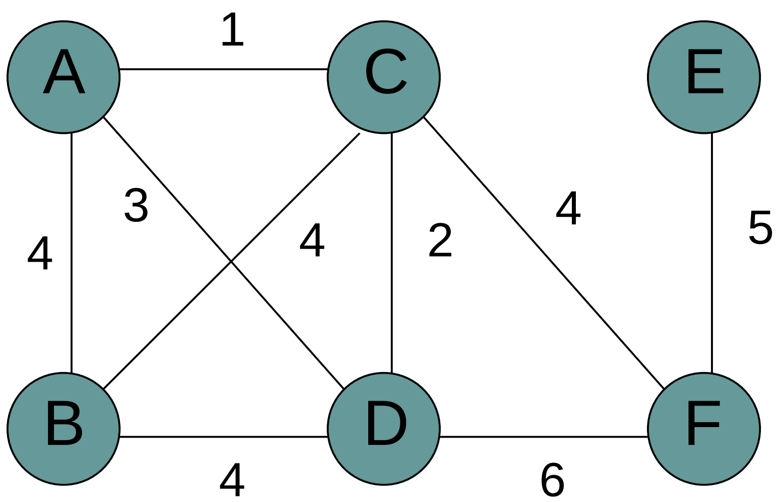
G

MST



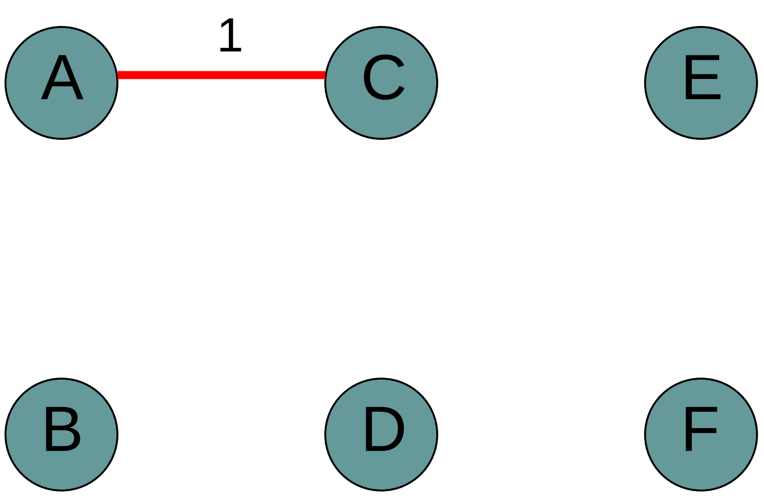
Kruskal's algorithm

Add smallest edge that connects
two sets not already connected



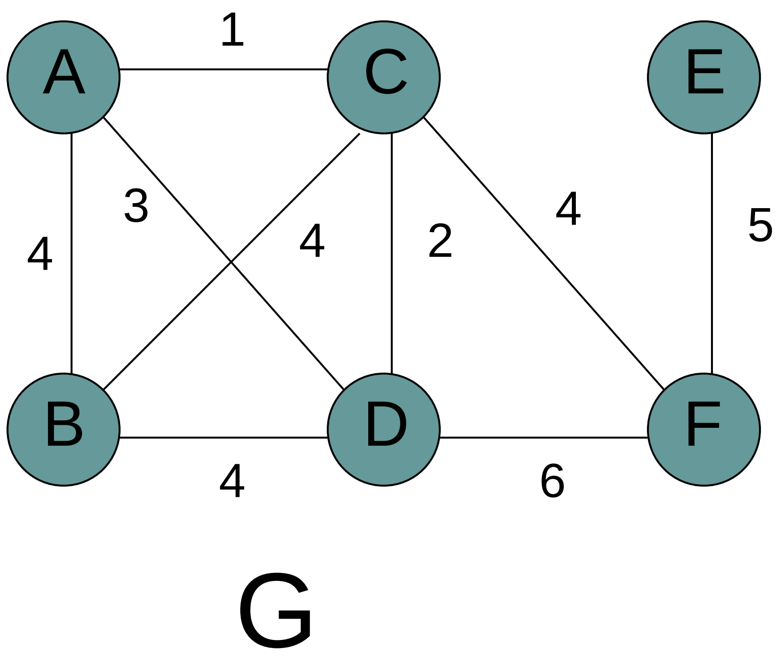
G

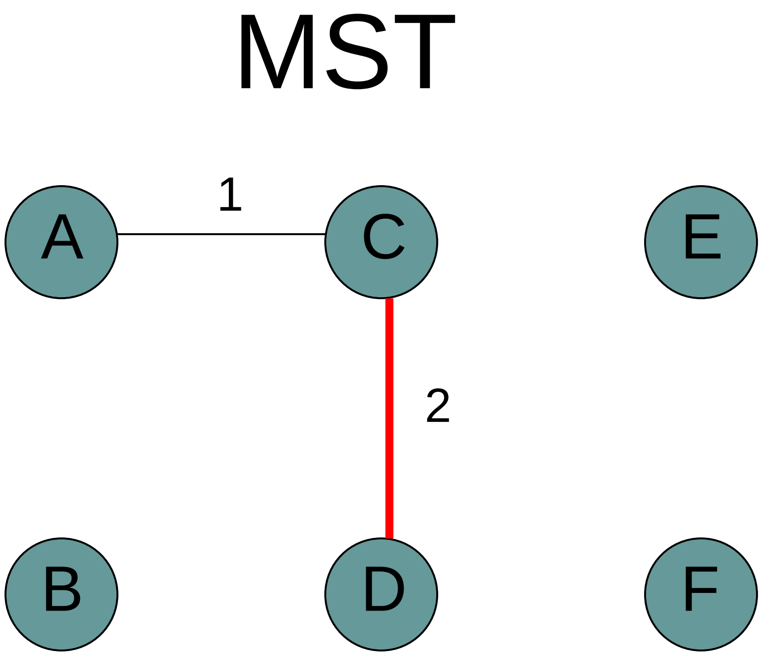
MST



Kruskal's algorithm

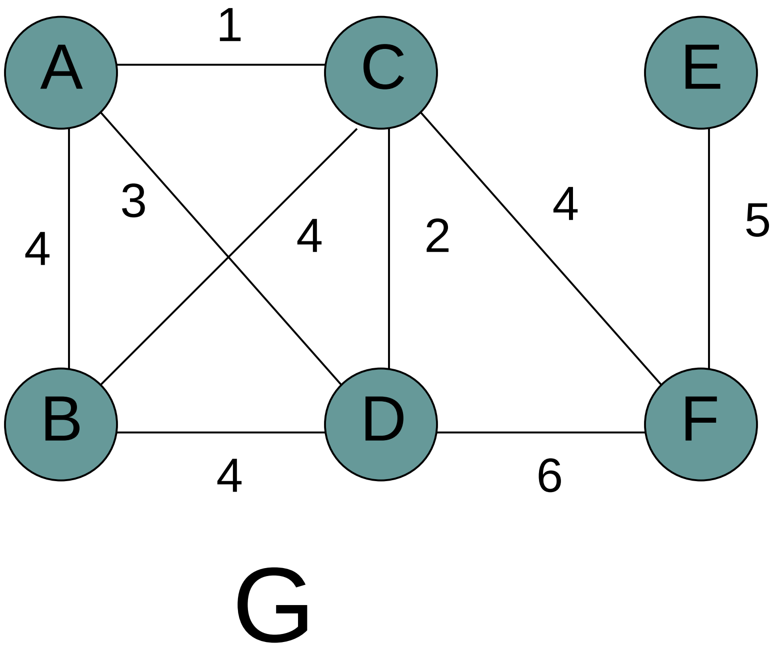
Add smallest edge that connects two sets not already connected

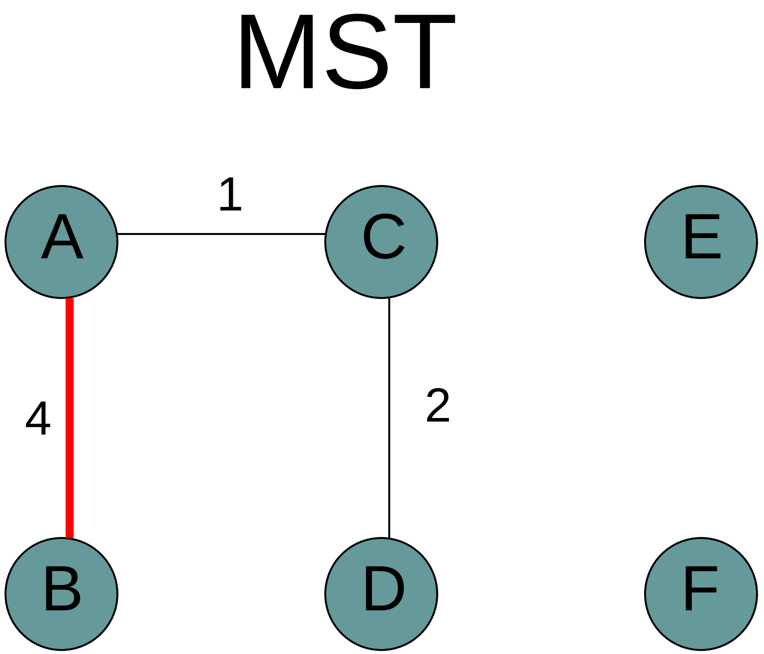




Kruskal's algorithm

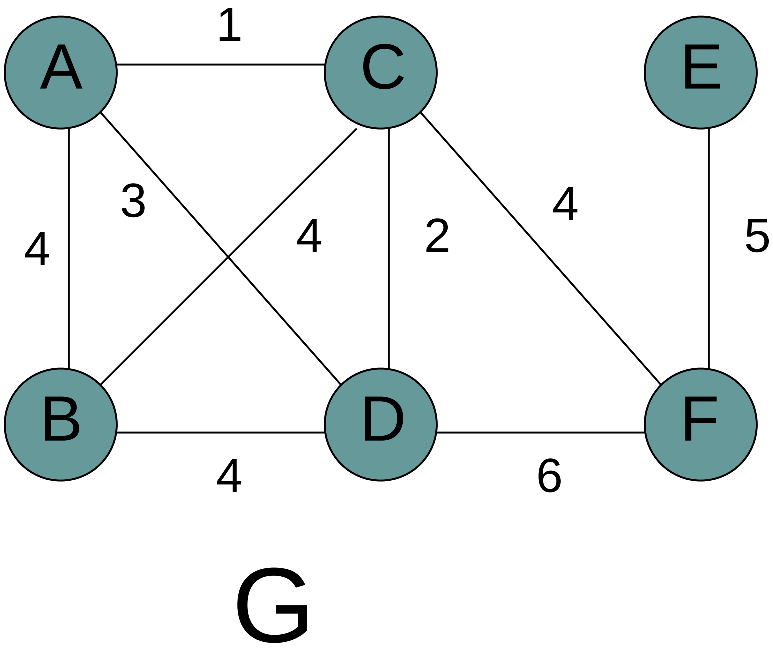
Add smallest edge that connects two sets not already connected

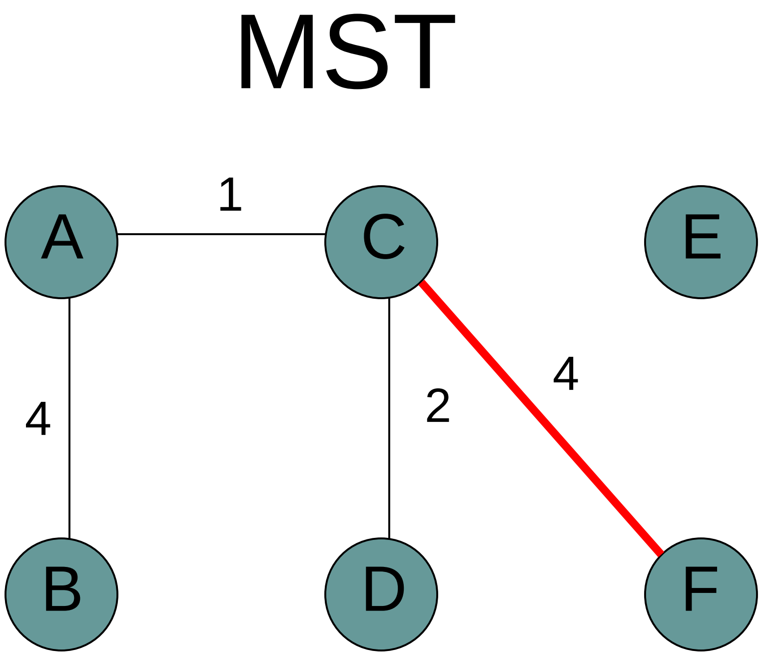




Kruskal’s algorithm

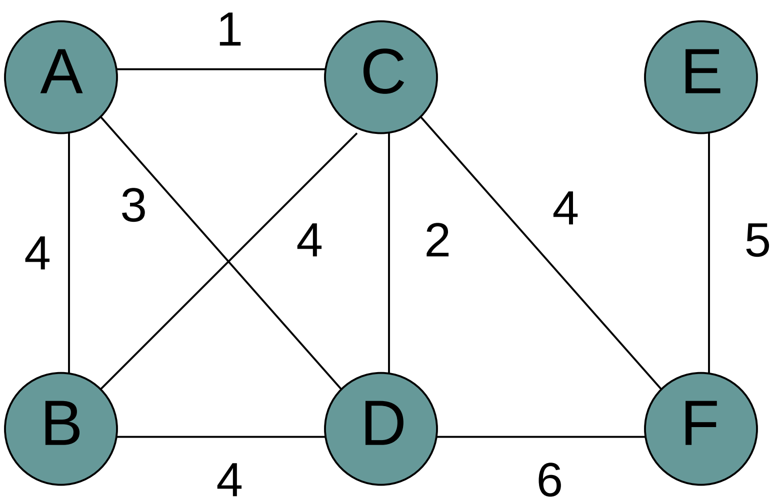
Add smallest edge that connects two sets not already connected





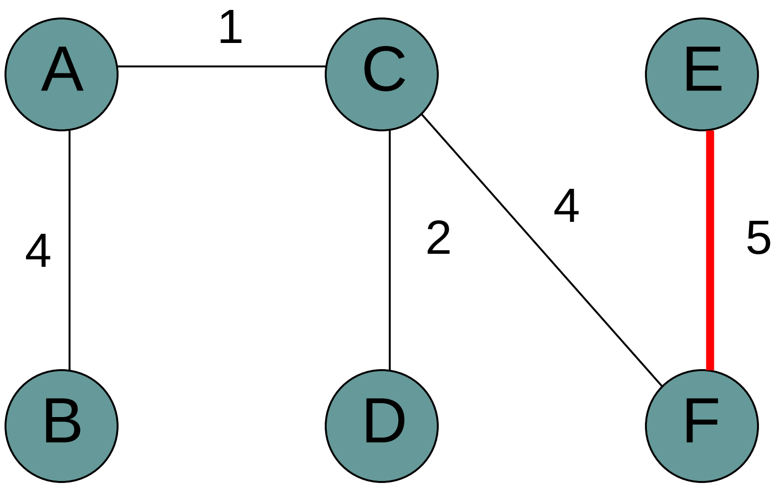
Kruskal's algorithm

Add smallest edge that connects two sets not already connected



G

MST



Correctness of Kruskal's

- Never adds an edge that connects already connected vertices
- Always adds lowest cost edge to connect two sets. By min cut property, that edge must be part of the MST

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

Running time of Kruskal's

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

|V| calls to MakeSet

Running time of Kruskal's

KRUSKAL(G)

for all $v \in V$
    MAKESET(v)
$T \leftarrow \{\}$
$\boxed{\text{sort the edges of } E \text{ by weight}}$
for all edges $(u, v) \in E$ in increasing order of weight
    if FIND-SET(u) $\neq$ FIND-SET(v)
        add edge to T
        UNION(FIND-SET(u),FIND-SET(v))

O(|E| log |E|)

Running time of Kruskal’s

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

2 |E| calls to FindSet

Running time of Kruskal's

```
KRUSKAL(G)
for all v ∈ V
        MAKESET(v)
T ← {}
sort the edges of E by weight
for all edges (u, v) ∈ E in increasing order of weight
        if FIND-SET(u) ≠ FIND-SET(v)
                add edge to T
                UNION(FIND-SET(u),FIND-SET(v))
```

|V| calls to Union

Running time of Kruskal's

- Disjoint set data structure

O(|E| log |E|) +

	MakeSet	FindSet |E| calls	Union |V| calls	Total
Linked lists	|V|	O(|V| |E|)	|V|	O(|V||E| + |E| log |E|) O(|V| |E|)
Linked lists + heuristics	|V|	O(|E| log |V|)	|V|	O(|E| log |V|+ |E| log |E|) O(|E| log |E|)

Prim's algorithm

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

Prim’s algorithm

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

```
DIJKSTRA(G, s)
for all v ∈ V
        dist[v] ← ∞
        prev[v] ← null
dist[s] ← 0
Q ← MAKEHEAP(V)
while !EMPTY(Q)
        u ← EXTRACTMIN(Q)
        for all edges (u, v) ∈ E
                if dist[v] > dist[u] + w(u, v)
                        dist[v] ← dist[u] + w(u, v)
                        DECREASEKEY(Q, v, dist[v])
                        prev[v] ← u
```

Prim's algorithm

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

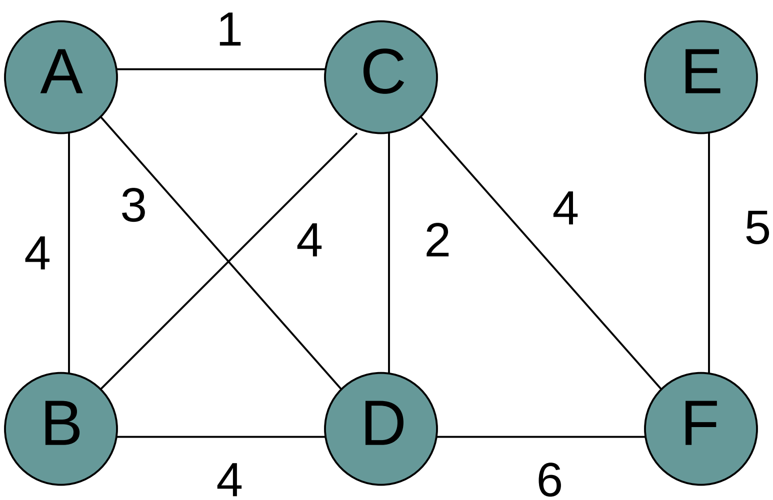
Prim's algorithm

- Start at some root node and build out the MST by adding the lowest weighted edge at the frontier

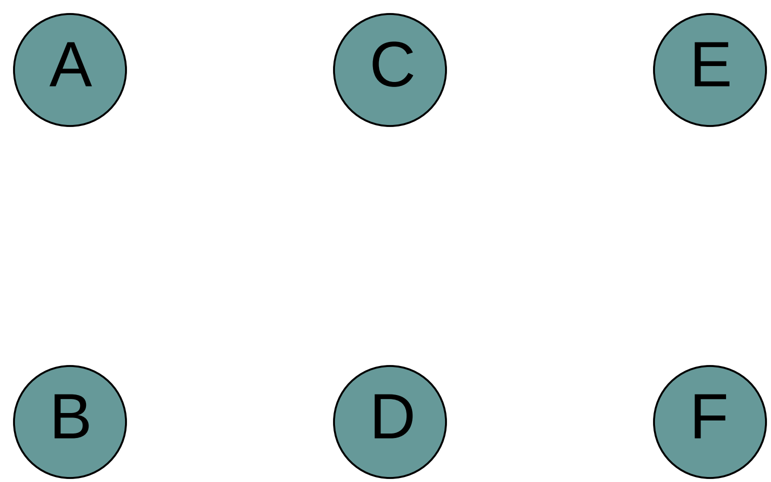
```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

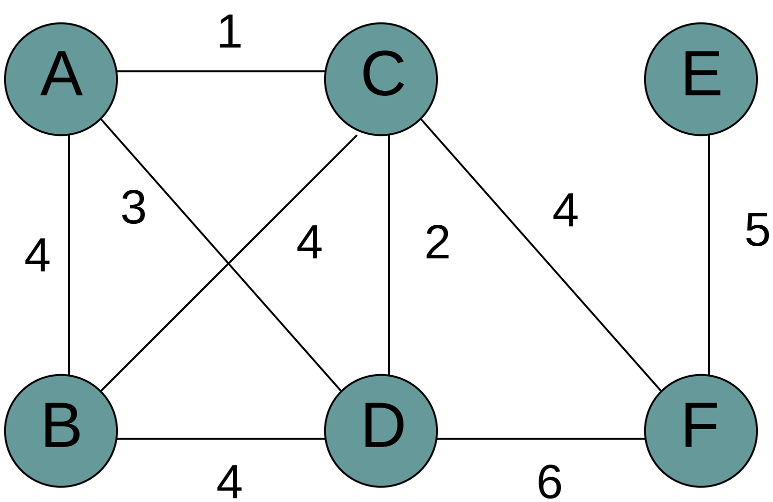


MST

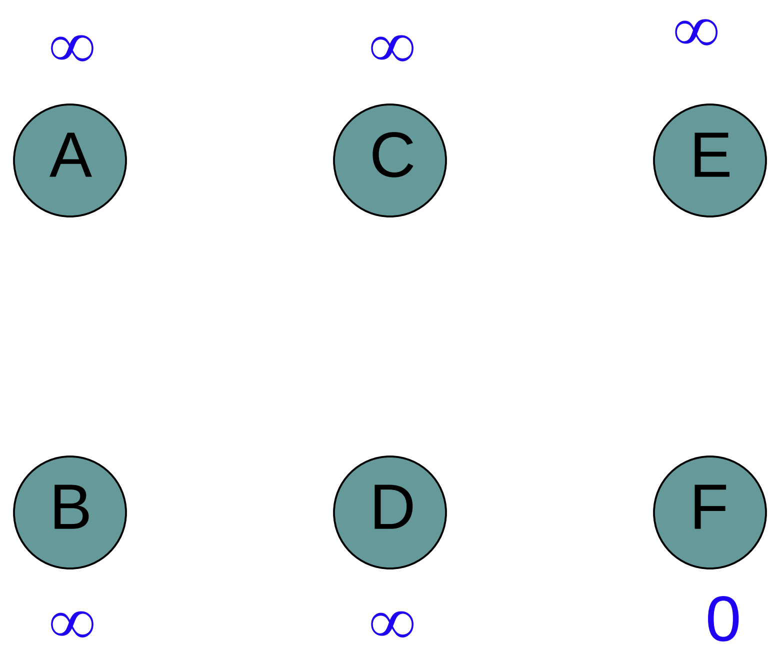


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

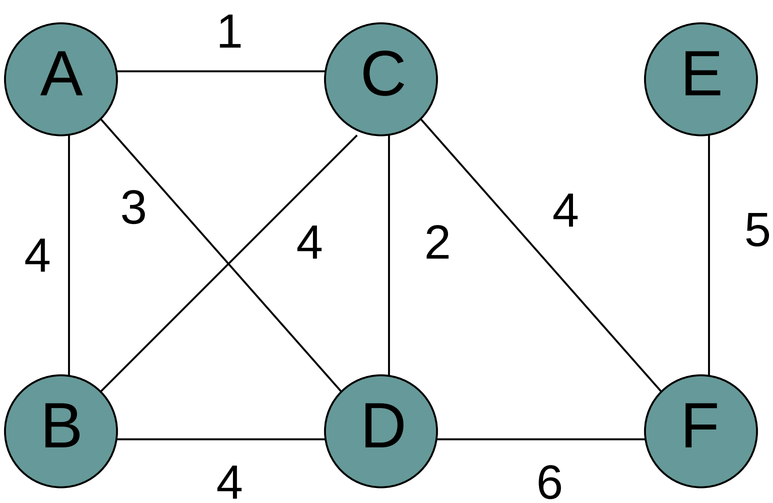


MST

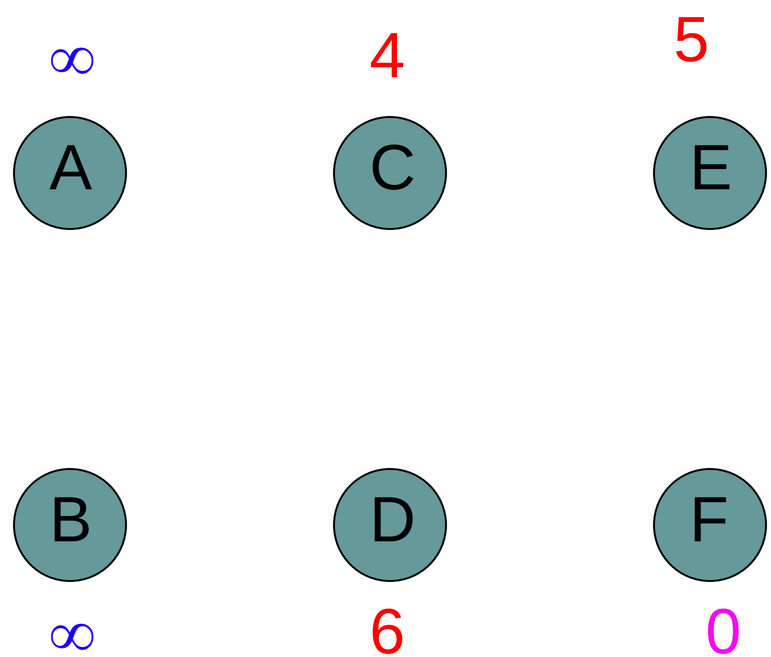


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

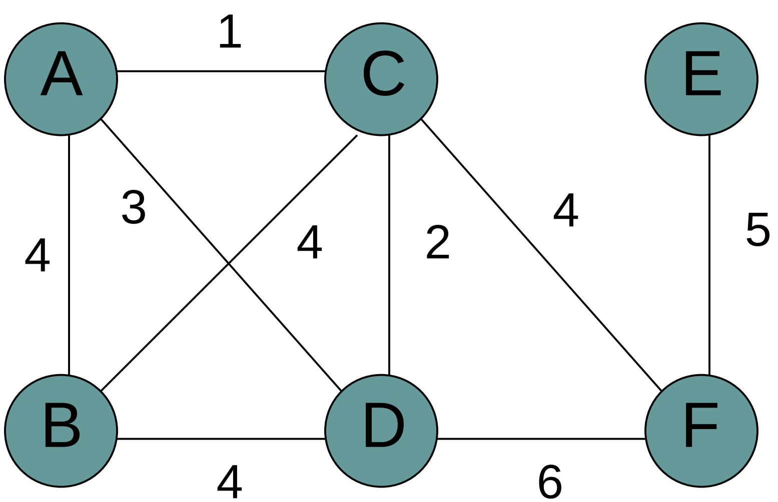


MST

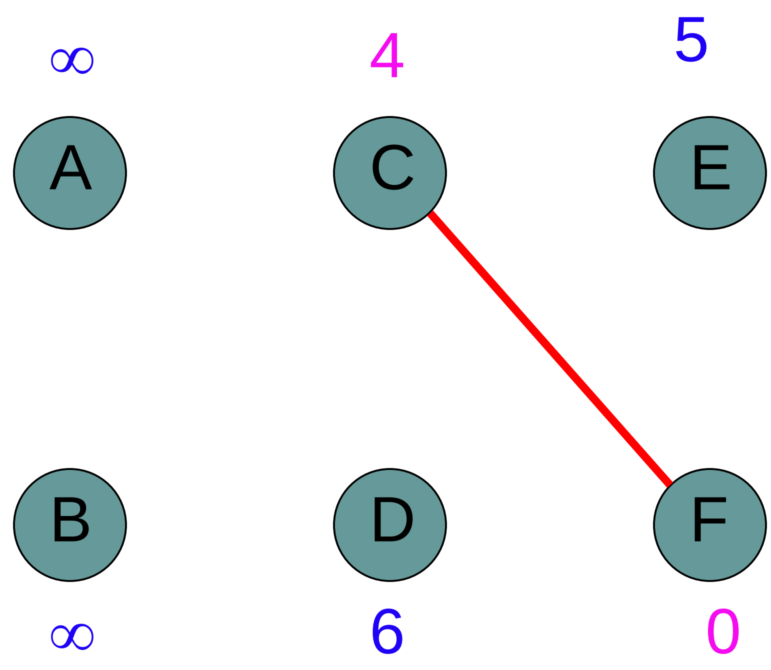


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

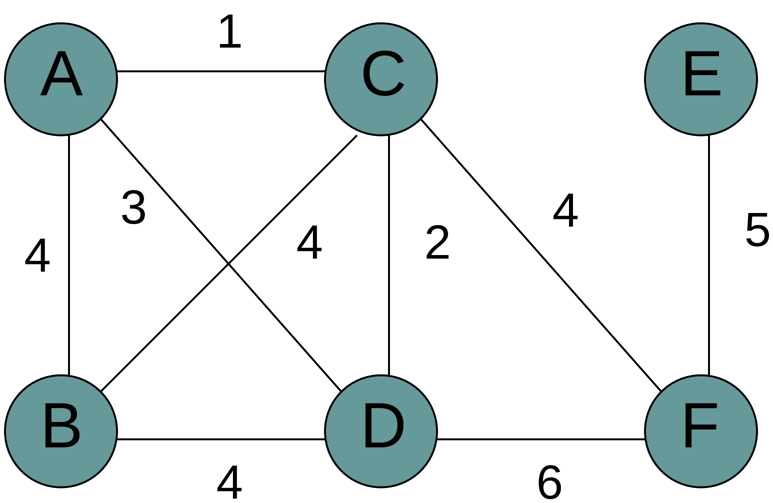


MST

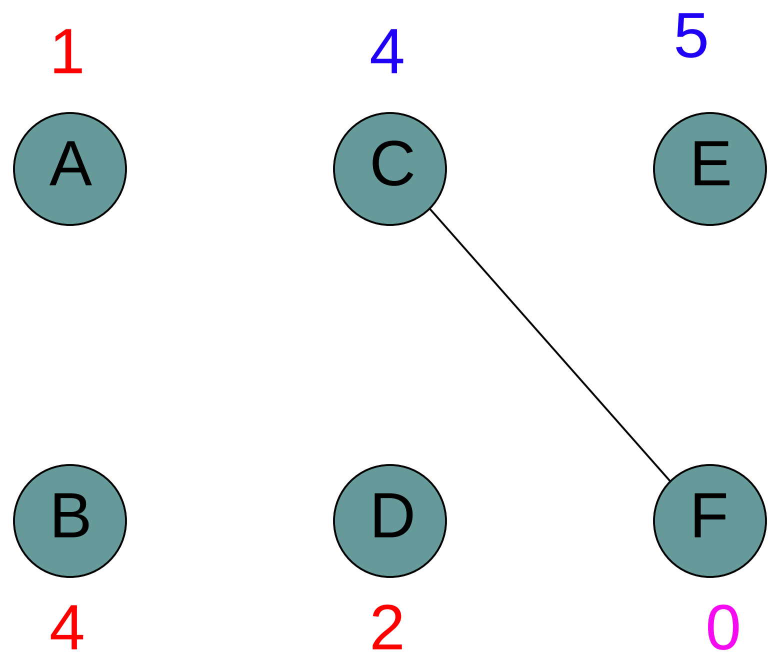


Prim's

```
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```

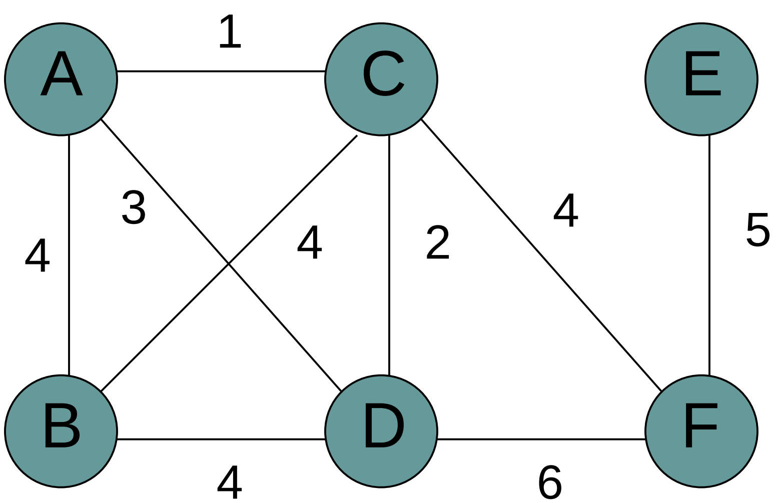


MST

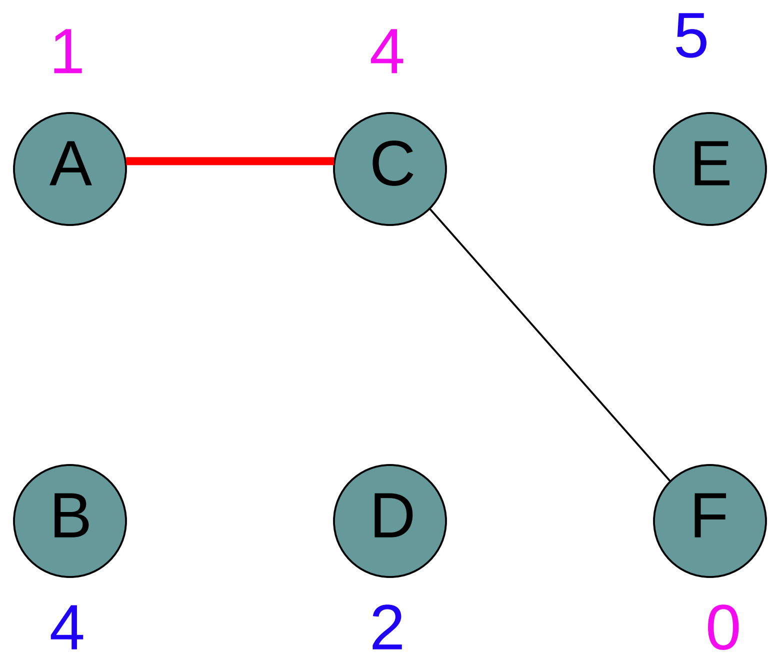


Prim's

```
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

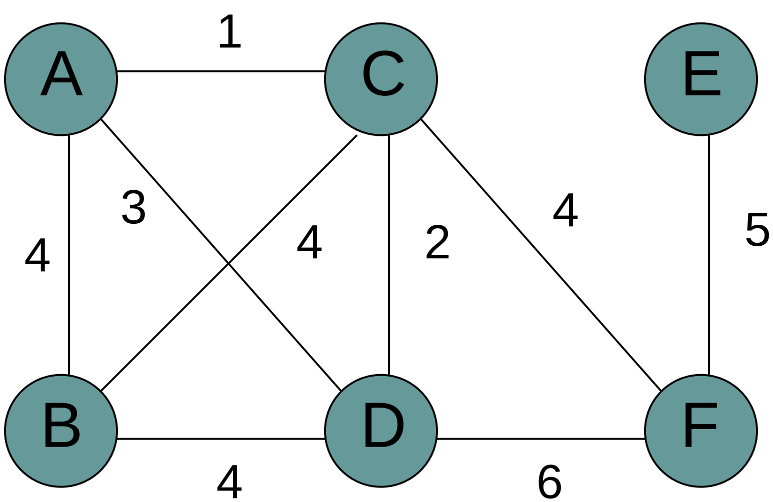


MST

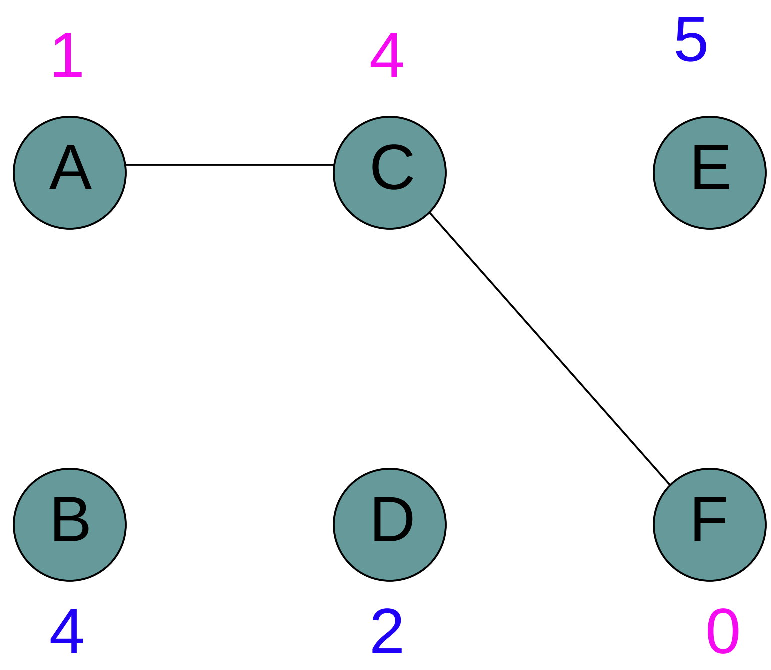


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

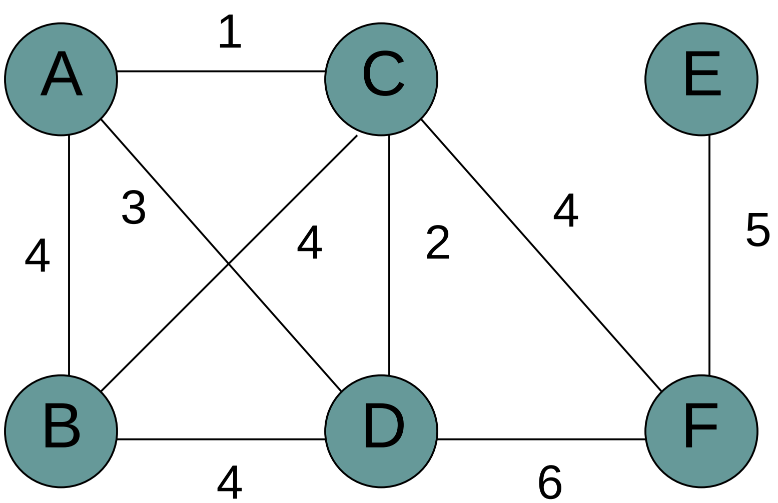


MST

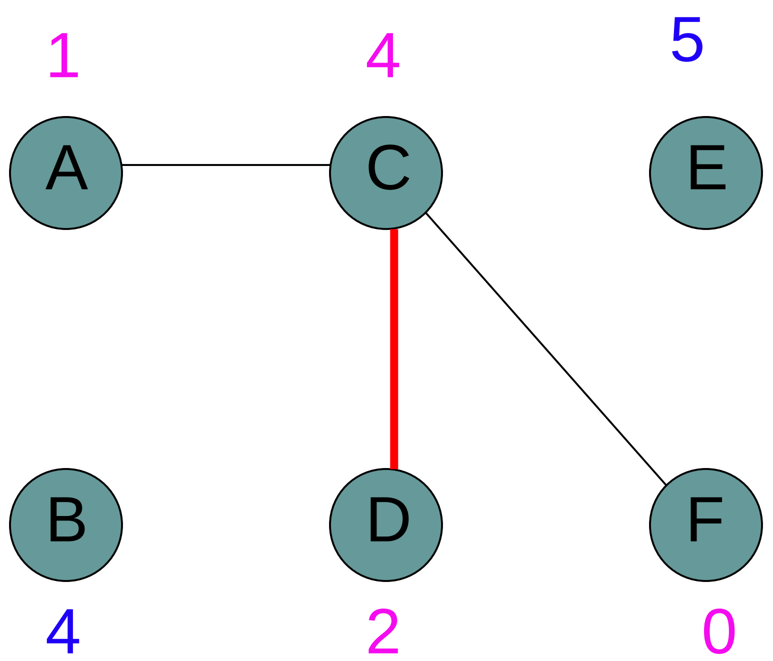


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

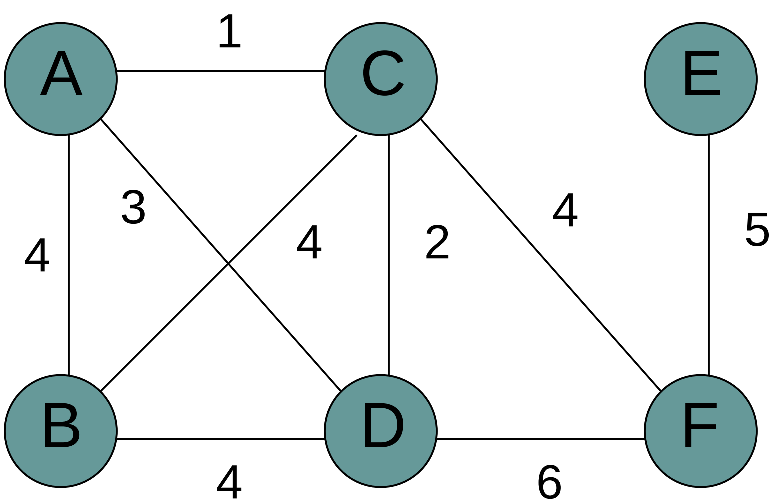


MST

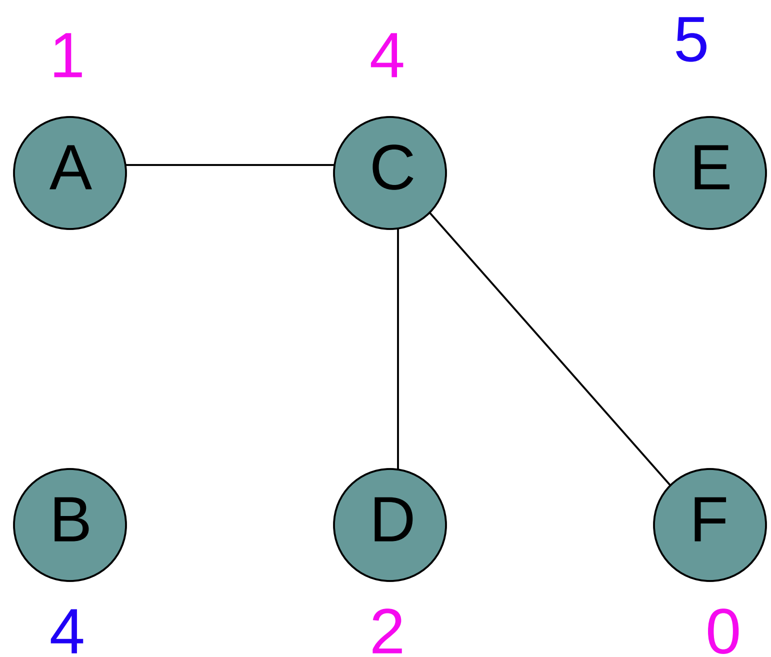


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

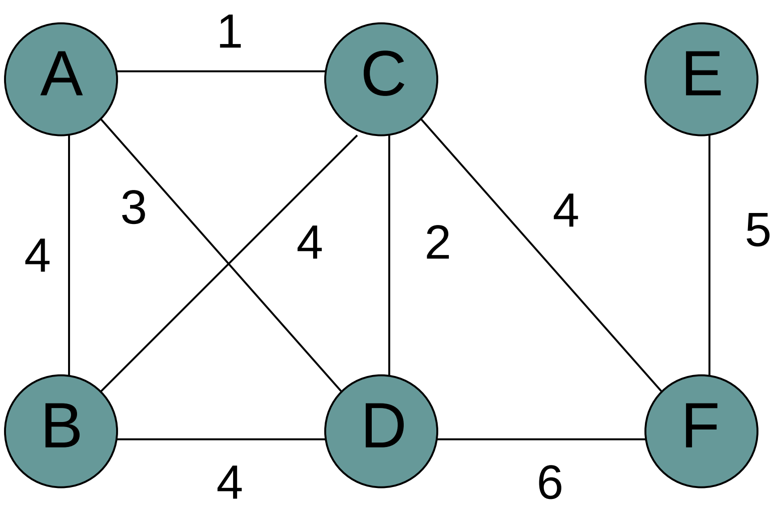


MST

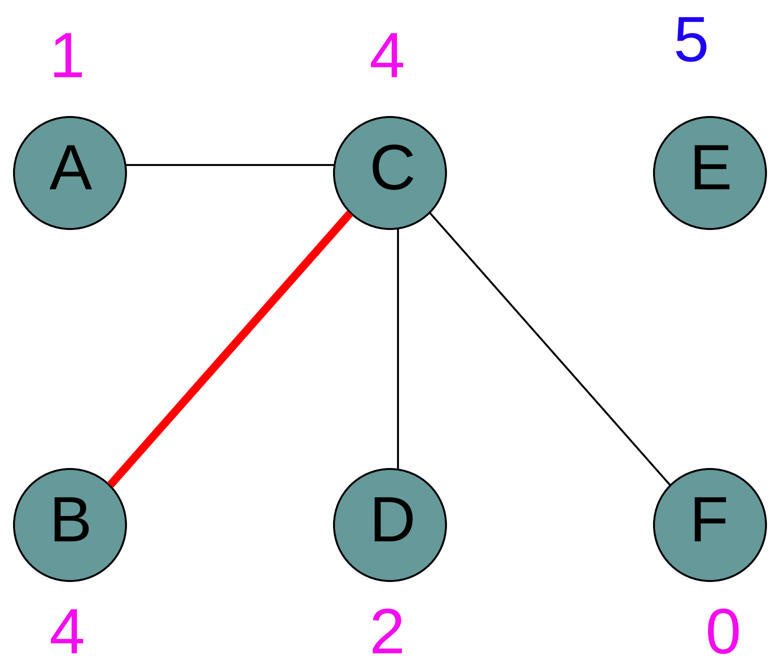


Prim's

```
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

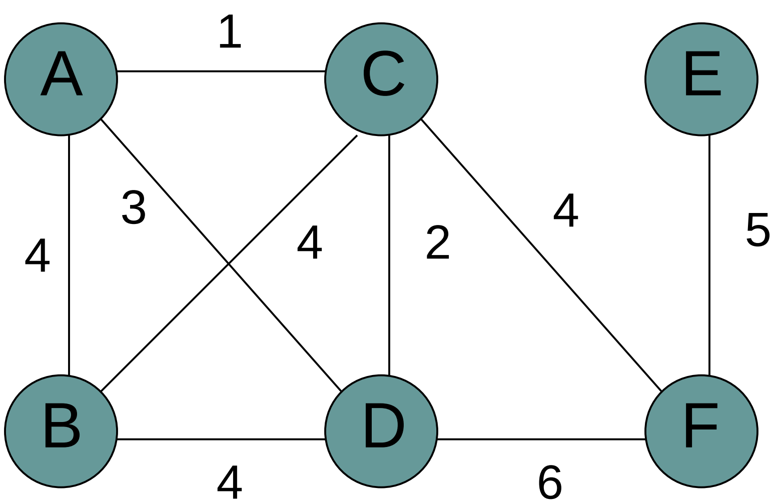


MST

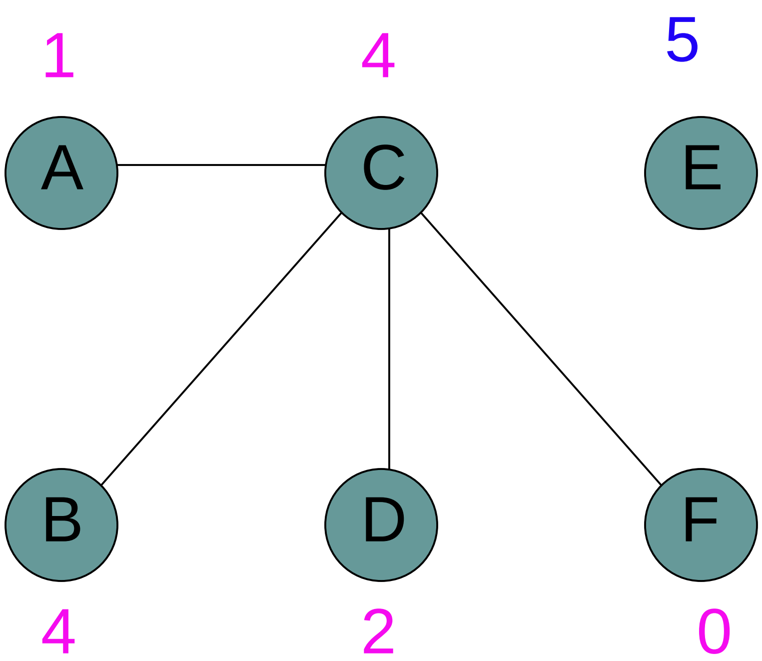


Prim's

```
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

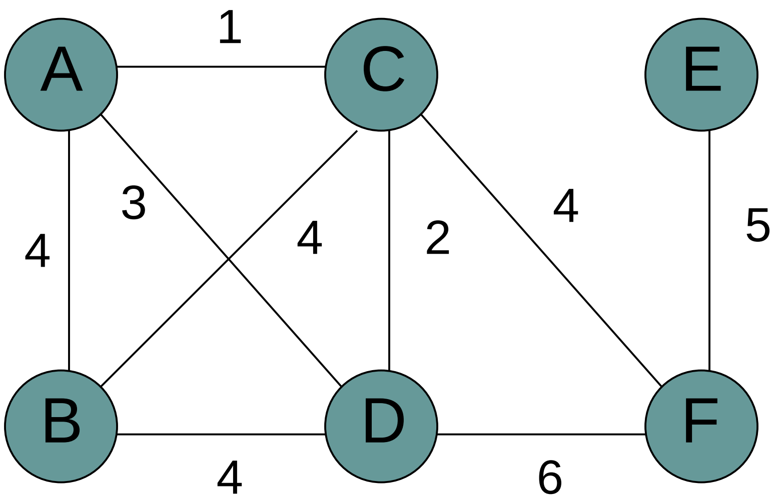


MST

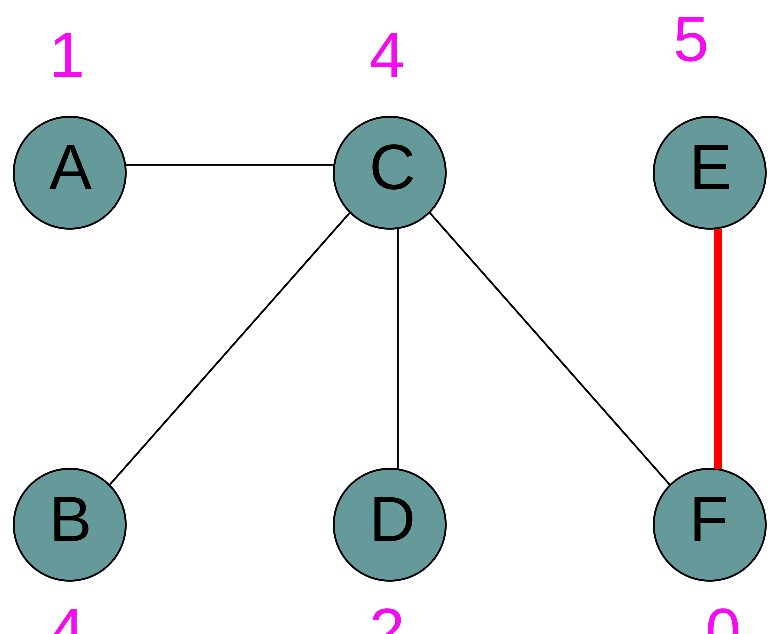


Prim's

```
while !Empty(H)
    u ← Extract-Min(H)
    visited[u] ← true
    for each edge (u, v) ∈ E
        if !visited[v] and w(u, v) < key(v)
            Decrease-Key(v, w(u, v))
            prev[v] ← u
```



MST



Correctness of Prim's?

- Can we use the min-cut property?
  - Given a partion S, let edge *e* be the minimum cost edge that **crosses** the partition. *Every* minimum spanning tree contains edge *e*.
- Let S be the set of vertices visited so far
- The only time we add a new edge is if it's the lowest weight edge from S to V-S

Running time of Prim's

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

$\Theta(|V|)$

Running time of Prim's

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

Θ(|V|)

Running time of Prim's

```
Prim(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MakeHeap(key)
while !Empty(H)
        u ← Extract-Min(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        Decrease-Key(v, w(u, v))
                        prev[v] ← u
```

|V| calls to Extract-Min

Running time of Prim's

```
PRIM(G, r)
for all v ∈ V
        key[v] ← ∞
        prev[v] ← null
key[r] ← 0
H ← MAKEHEAP(key)
while !Empty(H)
        u ← EXTRACT-MIN(H)
        visited[u] ← true
        for each edge (u, v) ∈ E
                if !visited[v] and w(u, v) < key(v)
                        DECREASE-KEY(v, w(u, v))
                        prev[v] ← u
```

|E| calls to Decrease-Key

Running time of Prim's

- Same as Dijksta's algorithm

	1 MakeHeap	$\|V\|$ ExtractMin	$\|E\|$ DecreaseKey	Total
Array	$O(\|V\|)$	$O(\|V\|^2)$	$O(\|E\|)$	$O(\|V\|^2)$
Bin heap	$O(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\| \log \|V\|)$	$O((\|V\|+\|E\|) \log \|V\|)$ $O(\|E\| \log \|V\|)$
Fib heap	$O(\|V\|)$	$O(\|V\| \log \|V\|)$	$O(\|E\|)$	**$O(\|V\| \log \|V\| + \|E\|)$**

Kruskal's: $O(|E| \log |E|)$