Chapter 2: Set Theory

2.1
Sets

2.2
Set Operations

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Agenda

- Ch 2.1
  - Set
  - The Power Set
  - Cartesian Products
  - Using Set Notation with Quantifiers
  - Truth Sets of Quantifiers
- Ch 2.2
  - Set Combination
  - Set Identifies
  - Generalized Unions and Intersections

Set

- **Definition**
  A set is an **unordered** collection of objects
- The objects in a set are called the **elements**, or **members**, of the set
- Notation:
  - a $\in$ A denote that a is an element of the set A
  - a $\notin$ A denotes that a is not an element of the set A

Set

- There are many ways to express the sets
  - **Listing all the elements**
  - **Set builder notation**
  - **Venn diagrams**

Set
Listing all the elements

$$S = \{e_1, e_2, e_3, \ldots, e_n\}$$

where e_i is **element** in the set

- Example
  - All vowels in the English alphabet: V = {a, e, i, o, u}
  - Odd positive integers < 10: O = {1, 3, 5, 7, 9}
  - Unrelated elements: U = {John, 3, *}
- **Ellipsis (…)** can be used to represent the general pattern of elements
  - Positive integers less than 100 can be denoted by {1, 2, 3, ..., 99}

Set
Set Builder

- Describe the properties the elements must have to be members

$$S = \{x \mid P(x)\}$$

S contains all the elements which make the predicate P true

- Example:
  - **R** = { x | x is integer < 100 and > 40}
  - **O** = { *x* | *x* is an odd positive integer less than 10}

    = $\{x \in Z^+ \mid x$ is odd and $x < 10\}$
    - Z^+ is the set of positive integers

Set Builder

- Important Sets:
  - **Real Numbers**

    R
  - **Natural Numbers**

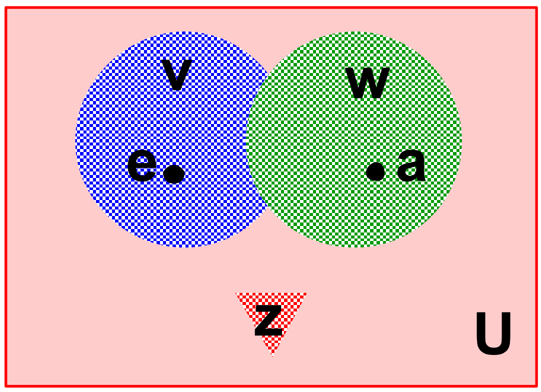
    N = {0, 1, 2, 3, ...}, counting numbers (sometimes not consider 0)
  - **Integers**

    Z = {..., -3, -2, -1, 0, 1, 2, 3, 4, ...}
    - **Positive / Negative Integers**: $\mathbf{Z}^{+}$ / $\mathbf{Z}^{-}$
  - **Rational Numbers**

    Q = $\{ p / q \mid p \in \mathbf{Z}, p \in \mathbf{Z}, \text{and } q \neq 0 \}$

Venn Diagrams

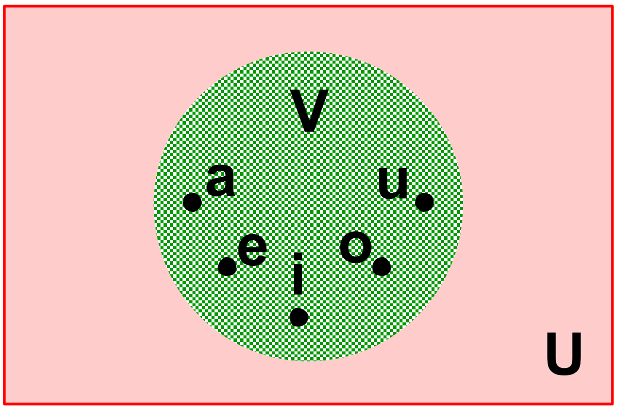
- Venn Diagrams are named after the English mathematician John Venn
- A rectangle represents the **universal set U**
  - Contains all the objects under consideration
  - U may varies depends on which objects are of interest



- Inside the rectangle, circles, or other geometrical figures are used to represent **sets**
  - Points may represents **elements**

Venn Diagrams

- Example
  - A Venn diagram that represents V, the set of vowels in the English alphabet
  - Rectangle : **U**
    - 26 letters of the English alphabet
  - Circle: **V**
    - the set of vowels
  - Elements: **a, e, i, o, u**



Set

- Two sets are equal if and only if they have the same elements
  - A and B are sets
  - A and B are equal if and only if

$$\forall x\ (x \in A \leftrightarrow x \in B)$$

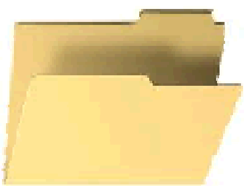
  - Notation (**=**)
    - We write A = B if A and B are equal sets

Empty Set and Singleton Set

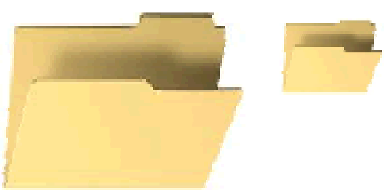
- **Empty set** (**null set**) is a special set that has no elements, denoted by $\varnothing$ or { }
- Example
  - The set of all positive integers that are greater than their squares is the null set
- A set with one element is called a **singleton set**

Set with Empty set

- A **common error** is to **confuse** with
  - $\varnothing$: the empty set
  - $\{\varnothing\}$: the set consisting of just the empty set
    - Singleton set:
      The single element is the empty set itself
- A useful analogy: **Folders**
  - **The empty set**
    - An empty folder



  - **The set consisting of just the empty set:**
    - A folder with exactly one folder inside, namely, the empty folder



Subset

- The set **A** is said to be a **subset** of **B** if and only if every element of A is also an element of B
- We use the notation **A** $\subseteq$ **B** to indicate that A is a subset of the set B
- We see that **A** $\subseteq$ **B** if and only if the quantification

$$\forall \mathbf{x}\ (\mathbf{x} \in \mathbf{A} \rightarrow \mathbf{x} \in \mathbf{B})$$

Subset

$\boxed{\textbf{Subset: } \forall \mathbf{x}\ (\mathbf{x} \in \mathbf{A} \rightarrow \mathbf{x} \in \mathbf{B})}$

- Every nonempty set S is guaranteed to have at least two subsets,
  - **Empty set ($\varnothing \subseteq$ S)**
    - x $\in \varnothing$ is always false
  - **Set S itself (S $\subseteq$ S)**
    - x $\in$ S $\rightarrow$ x $\in$ S must be true

Subset

- If A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$
- $\mathbf{A = B}$, where A and B are sets, if and only if

  $\forall x\ (x \in A \rightarrow x \in B)$ and $\quad A \subseteq B$

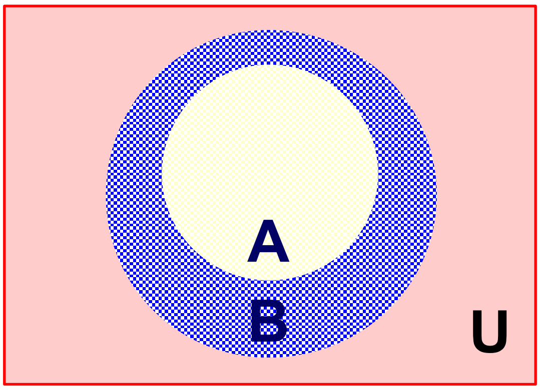
  $\forall x\ (x \in B \rightarrow x \in A)$, $\quad B \subseteq A$

  or equivalently if and only if

  $\forall x\ (x \in A \leftrightarrow x \in B) \quad A = B$

Subset: Proper Subset

- When we wish to emphasize that a set A is a **subset** of the set B but that $\mathbf{A \neq B}$, we write $\mathbf{A \subset B}$ and say that A is a **proper subset** of B
- For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A



- That is, A is a proper subset of B if

$$\forall x\ (x \in A \rightarrow x \in B) \boxed{\wedge\ \exists x\ (x \in B \rightarrow x \notin A)}$$

Subset

- Sets may have **other sets as members**
- Example:
  - **A** = {∅, {a}, {b}, {a, b}}
  - **B** = {x l x is a subset of the set {a, b}}
  - Note that **A = B**

    {a} ∈ **A**, but **a** ∉ **A**

Finite and Infinite Subset

- Let S be a set
- If there are exist n distinct elements in S
- **S** is a finite set and that **n** is the cardinality of S
- The cardinality of S is denoted by **|S|**
- Example:
  - A be the set of odd positive integers less than 10, **|A| = 5**
  - S be the set of letters in the English alphabet, **|S| = 26**
  - **|∅| = 0**
- A set is said to be **infinite** if it is not finite
  - The set of positive integers is infinite

Power Set

- Many problems involve testing all combinations of elements of a set to see if they satisfy some properties
- **Power set** of S is a set has as its members all the subsets of S
  - The power set of S is denoted by **P(S)**
- If a set has n elements, then its power set has 2^n elements

Power Set: Example

- What is the power set of {0, 1, 2}?
  - P({0, 1, 2}) =
    $\{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1, 2\}, \{0,1,2\}\}$
- What is the power set of {a}?
  - P({a}) = $\{\emptyset, \{a\}\}$
- What is the power set of $\emptyset$?
  - $P(\emptyset) = \{\emptyset\}$
- What is the power set of $\{\emptyset\}$?
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

Ordered n-tuple

- The **order** of elements in a collection is often important
- However, sets are unordered
- **Ordered n-tuple** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has
  - a_1 as its first element
  - a_2 as its second element
  - . . .
  - a_n as its n^{th} element

Ordered n-tuple

- Two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$

Ordered n-tuple

- Ordered 2-tuples are called **ordered pairs**
- The ordered pairs (a, b) and (c, d) are equal if and only if a = c and b = d
- Note that (a, b) and (b, a) are not equal unless a = b

Ordered n-tuple

Cartesian Products

- A subset **R** of the **Cartesian product** A **x** B is called a relation from the set A to the set B
- The elements of **R** are ordered pairs, where the first element belongs to A and the second to B

Ordered n-tuple
Cartesian Products

- Let A and B be sets
- The Cartesian product of A and B, denoted by **A x B**, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$

$$\mathbf{A \times B = \{(a, b) \mid a \in A \wedge b \in B\}}$$

Ordered n-tuple
Cartesian Products: Example 1

- Given A = {1, 2} and B = {a, b, c}
- What are A x B and B x A?

- A x B =
  {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}
- B x A =
  {(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)}

- A x B and B x A are not equal, unless
  - $A = \varnothing$ or $B = \varnothing$ (so that $A \times B = \varnothing$) or
  - A = B

Ordered n-tuple
Cartesian Products: Example 3

- Given
  - **A** represent the set of all students at a university
  - **B** represent the set of all courses offered at the university
- What is the meaning of A x B?
- A x B represents all possible enrollments of students in courses at the university

Ordered n-tuple
Cartesian Products

- Generally, the Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $\mathbf{A_1 \times A_2 \times \ldots \times A_n}$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for i = 1, 2, ..., n.

  $$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1,2,\ldots,n\}$$

Ordered n-tuple

Cartesian Products: Example 3

- What is $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?
- $A \times B \times C =$
  $\{(0,1,0), (0,1,1), (0,1,2),$
  $(0,2,0), (0,2,1), (0,2,2),$
  $(1,1,0), (1,1,1), (1,1,2),$
  $(1,2,0), (1,2,1), (1,2,2)\}$

Set Notation with Quantifiers

- Sometimes we restrict the domain of a quantified statement explicitly by making use of set
- Example
  - $\boxed{\forall x \in S}$ $(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S
    - $\forall x \in S\ (P(x))$ is shorthand for $\forall x\ (x \in S \rightarrow P(x))$
  - Similarly, $\boxed{\exists x \in S}$ $(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S
    - $\exists x \in S\ (P(x))$ is shorthand for $\exists x\ (x \in S \land P(x))$

Set Notation with Quantifiers

- Example
  - What do the statements
    $\forall x \in R\ (x^2 \geq 0)$ and $\exists x \in Z\ (x^2 = 1)$ mean?
  - $\forall x \in R\ (x^2 \geq 0)$
    - For every real number x, $x^2 \geq 0$
    - The square of every real number is nonnegative
  - $\exists x \in Z\ (x^2 = 1)$
    - There exists an integer x such that $x^2 = 1$
    - There is an integer whose square is 1

Truth Sets of Quantifiers

- We will now tie together concepts from set theory and from predicate logic
- Given a predicate P, and a domain D, we define **the truth set** of P to be the set of elements x in D for which P(x) is true
- The **truth set** of P(x) is denoted by
  $\{x \in D \mid P(x)\}$

Truth Sets of Quantifiers

- Given the domain is the set of integers, what is the truth set of the following predicate?
  - **P(x) is "$|x| = 1$"**
    - $|x| = 1$ when $x = 1$ or $x = -1$
    - The truth set of P is the set $\{-1, 1\}$
  - **Q(x) is "$x^2 = 2$"**
    - There is no integer x for which $x^2 = 2$
    - The truth set of Q is empty set
  - **R(x) is "$|x| = x$"**
    - $|x| = x$ if and only if $x \geq 0$
    - The truth set of R is N, the set of nonnegative integers

Truth Sets of Quantifiers

- Note that
  - $\forall x\, P(x)$ is true over the domain U if and only if the truth set of P is the set U
  - $\exists x\, P(x)$ is true over the domain U if and only if the truth set of P is non empty

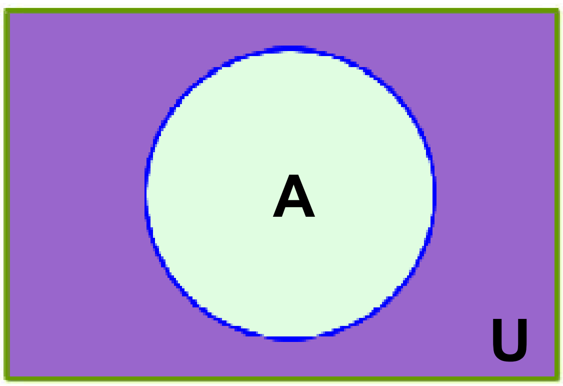
Set Combination

- Two sets can be combined in many different ways
  - Complement (‾)
  - Union (∪)
  - Intersection (∩)
  - Difference (-)
  - Symmetric Difference (⊕)

Complement

- Let U be the universal set
  The **complement** of the set **A**, denoted by $\overline{\mathbf{A}}$, is the complement of A with respect to U
- The complement of the set $\overline{A}$ is **U - A**.
- An element x belongs to $\overline{A}$ if and only if $x \notin A$

  $\overline{A} = \{x \mid x \notin A\}$

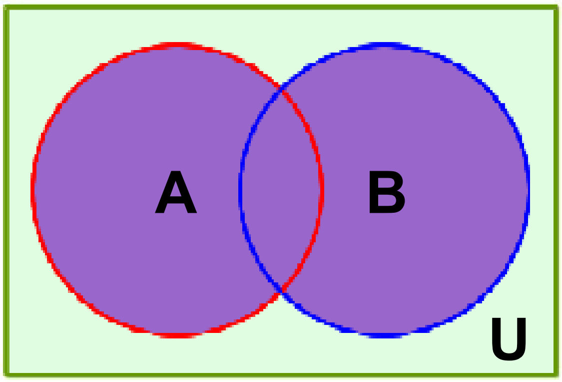


Union

- Let **A** and **B** be sets
  Union of the sets **A** and **B**, denoted by **A U B**, is the **set** that contains those elements that are either in A or in B, or in both
- An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B

  $A \cup B = \{ x \mid x \in A \vee x \in B \}$

- Notation: **U** (**U**nion)

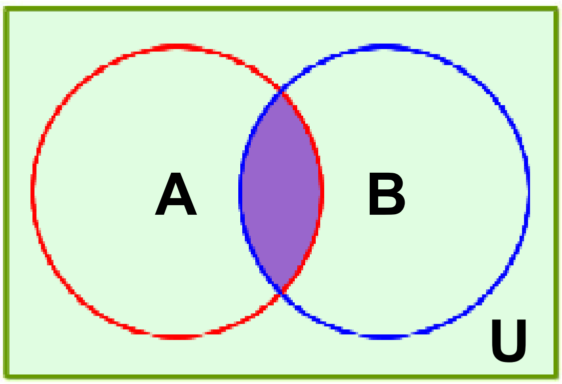


Intersection

- Let **A** and **B** be sets
  Intersection of the sets **A** and **B**, denoted by **A ∩ B**, is the set containing those elements in both A and B
- An element x belongs to the intersection of the sets A and B if and only if x belongs to A and B

  $A \cap B = \{ x \mid x \in A \wedge x \in B \}$

- Notation: ∩ (i∩teraction)

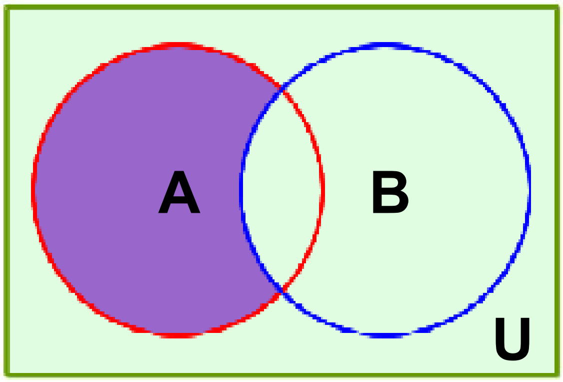


Difference

- Let **A** and **B** be sets
  Difference of **A** and **B**, denoted by **A - B**, is the set containing those elements that are in A but not in B
- The difference of A and B is also called the complement of B with respect to A
- An element x belongs to the difference of A and B if and only if
  $x \in A$ and $x \notin B$

  $A - B = \{x \mid x \in A \wedge x \notin B\}$

  $A - B = A \cap \overline{B}$



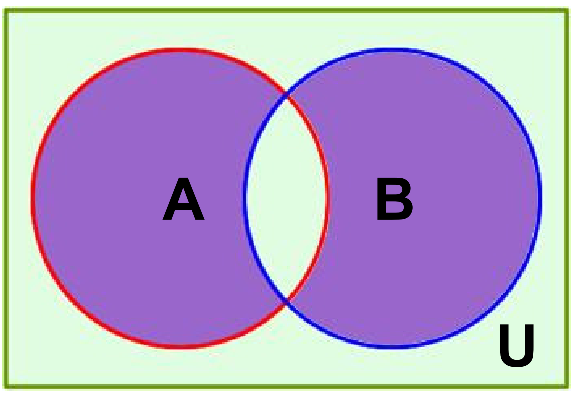
Symmetric Difference

- Let **A** and **B** be sets
  Symmetric Difference of **A** and **B**, denoted by **A ⊕ B**, is the set containing those elements is either in A or B, but not in both
- An element x belongs to the symmetric different of the sets A and B if and only if x belongs to A XOR B

$A \oplus B = \{ x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) \}$

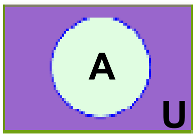
$A \oplus B = (A - B) \cup (B - A)$

$A \oplus B = (A \cup B) - (B \cap A)$

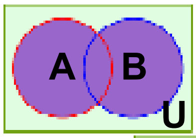


Set Combination
Summary

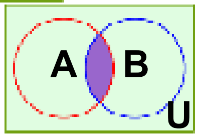
- $\overline{A} = \{x \mid x \notin A\}$



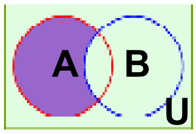
- $A \cup B = \{ x \mid x \in A \vee x \in B \}$



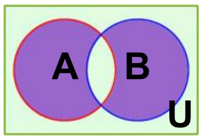
- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$



- $A - B = \{x \mid x \in A \wedge x \notin B\}$



- $A \oplus B = \{ x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B) \}$



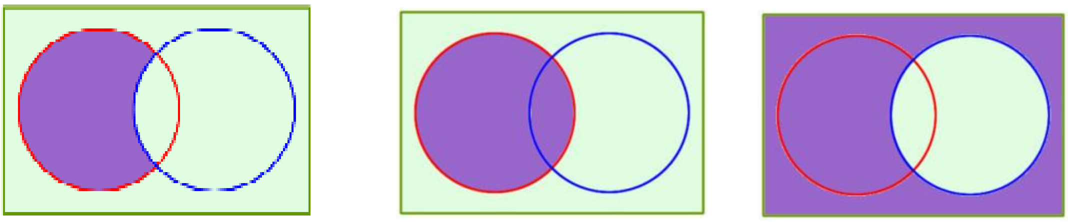
Set Combination: Example

- Universal set is {1…6},
- A = {1, 3, 5} and B = {1, 2, 3}
- $\overline{A}$ = {2, 4, 6}
- $A \cup B$ = {1, 2, 3, 5}
- $A \cap B$ = {1, 3}
- A – B = {5}
- B – A = {2}
- $A \oplus B$ = {2, 5}

Set Combination: Property

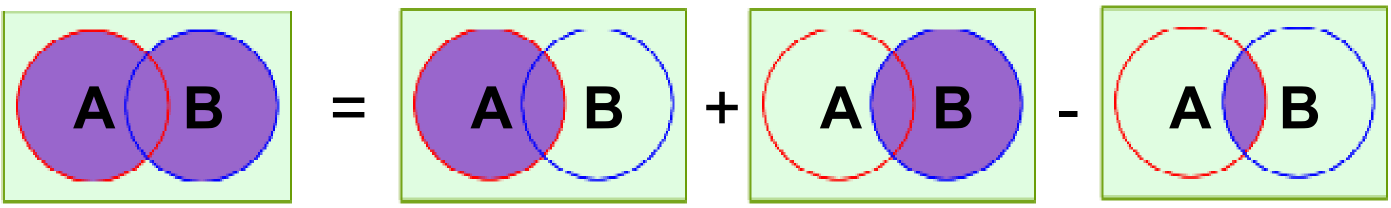
$$A - B = \{x \mid x \in A \land x \notin B\}$$
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

$$A - B = A \cap \overline{B}$$



Set Combination: Property

$$|A \cup B| = |A| + |B| - |A \cap B|$$

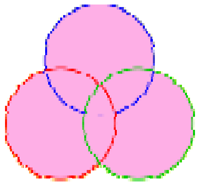


- The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion-exclusion**

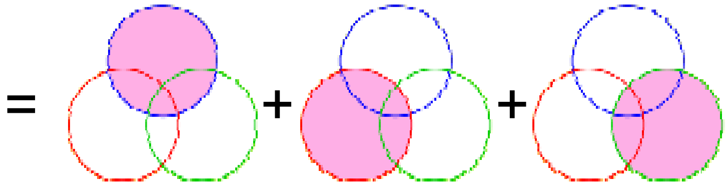
Set Combination: Property

- **Principle of Inclusion-Exclusion for three sets:**

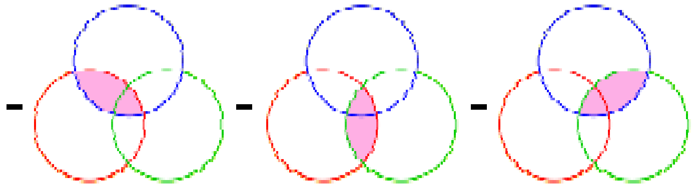
$$|A \cup B \cup C|$$



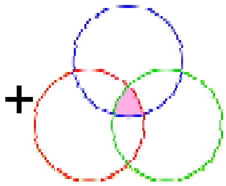
$$= |A| + |B| + |C|$$



$$- |A \cap B|$$
$$- |B \cap C|$$
$$- |A \cap C|$$

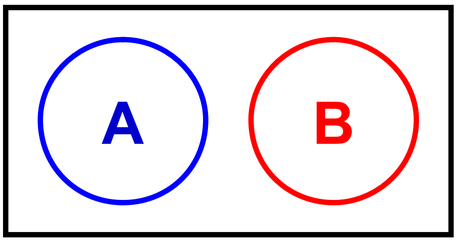


$$+ |A \cap B \cap C|$$



Set Combination: Property

- Two sets are called **disjoint** if their intersection is the empty set



- Example:
  - $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$
  - $A \cap B = \varnothing$
  - A and B are disjoint

Set Identi...

Recall…
In Chapter 1

Identify Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$p \vee (q \vee r) \equiv (p \vee q) \vee r$ $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$

For Set…

Identity Laws	$A \cup \varnothing = A$ $A \cap U = A$
Domination Laws	$A \cup U = U$ $A \cap \varnothing = \varnothing$
Idempotent Laws	$A \cup A = A$ $A \cap A = A$
Complementation Law	$(\overline{\overline{A}}) = A$
Commutative Laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative Laws	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's Laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Complement Laws	$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$

Set Identifies

- How to show two sets (A and B) are identical?
  - Membership Table
  - Builder Notation
  - Subset (i.e. $A \subseteq B$ and $B \subseteq A$)

Membership Table

- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Using membership table

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Builder Notation

- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Using Builder Notation and **equivalence rules**

$$\overline{A \cap B}$$
$$= \{ x \mid x \notin (A \cap B) \}$$
$$= \{ x \mid \neg((x \in A) \wedge (x \in B)) \}$$
$$= \{ x \mid \neg(x \in A) \vee \neg(x \in B)) \}$$
$$= \{ x \mid (x \notin A) \vee (x \notin B) \}$$
$$= \{ x \mid (x \in \overline{A}) \vee (x \in \overline{B}) \}$$
$$= \overline{A} \cup \overline{B}$$

Subset

- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- Using subset (**implication & equivalence rules**)
  - Show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$$\overline{A \cap B}$$
Let $x \notin (A \cap B)$
$$= \neg((x \in A) \wedge (x \in B))$$
$$= \neg(x \in A) \vee \neg(x \in B))$$
$$= (x \notin A) \vee (x \notin B)$$
$$= (x \in \overline{A}) \vee (x \in \overline{B})$$
Therefore, subset of $\overline{A} \cup \overline{B}$

  - Show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$$\overline{A} \cup \overline{B}$$
Let $(x \in \overline{A}) \vee (x \in \overline{B})$
$$= (x \notin A) \vee (x \notin B)$$
$$= \neg(x \in A) \vee \neg(x \in B))$$
$$= \neg((x \in A) \wedge (x \in B))$$
$$= x \notin (A \cap B)$$
Therefore, subset of $\overline{A \cap B}$

Generalized Unions and Intersections

- **Union** of a **collection of sets** is the set that contains those elements that are members of at least one set in the collection

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- **Intersection** of a **collection of sets** is the set that contains those elements that are members of all the sets in the collection

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

n maybe infinite

Generalized Unions and Intersections

- Another notation

Set of i, e.g. {1..n}

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I (x \in A_i)\}$$

x is union of all A_i

For any i, $x \in A_i$ is correct
x is an element in any A_i

Set of i, e.g. {1..n}

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I (x \in A_i)\}$$

x is intersection of all A_i

For all i, $x \in A_i$ is correct
x is an element in all A_i

Generalized Unions and Intersections

- Example 1
  - Let $A = \{0,2,4,6,8\}$, $B = \{0,1,2,3,4\}$, $C = \{0,3,6,9\}$
  - What are $A \cup B \cup C$ and $A \cap B \cap C$?
    - $A \cup B \cup C =$ **$\{0,1,2,3,4,6,8,9\}$**
    - $A \cap B \cap C =$ **$\{0\}$**

Generalized Unions and Intersections

- Example 2
  - Suppose that $A_i = \{1,2,3,\ldots,i\}$ for $i = 1,2,3,\ldots$

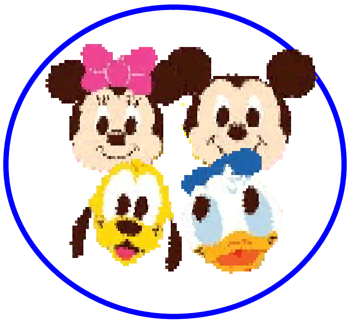
$$\bigcup_{i\in I} A_i = \bigcup_{i\in I} \{1,2,3,\ldots,i\} = \{1,2,3,\ldots i\}$$

$$\bigcap_{i\in I} A_i = \bigcap_{i\in I} \{1,2,3,\ldots,i\} = \{1\}$$

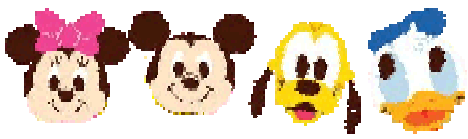
Computer Representation of Sets

- Many ways to represent sets in a computer
- One method is to store the elements of the set in an unordered fashion
  - E.g. in C++, we can use set to store set
    - `set <int> a;`
    - `a.insert(9);`
  - The operations of computing the union, intersection, or difference of two sets would be time-consuming
    - Including searching a large amount of element
- A easier way is discussed

Computer Representation of Sets

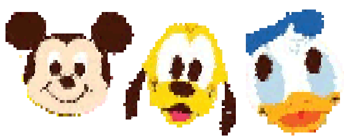


- Assume the universal set U is
  - Finite
  - Reasonable size
    - Smaller than the memory size
- **Methods**
  - First, specify an *arbitrary ordering* of the elements of U, for instance $a_1, a_2, \ldots, a_n$
  - Represent a subset A with the *bit string of length n*, where the i^{th} bit in this string is
    - **1** if a_i belongs to A
    - **0** if a_i does not belong to A



a_1 a_2 a_3 a_4

1010



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Computer Representation of Sets

- Equal =
- Union bitwise OR
- Intersection bitwise AND
- Complement bitwise NOT

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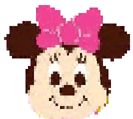

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Computer Representation of Sets

- Example
  - Let U = {1,2,3,4,5,6,7,8,9,10}
    A = {1, 3, 5, 7, 9}
    B = {1, 2, 3, 4, 5}
  - What is the **bit string** of
    - A **1010101010**
    - B **1111100000**
    - $\overline{B}$ **0000011111**
    - A ∩ B **1010100000**
    - A U B **1111101010**