Logical Implication & Equivalence

- **For Universal Quantifiers,**
  - $\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\,P(x) \wedge \forall x\,Q(x)$
  - $\forall x\,P(x) \vee \forall x\,Q(x) \rightarrow \forall x\,(P(x) \vee Q(x))$
- **For Existential Quantifiers,**
  - $\exists x\,(P(x) \wedge Q(x)) \rightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$
  - $\exists x\,(P(x) \vee Q(x)) \equiv \exists x\,P(x) \vee \exists x\,Q(x)$

Quantifiers: Logical Equivalence

- $\forall x\,(A \wedge P(x)) \equiv A \wedge \forall x\,P(x)$
- $\forall x\,(A \vee P(x)) \equiv A \vee \forall x\,P(x)$
- $\exists x\,(A \wedge P(x)) \equiv A \wedge \exists x P(x)$
- $\exists x\,(A \vee P(x)) \equiv A \vee \exists x P(x)$
- $\boxed{\forall x\,P(x) \rightarrow A \equiv \exists x\,(P(x) \rightarrow A)}$
- $A \rightarrow \forall x\,P(x) \equiv \forall x\,(A \rightarrow P(x))$
- $\boxed{\exists x\,P(x) \rightarrow A \equiv \forall x\,(P(x) \rightarrow A)}$
- $A \rightarrow \exists x\,P(x) \equiv \exists x\,(A \rightarrow P(x))$

* A does not consist of free variable x

$$\forall x P(x) \rightarrow A$$
$$\equiv \neg(\forall x P(x)) \vee A$$
$$\equiv \exists x\,(\neg P(x)) \vee A$$
$$\equiv \exists x\,(\neg P(x) \vee A)$$
$$\equiv \exists x\,(P(x) \rightarrow A)$$

$$A \rightarrow \forall x P(x)$$
$$\equiv \neg(A) \vee \forall x P(x)$$
$$\equiv \forall x(\neg(A) \vee P(x))$$
$$\equiv \forall x(A \rightarrow P(x))$$

Negating Quantifiers
Universal Quantification

- De Morgan's Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

Not all students are good   There is a student is bad

P(x): x is a good student

Negating Quantifiers
Existential Quantification

- De Morgan's Laws for Quantifiers

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

There is not exist a good student   All students are bad

P(x): x is a good student

☺ Small Exercise ☺

- What are the negation of the following statements?
  - $\forall x\ (x^2>x)$
    - $\neg\forall x(x^2>x) \equiv$
  - $\exists x\ (x^2=2)$
    - $\neg\exists x(x^2=2) \equiv$

☺ Small Exercise ☺

- Show that

$$\neg\forall x(P(x)\to Q(x)) \equiv \exists x(P(x)\wedge\neg Q(x))$$

$$\neg\forall x\ (P(x)\to Q(x))$$

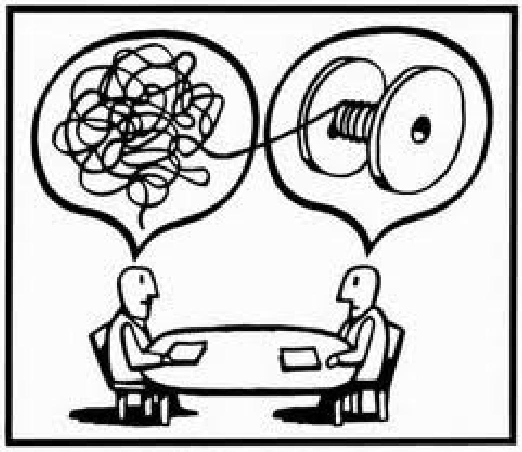
$$\equiv \neg\forall x\ (\neg P(x)\vee Q(x))$$

$$\equiv \exists x\ \neg(\neg P(x)\vee Q(x))$$

$$\equiv \exists x\ (P(x)\wedge\neg Q(x))$$

Translation Using Quantifiers

- Translating from **English** to **Logical Expressions with quantifiers**



Translation Using Quantifiers

Universal Quantification

- Using predicates and quantifiers, express the statement

Every student in this class is lazy

$\forall$ — x — Universe of Discourse — Predicate

- Quantifier: **Universal Quantifier**
- Variable: ***x***
- Universe of discourse: **the students in the class**
- Propositional Function: **P(x) : x is lazy**
- Answer: $\forall x\, P(x)$

Translation Using Quantifiers

Universal Quant

The universal quantifier connects with a implication

- Another way to express the statement:

Every student in this class is lazy

$\forall$ — x — Predicate (Q) — Predicate (P)

  - Quantifier: **Universal Quantifier**
  - Variable: ***x***
  - Universe of discourse: **Any person**
  - Propositional Function: **P(x): x is lazy**
    Q(x): x is a student in this class

- Answer:

  $\forall x\,(Q(x) \rightarrow P(x))$ ✓

  For every person, if he/she is in this class, he/she is lazy

  $\forall x\,(Q(x) \wedge P(x))$ ✗

  For every person, he/she is in this class and lazy

Translation Using Quantifiers

Existential Quantification

- Using predicates and quantifiers, express the statement

Some students in this class are lazy

$\exists$ — x — Universe of Discourse — Predicate

  - Quantifier: **Existential Quantifier**
  - Variable: ***x***
  - Universe of discourse: **the students in the class**
  - Propositional Function: **P(x) : x is lazy**
  - Answer: $\exists x\, P(x)$

Existential Quan

The existential quantifier connects with a conjunction

- Another way to express the statement:

$\boxed{\text{Some}}\ \boxed{\boxed{\text{students}}\ \text{in this class}}\ \boxed{\text{are lazy}}$

$\exists$ x Predicate (Q) Predicate (P)

  - Quantifier: **Existential Quantifier**
  - Variable: x
  - Universe of discourse: **Any person**
  - Propositional Function: **P(x): x is lazy**
    Q(x): x is a student in this class

- Answer:

Include the case which contains no person in this class

$\exists x\, (Q(x) \rightarrow P(x))$ ✘

For some persons, if he/she is in this class, he/she is lazy

$\exists x\, (Q(x) \wedge P(x))$ ✓

For some persons, he/she is in this class and lazy

☺ Small Exercise ☺

- Using predicates and quantifiers, set the domain as
  1. Staff in IBM company
  2. Any persons

  express the following statements:
  - Every staff in IBM company has visited Mexico
  - Some staff in IBM company has visited Canada or Mexico

☺ Small Exercise ☺

- Every staff in IBM company has visited Mexico
  - **Solution 1:**
    - Universal Quantifier
    - Variable: x
    - U.D.: Staffs in IBM company
    - Let $P(x)$: x has visited Mexico
    - $\forall x\, P(x)$
  - **Solution 2:**
    - Universal Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $Q(x)$: x is a staff in IBM company
    - Let $P(x)$: x has visited Mexico
    - $\forall x\, (Q(x) \rightarrow P(x))$

☺ Small Exercise ☺

- Some staff in IBM company has visited Canada or Mexico
  - **Solution 1:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Staffs in IBM company
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\, (P(x) \vee Q(x))$
  - **Solution 2:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a staff in IBM company
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\, (S(x) \wedge (P(x) \vee Q(x)))$

☺ Small Exercise ☺

- Some students in this class has visited Canada or Mexico
  - **Better Solution:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a student in this class
    - Let $P(x, loc)$: x has visited loc
    - $\exists x\ (S(x) \land (P(x, Canada) \lor P(x, Mexico)))$
  - **Solution 2:**
    - Existential Quantifier
    - Variable: x
    - U.D.: Any person
    - Let $S(x)$: x is a student in this class
    - Let $P(x)$: x has visited Mexico
    - Let $Q(x)$: x has visited Canada
    - $\exists x\ (S(x) \land (P(x) \lor Q(x)))$

Quantifiers with Restricted Domains

- An **abbreviated notation** is often used to restrict the domain of a quantifier
- Example
  - the square of any real number $\boxed{\text{which greater than 10}}$ is greater than 100
  - Using Domain
    $\forall x\ (x^2 > 100)$,
    U.D.s: the set of real number which is bigger than 10
  - Using Predicate
    $\forall x\ (x>10 \rightarrow x^2>0)$, U.D.s: the set of real number
  - Using Abbreviated Notation
    $\forall x{>}10\ (x^2>100)$, U.D.s: the set of real number

Quantifiers with Restricted Domains

- Example
  - Given that the domain in each case consists of the real number, what do the following statements mean?
    - $\forall x<0\ (x^2>0)$

      The square of negative real number is positive
    - $\forall y\neq 0\ (y^3\neq 0)$

      The cube of nonzero real number is nonzero
    - $\exists z>0\ (z^2=2)$

      There is a positive square root of 2

☺ Small Exercise ☺

- Using predicates and quantifiers, express the following statements:
  - Every mail message larger than one megabyte will be compressed
  - If a user is active, at least one network link will be available.

☺ Small Exercise ☺

- Every mail message larger than one megabyte will be compressed
- Solution:
  - Let S(***m***, ***y***) be "Mail message ***m*** is larger than ***y*** megabytes"
    - **Domain** of ***m*** :
    - **Domain** of ***y*** :
  - Let C(***m***) denote "Mail message ***m*** will be compressed"
  - **$\forall$m (S(m, 1) $\rightarrow$ C(m))**

☺ Small Exercise ☺

- If a user is active, at least one network link will be available.
- Solution
  - Let A(***u***) be "User ***u*** is active"
    - **Domain** of ***u*** :
  - Let S(***n***, ***x***) be "Network link ***n*** is in state ***x***"
    - **Domain** of ***n*** :
    - **Domain** of ***x*** :
  - **$\exists$u A(u) $\rightarrow$ $\exists$n S(n, available)**

Nested Quantifiers

- Two quantifiers are **nested** if one is within the scope of the other
- How to interpret it?
  - If quantifiers are **same** type, the **order is not a matter**
    - $\exists \mathbf{x}\ \exists \mathbf{y}$ "x+y=0"
    - $\exists \mathbf{y}\ \exists \mathbf{x}$ "x+y=0"

    Same meaning
  - If quantifiers are **different** types, read **from left to right**
    - $\forall \mathbf{x}\ \exists \mathbf{y}$ "x+y=0"
    - $\exists \mathbf{y}\ \forall \mathbf{x}$ "x+y=0"

    Different meaning

Nested Quantifiers

Different Type

- If quantifiers are different types, read from left to right
- Example 1:
  - **P(x, y) = "x loves y"**
    $\forall \mathbf{x}\ \exists \mathbf{y}$ **P(x, y)** **VS** $\exists \mathbf{y}\ \forall \mathbf{x}$ **P(x, y)**

- $\forall \mathbf{x}\ \exists \mathbf{y}$ **"x loves y"**
  - For all x, there is at least one y, to make P(x,y) happens
  - For all persons, there is a person they love
  - ALL people loves some people

- $\exists \mathbf{y}\ \forall \mathbf{x}$ **"x loves y"**
  - At least one y, all x, to make P(x,y) happens
  - There is a person who is loved by all persons
  - Some people are loved by ALL people

Different Type

- Example 2:
  - **P(x, y) = "x+y=0"**
    $\forall x\ \exists y$ P(x, y) VS $\exists y\ \forall x$ P(x, y)
    - **$\forall x\ \exists y\ (x+y=0)$** ✓
      - For all x, there is at least one y, to make P(x,y) happens
      - Every real number has an additive inverse
    - **$\exists y\ \forall x\ (x+y=0)$** ✗
      - At least one y, all x, to make P(x,y) happens
      - There is a real number which all real number are its inverse addition

Same Type

- If quantifiers are the same type, the order is not a matter
- Example:
  - Given
    - **Parent(x,y)** : "x is a parent of y"
    - **Child(x,y)** : "x is a child of y"
  - $\forall x\ \forall y$ (Parent(x,y) $\rightarrow$ Child(y,x))
  - $\forall y\ \forall x$ (Parent(x,y) $\rightarrow$ Child(y,x))
  - Two equivalent ways to represent the statement:
    - For all x and y, if x is a parent of y, y is a child of x

Nested Quantifiers: Example 1

- Let domain be the real numbers,
- **P(x,y): “xy = 0”**
- Which one(s) is correct?
  - $\forall \mathbf{x}\ \forall \mathbf{y}\ P(x, y)$ ✗
  - $\exists \mathbf{x}\ \exists \mathbf{y}\ P(x, y)$ ✓
  - $\forall \mathbf{x}\ \exists \mathbf{y}\ P(x, y)$ ✓
    e.g. y = 0
  - $\exists \mathbf{x}\ \forall \mathbf{y}\ P(x, y)$ ✓
    e.g. x = 0

Nested Quantifiers: Example 2

- Translate the statement

$$\forall \mathbf{x}\ (\mathbf{C(x)} \wedge \underbrace{\exists \mathbf{y}\ (\mathbf{C(y)} \wedge \mathbf{F(x,y)})})$$

  into English, where
  - C(x) is “x has a computer”,
  - F(x,y) is “x and y are friends” and
  - the universe of discourse for both x and y is the set of all students in your school

Every student in your school has a computer and has a friend who has a computer.

Nested Quantifiers: Example 3

- Translate the statement
  "If a person is female and is a parent, then this person is someone's mother"
  as a logical expression
- Let
  - F(x): x is female
  - P(x): x is a parent
  - M(x,y): x is y's mother

At least one y

$$(F(x) \wedge P(x)) \rightarrow M(x, y)$$

All x

- The domain is the set of all people

$$\forall x\ (\ (F(x) \wedge P(x)) \rightarrow \exists y\ M(x, y)\ ),\ \text{or}$$

$$\forall x\ \exists y\ (\ (F(x) \wedge P(x)) \rightarrow M(x, y)\ )$$

☺ Small Exercise ☺

- Translating the following statement into logic expression:

  "The sum of the two positive integers is always positive"

  - $\forall x\ \forall y\ (x+y > 0)$
    The domain for two variables consists of all positive integers
  - $\forall x\ \forall y\ ((x>0) \wedge (y>0) \rightarrow (x+y > 0))$
    The domain for two variables consists of all integers

☺ Small Exercise ☺

- $Q(x, y, z)$ be the statement "$x + y = z$"
- The domain of all variables consists of all real
- What are the meaning of the following statements?
  - $\forall x\ \forall y\ \exists z\ Q(x,y,z)$ ✓
    - For all real numbers x and for all real numbers y there is a real number z such that x + y = z
  - $\exists z\ \forall x\ \forall y\ Q(x,y,z)$ ✗
    - There is a real number z such that for all real numbers x and for all real numbers y it is true that x + y = z

☺ Small Exercise ☺

- Translate the statement

  $\exists x\ \forall y\ \forall z$

  $$(\ \underbrace{(F(x,y) \land F(x,z) \land (y \neq z))}\ \rightarrow \neg F(y,z)\ )$$

  into English, where
  - F(a,b) means a and b are friends and
  - the universe of discourse for x, y and z is the set of all students in your school

There is a student none of whose friends are also friends each other

Exactly One

- It also called **uniqueness quantification** of $P(x)$ is the proposition "There exists a unique x such that the predicate is true"
- In the book, you will see the notation: $\exists!\, xP(x)$, $\exists_1\, xP(x)$
- But we will try to express the concept of "exactly one" using the Universal and Existential quantifiers
- In next few slides, we assume L(x, y) be the statement "x loves y"
- Four cases will be discussed

L(x, y) : "x loves y"

Exactly One: Case 1

- Mary loves exactly one person
- It means…
  - Mary loves one person (x) $\exists x\, L(Mary, x)$
  - If any people who is not x, Mary must not love him/her $\forall z\, ((z \neq x) \rightarrow \neg L(Mary, z))$

$$\boxed{\exists x\, (L(Mary, x) \land \forall z\, ((z \neq x) \rightarrow \neg L(Mary, z)))}$$

L(x, y) : "x loves y"

Exactly One: Case 1 (v2)

- Mary loves exactly one person
- It means…
  - Mary loves one person x
  - If Mary must love any person, he/she must be x

$\exists x\, L(Mary, x)$

$\forall z\, (\, L(Mary, z) \rightarrow (z = x)\,)$

$$\boxed{\exists x\, (\, L(Mary, x) \wedge \forall z\, (\, L(Mary, z) \rightarrow (z = x)\,)\,)}$$

L(x, y) : "x loves y"

Exactly One: Case 1

- Mary loves exactly one person
- **Version 1**

$$\exists x\, (\, L(Mary, x) \wedge \forall z\, (\underbrace{(z \neq x)}_{\neg p} \rightarrow \underbrace{\neg L(Mary, z)}_{\neg q})\,)$$

- **Version 2**

$$\exists x\, (\, L(Mary, x) \wedge \forall z\, (\underbrace{L(Mary, z)}_{q} \rightarrow \underbrace{(z = x)}_{p}\,)\,)$$

- As $p \rightarrow q$ and its Contrapositive are equivalent, Version 1 and 2 are the same

L(x, y) : "x loves y"

Exactly One: Case 2

- Exactly one person loves Mary
- It means…
  - One person (x) loves Mary
  - If anyone loves Mary, he/she must be x

$\exists x\ L(x, Mary)$

$\forall z\ (\ L(z, Mary) \rightarrow (z = x)\ )$

$$\exists x\ (\ L(x, Mary) \wedge \forall z\ (\ L(z, Mary) \rightarrow (z = x)\ )\ )$$

L(x, y) : "x loves y"

Exactly One: Case 3

- All people love exactly one person
- It means…
  - Everyone (y) loves a person (x) $\forall y\ \exists x\ L(y, x)$
  - If y loves anyone, it must be x

$\forall z\ (\ L(y, z) \rightarrow (z = x)\ )$

$$\forall y\ \exists x\ (\ L(y, x) \wedge \forall z\ (\ L(y, z) \rightarrow (z = x)\ )\ )$$

L(x, y) : "x loves y"

Exactly One: Case 4

- Exactly one person loves all people
- It means…
  - A person x loves everyone y $\exists x\ \forall y\ L(x, y)$
  -
  - If anyone loves all people, it must be x

$$\forall z\ (\ \underbrace{\forall w\ L(z,\ w)}\ \rightarrow (z = x)\ )$$

$$\boxed{\exists x\ \forall y\ (\ L(x,\ y)\ \wedge \forall z\ (\forall w\ L(z,\ w) \rightarrow (z = x)\ )\ )}$$

L(x, y) : "x loves y"

Exactly One: Case 1 VS Case 3

- **Case 1**: Mary loves exactly one person

$$\exists x\ (\ L(\text{Mary},\ x) \wedge \forall z\ (\ L(\text{Mary},\ z) \rightarrow (z = x)\ )\ )$$

- **Case 3**: All people love exactly one person

$$\forall y\ \exists x\ (\ L(y,\ x) \wedge \forall z\ (\ L(y,\ z) \rightarrow (z = x)\ )\ )$$

$L(x, y)$: "x loves y"

Exactly One: Case 2 VS Case 4

- **Case 2**: Exactly one person loves Mary

$$\exists x \, (L(x, \boxed{Mary}) \wedge \forall z \, (L(z, \boxed{Mary}) \rightarrow (z = x)))$$

- **Case 4**: Exactly one person loves all people

$$\exists x \boxed{\forall y} (L(x, \boxed{y}) \wedge \forall z \, (\boxed{\forall w} \, L(z, \boxed{w}) \rightarrow (z = x)))$$

$L(x, y)$: "x loves y"

☺ Small Exercise ☺

- There is exactly one person whom everybody loves
- It means…
  - At least one person is loved by everyone $\exists x \, \forall y \, L(y, x)$
  - At most one person is loved by everyone
    - If anyone is loved by everyone, it must be x

$$\forall z \, (\forall w \, L(w, z) \rightarrow (z = x))$$

$$\boxed{\exists x \, \forall y \, (L(y, x) \wedge \forall z \, (\forall w \, L(w, z) \rightarrow (z = x)))}$$

L(x, y) : "x loves y"

☺ Small Exercise ☺

- Exactly two people love Mary
- It means… $\exists x\, \exists y\, (\ L(x, Mary) \wedge L(y, Mary) \wedge (x \neq y)\ )$
  - At least two persons love Mary
  - At most two persons love Mary
    - If anyone loves Mary, he/she must be x or y

$$\forall z\, (\, L(z, Mary) \rightarrow (\, (z = x) \vee (z = y)\,)\,)$$

$$\boxed{\begin{aligned} &\exists x\, \exists y\, (\ L(x, Mary) \wedge L(y, Mary) \wedge (x \neq y) \wedge \\ &\qquad \forall z\, (\, L(z, Mary) \rightarrow (\, (z = x) \vee (z = y)\,)\,)\,) \end{aligned}}$$

Nested Quantifiers

- Recall,
  - When all of the elements in the universe of discourse **can be listed one by one** (discrete) (e.g. $x_1, x_2, \ldots, x_n$),

$$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

$$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Nested Quan

$\forall x\, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
$\exists x\, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

- Example
  - Find an expression equivalent to

    $$\forall x\, \exists y\, P(x, y)$$

    where the universe of discourse consists of the positive integer not exceeding 3?

$$\forall x \exists y P(x, y) = \forall x\, (\exists y P(x, y))$$
$$= \exists y P(1, y) \wedge \exists y P(2, y) \wedge \exists y P(3, y)$$
$$= [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge [P(2,1) \vee P(2,2) \vee P(2,3)] \wedge [P(3,1) \vee P(3,2) \vee P(3,3)]$$

Negating Nested Quantifiers

- Recall, De Morgan's Laws for Quantifiers
  - $\neg \forall x\, P(x) \equiv$
  - $\neg \exists x\, P(x) \equiv$
- They also can be applied in Nested Quantifiers