Formal Foundations
Formal Approaches

Part 2: Introduction to set theory

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Overview

1. Introduction to set theory
2. Delving deeper...
3. Applying set theory
4. Exercises

Fundamentals

Definition 2.1 (What is a set?)

A set is a collection of objects. Those objects make up the elements of the set. Each object is known as a member.

We use curly brackets to denote the collection, and separate the members with commas:

Example 2.1

The set of my cats: $\mathcal{C} = \{\text{Billy}, \text{Tilly}, \text{Lizzy}, \text{Toby}\}$

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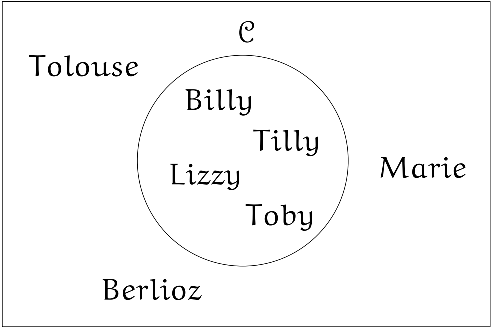
This same information can be presented as a Venn Diagram



Venn diagram of $\mathcal{C}$, the set of my cats

Neither the order of members in a set, nor their number of occurrences, is important, so the set $\{1, 2\}$ is exactly the same as the set $\{2, 1, 1, 1, 2\}$

Task 2.1

Which of the two facts above is obvious from looking at a set's Venn diagram?

Task 2.2

Which of the following sets are equal?

1. $\{a, b, c\} = \{c, b, a\}$
2. $\{a, a, c\} = \{c, b, a\}$
3. $\{a, a, a\} = \{a\}$
4. $\{a, b, c\} = \{c, b, b, c, a, d\}$

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Set membership

Sets can be defined by membership:

Definition 2.2 (Membership)

$x \in A$ *means that the element* x *is a member of the set* A

Example 2.2

Billy is a member of the set of my cats: Billy $\in \mathcal{C}$

Task 2.2

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 3, 5, 7\}$, $C = \{2, 4, 6, 8\}$.
What type of objects are the elements of A, B and C?

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What type of objects are the elements of A, B and C?

Task 2.4

Which of the following are true? And why? Draw Venn diagrams if it helps you

1. $a \in \{c, b, a\}$
2. $d \in \{a, b, c\}$
3. $\{a\} \in \{c, b, a\}$

When an element x is not a member of a set A, we write $x \notin A$

Task 2.5

Which of the following are true?

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Which of the following are true?

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The empty set

Do all sets have to have members?

Definition 2.3 (Empty Set)

The empty set is the unique set with no members, written as $\emptyset$

Since the empty set has no members, we can write $x \notin \emptyset$ for any x

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Draw the Venn diagram of the empty set

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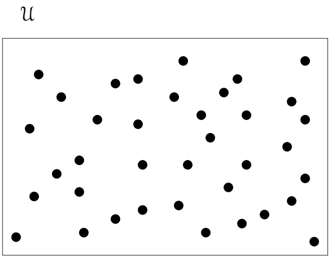
The universal set

Definition 2.4 (Universal set)

The universal set, usually written $\mathcal{U}$, is the set of all elements of the type we are discussing



Set operators

- If all we could do with sets was to define the elements in a set and test for membership, then they would not be very interesting or useful
- However, there is a range of operators that allow new sets to be constructed by combining existing sets in a variety of ways
- This is a familiar concept from arithmetic – we create new numbers by adding, subtracting, multiplying... other numbers
- We will see that set operators and arithmetic operators share some common properties...

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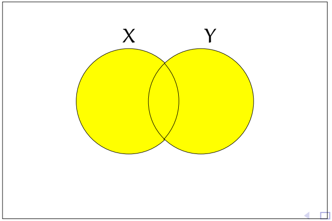
Set union

Definition 2.5 (Set Union)

From the two sets X, Y, a set $X \cup Y$ is formed by combining the elements of X and Y:

$$x \in (X \cup Y) \textit{ if } x \in X \textit{ or } x \in Y$$

In the Venn diagram below, the shaded area represents $X \cup Y$



Task 2.7

What do the following evaluate to?

1. $\{a, b, c\} \cup \{d, e, f\}$
2. $\{a, b, c\} \cup \{c, b, a\}$
3. $\{a, a, a\} \cup \{a\}$
4. $\{a, b, c\} \cup \emptyset$

Task 2.8

- What can we say about taking the union of any set with $\emptyset$?
- If $\cup$ is "similar" to $+$ in arithmetic, which number is "similar" to $\emptyset$?

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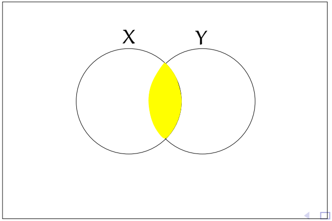
Set intersection

Definition 2.6 (Set Intersection)

From the two sets X, Y, *a set* $X \cap Y$ *is formed by taking those elements that are in both* X *and* Y*:*

$$x \in (X \cap Y) \textit{ if } x \in X \textit{ and } x \in Y$$

In the Venn diagram below, the shaded area represents $X \cap Y$



Task 2.9

What do the following evaluate to?

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2. $\{a, b, c\} \cap \{c, b, a\}$
3. $\{a, a, a\} \cap \{a\}$
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Task 2.10

If $\cap$ is "similar" to $\times$ in arithmetic, which set is "similar" to the number 1? Which is "similar" to 0?

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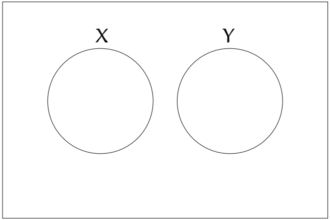
If $\cap$ is "similar" to $\times$ in arithmetic, which set is "similar" to the number 1? Which is "similar" to 0?

Disjoint sets

Definition 2.7 (Disjoint sets)

Two sets X, Y, are said to be disjoint if they no elements in common, i.e. their intersection is the empty set: $X \cap Y = \emptyset$

In the Venn diagram below, X and Y are disjoint



The concept of "disjointness" is used widely so it is useful to allocate it a name, rather than having to frequently write $X \cap Y = \emptyset$

Task 2.11

Which of the following sets are disjoint?

1. $\{a, b, c\}$
2. $\{d, f\}$
3. $\{a, e, c\}$
4. $\emptyset$

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1. $\{a, b, c\}$
2. $\{d, f\}$
3. $\{a, e, c\}$
4. $\emptyset$

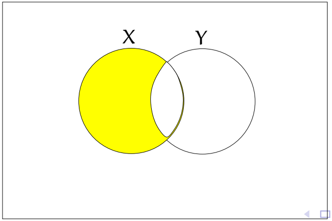
Set difference

Definition 2.8 (Set difference)

The difference between two sets, written $X \setminus Y$, is formed by "throwing away" the elements of Y from X:

$$x \in (X \setminus Y) \textit{ if } x \in X \textit{ and } x \notin Y$$

In the Venn diagram below the area $X \setminus Y$ is shaded



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What do the following evaluate to?

1. $\{a, b, c\} \setminus \{c, b\}$
2. $\{a, b, c\} \setminus \{d, e, f\}$
3. $\{a, a, a\} \setminus \{a\}$
4. $\{a, b, c\} \setminus \emptyset$

Task 2.13

If $\setminus$ is "similar" to $-$ (subtraction) in arithmetic, which set is "similar" to 0?

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If $\setminus$ is "similar" to $-$ (subtraction) in arithmetic, which set is "similar" to 0?

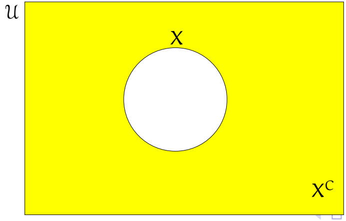
Set complement

Definition 2.9 (Complement)

If $\mathcal{U}$ is the universal set associated with a set X, then the complement *of X is written as X^C and is defined by*

$$x \in X^C \text{ if } x \in (\mathcal{U} \setminus X)$$

In the Venn diagram below, the shaded area represents X^C



Task 2.14

Let $\mathcal{U} = \{0, 1, 2, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{1, 2, 3, 4\}$. Find A^C and B^C.

Equivalent sets

Task 2.15

Draw Venn diagrams for the following expressions:

1. $\mathcal{U} \setminus A$
2. A^C

What can we say about these Venn diagrams?

Definition 2.10 (Equivalence)

Two set expressions are said to be equivalent if they have the same Venn diagram

Equivalent sets

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Subset

Definition 2.11 (Subset)

If every element of a set X belongs to a set Y then we say that X is a subset of Y:

$$X \subseteq Y \text{ if } x \in Y \text{ for all } x \in X$$

A Venn diagram illustrating subsets is shown below

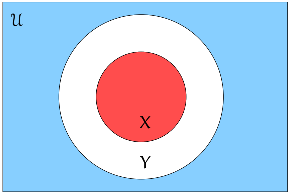

Figure: $X \subseteq Y$

- Occasionally we consider proper subsets, written $X \subset Y$, when $X \subseteq Y$ and $X \neq Y$
- We say that $\emptyset \subseteq X \subseteq \mathcal{U}$ for all sets X

Task 2.16

Why is $\emptyset \subseteq X$ true for all sets X? Does drawing a Venn diagram help explain this?

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Task 2.17

While of the following are always true for all sets A, B:

1. $A \cup B \subseteq A$
2. $A \cap B \subseteq A$
3. $A \cup B \subset A$
4. $A \cap B \subset A$
5. $A \setminus B \subseteq A$
6. $A \setminus B \subset A$

If it helps, draw Venn diagrams of the expressions.

Some special sets

The following sets are useful when describing systems:

- The set of all natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
- The set of all integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- The set of all real numbers, consisting of all numbers with a (possibly infinite) decimal expansion, e.g. 1.4, π, etc.

These sets are related:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$$

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Task 2.18

Answer the following – use Venn diagrams if you find it helps

1. $\mathbb{Z} \setminus \mathbb{N} =?$
2. $\mathbb{N} \cap \mathbb{R} =?$
3. Are $\mathbb{Z}$ and $\mathbb{N}$ disjoint?

Cardinality

Definition 2.12 (Cardinality)

The cardinality of a set is the number of distinct elements in that set

Example 2.3

If $A = \{2, 4, 6, 8\}$ then $\#A = 4$

- The cardinality operator is written $\#$ and is called a unary operator because it is applied to a single operand
- It is also a prefix operator because it is written before the operand

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Task 2.19

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 3, 3, 5, 7\}$, $D = \{5, 6, 7, 8, 9\}$. Answer the following:

1. $\#B$
2. $\#D$
3. $A \cup D$
4. $\#(A \cup D)$
5. $A \cap D$
6. $\#(A \cap D)$

Task 2.20

Use a Venn diagram to justify the statement:

If X and Y are disjoint sets, then

$$\#(X \cup Y) = \#X + \#Y$$

Task 2.19

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 3, 3, 5, 7\}$, $D = \{5, 6, 7, 8, 9\}$.
Answer the following:

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4. $\#(A \cup D)$
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Task 2.20

Use a Venn diagram to justify the statement:

If X and Y are disjoint sets, then

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In general, for arbitrary X and Y that may not be disjoint,

$$\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$$

Task 2.21

Can you see why it is necessary to subtract $\#(X \cap Y)$? If it helps, draw a Venn diagram to clarify the reason.

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Task 2.21

Can you see why it is necessary to subtract $\#(X \cap Y)$? If it helps, draw a Venn diagram to clarify the reason.

Task 2.22

Use a Venn diagram to justify the expression
$\#(A \setminus B) = \#A - \#(A \cap B)$

Task 2.23

- If $A \subseteq B$, what is the relationship between $\#A$ and $\#B$? What is the relationship between them when $A \subset B$?
- Which arithmetic relation is $\subseteq$ "similar" to? Which is $\subset$ "similar" to?

Task 2.22

Use a Venn diagram to justify the expression
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Distributive (and other) Laws

Remember the distributive law of multiplication over addition?

$$\begin{aligned} 4 \times (3 + 7) &= 4 \times 3 + 4 \times 7 \\ &= 12 + 28 \\ &= 40 \end{aligned}$$

The rule is usually written as:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

There are analogous rules in set theory

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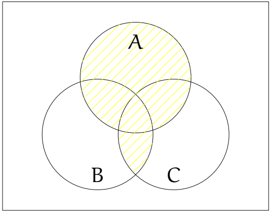
Definition 2.13 (Distributive laws)

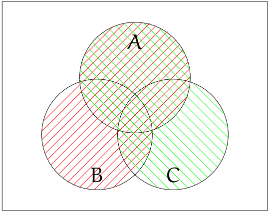
Set union distributes through set intersection, and vice versa:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The first rule is illustrated by a Venn diagram as follows:



$A \cup (B \cap C)$

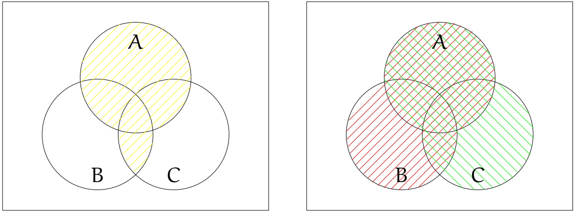


$(A \cup B) \cap (A \cup C)$

Task 2.24

Draw Venn diagrams to illustrate the second rule.

The first rule is illustrated by a Venn diagram as follows:



$A \cup (B \cap C)$ $(A \cup B) \cap (A \cup C)$

Task 2.24

Draw Venn diagrams to illustrate the second rule.

Definition 2.14 (de Morgan's Laws for Sets)

de Morgan's Laws for sets are shown below: notice that complementing the bracketed expression results in all sets being complemented and $\cup$ and $\cap$ being exchanged:

$$(A \cap B)^C = A^C \cup B^C$$
$$(A \cup B)^C = A^C \cap B^C$$

Task 2.25

Draw Venn diagrams to illustrate de Morgan's laws.

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Draw Venn diagrams to illustrate de Morgan's laws.

Sets by comprehension

- Sometimes we wish to define a subset of certain set, where each member of the subset possesses a given property: this is done with a set comprehension
- Recall the sets $\mathbb{N}$ and $\mathbb{Z}$; we can define subsets of these by writing:
  - $X = \{x : \mathbb{N} \mid x \text{ is even}\}$, pronounced "the set of even natural numbers"
  - $Y = \{y : \mathbb{Z} \mid y \text{ is odd}\}$, pronounced ". . . "
- The first part of the set comprehension declares the type of the members; the second part defines the property each member must satisfy

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Applying set theory

- An important use of set theory is to formalise natural language statements so they can be analysed and maybe implemented
- By itself, set theory can usefully describe a range of interesting scenarios, as we will see, but when combined with logic things become much more interesting...
- To illustrate how useful set theory may be, we will do a simple case study

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Darbdorf Language School

Students attending Darbdorf Language School may learn French, German or Italian – they have a free choice of which language(s) to learn.

Task 2.26

The universal set, $\mathcal{U}$, stands for all students in Darbdorf Language School, F stands for students studying French, G denotes students studying German, and I denotes students studying Italian.
Draw a Venn diagram describing students attending Darbdorf Language School

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Draw a Venn diagram describing students attending Darbdorf Language School

Currently the registers show that:

1. 60 students are studying French;
2. 46 students are studying German; and
3. 54 students are studying Italian.

Task 2.27

Use set theory to formalise the three statements above (i.e. write set expressions that convert the natural language statements into mathematical statements).

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Further analysis of the registers show that:

1. 15 students are studying French and Italian;
2. 10 students are studying French and German;
3. 4 students are studying German and Italian; and
4. 6 students are studying all three languages.

Task 2.28

Use set theory to formalise the four statements above

Task 2.29

The statements above are ambiguous, i.e. they can be interpreted in more than one way. Can you spot the source of the ambiguity? Does adding the information above to the Venn diagram you drew in task 2.26 help you find it?

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Task 2.30

Write set expressions to formalise the following highlighted statements, and then calculate the answers:

1. How many students are only studying French?
2. How many students are studying German or Italian, but not French?
3. How many students are studying French and German, but not Italian?
4. How many students are enrolled at Darbdorf Language School?

Exercises

These exercises are for you to do in your own time. I will not provide solutions to these exercises – that would only teach you to read the answers, not how to write them yourself. However, I would be delighted to give you feedback on your solutions, either in person or via email.

Exercise 2.1

Draw Venn diagrams to identify the following sets:

1. A^C
2. $A \cup A^C$
3. $A \cap A^C$
4. $(A \cup B)^C$
5. $A^C \cap B$
6. $A^C \cap B^C$
7. $(A \cap B)^C$

Exercise 2.2

Let A be a set of sets, $A = \{\{1,2,3\},\{4,5\},\{6,7,8\}\}$. Which of the following statements are true, and which are false?

1. $1 \in A$
2. $\{1,2,3\} \subseteq A$
3. $\{6,7,8\} \in A$
4. $\{\{4,5\}\} \subseteq A$
5. $\emptyset \in A$
6. $\emptyset \subseteq A$

Exercise 2.3

Let $A = \{1,2,\ldots,8,9\}$, $B = \{2,4,6,8\}$, $C = \{1,3,5,7,9\}$ and $D = \{3,4,5\}$

1. Can you find a set X such that $X \subseteq D$ but $X \not\subseteq B$?
2. Can you find a set Y such that $Y \subset C$ but $Y \not\subset A$?

Exercise 2.4

Define the following sets by comprehension:

1. The set of all positive integers.
2. The set of all even natural numbers.
3. The set of all square roots of 144.

Exercise 2.5

A survey on a sample of 25 new cars being sold at a local dealer was conducted to see which had air-conditioning (A), a radio (R), or electric windows (W) installed. The results showed that

- 15 had air-conditioning;
- 12 had a radio;
- 11 had electric windows;
- 5 had air-conditioning and electric windows;
- 9 had air-conditioning and a radio;
- 4 had a radio and electric windows; and
- 3 had all three options.

Exercise 2.5 (Cont.)

Find the number of cars that had:

1. only electric windows;
2. only air-conditioning;
3. only a radio;
4. a radio and electric windows, but not air-conditioning;
5. air-conditioning and a radio, but not electric windows;
6. only one option; and
7. none of the options.