University of Amsterdam
Korteweg de Vries Institute for Mathematics

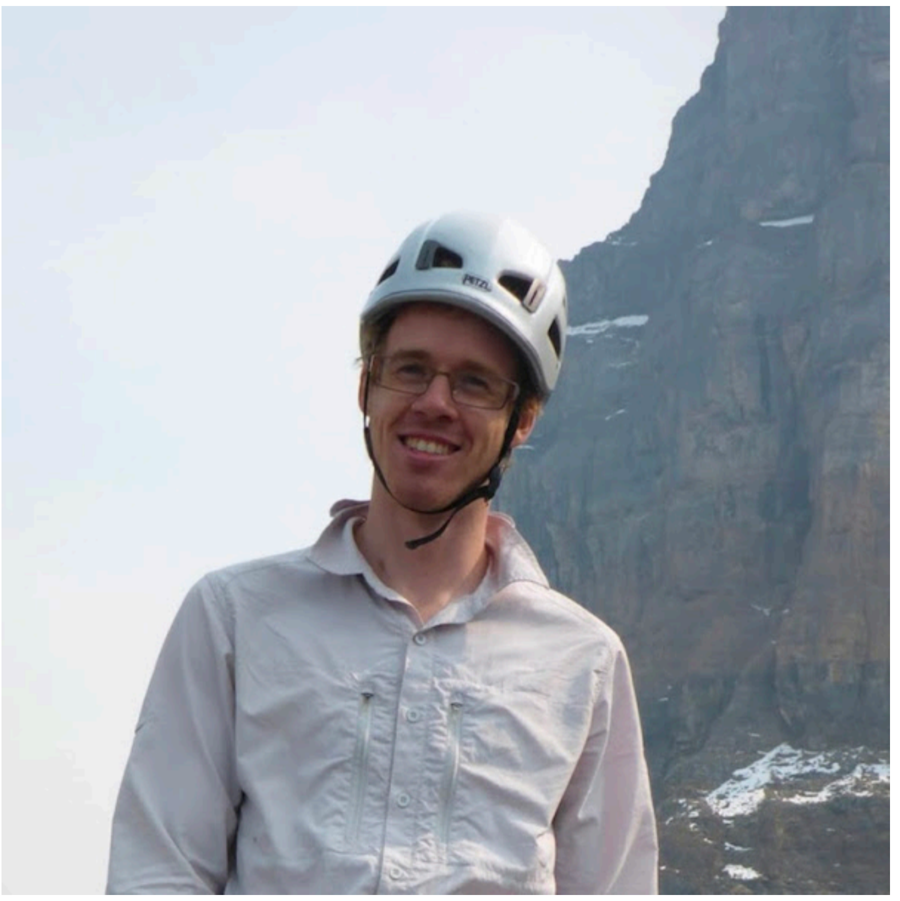
A Newton Method for Bandit Convex Optimisation

2024-03-15

Joint Work

- All work presented was created in collaboration with:

Jack Mayo
University of Amsterdam



Tor Lattimore
DeepMind

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Leiden University

Outline

- Introduction to Bandit Convex Optimisation
- Regret Result
- Gaussian Optimistic Smoothing
- Algorithm
- Sketch of Proof

Bandit Convex Optimisation

An introduction

Bandit Convex Optimisation

Motivation

- Multi-Armed Bandits, where the number of arms is too large:
  - Online Routing: Every arm is a path
  - Online Ranking
  - Online Ad-Placement

- Given a fixed budget, how to allocate grants amongst Research Projects, where the outcome is not known

Bandit Convex Optimisation

General setting

Let:

- $K \subseteq \mathbb{R}^d$
- $\ell_1, \ldots, \ell_n : K \to [0,1]$
- In each round $t = 1, \ldots, n$:
  - Learner chooses action $A_t \in K$
  - Suffers loss $\ell_t(A_t)$
  - Observes $Y_t = \ell_t(A_t) + \varepsilon_t$
  - ε_t conditionally Sub-Gaussian

Regret is measured as:

$$\mathrm{Reg}_n = \sum_{t=1}^{n} \ell_t(A_t) - \min_{x \in K} \sum_{t=1}^{n} \ell_t(x)$$

Bandit Convex Optimisation

Adversarial setting

Let:

- $K \subseteq \mathbb{R}^d$
- $\ell_1, \ldots, \ell_n : K \to [0,1]$
- In each round $t = 1, \ldots, n$:
  - Learner chooses action $A_t \in K$
  - Suffers loss $\ell_t(A_t)$
  - Observes $Y_t = \ell_t(A_t) + \varepsilon_t$
  - ε_t conditionally Sub-Gaussian
  - Or $\varepsilon_t = 0$ for all $t = 1$

Regret is measured as:

$$\text{Reg}_n = \sum_{t=1}^{n} \ell_t(A_t) - \min_{x \in K} \sum_{t=1}^{n} \ell_t(x)$$

Bandit Convex Optimisation

Stochastic setting

Let:

- $K \subseteq \mathbb{R}^d$
- $\ell : K \to [0,1]$
- In each round $t = 1, \ldots, n$:
  - Learner chooses action $A_t \in K$
  - Suffers loss $\ell(A_t)$
  - Observes $Y_t = \ell(A_t) + \varepsilon_t$
  - ε_t conditionally Sub-Gaussian

Regret is measured as:

$$\mathrm{Reg}_n = \sum_{t=1}^{n} \ell(A_t) - \min_{x \in K} \sum_{t=1}^{n} \ell(x)$$

Regret Result

Previous Regret Results

Paper	Losses	Regret Stochastic	Regret Adversarial	Running Time
[Abernethy et al., 2009]	linear	$\tilde{O}(d\sqrt{n})$	$\tilde{O}(d\sqrt{n})$	$O(d^2)$
[Hazan and Levy, 2014]	strongly convex, smooth	$\tilde{O}(d\sqrt{n})$	$\tilde{O}(d\sqrt{n})$	$O(d)$
[Suggala et al. 2021]	convex quadratic	$\tilde{O}(d^{16}\sqrt{n})$	$\tilde{O}(d^{16}\sqrt{n})$	$O(d^4)$
[Bubeck et al., 2021]	bounded convex	$\tilde{O}(d^{10.5}\sqrt{n})$	$\tilde{O}(d^{10.5}\sqrt{n})$	$\mathrm{poly}(d, T)$
[Lattimore, 2020]	convex	$\tilde{O}(d^{2.5}\sqrt{n})$	$\tilde{O}(d^{2.5}\sqrt{n})$	$\exp(d, T)$
[Lattimore and Gyorgy, 2021]	convex	$\tilde{O}(d^{4.5}\sqrt{n})$	$\times$	$\mathrm{poly}(d)$
[Lattimore and Gyorgy, 2023]	Lipschitz convex	$\tilde{O}(d^{1.5}\sqrt{n})$	$\times$	$O(d^3)$

Regret Guarantee

Theorem 1 & 2

There exists an algorithm such that with probability at least $1 - \delta$,

$$\mathrm{Reg}_n \leq d^{3.5}\sqrt{n}\,\mathrm{polylog}(n, d, 1/\delta)$$

In the Stochastic setting this can be improved to

$$\mathrm{Reg}_n \leq d^{2}\sqrt{n}\,\mathrm{polylog}(n, d, 1/\delta)$$

- Only Boundedness and Convexity needed
- For the proofs:
- We apply a reduction to
  - Lipschitz,
  - Smooth,
  - Strongly Convex
  - Twice Differentiable functions
- Reduction hides in the log terms

Previous Regret Results

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[Lattimore and Gyorgy, 2023]	Lipschitz convex	$\tilde{O}(d^{1.5}\sqrt{n})$	$\times$	$O(d^3)$
[Suggala et al., 2024]	κ-convex	$\tilde{O}(d^{2.5}\kappa^2\sqrt{n})$	$\tilde{O}(d^{2.5}\kappa^2\sqrt{n})$	$O(d^2)$
Ours	Bounded Convex	$\tilde{O}(d^2\sqrt{n})$	$\tilde{O}(d^{3.5}\sqrt{n})$	$O(d^3)$

Gaussian Optimistic Smoothing

Motivation

Full Information setting:

- Run OGD, Online Newton, etc.

To run Online Newton:

From 1 sample, Y_t, Estimate:

- $\ell_t(A_t)$
- $\nabla \ell_t(A_t)$
- $\nabla^2 \ell_t(A_t)$

Some Problems & Requirements:

- Twice Differentiable
- Ideally Strongly-Convex
- The obvious estimators are not reliable or need multiple queries

s_t - Estimator

Definition

Definition

Let f_t be a bounded convex function, and
$X_t \sim \mathcal{N}(\mu_t, \Sigma_t)$, given paramater $\lambda \in \left(0, \frac{1}{1+d}\right)$, define

$$s_t(x) = \mathbb{E}\left[\left(1 - \frac{1}{\lambda}\right) f_t(X_t) + \frac{1}{\lambda} f_t((1-\lambda)X_t + \lambda x)\right]$$

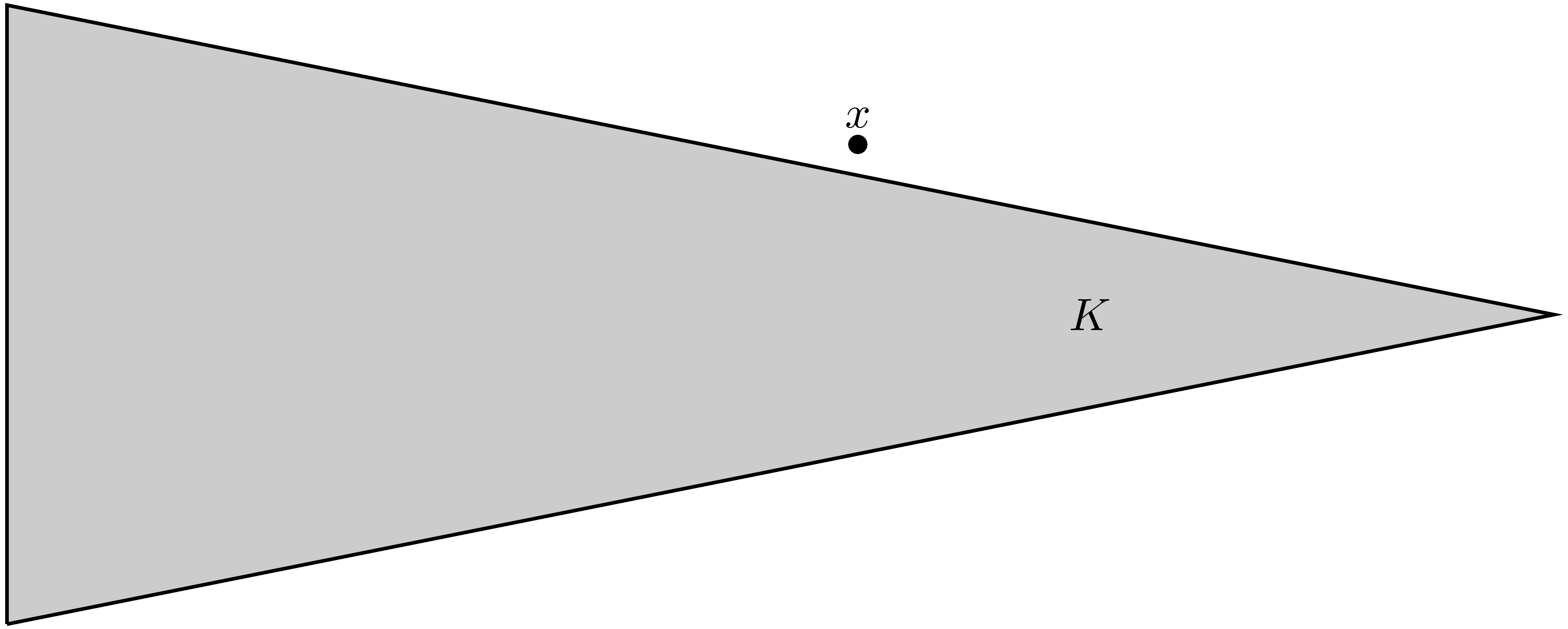
Also set $q(x) = \left\langle \nabla s_t(\mu), x - \mu_t \right\rangle + \frac{1}{6}\|x - \mu_t\|^2_{\nabla^2 s_t(\mu_t)}$

- ℓ_t is not defined on the whole of $\mathbb{R}^d$
- Need to extend ℓ_t!
- Difficult because ℓ_t can blow up at the boundary
- Convolve ℓ_t to make strongly convex
- Add quadratic term to make smooth

s_t - Estimator

ℓ_t extension

Utilise the Minkowski functional:
$\pi(x) = \inf\{t > 0 : x \in tK\}$



- $x \in K \iff \pi(x) \leq 1$
- $x \notin K \iff \pi(x) > 1$

Inspired by [Lattimore 2024, Figure 3.1]

s_t - Estimator

ℓ_t extension

Utilise the Minkowski functional:
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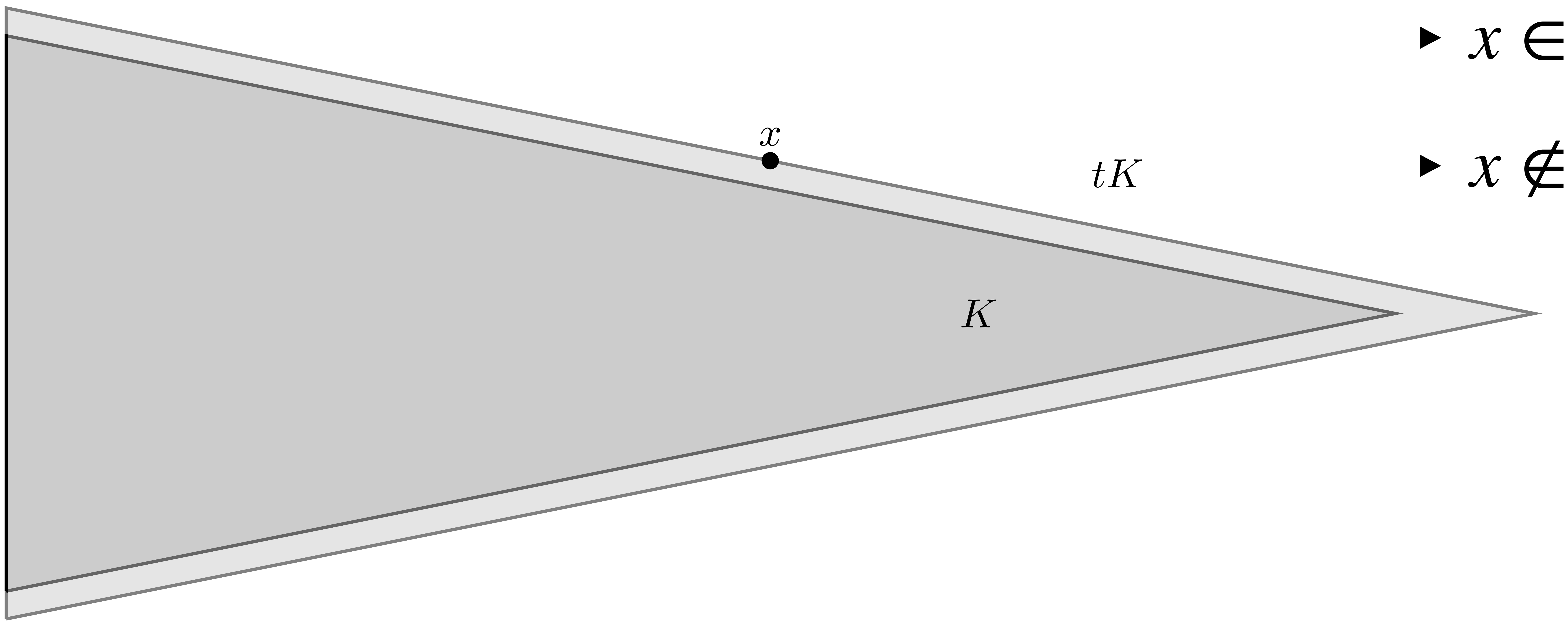

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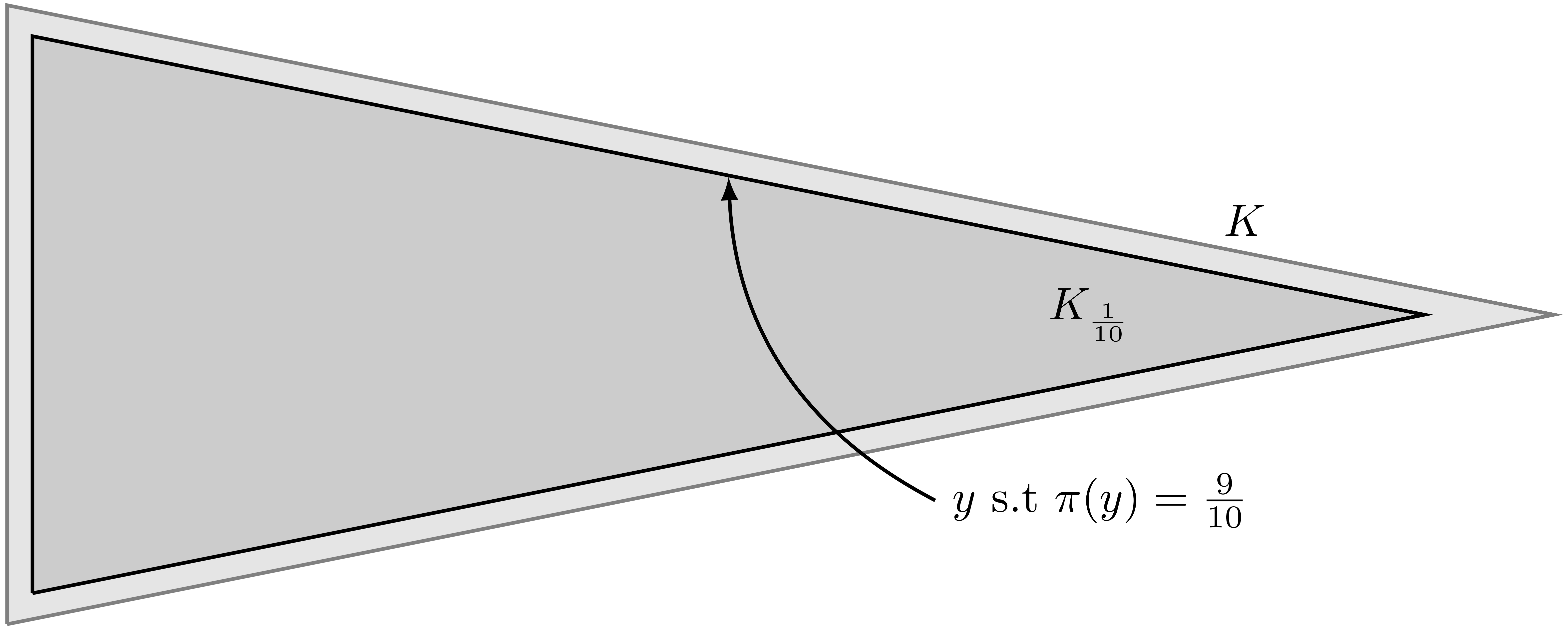
Inspired by [Lattimore 2024, Figure 3.1]

s_t - Estimator

ℓ_t extension

Utilise the Minkowski functional:
$\pi(x) = \inf\{t > 0 : x \in tK\}$

Shrink $K \to K_\varepsilon$ using $\pi_\epsilon(x) = \dfrac{\pi(x)}{1-\epsilon} \leq 1$



- $\varepsilon = \dfrac{1}{10}$

Inspired by [Lattimore 2024, Figure 3.1]

s_t - Estimator

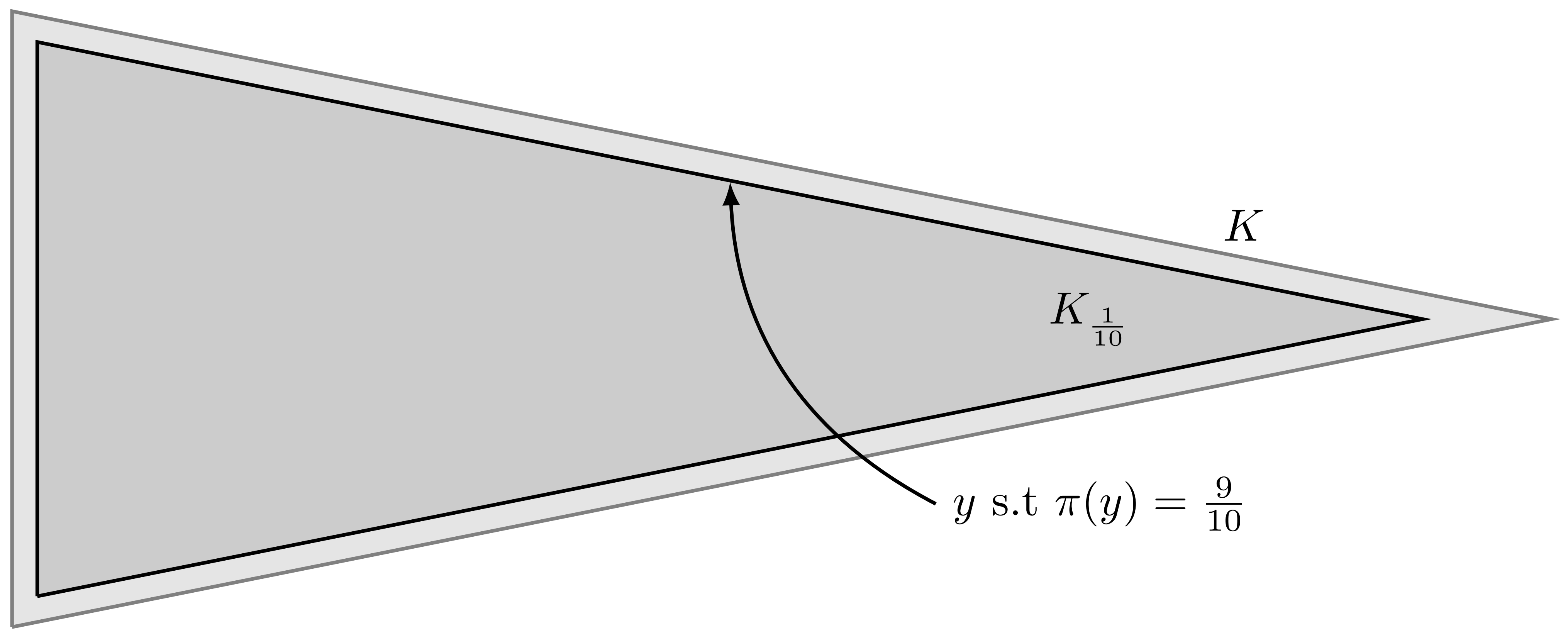
ℓ_t extension

Utilise the Minkowski functional:
$\pi(x) = \inf\{t > 0 : x \in tK\}$

Shrink $K \to K_\varepsilon$ using $\pi_\epsilon(x) = \dfrac{\pi(x)}{1-\epsilon} \leq 1$

Define the extension of ℓ_t from $K_\varepsilon \to \mathbb{R}^d$,

$$\mathrm{e}_t(x) = \pi_+(x)\ell_t\left(\frac{x}{\pi_+(x)}\right) + \frac{2(\pi_+(x)-1)}{\epsilon}.$$



Linearly extend from the boundary of K_ε

s_t - Estimator

Properties ℓ_t extension

$$\mathrm{e}_t(x) = \pi_+(x)\ell_t\left(\frac{x}{\pi_+(x)}\right) + \frac{2(\pi_+(x) - 1)}{\epsilon}$$

Lemma 4

- $\mathrm{e}_t(x) = \ell_t(x)$ for all $x \in K_\varepsilon$
- e_t is convex on $\mathbb{R}^d$
- $\partial_x \mathrm{e}_t(x) \geq 0$ for all $x \notin K_\varepsilon$
- $\frac{\pi(x) - 1}{\varepsilon} \leq \mathrm{e}_t(x) \leq 1 + \left(1 + \frac{1}{\epsilon}\right)[\pi_+(x) - 1]$

s_t - Estimator

Some Properties

Properties (Lemma 21)

If $f_t \colon \mathbb{R}^d \to \mathbb{R}$ is convex and α-strongly convex and β-smooth and let $X_t \sim \mathcal{N}(\mu_t, \Sigma_t)$

- s_t is optimistic: $s_t(z) \leq f_t(z)$
- s_t is convex and infinitely differentiable
- $\lambda\alpha I \leq \nabla^2 s(z) \leq \lambda\beta I$

Approximation (Lemma 24)

If $z \in K_t$ we also have

$$\mathbb{E}[f_t(X_t)] - f_t(z) \leq q_t(\mu_t) - q_t(z) + \frac{4}{\lambda}\mathrm{tr}(\nabla^2 s(\mu_t)\Sigma_t)$$

Define:

- $\mathcal{E}_t = \left\{ x \in K \mid \lambda \|x - \mu_t\|^2_{\Sigma_t^{-1}} \leq 1 \right\}$

- $K_0 = K_\epsilon$,
- $K_t = K_{t-1} \cap \mathcal{E}_t$

Focus regions, on which the approximation is good

s_t - Estimator

$$s_t(x) = \mathbb{E}\left[\left(1-\frac{1}{\lambda}\right)f_t(X_t)+\frac{1}{\lambda}f_t((1-\lambda)X_t+\lambda x)\right]$$

By magic:
Value, Derivative and Hessian can be estimated with single observation!

Magic = Radon-Nikodym!

$$R_t(z) = \frac{p_t\left(\frac{X_t-\lambda z}{1-\lambda}\right)}{(1-\lambda)^d p_t(X_t)}$$

$$\hat{s}_t(z) = Y_t\left[1-\frac{1}{\lambda}+\frac{R_t(z)}{\lambda}\right]$$

$$\nabla \hat{s}_t(z) = \frac{Y_tR_t(z)}{1-\lambda}\Sigma_t^{-1}\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]$$

$$\nabla^2 \hat{s}_t(z) = \frac{Y_tR_t(z)}{(1-\lambda)^2}\left(\Sigma_t^{-1}\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]^\top\Sigma_t^{-1}-\Sigma_t^{-1}\right)$$

s_t and q_t Estimator

$$s_t(x) = \mathbb{E}\left[\left(1-\frac{1}{\lambda}\right)f_t(X_t)+\frac{1}{\lambda}f_t((1-\lambda)X_t+\lambda x)\right]$$

$$\hat{s}_t(z) = Y_t\left[1-\frac{1}{\lambda}+\frac{R_t(z)}{\lambda}\right]$$

$$\nabla \hat{s}_t(z) = \frac{Y_t R_t(z)}{1-\lambda}\Sigma_t^{-1}\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]$$

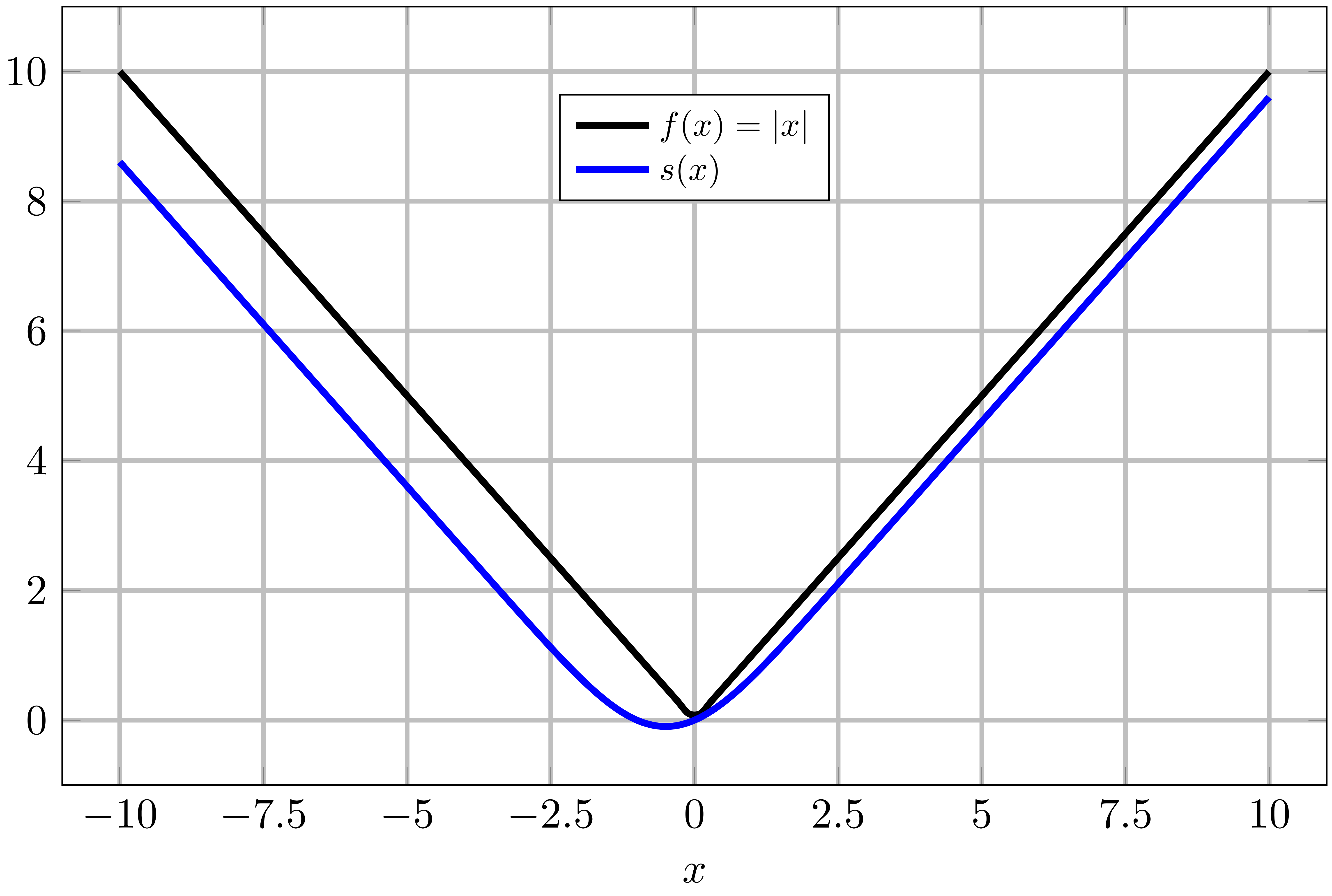
$$\nabla^2 \hat{s}_t(z) = \frac{Y_t R_t(z)}{(1-\lambda)^2}\left(\Sigma_t^{-1}\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]\left[\frac{X_t-\lambda z}{1-\lambda}-\mu_t\right]^\top\Sigma_t^{-1}-\Sigma_t^{-1}\right)$$

Estimator for the q_t function:

$$g_t = \nabla \hat{s}_t(\mu_t), \quad H_t = \nabla^2 \hat{s}_t(\mu_t)$$

$$\hat{q}_t(x) = \left\langle g_t, x-\mu_t\right\rangle+\frac{1}{6}\|x-\mu_t\|_{H_t}^2$$

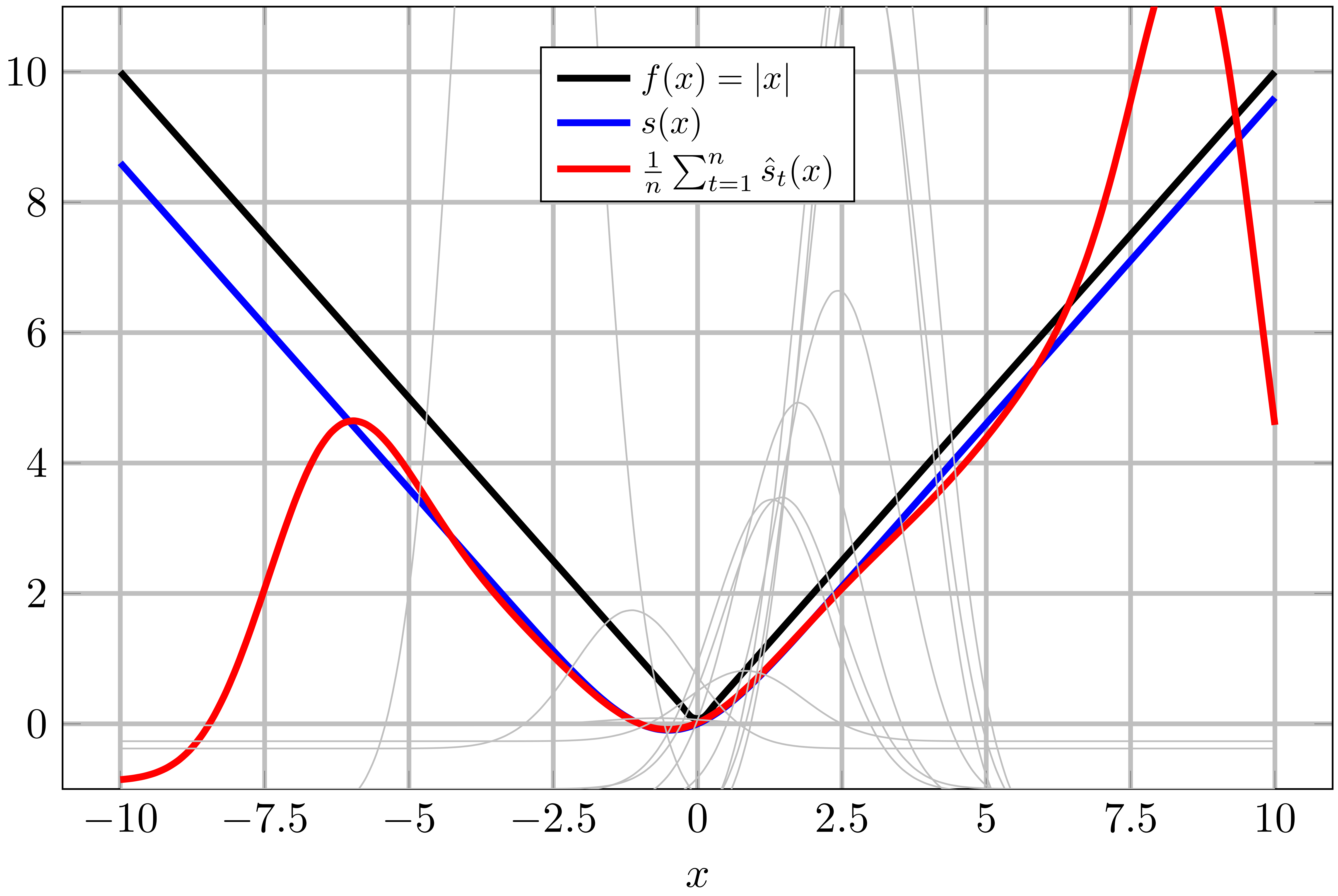
Example



- $\mu = \frac{1}{2}$, $\sigma^2 = 1$
- $\lambda = \frac{1}{2}$

- Approximation only good around μ
- Always Optimistic

Example



- $n = 10^4$
- $\mu = \frac{1}{2}$, $\sigma^2 = 1$
- $\lambda = \frac{1}{2}$

- Approximation only good around μ
- Estimation really only valid around μ
- Non-Convex!
- Not Optimistic everywhere

Summary

- Extend ℓ_t from K to $\mathbb{R}^d$
- Calculate s_t surrogate
- Estimate Quadratic Approximation
- Run FTRL (With 2 extra tricks…)

Algorithm

Algorithm

Pseudocode

input n, η, λ, γ and $K_0 = K_\varepsilon$
for $t = 1$ **to** n
    let $\Phi_{t-1}(x) = \frac{1}{2}\|x\|^2 + \sum_{u=1}^{t-1} \flat_u(x) + \eta \sum_{u=1}^{t-1} \hat{q}_u(x)$
    compute $\mu_t = \arg\min_{x \in K_{t-1}} \Phi_{t-1}(x)$ and $\Sigma_t^{-1} = \Phi''_{t-1}(\mu_t)$
    sample $X_t \sim \mathcal{N}(\mu_t, \Sigma_t)$ and observe $Y_t = f_t(X_t)$
    $K_t = K_{t-1} \cap \{x : \|x - \mu_t\|_t^2 \leq \mathrm{F}_{\max}\}$
    if in the adversarial setting:
    compute $z_t = \arg\min_{z \in \mathbb{R}^d} \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|z - \mu_s\|_s^2$

$$\flat_t(x) = \begin{cases} 0 & \text{if } \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|z_t - \mu_s\|_s^2 \geq \frac{\mathrm{F}_{\max}}{16} \\ -\gamma \|x - \mu_t\|_t^2 & \text{if } \|\cdot\|_t^2 \not\preceq \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|\cdot\|_s^2 \\ -\gamma \|x - \mu_t\|_t^2 & \text{if } \|\mu_t - z_t\|_t^2 \geq \frac{\mathrm{F}_{\max}}{8} \\ 0 & \text{otherwise.} \end{cases}$$

    if $\max_{y \in K_t} \eta \sum_{u=1}^{t} (\hat{s}_u(\mu_u) - \hat{s}_u(y)) \leq -\frac{\gamma \mathrm{F}_{\max}}{32}$
      then restart algorithm
    end if
end for

Algorithm

In words

input n, η, λ, γ and $K_0 = K_\varepsilon$
for $t = 1$ **to** n
  let $\Phi_{t-1}(x) = \frac{1}{2}\|x\|^2 + \sum_{u=1}^{t-1} \flat_u(x) + \eta \sum_{u=1}^{t-1} \hat{q}_u(x)$
  compute $\mu_t = \arg\min_{x \in K_{t-1}} \Phi_{t-1}(x)$ and $\Sigma_t^{-1} = \Phi''_{t-1}(\mu_t)$
  sample $X_t \sim \mathcal{N}(\mu_t, \Sigma_t)$ and observe $Y_t = f_t(X_t)$
  $K_t = K_{t-1} \cap \{x : \|x - \mu_t\|_t^2 \leq \mathrm{F}_{\max}\}$
  if in the adversarial setting:
  compute $z_t = \arg\min_{z \in \mathbb{R}^d} \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|z - \mu_s\|_s^2$

$$\flat_t(x) = \begin{cases} 0 & \text{if } \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|z_t - \mu_s\|_s^2 \geq \frac{\mathrm{F}_{\max}}{16} \\ -\gamma \|x - \mu_t\|_t^2 & \text{if } \|\cdot\|_t^2 \npreceq \sum_{s=1}^{t-1} \mathbf{1}(\flat_s \neq \mathbf{0}) \|\cdot\|_s^2 \\ -\gamma \|x - \mu_t\|_t^2 & \text{if } \|\mu_t - z_t\|_t^2 \geq \frac{\mathrm{F}_{\max}}{8} \\ 0 & \text{otherwise.} \end{cases}$$

  if $\max_{y \in K_t} \eta \sum_{u=1}^{t} (\hat{s}_u(\mu_u) - \hat{s}_u(y)) \leq -\frac{\gamma \mathrm{F}_{\max}}{32}$
    then restart algorithm
  end if
end for

- Run FTRL on Quadratic estimation + Bonus + Regularizer
- Determines μ_t and Σ_t^{-1}
- Sample action X_t and $Y_t = f_t(X_t)$
- Update Focus region
- Add bonus in Adversarial setting
- Check if the optimum is not moving away

Sketch of the Proof

Proof sketch

Assumptions

By the reduction we may assume:

- f_t is:
  - 1 - Lipschitz
  - $\rho = \dfrac{1}{n^2}$ - Strongly Convex
  - $\dfrac{(d+1)(d+6)}{\rho}$ - Smooth

α - Strongly Convex:

$$\alpha I \leq \nabla^2 f_t(x)$$

β - Smooth:

$$\|\nabla f_t(x) - \nabla f_t(y)\| \leq \beta \|x - y\|$$

Implied by

$$\nabla^2 f_t(x) \leq \beta I$$

Log factor:

- $L = C[1 + \log \max(n, d, 1/\delta)]$

Proof sketch

Some definitions

$$\mathrm{Reg}_\tau^f(x) = \sum_{t=1}^{\tau} \mathbb{E}_{t-1}[f_t(X_t)] - f_t(x)$$

$$\mathrm{Reg}_\tau^q(x) = \sum_{t=1}^{\tau} q_t(\mu_t) - q_t(x)$$

$$\mathrm{Reg}_\tau^s(x) = \sum_{t=1}^{\tau} s_t(\mu_t) - s_t(x)$$

$$\mathrm{Reg}_\tau^{\hat{q}}(x) = \sum_{t=1}^{\tau} \hat{q}_t(\mu_t) - \hat{q}_t(x)$$

$$x_{\star,t} = \arg\min_{x \in K_\epsilon} \sum_{u=1}^{t} f_u(x)$$

$$x_{\star,t}^s = \arg\min_{x \in K_\epsilon} \sum_{u=1}^{t} s_u(x)$$

$$x_{\star,t}^{\hat{s}} = \arg\min_{x \in K_t} \sum_{u=1}^{t} \hat{s}_u(x)$$

Proof sketch

Regret Reduction

$$
\begin{aligned}
\mathrm{Reg}_n &= \sum_{t=1}^{n} \ell_t(A_t) - \min_{x \in K} \sum_{t=1}^{n} \ell_t(x) \\
&\leq \sqrt{nL} + \sum_{t=1}^{n} \mathbb{E}_{t-1}[\ell_t(A_t)] - \min_{x \in K} \sum_{t=1}^{n} \ell_t(x) \\
&\leq \sqrt{nL} + 2\varepsilon n + \sum_{t=1}^{n} \mathbb{E}_{t-1}[\mathrm{e}_t(X_t)] - \min_{x \in K_\epsilon} \sum_{t=1}^{n} \mathrm{e}_t(x) \\
&\leq \sqrt{nL} + n(2\varepsilon + \rho) + \sum_{t=1}^{n} \mathbb{E}_{t-1}[f_t(X_t)] - \min_{x \in K_\epsilon} \sum_{t=1}^{n} f_t(x)
\end{aligned}
$$

- Hoeffding–Azuma
- Minkowski extension
- Strongly convex

Proof sketch

Regret Reduction + Estimation

$$
\begin{aligned}
\mathrm{Reg}_n &\le \sqrt{nL} + n(2\varepsilon + \rho) + \mathrm{Reg}_n^f(x_{\star,n}) \\
&\le \sqrt{nL} + n(2\varepsilon + \rho) + \mathrm{Reg}_n^s(x_{\star,n}^s) + \frac{4}{\lambda}\sum_{t=1}^{n} \mathrm{tr}(\nabla^2 s(\mu_t)\Sigma_t) \\
&\le \sqrt{nL} + n(2\varepsilon + \rho) + \mathrm{Reg}_n^q(x_{\star,n}^s) + \frac{4}{\lambda}\sum_{t=1}^{n} \mathrm{tr}(\nabla^2 s(\mu_t)\Sigma_t)
\end{aligned}
$$

- Appriximation bound
- q upper bound

Proof sketch

Reduction Regret Bound

Using concentration result of the estimators + standard FTRL analysis, with high probability:

$$\max_{x \in K_\epsilon} \mathrm{Reg}_n^q(x) \leq \begin{cases} \dfrac{F_{\max}}{\eta} & \text{Stochastic} \\ \dfrac{\gamma F_{\max}}{\eta} & \text{Adversarial}\,. \end{cases}$$

Proof sketch

Stopping time

Definition 8

We define a stopping time τ to be the first time that one of the following does not hold:

A. In the adversarial setting: $x^s_{\star,\tau} \in K_{\tau+1}$, In the Stochastic setting $x_\star \in K_{\tau+1}$
B. $\frac{1}{2}\overline{\Sigma}^{-1}_{\tau+1} \leq \Sigma^{-1}_{\tau+1} \leq \frac{3}{2}\overline{\Sigma}^{-1}_{\tau+1}$
C. The algorithm has not restarted at the end of round τ

In case none of these hold, then τ is defined to be n.

Proof sketch

Estimator concentration

s_t, Lemma 10

$$\max_{x \in K_\tau} \left| \sum_{t=1}^{\tau} (\hat{s}_t(x) - s_t(x)) \right| \leq \frac{11L^2}{\lambda}\sqrt{nd}$$

Hessian Covariance, Lemma 12

$$\sum_{t=1}^{\tau} \mathrm{tr}(\nabla^2 s_t(\mu_t)\Sigma_t) \leq \frac{dL}{\eta}$$

q_t, Lemma 10

$$\left| \sum_{t=1}^{\tau} (\hat{q}_t(x_\star) - q_t(x_\star)) \right| \leq \frac{11L^2}{\lambda}\sqrt{n}$$

$$\max_{x \in K_\tau} \left| \sum_{t=1}^{\tau} \hat{q}_t(x) - q_t(x) \right| \leq \frac{11L^2}{\lambda}\sqrt{nd}$$

Proof sketch

Estimated Regret bound

$$\mathrm{Reg}^s_\tau(x^s_{\star,\tau}) + \frac{4}{\lambda}\sum_{t=1}^{n} \mathrm{tr}(\nabla^2 s(\mu_t)\Sigma_t)$$

$$\leq \mathrm{Reg}^q_\tau(x^s_{\star,\tau}) + \frac{dL}{4\eta\lambda}$$

$$\leq \mathrm{Reg}^{\hat{q}}_\tau(x^s_{\star,\tau}) + \frac{dL}{4\eta\lambda} + \frac{11L^2}{\lambda}\sqrt{nd}$$

$$\leq \begin{cases} \dfrac{\gamma F_{\max}}{\eta} + \dfrac{dL}{4\eta\lambda} + \dfrac{11L^2}{\lambda}\sqrt{nd} & \text{Adversarial} \\[2ex] \dfrac{F_{\max}}{\eta} + \dfrac{dL}{4\eta\lambda} + \dfrac{11L^2}{\lambda}\sqrt{nd} & \text{Stochastic} \end{cases}$$

- Properties s_t and q_t
- Concentration
- FTRL analysis

Restart condition

Informally:

- If the optimum is leaving the Focus region -> Restart triggered
- If a Restart is triggered -> Negative regret -> Restart is Safe

Summary

- Reg_n -> Reg_n^f
- Reg_n^f can be bound in terms of Reg_n^s
- Reg_n^s can bound in terms of $\text{Reg}_n^{\hat{q}}$
- $\text{Reg}_n^{\hat{q}}$ has a $\sqrt{n}$ bound because of FTRL

Not Discussed:

- Exactly how the bonuses work
- Details of the Restart
- How to practically check the restart condition
- Complexity

Thank you for your attention!

Reduction

Reduction to Strongly convex/Smooth

Definition

Let e_t be the extension of ℓ_t, we make it Strongly convex + Smooth by convolving + regularising:

$$f_t(x) = \int_{\mathbb{B}(\rho)} \mathrm{e}_t(x+u)\phi_\rho(\mathrm{d}u) + \frac{\rho\|x\|^2}{2},$$

$$\phi_\rho(u) = \frac{\phi(u/\rho}{\rho^d}, \quad \phi(u) = \frac{1_{\mathbb{B}(1)}(1-\|u\|^2)^3}{\int_{\mathbb{B}(1)}(1-\|u\|^2)^3\mathrm{d}v}$$