Mathematics for Computer Science
Spring 2023
Due: 23:59, May 21, 2023

Homework Set 11

Reading Assignments: Read Chapter 15 of the textbook [LPV]. Try to solve some of the exercises in this chapter. (The solutions are sketched at the end of the book, if you have troubles solving them.)

Optional Readings:
The maxcut algorithm discussed in class is from Goemans and Williamson, JACM, 1995.
The *Unique Game Conjecture* was formulated by Subhash Khot, Proceedings of 34th ACM Symposium on Theory of Computing (2002). For a more recent popular article about the above conjecture, see: https://www.quantamagazine.org/computer-scientists-close-in-on-unique-games-conjecture-proof-20180424/

Written Assignments:

Recall that a *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \cdots, k\}$ such that $f(i) \neq f(j)$ for all $\{i, j\} \in E$. The *chromatic number* $\chi(G)$ is the smallest k such that a k-coloring exists for G. The *Greedy Coloring Algorithm* for coloring works as follows, given an input pair (G, σ) where $G = (V, E)$ is a graph and σ is an ordered list $(v_1, v_2, \cdots, v_n)$ of the vertices of G:

$s \leftarrow 1$;
$f(v_1) \leftarrow 1$;
for $i = 2$ **to** n **do**
$\quad f(v_i) \leftarrow \min\ \{s \,|\, s \neq f(v_t) \text{ for all } t < i,\ \{i, t\} \in E\,\}$

Problem 1 Prove that there exists a sequence of (G_n, σ_n), $n = 1, 2, 3, \cdots$, (where for each n, G_n has n vertices and σ_n is an ordering of the n vertices) such that the number of colors used by the Greedy Coloring Algorithm on G_n is more than $\Omega(n^c)$ times $\chi(G_n)$ (where $c > 0$ is some fixed constant independent of n).

In the *Radio Frequencies Assignment Problem* for *unit broadcast radius*, we are given n points $T = \{v_i = (x_i, y_i) \,|\, 1 \le i \le n\}$ on the plane. Let $G_T = (T, E)$, where E consists of all the pairs $\{i, j\}$ such that $0 < \|v_i - v_j\| \le 2$. We wish to analyze the performance of the greedy coloring algorithm on this class of graphs. (The points represent the locations of radio transmission stations. Each station is to be assigned a frequency (i.e. channel), with neighboring stations *not* using the same frequency. We wish to minimize the total number of channels used for this purpose.)

Problem 2 For the *Radio Frequencies Assignment Problem* for unit broadcast radius, show that the *Greedy Coloring Algorithm* performs reasonably well. That is, the number of colors used is $\le 5\chi(G) - 4$.

Problem 3 In the *Radio Frequencies Assignment Problem* for unit broadcast radius, consider the following modified algorithm: First sort the n points $v_i = (x_i, y_i)$ in decreasing lexicographical order, i.e. v_i precedes v_j if either (1) $x_i > x_j$ or (2) $x_i = x_j$ and $y_i > y_j$. Then apply the Greedy Coloring Algorithm according to the above list.

Question: Can you show that the number of colors used is $\le c \cdot \chi(G)$ for some $c \le 4$?

Problem 4 Consider the *Radio Frequencies Assignment Problem.* Instead of having a uniform broadcast radius for all the n points, assume that the n points $v_1, v_2, \cdots, v_n$ have, respectively, transmission radiuses $r_1, r_2, \cdots, r_n$. Let $G = (V, E)$ be the interference graph, where $V = \{v_1, v_2, \cdots, v_n\}$, and $\{v_i, v_j\} \in E$ $(i \neq j)$ if and only if $d(v_i, v_j) \le r_i + r_j$. Assume that the points are sorted in decreasing radius, i.e. $r_1 \geq r_2 \geq \cdots \geq r_n$. Let us apply the Greedy Coloring Algorithm to color the vertices of G, and let $t(G)$ be the number of colors used by the Algorithm on G.

Question: Does there exist a constant $c > 0$ (independent of n) such that $t(G) \leq c \cdot \chi(G)$? Give a rigorous proof of your answer.

Problem 1

For a complete bipartite graph $K_{n,n}$
if we denote the vertex as
$a_1, a_2 \cdots a_n$ and $b_1, b_2 \cdots b_n$
In fact we only need 2 colors,
but if we use Greedy Coloring Algorthim
with $G_n = \{a_1, b_1 \cdots a_n, b_n\}$
$\#$ of colors $= n$ which is more than
$\Omega(n^c)$ times $\chi(G_n)$

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