Lecture 5: Energy and Work

In this Chapter, we introduce the concept of energy and work and their applications in various physical processes. Energy is one of the most basic concepts in physics and appears in many different forms, such as potential energy and heat. Historically, people do not even know all these forms belong to the same term, energy. Moreover, work plays the role of converting one form of energy to another. In addition, we also show the conservation law of energy, which is one of the most universal laws in nature.

CONTENTS

I. ENERGY AND WORK

In physics, energy is a property of objects which can be transferred to other objects or converted into different forms. The 'ability of a system to perform work' is a common description, but it is difficult to give one single comprehensive definition of energy because of its various forms. For instance, in SI units, energy is measured in joules, and one joule is defined 'mechanically', being the energy transferred to an object by the mechanical work of moving it a distance of 1 meter against a force of 1 newton. However, there are many other definitions of energy, depending on the context, such as thermal energy, radiant energy, electromagnetic energy, nuclear energy, etc., where definitions are derived for the sake of convenience.

A. Work done by constant and non-constant forces

The physical concept of work can be mathematically described by the scalar product between the force and the displacement vectors (see Fig. 1),

$$\Delta W = \vec{F} \cdot \Delta \vec{r} = F \Delta r \cos\theta. \tag{1}$$

In the limit when $\Delta\vec{r} \to 0$, then the displacement becomes $\mathrm{d}\vec{r}$ and

$$đW = \vec{F} \cdot \mathrm{d}\vec{r}, \tag{2}$$

where $\mathrm{d}\vec{r}$ has the same direction of $\vec{v}$. Note that we use $đW$ instead of $\mathrm{d}W$ due to the fact that W is not a state function. Instead, W depends on the path of the movement.

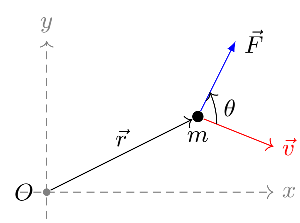

FIG. 1. Force and displacement are not in the same direction.

Question 1. *Is there any relation between $\vec{F} \cdot \mathrm{d}\vec{r}$ and $F\mathrm{d}r$, like Eq. (1)? What about $F|\mathrm{d}\vec{r}|$?*

For simplicity, we illustrate the work done by a constant/non-constant force in one dimension. Suppose a body moves from an initial point x_i to a final point x_f so that the displacement of the point the force acts on is $\Delta x = x_f - x_i > 0$. The **work done by a constant force** $F^a = F_x^a \hat{i}$ acting on the body is the product of the component of the force F_x^a and the displacement Δx,

$$W^a = F_x^a \Delta x. \tag{3}$$

Work is a scalar quantity; it is not a vector quantity. The SI unit for work is $[1N \cdot m] = [1kg \cdot m \cdot s^{-2}][1m] = [1kg \cdot m^2 \cdot s^{-2}] = [1J]$. The SI unit of energy is named for James Prescott Joule.

The displacement that appears in the above equation is not the displacement of the body but the displacement of the point of application of the force. For point-like objects, the displacement of the point of application of the force is equal to the displacement of the body. However, for an extended body, we need to focus on where the force acts and whether or not that point of application undergoes any displacement in the direction of the force.

Consider a body moving in the x-direction under the influence of a non-constant force in the x-direction, $\vec{F} = F_x(x)\hat{i}$. The body moves from an initial position x_i to a final position x_f. The **work done by this non-constant force** is given by

$$W = \int_{x_i}^{x_f} F_x(x)\mathrm{d}x. \tag{4}$$

Generally, the work done by a non-constant force along an arbitrary path is

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot \mathrm{d}\vec{r}. \tag{5}$$

Here, we give three examples: Example 1 is the scenario where the directions of the force and the displacement possess angles, Example 2 is about the work done by non-constant force, and Example 3 considers the case when a pair of forces are applied on two particles.

Example 1 (Work done by force applied at an angle to the direction of displacement)**.** *Suppose we push the cup by a constant force with the magnitude F^a and at an angle θ upwards with respect to the table, for a distance of Δx. Friction should not be ignored. Calculate the work done by the pushing force. Calculate the work done by the kinetic frictional force.*

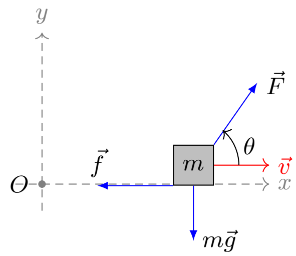

FIG. 2. Force diagram on the object.

Solution. The force diagram on the cup and coordinate system is shown in Figure 2.

The x-component of the pushing force is

$$F_x^a = F^a \cos\theta. \tag{6}$$

Suppose a body moves from an initial point x_0 to a final point x_f so that the displacement of the point the force acts on is positive $\Delta x \equiv x_f - x_0 > 0$. The **work done by a constant force** $\vec{F}^a = F_x^a \hat{i}$ acting on the body is the product of the component of the force F_x^a and the displacement Δx,

$$W^a = F_x^a \Delta x = F^a \cos\theta \Delta x. \tag{7}$$

The kinetic frictional force is

$$\vec{F}^f = -\mu_k N \hat{i} = F_x^a. \tag{8}$$

In this case, the magnitude of the normal force is not simply the same as the weight of the cup. To find the normal force, we need to find the y-component of the applied force and apply Newtons Second Law in the y-direction,

$$\begin{aligned} F_y^a &= F^a \sin\theta \\ F_y^a + N - mg &= 0 \end{aligned} \tag{9}$$

Then the work done by the kinetic frictional force is

$$W^f = -\mu_k \Delta x (mg - F^a \sin\theta) \tag{10}$$

□

Example 2 (Work done by the spring force)**.** *Fix one end of a spring to a wall and connect the other end of the spring to a body resting on a smooth (frictionless) table, as shown in Figure 3. Stretch the spring and release the spring-body system. How much work does the spring do on the body as a function of the stretched or compressed length of the spring?*

Solution. We first begin by choosing a coordinate system with origin at the position of the body when the spring is at rest in the equilibrium position. We choose the $\hat{i}$ unit vector to point in the direction that the body moves when the spring is being stretched and the coordinate x to denote the position of the body with respect to the equilibrium position, as in Fig. 3 (which indicates that in general the position x will be a function of time). The spring force on the body is given by

$$\vec{F}^s = F_x^s \hat{i} = -kx\hat{i} \tag{11}$$

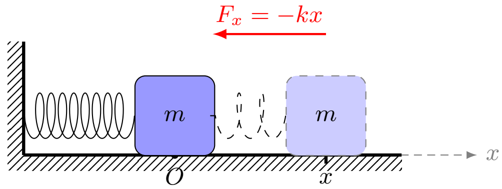

FIG. 3. Equilibrium position and position at time t.

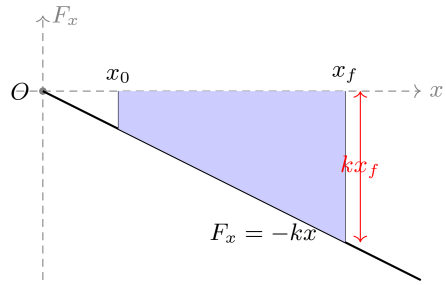

FIG. 4. The x-component of the spring force as a function of x.

In Fig. 4 we show the graph of the x-component of the spring force as a function of x for both positive values of x corresponding to stretching, and negative values of x corresponding to compressing of the spring. Note that x_0 and x_f can be positive, zero, or negative.

The work done is just the area under the curve for the interval x_0 to x_f,

$$W = \int_{x_0}^{x_f} F_x^s \mathrm{d}x = \int_{x_0}^{x_f} (-kx)\mathrm{d}x = -\frac{1}{2}k\left(x_f^2 - x_0^2\right) \tag{12}$$

When the absolute value of the final distance is less than the absolute value of the initial distance, $|x_f| < |x_0|$, the work done by the spring force is positive. This means that if the spring is less stretched or compressed in the final state than in the initial state, the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of greater tension to a state of lesser tension. $\square$

Example 3 (Work done by a pair of forces)**.** *Consider a pair of forces $\vec{F}_1$ and $\vec{F}_2$ acting on two particles A_1 and A_2. Suppose $\vec{F}_1 + \vec{F}_2 = 0$. Calculate the work caused by $\vec{F}_1$ and $\vec{F}_2$, and show that it only depends on the variance of relative distance $\vec{r}_{12}$.*

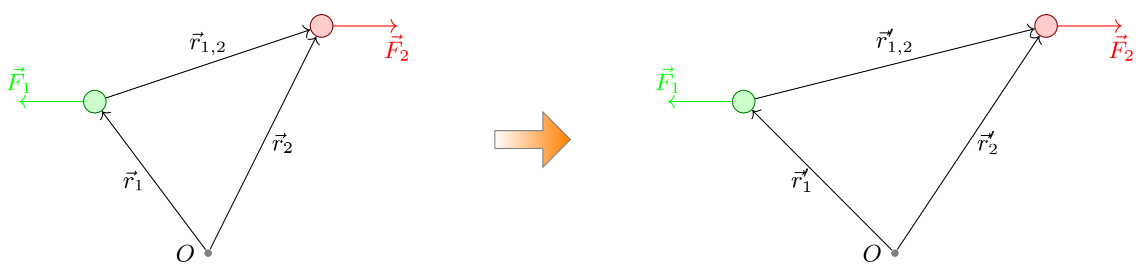

FIG. 5. Work done by a pair of force.

Solution. The work W is

$$\begin{aligned} W &= W_1 + W_2 = \vec{F}_1 \cdot (\vec{r}\,'_1 - \vec{r}_1) + \vec{F}_2 \cdot (\vec{r}\,'_2 - \vec{r}_2) \\ &= -\vec{F}_2 \cdot (\vec{r}\,'_1 - \vec{r}_1) + \vec{F}_2 \cdot (\vec{r}\,'_2 - \vec{r}_2) \\ &= \vec{F}_2 \cdot [(\vec{r}\,'_2 - \vec{r}\,'_1) - (\vec{r}_2 - \vec{r}_1)] = \vec{F}_2 \cdot (\vec{r}\,'_{12} - \vec{r}_{12}), \end{aligned} \tag{13}$$

from where we can see that W only depends on the change of $\vec{r}_{12}$. $\square$

B. Power applied by a force

Suppose that an applied force $\vec{F}^a$ acts on a body during a time interval Δt, and the displacement of the point of application of the force is in the x-direction by an amount Δx.

The **average power** of an applied force is defined to be the rate at which work is done,

$$P^a_{ave} = \frac{\Delta W^a}{\Delta t} = \frac{F^a_x \Delta x}{\Delta t} = F^a_x v_{ave.x}. \tag{14}$$

The **instantaneous power** at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P^a = \lim_{\Delta t \to 0} \frac{\Delta W^a}{\Delta t} = \lim_{\Delta t \to 0} \frac{F^a_x \Delta x}{\Delta t} = F^a_x \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = F^a_x v_x. \tag{15}$$

The average power delivered to the body is equal to the component of the force in the direction of motion times the component of the average velocity of the body. The instantaneous power of a constant applied force is the product of the component of the force in the direction of motion and the instantaneous velocity of the moving object. Power is a scalar quantity and can be positive, zero, or negative depending on the sign of work. The SI units of power are called watts $[W]$ and $[1W] = [1J \cdot s^{-1}]$.

Generally, the instantaneous power of a non-constant force at time t is

$$P(t) = \vec{F}(t) \cdot \vec{v}(t). \tag{16}$$

C. Kinetic energy and work-kinetic energy theorem

The **kinetic energy** K of a point particle of mass m moving with speed v is defined to be the non-negative scalar quantity

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}. \tag{17}$$

The kinetic energy is proportional to the square of the speed. The SI units for kinetic energy are $[kg \cdot m^2 \cdot s^{-2}]$. This combination of units is defined to be a joule and is denoted by $[J]$, thus $1J = 1kg \cdot m^2 \cdot s^{-2}$. The above definition of kinetic energy does not refer to any direction of motion, just the speed of the body.

There is a direct connection between the work done on a point particle and the change in kinetic energy the point particle undergoes. If the work done on the particle is non-zero, this implies that an unbalanced force has acted on the particle, and the particle will have undergone acceleration. For a particle undergoing one-dimensional motion, the work done on the particle by the component of the sum of the forces in the direction of displacement is given by

$$W = \int_{x_i}^{x_f} F_x(x) \mathrm{d}x = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = K_f - K_i. \tag{18}$$

This formula can be easily obtained from the second law of Newton for a point particle in one-dimensional motion.

In a general case, we consider the change of kinetic energy over a tiny time interval $\mathrm{d}t$. The changing rate of kinetic energy is given by

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}t} &= \frac{1}{2} m \frac{\mathrm{d}(\vec{v} \cdot \vec{v})}{\mathrm{d}t} \\ &= \frac{1}{2} m 2\vec{v} \cdot \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \\ &= m\vec{v} \cdot \vec{a} \\ &= \vec{v} \cdot \vec{F} \\ &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \vec{F} \\ &= \frac{đW}{\mathrm{d}t}. \end{aligned} \tag{19}$$

Here $đW$ is the work applied to the system over the time interval dt. The second equality can be derived from the Cartesian coordinates

$$\frac{\mathrm{d}(\vec{v}\cdot\vec{v})}{\mathrm{d}t}=\frac{\mathrm{d}(v_x^2+v_y^2+v_z^2)}{\mathrm{d}t}=2(v_x\frac{\mathrm{d}v_x}{\mathrm{d}t}+v_y\frac{\mathrm{d}v_y}{\mathrm{d}t}+v_z\frac{\mathrm{d}v_z}{\mathrm{d}t})=2\vec{v}\cdot\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}. \tag{20}$$

Integrate Eq. (19) over time, we obtain similar results as the one in Eq. (18).

When the work done on an object is positive, the object increases its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed. In fact, the work-energy relationship is quite precise; the work done by the applied force on an object is identically equal to the change in kinetic energy of the object.

Example 4 (Gravity and the work-energy theorem)**.** *Suppose a ball of mass $m = 0.2kg$ starts from rest at a height $y_0 = 15m$ above the surface of the Earth and falls down to a height $y_f = 5.0m$ above the surface of the Earth. What is the change in the kinetic energy? Find the final velocity using the work-energy theorem.*

Solution. As only one force acts on the ball, the change in kinetic energy is the work done by gravity,

$$W^g = -mg(y_f - y_0) = 20J. \tag{21}$$

The ball started from rest, $v_{y,0} = 0$. So the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_{y,f}^2 - \frac{1}{2}mv_{y,0}^2 = \frac{1}{2}mv_{y,f}^2. \tag{22}$$

We can solve Eq. (22) for the final velocity using Eq. (21)

$$v_{y,f} = \sqrt{\frac{2\Delta K}{m}} = \sqrt{\frac{2W^g}{m}} = 14m\cdot s^{-1} \tag{23}$$

For the falling ball in a constant gravitation field, the positive work of the gravitation force on the body corresponds to an increasing kinetic energy and speed. For a rising body in the same field, the kinetic energy and hence the speed decrease since the work done is negative. □

D. Kinetic energy of a system

For system of particles, the kinetic energy of the system is the summation of the kinetic energy of the particles:

$$K = \sum_i K_i = \sum_i \frac{1}{2}m_i|\vec{v}_i|^2. \tag{24}$$

Given a system of particles, the relationship between the total kinetic energy in two different reference frames is generally rather messy and unenlightening. But we know the COM frame is a very nice reference frame in the chapter of momentum, as the total momentum turns out to be 0 in the COM frame. Here we will find that if one of the reference frames is the COM frame, the relationship between the total kinetic energy will be quite neat.

Let S' be the COM frame, which moves at velocity $\vec{u}$ with respect to another reference frame S. Then the velocities of the particles in the two reference frames are related by

$$\vec{v}_i = \vec{v}_i' + \vec{u}. \tag{25}$$

The kinetic energy in the COM frame is

$$\mathrm{K_{COM}} = \frac{1}{2}\sum m_i|\vec{v}_i'|^2. \tag{26}$$

The kinetic energy in frame S is

$$\begin{aligned}\mathrm{K_S} &= \frac{1}{2}\sum m_i|\vec{v}_i'+\vec{u}|^2\\ &= \frac{1}{2}\sum m_i(\vec{v}_i'+\vec{u})\cdot(\vec{v}_i'+\vec{u})\\ &= \frac{1}{2}\sum m_i(\vec{v}_i'\cdot\vec{v}_i'+2\vec{v}_i'\cdot\vec{u}+\vec{u}\cdot\vec{u})\\ &= \frac{1}{2}\sum m_i|\vec{v}_i'|^2+\vec{u}\cdot\left(\sum m_i\vec{v}_i'\right)+\frac{1}{2}|\vec{u}|^2\sum m_i\\ &= \mathrm{K_{COM}}+\frac{1}{2}Mu^2.\end{aligned} \tag{27}$$

where M is the total mass of the system, and where we have used $\sum_i m_i\vec{v}'_i = 0$, by definition of the COM frame. Therefore, the kinetic energy in any frame equals the kinetic energy in the COM frame, plus the kinetic energy of the whole system treated like a point mass M located at the COM (which moves with velocity $\vec{u}$). An immediate corollary of this fact is that the kinetic energy in the COM frame is the smallest one among that in all the reference frames.

II. POTENTIAL ENERGY

A. Conservative forces and changes in potential energies of a system

Whenever the work done by a force in moving an object from an initial point to a final point is *independent* of the path, the force is called a **conservative force**. In other words, the work done by a conservative force $\vec{F}(\vec{r})$ in going around a closed path L is zero:

$$\oint_L \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} = 0. \tag{28}$$

Examples of conservative force are constant gravity near the surface of the Earth, the spring force and the universal gravitation force.

Consider a system consisting of two objects interacting through a conservative force. Denote $\vec{F}_{2,1}$ to be the force on object 1 due to the interaction with object 2 and

$$\mathrm{d}\vec{r}_{2,1} = \mathrm{d}\vec{r}_1 - \mathrm{d}\vec{r}_2. \tag{29}$$

to be the relative displacement of the two objects. The **change in internal potential energy of the system** is defined to be the negative of the work done by the conservative force when the objects undergo a relative displacement from the initial state A to the final state B along any displacement that changes the initial state A to the final state B,

$$\Delta U_{sys} = -W_c = -\int_A^B \vec{F}_{2,1} \cdot \mathrm{d}\vec{r}_{2,1} \tag{30}$$

Our definition of potential energy only holds for conservative forces, because the work done by a conservative force does not depend on the path but only on the initial and final positions.

Depending on the specific scenario, we shall choose a **zero reference potential** for the potential energy of the system, so that we can consider all changes in potential energy relative to this reference potential.

Consider the example of an object falling near the surface of the Earth. Choose the system consisting of the Earth and the object. The gravitational force is now an internal conservative force acting inside the system. The distance separating the object and the center of mass of the Earth, and the velocities of the Earth and the object specify the initial and final states.

Let us choose a coordinate system with the origin on the surface of the Earth and the $+y$-direction pointing away from the center of the Earth. Because the displacement of the Earth is negligible, we need only consider the displacement of the object in order to calculate the change in potential energy of the system.

Suppose the object starts at an initial height y_i above the surface of the Earth and ends at final height y_f. The gravitational force on the object is usually approximated by $\vec{F}^g = -mg\vec{j}$, the displacement is given by $\mathrm{d}\vec{r} = \mathrm{d}y\vec{j}$, and the scalar product is given by

$$\vec{F}^g \cdot \mathrm{d}\vec{r} = -mg\vec{j} \cdot \mathrm{d}y\vec{j} = -mgdy. \tag{31}$$

The work done by the gravitational force on the object is then

$$W^g = \int_{y_i}^{y^f} \vec{F}^g \cdot \mathrm{d}\vec{r} = -mg(y_f - y_i). \tag{32}$$

The change in potential energy is then given by

$$\Delta U^g = -W^g = mgy_f - mgy_i. \tag{33}$$

We introduce a potential energy function U so that

$$\Delta U^g = U^g_f - U^g_i. \tag{34}$$

Only differences in the function U^g have a physical meaning. We can choose a zero reference point for the potential energy anywhere we like. We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. Because the change in potential energy only depends on the vertical displacement, $\Delta y = y_f - y_i$. In the above expression for the change of potential energy, let $y_f = y$ be an arbitrary point and $y_i = 0$ denote the surface of the Earth. Choose the zero reference potential for the potential energy to be at the surface of the Earth corresponding to our origin $y = 0$, with $U^g(0) = 0$. Then

$$\Delta U^g = U^g(y) - U^g(0) = U^g(y). \tag{35}$$

Substitute $y_i = 0$, $y_f = y$ to above equations yielding a potential energy as a function of the height y above the surface of the Earth,

$$U^g(y) = mgy, \tag{36}$$

with $U^g(y = 0) = 0$.

Notice that until here, we are talking about the potential energy of a system or a pair of objects. Due to Newton's Third Law, we need to consider the work done by a pair of forces. But what if one of the objects, object 2, is always stable or can be regarded as stable? Just like the former system consisting of the Earth and the object, we can regard the Earth as a stable object. Then in Eq. (29) the relative displacement is $\mathrm{d}\vec{r}_1$ and

$$\Delta U = -\int_A^B \vec{F}_{2,1} \cdot \mathrm{d}\vec{r}_1, \tag{37}$$

which can be regarded as the potential energy of object 1, ΔU_1.

In one-dimensional case, with the relationship between integral and derivation we can get the conservation force from potential energy,

$$F_{2,1} = -\frac{\mathrm{d}U}{\mathrm{d}x}. \tag{38}$$

And in general, it should be written as a gradient,

$$\vec{F}_{2,1} = -\nabla U. \tag{39}$$

Example 5 (Energy diagram, Example 14.1 in the textbook)**.** *The potential energy function for a particle of mass m, moving in the x-direction is given by*

$$U(x) = -U_1 \left[\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2 \right], \tag{40}$$

where U_1 and x_1 are positive constants and $U(0) = 0$. (a) Sketch $U(x)/U_1$ as a function of x/x_1. (b) Find the points where the force on the particle is zero. Classify them as stable or unstable. (c) For energies E that lies in $0 < E < (4/27)U_1$ find an equation whose solution yields the turning points along the x-axis about which the particle will undergo periodic motion.

Solution. a) Figure 6 shows a graph of $U(x)$ vs. x, with the choice of values $x_1 = 1.5$ m and $U_1 = 27/4$ J.

b) The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{\mathrm{d}U}{\mathrm{d}x} = U_1 \left(\left(\frac{3}{x_1^3}\right) x^2 - \left(\frac{2}{x_1^2}\right) x \right) = 0, \tag{41}$$

which has two solutions

$$x = 2x_1/3 \quad \text{and} \quad x = 0. \tag{42}$$

The second derivative is given by

$$\frac{\mathrm{d}^2 U}{\mathrm{d}^2 x}(x) = -U_1 \left(\left(\frac{6}{x_1^3}\right) x - \left(\frac{2}{x_1^2}\right) \right). \tag{43}$$

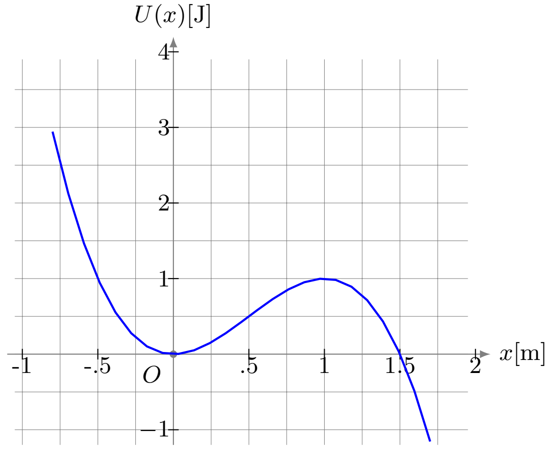

FIG. 6. A graph of $U(x)$ vs. x.

When $x = 2x_1/3$,

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2}\,(x = 2x_1/3) = -U_1\left(\left(\frac{6}{x_1^3}\right)\frac{2x_1}{3} - \left(\frac{2}{x_1^2}\right)\right) = -\frac{2U_1}{x_1^2} < 0, \tag{44}$$

indicating the solution $x = 2x_1/3$ represents a local maximum and hence is an unstable point. When $x = 0$,

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2}(x = 0) = -U_1\left(\left(\frac{6}{x_1^3}\right)0 - \left(\frac{2}{x_1^2}\right)\right) = \frac{2U_1}{x_1^2} > 0, \tag{45}$$

indicating the solution $x = 0$ represents a local minimum and is a stable point.

c) Consider a fixed value of the energy of the particle within the range

$$U(0) = 0 < E < U\left(2x_1/3\right) = \frac{4U_1}{27}. \tag{46}$$

If the particle at any time is found in the region $x_a < x < x_b < 2x_1/3$, where x_a and x_b are the turning points and are solutions to the equation

$$E = U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right). \tag{47}$$

then the particle will undergo periodic motion between the values $x_a < x < x_b$. Within this region $x_a < x < x_b$, the kinetic energy is always nonnegative because $K(x) = E - U(x)$. There is another solution x_c to Eq. (47) somewhere in the region $x_c > 2x_1/3$. If the particle at any time is in the region $x > x_c$ then it at any later time it is restricted to the region $x_c < x < +\infty$. □

B. Potential energy raised by gravitational force

Now we consider the potential energy originated by a common conservative force: gravitational force. For two particles A and B with mass m_A and m_B, the gravitational force of A acting on B is given by

$$\vec{F}_{AB} = -Gm_Am_B\frac{\vec{r}_{AB}}{r_{AB}^3}, \tag{48}$$

where G is the gravitational constant, $\vec{r}_{AB}$ is the displacement vector from A to B.

Now we fix the location of A, and move B from the position B_i to B_f, as shown in Fig. 7. We would like to calculate the work of $\vec{F}_{AB}$ acting on B.

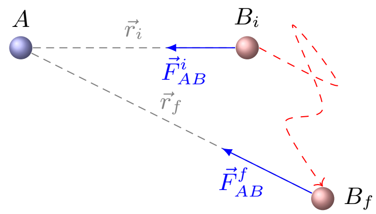

FIG. 7. Work done by the gravitational force and gravitational potential energy.

By the definition of work we have

$$\begin{aligned} W &= \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot \mathrm{d}\vec{r} = -Gm_A m_B \int_{\vec{r}_i}^{\vec{r}_f} \frac{\vec{r} \cdot \mathrm{d}\vec{r}}{r^3} \\ &= -\frac{1}{2} Gm_A m_B \int_{\vec{r}_i}^{\vec{r}_f} \frac{\mathrm{d}\left(r^2\right)}{r^3} = -Gm_A m_B \int_{\vec{r}_i}^{\vec{r}_f} \frac{\mathrm{d}r}{r^2} \\ &= Gm_A m_B \left(\frac{1}{r}\right)\Big|_{r_i}^{r_f} \\ &= -Gm_A m_B \left(\frac{1}{r_i} - \frac{1}{r_f}\right). \end{aligned} \tag{49}$$

From Eq. (49) we can see that, the gravitational force is conservative. If we set the gravitational potential $U_g(r \to +\infty) = 0$, then from Eq. (49) we have

$$U_g(r) = -Gm_A m_B \frac{1}{r} \tag{50}$$

Example 6 (Escape velocity of Toro, from MIT OpenCourseWare 8.01)**.** *The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0km$ and a mass of about $m_t = 2.0 \times 10^{15} kg$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass m on the surface of Toro? Could a person reach this speed (on the Earth) by running?*

Solution. The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U^G(\infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance r apart is given by

$$U^G(r) = -\frac{Gm_t m}{r}, \tag{51}$$

with $U^G(\infty) \equiv 0$. The expression *escape velocity* refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time t_i, when the object is at the surface of the asteroid with speed equal to the escape velocity v_{esc}. We choose our final state, at time t_f, to occur when the separation distance between the asteroid and the object is finite.

The initial kinetic energy is $K_i = \frac{1}{2} m v_{\text{esc}}^2$. The initial potential energy is $U_i = -\frac{Gm_t m}{R}$, and so the initial mechanical energy is

$$E_i = K_i + U_i = \frac{1}{2} m v_{\text{esc}}^2 - \frac{Gm_t m}{R}. \tag{52}$$

The final kinetic energy is $K_f = 0$, because this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ because we chose the zero point for potential energy at infinity. The final mechanical energy is then

$$E_f = K_f + U_f = 0. \tag{53}$$

There is no non-conservative work, so the change in mechanical energy is zero,

$$0 = W_{\rm nc} = \Delta E_m = E_f - E_i. \tag{54}$$

Therefore,

$$0 = -\left(\frac{1}{2}mv_{\rm esc}^2 - \frac{Gm_t m}{R}\right). \tag{55}$$

The escape velocity can be solved,

$$\begin{aligned} v_{\rm esc} &= \sqrt{\frac{2Gm_t}{R}} \\ &= \sqrt{\frac{2(6.67\times10^{-11}{\rm N}\cdot{\rm m}^2\cdot{\rm kg}^{-2})(2.0\times10^{15}{\rm kg})}{(5.0\times10^3{\rm m})}} \\ &= 7.3{\rm m}\cdot{\rm s}^{-1}. \end{aligned} \tag{56}$$

Considering that Olympic sprinters typically reach velocities of $12\text{m}\cdot s^{-1}$, this is an easy speed to attain by running on Earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on Earth to come back down for another push off the ground. □

III. CONSERVATION OF MECHANICAL ENERGY

A. The principle of conservation of mechanical energy

Mechanical energy is the energy associated with the motion and position of an object. It can be divided into two parts: potential energy and kinetic energy.

Consider an isolated system that is only subject to conservation force, i.e., there is no external force and the internal force is conservation force. Then only the work done by internal conservation force can increase the kinetic energy:

$$K_f - K_i = W_c, \tag{57}$$

and the work done by internal conservation force lead to the decreasing of the potential energy of the system:

$$-W_c = \Delta U_{sys} = U_f - U_i. \tag{58}$$

By combining them we have

$$K_f - K_i = -(U_f - U_i), \tag{59}$$

and the following principle.

The principle of conservation of mechanical energy states that in an *isolated* system that is only subject to conservative forces, the mechanical energy is constant,

$$\Delta E_m = \Delta K_{sys} + \Delta U_{sys} = 0. \tag{60}$$

Recall that the work done by a conservative force in going around a closed path is zero, therefore both the changes in kinetic energy and potential energy are zero when a closed system with only conservative internal forces returns to its initial state. Throughout the process, the kinetic energy may change into internal potential energy but if the system returns to its initial state, the kinetic energy is completely recoverable. We shall refer to a closed system in which processes take place in which only conservative forces act as **completely reversible processes**.

B. Change of mechanical energy for closed system with internal non-conservative forces

Whenever the work done by a force in moving an object from an initial point to a final point *depends* on the path, the force is called a **non-conservative force**.

Suppose the internal forces are both conservative and non-conservative. The work W done by the forces is a sum of the conservative work W_c , which is path-independent, and the non-conservative work W_{nc} , which is path-dependent,

$$W = W_c + W_{nc}. \tag{61}$$

The work done by the conservative forces is equal to the negative of the change in the potential energy

$$\Delta U = -W_c. \tag{62}$$

The work done is equal to the change in the kinetic energy,

$$W = \Delta K. \tag{63}$$

Thus we have

$$W_{nc} = \Delta K + \Delta U = \Delta E_m, \tag{64}$$

the mechanical energy is no longer constant. The total change in energy of the system is zero,

$$\Delta E_{sys} = \Delta E_m - W_{nc} = 0. \tag{65}$$

Energy is conserved but some mechanical energy has been transferred into nonrecoverable energy. We shall refer to processes in which there is non-zero nonrecoverable energy as **irreversible processes**.

Example 7 (Bernoulli equation)**.** *The Bernoulli equation shows how the pressure and velocity vary from one point to another within a flowing (ideal) fluid. It states the conservation of energy principle for flowing fluids and can be summarized in the following memorable word equation: static pressure + dynamic pressure = total pressure. Denote p to be the pressure of the fluid at one point, v to be its flowing velocity, and h to be the height, then,*

$$\frac{p}{\rho g} + \frac{v^2}{2g} + h = const, \tag{66}$$

where ρ is the density of the fluid and g is the acceleration due to gravity.

Solution. We study the flow of the fluid in a tube, as shown in Figure 8. The work done by the pressure is given by

$$đW = (p_1 A_1 v_1 - p_2 A_2 v_2)\mathrm{d}t. \tag{67}$$

The fluid cannot be compressed: $A_1 v_1 \mathrm{d}t = p_2 A_2 v_2 \mathrm{d}t \equiv \mathrm{d}V$. Only this $\mathrm{d}V$ part of fluid would cause the change of mechanical energy,

$$\begin{aligned} \mathrm{d}E &= (\frac{1}{2}v_2^2 \mathrm{d}m + gh_2 \mathrm{d}m) - (\frac{1}{2}v_1^2 \mathrm{d}m + gh_1 \mathrm{d}m) \\ &= \rho dV[(\frac{1}{2}v_2^2 + gh_2) - (\frac{1}{2}v_1^2 + gh_1)]. \end{aligned} \tag{68}$$

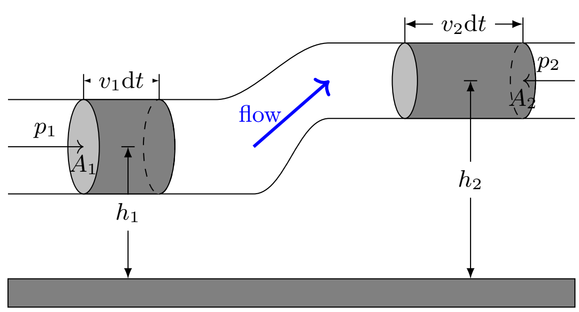

FIG. 8. Flow of a fluid through a tube.

The change of mechanical energy in Eq. (68) equals to the external work given by Eq. (67),

$$(p_1 - p_2)\mathrm{d}V = \rho \mathrm{d}V[(\frac{1}{2}v_2^2 + gh_2) - (\frac{1}{2}v_1^2 + gh_1)], \tag{69}$$

and hence,

$$p_1 + \rho(\frac{1}{2}v_1^2 + gh_1) = p_2 + \rho(\frac{1}{2}v_2^2 + gh_2), \tag{70}$$

which leads to Eq. (66).

□

Example 8 (Object sliding down an inclined plane). *An object of mass $m = 4.0kg$, starting from rest, slides down an inclined plane of length $I = 3.0m$. The plane is inclined by an angle of $\theta = 30°$ to the ground. The coefficient of kinetic friction is $\mu_k = 0.2$. (a) What is the work done by each of the three forces while the object is sliding down the inclined plane? (b) For each force, is the work done by the force positive or negative? (c) What is the sum of the work done by the three forces? ls this positive or negative?*

Solution. (a) and (b) Choose a coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, and the positive y-direction pointing towards the upper right as shown in Figure 9. While the object is sliding down the inclined plane, three uniform forces act on the object, the gravitational force which points downward and has magnitude $F_g = mg$, the normal force N which is perpendicular to the surface of the inclined plane, and the frictional force which opposes the motion and is equal in magnitude to $f_k = \mu_k N$. A force diagram on the object is shown in Figure 10.

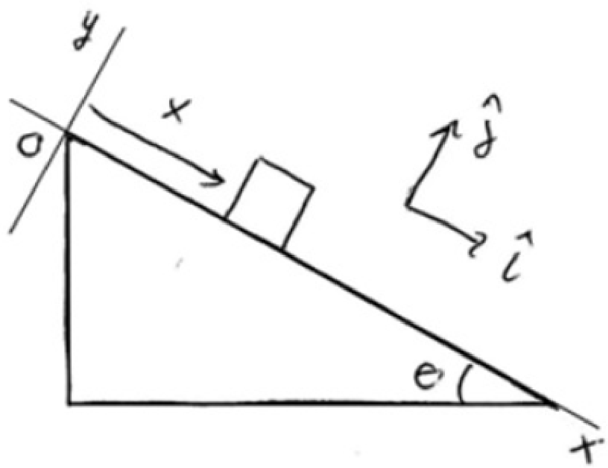

FIG. 9. Coordinate system for object sliding down inclined plane.

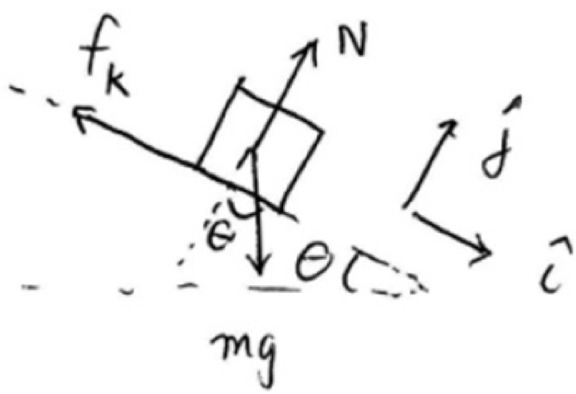

FIG. 10. Free-body force diagram for object.

In order to calculate the work we need to determine which forces have a component in the direction of the displacement. Only the component of the gravitational force along the positive x-direction $F_{gx} = mgsin\theta$ and the frictional force are directed along the displacement and therefore contribute to the work. We need to use Newton's Second Law to determine the magnitudes of the normal force. Because the object is constrained to move along the positive x-direction, $a_y = 0$, Newton's Second Law in the $\hat{j}$-direction shows that $N - mg\cos\theta = 0$. Therefore $N = mg\cos\theta$ and the magnitude of the frictional force is $f_k = \mu_k mg\cos\theta$.

With our choice of coordinate system with the origin at the top of the inclined plane and the positive x-direction pointing down the inclined plane, the displacement of the object is given by the vector $\Delta\vec{r} = \Delta x\hat{i}$, as shown in Figure 11.

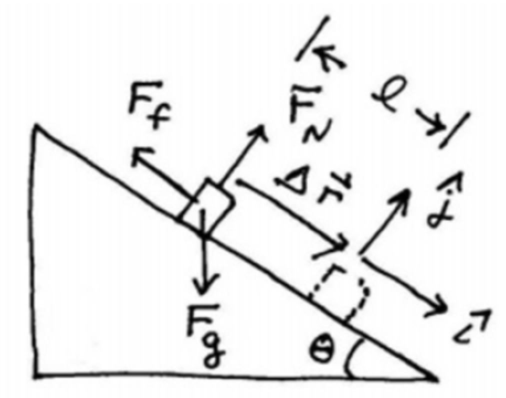

FIG. 11. Force vectors and displacement vector for object.

The vector decomposition of the three forces are $\vec{F}^g = mg\sin\theta\hat{i} - mg\cos\theta\hat{j}$, $\vec{F}^f = -\mu_k mg\cos\theta\hat{i}$, and $\vec{F}^N = mg\cos\theta\hat{j}$. The work done by the normal force is zero because the normal force is perpendicular the displacement

$$W^N = \vec{F}^N \cdot \Delta\vec{r} = mg\cos\theta\hat{j} \cdot l\hat{i} = 0. \tag{71}$$

The work done by the frictional force is negative and is given by

$$W^f = \vec{F}^f \cdot \Delta\vec{r} = -\mu_k mg\cos\theta\hat{i} \cdot l\hat{i} = -\mu_k mg\cos\theta l < 0. \tag{72}$$

Substituting in the appropriate values yields

$$W^f = -\mu_k mg\cos\theta l = -(0.2)(4.0\text{kg})(9.8\text{m}\cdot\text{s}^{-2})(3.0\text{m})(\cos(30°))(3.0\text{m}) = -20.4\text{J}. \tag{73}$$

The work done by the gravitational force is positive and given by

$$W^g = \vec{F}^g \cdot \Delta\vec{r} = (mg\sin\theta\hat{i} - mg\cos\theta\hat{j}) \cdot l\hat{i} = mgl\sin\theta > 0. \tag{74}$$

Substituting in the appropriate values yields

$$W^g = mgl\sin\theta = (4.0\text{kg})(9.8\text{m}\cdot\text{s}^{-2})(3.0m)(sin(30°)) = 58.8J. \tag{75}$$

(c) The scalar sum of the work done by the three forces is then

$$W = W^g + W^f = mgl(\sin\theta - \mu_k\cos\theta) \tag{76}$$

$$W = (4.0\text{kg})(9.8\text{m}\cdot\text{s}^{-2})(3.0\text{m})(\sin(30°) - (0.2)(\cos(30°)) = 38.4J. \tag{77}$$

□

Example 9 (Cart-spring on an inclined plane). *An object of mass m slides down a plane that is inclined at an angle θ from the horizontal (Figure 12). The object starts out at rest. The center of mass of the cart is a distance d from an unstretched spring that lies at the bottom of the plane. Assume the spring is massless, and has a spring constant k. Assume the inclined plane to be frictionless. (a) How far will the spring compress when the mass first comes to rest? (b) Now assume that the inclined plane has a coefficient of kinetic friction μk. How far will the spring compress when the mass first comes to rest? The friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction μk. (c) In case (b), how much energy has been lost to friction?*

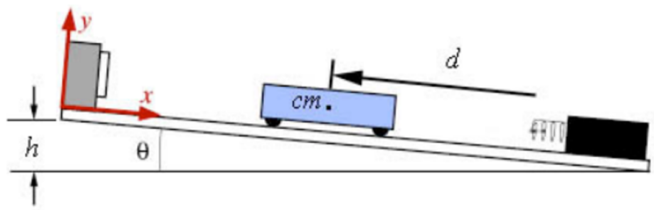

FIG. 12. Cart on inclined plane.

Solution. Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U^g(0) = 0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy where the spring is at its equilibrium position, $U^s(0) = 0$.

a) Choose the initial state as the instant when the object is released (Figure 13). The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = mgd\sin\theta$. The initial mechanical energy is then

$$E_i = K_i + U_i = mgd\sin\theta. \tag{78}$$

Choose the final state as the instant when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane (Figure 13). The final kinetic energy is $K_f = 0$ since the mass is not in motion. The final potential energy is non-zero, $U_f = kx^2/2 - xmg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.

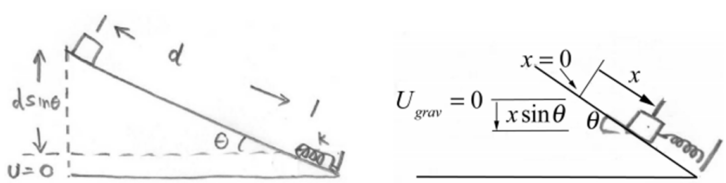

FIG. 13. Initial state and final state

The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}kx^2 - xmg\sin\theta. \tag{79}$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{nc} = \Delta E_m = E_f - E_i. \tag{80}$$

Therefore,

$$dmg\sin\theta = \frac{1}{2}kx^2 - xmg\sin\theta. \tag{81}$$

This is a quadratic equation in x,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2dmg\sin\theta}{k} = 0. \tag{82}$$

In the quadratic formula, we want the positive choice of square root for the solution to ensure a positive displacement of the spring from equilibrium,

$$x = \frac{mg\sin\theta}{k} + \left(\frac{m^2g^2\sin^2\theta}{k^2} + \frac{2dmg\sin\theta}{k}\right)^{1/2} \tag{83}$$

$$= \frac{mg}{k}\left(\sin\theta + \sqrt{1 + 2(kd/mg)\sin\theta}\right). \tag{84}$$

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d+x$, given by

$$W_{nc} = -f_k(d+x) = -\mu_k N(d+x) = -\mu_k mg\cos\theta(d+x). \tag{85}$$

Note the normal force is found by using Newton's Second Law in the perpendicular directito the inclined plane,

$$N - mg\cos\theta = 0. \tag{86}$$

The change in mechanical energy is therefore

$$W_{nc} = \Delta E_m = E_f - E_i, \tag{87}$$

which becomes

$$-\mu_k mg\cos\theta(d+x) = (\frac{1}{2}kx^2 - xmg\sin\theta) - dmg\sin\theta. \tag{88}$$

Equation. 88 simplifies to

$$0 = (\frac{1}{2}kx^2 - xmg(\sin\theta - \mu_k\cos\theta)) - dmg(\sin\theta - \mu_k\cos\theta). \tag{89}$$

This is the same as Equation. 81 above, but with $\sin\theta \to \sin\theta - \mu_k\cos\theta$. The maximum displacement of the spring when there is friction is then

$$x = \frac{mg}{k}((\sin\theta - \mu_k\cos\theta) + \sqrt{1 + 2(kd/mg)(\sin\theta - \mu_k\cos\theta)}). \tag{90}$$

c) The energy lost to friction is given by $W_{nc} = -\mu_k mg\cos\theta(d+x)$, where x is given in part b). □

C. Change of mechanical energy for a non-closed System

When the system is no longer closed but in contact with its surroundings, the change in the energy of the system is equal to the negative of the change in energy of the surroundings,

$$\Delta E_{sys} = -\Delta E_{sur}. \tag{91}$$

The change in energy of the system can be the result of external work done by the surroundings on the system (which can be positive or negative),

$$W_{ext} = \int_A^B \vec{F}_{ext} \cdot \mathrm{d}\vec{r}. \tag{92}$$

If the system is in thermal contact with the surroundings, then energy can flow into or out of the system. This energy flow due to thermal contact is often denoted by Q with the convention that $Q > 0$ if the energy flows into the system ($\Delta E_{sur} < 0$) and $Q < 0$ if the energy flows out of the system ($\Delta E_{sur} > 0$). Then $\Delta E_{sys} = -\Delta E_{sur}$ can be rewritten as

$$\Delta E_{sys} = W_{ext} + Q. \tag{93}$$

This is also called **the first law of thermodynamics**.

D. Conservation of energy

Any physical process can be characterized by two states, initial and final, between which energy transformations can occur. Each form of energy E_i, where i is an arbitrary label identifying one of the N forms of energy, may undergo a change during this transformation,

$$\Delta E_i = E_{final,i} - E_{initial,i}. \tag{94}$$

Conservation of energy means that the sum of these changes is zero,

$$\sum_{i=1}^{N} \Delta E_i = 0. \tag{95}$$

Two important points emerge from this idea. First, we are interested primarily in changes in energy and so we search for relations that describe how each form of energy changes. Second, we must account for all the ways energy can change. If we observe a process, and the sum of the changes in energy is not zero, either our expressions for energy are incorrect, or there is a new type of change of energy that we had not previously discovered. This is our first example of the importance of conservation laws in describing physical processes, as energy is a key quantity conserved in all physical processes. If we can quantify the changes of different forms of energy, we have a very powerful tool to understand nature.

Particularly, as shown in Figure 14, when a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,

$$\Delta E = \Delta E_{system} + \Delta E_{surroundings} = 0. \tag{96}$$

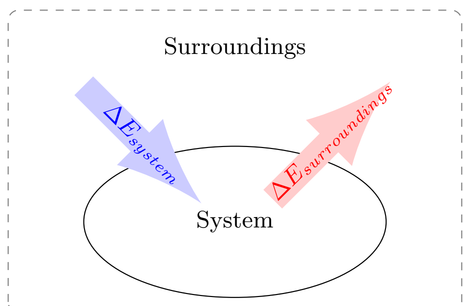

FIG. 14. Energy exchange between a system and its surroundings.