体論 (第 8 回) の解答

問題 8-1 の解答

定義 8-1 の (1-i), (1-ii), (2) を示す. $x = a + b\sqrt{-1}$, $y = c + d\sqrt{1} \in \mathbb{C}$ $(a, b, c, d \in \mathbb{R})$ をとる.

(1-i) について.

$$\begin{aligned}\sigma(x+y) &= \sigma((a+c)+(b+d)\sqrt{-1}) \\ &= (a+c)-(b+d)\sqrt{-1} \\ &= (a-b\sqrt{-1})+(c-d\sqrt{-1}) \\ &= \sigma(x)+\sigma(y).\end{aligned}$$

(1-ii) について.

$$\begin{aligned}\sigma(xy) &= \sigma((a+b\sqrt{-1})(c+d\sqrt{-1})) \\ &= \sigma((ac-bd)+(ad+bc)\sqrt{-1}) \\ &= (ac-bd)-(ad+bc)\sqrt{-1} \\ &= (a-b\sqrt{-1})(c-d\sqrt{-1}) \\ &= \sigma(x)\sigma(y).\end{aligned}$$

(2) $a \in \mathbb{R}$ に対して, $\sigma(a) = \bar{a} = a$. 従って, $\sigma\mid_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$.

問題 8-2 の解答

(1) について.

$$\begin{aligned}I_1 &= \sigma_4(1+\alpha+3\alpha^2) = \sigma_4(1)+\sigma_4(\alpha)+\sigma_4(3)\sigma_4(\alpha)^2 = 1-\alpha i+3(-\alpha i)^2 = 1-\alpha i-3\alpha^2, \\ I_2 &= \sigma_4\left(\frac{1+\alpha}{1-\alpha}\right) = \frac{1+\sigma_4(\alpha)}{1-\sigma_4(\alpha)} = \frac{1-\alpha i}{1+\alpha i}.\end{aligned}$$

(2) について.

$$\begin{aligned}\sigma_1(\beta) &= \sigma_1(1+\alpha^2) = 1+\alpha^2 \\ \sigma_2(\beta) &= \sigma_2(1+\alpha^2) = 1+(-\alpha)^2 = 1+\alpha^2 \\ \sigma_3(\beta) &= \sigma_3(1+\alpha^2) = 1+(i\alpha)^2 = 1-\alpha^2 \\ \sigma_4(\beta) &= \sigma_4(1+\alpha^2) = 1+(-i\alpha)^2 = 1-\alpha^2\end{aligned}$$

従って

$$\sigma_1(\beta)\sigma_2(\beta)\sigma_3(\beta)\sigma_4(\beta) = (1+\alpha^2)^2(1-\alpha^2)^2 = 1.$$

(3) $\gamma = a + b\alpha + c\alpha^2 + d\alpha^3$ $(a, b, c, d \in \mathbb{Q})$ と表す. $\gamma = \sigma_4(\gamma)$ より

$$a + b\alpha + c\alpha^2 + d\alpha^3 = (a - c\alpha^2) + (-b\alpha + d\alpha^3)i.$$

両辺の実部を比較すると,

$$a + b\alpha + c\alpha^2 + d\alpha^3 = a - c\alpha^2.$$

$\{1, \alpha, \alpha^2, \alpha^3\}$ は $K/\mathbb{Q}$ の基底より $b = c = d = 0$. よって $\gamma = a \in \mathbb{Q}$ である.

$\square$