Covering Numbers

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概 要

本稿は, Covering Numbers について解説する.

1 Covering Numbers

We introduce the Covering Numbers[1].

Let $\mathbb{E}$ be finite dimensional Banach Space. Let $\mathcal{N}(B_R, \eta)$ be the covering number. Let B_R be the closed ball of radius R centered at the origin.

$$B_R = \{x \in \mathbb{E} | \|x\| \leq R\} \tag{1}$$

Let S be a metric space. For $k \geq 1$ define

$$\varepsilon_k(S) = \inf\{\varepsilon > 0 \mid \exists \text{closed balls } D_1, \cdots D_k \text{ with radius } \varepsilon \text{ covering } S\} \tag{2}$$

Note that

$$\varepsilon_k(S \leq \eta) \iff \log \mathcal{N}(S, \eta) \leq k \tag{3}$$

Also note that ε_k scales well in the sense that, for all $R > 0$, $\varepsilon_k(RS) = R\varepsilon_k(S)$.

Here $RS = \{Rx | x \in S\}$.

Also, for $k \geq 1$, define

$$\varphi_k(S) = \sup\{\delta > 0 \mid \exists x_1, \cdots, x_{k+1} \in S \text{ s.t. for } i \neq j, d(x_i, x_j) > 2\delta\}. \tag{4}$$

補題 1.1. .

- *For all $k \geq 1$, $\varphi_k(S) \leq \varepsilon_k(S) \leq 2\varphi_k(S)$.*
- *Let $N = dim\ \mathbb{E}$. Let B_1 be the unit ball in $\mathbb{E}$. For all $k \geq 1$,*

$$k^{-\frac{1}{N}} \leq \varepsilon_k(B_1) \leq 4(k+1)^{-\frac{1}{N}} \tag{5}$$

証明. .

- It is easy to prove.
- Note that $\varphi_k(B_1) \leq 1$ for all $k \in \mathbb{N}$. Let $\rho < \varphi_k(B_1)$. Then there exists $x_1, \cdots, x_{k+1}$ such that $d(x_i, x_j) > 2\rho$ for $1 \leq i \neq j \leq k+1$. Let $D_j = x_j + \rho B_1$, $j = 1, \cdots, k+1$. Clearly, $D_i \bigcap D_j =$ if $i \neq j$. In addition, for all $x \in D_j$,

$$\begin{aligned} \|x\| &\leq \|x - x_j\| + \|x_j\| && (6)\\ &\leq \rho + 1 && (7)\\ &< 2. && (8) \end{aligned}$$

  Therefore, $D_j \subset B_2$.

Using measure, we get

$$
\begin{aligned}
& \sum_{i=1}^{k+1} \nu(D_j) \leq \nu(B_2) && (9)\\
\Longrightarrow \quad & \sum_{i=1}^{k+1} \rho^N \nu(B_1) \leq 2^N \nu(B_1) && (10)\\
\Longrightarrow \quad & (k+1)\rho^N \leq 2^N && (11)\\
\Longrightarrow \quad & \rho \leq 2(k+1)^{-\frac{1}{N}}. && (12)
\end{aligned}
$$

From here, it follows $\varepsilon_k(B_1) \leq 4(k+1)^{-\frac{1}{N}}$

For the other inequality consider any $\varepsilon > \varepsilon_k(B_1)$. Then there exist closed balls $D_1, \cdots, D_k$ of radius ε covering B_1.

$$
\begin{aligned}
\nu(B_1) \quad & \leq \quad \sum_{j=1}^{k} \nu(D_j) && (13)\\
& = \quad \sum_{j=1}^{k} \nu(\varepsilon B_1) && (14)\\
& = \quad k\varepsilon^N \nu(B_1) && (15)
\end{aligned}
$$

Consequently, it implies $k^{-\frac{1}{N}} \leq \varepsilon$ □

定理 1.2. *Let $N = dim\, \mathbb{E}$. Then* $\log \mathcal{N}(B_R, \eta) \leq N \log \dfrac{4R}{\eta}$.

証明. Let $x \in \mathbb{R}$. We denote by $\lceil x \rceil$ the largest integer smaller than or equal to x.

Let $k = \left\lceil \left(\dfrac{4R}{\eta}\right)^N - 1 \right\rceil$. Then $k+1 \geq \left(\dfrac{4R}{\eta}\right)^N$.

$$
\begin{aligned}
& 4(k+1)^{-\frac{1}{N}} \leq \frac{\eta}{R} && (16)\\
\Longrightarrow \quad & \varepsilon_k(B_1) \leq \frac{\eta}{R} && (17)\\
\Longleftrightarrow \quad & \varepsilon_k(B_R) \leq \eta && (18)\\
\Longleftrightarrow \quad & \mathcal{N}(B_R, \eta) \leq k \leq \left(\frac{4R}{\eta}\right)^N && (19)
\end{aligned}
$$

□

参考文献

[1] F. Cucker and S. Smale, “On the mathematical foundations of learning,” *Bulletin of the American Mathematical Society*, vol. 39, pp. 1–49, 2001.