Power Iteration Method

- **Given a web graph with *N* nodes, where the nodes are pages and edges are hyperlinks**
- **Power iteration:** a simple iterative scheme
  - Suppose there are ***N*** web pages
  - Initialize: $\mathbf{r}^{(0)} = [1/N,....,1/N]^T$
  - Iterate: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$
  - Stop when $|\mathbf{r}^{(t+1)} - \mathbf{r}^{(t)}|_1 < \varepsilon$

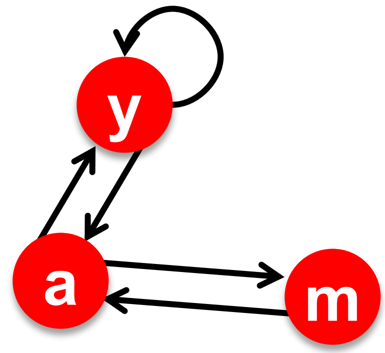
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d}_i}$$

d_i . out-degree of node i

$|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ is the $\mathbf{L_1}$ norm
Can use any other vector norm, e.g., Euclidean

PageRank: How to solve?

- **Power Iteration:**
  - Set $r_j = 1/N$
  - **1:** $r'_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - **2:** $r = r'$
  - If not converged: goto **1**
- **Example:**



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 \\ 1/3 \\ 1/3 \end{matrix}$$

Iteration 0, 1, 2,

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  - Set $r_j = 1/N$
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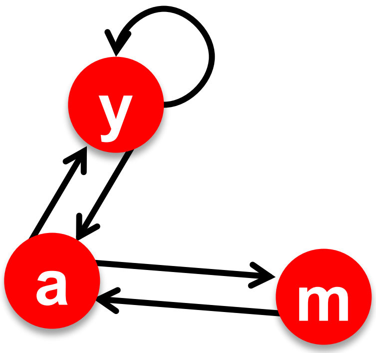

	y	a	m
y	½	½	0
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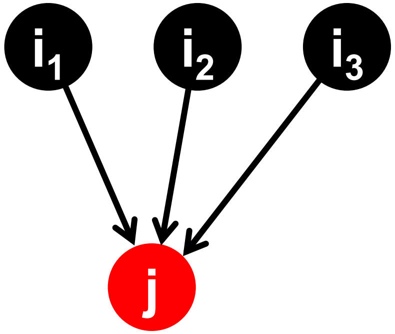
$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{matrix}$$

Iteration 0, 1, 2,

Random Walk Interpretation

- **Imagine a random web surfer:**
  - At any time $\boldsymbol{t}$, surfer is on some page $\boldsymbol{i}$
  - At time $\boldsymbol{t+1}$, the surfer follows an out-link from $\boldsymbol{i}$ uniformly at random
  - Ends up on some page $\boldsymbol{j}$ linked from $\boldsymbol{i}$
  - Process repeats indefinitely

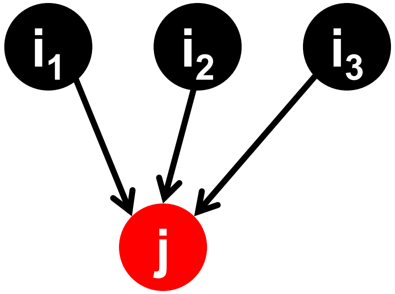


$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_{\mathrm{out}}(\mathrm{i})}$$

- **Let:**
  - $\boldsymbol{p(t)}$ … vector whose $\boldsymbol{i}^{\text{th}}$ coordinate is the prob. that the surfer is at page $\boldsymbol{i}$ at time $\boldsymbol{t}$
  - So, $\boldsymbol{p(t)}$ is a probability distribution over pages

The Stationary Distribution

- **Where is the surfer at time $t+1$?**
  - Follows a link uniformly at random
    $$\boldsymbol{p}(\boldsymbol{t}+1) = \boldsymbol{M} \cdot \boldsymbol{p}(\boldsymbol{t})$$



$$p(t+1) = \mathrm{M} \cdot p(t)$$

- Suppose the random walk reaches a state
  $$\boldsymbol{p}(\boldsymbol{t}+1) = \boldsymbol{M} \cdot \boldsymbol{p}(\boldsymbol{t}) = \boldsymbol{p}(\boldsymbol{t})$$
  then $\boldsymbol{p}(\boldsymbol{t})$ is **stationary distribution** of a random walk
- **Our original rank vector** $\boldsymbol{r}$ satisfies $\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$
  - **So, r is a stationary distribution for the random walk**

Existence and Uniqueness

- **A central result from the theory of random walks (a.k.a. Markov processes):**

For graphs that satisfy **certain conditions**, the **stationary distribution is unique** and eventually will be reached no matter what the initial probability distribution at time **t = 0**