EXAMINATION

April 2005

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

15 June 2005

1 $f = S - I - Ke^{-r(T-t)}$

where:

t is the present time
T is the time of maturity of the forward contract
r is the continuously compounded risk-free rate of interest for the interval from t to T
S is the spot price of the security at time t
I is the present value, at the risk-free interest rate, of the income generated by the security during the interval from t to T
K is the delivery price of the forward contract
f is the value of a long position in the forward contract

Here, working with £100 nominal,

$S = 95$, $K = 98$, $T - t = 1$, $r = 0.052$

$$I = 2.5\left(e^{-0.046\times0.5} + e^{-0.052\times1}\right) = 4.81648$$

$$\Rightarrow f = 95 - 4.81648 - 98e^{-0.052} = -2.85071$$

The value of the investor's short position in a forward contract on £1 million is therefore

$$\left(\frac{1{,}000{,}000}{100}\right)\times -f = 10{,}000\times 2.85071$$

$$= £28{,}507$$

2 MWRR: $2.2\ (1+i)^3 + 1.44(1+i) = 4.2$

Estimate $i = 6\%$, $LHS = 4.1466$

$i = 7\%$, $LHS = 4.2359$

$$\Rightarrow i = 0.06 + \frac{4.2 - 4.1466}{4.2359 - 4.1466} * 0.01$$

= 6.60% p.a. to two decimal places

Let F = Fund value before net cashflow on 31 December 2003

Then,

TWRR = 6.60% p.a. means that

$$1.066^3 = \frac{F}{2.2} * \frac{4.2}{(F+1.44)}$$

$$\Rightarrow 0.63452 = \frac{F}{F+1.44}$$

$$\Rightarrow 0.63452\,F + 0.63452 \text{ x } 1.44 = F$$

$$\Rightarrow F = £2.5\text{m}$$

3 (i) Work in millions:

PV of liabilities $= 9 + 12v\,\bar{a}_{\overline{1}|}$ at 9%

$$= 9 + 12v.\frac{i}{\delta}v$$

$$= 9 + 12 \times 0.91743^2 \times 1.044354$$

= 19.54811

The assets up to $(k+2)$ years from 1 January 2006 have:

$$PV = 5v^2 a^{(2)}_{\overline{k}|} = 5v^2 \frac{i}{i^{(2)}}\, a_{\overline{k}|}$$

$$= 5 \times 0.84168 \times 1.022015 \times a_{\overline{k}|}$$

$$= 4.301048\, a_{\overline{k}|}$$

With $k = 6, PV = 4.301048 \times 4.4859$

= 19.2941

The next payment of 2.5 million at k = 6.5 is made at time 8.5 and has present value = $2.5 \times v^{8.5} = 1.2018$

This would make PV of assets (20.5m) > PV of liabilities (19.5m)

$\Rightarrow$ Discounted payback period = 8.5 years.

(ii) The income of the development is received later than the costs are incurred. Hence an increase in the rate of interest will reduce the present value of the income more than the present value of the outgo. Hence the DPP will increase.

4 (i) Accumulation $= 500\, e^{\int_0^{10} \delta(s)ds}$

$$= 500\, e^{\left[\int_0^8 (0.07 - 0.005s)ds + \int_8^{10} 0.06ds\right]}$$

$$= 500\, e^{\left\{\left[0.07S - \frac{0.005}{2}S^2\right]_0^8 + \left[0.06S\right]_8^{10}\right.}$$

$$= 500e^{0.40+0.12}$$

$$= 841.01$$

(ii) $PV = \int_{10}^{18} 200e^{0.1t}.e^{-\int_0^t \delta(s)ds}\; dt$

$$= \int_{10}^{18} 200e^{0.1t}.\, e^{-\left[\int_0^8 0.07 - 0.005s)ds + \int_8^t 0.06ds\right]}$$

$$= \int_{10}^{18} 200e^{0.1t}.\, e^{-0.40}.\, e^{0.48-0.06t}\; dt$$

$$= 200e^{0.08} \int_{10}^{18} e^{0.04t}\; dt$$

$$= \frac{200e^{0.08}}{0.04}\left[e^{0.04t}\right]_{10}^{18}$$

$$= 5000\, e^{0.08}\left[e^{0.72} - e^{0.40}\right] = 3047.33$$

5 Present Value = $5000\left(\bar{a}_{\overline{1|}} + v.\ddot{a}^{(12)}_{\overline{1|}} + v^2.\ddot{a}^{(2)}_{\overline{1|}}\right)$ at i%

where $1+i = (1.02)^4 \Rightarrow i = 8.24322\%$ p.a. effective

$$\bar{a}_{\overline{1|}} = \frac{i}{\delta}.v = \frac{0.0824322}{Ln\,1.0824322}.\frac{1}{1.0824322}$$

$$= 0.9614201$$

and $\ddot{a}^{(12)}_{\overline{1|}} = (1.0824322)^{1/12}.\ \dfrac{1-v}{i^{(12)}}$

where

$1.0824322 = \left(1+\dfrac{i^{(12)}}{12}\right)^{12} \Rightarrow i^{(12)} = 0.0794725$

$\Rightarrow \ddot{a}^{(12)}_{\overline{1|}} = 0.9645970$

and $\ddot{a}^{(2)}_{\overline{1|}} = (1.0824322)^{\frac{1}{2}}.\ \dfrac{1-v}{i^{(2)}}$

where $1.0824322 = \left(1+\dfrac{i^{(2)}}{2}\right)^{2} \Rightarrow i^{(2)} = 0.0808000$

$\Rightarrow \ddot{a}^{(2)}_{\overline{1|}} = 0.9805844$

So $PV = 5000\left(0.9614201 + v * 0.9645970 + v^2 * 0.9805844\right) = 13,447.39$

Examiners' Comment: There are other valid methods for obtaining the required answer which also received full credit.

6 (i) (a) Work in $t = 0$ monetary values

$$25000 = 10000\left(v \times \frac{170.7}{183.3} + v^2 \times \frac{170.7}{191.0} + v^3 \times \frac{170.7}{200.9}\right)$$

where $v = \dfrac{1}{1+i'}$ with i' = real rate of return

Try 4% RHS = 24770.94

3% RHS = 25241.25

$$\Rightarrow i = 0.03 + \frac{25241.25 - 25000}{25241.25 - 24770.94} \times 0.01$$

$= 0.0351$ i.e. 3.5%

(b) $25000 = 10000\ a_{\overline{3|}}$ at $i\%$ p.a.

$$\Rightarrow a_{\overline{3|}} = 2.5$$

From tables, $a_{\overline{3|}} = 2.5313$ at 9%

$= 2.4869$ at 10%

$$\Rightarrow i = 0.09 + \frac{2.5313 - 2.5}{2.5313 - 2.4869} * 0.01$$

$= 0.097$

i.e. 9.7% p.a.

(ii) We should find that $\dfrac{1+i}{1+i'} \simeq 1 + e$

where e = average annual rate of inflation over the period.

Hence $\dfrac{1+i}{1+i^1} = \dfrac{1.097}{1.035} = 1.06$

which implies 6% p.a. inflation over the period

The actual average inflation rate was:

$$(1+e)^3 = \frac{200.9}{170.7} \Rightarrow e = 5.6\% \text{ p.a.}$$

The inflation rate would not be expected to be exactly 6% p.a. since the Retail Price Index is not increasing by a constant amount each year.

7 $$\left(1+\frac{i^{(4)}}{4}\right)^4 = 1.04 \Rightarrow i^{(4)} = 0.039414$$

$$g\left(1-t_1\right) = \frac{0.05}{1.03} \times 0.80 = 0.038835$$

$$\Rightarrow i^{(4)} > \left(1-t_1\right) g$$

$\Rightarrow$ Capital gain on contract

$\Rightarrow$ Assume redeemed as late as possible (ie: after 20 years) to obtain minimum yield.

Price of stock, *P*:

$$P = 100000 \times 0.05 \times 0.80 \times a_{\overline{20}|}^{(4)}$$

$$+\left(103000 - 0.25\left(103000 - P\right)\right) v^{20} \text{at } 4\%$$

$$\Rightarrow P = \frac{4000\, a_{\overline{20}|}^{(4)} + 77250 v^{20}}{1-0.25 v^{20}}$$

$$= \frac{4000 \times 1.014877 \times 13.5903 + 77250 \times 0.45639}{1-0.25 \times 0.45639}$$

= 102,072.25

8 (i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets $\Rightarrow$ 3rd Redington condition would never be satisfied.

(ii) Conditions required: (a) $V_A = V_L$

(b) $V_A' = V_L'$

(c) $V_A'' > V_L''$

where differentiation can be in respect of delta or *i*. In this solution, it is in respect of delta.

(a) $V_A = 3.43v^{15} + 7.12v^{25}$ @ 7%

= 2.5550

$V_L = 4v^{19} + 6v^{21}$

= 2.5551

$\Rightarrow V_A = V_L$ (ignoring rounding)

(b) $-V'_A = 3.43 \times 15v^{15} + 7.12 \times 25v^{25}$

= 51.444

$-V'_L = 4 \times 19v^{19} + 6 \times 21v^{21}$

= 51.445

$\Rightarrow V'_A = V'_L$ (ignoring rounding)

(c) $V''_A = 3.43 \times 15^2 v^{15} + 7.12 \times 25^2 v^{25}$

= 1099.627

$V''_L = 4 \times 19^2 v^{19} + 6 \times 21^2 v^{21}$

1038.322

$\Rightarrow V''_A > V''_L$

$\Rightarrow$ all 3 conditions are satisfied.

Examiners' Comment: There are other valid methods for obtaining the required answer which also received full credit.

9 (i) From two–year stock information:

Price = $3a_{\overline{2|}} + 102v^2$ at 5.5%

= $3 * 1.84632$ + $102 * 0.89845$

= 97.1811

Therefore, from one-year forward rate information,

$$97.1811 = \frac{3}{1+i_1} + \frac{3+102}{(1+i_1)(1+f_{1,1})}$$

where i_1=one-year spot rate

$f_{1,1}$= one-year forward rate from t = 1

$$\Rightarrow 97.1811 = \frac{3}{1+i_1} + \frac{105}{(1+i_1)1.05}$$

$$\Rightarrow 97.1811 = \frac{103}{1+i_1}$$

$$\Rightarrow i_1 = 5.9877\% \text{ p.a.}$$

(ii) From three-year stock information:

$$108.9 = \frac{10}{1+i_1} + \frac{10}{(1+i_i)1.05} + \frac{110}{(1+i_1)1.05(1+f_{2,1})}$$

where $f_{2,1}$=one-year forward rate from t = 2

Hence

$$108.9 = \frac{10}{1.059877} + \frac{10}{1.059877 \times 1.05} + \frac{110}{1.059877 \times 1.05 \times (1+f_{2,1})}$$

$$\Rightarrow 108.9 = 9.4351 + 8.9858 + \frac{110}{1.11287 * (1+f_{2,1})}$$

$\Rightarrow f_{2,1} = 9.245\%$ p.a.

(iii) Let $y_2\%$ p.a. be the two-year par yield

$$\Rightarrow 100 = y_2\left(\frac{1}{1+i_1}+\frac{1}{(1+i_1)(i+f_{1,1})}\right)+\frac{100}{(1+i_1)(1+f_{1,1})}$$

$$\Rightarrow 100 = y_2\left(\frac{1}{1.059877}+\frac{1}{1.059877*(1.05)}\right)+\frac{100}{1.059877\times 1.05}$$

$$\Rightarrow 100 = y_2(1.84208)+89.8577$$

$$\Rightarrow y_2 = 5.506\% \text{ p.a.}$$

10 (i) (a) Let i_t be the (random) rate of interest in year t. Let S_n be the accumulation of a single investment of 1 unit after n years:

$$E(S_n) = E\left[(1+i_1)(1+i_2)\ldots(1+i_n)\right]$$

$$E(S_n) = E[1+i_1]E[1+i_2]\ldots E[1+i_n] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j$$

$$\therefore \quad E(S_n) = (1+j)^n$$

(b)
$$E\left(S_n^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_n)\right]^2\right]$$

$$= E(1+i_1)^2\,E(1+i_2)^2\ldots E(1+i_n)^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_n+i_n^2\right)$$

$$= \left(1+2j+s^2+j^2\right)^n$$

as $E\left[i_i^2\right] = V\left[i_t\right] + E\left[i_t\right]^2 = s^2 + j^2$

$\therefore \ \text{Var}\left[S_n\right] = \left(1 + 2j + s^2 + j^2\right)^n - (1 + j)^{2n}$

(ii) (a) $E\left[\text{Interest}\right] = j = \frac{1}{2}(i_1 + i_2)$

$Var\left[\text{Interest}\right] = s^2 = E\left[\text{Interest}^2\right] - \left[E(\text{Interest})\right]^2$

$$= \frac{1}{2}\left(i_1^2 + i_2^2\right) - \left[\frac{1}{2}(i_1 + i_2)\right]^2$$

$$= \frac{1}{4}\left(i_1^2 + i_2^2\right) - \frac{1}{2} i_1.i_2$$

$$= \left[\frac{1}{2}(i_1 - i_2)\right]^2$$

(b) $E\left[S_{25}\right] = (1 + j)^{25} = 5.5$

$\Rightarrow j = 0.0705686$

$\text{Var}\ \left[S_{25}\right] = \left(1 + 2j + j^2 + s^2\right)^{25} - (1 + j)^{50} = (0.5)^2$

$\Rightarrow \left(1 + 2 * 0.0705686 + 0.0705686^2 + s^2\right)^{25} - (1.0705686)^{50} \qquad = 0.25$

$\Rightarrow s^2 = 0.000377389$

Hence, $s^2 = 0.000377389 = \frac{1}{4}(i_1 - i_2)^2$

$\Rightarrow \quad i_1 - i_2 = 0.0388530$ (taking positive root since $i_1 > i_2$)

$i_1 + i_2 = 2 \times 0.07056862 = 0.1411372$

$\Rightarrow \quad 2i_1 = 0.0388530 + 0.1411372$

$i_1 = 0.089995 \quad (8.9995\%\text{p.a.})$

and $i_2 = 0.051142 \quad (5.1142\%\text{p.a.})$

11 (i) Loan $= 1000\left(a_{\overline{10}|}^{5\%} + v_{5\%}^{10}\, a_{\overline{10}|}^{7\%}\right)$

$= 1000(7.7217 + 0.61391 \times 7.0236)$

$= 12033.56$

(ii) Note $\dfrac{439.52}{8790.48} = 0.05 \Rightarrow x \le 10$

$\Rightarrow 8790.48 = 1000\left(a_{\overline{11-x}|}^{5\%} + v_{5\%}^{11-x} a_{\overline{10}|}^{7\%}\right)$

$\Rightarrow 8.79048 = \dfrac{\left(1 - v^{11-x}\right)}{0.05} + v^{11-x} * 7.0236$

$\Rightarrow 8.79048 = 20 - (20 - 7.0236)v^{11-x}$

$\Rightarrow v^{11-x} = \dfrac{11.20952}{12.9764} = 0.86384$ at 5%

$\Rightarrow x = 8$

(iii) Let Y = reduced final payment
n = new total term of loan

Loan outstanding after 10 years = $1000\, a_{\overline{10}|}^{7\%}$ = £7,023.60

After change is made:

$7023.60 = 1000\, a_{\overline{n-11}|} + Yv^{n-10}$ at 5%

try n = 20 (i.e., keep to original term)

RHS = $1000 \times 7.1078 + Y \times 0.61391$

$\Rightarrow Y = -137.15$

$\Rightarrow$ doesn't work

try n = 19

RHS = $1000 \times 6.4632 + Y \times 0.64461$

$\Rightarrow Y = 869.36$

Hence:

(a) Term shortened by 1 year

(b) Final instalment = £869.36

(c) Under original terms, total interest paid is:

$20 \times 1000 - 12033.56 = 7966.44$

Under changed terms, total interest paid is:

$18 \times 1000 + 869.36 - 12033.56 = 6835.80$

$\Rightarrow$ difference = £1,130.64

END OF EXAMINERS' REPORT

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September 2005

Subject CT1 — Financial Mathematics
Core Technical

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15 November 2005

As is in some recent diets, the questions requiring descriptions of concepts, definitions or verbal reasoning (such as Q1, Q8(ii) and Q9(i)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points. It is important that candidates understand the subject well enough to express important topics and issues in their own words as well as in mathematical language. In 'show that' questions or questions where students are asked to derive formulae (such as Q8 part (i)) candidates are required to show detailed steps in deriving the results required in order to obtain full marks.

Please note that differing answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates were not penalised for this. However, candidates were penalised where excessive rounding had been used or where insufficient working had been shown.

1 One party agrees to pay to the other a regular series of fixed amounts for a certain term. In exchange the second party agrees to pay a series of variable amounts based on the level of a short term interest rate.

2 If f = the rate of inflation; j = the real rate of return and i = the money rate of return, then $j = (i - f)/(1 + f)$. In this case, $f = -2\%$, $i = 1\%$ and therefore $j = 3.061\%$.

3 (a) Let the answer be t days

$1{,}500(1 + 0.05 \times t/365) = 1{,}550$

$t = 243.333$ days

(b) Let the answer be t days

$1{,}500e^{0.05(t/365)} = 1{,}550$

$0.05\,(t/365) = \ln\,(1{,}550/1500)$

$t = 239.366$ days

4
$$210 = 200\exp\left\{\int_0^5 \left(a + bt^2\right)dt\right\} = 200\exp\left[at + \tfrac{1}{3}bt^3\right]_0^5 = 200\left[5a + 41.667b\right]$$
$$230 = 200\exp\left\{\int_0^{10} \left(a + bt^2\right)dt\right\} = 200\exp\left[at + \tfrac{1}{3}bt^3\right]_0^{10} = 200\left[10a + 333.333b\right]$$

$$\ln(1.05) = 5a + 41.667b$$
$$\ln(1.15) = 10a + 333.333b$$

The second expression less twice the first expression gives:

$$\ln(1.15) - 2\ln(1.05) = 250b \Rightarrow b = 0.0001687$$
$$a = \frac{\ln(1.15) - 333.333 \times 0.0001687}{10} = 0.0083520$$

5 (i) (a) $100 \times (1 + 0.05/12)^{-12\times10} = £60.716$

(b) $100 \times (1 - 0.05/12)^{12\times10} = £60.590$

(c) $100 \times e^{-10\delta} = £60.6531$

(ii) $98.91 = 100(1 + i)^{-91/365}$

$\ln(1 + i) = (-365/91) \times \ln(98.91/100) = 0.04396$

therefore $i = 0.04494$

6 (i)

- Used for medium or long-term borrowing
- Unsecured
- Regular annual coupon payments
- Generally repayable at par
- Generally issued by large companies and on behalf of governments
- Yields depend on risk and marketability
- Generally innovative market designed to attract different types of investor
- Issued internationally (normally by a syndicate of banks)
- Can be issued in any currency (not necessarily the domestic currency of the borrower)

(ii) (a) $97 = g a_{\overline{20|}} + 100v^{20}$ at 5% per annum effective

$a_{\overline{20|}} = 12.4622$; $v^{20} = 0.37689$ therefore $97 = 12.4622g + 100 \times 0.37689$

$g = (97 - 37.689)/12.4622 = 4.75927$

(b) Duration $= \sum C_t tv^t / \sum C_t v^t$ where C_t is the amount of the cash flow at time t

$(Ia)_{\overline{20|}} = \sum tv^t$ Therefore duration of the eurobond is:

$(4.75927\,(Ia)_{\overline{20|}} + 100 \times 20v^{20})/(4.75927\,a_{\overline{20|}} + 100v^{20})$

$(Ia)_{\overline{20|}} = 110.9506$ all other values have been used in (a) above

therefore duration is:

$(4.75927 \times 110.9506 + 100 \times 20 \times 0.37689)/(4.75927 \times 12.4622 + 100 \times 0.37689) = 1281.8239/97 = 13.2147$

7 (i) Value of loan = $50v + 48v^2 + 46v^3 + 44v^4 + \ldots + 22v^{15}$

$= 52(v + v^2 + v^3 + \ldots + v^{14} + v^{15}) - 2(v + 2v^2 + 4v^3 + \ldots + 28v^{14} + 30v^{15})$

$= 52\, a_{\overline{15}|} - 2\,(Ia)_{\overline{15}|}$

$(Ia)_{\overline{15}|} = 67.2668$
$a_{\overline{15}|} = 9.7122$

Therefore amount of the loan is $52 \times 9.7122 - 2 \times 67.2668 = 370.501$

Candidates who derived an appropriate formula for a decreasing annuity directly or who calculated the value of the loan by summing the individual terms received full credit.

(ii) Interest component in first year is $0.06 \times 370.504 = 22.23024$; therefore capital component is $50 - 22.23024 = 27.76976$.

Capital remaining after first instalment is $370.504 - 27.76976 = 342.73424$.
Interest paid in second instalment is $0.06 \times 342.73424 = 20.56405$

Capital in second instalment is $48 - 20.56405 = 27.43595$.

(iii) At the end of the thirteenth year, the capital outstanding is:

$24v + 22v^2 = 24 \times 0.94340 + 22 \times 0.89000 = 42.2216$

The interest due in the fourteenth instalment $0.06 \times 42.2216 = 2.53330$

The capital payment is therefore $24 - 2.53330 = 21.46670$

8 (i) Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after 5 years:

$$E(S_5) = E\left[\prod_{t=1}^{5}(1+i_t)\right]$$

$$= \prod_{t=1}^{5} E\left[(1+i_t)\right]$$

as $\{i_t\}$ are independent

$$E(S_5) = E[1+i_t]^5$$

$$E[1+i_t] = (1+E[i_t]) = 1.035$$

$$\therefore \quad E(S_5) = (1.035)^5 = 1.187686$$

$$E\left(S_5^2\right) = E\left[\prod_{t=1}^{5}(1+i_t)^2\right] = \prod_{t=1}^{5} E\left[(1+i_t)^2\right] \text{ (using independence)}$$

$$= \left(E(1+i_t)^2\right)^5 = \left(E\left[1+2i_t+i_t^2\right]\right)^5 = \left(1+2E[i_t]+E\left[i_t^2\right]\right)^5$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5$$

$$Var(S_5) = E\left(S_5^2\right) - E(S_5)^2$$

$$= \left(1+2E[i_t]+Var[i_t]+E[i_t]^2\right)^5 - E[1+i_t]^{10}$$

$$E(i_t) = 0.035$$

$$Var(i_t) = 0.03^2$$

$$\therefore \quad Var(S_5) = \left(1+2\times 0.035+0.03^2+0.035^2\right)^5 - (1.035)^{10}$$

$$= 1.416534 - 1.410598$$

$$= 0.0059356$$

Mean value of the accumulation of premiums is:

$$425000E(S_5) + 425000(1.03)^5 = (425000\times 1.187686) + (425000\times 1.15927)$$

$$= 997458$$

Standard deviation is $425000SD(S_5) = 425000\times\sqrt{0.0059356} = 32743.21$

Candidates who obtained slightly different answers by first deriving the parameters of the lognormal distribution received full credit.

(ii) Investing all premiums in the risky assets is likely to be more risky because, although there may be a higher probability of the assets accumulating to more than £1 million, the standard deviation would be twice as high so the probability of a large loss would be greater.

9 (i) Bond yields are determined by investors' expectations of future short-term interest rates, so that returns from longer-term bonds reflect the returns from making an equivalent series of short-term investments

(ii) (a) Let i_t be the spot yield over t years:

One year: yield is 8% therefore $i_1 = 0.08$
two years: $(1 + i_2)^2 = 1.08 \times 1.07$ therefore $i_2 = 0.074988$
three years: $(1 + i_3)^3 = 1.08 \times 1.07 \times 1.06$ therefore $i_3 = 0.06997$
four years: $(1 + i_4)^4 = 1.08 \times 1.07 \times 1.06 \times 1.05$ therefore $i_4 = 0.06494$

(b) Price of the bond is $5[(1.08)^{-1} + (1.074988)^{-2} + (1.06997)^{-3}]$
$+ 105 \times (1.06494)^{-4} = 13.03822 + 81.6373 = 94.67552$

Find gross redemption yield from

$94.67552 = 5\,a_{\overline{4}|} + 100v^4$

try 7%; $a_{\overline{4}|} = 3.3872$; $v^4 = 0.76290$
gives RHS = 93.226

GRY must be lower, try 6%; $a_{\overline{4}|} = 3.4651$; $v^4 = 0.79209$
gives RHS = 96.5345

interpolate between 6% and 7%.
$i = 0.07 - 0.01 \times (94.67552 - 93.226)/(96.5345 - 93.226)$
$i = 0.07 - 0.0043812 = 0.06562$

(c) Present value of the dividend is $4v$ calculated at 8% per annum effective = 3.70370.

Therefore forward price is
$F = (400 - 3.70370) \times 1.08 \times 1.07 = 457.9600$

10 (i) Price paid by first investor is P_1

$$P_1 = 4a^{(2)}_{\overline{15}|5\%} + 100v^{15}$$

$$\frac{i}{i^{(2)}} = 1.012348$$

$$v^{15} = 0.48102$$

$$a_{\overline{15}|} = 10.3797$$

$$\therefore P_1 = \left(4 \times 1.012348 \times 10.3797\right) + \left(100 \times 0.48102\right)$$

$$= 42.0315 + 48.1020 = 90.1335$$

(ii) (a) $\left(1 + \dfrac{i^{(2)}}{2}\right)^2 = 1.06 \Rightarrow i^{(2)} = 0.059126$

$$g\left(1 - t_1\right) = 0.04 \times 0.75 = 0.03$$

$$\Rightarrow i^{(2)} > \left(1 - t_1\right) g$$

$\Rightarrow$ Capital gain on contract

Price paid by second investor is P_2

$$P_2 = 0.75 \times 4a^{(2)}_{\overline{7}|6\%} + 100v^7_{6\%} - 0.4\left(100 - P_2\right)v^7_{6\%}$$

$$P_2\left(1 - 0.4v^7_{6\%}\right) = 0.75 \times 4a^{(2)}_{\overline{7}|6\%} + 0.6 \times 100v^7_{6\%}$$

$$\frac{i}{i^{(2)}} = 1.014782$$

$$v^7 = 0.66506$$

$$a_{\overline{7}|} = 5.5824$$

$$\therefore P_2 = \frac{\left(0.75 \times 4 \times 1.014782 \times 5.5824\right) + \left(60 \times 0.66506\right)}{1 - 0.4 \times 0.66506}$$

$$= 77.5207$$

(b) Rate of return earned by the first investor is the solution to:

$$90.1335 = 0.75 \times 4a_{\overline{8}|}^{(2)} + 77.5207v^8$$

$$i = 2\%$$

$$\frac{i}{i^{(2)}} = 1.004975$$

$$v^8 = 0.85349$$

$$a_{\overline{8}|} = 7.3255$$

$$RHS = 88.2490$$

$$i = 1.5\%$$

$$\frac{i}{i^{(2)}} = 1.003736$$

$$v^8 = 0.88771$$

$$a_{\overline{8}|} = 7.4859$$

$$RHS = 91.3575$$

$$i = 0.02 - \left(\frac{90.1335 - 88.2490}{91.3575 - 88.2490}\right) \times 0.005 = 1.697\% \approx 1.7\%$$

11 (i) (a) An equation of value expresses the equality of the present value of positive and negative (or incoming and outgoing) cash flows that are connected with an investment project, investment transaction etc.

(b) The discounted payback period from an investment project is the first time at which the net present value of the cash flows from the project is positive.

(ii) Consider first the NPV at 9% per annum effective. Working in £million.

Present value of cash outflows:

$$1.5\bar{a}_{\overline{3}|9\%} + 0.3\ddot{a}^{(4)}_{\overline{12}|9\%}v^3_{9\%} + v^3_{9\%} + 1.05v^4_{9\%} + 1.05^2 v^5_{9\%} + \ldots + 1.05^{11}v^{14}_{9\%}$$

$$= 1.5 \times 1.044354 \times 2.5313 + 0.3 \times 1.055644 \times 7.1607 \times 0.77218$$

$$+ 0.77218 \times \left(\frac{1 - 1.05^{12}v^{12}}{1 - 1.05v}\right) = 5.71647 + 7.60679 = 13.32326$$

Present value of cash inflows:

$$\left(\bar{a}_{\overline{6}|9\%} - \bar{a}_{\overline{3}|9\%}\right) + 1.9\left(\bar{a}_{\overline{9}|9\%} - \bar{a}_{\overline{6}|9\%}\right) + 2.5\left(\bar{a}_{\overline{15}|9\%} - \bar{a}_{\overline{9}|9\%}\right) + 8v^{15}_{9\%}$$

$$= 2.5\bar{a}_{\overline{15}|} - 0.6\bar{a}_{\overline{9}|} - 0.9\bar{a}_{\overline{6}|} - \bar{a}_{\overline{3}|} + 8v^{15}$$

$$= 1.044354\left(2.5 \times 8.0607 - 0.6 \times 5.9952 - 0.9 \times 4.4859 - 2.5313\right) + 8 \times 0.27454$$

$$= 12.6253$$

Hence NPV of project @ 9% = 12.6253 − 13.3233 = −£0.698 million
so the IRR is less than 9% p.a. effective

To find whether the discounted payback period is less than 12 years at 7% per annum effective, we need to find the NPV @ 7% of first twelve years cashflows

Present value of cash outflows:

$$1.5\bar{a}_{\overline{3}|7\%} + 0.3\ddot{a}^{(4)}_{\overline{9}|7\%}v^3_{7\%} + v^3_{7\%} + 1.05v^4_{7\%} + 1.05^2 v^5_{7\%} + \ldots + 1.05^{8}v^{11}_{7\%}$$

$$= 1.5 \times 1.034605 \times 2.6243 + 0.3 \times 1.043380 \times 6.5152 \times 0.81630$$

$$+ 0.81630 \times \left(\frac{1 - 1.05^{9}v^{9}}{1 - 1.05v}\right) = 5.73739 + 6.82096 = 12.55835$$

Present value of cash inflows:

$$\left(\bar{a}_{\overline{6}|7\%} - \bar{a}_{\overline{3}|7\%}\right) + 1.9\left(\bar{a}_{\overline{9}|7\%} - \bar{a}_{\overline{6}|7\%}\right) + 2.5\left(\bar{a}_{\overline{12}|7\%} - \bar{a}_{\overline{9}|7\%}\right)$$

$$= 2.5\bar{a}_{\overline{12}|} - 0.6\bar{a}_{\overline{9}|} - 0.9\bar{a}_{\overline{6}|} - \bar{a}_{\overline{3}|}$$

$$= 1.034605\left(2.5 \times 7.9427 - 0.6 \times 6.5152 - 0.9 \times 4.7665 - 2.6243\right)$$

$$= 9.3461$$

NPV is negative so the discounted payback period is more than 12 years.

Project fulfils neither the discounted payback period criterion nor the internal rate of return criterion.

END OF EXAMINERS' REPORT

EXAMINATION

April 2006

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M Flaherty
Chairman of the Board of Examiners

June 2006

Comments

Individual comments are shown after each question.

General comments

As is in some recent diets, the questions requiring verbal reasoning (such as Q3(c), Q7(b), Q10(iv) and Q11(iv)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

1 Annual rate of interest is i where $\left(1-\dfrac{28d}{365}\right)^{-1} = (1+i)^{28/365}$

This gives $i = \left(1-\dfrac{28\times 0.045}{365}\right)^{-365/28} -1 = 4.611\%$

Comments on question 1: *This was generally well answered.*

2 We require X where:

$$600a_{\overline{n}|}^{(4)} = 12X\ddot{a}_{\overline{n}|}^{(12)} \Rightarrow X = 50\frac{a_{\overline{n}|}^{(4)}}{\ddot{a}_{\overline{n}|}^{(12)}} = 50\frac{d^{(12)}}{i^{(4)}}$$

$$d^{(12)} = 12\left(1-(1-d)^{1/12}\right) = 12\left(1-e^{-\delta/12}\right) = 0.099584$$

$$i^{(4)} = 4\left((1+i)^{1/4}-1\right) = 4\left(e^{\delta/4}-1\right) = 0.101260$$

Hence $X = 49.1724$ or £49.17

Comments on question 2: *Candidates were not penalised for assuming that the annuities were for a specific term even though this was not needed for the calculations.*

3 (a) $$\left(1+f_{3,2}\right)^2 = \frac{(1+y_5)^5}{(1+y_3)^3} = \frac{(1.035)^5}{(1.03)^3} \Rightarrow f_{3,2} = 4.255\%$$

(b) Par yield is yc_4 where $yc_4\left(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + v_{y_4}^4\right) + v_{y_4}^4 = 1$

Thus $yc_4\left(1.025^{-1} + 1.0275^{-2} + 1.03^{-3} + 1.0325^{-4}\right) + 1.0325^{-4} = 1$

$$yc_4 = \frac{0.12009}{3.71785} = 3.230\%$$

(c) The par yield is equal to the gross redemption yield for a par yield bond. Coupons for the 3.5% bond are higher than for the par yield bond. Thus a lower proportion of the total proceeds are included within the redemption payment which is when spot yields/discount rates are highest. The present value of the proceeds of the 3.5% bond will be higher and so the gross redemption yield will be lower than that of the par yield bond and thus less than the par yield.

Comments on question 3: *Part (a) was answered well but some candidates struggled with the calculation of the par yield in part (b). In part (c) the marks were awarded for a clear explanation. Many candidates, who just stated their conclusion, were unable to explain their reasoning clearly and so failed to score full marks on this part.*

4 $i^{(2)} = 0.049390$

$g\left(1 - t_1\right) = 0.0625 \times 0.80 = 0.05$

$\Rightarrow i^{(2)} < \left(1 - t_1\right) g$

$\Rightarrow$ Capital loss on contract

$\Rightarrow$ Assume redeemed as early as possible (i.e.: after 10 years) to obtain minimum yield.

Price of stock per £100 nominal, *P*:

$P = 100 \times 0.0625 \times 0.80 \times a_{\overline{10}|}^{(2)} + 100v^{10}$ at 5%

$\Rightarrow P = 5\, a_{\overline{10}|}^{(2)} + 100v^{10}$

$= \left(5 \times 1.012348 \times 7.7217\right) + \left(100 \times 0.61391\right)$

$= 39.0852 + 61.3910 = £100.4762$

Comments on question 4: *Well answered although some candidates who recognised that the investor faced a capital loss did not recognise that this meant that the minimum yield would be obtained if the bond was redeemed at the earliest possible date.*

5 An investor can borrow £10 at the risk-free rate, buy one share for £10, enter into the forward contract to sell the share in six months time.

The initial cashflow is zero.

After one month the 50*p* dividend from the share is invested at the risk-free rate. After six months the share can be sold for £9.70, the dividend proceeds are worth $0.5e^{0.03\times 5/12}$ and the borrowing is repaid at $10e^{0.015}$. This gives a net cashflow of 9.7 $+0.5e^{0.03\times 5/12} - 10e^{0.015}$ = 0.0552

The investor has made a deal with zero initial cost, no risk of future loss and a risk-free future profit.

Comments on question 5: *The majority of candidates were able to calculate the non-arbitrage forward price by use of the appropriate formula. However, marks were lost for not clearly explaining how a risk-free profit could thus be made.*

6 (a) Expected accumulated value

$$= 800\left(0.25\ddot{s}_{\overline{10}|0.02} + 0.55\ddot{s}_{\overline{10}|0.04} + 0.2\ddot{s}_{\overline{10}|0.07}\right)$$

$$= 800\left(0.25\left(s_{\overline{11}|0.02} - 1\right) + 0.55\left(s_{\overline{11}|0.04} - 1\right) + 0.2\left(s_{\overline{11}|0.07} - 1\right)\right)$$

$$= 800\left(\left(0.25\times 11.1687\right) + \left(0.55\times 12.4864\right) + \left(0.2\times 14.7836\right)\right)$$

$$= \left(0.25\times 8934.96\right) + \left(0.55\times 9989.12\right) + \left(0.2\times 11826.88\right)$$

$$= £10{,}093.13$$

(b) Accumulation is only over £10,000 if the interest rate is 7% p.a. which has probability 0.2

Comments on question 6: *The most poorly answered question on the paper. This model of interest rates had not been examined recently and the majority of candidates assumed instead that the interest rate changed each year (in line with previous examination questions on this topic).*

7 (a) The counterparty faces *market risk* which is the risk that market conditions will change so that the present value of the net outgo under the agreement increases.

The counterparty also faces *credit risk* which is the risk that the other counterparty will default on its payments.

(b) The company still faces the market risk since the interest rates could fall further which will make the value of the swap even more negative to the company.

The company does not currently face a credit risk since the value of the swap is positive to the other counterparty.

Comments on question 7: *Part (a) was answered well but many candidates failed to recognise in (b) that the company would not currently face credit risk in this example.*

8 (i) Main characteristics of ordinary shares:

- Issued by commercial undertakings and other bodies.
- Entitle holders to receive all net profits of the company in the form of dividends after interest on loans and other fixed interest stocks has been paid.
- Higher expected returns than for most other asset classes …
- …but risk of capital losses
- … and returns can be variable.
- Lowest ranking form of finance.
- Low initial running yield but dividends should increase with inflation.
- Marketability varies according to size of company.
- Voting rights in proportion to number of shares held.

(ii) Present value of future dividends

$$= 100 \times 0.25\left(1.02v + 1.02 \times 1.04v^2 + 1.02 \times 1.04 \times 1.06v^3 + 1.02 \times 1.04 \times 1.06^2 v^4 + \ldots\right)$$

$$= 25 \times 1.02v + 25 \times 1.02 \times 1.04v^2\left(1 + 1.06v + 1.06^2 v^2 + \ldots\right)$$

$$= 25 \times 1.02v + 25 \times 1.02 \times 1.04v^2\left(\frac{1.09}{0.03}\right)$$

$$= 23.3945 + 811.0092 = 834.4037 = £834.40$$

(iii) Real rate of return is i such that:

$$820 = 100 \times 0.25 \times 1.02 \times \frac{100}{103} v + 100 \times 0.25 \times 1.02 \times 1.04 \times \frac{100 \times 100}{103 \times 103.5} v^2$$

$$+900 \times \frac{100 \times 100}{103 \times 103.5} v^2$$

$$= 24.7573 v + 869.1150 v^2$$

$$v = \frac{-24.7573 \pm \sqrt{24.7573^2 + 4 \times 869.1150 \times 820}}{2 \times 869.1150} = 0.95719$$

(taking positive root)

Hence i = 4.47%

Comments on question 8: *Despite being a bookwork question, part (i) was answered patchily with few students getting all of the required points. Part (ii) was answered well. In part (iii), it was expected that students would solve the quadratic equation. However, full credit was given to students who used interpolation methods.*

9 (i) $A(0,5) = e^{\int_0^5 0.04 dt} = e^{[0.04t]_0^5} = e^{0.2} = 1.22140$

$$A(5,10) = e^{\int_5^{10} 0.008 t dt} = e^{\left[0.004t^2\right]_5^{10}} = e^{0.3} = 1.34986$$

$$A(10,12) = e^{\int_{10}^{12}\left(0.005t + 0.0003t^2\right)dt} = e^{\left[0.0025t^2 + 0.0001t^3\right]_{10}^{12}} = e^{0.1828} = 1.20057$$

Required present value

$$= \frac{1}{A(0,5)\,A(5,10)\,A(10,12)} = \frac{1}{1.22140 \times 1.34986 \times 1.20057} = \frac{1}{1.97941}$$

= 0.50520

(ii) Equivalent effective annual rate is i where $(1+i)^{12} = 1.97941 \Rightarrow i = 5.855\%$

(iii) Present Value at time $t = 0$

$$= \int_2^5 e^{-0.05t}\left(e^{-\int_0^t 0.04ds}\right)dt = \int_2^5 e^{-0.05t}\left(e^{-0.04t}\right)dt$$

$$= \int_2^5 e^{-0.09t}dt = \left[\frac{e^{-0.09t}}{-0.09}\right]_2^5 = \frac{e^{-0.18} - e^{-0.45}}{0.09} = 2.1960$$

Comments on question 9: *Well answered.*

10 (i) Net present value of costs

$$= 5,000,000 + 3,500,000\bar{a}_{\overline{2}|} = 5,000,000 + 3,500,000\frac{i}{\delta}a_{\overline{2}|}$$

$$= 5,000,000 + 3,500,000 \times 1.073254 \times 1.6257 = 11,106,762$$

Net present value of benefits

$$= 450,000v^2\ddot{a}^{(4)}_{\overline{n-2}|} + 50,000v^2\left(I\ddot{a}\right)^{(4)}_{\overline{n-2}|} + S_n v^n$$

$$= 450,000v^2\frac{i}{d^{(4)}}a_{\overline{n-2}|} + 50,000v^2\frac{i}{d^{(4)}}\left(Ia\right)_{\overline{n-2}|} + S_n v^n$$

where n is the year of sale and S_n are the sale proceeds if the sale is made in year n.

If $n = 3$ the NPV of benefits

$$= \left(450,000 \times 0.75614 \times 1.092113 \times 0.86957\right)$$
$$+ \left(50,000 \times 0.75614 \times 1.092113 \times 0.86957\right)$$
$$+ \left(16,500,000 \times 0.65752\right)$$
$$= 323,137 + 35,904 + 10,849,080 = 11,208,121$$

Hence net present value of the project is 11,208,121 – 11,106,762 = 101,359

Note that if $n = 4$ the extra benefits in year 4 consist of an extra £1.5 million on the sale proceeds and an extra £650,000 rental income. This is clearly less than the amount that could have been obtained if the sale had been made at the end of year 3 and the proceeds invested at 15% per annum. Hence selling in year 4 is not an optimum strategy.

If n = 5 the NPV of benefits

$$\begin{aligned}&=\left(450,000\times 0.75614\times 1.092113\times 2.2832\right)\\&+\left(50,000\times 0.75614\times 1.092113\times 4.3544\right)\\&+\left(20,500,000\times 0.49718\right)\\&=848,450+179,791+10,192,190=11,220,431\end{aligned}$$

Hence net present value of the project is 11,220,431 – 11,106,762 = 113,669

Hence the optimum strategy if net present value is used as the criterion is to sell the housing after 5 years.

(ii) If the discounted payback period is used as the criterion, the optimum strategy is that which minimises the first time when the net present value is positive. By inspection, this is when the housing is sold after 3 years.

(iii) We require

$$5,000,000+3,500,000\frac{i}{\delta}a_{\overline{2}|}=450,000v^2\ddot{a}^{(4)}_{\overline{n-2}|}+50,000v^2\left(I\ddot{a}\right)^{(4)}_{\overline{n-2}|}+S_nv^n \text{ at } 17.5\%$$

$$\text{LHS}=5,000,000+3,500,000\left(\frac{1-v_{0.175}^2}{\delta_{0.175}}\right)=5,000,000+3,500,000\left(\frac{1-0.72431}{0.16127}\right)$$

$$=10,983,227$$

$$\text{RHS}\ \ =450,000v_{0.175}^2\left(\frac{1-v_{0.175}^4}{d^{(4)}}\right)+50,000v_{0.175}^2\left(\frac{\ddot{a}_{\overline{4}|}-4v_{0.175}^4}{d^{(4)}}\right)+S_6v_{0.175}^6$$

$$d_{0.175}^{(4)}=4\left(1-v^{1/4}\right)=0.15806$$

$$\ddot{a}_{\overline{4}|}=\frac{1-v^4}{d}=3.1918$$

Therefore we have on the RHS

$$450,000\times 0.72431\times 3.0076+50,000\times 0.72431\times\left(\frac{3.1918-2.0985}{0.15806}\right)+0.37999S_6$$

$$=980,296+250,502+0.37999S_6$$

For equality $S_6 = \dfrac{10{,}983{,}227 - 1{,}230{,}798}{0.37999} = £25{,}665{,}000$

(iv) Reasons investor may not achieve the internal rate of return:

- Allowance for expenses when buying/selling which may be significant.
- There may be periods when the property is unoccupied and no rental income is received.
- Rental income may be reduced by maintenance expenses.
- Tax on rental income and/or sale proceeds

Comments on question 10: *A significant number of candidates assumed that the development costs amounted to £7 million per annum and subsequently found that no strategy would lead to a profit. Otherwise the calculations were performed well. In part (iv), credit was given for other valid answers. Despite this, few students scored full marks on this part.*

11 (i) Let X_A, X_B be the monthly repayments under Loans A and B respectively.

For loan A:

Flat rate of interest = 10.715%

$$= \frac{60X_A - L_A}{5L_A} = \frac{60X_A - 10000}{50000} \Rightarrow X_A = £255.96$$

For loan B:

$$L_B = 15000 = 12X_B\left(a^{(12)}_{\overline{2}|12\%} + v^2_{12\%}a^{(12)}_{\overline{3}|10\%}\right)$$

$$X_B = \frac{1{,}250}{\left(\dfrac{i}{i^{(12)}}a_{\overline{2}|}\right)_{12\%} + v^2_{12\%}\left(\dfrac{i}{i^{(12)}}a_{\overline{3}|}\right)_{10\%}}$$

$$= \frac{1{,}250}{(1.053875 \times 1.6901) + 0.79719(1.045045 \times 2.4869)}$$

$$\Rightarrow X_B = £324.43$$

Hence student's overall surplus = 600 – $X_A - X_B$ = £19.61

(ii) Effective rate of interest under loan A is *i* % where

$$=12\times 255.96 a_{\overline{5}|}^{(12)}=10000\Rightarrow a_{\overline{5}|}^{(12)}=3.2557$$

Try i = 20%: $a_{\overline{5}|}^{(12)}$ = 3.2557

So capital outstanding after 24 months is $12\times 255.96\ a_{\overline{3}|}^{(12)}$ at 20%

$$=12\times 255.96\times 1.088651\times 2.1065=7043.74$$

Capital outstanding under B is $12\times 324.43\ a_{\overline{3}|}^{(12)}$ at 10%

$$=12\times 324.43\times 1.045045\times 2.4869=10118.02$$

So interest paid in month 25 under loans A and B

$$=7043.74\frac{i_{20\%}^{(12)}}{12}+10118.02\frac{i_{10\%}^{(12)}}{12}=107.84+80.68=£188.52$$

and capital repaid

$$=(255.96-107.84)+(324.43-80.68)=148.12+243.75=£391.87$$

(iii) Under the new loan the capital outstanding is the same as under the original arrangement = 17161.76.

The monthly repayment $=\left(\dfrac{255.96+324.43}{2}\right)=£290.20$

The effective rate of interest on the new loan A is *i* where

$$=12\times 290.20 a_{\overline{10}|}^{(12)}=17161.76\Rightarrow a_{\overline{10}|}^{(12)}=4.9281$$

Try i = 20%: $a_{\overline{10}|}^{(12)}=4.5642$

Try i = 15%: $a_{\overline{10}|}^{(12)}=5.3551$

By interpolation $i=15\%+\left(\dfrac{5.3551-4.9281}{5.3551-4.5642}\right)(20\%-15\%)\approx 17.7\%$

Hence interest paid in month 25

$$= 17161.76 \frac{i^{(12)}_{17.7\%}}{12} = 234.66$$

and capital repaid is £290.20 – £234.66 = £55.54

(iv) The new strategy reduces the monthly payments but repays the capital more slowly. The student could consider the following options:

- Keeping loan B and taking out a smaller new loan to repay loan A (which has the highest effective interest rate).

- Taking out the new loan for a shorter term to repay the capital more quickly.

Comments on question 11: *In part (i) some candidates struggled to deal with the flat rate of Loan A whilst others failed to deal with the change in interest rate of Loan B. Part (ii) was answered well. In part (iii), different answers for the effective rate of interest (and hence the interest paid) for the new loan could be obtained according to the actual interpolation used and full credit was given for a range of answers. If calculated exactly, the effective rate of interest is actually 17.5%. In part (iv), credit was again given for any valid strategy suitably explained.*

12 (i) We will consider three conditions necessary for immunisation

(1) $V_A = V_L$ (all expressions in terms of £m)

$$V_A = a_{\overline{5|}} + Rv^n \text{ at } 8\%$$
$$= 3.9927 + Rv^n$$

$$V_L = 3v^3 + 5v^5 + 9v^9 + 11v^{11} \text{ at } 8\%$$
$$= 15.0044$$

$$\Rightarrow Rv^n = 11.0117$$

(2) $V_A^{'} = V_L^{'}$ where $V_A^{'} = \dfrac{\partial V_A}{\partial \delta}$ & $V_L^{'} = \dfrac{\partial V_L}{\partial \delta}$

$$V_A^{'} = -(Ia)_{\overline{5|}} - nRv^n$$
$$= -11.3651 - nRv^n$$

$$V_L^{'} = -9v^3 - 25v^5 - 81v^9 - 121v^{11}$$
$$= -116.5741$$

$$\Rightarrow nRv^n = 105.2090$$

$$\Rightarrow n = \frac{105.2090}{11.0117} = 9.5543$$

$$\Rightarrow R = 11.0117 \times (1.08)^{9.5543} = £22.9720m$$

Alternatively:

$$V_A^{'} = V_L^{'} \text{ where } V_A^{'} = \frac{\partial V_A}{\partial i} \& V_L^{'} = \frac{\partial V_L}{\partial i}$$

$$\begin{aligned} V_A^{'} &= -v(Ia)_{\overline{5|}} - nRv^{n+1} \\ &= -11.3651v - nRv^{n+1} \\ &= -10.5233 - nRv^{n+1} \end{aligned}$$

$$\begin{aligned} V_L^{'} &= -9v^4 - 25v^6 - 81v^{10} - 121v^{12} \\ &= -107.9389 \end{aligned}$$

$$\Rightarrow nRv^{n+1} = 97.4156$$

$$\Rightarrow n = \frac{97.4156}{11.0117v} = 9.5543$$

$$\Rightarrow R = 11.0117 \times (1.08)^{9.5543} = £22.9720m$$

(3) $V_A^{"} > V_L^{"}$ (where $V_A^{"} = \dfrac{\partial^2 V_A}{\partial \delta^2} \& V_L^{"} = \dfrac{\partial^2 V_L}{\partial \delta^2}$)

$$\begin{aligned} V_A^{"} &= \sum_{t=1}^{5} t^2 v^t + n^2 R v^n \\ &= 40.275 + (9.5543)^2 \times 22.9720 \times v^{9.5543} \\ &= 1045.483 \end{aligned}$$

$$\begin{aligned} V_L^{"} &= 27v^3 + 125v^5 + 729v^9 + 1331v^{11} \\ &= 1042.031 \end{aligned}$$

Alternatively (differentiating with respect to i):

$$V_A'' = \sum_{t=1}^{5} t\left(t+1\right)v^{t+2} + n\left(n+1\right)Rv^{n+2}$$
$$= v^2\sum_{t=1}^{5} t^2 v^t + v^2\left(Ia\right)_{\overline{5|}} + n\left(n+1\right)Rv^{n+2}$$
$$= 0.85734\times 40.275 + 0.85734\times 11.3651 + 9.5543\times 10.5543\times 22.9720\times v^{11.5543}$$
$$= 34.53 + 9.74 + 952.00 = 996.27$$

$$V_L'' = 3\times 3\times 4\times v^5 + 5\times 5\times 6\times v^7 + 9\times 9\times 10\times v^{11} + 11\times 11\times 12\times v^{13}$$
$$= 993.32$$

Thus $n = 9.5543, R = £22.9720m$ will satisfy all three conditions and so will achieve immunisation.

(ii) (a) Value of assets at 3% $= a_{\overline{5|}} + Rv^n = 4.5797 + 22.9720v^{9.5543} = £21.900m$

Value of liabilities at 3% = $3v^3 + 5v^5 + 9v^9 + 11v^{11} = £21.903m$

Hence fund has a deficit of approximately £3,000.

(b) Immunisation will only enable to be a fund to be protected against a *small* change in interest rates. It will not be necessarily protected against sudden large changes as in this case.

Comments on question 12: *Part (i) was answered surprisingly poorly, given that it required the same techniques as those required in previous examination questions on the same topic. Full credit was given to students who observed directly that the spread of the assets around the mean term was greater than the spread of the liabilities. Few students answered part (ii) fully and the examiners felt that students should have recognised that immunisation would not protect the fund against such a large change in interest rates even if they had not answered part (i) correctly.*

END OF EXAMINERS' REPORT

EXAMINATION

September 2006

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

November 2006

Comments

As in many recent diets, the questions requiring verbal reasoning (e.g. Question 4(i)) tended not to be well answered with candidates producing vague statements which did not demonstrate that they understood the relevant points

Please note that differing answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on solutions presented to individual questions for this September 2006 paper are given below.

Question 1

Generally well answered. To gain full marks candidates were required to specify the difference between futures and options rather than just defining each contract separately.

Question 2

Well answered. This was a question where some candidates were penalised if answers had been rounded excessively.

Question 3

Generally well answered. Another possible solution is to use $1+j=\dfrac{1+0.6i}{1+f}=\dfrac{1+0.6\times 0.11}{1.07143}$ *which leads to the same answer.*

Question 4

For full marks in part (i), an answer should have included a description of the 'risk-free' concept (rather than just saying arbitrage profits are impossible). Many students had difficulty with part (ii).

Question 5

Full marks were given if either 365 or 365.25 days were used in the calculation. Most students scored well on this question.

Question 6

This question was well answered. For full marks, candidates were required to show detailed steps in deriving the result required including a definition of the initial terms used and a correct explanation of the relevance of the independence assumption.

Question 7

This question was poorly answered to the surprise of the examiners. Many candidates struggled to deal with the linked internal rate of return.

Question 8

Well answered.

Question 9

This question appeared to reward candidates who had a good understanding of the topic. Whilst the best candidates usually scored close to full marks on this question, weaker or less-prepared candidates often scored very badly.

Whilst the question did state that payments were made monthly, the examiners recognised that there was some potential for misinterpretation as to the frequency of the loan repayments in part (e) and took this into account. Thus students who used the formula $Xa_{\overline{30|}} = 100,000$ *with* $i^{(12)}$ *= 6% &* i *=6.168% to get an answer of £7,396 in this part were awarded full marks.*

Question 10

Generally well answered.

Question 11

This was the worst answered question on the paper by some margin with very few candidates scoring close to full marks. This may be because this type of question has not appeared in recent diets. Candidates needed to show that they could derive logically the amounts that will be paid, the real values of those amounts and their present values in real terms. Appropriate formulae then needed to be developed.

Question 12

Many candidates answered this question well although a minority scored very badly (possibly due to time pressure).

1 (i) A future is a contract binding buyer and seller to deliver or take delivery of an asset at a given price at a given time in the future. An option is a contract that gives the buyer the option to deliver or take delivery of the asset at the given price. The seller of the option must deliver/take delivery if the buyer of the option wishes to exercise the option.

(ii) Convertibles have option-like characteristics because they give the holder the option to purchase equity in a company on pre-arranged terms.

2 The accumulated value is

$$4\bar{s}_{\overline{3}|} + 2\bar{s}_{\overline{2}|} + 2\bar{s}_{\overline{1}|}$$

$$= \frac{i}{\delta}\left(4s_{\overline{3}|} + 2s_{\overline{2}|} + 2s_{\overline{1}|}\right)$$

$$= \frac{0.04}{0.039221}\left(4\times 3.1216 + 2\times 2.0400 + 2\right)$$

$$= 18.9352$$

3 (a) The money rate of return is i where $(1+i) = 11.1/10$
$i = 0.11$ or 11%

(b) The rate of inflation is f where $(1+f) = 120/112$
$f = 0.07143$ or 7.143%

(c) The net real rate of return per annum is j
where $j = \dfrac{0.6i - f}{1+f} = \dfrac{0.6\times 0.11 - 0.07143}{1.07143}$ = –0.005068 or –0.5068%

4 (i) The no arbitrage assumption means that it is assumed that an investor is unable to make a risk-free trading profit.

(ii) In all states of the world, security B pays 80% of A. Therefore its price must be 80% of A's price, or the investor could obtain a better payoff by only purchasing one security and make risk-free profits by selling one security short and buying the other. The price of B must therefore be 16p.

5 (i) (a) Let the answer be t days

$$3{,}600(1 + 0.06 \times t/365) = 4{,}000$$

$$t = 675.9 \text{ days}$$

(b) Let the answer be t days

$$3{,}600\left(1+\tfrac{0.06}{4}\right)^{4t/365} = 4{,}000$$

$$(4t/365)\ln(1.015) = \ln(4{,}000/3{,}600)$$

$$t = 645.7 \text{ days}$$

(c) Let the answer be t days

$$3{,}600\left(1+\tfrac{0.06}{12}\right)^{12t/365} = 4{,}000$$

$$(12t/365)\ln(1.005) = \ln(4{,}000/3{,}600)$$

$$t = 642.5 \text{ days}$$

(ii) (i)(a) takes longest because, under conditions of simple interest, interest does not earn interest.

6 (i) Let i_t be the (random) rate of interest in year t. Let S_{10} be the accumulation of the unit investment after 10 years:

$$E(S_{10}) = E\left[(1+i_1)(1+i_2)\ldots(1+i_{10})\right]$$

$$E(S_{10}) = E[1+i_1]E[1+i_2]\ldots E[1+i_{10}] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j$$

$$\therefore \quad E(S_{10}) = (1+j)^{10} = 1.07^{10} = 1.96715$$

(ii)
$$E\left(S_{10}^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_{10})\right]^2\right]$$

$$= E(1+i_1)^2\, E(1+i_2)^2 \ldots E(1+i_{10})^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_{10}+i_{10}^2\right)$$

$$= \left[E\left(1+2i_t+i_t^2\right)\right]^{10} = \left(1+2j+s^2+j^2\right)^{10}$$

as $E\left[i_i^2\right] = V\left[i_t\right] + E\left[i_t\right]^2 = s^2 + j^2$

$$\therefore\ \text{Var}\left[S_n\right] = \left(1+2j+s^2+j^2\right)^{10} - \left(1+j\right)^{20}$$

$$= \left(1+2\times 0.07+0.016+0.07^2\right)^{10} - \left(1.07\right)^{20} = 0.5761$$

(iii) If 1,000 units had been invested, the expected accumulation would have been 1,000 times bigger. The variance would have been 1,000,000 times bigger.

7 (i) (a) $\left(1+i\right)^2 = \dfrac{450}{600}\dfrac{500}{450+40}\dfrac{800}{500+100} \Rightarrow i = 1.015\%$

(b) First sub-interval is first year. Money weighted rate of return is i_1 where $\left(1+i_1\right) = \dfrac{450}{600} \Rightarrow i_1 = -25\%$

Second sub-interval is second year. Money weighted rate of return is i_2 where $490\left(1+i_2\right) + 100\left(1+i_2\right)^{1/2} = 800$

Then $\left(1+i_2\right)^{1/2} = \dfrac{-100 \pm \sqrt{100^2 - 4\times 490 \times (-800)}}{2\times 490} = \dfrac{-100 \pm 1256.1847}{980}$

= 1.17978 (taking positive root)

$\left(1+i_2\right) = 1.39188 \Rightarrow i_2 = 39.188\%$

Linked internal rate of return is i where $\left(1+i\right)^2 = 0.75\times 1.39188 \Rightarrow i = 2.1719\%$

(ii) The linked IRR is higher because it relies on two money weighted rates of return. With the calculation of the second money weighted rate of return, there is more money in the fund when the fund is performing well (in the second half of the year).

8 (i)

$$150 = 100\exp\left\{\int_0^5 \left(at+bt^2\right)dt\right\} = 100\exp\left[\tfrac{1}{2}at^2+\tfrac{1}{3}bt^3\right]_0^5 = 100\exp\left[12.5a+41.667b\right]$$

$$230 = 100\exp\left\{\int_0^{10} \left(at+bt^2\right)dt\right\} = 100\exp\left[\tfrac{1}{2}at^2+\tfrac{1}{3}bt^3\right]_0^{10} = 100\exp\left[50a+333.333b\right]$$

$$\ln(1.5) = 12.5a + 41.667b$$
$$\ln(2.3) = 50a + 333.333b$$

The second expression less four times the first expression gives:

$$\ln(2.3) - 4\ln(1.5) = 166.667b \Rightarrow b = -0.0047337$$

$$a = \frac{\ln(2.3) - 333.333 \times -0.0047337}{50} = 0.0482162$$

(ii) $100e^{10\delta} = 230 \Rightarrow 10\delta = \ln 2.3 \Rightarrow \delta = 0.08329$

(iii) Present Value $= \int_0^{10} 20e^{0.05t}e^{-0.08329t}dt$

$$= \int_0^{10} 20e^{-0.03329t}dt$$

$$= 20\left[\frac{e^{-0.03329t}}{-0.03329}\right]_0^{10}$$

$$= 20 \times 8.5058 = 170.116$$

9 (a) Premiums were expected to accumulate to

$$1,060\ddot{s}_{\overline{30|}}^{(12)} \text{ at } 7\% = 1,060\frac{i}{d^{(12)}}s_{\overline{30|}} = 1,060 \times 1.037525 \times 94.4608 = £103,885.77$$

(b) Premiums would have accumulated to

$$1,060\ddot{s}_{\overline{30|}}^{(12)} \text{ at } 4\% = 1,060\frac{i}{d^{(12)}}s_{\overline{30|}} = 1,060 \times 1.021537 \times 56.0849 = £60,730.37$$

The shortfall is 100,000 – 60,730.37 = £39,269.63

(c) Accumulation will be

$$1,060\ddot{s}^{(12)}_{\overline{20}|4\%}(1.04)^{10}+5,000\ddot{s}^{(12)}_{\overline{10}|4\%}$$

$$=1,060\frac{i}{d^{(12)}}s_{\overline{20}|}(1.04)^{10}+5,000\frac{i}{d^{(12)}}s_{\overline{10}|}$$

$$=1,060\times1.021537\times29.7781\times1.48024+5,000\times1.021537\times12.0061$$

$$=\text{£}109,053.12$$

Therefore the excess is £9,053.12

(d) The investor has earned a return of 4 % by investing extra premiums in the investment policy. The investor could have obtained a lower present value of total payments on the loan by paying off part of the loan instead. This is because the interest being paid on the loan was greater than the interest he was earning on his premiums.

(e) If he had repaid the loan by a level annuity, the annual instalment would have been X where

$\frac{X}{12}a_{\overline{360}|}=100,000$ at 0.5% (or $Xa^{(12)}_{\overline{30}|}=100,000$ with $i^{(12)}$ = 6% & i = 6.168%)

$$X=\frac{12\times100,000}{a_{\overline{360}|}}=\frac{1,200,000}{166.7916}=\text{£}7,194.61$$

10 Present value of companies' and consumers' costs is (in £ million)

$$\frac{i}{\delta}(50+10)\left(v+1.03v^2+1.03^2v^3+\ldots+1.03^{19}v^{20}\right)$$

$$=\frac{i}{\delta}60v\left(1+1.03v+(1.03v)^2+\ldots+(1.03v)^{19}\right)$$

$$=\frac{i}{\delta}60v\frac{\left(1-(1.03v)^{20}\right)}{1-1.03v}=1.019869\times60\times0.96154\times\left(\frac{1-1.80611\times0.45639}{1-1.03\times0.96154}\right)$$

$$=1.019869\times60\times0.96154\times18.27680=1075.383$$

Present value of costs to financial advisors (in £ million)

$$\frac{i}{\delta}\left(60v+19v^2+18v^3+\ldots+v^{20}\right)$$

$$=40\frac{iv}{\delta}+\frac{i}{\delta}\left(20v+19v^2+18v^3+\ldots+v^{20}\right)$$

$$=40\frac{iv}{\delta}+\frac{i}{\delta}\left(21a_{\overline{20}|}-Ia_{\overline{20}|}\right)=\frac{i}{\delta}\left(40v+21a_{\overline{20}|}-Ia_{\overline{20}|}\right)$$

$$=1.019869\times\left(40\times0.96154+21\times13.5903-125.1550\right)$$

$$=1.019869\times198.7029=202.651$$

Total PV of all costs = £1278.034 million

Present value of benefits (in £ million)

$$\frac{i}{\delta}\left(30v+33v^2+36v^3+\ldots+87v^{20}\right)+\frac{i}{\delta}12a_{\overline{20}|}$$

$$\frac{i}{\delta}\left(27a_{\overline{20}|}+3v+6v^2+9v^3+\ldots+60v^{20}+12a_{\overline{20}|}\right)$$

$$=\frac{i}{\delta}\left(3\left(Ia\right)_{\overline{20}|}+39a_{\overline{20}|}\right)$$

$$=1.019869\left(3\times125.1550+39\times13.5903\right)$$
$$=1.019869\times905.4867$$

= 923.478

Net present value of costs = PV(costs) – PV(benefits)
= 1278.034 – 923.478 = £354.556 million

11 (i)

- Payments guaranteed by government.
- Can be various different indexation provisions but, in general, protection is given against a fall in the purchasing power of money.
- Fairly liquid (i.e. large issue size and ability to deal in large quantities) compared with corporate issues, but not compared with conventional issues.
- Normally coupon and capital payments both indexed to increases in a given price index with a lag.
- Low volatility of return and low expected real return.
- More or less guaranteed real return if held to maturity (can vary due to indexation lag).
- Nominal return is not guaranteed.

(ii) The first coupon the investor will receive will be on 31st December 2003. The net coupon per £100 nominal will be:

$$0.8\times 1\times \text{ (Index May 2003/Index November 2001)} = 0.8\times 1\times \frac{113.8}{110}$$

In real present value terms, this is $0.8\dfrac{113.8}{110}\dfrac{v}{(1+r)^{0.5}}$

where r = 2.5% per annum and v is calculated at 1.5% (per half year)

The second coupon on 30th June 2004 per £100 nominal will be

$$0.8\times 1\times \frac{113.8}{110}(1+r)^{0.5}$$

In real present value terms, this is $0.8(1+r)^{0.5}\dfrac{113.8}{110}\dfrac{v^2}{(1+r)}$

The third coupon on 31st December 2004 per £100 nominal will be

$$0.8\times 1\times \frac{113.8}{110}(1+r)$$

In real present value terms, this is $0.8(1+r)\dfrac{113.8}{110}\dfrac{v^3}{(1+r)^{1.5}}$

Continuing in this way, the last coupon payment on 30 June 2009 per £100 nominal will be $0.8\times 1\times \dfrac{113.8}{110}(1+r)^{5.5}$

In real present value terms, this is $0.8(1+r)^{5.5}\dfrac{113.8}{110}\dfrac{v^{12}}{(1+r)^6}$

By similar reasoning, the real present value of the redemption payment is

$$100(1+r)^{5.5}\frac{113.8}{110}\frac{v^{12}}{(1+r)^{6}}$$

The present value of the succession of coupon payments and the capital payment can be written as:

$$P=\frac{1}{(1+r)^{0.5}}\frac{113.8}{110}\left(0.8\left(v+v^2+\ldots+v^{12}\right)+100v^{12}\right)$$

$$=\frac{1}{1.0124224}\frac{113.8}{110}\left(0.8a_{\overline{12}|1.5\%}+100v^{12}_{1.5\%}\right)$$

$$=1.02185\times\left(0.8\times 10.9075+100\times 0.83639\right)$$

$$=94.3833$$

12 (i) Present value of liabilities is $160,000a_{\overline{15}|}+200,000v^{10}$ at 7%

$=160,000\times 9.1079+200,000\times 0.50835$

$=£1,558,934$

(ii) Discounted mean term (DMT) of liabilities is

$$=\frac{\left(1\times 160,000\times v+2\times 160,000\times v^2+\ldots+15\times 160,000\times v^{15}\right)+200,000\times 10\times v^{10}}{160,000a_{\overline{15}|}+200,000v^{10}}$$

$$=\frac{160,000\times\left(Ia_{\overline{15}|}\right)+200,000\times 10\times v^{10}}{160,000a_{\overline{15}|}+200,000v^{10}}$$

$$=\frac{160,000\times 61.5540+200,000\times 10\times 0.50835}{1,558,934}$$

$=\dfrac{10,865,340}{1,558,934}$ = 6.9697 years (½ mark deducted for no units)

(iii) Let the nominal amounts in each security equal *A* and *B* respectively.

If the present values of assets and liabilities are to be equal then:

$$A\left(0.08a_{\overline{8}|}+v^{8}\right)+B\left(0.03a_{\overline{25}|}+v^{25}\right)=1{,}558{,}934 \quad (1)$$

If the DMTs of the assets and liabilities are equal, then:

$$\frac{A\left(0.08(Ia)_{\overline{8}|}+8v^{8}\right)+B\left(0.03(Ia)_{\overline{25}|}+25v^{25}\right)}{1{,}558{,}934}=6.9697$$

or $A\left(0.08(Ia)_{\overline{8}|}+8v^{8}\right)+B\left(0.03(Ia)_{\overline{25}|}+25v^{25}\right)=10{,}865{,}340 \quad (2)$

From (1)

$$A\left(0.08\times 5.9713+0.58201\right)+B\left(0.03\times 11.6536+0.18425\right)=1{,}558{,}934$$
$$\Rightarrow 1.059714A+0.533858B=1{,}558{,}934$$

From (2)

$$A\left(0.08\times 24.7602+8\times 0.58201\right)+B\left(0.03\times 112.3301+25\times 0.18425\right)=10{,}865{,}340$$
$$\Rightarrow 6.636896A+7.976153B=10{,}865{,}340$$

Therefore

$$6.636896\left(\frac{1{,}558{,}934-0.533858B}{1.059714}\right)+7.976153B=10{,}865{,}340$$

$$\Rightarrow B\left(7.976153-\frac{6.636896\times 0.533858}{1.059714}\right)=10{,}865{,}340-\frac{6.636896\times 1{,}558{,}934}{1.059714}$$

$$\Rightarrow B=\frac{1{,}101{,}872.85}{4.632647}=\pounds 237{,}850$$

$$A=\left(\frac{1{,}558{,}934-0.533858B}{1.059714}\right)=\pounds 1{,}351{,}266$$

(iv) It appears that the asset payments are more spread out than the liability payments. The third condition for immunisation is that that convexity of the assets is greater than that of the liabilities, or that the asset times are more spread around the discounted mean term than the liability times. From observation is appears likely that this condition is met.

END OF EXAMINERS' REPORT

EXAMINATION

April 2007

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

June 2007

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Q1.

Whilst most candidates made a good attempt at this question on basic compound interest accumulation, comparatively few students completed the question without error.

Q2.

Well answered.

Q3.

Most students answered this question well although candidates were expected to note that the sum of the geometric progression would only converge if the rate of return was below the dividend growth rate. Depending on the interpolation used, the final answer can justifiably vary from that given.

Q4.

This proved to be the most difficult question on the paper. Other related methods to determine the answers were available e.g. calculating the forward price of each contract and working out the present value of the difference in these prices.

Q5.

Well answered.

Q6.

The calculations in parts (i) and (ii) were generally well done. Again, depending on the interpolation used, the final answer can justifiably vary from that given although the examiners penalised the use of too wide a range of interpolation.
The explanation in part (iii) was very poorly handled. In such cases, the examiners are not simply looking for a statement lifted directly from the Core Reading. Instead, candidates are expected to apply the relevant theory to the actual situation described in the question.

Q7.

Generally well answered.

Q8.

This was the best answered question on the paper.

Q9.

Many candidates struggled with this question, firstly in determining when the various costs/payments would be made and then in manipulating the resulting equation(s). A common error was not to recognise that the DPP should be expressed as a whole number of months since payments at the relevant time were being made at monthly intervals. In part (ii) little credit was given for a correct conclusion without any accompanying explanation.

Q10.

This question seemed to provide a significant differentiation between candidates with many scoring well and a sizeable minority scoring very badly. This seemed surprising given that this topic is regularly examined. A common omission on part (ii)(b) was not to state whether a capital gain had been made.

Q11.

The workings for parts (i) and (ii) were often too brief (the questions said 'Derive…'). Note that the final answer in part (ii) can justifiably vary significantly according to the rounding used in intermediate calculations. Part (iii) was poorly done with many candidates assuming a lognormal distribution for this discrete example.

1 Fund after 25 years =

$$400\ddot{S}_{\overline{30|}}^{\,i^{*}\%}\times(1.03)^{20}+400\ddot{S}_{\overline{20|}}^{\,3\%}$$

where $1+i^{*}=(1.005)^{6}$

$\Rightarrow i^{*}=3.03775\%$ per ½-year

$$\ddot{s}_{\overline{30|}}\ @\ 3.03775\%=1.0303775\times\left[\frac{(1.0303775)^{30}-1}{0.0303775}\right]$$

= 49.3215

$$\ddot{S}_{\overline{20|}}\ @\ 3\%=1.03\times\left[\frac{(1.03)^{20}-1}{0.03}\right] = 27.6765$$

Hence fund =

$$400\times 49.3215\times(1.03)^{20}+400\times 27.6765$$

= 35632.06 + 11070.60

= £46,702.66

2 (i) $PV=\int_{9}^{12}50\,e^{0.01t}.\ e^{-\int_0^t \delta(t)dt}\,dt$

where

$$\int_0^t \delta(t)\,dt=\int_0^4(0.04+0.01t)\,dt+\int_4^8(0.12-0.01t)\,dt+\int_8^t 0.06dt$$

$$=\left[0.04t+0.005t^2\right]_0^4+\left[0.12t-0.005t^2\right]_4^8+\left[0.06t\right]_8^t$$

$$=\left[0.24\right]+\left[0.64-0.40\right]+\left[0.06t-0.48\right]$$

$= 0.06t$

Hence

$$PV = \int_9^{12} 50e^{0.01t}.\ e^{-0.06t}\ dt$$

$$= \int_9^{12} 50\, e^{-0.05t}\ dt$$

$$= \left[\frac{-50}{0.05} e^{-0.05t}\right]_9^{12}$$

$= \ -548.812 + 637.628$

$= \ 88.816$

3 Let i = money rate of return

i' = real rate of return

$\Rightarrow 1+i = (1+i')(1.03)$ here

$$21.50 = (1+i)^{4/12} \cdot \left(1.10v + 1.05 \times 1.10v^2 + (1.05)^2 \times 1.10v^3 + \cdots\cdots\right)$$

$$= (1+i)^{4/12} \times 1.10v \left(\frac{1-\left(\frac{1.05}{1+i}\right)^{\infty}}{\left(1-\frac{1.05}{1+i}\right)}\right)$$

$$= 1.10\ \frac{1}{(1+i)^{8/12}} \times \frac{1}{\left(1-\frac{1.05}{1+i}\right)} \quad \text{assuming } i > 0.05$$

$$19.5455 = \frac{1}{(1+i)^{8/12}} \times \frac{1}{1-\frac{1.05}{1+i}}$$

Try i = 10% RHS = 20.6456
11% RHS = 17.2566

$$\Rightarrow i = 0.10 + \frac{20.6456 - 19.5455}{20.6456 - 17.2566} \times 0.01 = 0.10325$$

$$\Rightarrow i' \text{ comes from } 1 + i' = \frac{1.10325}{1.03} \Rightarrow i' = 7.1\% \text{ p.a.}$$

4 (i) The current value of the forward price of the old contract is:

$$95\times(1.03)^{5}-5(1.03)^{-2}-6(1.03)^{-4}$$

whereas the current value of the forward price of a new contract is:

$$145-5(1.03)^{-2}-6(1.03)^{-4}$$

Hence, current value of old forward contract is:

$$145-95(1.03)^{5}=\pounds 34.87$$

(ii) The current value of the forward price of the old contract is:

$$95(1.02)^{-12}(1.03)^{5}=86.8376$$

whereas the current value of the forward price of a new contract is:

$$145(1.02)^{-7}=126.2312$$

$\Rightarrow$ current value of old forward contract is:

126.23 - 86.84 = £39.39

5 (i) Let $Y_k =$ spot rate for k year term

$P_k =$ Price per unit nominal for k year term

$Y_9 = 0.063737$

$$P_9 = \left(\frac{1}{1+Y_9}\right)^9 = 0.57344$$

(ii) $Y_7 = 0.08 - 0.04\,e^{-0.1(7)} = 0.060137$

$Y_{11} = 0.08 - 0.04e^{-0.1(11)} = 0.066685$

$$\left(1+f_{7,4}\right)^4 = \frac{\left(1+Y_{11}\right)^{11}}{\left(1+Y_7\right)^7} = \frac{\left(1.066685\right)^{11}}{\left(1.060137\right)^7}$$

$= 1.35165$

$\therefore$ 4-year forward rate is 7.824% at time 7.

(iii) $Y_1 = 0.04381, Y_2 = 0.04725, Y_3 = 0.05037$

$$1 = \left(Y_{c_3}\right)\left(v_{Y_1}^1 + v_{Y_2}^2 + v_{Y_3}^3\right) + v_{Y_3}^3$$

$Y_{c_3} = 0.05016$ i.e. 5.016% p.a.

6 (i) Work in £000's

MWRR is *i* such that:

$$21(1+i)^3 + 5(1+i)^2 + 8(1+i) = 38$$

Try i = 5%, LHS = 38.223
i = 4%, LHS = 37.350

By interpolation $i = 4.74\%$ p.a.

(ii) TWRR is *i* such that:

$$(1+i)^3 = \frac{24}{21} \times \frac{32}{29} \times \frac{38}{40} \Rightarrow i = 6.21\% \text{ p.a.}$$

(iii) MWRR is lower than TWRR because of the large cash flow on 1/7/05; the overall return in the final year is much lower than in the first 2 years, and the payment at 1/7/05 gives this final year more weight in the MWRR, but does not affect the TWRR.

7 Let PV_L be PV of liabilities, DMT_L be DMT of liabilities, C_L be convexity of liabilities.

(i) $PV_L = 87,500v^8 + 157,500v^{19}$ at 7%

$= 94,475.86$

$$\Rightarrow DMT_L = \frac{87,500 \times 8v^8 + 157,500 \times 19v^{19}}{94,475.86} \text{ at } 7\%$$

$$= \frac{1,234,857.56}{94,475.86}$$

= 13.070615 years

$$C_L = \frac{87,500 \times 8 \times 9v^{10} + 157,500 \times 19 \times 20v^{21}}{94,475.86} \text{ at } 7\%$$

$$= \frac{17,657,158.78}{94,475.86}$$

= 186.895985

(ii) Firstly, PVs should be equal:

$$\Rightarrow 66,850v^4 + Xv^n = 94,475.86 \text{ at } 7\%$$
$$\Rightarrow Xv^n = 43,476.31507$$

Secondly, DMTs should be equal

$$\Rightarrow 66,850 \times 4v^4 + Xnv^n = 1,234,857.56$$
$$\Rightarrow Xnv^n = 1,030,859.38$$
$$\Rightarrow n = 23.710827 \text{ years}$$
$$\Rightarrow X = 43,476.31507 \times 1.07^n$$
$$= 216,255.12$$

Lastly, verify 3rd condition

$$C_A = \left(66,850 \times 4 \times 5v^6 + 216,255.12n\left(n+1\right)v^{(n+2)}\right) / 94,475.86$$

= 23,140,343.20/94,475.86
= 244.93393

> C_L

Hence, immunisation is achieved.

8 (i) $800,000 = P\, a_{\overline{10}|}^{8\%} = P \times 6.7101$

$\Rightarrow P = 119,223.26$

Total amount of interest = 10 × 119,223.26 – 800,000

= £392,232.60

(ii) (a) Capital o/s at start of 8th year

= 119,223.26 $a_{\overline{3}|}^{8\%} = 119,223.26 * 2.5771 = 307,250.26$

Let new payment be P' per annum, then

$$P' \underset{12\%}{a_{\overline{5}|}^{(4)}} = P' * 1.043938 * 3.6048 = 307,250.26$$

$\Rightarrow P' = 81,646.28$

$\Rightarrow$ q'ly payment $= 20,411.57$

(b) Capital o/s after 7 years = 307,250.26

$\Rightarrow$ Interest in 1st q'ly payment $= 30,7250.26 * \left(\left(1.12\right)^{\frac{1}{4}} - 1\right) = 8,829.56$

$\Rightarrow$ capital component = 20,411.57 – 8,829.56 = 11,582.01

9 (i) The discounted payback period is the first point at which the present value of the income exceeds the present value of the outgoings. The present value of all payments and income up to time *t* is given by (working in £m)

$$PV = -40 - 36a^{(12)}_{\overline{\frac{1}{2}}|} - 2v^{\frac{1}{2}}a^{(12)}_{\overline{t-\frac{1}{2}}|} + 12v^{\frac{1}{2}}\ddot{a}^{(12)}_{\overline{t-\frac{1}{2}+\frac{1}{12}}|}$$

$$= -40 - 36a^{(12)}_{\overline{\frac{1}{2}}|} - 2v^{\frac{1}{2}}a^{(12)}_{\overline{t-\frac{1}{2}}|} + 12v^{\frac{1}{2}}\left(\frac{1}{12} + a^{(12)}_{\overline{t-\frac{1}{2}}|}\right)$$

$$= -40 - 36a^{(12)}_{\overline{\frac{1}{2}}|} + v^{\frac{1}{2}} + 10v^{\frac{1}{2}}\frac{1-v^{t-0.5}}{i^{(12)}}$$

$$a^{(12)}_{\overline{\frac{1}{2}}|} = \frac{1-v^{\frac{1}{2}}}{i^{(12)}} \text{ at } 10\% = \frac{1-0.9534626}{0.0956897} = 0.48634$$

$$\Rightarrow 0.56758 = 1 - v^{t-0.5}$$
$$\Rightarrow v^{t-0.5} = 0.43242$$
$$\Rightarrow t = \frac{\log(0.43242)}{\log(0.90909)} + 0.5$$
$$\Rightarrow t \geq 9.296$$

Hence, the discounted pay back period is 9 years and 4 months.

(ii) If the effective rate of interest were less than 10% p.a. then the present values of the income and outgo would both increase. However, the bigger impact would be on the present value of the income since the bulk of the outgo occurs in the early years when discounting has less effect. Hence, the DPP would decrease.

10 (i) $i^{(2)} = 0.059126$

$$g(1-t_1) = \frac{0.09}{1.10} \times 0.75 = 0.06136$$

$$\Rightarrow i^{(2)} < (1-t_1)g$$

$\Rightarrow$ No capital gain

Price of £100 nominal stock

$$= 0.75 \times 9\, a_{\overline{13}|}^{(2)} + 110 v^{13} \text{ at } 6\%$$

$$= 0.75 \times 9 \times 1.014782 \times 8.8527 + 110 \times 0.46884$$

$$= 60.639 + 51.572$$

$$= £112.21$$

(ii) (a) $i^{(2)} = 0.078461$

$$g\left(1-t_1\right) = \frac{0.09}{1.10} \times 0.90 = 0.073636$$

$$\Rightarrow i^{(2)} > \left(1-t_1\right) g$$

$\Rightarrow$ Capital gain

Price, $P = 0.90 \times 9 \times a_{\overline{11}|}^{(2)} + \left(110 - \left(110 - P\right) \times 0.35\right) v^{11}$ at 8%

$$\Rightarrow P = \frac{0.90 \times 9 \times 1.019615 \times 7.1390 + 0.65 \times 110 \times 0.42888}{1 - 0.35 \times 0.42888}$$

$$= \frac{89.62508}{0.849892} = 105.455$$

(b) No capital gain made

$$112.21 = 0.75 \times 9 \times a_{\overline{2}|}^{(2)} + 105.455 v^2$$

Try i = 3%, RHS = 112.41

i = 4%, RHS = 110.36

$\Rightarrow$ yield = 3% p.a. to nearest 1%

11 (i) Let $S_3 =$ Accumulated fund after 3 years of investment of 1 at time 0

i_t = Interest rate for year t

Then, fund after 3 years

$$= 80{,}000\ S_3 = 80000(1+i_1)(1+i_2)(1+i_3)$$

$$E(i_1) = \tfrac{1}{3}(0.04+0.06+0.08) = 0.06$$

$$E(i_2) = 0.75\times 0.07 + 0.25\times 0.05 = 0.065$$

$$E(i_3) = 0.7\times 0.06 + 0.3\times 0.04 = 0.054$$

Then:

$$E[80000S_3] = 80{,}000\ E[S_3]$$

$$= E\left[80{,}000(1+i_1)(1+i_2)(1+i_3)\right]$$

$$= 80{,}000\ E(1+i_1).\ E(1+i_2).\ E(1+i_3)$$

since i_t's are independent

$$= 80{,}000 \times 1.06 \times 1.065 \times 1.054 = £95{,}188.85$$

(ii) $\text{Var}[80000S_3] = 80{,}000^2 \times \text{Var}[S_3]$

where $\text{Var}[S_3] = E\left[S_3^2\right] - \left(E[S_3]\right)^2$

$$E\left[S_3^2\right] = E\left[(1+i_1)^2(1+i_2)^2(1+i_3)^2\right]$$

$$= E\left[(1+i_1)^2\right].\ E\left[(1+i_2)^2\right].\ E\left[(1+i_3)^2\right]$$

using independence

$$= \left(1+2E[i_1]+E\left[i_1^2\right]\right).\left(1+2E[i_2]+E\left[i_2^2\right]\right)\cdot\left(1+2\ E[i_3]+E\left[i_3^2\right]\right)$$

Now,

$$E\left(i_1^2\right) = \tfrac{1}{3}\left(0.04^2 + 0.06^2 + 0.08^2\right) = 0.0038667$$

$$E\left(i_2^2\right) = 0.75 \times 0.07^2 + 0.25 \times 0.05^2 = 0.0043$$

$$E\left(i_3^2\right) = 0.7 \times 0.06^2 + 0.3 \times 0.04^2 = 0.0030$$

Hence, $E\left[S_3^2\right]$

$$= \left(1 + 2 \times 0.06 + 0.0038667\right) \times \left(1 + 2 \times 0.065 + 0.0043\right) \times \left(1 + 2 \times 0.054 + 0.003\right)$$

=1.41631

Hence $\text{Var}\left[80,000 S_3\right]$

$$= 80,000^2 \ \text{Var}\left[S_3\right]$$

$$= 80,000^2 \left(1.41631 - \left(1.18986\right)^2\right)$$

= 3,476,355

(iii) *Note:* 80,000 × 1.08 × 1.07 × 1.06 = 97,995 > 97,000

But, if in any year, the highest interest rate for the year is not achieved then the fund after 3 years falls below £97,000.

Hence, answer is probability that highest interest rate is achieved in each year

$$= \frac{1}{3} \times 0.75 \times 0.7 = 0.175$$

END OF EXAMINERS' REPORT

EXAMINATION

September 2007

Subject CT1 — Financial Mathematics
Core Technical

MARKING SCHEDULE

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

December 2007

Comments

Please note that different answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this.

However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

It should be noted that the rubric of the examination paper does ask for candidates to show their calculations where this is appropriate. Candidates often failed to show sufficient clarity and detail in their working and lost marks as a result.

Q1.

Well answered.

Q2.

Well answered.

Q3.

Whilst this question was generally answered well, some candidates lost marks by not stating the conclusions that arose from their calculations i.e. that neither deal was acceptable.

Q4.

This question was very poorly answered which was disappointing given that this was a bookwork question.

Q5.

Reasonably well answered but some candidates failed to obtain full marks by not stating the required assumption.

Q6.

Parts (i) and (ii) were well answered but part (iii) was a good differentiator with weaker candidates failing to recognise the correct method for calculating the gross redemption yield. As with many previous diets, many candidates in part (iv) had great difficulty in giving a clear explanation of their calculations.

Q7.

Generally well answered. Some candidates lost marks by not giving an explicit formula for $v(t)$ *when* $t \leq 10$.

Q8.

This question was very poorly answered to the surprise of the examiners who felt that the question should have been relatively straightforward.

Q9.

Part (i) can be done much more simply than by using the method given in this report but the calculations given would still need to be done for part (ii).

Q10.

This question was the worst answered on the paper. Part (ii) did successfully differentiate between candidates with weaker candidates appearing to struggle to apply the theory to a real-life situation.

Q11.

The first three parts were generally answered well by the candidates who attempted the question. Many struggled to complete part (iv) although it is possible that this was due to time pressure. When calculating DMTs, candidates were expected to give the answer in terms of the correct units.

1 The first investor receives the higher rate of return if:

$$\frac{97.9}{96} > \frac{100}{97.9}$$

This inequality does not hold, therefore the second investor receives the higher rate of return.

2 Start by working in half years. The half yearly effective return is i such that:

$769 = 4v + 800v - 0.3(800 - 769)v$

$769 = (804 - 240 + 230.7)v$

$v = \dfrac{769}{794.7} = 0.967661$ therefore $i = 3.3420\%$

Annual effective rate is $(1.03342^2 - 1) = 6.7957\%$

3 The annual rate of payment for the first deal is 240.

This deal is acceptable if:

$240\,\ddot{a}_{\overline{2}|}^{(12)} < 456$ at a rate of interest of 5%

$240\,\ddot{a}_{\overline{2}|}^{(12)} = 240 \times 1.8594 \times 1.026881 = 458.252$

Therefore first deal is not acceptable

The annual rate of payment on the second deal is 246.

This deal is acceptable if:

$246\,a_{\overline{2}|}^{(12)} = 246 \times 1.8594 \times 1.022715 = 467.803$

Therefore second deal is also not acceptable

4 Main characteristics of equity investments:

- Issued by commercial undertakings and other bodies.
- Entitle holders to receive all net profits of the company in the form of dividends after interest on loans and other fixed interest stocks has been paid.
- Higher expected returns than for most other asset classes …

- …but risk of capital losses
- … and returns can be variable.
- Lowest ranking form of finance.
- Low initial running yield…
- … but dividends should increase with inflation.
- Marketability varies according to size of company.
- Voting rights in proportion to number of shares held.

5 Assuming no arbitrage:
Present value of dividends is (in£):
$0.5v^{1/2}$ (at 5%) + $0.5v$ (at 6%) = 0.5(0.97590+0.94340) = 0.95965

Hence forward price is: $F = (9-0.95965)\times 1.06 = £8.5228$

6 (i) $f_{3,1}$ is such that $1.045\times f_{3,1} = 1.05^2$. Therefore $f_{3,1} = 5.5024\%$

(ii) One-year spot rate is same as one-year forward rate = 4%

Two-year spot rate is i_2 such that $(1+i_2)^2 = 1.04\times 1.0425$.
Therefore $i_2 = 4.1249\%$

Three-year spot rate is i_3 such that $(1+i_3)^3 = 1.04\times 1.0425\times 1.045$.
Therefore $i_3 = 4.2498\%$

Four year spot rate is such that $(1+i_4)^4 = 1.04\times 1.0425\times 1.045\times 1.055024$
Therefore $i_4 = 4.5615\%$

(iii) Present value of the payments from the bond is:

$P = 3(1.04^{-1} + 1.041249^{-2} + 1.042498^{-3} + 1.045615^{-4})$
$+ 100\times 1.045615^{-4}$

Therefore $P = 3(0.96154 + 0.92234 + 0.88262 + 0.83659)$
$+ 100\times 0.83659 = 94.468$

Equation of value to find the gross redemption yield from the bond is such that:

$94.468 = 3a_{\overline{4}|} + 100v^4$

Try $i = 4.5\%$

$v^4 = 0.83856$, $a_{\overline{4}|} = 3.58753$, RHS = 94.619

Try i = 5%

$v^4 = 0.82270$, $a_{\overline{4|}} = 3.5460$, RHS = 92.908

Interpolation:

Yield = 0.045 + 0.005 × (94.619 – 94.468) /(94.619 – 92.908)

= 4.544%

(iv) The yield from the bond is lower than the one-year forward rate up to time 4 because the bond can be seen to be a series of zero coupon bonds (1 year, 2 years etc.) each with lower yields than the forward rate. The gross redemption yield from the bond is, in effect, an average of spot rates that are themselves a weighted average of earlier forward rates.

7 (i) For $t \leq 10$

$$v(t) = e^{-\int_0^t 0.04+0.01s\,ds} = e^{-\left[0.04s+0.005s^2\right]_0^t} = e^{-0.04t-0.005t^2}$$

For $t > 10$

$$v(t) = v(10)e^{-\int_{10}^t 0.05\,ds} = e^{-0.9}e^{-\left[0.05s\right]_{10}^t} = e^{-0.9}e^{-0.05(t-10)} = e^{-(0.4+0.05t)}$$

(ii) (a) Present value $= 1000e^{-(0.4+0.05\times 15)} = 1000e^{-1.15} = 316.637$

(b) $1000(1-d)^{15} = 316.637 \Rightarrow d = 7.380\%$

(iii) Present value $= \int_{10}^{15} e^{-(0.4+0.05t)} 20e^{-0.01t}\,dt$

$$= 20\int_{10}^{15} e^{-0.4}e^{-0.06t}\,dt$$

$$= 20e^{-0.4}\left[\frac{e^{-0.06t}}{-0.06}\right]_{10}^{15} = 20e^{-0.4}\left(-6.77616 + 9.14686\right) = 31.783$$

8 (i) Linked internal rate of return is found by linking the money weighted rate of return from the sub-periods.

$(\text{LIRR})^3 = 1.05 \times 1.06 \times 1.065 \times 1.03$

Therefore LIRR = 0.06879 or 6.879%

(ii) The TWRR requires the value of the fund every time a payment is made.

Size of the fund after six months is: $12.5 \times (1.05) = 13.125$
Size of the fund after one year is: $(13.125 + 6.6) \times 1.06 = 20.909$
Size of the fund after two years is: $(20.909 + 7) \times 1.065 = 29.723$
Size of the fund after three years is: $(29.723 + 8) \times 1.03 = 38.855$

The TWRR is i where i is the solution to:

$$(1+i)^3 = (13.125/12.5) \times [20.909/(13.125+6.6)] \times [29.723/(20.909+7)] \times [38.855/(29.723+8)]$$

or just use the rates of return given to give:

$$(1+i)^3 = 1.05 \times 1.06 \times 1.065 \times 1.03$$

giving $i = 6.879\%$

(iii) For MWRR, we need to know the size of the fund at the end of the period. We can use the values above to give:

MWRR is solution to: $12.5(1+i)^3 + 6.6(1+i)^{2.5} + 7(1+i)^2 + 8(1+i) = 38.855$

Solve by iteration and interpolation, starting with $i = 7\%$.

$i = 7\%$ gives LHS = 39.704
$i = 6\%$ gives LHS = 38.868
$i = 5.5\%$ gives LHS = 38.454

Interpolate between 5.5% and 6%.

$$i = 0.055 + 0.005 \times (38.855 - 38.454)/(38.868 - 38.454) = 5.98\%$$

(iv) (i) and (ii) are the same because there are no cash flows within sub-periods to "distort" the LIRR away from the TWRR. The MWRR is lower because the fund has a smaller amount of money in it at the beginning when rates of return are higher.

9 (i) $(1+i_t) \sim Lognormal\left(\mu,\sigma^2\right)$

$\ln(1+i_t) \sim N\left(\mu,\sigma^2\right)$

$\ln(1+i_t)^{10} = \ln(1+i_t) + \ln(1+i_t) + \ldots + \ln(1+i_t) \sim N\left(10\mu,10\sigma^2\right)$

since i_t's are independent

$(1+i_t)^{10} \sim Lognormal\left(10\mu,10\sigma^2\right)$

[½] for correct use of independence assumption

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.06$$

$$Var(1+i_t) = \exp\left(2\mu+\sigma^2\right)\left[\exp\left(\sigma^2\right)-1\right] = 0.08^2$$

$$\frac{0.08^2}{1.06^2} = \left[\exp\left(\sigma^2\right)-1\right] \therefore \sigma^2 = 0.0056798$$

$$\exp\left(\mu + \frac{0.0056798}{2}\right) = 1.06 \Rightarrow \mu = \ln 1.06 - \frac{0.0056798}{2} = 0.055429$$

$10\mu = 0.55429, 10\sigma^2 = 0.056798$

Let S_{10} be the accumulation of one unit after 10 years:

$$E(S_{10}) = \exp\left(0.55429 + \frac{0.056798}{2}\right) = 1.790848$$

Expected value of investment $= 2{,}000{,}000E(S_{10}) = £3.5817m$

(ii) We require $P\left[S_{10} < 0.8 \times 1.790848 = 1.4327\right]$

$P\left[\ln S_{10} < \ln 1.4327\right]$ where $\ln S_{10} \sim N(0.55429, 0.056798)$

$$\Rightarrow P\left[N(0,1) < \frac{\ln 1.4327 - 0.55429}{\sqrt{0.056798}}\right]$$

$$\Rightarrow P\left[N(0,1) < -0.8171\right] = 0.207 \approx 21\%$$

10 (i) (a) The flat rate of interest is: $(2\times 2{,}400 - 2{,}000)/(2\times 2{,}000) = 70\%$

(b) The flat rate of interest is not a good measure of the cost of borrowing because it takes no account of the timing of payments and the timing of repayment of capital.

(ii) If the consumers' association is correct, then the present value of the repayments is greater than the loan at 200%

i.e. $2{,}000 < 2{,}400\dfrac{i}{d^{(12)}}a_{\overline{2}|}$

$i = 2$; $a_{\overline{2}|} = 0.44444$; $d^{(12)} = 1.04982$ gives RHS = 2,032

The consumers' association is correct.

If the banks are correct, then the present value of the payments received by the bank, after expenses, is less than the amount of the loan at a nominal (before inflation) rate of interest of $(1.01463\times 1.025 - 1)$ per annum effective = 0.04.

i.e. $2{,}000 > 720\dfrac{i}{d^{(12)}}a_{\overline{2}|} + 720\dfrac{i}{d^{(12)}}a_{\overline{1.5}|} + 960\dfrac{i}{d^{(12)}}a_{\overline{1}|} - 0.3\times 2{,}400\dfrac{i}{d^{(12)}}a_{\overline{2}|}$

$\dfrac{i}{d^{(12)}} = 1.021529$; $a_{\overline{2}|} = 1.8861$; $a_{\overline{1}|} = 0.9615$; $a_{\overline{1.5}|} = \dfrac{1-1.04^{-1.5}}{0.04} = 1.4283$

So RHS = $720\times 1.021529\times 1.8861 + 720\times 1.021529\times 1.4283 + 960\times 1.021529\times 0.9615 - 0.3\times 2{,}400\times 1.021529\times 1.8861$

= 1,387.23+ 1,050.52 + 942.91 – 1,387.23 = 1,993.43

Therefore, the banks are also correct.

11 (i) Present value of the fund's liabilities (in £m) is:

$$100\left(v + 1.05v^2 + 1.05^2v^3 + \ldots + 1.05^{59}v^{60}\right)$$

$$= 100v\left(1 + 1.05v + (1.05v)^2 + \ldots + (1.05v)^{59}\right)$$

$$= 100v\left(\frac{1-(1.05v)^{60}}{1-1.05v}\right) = 100\times 0.97087\left(\frac{1-\left(\frac{1.05}{1.03}\right)^{60}}{1-\left(\frac{1.05}{1.03}\right)}\right)$$

$$= 97.087\times 111.7795 = £10{,}852m$$

(ii) Let the nominal holding of bonds = N in £m

The present value of the bonds must equal £10,852*m*

Therefore $0.04Na_{\overline{20|}} + Nv^{20} = 10{,}852$ at 3%

$a_{\overline{20|}} = 14.8775$, $v^{20} = 0.55368$

So $10{,}852 = 0.04N \times 14.8775 + N \times 0.55368$

$N = 10{,}852 / (0.04 \times 14.8775 + 0.55368) = £9{,}446.54m$

(iii) The numerator for the duration of the liabilities can be expressed as follows:

$100v\,(1\times 1 + 1.05v\times 2 + 1.05^2v^2 \times 3 + \ldots + 1.05^{59}v^{59}\times 60)$

$= \dfrac{1.03}{1.05} 100\,v\,(1.05v\times 1 + 1.05^2v^2\times 2 + 1.05^3v^3 \times 3 + \ldots + 1.05^{60}v^{60}\times 60)$

The part inside the brackets can be regarded as $(Ia)_{\overline{60|}}$ evaluated at a rate of interest i such that $v = 1.05/1.03$; the discount factor outside the brackets should be evaluated at 3%

$$\frac{1.03}{1.05} 100\,v = \frac{100}{1.05} = 95.2381$$

For the $(Ia)_{\overline{60|}}$ function, $v = 1.019417$; $i = -0.019048$; $(1+i)a_{\overline{60|}} = 111.7727$

$$(Ia)_{\overline{60|}} = \frac{111.7727 - 60\times 1.019417^{60}}{-0.019048} = 4118.567$$

Therefore numerator for duration is: $95.2381 \times 4118.567 = 392{,}244$
Therefore the duration is: 392,244/10,852 = 36.1 years.

(iv) The duration of the assets can be expressed as the sum of payments times time of receipt times present value factors divided by total present value.

The equation for the numerator is

$0.04\times 9{,}446.54\,(Ia)_{\overline{20|}} + 9{,}446.54\times 20\times v^{20}$ at 3%

$(Ia)_{\overline{20|}} = 141.6761$, $v^{20} = 0.55368$

Numerator is: 158,141
Therefore the duration is: 158,141/10,852 = 14.6 years.

(v) Duration of the liabilities is 36.1 years. Therefore volatility of the liabilities is: 36.1/1.03 = 35. If there were a reduction in interest rates to 1.5%, the liabilities would increase in value by approximately $35\times 1.5 = 52.5\%$

Duration of the assets is 14.6 years. Therefore volatility of the assets is: 14.6/1.03 = 14.2. If there were a reduction in interest rates to 1.5%, the assets would increase in value by approximately $14.2 \times 1.5 = 21.3\%$.

The liabilities would increase in value by an additional 31.2% of their original value i.e. by £3,386 more than the value of the assets.

END OF EXAMINERS' REPORT

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

April 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

M A Stocker
Chairman of the Board of Examiners

June 2008

Comments

Comments on solutions presented to individual questions for this April 2008 paper are given below.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Question 1 *Well answered.*

Question 2 *As has often been the case when words rather than numbers have been required, this bookwork question was answered poorly.*

Question 3 *Generally well answered, although some students treated the fees on Product B paid by the customer as a cost to the mortgage company.*

Question 4 *Well answered although many candidates' working was unclear when performing the CGT test.*

Question 5 *Part (i) was answered well but in part (ii) many candidates failed to recognise the need to calculate the 4-year spot rate before calculating the bond price.*

Question 6 *Part (i) of this question did appear to differentiate between stronger candidates who often scored very well and weaker candidates who often failed to score at all. As with many previous diets, many candidates in part (ii) had difficulty in giving a clear explanation of their results.*

Question 7 *This question was answered relatively poorly with, particularly in part (ii), candidates often appearing confused between real and money rates of interest.*

Question 8 *Most candidates managed to make a reasonable attempt at this question although marks were often lost in part (i) through a combination of calculation errors and insufficient working being shown. Candidates generally made a better attempt at the explanation required in part (ii) when compared to similar questions both on this paper and in previous diets.*

Question 9 *Well answered.*

Question 10 *Part (i) (for Option A) can be done much more simply than by using the method given in this report but the calculations given would still need to be done for part (ii). It was disappointing to see many candidates incorrectly calculate the mean accumulated value for Option B by using the mean rate of interest. Few candidates brought together the answers from (i) and (ii) to fully answer part (iii).*

1 The present value of the dividends, I, is:

$$I = 0.5v^{1/12} + 0.5v^{7/12} = 0.5\left(0.99594 + 0.97194\right) = 0.98394 \text{ calculated at } i = 5\%$$

Hence forward price is (again calculated at i = 5%):

$$F = \left(10 - 0.98394\right)\left(1+i\right)^{11/12} = 9.42845$$
$$= £9.43$$

2 (a) Eurobonds

- form of unsecured medium or long-term borrowing
- issued in a currency other than the issuer's home currency outside the issuer's home country
- pay regular interest payments and a final capital repayment at par.
- issued by large companies, governments and supra-national organisations.
- yields depend upon the issuer and issue size but will typically be slightly lower than for the conventional unsecured loan stocks of the same issuer.
- issuers have been free to add novel features to their issues in order to make them appeal to different investors.
- usually issued in bearer form

(b) Certificates of Deposit

- a certificate stating that some money has been deposited
- issued by banks and building societies
- terms to maturity are usually in the range 28 days to 6 months.
- interest is payable on maturity
- security and marketability will depend on the issuing bank
- active secondary market

3 For Product A, the annual rate of return satisfies the equation:

$$7,095.25a_{\overline{25|}} = 100,000$$

$$\Rightarrow a_{\overline{25|}} = 14.0939$$

This equates to the value of $a_{\overline{25|}}$ at 5%. Hence the annual effective rate of return is 5%.

For Product B, the annual rate of payment is X such that:

$$X\ddot{a}^{(12)}_{\overline{25|}} = 100,000 \text{ at } 4\%$$

$$\ddot{a}^{(12)}_{\overline{25|}} = \frac{i}{d^{(12)}}a_{\overline{25|}} = 1.021537 \times 15.6221 = 15.95855$$

$$\Rightarrow X = \frac{100,000}{15.95855} = 6,266.23$$

The equation of value to calculate the rate of return from Product B is:

$$6{,}000 + 5{,}000v^{25} + 6,266.23\frac{i}{d^{(12)}}a_{\overline{25}|} = 100,000$$

Clearly the rate of return must be greater than 4%. Try 5%.

$LHS = 6,000 + 5,000 \times 0.29530 + 6{,}266.2335 \times 1.026881 \times 14.0939 = 98{,}166$

At 5% the present value of the payments is less than the amount of the loan at 5% so the rate of return must be less than 5%. Try 4%:

$LHS = 6,000 + 5,000 \times 0.37512 + 100,000 = 107{,}876$

Interpolate between 4% and 5% to get the effective rate of return, *i*:

$$i = 0.04 + 0.01\left(\frac{107{,}876 - 100{,}000}{107{,}876 - 98{,}166}\right) \approx 4.81\% \text{ (actual answer is 4.80\%)}$$

Therefore Product B charges a lower effective annual return than Product A.

4 $\left(1 + \frac{i^{(4)}}{4}\right)^4 = 1.05 \Rightarrow i^{(4)} = 0.049089$

$$g\left(1 - t_1\right) = \frac{0.07}{1.08} \times 0.75 = 0.04861$$

$$\Rightarrow i^{(4)} > \left(1 - t_1\right)g$$

$\Rightarrow$ Capital gain on contract and we assume loan is redeemed as late as possible (i.e. after 20 years) to obtain minimum yield.

Let Price of stock = P

$$P = 0.07 \times 100,000 \times 0.75 \times a_{\overline{20}|}^{(4)}$$

$$+\left(108,000 - 0.35\left(108,000 - P\right)\right)v^{20} \text{at } 5\%$$

$$\Rightarrow P = \frac{5250a_{\overline{20}|}^{(4)} + 70,200v^{20}}{1 - 0.35v^{20}}$$

$$= \frac{5250 \times 1.018559 \times 12.4622 + 70,200 \times 0.37689}{1 - 0.35 \times 0.37689}$$

= 107,245.38

5 Assuming no arbitrage.

(i) $i_1 = f_0$ and $(1+i_2)^2 = (1+i_1)(1+f_1)$.

Hence $a - b = 0.061$

$\Rightarrow a = b + 0.061$

$(1+a-2b)^2 = 1.061 \times 1.065$

$\Rightarrow 1 + a - 2b = \sqrt{1.061 \times 1.065}$

$\Rightarrow b = -0.002$

$\Rightarrow a = 0.059$

(ii) Firstly, find the 4-year spot rate. Consider £1 nominal:

$1 = 0.07 \left(v_{i_1} + v_{i_2}^2 + v_{i_3}^3 + v_{i_4}^4\right) + v_{i_4}^4$

$= 0.07\left(1.061^{-1} + 1.063^{-2} + 1.065^{-3}\right) + 1.07 \times v_{i_4}^4$

$\Rightarrow (1+i_4)^4 = 1.31429212$

$\Rightarrow i_4 = 7.0713\%$ *p.a*

Let bond price per £1 nominal be P. Then

$P = 0.05\left(v_{i_1} + v_{i_2}^2 + v_{i_3}^3 + v_{i_4}^4\right) + 1.03v_{i_4}^4$

$= 0.05\left(1.061^{-1} + 1.063^{-2} + 1.065^{-3}\right) + 1.08 \times 1.070713^{-4}$

$= 0.9545$

i.e. 95.45 pence per £1 nominal

6 (i) (a) The duration is:

$$\frac{500\left(v + 2v^2 + 3v^3 + \ldots + 20v^{20}\right)}{500(v + v^2 + v^3 + \ldots + v^{20})} \text{ at } 8\%$$

$$= \frac{(Ia)_{\overline{20|}}}{a_{\overline{20|}}} = \frac{78.9079}{9.8181} = 8.037 \text{ years}$$

(b) The duration is:

$$\frac{500\left[v+\left(1.08\times 2v^2\right)+\left(1.08^2\times 3v^3\right)+\ldots+\left(1.08^{19}\times 20v^{20}\right)\right]}{500\left[v+\left(1.08v^2\right)+\left(1.08^2v^3\right)+\ldots+\left(1.08^{19}v^{20}\right)\right]} \text{ at } 8\%$$

$$=\frac{v\left(1+2+3+\ldots+20\right)}{20v}=\frac{\frac{1}{2}\left(20\times 21\right)}{20}=10.5 \text{ years}$$

(ii) The duration in (i)(b) is higher because the payments increase over time so that the weighting of the payments is further towards the end of the series.

7 (i) $260=12\left(v\left(1+e\right)+v^2\left(1+e\right)^2+v^3\left(1+e\right)^3+......\right)$

where $v = \dfrac{1}{1.06}$ and e denotes inflations rate.

Then,

$260 = 12a_{\overline{\infty}|}$ *at* $j\%$ where $\dfrac{1}{1+j}=\dfrac{1+e}{1+i}$ i.e. $j=\dfrac{0.06-e}{1+e}$

$\Rightarrow 260=\dfrac{12}{j}$

$\Rightarrow j=0.046153846$

$\Rightarrow e=0.01324$ i.e 1.324% pa

(ii) $260=12\left(1.03v+1.03^2v^2+.....+1.03^{12}v^{12}\right)+500v^{12}$

$=12\,\underset{j\%}{a_{\overline{12}|}}+500\,\underset{i\%}{v^{12}}$ where $j=\dfrac{i-0.03}{1.03}$

Try $i=10\%$, RHS $=255.67$

$i=9\%$, RHS $=279.35$

Hence, $i=0.09+\dfrac{279.35-260}{279.35-255.67}\times 0.01$

$= 0.098$

Let i' = real return

Then $\left(1+i'\right)\left(1+e\right)=1+i$

$\Rightarrow 1+i'=\dfrac{1.0982}{1.03}\Rightarrow i'=6.62\%$ *pa*

8 (i) Working in £000s

Outlay

$$Pv = 500 + 90a_{\overline{5}|} + 10(Ia)_{\overline{5}|} \ @11\%$$

$$a_{\overline{5}|} = \frac{1-v^5}{0.11} = 3.695897$$

$$(Ia)_{\overline{5}|} = \frac{\ddot{a}_{\overline{5}|} - 5v^5}{0.11} = \frac{1.11 \times 3.695897 - 5v^5}{0.11}$$

= 10.319900

$$\Rightarrow PV = 500 + 90 \times 3.695897 + 10 \times 10.3199$$

= 935.8297

Income

$$PV = 80\left(\bar{a}_{\overline{1}|} + 1.04v\,\bar{a}_{\overline{1}|} + (1.04)^2 v^2 \bar{a}_{\overline{1}|} + \cdots \ \cdots + (1.04)^{24} v^{24} \bar{a}_{\overline{1}|}\right)$$

$$= 80\bar{a}_{\overline{1}|} \times \left[\frac{1-(1.04v)^{25}}{1-(1.04v)}\right]$$

where $\bar{a}_{\overline{1}|} = \frac{i}{\delta}.v = \frac{0.11}{\ln 1.11}.\frac{1}{1.11} = 0.949589$

$$\Rightarrow PV = 80 \times 0.949589 \times 12.74554 = 968.2421$$

PV of cost of further investment
$= 300v^{15} = 62.7013$

PV of sale $= 700v^{25} = 51.5257$

Hence *NPV* = 968.2421 + 51.5257 - 935.8297 - 62.7013

= 21.2368 (£21,237)

(ii) If interest > 11% then $\frac{1}{1+i}$ decreases.

$\Rightarrow PV$ of both income and outgo $\downarrow$

However, PV of outgo is dominated by initial outlay of £500k at time 0 which is unaffected.

$\Rightarrow PV$ of income decreases by more than decrease in PV of outgo

$\Rightarrow NPV = PV$ of income $-$ PV of outgo

would reduce (and possibly become negative)

9 (i) $pv = 1{,}000 * \exp\left[-\int_7^{10}(0.01t - 0.04)\,dt\right] * \exp\left[-\int_5^{7}(0.10 - 0.01t)\,dt\right]$

$$= 1000 * \exp\left(-\left[\frac{0.01t^2}{2} - 0.04t\right]_7^{10}\right) * \exp\left(-\left[0.10t - \frac{0.01t^2}{2}\right]_5^{7}\right)$$

$$= 1000 * \exp\left(-\left[\frac{0.01 * 51}{2} - 0.04 \times 3\right]\right) * \exp\left(-\left[0.10 * 2 - \frac{0.01 * 24}{2}\right]\right)$$

$$= 1000 * \exp(-0.255 + 0.12 - 0.20 + 0.12)$$

$$= 1000 * \exp(-0.215)$$

$= 806.54$

(ii) Required interest rate p.a. convertible monthly is given by

$$806.54\left(1 + \frac{i^{(12)}}{12}\right)^{12\times 5} = 1{,}000$$

$\Rightarrow i^{(12)} = 4.3077\%\, p.a.$ convertible monthly

(iii) Accumulated amount $= \int_0^4 100e^{0.02t} \times e^{\int_t^4 0.06dr} \times e^{\int_4^7 (0.10 - 0.01r)dr} \times e^{\int_7^{12}(0.01r - 0.04)dr}\,dt$

$$= 100\int_0^4 e^{0.02t} \times e^{[0.06r]_t^4} \times e^{\left[0.10r - \frac{0.01r^2}{2}\right]_4^7} \times e^{\left[\frac{0.01r^2}{2} - 0.04r\right]_7^{12}}\,dt$$

$$= 100\int_0^4 e^{0.02t} e^{(0.24 - 0.06t)} e^{(0.30 - 0.165)} e^{(0.475 - 0.200)}\,dt$$

$$= 100e^{0.24} e^{0.135} e^{0.275} \int_0^4 e^{-0.04t}\,dt$$

$$= 100e^{0.65}\left[\frac{-e^{-0.04t}}{0.04}\right]_0^4$$

$$= 2{,}500e^{0.65}\left(1 - e^{-0.16}\right)$$

= 2,500 * 1.915540829 * 0.1478562

= 708.06

10 (i) Option A:

$$(1+i_t) \sim Lognormal\left(\mu, \sigma^2\right)$$

$$\ln(1+i_t) \sim N\left(\mu, \sigma^2\right)$$

$$\ln(1+i_t)^{10} = \ln(1+i_t) + \ln(1+i_t) + \ldots + \ln(1+i_t) \sim N\left(10\mu, 10\sigma^2\right)$$

since i_t's are independent

$$(1+i_t)^{10} \sim Lognormal\left(10\mu, 10\sigma^2\right)$$

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.055$$

$$Var(1+i_t) = \exp\left(2\mu + \sigma^2\right)\left[\exp\left(\sigma^2\right) - 1\right] = 0.07^2$$

$$\frac{0.07^2}{1.055^2} = \left[\exp\left(\sigma^2\right) - 1\right] \therefore \sigma^2 = 0.0043928$$

$$\exp\left(\mu + \frac{0.0043928}{2}\right) = 1.055 \Rightarrow \mu = \ln 1.055 - \frac{0.0043928}{2} = 0.051344$$

$$10\mu = 0.51344, 10\sigma^2 = 0.043928$$

Let S_{10} be the accumulation of one unit after 10 years:

$$E(S_{10}) = \exp\left(0.51344 + \frac{0.043928}{2}\right) = 1.70814$$

Accumulated sum is $100E(S_{10}) = £170.81$

Option B:

The accumulated sum at the end of five years is:
$100 \times 1.06^5 = 100 \times 1.33823 = £133.823$

The expected value of the accumulated sum at the end of ten years is:

$133.823\left(0.2\times1.01^5+0.3\times1.03^5+0.2\times1.06^5+0.3\times1.08^5\right)$

$=133.823\left(0.2\times1.05101+0.3\times1.15927+0.2\times1.33823+0.3\times1.46933\right)$

$=£169.48$

Option A:

$Var\left(S_{10}\right)=\exp\left(2\times0.51344+0.043928\right)\left[\exp\left(0.043928\right)-1\right]$

$=2.91776\times0.04491=0.13103$

Therefore standard deviation of £100 is $100\sqrt{0.13103}=£36.20$

Option B:

Here we need to find the expected value of the square of the accumulation as follows:

$133.823^2\left(0.2\times1.05101^2+0.3\times1.15927^2+0.2\times1.33823^2+0.3\times1.46933^2\right)$

= 29,189.86

The variance of the accumulation is therefore:

$29{,}189.86-169.48^2=£^2 467.54$

and the standard deviation is £21.62

(ii) For option A we require $P\left[S_{10}<1.15\right]$

$P\left[\ln S_{10}<\ln 1.15\right]$ where $\ln S_{10}\sim N(0.51344, 0.043928)$

$\Rightarrow P\left[N\left(0,1\right)<\dfrac{\ln 1.15-0.51344}{\sqrt{0.043928}}\right]$

$\Rightarrow P\left[N\left(0,1\right)<-1.7829\right]=0.0373\approx4\%$

For option B we first examine the lowest payout possible.

There is a probability of 0.2 that the amount will be $100\times1.06^5\times1.01^5$ or less which equals $133.823\times1.05101=£140.65$. Therefore the probability of a payment of less than £115 is zero.

(iii) Option A is riskier both from the perspective of having a higher standard deviation of return and also a higher probability of a very low value.

END OF EXAMINERS' REPORT

Subject CT1 — Financial Mathematics Core Technical

EXAMINERS' REPORT

September 2008

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

November 2008

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Candidates appeared to be less well prepared than in previous recent diets. As has often been the case when words rather than numbers have been required, Q4 was answered relatively poorly despite only involving bookwork with a wide range of available points that could be made. Many candidates also struggled with the first part of Q2 where explanation rather than calculation was required. The remainder of the shorter questions were answered well with candidates scoring particularly highly on Q7.

The more application styled questions (especially Qs 8, 11 and 12) tended to act as a clear discriminator between stronger and weaker candidates with a significant minority of candidates scoring very few marks on these questions. By contrast, Q9 on spot and forward yields was answered relatively well compared to questions in previous diets on this topic.

1 If j = real rate of return then equation of value in real terms is:

$$95(1+j)^{91/365} = 100\frac{220}{222}$$

$$(1+j)^{91/365} = 1.04315$$

therefore j = 18.465%

2 (i) MWRR

- Requires less information compared to TWRR

But

- Affected by amount and timing of net cashflows, which may not be in the manager's control and less fair measure than TWRR
- More difficult equation to solve than TWRR
- Also: equation may not have unique (or any) solution

(ii) Let TWRR = i

Then

$$(1+i)^2 = \frac{45}{41}\times\frac{72}{57}$$
$$= 1.386392811$$
$$\Rightarrow i = 17.745\% \text{ p.a.}$$

3 (i) Consider two portfolios A and B at time 0.

Portfolio A: - buy forward at price of K
- deposit $Ke^{-\delta T}$ in risk-free asset

Portfolio B: - buy asset at price of B

Then, at maturity, both portfolios have the same value (i.e. hold the underlying asset).

Thus, by the no-arbitrage principle, both portfolios must have same value at time 0.

$$\Rightarrow \quad Ke^{-\delta T} = B \quad \Rightarrow K = Be^{\delta T}$$

(ii) $i = 2\%$ per quarter

$$\Rightarrow K = 200 \times (1.02)^2 - 10 \times 1.02 = 197.88$$

$$\left(\text{using } K = Be^{\delta T} - Ce^{\delta(T - t_1)}\right)$$

4 Main characteristics of commercial property investments:

- Many different types of properties available for investment, e.g. offices, shops and industrial properties.
- Return comes from rental income and from the proceeds on sale.
- Total expected return higher than for gilts
- Rents and capital values are expected to increase broadly with inflation in the long term
- Neither rental income nor capital values are guaranteed – capital values in particular can fluctuate in the short term…
- …but rental income more secure than dividends
- Rents and capital values expected to increase when the price level rises (though the relationship is far from perfect).
- Rental terms are specified in lease agreements. Typically, rents increase every three to five years, Some leases have clauses which specify upward-only adjustments of rents.
- Large unit sizes, leading to less flexibility than investment in shares
- Each property is unique…
- …. so can be difficult to value.
- Valuation is expensive, because of the need to employ an experienced surveyor
- Marketability and liquidity are poor because of uniqueness …
- …and because buying and selling incurs high costs.
- Rental income received gross of tax.
- Net rental income may be reduced by maintenance expenses
- There may be periods when the property is unoccupied, and no income is received.
- The running yield from property investments will normally be higher than that for ordinary shares.

5 Present value in first case is

$$1,200 \times \frac{i}{d^{(4)}} \times a_{\overline{10|}} = 1200 \times 1.024877 \times 8.1109 = £9,975.210$$

Present value in second case is:

$$2,520 \times (v^2 + v^4 + \ldots + v^{10}) = 2,520 \times v^2 \times \frac{\left(1 - v^{10}\right)}{\left(1 - v^2\right)}$$

$$= 2,520 \times 0.92456 \times \frac{(1-0.67556)}{(1-0.92456)} = £10,020.01$$

Therefore first option is better for the borrower.

6 (i) Let $i_t =$ investment return for year t

Then, the expected value of the accumulation (S_{10}) is given by (in £ millions):

$$E(S_{10}) = E\left(\prod_{t=1}^{10}(1+i_t)\right)$$

$$= \prod_{t=1}^{10} E(1+i_t) \text{ using independence}$$

$$= \prod_{t=1}^{10}(1+E(i_t))$$

Now, $E(i_1) = 0.5 \times (0.07 + 0.03) = 0.05$
and for $t \neq 1$, $E(i_t) = (0.3 \times 0.02 + 0.4 \times 0.04 + 0.3 \times 0.06)$
$= 0.04$

So the expected value of the accumulation is

$1.05 \times 1.04^9 = 1.494477$ (i.e. £1,494,477)

(ii) The variance of the accumulation is

$$1,000,000^2 \times \left(E\left(S_{10}^2\right) - E(S_{10})^2\right)$$

where $E\left(S_{10}^2\right) = E\left(\prod_{t=1}^{10}(1+i_t)^2\right)$

$$= E\left(\prod_{t=1}^{10}\left(1+2i_t+i_t^2\right)\right)$$

$$= \prod_{t=1}^{10}\left(1+2E(i_t)+E\left(i_t^2\right)\right) \text{ from independence}$$

Now $E\left(i_1^{\,2}\right) = 0.5 \times \left(0.07^2 + 0.03^2\right) = 0.0029$

for $t \neq 1, E\left(i_t^{\,2}\right) = 0.3 \times 0.02^2 + 0.4 \times 0.04^2 + 0.3 \times 0.06^2$

= 0.00184

Hence,

$$E\left(S_{10}^2\right) = (1 + 0.1 + 0.0029) \times (1 + 0.08 + 0.00184)^9$$

= 2.238739

Standard deviation of the accumulation is

$$1,000,000 \times \left(2.238739 - 1.494477^2\right)^{1/2} = £72,646$$

(iii) The mean would remain unchanged as the expected rate of return in years 2-10 is unchanged. The variance of the rate in years 2-10 has increased and this will lead to an increase in the variance of the 10 year accumulation.

7 (i) Discounting from t = 12 to t = 5

$$v(12,5) = \exp\left(-\int_5^{12} 0.15ds\right)$$

$$= \exp\left[-0.15s\right]_5^{12} = e^{-1.05} = 0.34994$$

Discounting from t = 5 to t = 0

$$v(5,0) = \exp\left(-\int_0^5 0.05 + 0.02sds\right)$$

$$= \exp\left[-0.05s - 0.01s^2\right]_0^5 = e^{-0.5} = 0.60653$$

Hence present value of £1,000 at time t = 12

$$= 1,000v(12,5)v(5,0) = 1,000 \times 0.34994 \times 0.60653 = £212.25$$

(ii) The annual effective rate of discount is d such that:

$$1000(1-d)^{12} = 212.25$$
$$\Rightarrow d = 1 - 0.21225^{1/12} = 12.117\%$$

8 (i) Investment A: the gross rate of return per annum effective is clearly 10%. The net return is therefore $(1\text{-}0.4)\times 10\% = 6\%$ per annum effective.

Investment B: the investment will accumulate to $£1m\times 1.1^{10} = £2.5937m$ at the end of the ten years. The equation of value is:

$$1 = 2.59374(1+i)^{-10} - 0.4(2.59374\text{-}1)(1+i)^{-10}$$
$$= 1.95625(1+i)^{-10}$$
$$\Rightarrow (1+i)^{10} = 1.95625$$
$$\Rightarrow i = 6.94\%$$

Investment C: again the investment will accumulate to £2.5937m at the end of ten years. However, the indexed purchase price is subtracted from the value of the investment in this case. Thus the equation of value is:

$$1 = 2.59374(1+i)^{-10} - 0.4\left(2.59374\text{-}1\times 1.04^{10}\right)(1+i)^{-10}$$
$$= 2.5937(1+i)^{-10} - 0.4\times 2.59374(1+i)^{-10} + 0.4\times 1.04^{10}\times(1+i)^{-10}$$
$$= 2.14834(1+i)^{-10}$$
$$\Rightarrow (1+i)^{10} = 2.14834$$
$$\Rightarrow i = 7.95\%$$

(ii) All investments give a gross return of 10% per annum effective. Investment B gives a higher return than A because the tax is deferred until the end of the investment as capital gains tax is paid and not income tax. [However, candidates might note that tax is paid on the interest earned by deferral of tax]. Investment C gives a higher return than investment B because the tax is only paid on the real return over the ten year period which is lower than the nominal return.

9 (i) $103 = 6a_{\overline{3|}} + 105v^3$

try i = 6%: $a_{\overline{3|}} = 2.6730 \quad v^3 = 0.83962$
RHS = 104.1981

try i = 7%: $a_{\overline{3|}} = 2.6243 \quad v^3 = 0.81630$
RHS = 101.4573

Using linear interpolation:

$$i = 0.06 + \frac{(104.1981 - 103)}{(104.1981 - 101.4573)} \times 0.01 = 0.06437 = 6.44\%$$

(ii) Let i_n = spot yield for term n

Then

$$103(1+i_1) = 111 \Rightarrow i_1 = 7.767\%$$
$$103 = 6(1.07767)^{-1} + 111(1+i_2)^{-2} \Rightarrow i_2 = 6.736\%$$
$$103 = 6(1.07767)^{-1} + 6(1.06736)^{-2} + 111(1+i_3)^{-3} \Rightarrow i_3 = 6.394\%$$

(iii) First year forward rate is 7.767% (same as spot rate).

Forward rate from time one to time two is i such that:
$$1.07767(1+i) = 1.06736^2 \Rightarrow i = 5.715\%$$

Forward rate from time two to time three is i such that:
$$1.06736^2(1+i) = 1.06394^3 \Rightarrow i = 5.713\%$$

Forward rate from time one to time three is i such that:
$$1.07767(1+i)^2 = 1.06394^3 \Rightarrow i = 5.714\%$$

Forward rate from time zero to two and from time zero to three are the same as the respective spot rates (no additional marks for this point).

10 (i) NPV of first project in £*m* is:

$$0.5\left(a_{\overline{27|}} - a_{\overline{7|}}\right) + 5v^{27} - 0.1(Ia)_{\overline{10|}} - 0.25 \text{ at } 7\%$$
$$= 0.5(11.9867 - 5.3893) + 5 \times 0.16093 - 0.1 \times 34.7391 - 0.25$$
$$= £0.379m$$

The NPV of second project in £*m* is:

$$0.21v+0.21(1.05)v^2+0.21(1.05)^2v^3+\ldots+0.21(1.05)^9v^{10}+5.64v^{10}-4.2$$
$$=\ 0.21v\left(\frac{1-1.05^{10}v^{10}}{1-1.05v}\right)+5.64v^{10}-4.2$$
$$v=0.93458 \quad v^{10}=0.50835$$

Therefore NPV $= 1.8055+5.64\times 0.50835-4.2\ =\ £0.473m$
The second project has the higher net present value at 7% per annum effective.

(ii) The second project clearly has a discounted mean term of less then ten years. However, the discounted mean term of the first project must be greater than ten years because the undiscounted incoming cash flows are less than the undiscounted outgoing cash flows after ten years.

11 (i) Working in '000s

Let X = Nominal amount of Zero Coupon Bond

Y = Nominal amount of 8% bond

$$V_L=400v^{10}=185.2774$$

$$V_A=18.52774+Xv^{12}+0.08Ya_{\overline{16|}}+1.1\,Yv^{16}$$

Then, since $V_A=V_L$ (1st condition)

$$\Rightarrow 166.74966=0.39711\,X+0.08\times 8.8514\,Y+0.32108\,Y$$

$$\Rightarrow 166.74966=0.39711\,X+1.02919\,Y\ldots\ldots(1)$$

2nd condition is $V_A'=V_L'$

$$V_L'=4000\,v^{10}=1852.7740$$

$$V_A'=12\,X\,v^{12}+0.08\left(Ia\right)_{\overline{16|}}.Y\quad+1.1*16\,Y\,v^{16}$$

$$=4.76537\,X+0.08*61.1154\,Y+5.13727\,Y$$

$$\Rightarrow 1852.7740=4.76537\,X+10.0265\,Y\ldots\ldots(2)$$
$$\Rightarrow 148.2429=2.32391\,Y$$

$$\left[\text{from } (2) - \frac{4.76537}{0.39711} * (1)\right]$$

Hence Y = 63,790

X = 254,583

(ii) Amount invested in X is 254,583 v^{12}

= 101,098

and amount invested in Y is:

185,277 - 18,528 - 101,098 = 65,651

(iii) The spread of the assets is clearly greater than the spread of the liability (which is a single point).

Hence, Redington's 3rd condition is satisfied and the fund is immunised.

12 (i) First 15 years:

Interest paid each month

$$= \frac{i^{(12)}}{12} \times 300{,}000 \text{ where } 1.085 = \left(1 + \frac{i^{(12)}}{12}\right)^{12}$$

$$\Rightarrow \frac{i^{(12)}}{12} = 0.0068215$$

$\Rightarrow$ monthly interest = 0.0068215 × 300,000 = £2,046.45

After repayment of £150,000 after 15 years:

Interest paid each quarter

$$= \frac{i^{(4)}}{4} \times 150{,}000 \text{ where } 1.085 = \left(1 + \frac{i^{(4)}}{4}\right)^{4}$$

$$\Rightarrow \frac{i^{(4)}}{4} = 0.020604$$

$\Rightarrow$ Quarterly interest = 0.020604 × 150,000 = £3,090.66

Total interest paid over the 25 years

= (2046.45 × 12 × 15) + (3090.66 × 4 × 10) = £491,987.40

(ii) 150,000 = $X\ \ddot{s}_{\overline{30|}}^{(6)}$ @ 4½ %

where $X =$ Amount paid in each 6 month period

$$\ddot{s}_{\overline{30|}}^{(6)} = \frac{(1.045)^{30} - 1}{d^{(6)}}$$

where $\dfrac{1}{1.045} = \left(1 - \dfrac{d^{(6)}}{6}\right)^6$

$\Rightarrow d^{(6)} = 0.043856$

Hence $X = \dfrac{150000}{\left[\frac{(1.045)^{30}-1}{0.043856}\right]} = \dfrac{150000}{62.5985}$ = 2396.23

$\Rightarrow$ Monthly contribution = $\dfrac{2396.23}{6}$ = £399.37 per month

(iii) Savings proceeds after 15 years:

$12 \times 399.37\ \ddot{s}_{\overline{15|}\ 10\%}^{(12)}$

where $\ddot{s}_{\overline{15|}}^{(12)} = \dfrac{i}{d^{(12)}} \times s_{\overline{15|}}$

$= 1.0533781 \times 31.7725$

$= 33.46845$

Hence, savings proceeds

= 4792.44 × 33.46845 = 160,395.56

$\Rightarrow$ Loan o/s after 15 years

= 300,000 - 160,395.56 = 139,604.44

Let Y = new monthly payment

$$139{,}604.44 = 12\,Y\, a^{(12)}_{\overline{10}|\,7\%}$$

$$= 12Y\,\frac{0.07}{0.06785} \times 7.02358$$

$$\Rightarrow Y = £1{,}605.50 \text{ per month}$$

END OF EXAMINERS' REPORT

Subject CT1 — Financial Mathematics
Core Technical

EXAMINERS' REPORT

April 2009

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

June 2009

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

There were some excellent performances and well-prepared candidates scored well across the whole paper. However, the comments below on each question concentrate on areas where candidates could have improved their performance.

Q1, Q2.

As has often been the case when words rather than numbers have been required, these bookwork questions were answered relatively poorly (although Q2 was answered better than Q1).

Q3.

Well answered.

Q4.

Defining an arbitrage profit correctly was also acceptable as an answer to (i) although a description of both possible arbitrage scenarios was required for full marks. Many candidates performed the calculations well although the methodology being used was not always clear.

Q5.

The question required an ability to bring together two separate elements of the syllabus and less well-prepared candidates seemed to struggle with this.

Q6.

This was another question where students scored relatively poorly with many candidates having difficulty with the income calculation. A common error was to assume that the income rose by 4% every three years.

Q7.

This was answered much better than questions on the same topic in previous exams. However, some candidates did confuse the money-weighted and time-weighted rates of return.

Q8.

It was particularly disappointing to see many candidates using the wrong formula for DMT in part (i) but ending their proof with'=14.42 QED' in the final line. This suggests a lack of professionalism, honesty and integrity which are key attributes of the actuarial profession.

Part (ii) was well-answered with various different methods leading to the correct answer.

Q9.

This was the worst-answered question on the paper although it was still possible to score significant marks by calculating forward rates using the correct formula even if the spot rates had been calculated incorrectly.

Q10.

Part (i) was answered well but many candidates lost marks in part (ii) by not realising that a separate test was required to ascertain the worst time to redemption. Many candidates calculated the annual effective yield rather than the yield per annum convertible quarterly in part (iii).

Q11.

Many candidates seemed confused as to what to calculate in part (i) and failed to distinguish between the premium needed in 10 years' time and the present value of that premium. Part (ii) was answered well (although some candidates appeared to be short of time at this stage). Part (iii) was answered very poorly with many candidates not appreciating the effects of the high variance.

1 Characteristics of government bills:

- short-dated securities issued by governments to fund their short-term spending requirements.
- issued at a discount and redeemed at par with no coupon.
- mostly denominated in the domestic currency, although issues can be made in other currencies.
- yield is typically quoted as a simple rate of discount for the term of the bill
- absolutely secure
- often highly marketable despite being unquoted.
- often used as a benchmark risk-free short-term investment.

2 (a) An interest-only loan requires the borrower only to pay interest on the entire loan in each time period. The loan does not reduce over time so the interest remains constant. A separate investment or savings account can be established in which payments are made to extinguish the whole loan at the end of the term.

(b) A repayment loan involves level repayments of capital and interest. The first part of the payment is used to pay interest on any remaining capital. The remaining part of the payment is then used to repay capital so that the capital gradually reduces over the term of the loan.

3 (i) $300a_{\overline{20}|} + 30v\left(\mathrm{I}a\right)_{\overline{19}|}$ at 7%

$= 300\left(10.594\right) + 30 \times \frac{1}{1.07} \times 82.9347 = 5503.47$

(ii) Capital outstanding after 5 payments:

$420a_{\overline{15}|} + 30\left(\mathrm{I}a\right)_{\overline{15}|}$

$= 420 \times 9.1079 + 30 \times 61.5540 = 5671.94$

(iii) Cap o/s after 19 payments = 870v @ 7% = £813.08

= Capital in the final payment

Interest in the final payment = 870 – 813.08 = £56.92

4 (i) The "no arbitrage" assumption means that **neither** of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The forward price at the outset of the contract was:

$$\left(94.5 - 9v_{5\%}^{10} - 10v_{5\%}^{11}\right) \times (1.05)^{12} = 149.29$$

The forward price that should be offered now is:

$$\left(143 - 9v_{5\%}^{2} - 10v_{5\%}^{3}\right) \times (1.05)^{4} = 153.39$$

Hence the value of the contract now is:

$$(153.39 - 149.29)\, v_{5\%}^{4} = 3.37$$

Note:

This result can also be obtained directly from:

$$143 - 94.5 \times (1.05)^{8} = 3.38$$

since the coupons are irrelevant in this calculation.

5 Working in £000's

$$\text{PV of outgo} = 100 + 80e^{-\int_0^2 (0.05 + 0.002t)dt}$$

$$= 100 + 80e^{-\left[0.05t + 0.001t^2\right]_0^2}$$

$$= 100 + 80e^{-0.104} = 172.10$$

DPP is value of T for which:

PV (income paid up to T) = PV (outgo)

Where

PV (income paid up to T) = $\int_8^T 100e^{0.001t}\ v(t)\,dt$

and $v(t) = e^{-\left[\int_0^5 (0.05+0.002t)dt + \int_5^t 0.06dt\right]}$

$$= e^{-\left[0.05t+0.001t^2\right]_0^5} . e^{-(0.06t-0.30)}$$

$$= e^{-0.275} . e^{-0.06t}\ e^{0.30}$$

$$= e^{0.025}\ e^{-0.06t}$$

$$\Rightarrow PV\left(\text{income paid up to } T\right) = \int_8^T 100e^{0.001t}\ e^{0.025}\ e^{-0.06t}dt$$

$$= \int_8^T 100e^{0.025}\ e^{-0.059t}dt$$

$$= \frac{100}{-0.059}e^{0.025}\left[e^{-0.059T} - e^{-0.059\times 8}\right]$$

$$= -1737.8222\ e^{-0.059T} + 1083.97$$

$\Rightarrow$ DPP is T such that

$$172.10 = -1737.8222e^{-0.059T} + 1083.97$$

$$\Rightarrow e^{-0.059T} = 0.52472$$
$$\Rightarrow -0.059T = Ln(0.52472) \Rightarrow T = 10.93 \text{ years}$$

6 Working in 000's

PV of costs $= 5000 + 900v^{11/12}$ at 8%

$= 5838.695$

PV of income $= 800v^{1\,3/12}\left(\ddot{a}_{\overline{3}|}^{(4)} + 1.04^3 v^3 \ddot{a}_{\overline{3}|}^{(4)} + \cdots + (1.04)^{12} v^{12}\ \ddot{a}_{\overline{3}|}^{(4)}\right)$

$= 800v^{1\,3/12}\ \ddot{a}_{\overline{3}|}^{(4)}\left(1 + (1.04v)^3 + \cdots (1.04v)^{12}\right)$

$$= 800 \times 0.908281 \times 1.049519 \times 2.5771 \times \left(\frac{1-\left(\frac{1.04}{1.08}\right)^{15}}{1-\left(\frac{1.04}{1.08}\right)^{3}}\right)$$

$= 1965.3133 \times 4.038121$

$= 7936.173$

PV of proceeds from sale $= 6000v^{16\,3/12} = 1717.969$

NPV of project = 7936.173+1717.969 – 5838.695

= 3815.447 (i.e. £3,815,447)

7 Working in 000's

(i) TWRR is i such that

$$(1+i)^{2\frac{1}{2}} = \frac{175}{150} \times \frac{225}{175+30} \times \frac{280}{225+40}$$

$$= \frac{175}{150} \times \frac{225}{205} \times \frac{280}{265} = 1.352968$$

$\therefore \quad i = 12.85\%$ p.a.

(ii) MWRR is i such that

$$150(1+i)^{2\frac{1}{2}} + 30(1+i)^{1\frac{1}{2}} + 40(1+i)^{\frac{1}{2}} = 280$$

Try: i = 12%, LHS = 277.02
i = 12.5%, LHS = 279.58
i = 13%, LHS = 282.16

$$\therefore i = 12.5\% + \frac{(28-27.958)}{(28.216-27.958)} \times 0.5\%$$

= 12.58% p.a.

(iii) The TWRR is better for comparing 2 investment manager's performances as it is not sensitive to cash flow amounts and timing of payments. The MWRR is sensitive to both.

8 (i) Working in £m

Discounted mean term =

$$\frac{10v^{10} + 11v^{11} + 12v^{12} + \ldots\ldots\ldots\ldots\ldots + 20v^{20}}{v^{10} + v^{11} + v^{12} + \ldots\ldots\ldots\ldots\ldots\ldots + v^{20}}$$

$$= \frac{10v + 11v^2 + 12v^3 + \ldots\ldots\ldots\ldots + 20v^{11}}{v + v^2 + v^3 + \ldots\ldots\ldots\ldots\ldots + v^{11}}$$

$$= \frac{9a_{\overline{11|}} + (Ia)_{\overline{11|}}}{a_{\overline{11|}}} = 9 + \frac{(Ia)_{\overline{11|}}}{a_{\overline{11|}}} \quad \text{at 6\%}$$

$$(Ia)_{\overline{11|}} = 42.7571$$

$$\Rightarrow DMT = \quad 9 + \frac{42.7571}{7.8869} = 14.42128$$

$\Rightarrow$ to 4 significent figures DMT = 14.42

(ii) First condition: pv assets = pv liabilities

$$\Rightarrow \quad Xv^{10} + Yv^{20} = v^9 a_{\overline{11|}} *1 \qquad \text{at 6\%.}$$

$X * 0.55839 + Y * 0.31180 = 0.59190 * 7.8869$ (using tables)

= 4.668256 ………….(1)

2nd condition: DMT assets = DMT liabilities

$\Rightarrow \dfrac{X*10v^{10}+Y*20v^{20}}{Xv^{10}+Yv^{20}}$ = 14.42128 (use of 14.42 from (i) will be accepted)

$\Rightarrow X*5.5839+Y*6.236 = 14.42128*\left(Xv^{10}+Yv^{20}\right)$

= $14.42128*4.668256$ from (1)

= 67.3222 (or 67.3163 if DMT of 14.42 is used)…………(2)

Equ^{n} (2) – 10* Equ^{n} (1) $\Rightarrow$

$Y*6.236-Y*3.1180=67.3222-10*4.668256$

$\Rightarrow Y=\dfrac{20.639667}{3.1180}=6.6195$ (or 6.6176 if DMT of 14.42 is used)

[or $V_A^{'}=V_L^{'}$ (differentiating with respect to *i*)

$$10Xv^{11}+20Yv^{21}=10v^{11}+11v^{12}+\ldots+20v^{21}$$
$$=v^{10}\left(9a_{\overline{11}|}+\left(Ia\right)_{\overline{11}|}\right)$$

$\Rightarrow 5.2679X+5.8831Y = 63.5112$ …………(2)

Equ^{n} (2) – $\dfrac{5.2679}{5.8831}\times$Equ^{n} (1)

$\Rightarrow 2.94155Y=19.4711\Rightarrow Y=6.6193$]

Equ^{n} (1) $\Rightarrow$ X * 0.55839 = 4.668256 – 6.6195 * 0.31180

$\Rightarrow$ X = 4.6639 (or 4.6650 if DMT of 14.42 is used)

[check, in equ^{n} (2). 4.6639 * 5.5839 + 6.6195 * 6.236 = 67.3222]

(iii) For the third condition to be satisfied, it is necessary for the spread of the assets to exceed the spread of the liabilities. This appears to be the case given that the liabilities occur in equal annual amounts at durations from 10 years to 20 years, whereas the assets are concentrated in two lumps at the two most extreme durations, 10 years and 20 years.

9 Let the 1-year and 2-year zero-coupon yields (spot rates) be i_i and i_2 respectively.

$$\frac{105}{1+i_1} = 105v \text{ @ } 4.5\%$$

$$\therefore i_1 = 0.045$$

For the 2-year spot rate:

$$\frac{5}{1+i_1} + \frac{105}{(1+i_2)^2} = 5a_{\overline{2|}5.3\%} + 100v_{5.3\%}^2$$

$$\frac{5}{1.045} + \frac{105}{(1+i_2)^2} = 5\frac{\left(1 - \frac{1}{1.053^2}\right)}{0.053} + \frac{100}{1.053^2}$$

= 9.257681 + 90.186858

= 99.444539

$$\frac{105}{(1+i_2)^2} = 99.444539 - \frac{5}{1.045}$$

$$\Rightarrow (1+i_2)^2 = \frac{105}{94.659850}$$

$$\Rightarrow i_2 = 5.3202\% \text{ p.a.}$$

For the 3-year spot rate:

The 3-year par yield is 5.6% p.a.

$$\Rightarrow 1 = 0.056\left(\frac{1}{1+i_1} + \frac{1}{(1+i_2)^2} + \frac{1}{(1+i_3)^3}\right) + \frac{1}{(1+i_3)^3}$$

$$\Rightarrow \frac{1.056}{(1+i_3)^3} = 1 - \frac{0.056}{1.045} - \frac{0.056}{(1.053202)^2}$$

$$\Rightarrow (1+i_3)^3 = \frac{1.056}{0.895926}$$

$$\Rightarrow i_3 = 5.6324\% \text{ p.a.}$$

1-year forward rates:

$f_0 = i_1 = 4.5\%$ p.a.

$(1+i_1)(1+f_1) = (1+i_2)^2$

$\Rightarrow 1+f_1 = \dfrac{1.053202^2}{1.045}$

$\Rightarrow f_1 = 6.1468\%$ p.a.

$(1+i_2)^2(1+f_2) = (1+i_3)^3$

$\Rightarrow 1+f_2 = \dfrac{(1.056324)^3}{(1.053202)^2}$

$\Rightarrow f_2 = 6.2596\%$ p.a.

10 (i) check for capital gain:

$g(1-t_1) = \dfrac{0.11}{1.15} * (1-0.3)$

$= 0.06696$

$i = 8\% \Rightarrow i^{(4)} = 0.077706$

$\Rightarrow i^{(4)} > g(1-t_1)$

$\Rightarrow$ There's a capital gain and thus loan should be assumed to be redeemed at the latest possible date.

Let P be price at which the investor bought the loan.

Then

$P = 11 \times 0.7 a_{\overline{15}|}^{(4)} + 115v^{15} - 0.25(115-P)v^{15}$ at 8%

$\Rightarrow P = \dfrac{7.7 \times 1.029519 \times 8.5595 + 0.75 \times 115 \times 0.31524}{1 - 0.25 \times 0.31524}$

= £103.17 per £100 nominal

(ii) check for capital gain:

$$g\left(1-t_1\right)=\frac{0.11}{1.15}=0.095652$$

$$i=9\% \;\Rightarrow i^{(4)}=0.087113$$
$$\Rightarrow i^{(4)}<g\left(1-t_1\right)$$

$\Rightarrow$ There's no capital gain and thus loan should be assumed to be redeemed at the earliest possible date.

Let P' be the price at which the investor sold the loan. Then

$$P'=11a_{\overline{7}|}^{(4)}+115v^7 \text{ at } 9\%$$

$$=11\times 1.033144\times 5.033+115\times 0.54703$$

$$=\text{£}120.1064 \text{ per £}100 \text{ nominal}$$

(iii) Let *j* be the yield per quarter. Then

$$103.17=\frac{11}{4}\times 0.7a_{\overline{12}|}+120.1064v^{12}-0.25\left(120.1064-103.17\right)v^{12} \text{ at } j\ \%$$

$$\Rightarrow 103.17=1.925\,a_{\overline{12}|}+115.8723\,v^{12}$$

Try

j = 3%: *RHS* = 100.4319638

j = 2.5%: *RHS* = 105.9042724

Linear interpolation:

$$j=0.025+0.005\times\frac{\left(103.17-105.9042724\right)}{\left(100.4319638-105.9042724\right)}$$

$$=0.02749828$$

Hence, net yield is 11% p.a. (or 10.99931% p.a.) payable quarterly.

11 (i) In 10 years' time the single premium P is

$$P = 12000\left(a_{\overline{1}|}^{(12)} + 1.03a_{\overline{1}|}^{(12)}v + (1.03)^2 a_{\overline{1}|}^{(12)}v^2 + \ldots + (1.03)^{14} v^{14} a_{\overline{1}|}^{(12)}\right)$$

$$= 12000a_{\overline{1}|}^{(12)}\left(1 + \frac{1.03}{1.06} + \left(\frac{1.03}{1.06}\right)^2 + \ldots + \left(\frac{1.03}{1.06}\right)^{14}\right)$$

$$= 12000a_{\overline{1}|}^{(12)}\left(\frac{1-\left(\frac{1.03}{1.06}\right)^{15}}{1-\frac{1.03}{1.06}}\right)$$

where $a_{\overline{1}|}^{(12)} = \frac{i}{i^{(12)}}v$

$$= \frac{1.027211}{1.06} = 0.969067$$

$$\Rightarrow P = 12000 \times 0.969067 \times \frac{0.3499146}{0.0283019}$$

$$= 143{,}774.45$$

(ii) $E(1+i_t) = 1.06 = e^{\mu+\sigma^2/2}$

$$Var(1+i_t) = (0.15)^2 = e^{2\mu+\sigma^2}.\left(e^{\sigma^2} - 1\right)$$

Then $\frac{0.15^2}{(1.06)^2} = e^{\sigma^2} - 1$

$$\Rightarrow \sigma^2 = 0.01982706$$

$$\therefore \mu = \ell n\, 1.06 - \frac{0.01982706}{2}$$

$$= 0.04835538$$

$$\Rightarrow S_{10} \sim LN(0.4835538, 0.1982706)$$

Let X be the amount to be invested at time 0

We want $\Pr\left(X.S_{10} \geq 143,774.45\right) = 0.98$

so $\Pr\left(S_{10} \geq \dfrac{143,774.45}{X}\right) = 0.98$

so $1 - \Phi\left(\dfrac{Ln\frac{143774.45}{X} - 10\mu}{\sqrt{10\sigma^2}}\right) = 0.02$

$\Rightarrow \dfrac{Ln\frac{143774.45}{X} - 10\mu}{\sqrt{10\sigma^2}} = -2.0537$

So $Ln\dfrac{143774.45}{X} = -2.0537 \times \sqrt{0.1982706} + 0.4835538$

$= -0.430909$

$\Rightarrow \dfrac{143774.45}{X} = 0.6499179$

$\Rightarrow \quad X = £221,219.41$

(iii) It might seem odd that the initial investment needs to be substantially higher than the single premium required in 10 years' time to have a 98% probability of accumulating to the single premium.

This strange result is explained by the fact that the variance of the interest rate is so high relative to the mean. There is therefore a significant risk that the investment will decrease in value over the next 10 years.

END OF EXAMINERS' REPORT

Subject CT1 — Financial Mathematics. Core Technical.

September 2009 examinations

EXAMINERS REPORT

Introduction

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Chairman of the Board of Examiners

December 2009

Comments for individual questions are given with the solutions that follow.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Well-prepared candidates scored well across the whole paper. However, the comments below on each question concentrate on areas where candidates could have improved their performance.

1

a. $96\ 1.05^{\,t} = 97.89 \Rightarrow\ 1.05^{\,t} = \dfrac{97.89}{96}$

$$t = \frac{\ln\ {}^{97.89}\!/_{96}}{\ln\ 1.05} = 0.400 \text{ years or } 146 \text{ days}$$

b. Second investor held the bill for 36 days. Therefore

$$97.89\left(1+\frac{36}{365}i\right) = 100 \Rightarrow i = \frac{365}{36}\left(\frac{100}{97.89}-1\right) = 21.854\%$$

This was answered well except by the very weakest candidates.

2

- Issued by corporations.
- Holders entitled to a distribution (dividend) declared from profits.
- Potential for high returns relative to other asset classes.
- Commensurate risk of capital losses.
- Lowest ranking finance issued by companies.
- Initial running yield low but has potential to increase with dividend growth.
- Dividends and capital values have the potential to grow in nominal terms during times of inflation.
- Return made up of income return and capital gains.
- Marketability depends on the size of the issue.
- Ordinary shareholders receive voting rights in proportion to their holding.

This question was not answered as well as the examiners would have expected given that the topic is standard bookwork.

3

We convert all cash flow to amounts in time 0 values:

Dividend paid at $t = 1: 10000 \times 0.041 \times \frac{147.7}{153.4} = 394.77$

Dividend paid at $t = 2: 10000 \times 0.046 \times \frac{147.7}{158.6} = 428.39$

Dividend paid at $t = 3: 10000 \times 0.051 \times \frac{147.7}{165.1} = 456.25$

Sale proceeds at $t = 3: 10000 \times 0.93 \times \frac{147.7}{165.1} = 8319.87$

$\Rightarrow$ Equation of value involving v where $v = \frac{1}{1+r}$
and r = real rate of return:

$$7800 = 394.77v + 428.39v^2 + 8776.17v^3 ..... (1)$$

[To estimate r:

Approx nominal rate of return is

$$\left(4.6 + \frac{93-78}{3}\right) / 78 = 12.3\% \text{ p.a.}$$

Average inflation over 3 year period comes from

$$\left(\frac{165.1}{147.7}\right)^{1/3} - 1 = 3.8\,\% \text{ p.a.}$$

$\Rightarrow$ Approx real return: $\frac{1.123}{1.038} - 1 = 8.2\,\%$ p.a.]

Try $r = 8\%$, RHS of $(1) = 7699.61$

$r = 7\%$, RHS of $(1) = 7907.09$

$$r = 7\% + \frac{7907.09 - 7800}{7907.69 - 7699.61} \times 1\%$$

= 7.52 % p.a.

Some candidates seemed to struggle to derive the equation of value based on a real rate of return and multiplied (rather than divided) the payments by the increase in the inflation index.

4

(i) Let required price = *P*:

$$P = 1-0.2\ \ 5a_{\overline{20|}}^{2} + 100v^{20} \text{ at } 6\%$$

$$a_{\overline{20|}}^{2} = \frac{i}{i^{2}} a_{\overline{20|}} = \frac{0.06}{0.059126} 11.4699 = 11.6394;\ v^{20} = 0.311805$$

Therefore

$$P = 1-0.2\ \ 5\times 11.6394 + 100\times 0.311805$$
$$= 46.5576 + 31.1805 = 77.7381$$

(ii) The equation of value for the gross rate of return is:

$$77.7381 = 5a_{\overline{20|}}^{2} + 100v^{20}$$

If i = 8%

$$a_{\overline{20|}}^{2} = \frac{i}{i^{2}} a_{\overline{20|}} = 1.019615\times 9.8181 = 10.0107;\ v^{20} = 0.21455$$

RHS = 50.0534 + 21.4550 = 71.5084

If i = 7%

$$a_{\overline{20|}}^{2} = \frac{i}{i^{2}} a_{\overline{20|}} = 1.017204\times 10.5940 = 10.7763;\ v^{20} = 0.25842$$

RHS = 53.8813 + 25.8420 = 79.7233

Interpolating gives $i \approx 0.07 + \frac{79.7233 - 77.7381}{79.7233 - 71.5084}\times 0.01 = 7.24\% = 7.2\%$ say

(iii) If the nominal rate of return is 7.2% per annum effective and inflation is 3% per annum effective, then the real rate of return is calculated from:

$$\left(\frac{1.072}{1.03}\right) - 1 = 4.1\%$$

This question was answered very well.

5

(i) $130 = 100\exp\left\{\int_0^5 (a+bt^2)\,dt\right\} = 100\exp\left[at + \tfrac{1}{3}bt^3\right]_0^5 = 100\exp(5a + 41.667b)$

$$200 = 100\exp\left\{\int_0^{10} (a+bt^2)\,dt\right\} = 100\exp\left[at + \tfrac{1}{3}bt^3\right]_0^{10} = 100\exp(10a + 333.333b)$$

$\ln(1.3) = 5a + 41.667b$

$\ln(2) = 10a + 333.333b$

The second expression less twice times the first expression gives:

$\ln(2) - 2\ln(1.3) = 250b \Rightarrow b = 0.0006737$

$a = \dfrac{\ln(2) - 333.333 \times 0.0006737}{10} = 0.04686$

(ii) $100\left(1 + \dfrac{i^{(12)}}{12}\right)^{60} = 130 \Rightarrow i^{(12)} = 12\left[\left(\dfrac{130}{100}\right)^{1/60} - 1\right] \Rightarrow i^{(12)} = 5.259\%$ p.a.

(iii) $130e^{5\delta} = 200 \Rightarrow 5\delta = \ln\dfrac{200}{130} \Rightarrow \delta = 8.616\%$ p.a.

This question was answered very well.

6

(i) A future is a contract which obliges the parties to deliver/take delivery of a particular quantity of a particular asset at a particular time at a fixed price.

An option is the right to buy or sell a particular quantity of a particular asset at (or before) a particular time at a given price.

(ii) Assume no arbitrage

a. Buying the forward is exactly the same as buying the bond except that the forward will not pay coupons and the forward does not require immediate settlement.

Let the forward price = F. The equation of value is:

$$F = 97(1.06) - 3.5 \times \frac{1.06}{(1.05)^{\frac{1}{2}}} - 3.5$$

$$= 102.82 - 3.62059 - 3.5 = 95.6994$$

b. Let six month forward interest rate $= f_{0.5,0.5} = \frac{1.06}{(1.05)^{\frac{1}{2}}} - 1 = 3.4454\%$

This does not have to be expressed as a rate of interest per annum effective, though it could be.

c. $P = 2(1.05)^{-0.5} + 102(1.06)^{-1} = 1.9518 + 96.2264 = 98.1782$

d. Gross redemption yield is i such that

$$98.1782 = 2(1+i)^{-0.5} + 102(1+i)^{-1}$$

Using the formula for solving a quadratic (interpolation will do):

$(1+i)^{-0.5} = 0.97133$. Therefore, $i \approx 6\%$ (in fact 5.99%).

e. Answer is very close to 6% (the one-year spot rate) because the payments from the bond are so heavily weighted towards the redemption time in one year.

This was generally well-answered apart from part (e). A common error in parts (c) and (d) was to assume that the coupon payments were 4% per half-year.

7 .

(i) The accumulation is $1200\ddot{s}_{\overline{20|}}^{(12)}(1.06)^{20} + 2300\ddot{s}_{\overline{20|}}^{(12)} + 100(I\ddot{a})_{\overline{20|}}^{(12)}(1.06)^{20}$

$$= \frac{i}{d^{(12)}}\left[1200 s_{\overline{20|}}(1.06)^{20} + 2300 s_{\overline{20|}} + 100(Ia)_{\overline{20|}}(1.06)^{20}\right]$$

$$= 1.032211\begin{pmatrix}1,200 \times 36.7856 \times 3.20714 + 2,300 \times 36.7856 \\ +100 \times 98.7004 \times 3.20714\end{pmatrix}$$

$$= 1.032211(141,571.88 + 84,606.88 + 31,654.60)$$

$$= 266,138$$

(ii) Let half-yearly payment = X

$$Xa_{\overline{40}|} = 266{,}138 \text{ at } 2.5\%$$

$$\Rightarrow X = \frac{266{,}138}{25.1028} = 10{,}601.94$$

Therefore, annual rate of payment = £21,203.88

(iii) Work in half-years. Discounted mean term is:

$$10{,}601.94\ \ v + 2v^2 + \ldots + 40v^{40}\ \ /266{,}138$$

Numerator = $10{,}601.94\ \ Ia_{\overline{40}|}$ at 2.5% per half year effective.

$$= 10{,}601.94 \times 433.3248 = 4{,}584{,}075$$

Therefore DMT = 17.26 half years or 8.63 years.

In part (i), many candidates developed the correct formula although calculation errors were common. In such cases, candidates also lost marks for not showing and explaining their working fully. Part (ii) was answered well but many candidates surprisingly had trouble calculating the DMT in part (iii). In this part, candidates often lost marks for not showing the units properly at the end of the answer; indeed, in many cases, showing the units may well have alerted candidates to possible mistakes.

2

(i) The equation of value for the borrower is $4{,}012.13a_{\overline{20}|} = 50{,}000$.

Therefore $a_{\overline{20}|} = \dfrac{50{,}000}{4{,}012.13} = 12.4622$

From inspection of tables, i = 5%

(ii) The second customer pays interest of $0.055 \times 50{,}000$ = £2,750 per annum, annually in arrear.

The annual rate of monthly payments in advance from the savings policy is X such that:

$$X\ddot{s}_{\overline{20|}}^{(12)} = 50,000 \text{ at } 4\%$$

$$\Rightarrow X s_{\overline{20|}} \frac{i}{d^{(12)}} = 50,000$$

$$\Rightarrow X = \frac{50,000}{29.7781 \times 1.021537} = £1,643.69$$

The equation of value for this borrower is:

$$50,000 = 2,750 a_{\overline{20|}} + 1,643.686 \ddot{a}_{\overline{20|}}^{(12)}$$
$$= 2,750 a_{\overline{20|}} + 1,643.686 \frac{i}{d^{(12)}} a_{\overline{20|}}$$

Try i = 6%: RHS = 51,002.41

Try i = 7%: RHS = 47,200.14

By interpolation i = 6.3%

Part (i) was well answered but weaker candidates failed to recognise the need to calculate separately the payments into the savings policy in part (ii).

3

(i) The expected annual interest rate in the first ten years is $0.3 \times 0.04 + 0.7 \times 0.06 = 0.054$. The expected interest rate in the second ten years is clearly 5.5%.

If the premium is calculated on the basis of these interest rates, then the premium will be P such that:

$$20,000 = P (1.054)^{10} (1.055)^{10}$$
$$\Rightarrow 20,000 = 2.89022P \Rightarrow P = 6,919.89$$

(ii) The expected accumulation factor in the first ten years is:

$$0.3 \times 1.04^{10} + 0.7 \times 1.06^{10} = 1.69767$$

The expected accumulation factor in the second ten years is:

$$0.5\left[(1.05)^{10} + (1.06)^{10}\right] = 1.70987$$

As they are independent, we can multiply the accumulation factors together and multiply by the premium to give an expected accumulation of: $6,919.89 \times 1.69767 \times 1.70987 = 20,087.04$.

The expected profit is 87.04.

(iii) There is an expected profit because (in general) the accumulation of a sum of money at the expected interest rate is not equal to the expected accumulation when the interest rate is a random variable.

(iv) The highest possible outcome for the accumulation factor is:

$1.06^{10} \times 1.06^{10} = 3.20714$ with probability $0.7 \times 0.5 = 0.35$

The lowest possible outcome is:

$1.04^{10} \times 1.05^{10} = 2.41116$ with probability $0.3 \times 0.5 = 0.15$.

The range is therefore: 6,919.89 (3.20714 – 2.41116) = 5,508.05.

The other two possible outcomes are:

$1.06^{10} \times 1.05^{10} = 2.91710$ with probability $0.7 \times 0.5 = 0.35$

and $1.04^{10} \times 1.06^{10} = 2.65089$ with probability $0.3 \times 0.5 = 0.15$

The mean accumulation factor is: $1.69767 \times 1.70987 = 2.90280$

The variance of the accumulation from one unit of investment is:

$0.35(3.20714-2.90280)^2 + 0.15(2.41116-2.90280)^2$
$+ 0.35\,(2.91710-2.90280)^2 + 0.15\,(2.65089-2.90280)^2$

$= 0.03241 + 0.03626 + 0.00007 + 0.00952 = 0.07826.$

Standard deviation is $\sqrt{0.07826} = 0.27976$.

Standard deviation of the accumulation of the whole premium is: $6{,}919.89 \times 0.27976 = £1{,}935.88$ which is also the standard deviation of the profit.

This was the worst answered question on the paper with many candidates not recognising that the accumulation of a sum of money at the expected interest rate is not equal to the expected accumulation when the interest rate is a random variable. The calculation of the standard deviation of the accumulation was generally only calculated correctly by the strongest candidates.

4

(i) The discounted payback period is the first time at which the accumulated profit from/net present value of the cash flows from a project is positive at a given interest rate.

It is an inappropriate decision criterion because it does not tell us anything about the overall profitability of the project.

(ii) If the internal rate of return were greater than 1.5% then the net present value of the project at 1.5% must be greater than zero. As such, there must be a discounted payback period as the discounted payback period is the first time at which the net present value is greater than zero: such a time must exist.

(iii) Returns are real rates of return and figures are in 2009 dollar terms so we are automatically working with real rather than nominal values. All figures below are in \$bn.

The net benefits from using the technology are the \$30 every three years; \$20 incurred continuously increasing at 1% per annum and \$30 per annum incurred annually in arrears.

The costs of the technology are \$440 incurred immediately and \$50 incurred annually in arrears.

The net present value of the project at 1% per annum effective is:

$$30\left(v^3+v^6+\ldots+v^{48}\right)+50\times 20+30a_{\overline{50|}}-\left(440+50a_{\overline{50|}}\right)$$

The 20 does not need to be discounted because the cash flows are growing at the same rate as they are being discounted.

$$=30v^3\left(\frac{1-v^{48}}{1-v^3}\right)+560-20a_{\overline{50|}} \text{ calculated at 1\%}$$

$$=30\times 0.97059\frac{1-0.62026}{1-0.97059}+560-20\times 39.1961$$

$$=375.967+560-783.922$$

$$=152.045$$

(iv) The net present value of the project at 4% per annum effective is:

$$30\left(v^3+v^6+\ldots+v^{48}\right)+20\bar{a}'_{\overline{50|}}+30a_{\overline{50|}}-\left(440+50a_{\overline{50|}}\right)$$

All are calculated at 4% except $\bar{a}'_{\overline{50|}}$ which is calculated at

$$i=\frac{1.04}{1.01}-1=2.97\%$$

$$=30v^3\left(\frac{1-v^{48}}{1-v^3}\right)+20\frac{i}{\delta}a'_{\overline{50|}}-440-20a_{\overline{50|}}$$

$$=30\times 0.88900\frac{1-0.15219}{1-0.88900}+20\times 1.014779\times 25.8755-440-20\times 21.4822$$

$$= 203.704 + 525.158 - 440 - 429.644$$
$$= -140.790$$

(v) Whether the investment should go ahead would depend on the choice of the interest rate – it is clearly a crucial assumption (students could make a choice themselves and indicate whether it should go ahead on the basis of that rate but there must be some justification for the choice).

This question was also poorly answered possibly because project appraisal using real interest rates has rarely been examined in the past (and also possibly because of time pressure). Whilst some parts of the question were challenging (e.g. the treatment of the increasing costs of climate change), it was disappointing that many candidates failed to recognise that the costs of climate change no longer incurred would be a benefit of the carbon storing technology project and so failed to score many marks.

END OF EXAMINERS' REPORT

EXAMINERS' REPORT

April 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

R D Muckart
Chairman of the Board of Examiners

July 2010

Comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Well-prepared candidates scored well across the whole paper and the examiners were pleased with the general standard of answers. However, questions that required an element of explanation or analysis were less well answered than those which just involved calculation. The comments below concentrate on areas where candidates could have improved their performance.

Q2.

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

Q3.

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

Q6.

Many candidates assumed that the accumulation in part (i) was for a single payment.

Q7.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

Q9.

A common error was to assume that income only started after three years rather than 'starting from the beginning of the third year'.

Q10.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

1 (i) (a) Options – holder has the right but not the obligation to trade
Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset at specified price at specified future date.

Put Option – right but not the obligation to SELL specified asset at specified price at specified future date.

(ii) $K = 60e^{0.06\times 3/12} - 2.80e^{0.06\times 1/12} = 60.90678 - 2.81404 = £58.09$

2 (i) Cash flows:

Issue price:	Jan 08	$-0.98\times 100,000$	$= -£98,000$
Interest payments:	July 08	$0.02\times 100,000\times\dfrac{112.1}{110.5}$	$= £2,028.96$
	Jan 09	$0.02\times 100,000\times\dfrac{115.7}{110.5}$	$= £2,094.12$
	July 09	$0.02\times 100,000\times\dfrac{119.1}{110.5}$	$= £2,155.66$
	Jan 10	$0.02\times 100,000\times\dfrac{123.2}{110.5}$	$= £2,229.86$
Capital redeemed:	Jan 10	$100,000\times\dfrac{123.2}{110.5}$	$= £111,493.21$

(ii) Equation of value is:

$$98000 = 2028.96v^{\frac{1}{2}} + 2094.12v + 2155.66v^{1\frac{1}{2}} + 2229.86v^2 + 111493.21v^2$$

At 11%, RHS = 97955.85 ≈ 98000

3 Purchase price = 0.45 × 1,000,000 = £450,000

PV of dividends = $50000\times(1-0.2)\times\left[\left(v^2+v^{2\frac{1}{2}}\right)+1.03\left(v^3+v^{3\frac{1}{2}}\right)+1.03^2\left(v^4+v^{4\frac{1}{2}}\right)+\cdots\right]$

$$= 40000\left(v^2+v^{2\frac{1}{2}}\right)\left[1+1.03v+1.03^2v^2+\cdots\right] \text{ @ } 8\%$$

$$= 40000\times 1.68231\times\left(\frac{1}{1-1.03/1.08}\right)=1,453,516$$

⇒ NPV = 1,453,516 – 450,000 = £1,003,516

4 Let i = money yield

$\Rightarrow 1+i=1.0285714\times 1.05=1.08 \quad \Rightarrow i=8\%$ p.a.

Check whether CGT is payable: compare $i^{(2)}$ with $(1-t)g$

$$(1-t)g=0.8\times\frac{6}{105}=0.04571$$

From tables, $i^{(2)}=7.8461\% \;\Rightarrow i^{(2)}>(1-t)g$

⇒ CGT is payable

$P=0.8\times 6a_{\overline{10|}}^{(2)}+105v^{10}-0.25(105-P)v^{10}$ @ 8%

$$=\frac{0.8\times 6a_{\overline{10|}}^{(2)}+0.75\times 105v^{10}}{1-0.25v^{10}}$$

$$=\frac{4.8\times 1.019615\times 6.7101+78.75\times 0.46319}{1-0.25\times 0.46319}$$

= £78.39

5 (i) Let P denote the current price (per £100 nominal) of the security.

Then, we have:

$$P = \frac{7}{1.044} + \frac{7}{1.044 \times 1.047} + \frac{7}{1.044 \times 1.047 \times 1.049} + \frac{107}{1.044 \times 1.047 \times 1.049 \times 1.05} = 108.0872$$

(ii) The gross redemption yield, i, is given by:

$$108.09 = 7 \times a_{\overline{4}|}^{i\%} + 100 \times v_{i\%}^{4}$$

Then, we have:

$$\left.\begin{array}{lll} i = 5\% & \Rightarrow & RHS = 107.0919 \\ i = 4.5\% & \Rightarrow & RHS = 108.9688 \end{array}\right\} \Rightarrow i \approx 0.045 + (0.05 - 0.045) \times \left(\frac{108.0872 - 108.9688}{107.0919 - 108.9688} \right) = 0.0473$$

(iii) The gross redemption yield represents a weighted average of the forward rates at each duration, weighted by the cash flow received at that time.

Thus, increasing the coupon rate will increase the weight applied to the cash flows at the early durations and, as the forward rates are lower at early durations, the gross redemption yield on a security with a higher coupon rate will be lower than above.

Note to markers: no marks for simply plugging 9% pa in, and providing no explanation for result.

6 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$$= e^{0.05 + \frac{1}{2} \times 0.004}$$

$$= 1.0533757$$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation at the end of 25 years of £3,000 paid annually in advance for 25 years.

Then $E\left[A\right] = 3000\ddot{s}_{\overline{25|}}$ at rate $j = 0.0533757$

$$= 3000\frac{\left((1+j)^{25}-1\right)}{j}\times(1+j)$$

$$= 3000\frac{\left(1.0533757^{25}-1\right)}{0.0533757}\times 1.0533757$$

$$= £158,036.43$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$\therefore E\left[S_{20}\right] = e^{20\mu+\frac{1}{2}\times 20\sigma^2}$

$\left\{\text{or}\,(1+j)^{20}\right\}$

$= \exp\left(20\times 0.05 + 10\times 0.004\right)$

$= e^{1.04} = 2.829217$

In $S_{20} \sim N\left(20\mu, 20\sigma^2\right)$

$\Rightarrow$ In $S_{20} \sim N\left(1, 0.08\right)$

$\Pr\left(S_{20} > 2.829217\right) = \Pr\left(\ln S_{20} > \ln 2.829217\right)$

$$= \Pr\left(Z > \frac{\ln 2.829217\text{-}1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= \Pr\left(Z > 0.14\right) = 1 - \Phi\left(0.14\right)$$

= 1 – 0.55567

= 0.44433 i.e. 44.4%

7 (i) DMT of liabilities is given by:

$$\frac{1\times 1\times v_{7\%}+2\times(1.038835)\times v_{7\%}^{2}+3\times(1.038835)^{2}\times v_{7\%}^{3}+\ldots+40\times(1.038835)^{39}\times v_{7\%}^{40}}{1\times v_{7\%}+(1.038835)\times v_{7\%}^{2}+(1.038835)^{2}\times v_{7\%}^{3}+\ldots+(1.038835)^{39}\times v_{7\%}^{40}}$$

$$=\frac{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+2\times\left(\frac{1.038835}{1.07}\right)^{2}+3\times\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+40\times\left(\frac{1.038835}{1.07}\right)^{40}\right]}{(1.038835)^{-1}\times\left[\left(\frac{1.038835}{1.07}\right)+\left(\frac{1.038835}{1.07}\right)^{2}+\left(\frac{1.038835}{1.07}\right)^{3}+\ldots+\left(\frac{1.038835}{1.07}\right)^{40}\right]}$$

$$=\frac{v_{i^*}+2\times v_{i^*}^{2}+3\times v_{i^*}^{3}+\ldots+40\times v_{i^*}^{40}}{v_{i^*}+v_{i^*}^{2}+v_{i^*}^{3}+\ldots+v_{i^*}^{40}}$$

$$=\frac{(Ia)_{\overline{40}|}^{i^*}}{a_{\overline{40}|}^{i^*}}$$

where $v_{i^*}\equiv\frac{1}{1+i^*}=\frac{1.038835}{1.07}\Rightarrow i^*=\frac{1.07}{1.038835}-1=\frac{0.07-0.038835}{1.038835}=0.03$.

Hence, DMT of liabilities is:

$$\frac{(Ia)_{\overline{40}|}^{3\%}}{a_{\overline{40}|}^{3\%}}=\frac{384.8647}{23.1148}=16.65\text{ years}$$

(Alternative method for DMT formula

$$DMT=\frac{v(1+2gv+3g^2v^2+\cdots+40g^{39}v^{39})}{v(1+gv+g^2v^2+\cdots+g^{39}v^{39})}=\frac{v(I\ddot{a})_{\overline{40}|}^{3\%}}{v\ddot{a}_{\overline{40}|}^{3\%}}=\frac{(I\ddot{a})_{\overline{40}|}^{3\%}}{\ddot{a}_{\overline{40}|}^{3\%}}=\frac{(Ia)_{\overline{40}|}^{3\%}}{a_{\overline{40}|}^{3\%}}$$

where $g=1.038835$.)

(ii) Even if the fund manager invested entirely in the 15-year zero-coupon bond, the DMT of the assets will be only 15 years (and, indeed, any other portfolio of securities will result in a lower DMT).

Thus, it is not possible to satisfy the second condition required for immunisation (i.e. DMT of assets = DMT of liabilities).

Hence, the fund cannot be immunised against small changes in the rate of interest.

(iii) The other problems with implementing an immunisation strategy in practice include:

- the approach requires a continuous re-structuring of the asset portfolio to ensure that the volatility of the assets remains equal to that of the liabilities over time

- for most institutional investors, the amounts and timings of the cash flows in respect of the liabilities are unlikely to be known with certainty

- institutional investor is only immunised for small changes in the rate of interest

- the yield curve is unlikely to be flat at all durations

- changes in the term structure of interest rates will not necessarily be in the form of a parallel shift in the curve (e.g. the shape of the curve can also change from time to time)

8 (i) Loan = $4500a_{\overline{20|}} + 150(Ia)_{\overline{20|}}$ at 9%

$\Rightarrow$ Loan = $4500 \times 9.1285 + 150 \times 70.9055$

$= 41,078.25 + 10,635.83 = 51,714.08$

(ii) Loan o/s after 9th year = $(4500 + 1350)a_{\overline{11|}} + 150(Ia)_{\overline{11|}}$ at 9%

Loan o/s = $5,850 \times 6.8052 + 150 \times 35.0533$

$= 39,810.42 + 5258.00 = 45,068.42$

Repayment = $6000 - 45,068.42 \times 0.09 = £1,943.84$

(Alternative solution to (ii)

(ii) Loan o/s after 9th year = $(4500 + 1350)a_{\overline{11|}} + 150(Ia)_{\overline{11|}}$ at 9%

$= 5,850 \times 6.8052 + 150 \times 35.0533 = 45,068.42$ as before

Loan o/s after 10th year = $(4500 + 1500)a_{\overline{10|}} + 150(Ia)_{\overline{10|}}$ at 9%

$= 6,000 \times 6.4177 + 150 \times 30.7904 = 43,124.76$

Repayment = $45,068.42 - 43,124.76 = £1,943.66$)

(iii) Last instalment = $4650+19\times150 = 7500$

Loan o/s = $7500a_{\overline{1}|} = 7500v$

Interest = $7500\times0.91743\times0.09 = £619.27$

(iv) Total payments $= 20\times4650+\dfrac{1}{2}\times19\times20\times150$

$= 93,000+28,500 = 121,500$

Total interest = 121,500 – 51,714.08 = £69,785.92

9 (i) NPV $= -5-3v^{1/4}+1.7\bar{a}_{\overline{15}|}v^2$ @10%

NPV = $-5-3\times0.976454+1.7\times0.82645\times\dfrac{i}{\delta}a_{\overline{15}|}$ @10%

$= -5-2.929362+1.404965\times1.049206\times7.6061$

$= -7.929362+11.21213458$

$= 3.282772575$

NPV = £3.283m

(ii) DPP is $t+2$ such that

$1.7\bar{a}_{\overline{t}|}v^2 = 5+3v^{\frac{1}{4}} \Rightarrow 1.474097708a_{\overline{t}|} = 7.929362$ @10%

$\dfrac{1-1.1^{-t}}{0.1} = 5.379129 \Rightarrow 1-1.1^{-t} = 0.5379129$

$\Rightarrow 0.4620871 = 1.1^{-t} \Rightarrow \ln 0.4620871 = -t\ln 1.1$

$\Rightarrow t = 8.100$

$\therefore DPP = 10.1$ years

(iii) Accumulated profit 17 years from start of project:

$$= 1.7 s_{\overline{6.9|}7\%} = 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{\delta} \;\; @\, 7\%$$

$$= 1.7 \times \frac{\left(1.07^{6.9} - 1\right)}{0.067659}$$

$$= 1.7 \times 8.79346$$

$$= £14.95m$$

10 (i) The values of the funds before and after the cash injections are:

	Manager A		*Manager B*	
1 January 2007	120,000		100,000	
31 December 2007	130,000	140,000	140,000	150,000
31 December 2008	135,000	145,000	145,000	155,000
31 December 2009	180,000		150,000	

Thus, TWRR for Manager A is given by:

$$(1+i)^3 = \frac{130}{120} \times \frac{135}{140} \times \frac{180}{145} \Rightarrow i = 0.0905 \text{ or } 9.05\%$$

And, TWRR for Manager B is given by:

$$(1+i)^3 = \frac{140}{100} \times \frac{145}{150} \times \frac{150}{155} \Rightarrow i = 0.0941 \text{ or } 9.41\%$$

(ii) MWRR for Manager A is given by:

$$120 \times (1+i)^3 + 10 \times (1 \times i)^2 + 10 \times (1+i) = 180$$

Then, putting $i = 0.094$ gives $LHS = 180.03$ which is close enough to 180.

(iii) Both funds increased by 50% over the three year period and received the same cashflows at the same times.

Since the initial amount in fund B was lower, the cash inflows received represent a larger proportion of fund B and hence the money weighted return earned by fund B over the period will be lower, particularly since the returns were negative for the 2nd and 3rd years.

[Could also note that for fund B:

$$100\times(1+i)^3+10\times(1\times i)^2+10\times(1+i)=150$$

So by a proportional argument $120\times(1+i)^3+12\times(1\times i)^2+12\times(1+i)=180$

which when compared with the equation for fund A in (ii) clearly shows that the return for B is lower.]

(iv) The money weighted rate of return is higher for fund A, whilst the time weighted return is higher for fund B.

When comparing the performance of investment managers, the time weighted rate of return is generally better because it ignores the effects of cash inflows or outflows being made which are beyond the manager's control.

In this case, Manager A's best performance is in the final year, when the fund was at its largest, whilst Manager B's best performance was in the first year, where his fund was at its lowest.

Overall, it may be argued that Manager B has performed slightly better than Manager A since Manager B achieved the higher time weighted return.

11 (i) $t<5$

$$v(t) = e^{-\int_0^t (0.04+0.02s)ds}$$

$$= e^{-\left[0.04s+0.01s^2\right]_0^t}$$

$$= e^{-\left[0.04t+0.01t^2\right]}$$

$t\geq 5$

$$v(t) = e^{-\left\{\int_0^5 (0.04+0.02s)ds+\int_5^t 0.05ds\right\}}$$

$$= v(5)\times e^{-\left[0.05(t-5)\right]}$$

$$= e^{-0.45}\times e^{-\left[0.05(t-5)\right]} = e^{-\left[0.05t+0.2\right]}$$

(ii) (a) $PV = 1,000e^{-[0.05\times 17+0.2]} = e^{-1.05}$

$= 349.94$

(b) $1000\left(1+\dfrac{i^{(12)}}{12}\right)^{-204} = 349.94$

$\Rightarrow i^{(12)} = 6.1924\%$

(iii) $PV = \int_6^{10} e^{-0.45}e^{-[0.05t-0.25]}10e^{0.01t}dt$

$= 10e^{-0.2}\int_6^{10} e^{-0.04t}dt$

$= 10e^{-0.2}\left[-\dfrac{e^{-0.04t}}{0.04}\right]_6^{10}$

$= 8.18733\times 2.90769$

$= 23.806$

(Alternative Solution to (iii)

Accumulated value at time $t = 10$

$$= \int_6^{10} 10e^{0.01t}\left(\exp\int_t^{10} 0.05ds\right)dt$$

$$= \int_6^{10} 10e^{0.01t}\left(\exp\left[0.05s\right]_t^{10}\right)dt$$

$$= \int_6^{10} 10e^{0.01t}e^{0.5-0.05t}dt = \int_6^{10} 10e^{0.5-0.04t}dt$$

$$= \left[\frac{10e^{0.5-0.04t}}{-0.04}\right]_6^{10} = -276.293+324.233 = 47.940$$

Present value = $v(10)\times 47.940 = 0.63763e^{-[0.05\times 10-0.25]}\times 47.940 = 23.806$

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2010 Examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

December 2010

Comments

Please note that different answers may be obtained from those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown. Candidates also lose marks for not showing their working in a methodical manner which the examiner can follow. This can particularly affect candidates on the pass/fail borderline when the examiners have to make a judgement as to whether they can be sure that the candidate has communicated a sufficient command of the syllabus to be awarded a pass.

The general standard of answers was noticeably lower than in previous sessions and there were a significant number of very ill-prepared candidates. As in previous exams, questions that required an element of explanation or analysis were less well answered than those which just involved calculation.

Comments on individual questions, where relevant, can be found after the solution to each question. These comments concentrate on areas where candidates could have improved their performance.

1 Working in half years:

The present value of the security on 1st June would have been $\dfrac{3.5}{i^{(2)}}$

20 August is 80 days later so the present value is $\dfrac{3.5}{i^{(2)}}(1+i)^{80/365}$

Hence the price per £100 nominal is $\dfrac{3.5}{0.097618}(1.1)^{80/365} = £36.611$

2 (i) (a) Gross rate of return convertible half yearly is simply 4/110 = 0.03636 or 3.636%.

(b) Gross effective rate of return is $\left(1+\dfrac{0.03636}{2}\right)^2 - 1 = 0.03669$ or 3.669%

(ii) The net effective rate of return per half year is $0.75 \times \dfrac{0.03636}{2} = 0.013635$.

The net effective rate of return per annum is therefore:

$$(1.013635)^2 - 1 = 0.02746 \text{ or } 2.746\%.$$

A common error was to divide the nominal payments by the increase in the index factor (rather than multiplying).

3 (a) Let S_{20} be the accumulation of the unit investment after 20 years:

$$E(S_{20}) = E\left[(1+i_1)(1+i_2)\ldots(1+i_{20})\right]$$

$$E(S_{20}) = E[1+i_1]E[1+i_2]\ldots E[1+i_{20}] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] = j \quad \therefore \quad E(S_{20}) = (1+j)^{20} = 2$$

$$\Rightarrow j = 2^{1/20} - 1 = 3.5265\%$$

(b) The variance of the effective rate of return per annum is s^2 where

$$\text{Var}\left[S_n\right] = \left(\left(1+j\right)^2 + s^2\right)^{20} - \left(1+j\right)^{40} = 0.6^2$$

$$s^2 = \left[0.6^2 + \left(\left(1+j\right)^{20}\right)^2\right]^{1/20} - \left(1+j\right)^2$$

$$= \left(0.6^2 + 2^2\right)^{1/20} - 2^{1/10} = 0.004628$$

Many candidates made calculation errors in this question but may have scored more marks if their working had been clearer.

4 (i) Assuming no arbitrage, buying the share is the same as buying the forward except that the cash does not have to be paid today and a dividend will be payable from the share.

Therefore, price of forward is:

$$700\left(1.05\right)^{1/12}\left(1.03\right)^{5/12} - 20\left(1.03\right)^{5/12}$$
$$= 711.562 - 20.248 = 691.314$$

(ii) The no arbitrage assumption means that we can compare the forward with the asset from which the forward is derived and for which we know the market price. As such we can calculate the price of the forward from this, without knowing the expected price at the time of settlement. [It could also be mentioned that the market price of the underlying asset does, of course, already incorporate expectations].

(iii) If it was not known with certainty that the dividend would be received we could not use a risk-free interest rate to link the cash flows involved with the purchase of the forward with all the cash flows from the underlying asset.

5 (a) Eurobonds

- Medium-to-long-term borrowing.
- Pay regular coupon payments and a capital payment at maturity.
- Issued by large corporations, governments or supranational organisations.
- Yields to maturity depend on the risk of the issuer.
- Issued and traded internationally (not in core reading).
- Often have novel features.
- Usually unsecured
- Issued in any currency
- Normally large issue size
- Free from regulation of any one government

(b) Convertible Securities

- Generally unsecured loan stocks.
- Can be converted into ordinary shares of the issuing company.
- Pay interest/coupons until conversion.
- Provide levels of income between that of fixed-interest securities and equities.
- Risk characteristics vary as the final date for convertibility approaches.
- Generally less volatility than in the underlying share price before conversion.
- Combine lower risk of debt securities with the potential for gains from equity investment.
- Security and marketability depend upon issuer
- Generally provide higher income than ordinary shares and lower income than conventional loan stock or preference shares

6 (i) Net present value (all figures in £*m*)

$$= -2,000 + 0.1 \times 200 \times v^3 \left(v^{0.5} + 1.1v^{1.5} + 1.1^2 v^{2.5} + \ldots + 1.1^5 v^{5.5} \right)$$
$$+0.2 \times 200 \times v^3 \left(v + 1.1v^2 + 1.1^2 v^3 + \ldots + 1.1^5 v^6 \right) + 3,500v^9$$

at 8% per annum effective.

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right)\left(1.1v + (1.1v)^2 + (1.1v)^3 + \ldots + (1.1v)^6\right) + 3,500v^9$$

$$= -2,000 + \frac{200}{1.1}\left(0.1v^{2.5} + 0.2v^3\right)a'_{\overline{6}|} + 3,500v^9$$

where the annuity is evaluated at a rate of $\frac{0.08 - 0.1}{1 + 0.1} = -1.818\%$ per annum effective.

$$a'_{\overline{6}|} = \frac{1 - (1 - 0.018181)^{-6}}{-0.018181} = 6.4011$$

and so net present value is

$$-2,000 + \frac{200}{1.1}\left(0.1 \times 1.08^{-2.5} + 0.2 \times 1.08^{-3}\right) \times 6.4011 + 3,500 \times 1.08^{-9} = £31.66m$$

(ii) Accumulated profit at the time of sale is $31.66 \times 1.08^9 = £63.30m$

Many candidates assumed that the accumulation in part (i) was for a single payment.

7 (i) The present value of the assets is equal to the present value of the liabilities. The duration of the assets is equal to the duration of the liabilities.

The spread of the asset terms around the duration is greater than that for the liability terms (or, equivalently, convexity of assets is greater).

(ii) (a) Present value of liabilities (in £m)
$= 10a_{\overline{10|}}$ at 4% $= 10 \times 8.1109 = 81.109$

(b) Duration is equal to $\dfrac{10(Ia)_{\overline{10|}}}{10a_{\overline{10|}}}$ at 4% $= \dfrac{41.9922}{8.1109} = 5.1773$ years

(c) Let the amounts to be invested in the two zero coupon bonds be X and Y.

$$Xv^3 + Yv^{12} = 81.109 \qquad (1)$$

$$3Xv^3 + 12Yv^{12} = 419.922 \qquad (2)$$

(2) less 3 times (1) gives:

$$9Yv^{12} = 176.595$$

$$\Rightarrow Y = \frac{176.595}{9 \times 0.62460} = £31.415m$$

Substituting back into (1) gives:

$$X = \frac{(81.109 - 31.415 \times 0.62460)}{0.88900} = £69.164m$$

(iii) (a) In one year, the present value of the liabilities is:

$$10 + 10a_{\overline{9|}} \text{ at } 5\% = 10 + 10 \times 7.1078 = 81.078$$

Numerator of duration is $10 \times 0 + 10(Ia)_{\overline{9|}} = 332.347$

Duration of liabilities is therefore $\dfrac{332.347}{81.078} = 4.0991$ years

Present value of assets is:

$69.164 \times v^2 + 31.415 \times v^{11} = 69.164 \times 0.90703 + 31.415 \times 0.58468$
$= 81.101$

Duration of assets will be:

$$\frac{2\times 69.164\times v^2 + 11\times 31.415\times v^{11}}{81.101}$$

$$= \frac{2\times 69.164\times 0.90703 + 11\times 31.415\times 0.58468}{81.101} = 4.0383 \text{ years}$$

(b) One of the problems of immunisation is that there is a need to continually adjust portfolios. In this example, a change in the interest rate means that a portfolio that has a present value and duration equal to that of the liabilities at the outset does not have a present value and duration equal to that of the liabilities one year later.

The calculation was often performed well. In part (ii), many explanations were unclear and some candidates seemed confused between DMT and convexity although a correct explanation could involve either of these concepts.

8 (i) $t \le 20$:

$$v(t) = \exp\left(-\int_0^t 0.05 + 0.001s\,ds\right)$$

$$= \exp\left\{-\left[0.05s + \frac{0.001s^2}{2}\right]_0^t\right\}$$

$$= e^{-0.05t - 0.0005t^2}$$

$t > 20$:

$$v(t) = \exp\left\{-\left(\int_0^{20} \delta(s)\,ds + \int_{20}^t 0.05\,ds\right)\right\}$$

$$= v(20)\exp\left\{-[0.05s]_{20}^t\right\}$$

$$= e^{-1.2}e^{1-0.05t} = e^{-0.2-0.05t}$$

(ii) (a) PV = $100v(25) = 100e^{-0.2-0.05\times 25}$

$$= 100e^{-1.45} = £23.46$$

(b) $$100\left(1-\frac{d^{(4)}}{4}\right)^{4\times 25}=100v(25)=23.46$$

$$\Rightarrow d^{(4)}=4\left(1-0.2346^{1/100}\right)=0.05758$$

(iii) PV = $\int_{20}^{25} 30e^{-0.015t}e^{-(0.2+0.05t)}dt$

$$=30e^{-0.2}\int_{20}^{25} e^{-0.065t}dt=\frac{30e^{-0.2}}{-0.065}\left[e^{-0.065t}\right]_{20}^{25}$$

$$=\frac{30e^{-0.2}}{-0.065}\left(e^{-1.625}-e^{-1.3}\right)=28.575$$

Accumulated value = $\dfrac{28.575}{v(25)}=28.575e^{0.2+0.05\times 25}=28.575e^{1.45}=121.82$

9 (i) The one-year spot rate of interest is simply 4% per annum effective.

For two-year spot rate of interest

First we need to find the price of the security, P:

$P=8a_{\overline{2}|}+100v^2$ at 3% per annum effective.

$a_{\overline{2}|}=1.91347 \quad v^2=0.942596$

$$\Rightarrow P=8\times 1.91347+100\times 0.942596=109.5673$$

Let the t-year spot rate of interest be it.

We already know that i_1 = 4%. i_2 is such that:

$$109.56736=\frac{8}{1.04}+\frac{108}{(1+i_2)^2}$$

$$\Rightarrow (1+i_2)^{-2}=0.943287$$

$$\Rightarrow i_2=0.029623 \text{ or } 2.9623\%.$$

For three-year spot rate of interest we need to find the price of the security P:

$P = 8a_{\overline{3}|} + 100v^3$ at 3% per annum effective.

$a_{\overline{3}|} = 2.8286 \quad v^3 = 0.91514$

$\Rightarrow P = 8 \times 2.8286 + 100 \times 0.91514 = 114.1428$

i_3 is such that:

$$114.1428 = \frac{8}{1.04} + \frac{8}{(1.029623)^2} + \frac{108}{(1+i_3)^3}$$

$$\Rightarrow \frac{108}{(1+i_3)^3} = 114.1428 - 15.23860 = 98.9042$$

$\Rightarrow i_3 = 0.02976$ or 2.976%.

(ii) The one year forward rate of interest beginning at the present time is clearly 4%.

The forward rate for one year beginning in one year is $f_{1,1}$ such that:

$$1.04(1 + f_{1,1}) = 1.029623^2 \Rightarrow f_{1,1} = 0.01935 = 1.935\%.$$

The forward rate for one year beginning in two years is $f_{2,1}$ such that:

$$1.029623^2 (1 + f_{2,1}) = 1.02976^3 \Rightarrow f_{2,1} = 0.03003 = 3.003\%.$$

The forward rate for two years beginning in one year is $f_{1,2}$ such that:

$$1.02976^3 = 1.04(1 + f_{1,2})^2$$

$$\Rightarrow f_{1,2} = 0.02468 = 2.468\%$$

(iii) Let the *t*-year "spot rate of inflation" be e_t

For each term $\dfrac{(1+i_t)^t}{(1+e_t)^t} = 1.02^t \;\Rightarrow (1+e_t)^t = \left(\dfrac{1+i_t}{1.02}\right)^t$

$$(1+e_1) = \frac{1.04}{1.02} \Rightarrow e_1 = 1.96\%$$

and so the value of the retail price index after one year would be 101.96

$$(1+e_2)^2 = \left(\frac{1.029623}{1.02}\right)^2 \Rightarrow e_2 = 0.943\%$$

and so the value of the retail price index after two years would be $100(1.00943)^2 = 101.90$

$$(1+e_3)^3 = \left(\frac{1.02976}{1.02}\right)^3 \Rightarrow e_3 = 0.9569\%$$

and so the value of the retail price index after three years would be $100(1.009569)^3 = 102.90$

(iv) The "spot" rates of inflation or the price index values could be used.

Clearly the expected rate of inflation in the first year is 1.96%.

The expected rate of inflation in the second year is:

$$\frac{101.90 - 101.96}{101.96} = -0.06\%.$$

The expected rate of inflation in the third year is:

$$\frac{102.90 - 101.90}{101.90} = 0.98\%$$

A common error was to assume that income only started after three years rather than "starting from the beginning of the third year".

10 (i) The price of the securities might have fallen because interest rates have risen or because their risk has increased (for example credit risk).

(ii)

Date	*Market price of securities (£)*	*X*		*Y*	
		No of securities held before purchases	*Market value of holdings before purchases (£)*	*No of securities held before purchases*	*Market value of holdings before purchases (£)*
1 April 2003	64	–	–	–	–
1 April 2004	65	100	6,500	100	6,500
1 April 2005	60	100	6,000	200	12,000
1 April 2006	65	1,100	71,500	300	19,500
1 April 2007	68	1,100	74,800	400	27,200
1 April 2008	70	1,100	77,000	500	35,000

(iii) (a) Money weighted rate of return is i where:

$$6,400(1+i)^5 + 60,000(1+i)^3 = 77,000$$

try i = 5% LHS = 77,625.70
try i = 4% LHS = 75,278.42

interpolation implies that

$$i = 0.05 - 0.01 \times \frac{77,625.70 - 77,000}{77,625.70 - 75,278.42} = 4.73\%$$

(*Note true answer is 4.736%*)

(b) Time weighted rate of return is i where using figures in above table:

$$(1+i)^5 = \frac{6,000}{6,400}\frac{77,000}{6,000+60,000} = 1.09375.$$

$$\Rightarrow i = 1.808\%$$

(iv) (a) Money weighted rate of return is *i* where:

$$6,400(1+i)^5 + 6,500(1+i)^4 + 6,000(1+i)^3 + 6,500(1+i)^2 + 6,800(1+i)$$
$$= 35,000$$

Put in *i* = 4.73%; LHS = 37,026.95

Therefore the money weighted rate of return for Y is less to make LHS less.

(b) Time weighted rate of return for Y uses the figures in the above table:

$$(1+i)^5 = \frac{6,500}{6,400}\frac{12,000}{6,500+6,500}\frac{19,500}{12,000+6,000}\frac{27,200}{19,500+6,500}\frac{35,000}{27,200+6,800}$$

= 1.09375.

$\Rightarrow i = 1.808\%$

(*Student may reason that the TWRRs are the same and can be derived from the security prices in which case, time would be saved.*)

(v) The money weighted rate of return was higher for X than for Y because there was a much greater amount invested when the fund was performing well than when it was performing badly.

The money weighted rate of return for X (and probably for Y) was more than the time weighted rate of return because the latter measures the rate of return that would be achieved by having one unit of money in the fund from the outset for five years: both X and Y has less in the fund in the years it performed badly.

This question was answered well but examiners were surprised by the large number of candidates who used interpolation or other trial and error methods in part (ii) when the answer had been given in the question. The examiners recommend that students pay attention to the details given in the solutions to parts (iii) and (iv). For such questions, candidates should be looking critically at the figures given/calculated and making points specific to the scenario rather than just making general statements taken from the Core Reading.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2011 examinations

Subject CT1 — Financial Mathematics
Core Technical

Introduction

The attached subject report has been written by the Principal Examiner with the aim of helping candidates. The questions and comments are based around Core Reading as the interpretation of the syllabus to which the examiners are working. They have however given credit for any alternative approach or interpretation which they consider to be reasonable.

T J Birse
Chairman of the Board of Examiners

July 2011

General comments

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

The general performance was slightly worse than in April 2010 but well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii) and Q6(iii) were less well answered than those that just involved calculation. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) We want $1000e^{\int_3^7 \delta(s)\,ds}$

$$=1000\,e^{\left[\int_3^5\left(0.04+0.003s^2\right)ds+\int_5^7(0.01+0.03s)ds\right]}$$

where $\int_3^5\left(0.04+0.003s^2\right)ds=\left[0.04s+0.001\,s^3\right]_3^5$

$= 0.325 - 0.147 = 0.178$

and $\int_5^7\left(0.01+0.03s\right)ds=\left[0.01s+\frac{0.03}{2}s^2\right]_5^7$

$=0.805-0.425=0.380$

$\Rightarrow$ accumulation at $t=7$ is

$1000e^{(0.178+0.380)}=1000e^{0.558}=1{,}747.17$

(ii) $1747.17\left(1-\frac{d^{(12)}}{12}\right)^{4\times 12}=1000$

$\Rightarrow d^{(12)}=0.138692$

2 Forward price of the contract is $K_0=\left(S_0-I\right)e^{\delta T}=\left(68-I\right)e^{0.14\times 1}$

where I is the present value of income during the term of the contract $=2.5e^{-0.14\times 8/12}$

$\Rightarrow K_0=\left(68-2.5e^{-0.14\times 8/12}\right)e^{0.14}=75.59919$

Forward price a new contract issued at time r (3 months) is
$K_r=\left(S_r-I^*\right)e^{\delta(T-r)}=\left(71-I^*\right)e^{0.12\times 9/12}$

(where I^* is the present value of income during the term of the contract)
$=2.5e^{-0.12\times 5/12}\Rightarrow K_{0.25}=\left(71-2.5e^{-0.05}\right)e^{0.09}=75.08435$

Value of original contract = $\left(K_r - K_0\right)e^{-\delta(T-r)}$

$$=\left(75.08435-75.59919\right)e^{-0.12\times 9/12}$$

$$=-0.47053=-47.053p$$

Many candidates failed to incorporate the change in the value of δ. Another common error was in counting the number of months.

3 (i) $$135,000=7,900\times\frac{121.4}{125.6}.\,\nu+8,400\times\frac{121.4}{131.8}\nu^2+8,800\times\frac{121.4}{138.7}\nu^3$$

$$+9,400\times\frac{121.4}{145.3}\nu^4+\left(10,100+151,000\right)\times\frac{121.4}{155.2}\nu^5$$

at i' % where i' = real yield

Approx yield:

$$135,000 = (7635.828+7737.178+7702.379+7853.820+126015.077)\,\nu^5$$

$\Rightarrow i' \simeq 3.1\%$ p.a.

Try $i'=3\%$, RHS $=137434.955$

Try $i'=3.5\%$, RHS $=134492.919$

$$i'=0.035-0.005\times\frac{135000-134492.919}{137434.955-134492.919}$$

$=0.03414$ (i.e. 3.4% p.a.)

(ii) The term:

$$8,800\times\frac{121.4}{RPI\ \text{(June 2008)}}$$

would have a lower value (i.e. the dividend paid on 30 June 2008 would have a lower value when expressed in June 2005 money units). The real yield would therefore be lower than 3.4% p.a.

The most common error on this question was incorrect use of the indices, e.g. many candidates inverted them. Several candidates also had difficulty in setting up the equation of *value. The examiners noted that a large number of final answers were given to excessive levels of accuracy given the approximate methods used.*

4 (i) We can find forward rates $f_{2,1}$ and $f_{2,2}$ from:

$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1})$ and

$(1+y_4)^4 = (1+y_2)^2(1+f_{2,2})^2$

$\Rightarrow (1.033)^3 = (1.032)^2(1+f_{2,1})$

$\Rightarrow f_{2,1} = 3.50029\,\%$ p.a.

and $(1.034)^4 = (1.032)^2(1+f_{2,2})^2$

$\Rightarrow f_{2,2} = 3.60039\,\%$ p.a.

(ii) (a) Price per £100 nominal

$$4\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + 115\,v^3_{3.3\%}$$

$$= 4(0.969932 + 0.938946 + 0.907192) + 115 \times 0.907192$$

$= 115.59$

(b) Let $yc_3 = 3-$year par yield

$$1 = yc_3\left(v_{3.1\%} + v^2_{3.2\%} + v^3_{3.3\%}\right) + v^3_{3.3\%}$$

$$1 = yc_3(0.969932 + 0.938946 + 0.907192) + 0.907192$$

$$\Rightarrow yc_3 = 0.032957$$

i.e. 3.2957% p.a.

5 (i) $\left(1+\dfrac{i^{(2)}}{2}\right)^2 = 1.05 \Rightarrow i^{(2)} = 4.939\%$ (or use tables)

$$g(1-t_1) = \frac{0.06}{1.05} \times 0.80 = 0.0457$$

So $i^{(2)} > g(1-t_1) \Rightarrow$ there is a capital gain on the contract

(ii) Since there is a capital gain, the loan is least valuable to the investor if the repayment is made by the borrower at the latest possible date. Hence, we assume redemption occurs 25 years after issue in order to calculate the minimum yield achieved.

(iii) If A is the price per £100 of loan:

$$A = 100 \times 0.06 \times 0.80\, a_{\overline{25}|}^{(2)} (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) v^{24\frac{10}{12}} \text{ at } 5\%$$

$$= 4.8 \times 1.012348 \times 14.0939 \times (1.05)^{\frac{2}{12}} + \left(105 - 0.35(105 - A)\right) \times 0.29771$$

Hence $A = \dfrac{69.0452 + 20.3187}{1 - 0.35 \times 0.29771} = 99.759$

$\Rightarrow$ Price of loan = £99,759

The majority of this question was well-answered but most candidates struggled with the two month adjustment. This adjustment needs to be directly incorporated into the equation of value. Calculating the price first without adjustment and then multiplying by $(1+i)^{1/6}$ will lead to the wrong answer.

6 (i) MWRR is given by:

$$10.0 \times (1+i) + 5.5 \times (1+i)^{8/12} = 17.1$$

Try 11%, LHS = 16.996

Try 12%, LHS = 17.132

$$\text{MMRR} = 0.11 + 0.01 \times \frac{17.1 - 16.996}{17.132 - 16.996} = 11.8\% \text{p.a.}$$

(ii) TWRR is given by:

$$\frac{8.5}{10.0}\times\frac{17.1}{8.5+5.5}=1+i\Rightarrow i=3.821\%\text{p.a.}$$

(iii) MWRR is higher since fund received a large (net) cash flow at a favourable time (i.e. just before the investment returns increased).

(iv) TWRR is more appropriate. Cash flows into and out of the fund are outside the control of the fund manager, and should not influence the level of bonus payable. TWRR is not distorted by amount and/or timing of cash flows whereas MWRR is.

The calculations in parts (i) and (ii) were generally well done but parts (iii) and (iv) were poorly answered (or not answered at all) even by many of the stronger candidates. In (iii) for example, candidates were expected to comment on the timing of the cashflows for this particular year.

7 (i) Let initial quarterly amount be X. Work in time units of one quarter. The effective rate of interest per time unit is

$$\frac{0.08}{4}=0.02\text{ (i.e.2\% per quarter)}$$

So

$$60,000=X\,a_{\overline{80|}}+100v^{16}a_{\overline{64|}}+100v^{32}a_{\overline{48|}}+100v^{48}a_{\overline{32|}}+100v^{64}a_{\overline{16|}}\ \text{ at }2\%$$

$$\text{(where }a_{\overline{64|}}^{2\%}=\frac{1-v^{64}}{0.02}=35.921415)$$

$$=39.7445X+2,616.695465+1,627.606705+907.1436682+382.3097071$$

$$\Rightarrow X=\frac{60,000-5,533.756}{39.7445}$$

$$=£1,370.41\text{ per quarter}$$

(ii) Interest paid at the end of the first quarter (i.e. on 1 October 1998) is

$$60,000\times0.02=£1,200$$

Hence, capital repaid on 1 October 1998 is

$$1370.41-1200=£170.41$$

Therefore, interest paid on 1 January 1999 is

$$(60000-170.41)\times 0.02 = 1196.59$$

$\Rightarrow$ capital repaid on 1 January 1999 is

$$1370.41-1196.59 = 173.82$$

(iii) Loan outstanding at 1 July 2011 (after repayment of instalment)

$$= 1670.41\, a_{\overline{12|}} + 1770.41\, v^{12}\, a_{\overline{16|}} \text{ at } 2\%$$

$$= 1670.41\times 10.5753 + 1770.41\times 0.78849\times 13.5777$$

= £36,619

Candidates found this to be the most challenging question on the paper. The easiest method was to work in quarters with an effective rate of 2% per quarter. Where candidates worked using a year as the time period the most common error was to allow for an increase to payments of £100 pa when the increases were £400pa when they occurred. In part (i), the examiners were disappointed to see many attempts with incorrect and/or insufficient working end with the numerical answer that had been given in the question. A candidate who claims to have obtained a correct answer after making obvious errors in the working is not demonstrating the required level of skill and judgement and, indeed, is behaving unprofessionally.

Part (iii) was very poorly answered with surprisingly few candidates recognising the remaining loan was simply the present value of the last 28 payments.

8 (i) No, because the spread (convexity) of the liabilities would always be greater than the spread (convexity) of the assets then the 3rd Redington condition would never be satisfied.

(ii) Work in £millions

Let proceeds from four-year bond = X
Let proceeds from 20-year bond = Y

Require PV Assets = PV Liabilities

$$Xv^4 + Yv^{20} = 10v^3 + 20v^6 \qquad (1)$$

Require DMT Assets = DMT Liabilities

$$\Rightarrow 4Xv^4 + 20Yv^{20} = 30v^3 + 120v^6 \qquad (2)$$

(2) − 4 × (1)

$\Rightarrow 16Y\nu^{20} = 40\nu^6 - 10\nu^3$

$$\Rightarrow Y = \frac{40\nu^6 - 10\nu^3}{16\nu^{20}} = \frac{31.61258 - 8.88996}{7.30219} = £3.11175\text{m}$$

From (1):

$$X = \frac{10\nu^3 + 20\nu^6 - Y\nu^{20}}{\nu^4} = \frac{8.88996 + 15.80629 - 1.42016}{0.8548042} = £27.22973\text{m}$$

So amount to be invested in 4-year bond is

$X\nu 4$=£23.27609m

And amount to be invested in 20-year bond is

$Y\nu^{20}$ = £1.42016m

Require Convexity of Assets > Convexity of Liabilities

$$\Rightarrow 20X\nu^6 + 420Y\nu^{22} > 120\nu^5 + 840\nu^8$$

LHS = 981.869 > 712.411 = RHS

Therefore condition is satisfied and so above strategy will immunise company against small changes in interest rates.

<u>Or</u> state that spread of assets (t = 4 to t = 20) is greater than spread of liabilities (t = 3 to t = 6).

Part (i) was poorly answered. In part (ii) many candidates correctly derived X and Y as the proceeds from the two bonds. However, only the better candidates recognised that the amounts to be invested (as required by the question) were therefore Xv^4 *and* Yv^{20}.

9 (i) PV of outgo (£000s)

$$105\left(1 + v^{\frac{1}{2}} + v\right) + 200v^{15} = 366.31 \qquad \text{at } 8\%$$

PV of income

$$\bar{a}_{\overline{1|}}\left[\begin{array}{l}20v+23v^2+26v^3+29v^4 \\ \qquad +29v^5 1.03\left(1+(1.03v)+(1.03v)^2+\ldots+(1.03v)^{24}\right)\end{array}\right]$$

$$=\bar{a}_{\overline{1|}}\left[20v+23v^2+26v^3+29v^4+29v^5 1.03\times\left(\frac{1-(1.03)^{25}v^{25}}{1-1.03v}\right)\right]$$

PV of income

$$=\bar{a}_{\overline{1|}}\{80.193+20.329\times 14.996\}=370.61$$

So NPV is 4.30 (=£4,300)

(ii) The NPV is very small. It is considerably less than the PV of the final year's income $\left(29\times(1.03)^{25}\times\bar{a}_{\overline{1|}}\times v^{29}=6.272\right)$; therefore the DPP must fall in the final year.

We know the DPP exists as the NPV > 0.

So DPP is $29+r$ where

$$366.31=\bar{a}_{\overline{1|}}\times\left\{80.193+20.329\times\left(\frac{1-(1.03)^{24}v^{24}}{1-1.03v}\right)\right\}$$

$$+29\times 1.03^{25}\times v^{29}\times\bar{a}_{\overline{r|}} \qquad \text{at } 8\%$$

$$\Rightarrow \quad 366.31=364.335+6.5169\bar{a}_{\overline{r|}}$$

$$\Rightarrow \quad \bar{a}_{\overline{r|}}=0.3031$$

$$\Rightarrow \quad v^r=0.97668 \quad \Rightarrow \quad r=0.307$$

So the DPP is 29.31.

This question tended to separate out the stronger and weaker candidates. The most common errors in part (i) were discounting for an extra year, not including the one-year annuity factor and incorrectly calculating the geometric progression. Many candidates also lost marks through poorly presented or illegible methods that were therefore difficult for the examiners to follow. Part (ii) was poorly attempted with few candidates completing the question.

10 (i)

$$E(1+i_t)=1.06$$

$$Var(1+i_t)=0.03^2=0.0009$$

$$\Rightarrow 1.06=e^{\left(\mu+\frac{\sigma^2}{2}\right)} \qquad (1)$$

$$0.0009=e^{\left(2\mu+\sigma^2\right)}\left(e^{\sigma^2}-1\right) \qquad (2)$$

$$\Rightarrow \frac{(2)}{(1)^2}=\frac{0.0009}{(1.06)^2}=e^{\sigma^2}-1$$

$$\Rightarrow \sigma^2=Ln\left(\frac{0.0009}{(1.06)^2}+1\right)$$

$$=0.000800676 \quad (\text{and } \sigma=0.0282962)$$

$$\Rightarrow 1.06=e^{\left(\mu+\frac{0.000800676}{2}\right)}$$

$$\therefore \mu=Ln(1.06)-\frac{0.000800676}{2}$$

$$=0.0578686$$

(ii) (a) Working in £m. Assets would accumulate to $14\times1.04=14.56<15$

$\Rightarrow$ Probability = 1.00

(b) The guaranteed portion of the fund would accumulate to

$0.25\times14\times1.04=3.64.$

$\therefore$ non-guaranteed portion needs to accumulate to

$$15 - 3.64 = 11.36$$

$\therefore$ we require probability that

$$(0.75 \times 14)(1 + i_t) < 11.36$$

$$= \Pr(1 + i_t) < 1.081905$$

$$= \Pr(\ln(1 + i_t) < \ln 1.081905)$$

$$= \Pr\left(\frac{\ln(1+i_t) - 0.0578686}{0.0282962} < \frac{\ln 1.081905 - 0.0578686}{0.0282962}\right)$$

$$= \Pr(Z < 0.7370169) \text{ where } Z \sim N(0,1).$$

$$= 0.77$$

(iii) (a) Return is fixed (= 4% p.a.) $\Rightarrow$ variance of return = 0

(b) Return from portfolio = $0.25 \times 0.04 + 0.75\, i_t$

$\therefore$ Variance of return = $0.75^2 Var(i_t)$

$$= 0.75^2 \times 0.0009 = 0.00050625$$

[In monetary terms the variance of return for (iii)(b) will be $(£14m)^2 \times 0.00050625 = £^2 99,225m$ which is equivalent to a standard deviation of £315,000]

This question was generally well answered by those candidates who had left enough time to fully attempt the question. In part (i) the common errors were equating the mean to 0.06 instead of 1.06 and using 0.03 as the variance instead of 0.03^2. Part (ii) was also well answered although many candidates quoted the probability of meeting liabilities when the probability of not meeting the liabilities was asked for. Part (iii) a) was answered well by the candidates who attempted it, while part b) was not answered well. In part (iii) answers given in terms of the annual return and in terms of the monetary amounts were both fully acceptable.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2011 examinations

Subject CT1 — Financial Mathematics
Core Technical

Purpose of Examiners' Reports

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

December 2011

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2011 paper

The general performance was considerably better than in September 2010 and also slightly better than in April 2011. Well-prepared candidates scored well across the whole paper.
As in previous diets, questions that required an element of explanation or analysis, such as Q5(ii) and Q9(iv) were less well answered than those that just involved calculation. Marginal candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 $\left(1-\frac{91}{365}\times 0.08\right)=\left(1+i\right)^{-91/365}$

$0.980055=\left(1+i\right)^{-91/365}$

$1+i=1.08416 \Rightarrow i=8.416\%$

2 Issued by the government
Pay regular interest
Redeemable at a given redemption date
Normally liquid/marketable
More or less risk-free relative to inflation
Low expected return
Low default risk
Coupon and capital payments linked to an index of prices…
… with a time lag.

This type of bookwork question is common in CT1 exam papers. As such, it was disappointing that only about one-sixth of candidates obtained full marks here (which could be achieved by listing six distinct features).

3 Let the annual rate of payment = X

Present value of the payments = $X\ddot{a}_{\overline{4}|}^{(4)}$

Present value of the payments needed from the annuity is:

$8,000\ddot{a}_{\overline{3}|}^{(12)}v^4+3,000\ddot{a}_{\overline{10}|}^{(12)}v^7$

$X\ddot{a}_{\overline{4}|}^{(4)}=8,000\ddot{a}_{\overline{3}|}^{(12)}v^4+3,000\ddot{a}_{\overline{10}|}^{(12)}v^7$

$a_{\overline{3}|}=2.7232 \quad i/d^{(4)}=1.031059$

$a_{\overline{4}|}=3.5460 \quad a_{\overline{10}|}=7.7217 \quad i/d^{(12)}=1.026881 \quad v^4=0.82270 \quad v^7=0.71068$

$X\frac{i}{d^{(4)}}a_{\overline{4}|}=8,000\frac{i}{d^{(12)}}a_{\overline{3}|}v^4+3,000\frac{i}{d^{(12)}}a_{\overline{10}|}v^7$

$$\begin{aligned}X\times 1.031059\times 3.5460 &= 8,000\times 1.026881\times 2.7232\times 0.82270\\ &\quad +3,000\times 1.026881\times 7.7217\times 0.71068\end{aligned}$$

$3.65614X=18,404.80+16.905.51$

$X = £9,657.81$

$\therefore$ Quarterly payment is: £2,414.45.

Many candidates struggled to allow correctly for the Government pension. In some cases, candidates would have scored more marks if they had explained their methodology and their workings more clearly.

4 (i) The fund value on 30 June 2009 will be:

$$1.5 \times 1.01 = 1.515$$

The fund value on 31 December 2009 will be:

$$(1.5 \times 1.01 + 6) \times 1.02 = 7.6653$$

The fund value on 31 December 2010 will be:

$$\left[(1.5 \times 1.01 + 6) \times 1.02 + 4\right] \times 1.05 = 12.2486$$

TWRR is i such that:

$$\frac{1.515}{1.5} \times \frac{7.6653}{7.515} \times \frac{12.2486}{11.6653} = (1+i)^2 = 1.0817$$

$$\therefore i = 4.005\%$$

(This can also be calculated directly from the rates of return for which no marks would be lost).

(ii) The equation of value is:

$$1.5(1+i)^2 + 6.0(1+i)^{1\frac{1}{2}} + 4(1+i) = 12.2486$$

Try i = 4% LHS = 12.146

Try i = 4.5% LHS = 12.22754

Try i = 5% LHS = 12.3094

Interpolate:

$$i = 0.045 + \frac{12.2486 - 12.22754}{12.3094 - 12.22754} \times 0.005$$

= 0.04629 or 4.63%

A common error was to assume that the 1% and 2% rates of return were annualised figures rather than returns over a six-month period.

5 (i) Forward price is accumulated value of the share less the accumulated value of the expected dividends:

$$F = 9.56(1.03)^{9/12} - 0.2(1.03)^{8/12} - 0.2(1.03)^{2/12}$$

= 9.7743 – 0.20398 – 0.20099

= £9.3693

(ii) (a) Although the share will be bought in nine months, it is not necessary to take into account the expected share price. The current share price already makes an allowance for expected movements in the price and the investor is simply buying an instrument that is (more or less) identical to the underlying share but with deferred payment. As such, under given assumptions, the forward can be priced from the underlying share.

(b) An option does not have to be exercised. As such, movements in the share price in one direction will benefit the holder whereas movements in the other direction will not harm him. The more volatile is the underlying share price, the more potential there is for gain for the holder of the option (with limited risk of loss), compared with holding the underlying share. This is not the case for a forward which has to be exercised.

Part (i) was well-answered but part (ii) was very poorly answered. The examiners anticipated that many candidates would find part (ii)(b) challenging but it was pleasing to see some of the strongest candidates give some well-reasoned explanations for this part.

6 (i) $45e^{\int_0^5 (a+bt)dt} = 55 \quad (1)$

$45e^{\int_0^{10} (a+bt)dt} = 120 \quad (2)$

From (1)

$$45\exp\left[at + \frac{bt^2}{2}\right]_0^5 = 55$$

$$\ln\left(\frac{55}{45}\right) = 5a + 12.5b = 0.2007 \qquad (1a)$$

From (2)

$$45\exp\left[at + \frac{bt^2}{2}\right]_0^{10} = 120$$

$$\ln\left(\frac{120}{45}\right) = 10a + 50b = 0.98083$$

From (1a)

$$10a = 0.4014 - 25b \qquad (2a)$$

Substituting into (2a)

$$0.4014 + 25b = 0.98083$$

$$\therefore b = \frac{0.98083 - 0.4014}{25} = 0.02318$$

Substituting into (1a)

$$5a + 12.5 \times 0.02318 = 0.2007$$

$$\therefore a = \frac{0.2007 - 12.5 \times 0.0231772}{5} = -0.01781$$

(ii) $45e^{10\delta} = 120$

$$e^{10\delta} = \frac{120}{45}; 10\delta = \ln\left(\frac{120}{45}\right) = 0.98083$$

$$\therefore \delta = 0.09808 \text{ or } 9.808\ \%$$

7 (i) Expected price of the shares in five years is:

$$X = 2v + 2.5v^2 + 2.5 \times 1.01 \times v^3 + 2.5 \times 1.01^2 v^4 + \ldots$$

$$= 2v + 2.5v^2 + 2.5v^2 \left(1.01v + 1.01^2 v^2 + \ldots.\right)$$

$1.01v + 1.01^2 v^2 + \ldots$ at 8% $= \dfrac{1}{i'}$

where $i' = \dfrac{1.08}{1.01} - 1 = 0.069307$

$$X = 2 \times 0.92593 + 2.5 \times 0.85734 + \frac{2.5 \times 0.85734}{0.069307}$$

= 3.9952 + 30.9254 = 34.9206

Equation of value for the investor is:

$$12(1+i)^5 = 34.9206$$

$i = 0.23817$ or 23.817%

(ii) $12(1+i)^5 = 34.9206 - (34.9206 - 12) \times 0.25$

where i is the net rate of return.

$$12(1+i)^5 = 29.1905$$

$i = 0.1946$ or 19.46%

(iii) The cash flow received in nominal terms is still the same: 29.190495

The equation of value expressed in real terms is:

$12 = \dfrac{29.1905}{(1+f)^5} v^5$ where $f = 0.04$

$$v^5 = \frac{12 \times (1.04)^5}{29.1905} = 0.50016$$

$$\therefore v = 0.50016^{1/5} = 0.87061$$

i = 14.86%

8 (i) The present value of the assets is equal to the present value of the liabilities at the starting rate of interest.

The duration /discounted mean term/volatility of the assets is equal to that of the liabilities.

The convexity of the assets (or the spread of the timings of the asset cashflows) around the discounted mean term is greater than that of the liabilities.

(ii) (a) PV of liabilities is: $£100m\ a_{\overline{40}|}$ at 4%

$= £100\text{m} \times 19.7928$

$= £1{,}979.28\text{m}$

(b) The duration of the liabilities is:

$$\sum_{t=1}^{t=40} 100t\ v^t \Big/ \sum_{t=1}^{t=40} 100v^t \quad \text{(working in £m)}$$

$$= \frac{100 \sum_{t=1}^{t=40} t\ v^t}{1{,}979.28} = \frac{100\left(\mathrm{I}a\right)_{\overline{40}|}}{1{,}979.28} \text{ at 4\%}$$

$$= \frac{100 \times 306.3231}{1{,}979.28} = 15.4765 \text{ years}$$

(iii) Let x = nominal amount of five-year bond
y = nominal amount of 40-year bond.

working in £m

$$1{,}979.28 = xv^5 + yv^{40} \qquad \text{— (1)}$$
$$30{,}632.31 = 5xv^5 + 40yv^{40} \qquad \text{— (2)}$$

multiply equation (1) by 5.

$$9{,}896.4 = 5xv^5 + 5yv^{40} \qquad \text{— (1a)}$$

subtract (1a) from (2) to give

$$20735.91 = 35yv^{40}$$

$$\frac{20{,}735.91}{35 \times v^{40}} = y$$

with $v^{40} = 0.20829$

$y = 2,844.38$

Substitute into (1) to give:

$1,979.28 = Xv^5 + 2,844.38 \times 0.20829$

$v^5 = 0.82193$

$$\frac{1,979.28 - 2,844.38 \times 0.20829}{0.82193} = x = 1,687.28$$

Therefore £1,687.28m nominal of the five-year bond and £2,844.38*m* nominal of the 40-year bond should be purchased.

(iv) (a) The duration of the liabilities is 15.4765

Therefore the volatility of the liabilities is $\dfrac{15.4765}{1.04}$ =14.88125%

The value of the liabilities would therefore change by:

$1.5 \times 0.1488125 \times 1,979.28m = £441.81m$

and the revised present value of the liabilities will be £2,421.09*m*.

(b) PV of liabilities is: £100*m* $a_{\overline{40|}}$ at 2.5%

= £100m× $\dfrac{1-1.025^{-40}}{0.025}$

= £2,510.28*m*.

(c) The PV of liabilities has increased by £531*m*. This is significantly greater than that estimated in (iv) (a). This estimation will be less valid for large changes in interest rates as in this case.

The first three parts were generally well-answered but, in part (iv), the examiners were surprised that so few candidates were able to use the duration to estimate the change in the value of the liability.

9 (i) (a) The theoretical rate of return that could be achieved over a given time period in the future from investment in government bonds today.

(b) The theoretical rate of return that could be achieved between the current time and a given future time from investment in government bonds.

(c) The gross redemption yield that could be theoretically achieved by investing in government bonds of different terms to redemption. The yield curve represents a statistical average gross redemption yield.

(ii)

Time	*Government bond yield*	*Valuation rate of interest*	*P.V factor*
1	0.02	0.031	0.96993
2	0.04	0.052	0.90358
3	0.06	0.073	0.80947
4	0.08	0.094	0.69812
5	0.1	0.115	0.58026

$$PV = 10(0.96993 + 0.90358 + 0.80947 + 0.69812 + 0.58026) + 100 \times 0.58026$$
= 97.6396.

(iii) GRY is such that: $97.6396 = 10a_{\overline{5}|} + 100v^5$

Try 11% $a_{\overline{5}|} = 3.69590$ $v^5 = 0.59345$ RHS = 96.30397

Try 10% $a_{\overline{5}|} = 3.7908$ $v^5 = 0.62092$ RHS = 100 [calculation not necessary]

Interpolate to find i:

$$i = -\frac{97.6396 - 96.30397}{100 - 96.30397} \times 0.01 + 0.11$$

$\Rightarrow i = 0.10639$ or 10.64%

(iv) It is reasonable for the investor to price a corporate bond with reference to the rates of return from government bonds which may be (more or less) risk free.

A risk premium will then need to be added.

It is also not unreasonable that this risk premium rises with term as the uncertainty regarding credit risk rises.

This question proved to be the most difficult on the paper. The examiners had anticipated that some candidates would have difficulty with part (i) but it was disappointing to see the number of candidates who were unable to give even a basic description of a spot rate and a forward rate. Part (iv) was also very poorly answered and whilst it had been anticipated that only the *strongest candidates would make all the relevant points, the examiners were surprised at how many candidates failed to score any marks on this part.*

10 (i) The payback period measures the earliest time at which the project breaks even but takes no account either of interest on borrowings or on cash flows received after the payback period. It is therefore a poor measure of ultimate profitability.

(ii) The present value of preparation costs is (in £m):

$2\bar{a}_{\overline{2|}} @ 4\%$ per annum effective.

$$= 2.\frac{i}{\delta}.a_{\overline{2|}} \qquad \frac{i}{\delta} = 1.019869 \quad a_{\overline{2|}} = 1.8861$$

$$= 2 \times 1.019869 \times 1.8861 = 3.847$$

The present value the stadium building costs is (in £m):

$$200v^{4\frac{1}{2}} + 200 \times 1.05v^{5\frac{1}{2}} + 200 \times 1.05^2 v^{6\frac{1}{2}} + \ldots + 200 \times 1.05^9 v^{13\frac{1}{2}}$$

$$200v^{4\frac{1}{2}}\left(1 + 1.05v + 1.05^2 v^2 + \ldots + 1.05^9 v^9\right)$$

$$= 200v^{4\frac{1}{2}}\left[\frac{1 - 1.05^{10} v^{10}}{1 - 1.05v}\right]$$

with $v = 0.96154 \quad v^{10} = 0.67556 \quad 1.05^{10} = 1.62889 \quad v^{4\frac{1}{2}} = 0.83820$

$$= 200 \times 0.83820 \times \left(\frac{1 - 1.62889 \times 0.67556}{1 - 1.05 \times 0.96154}\right)$$

$$= £1,750.837$$

Present value of admin. costs is (£m):

$$100\ddot{a}^{(12)}_{\overline{2|}} v^{13} \qquad @ 4\%$$

with $\frac{i}{d^{(12)}} = 1.021537 \; v^{13} = 0.60057 \; a_{\overline{2|}} = 1.8861$

$$= 100 \times 1.021537 \times 1.8861 \times 0.60057$$

$$= 115.714$$

Present value of revenue (£m):

$3,300\bar{a}_{\overline{1}|}v^{14}$ with $\dfrac{i}{\delta}=1.019869$ $\quad a_{\overline{1}|}=0.9615 \quad v^{14}=0.57748$

$=3,300\times 1.019869\times 0.9615\times 0.57748$

= 1,868.781

NPV = 1,868.781 – 115.714 – 1,750.837 – 3.847 = –£1.617m.

Therefore should not make a bid.

(iii) One way of dealing with this would be to multiply the NPV of all the revenues and costs that are only received if the bid is won by 0.1.

The costs of preparing the bid would be incurred for certain and therefore not multiplied by 0.1. This adjustment would make it less likely the bid will go ahead because the only certain item is a cost.

This question contained a potential ambiguity regarding the timing of the administration costs. Although the examiners felt that the approach given in the model solution was the most logical, candidates who assumed that the administration costs were only payable during 2025 were given full credit. This question was answered well and it was very pleasing to see that (a) candidates managed their time efficiently and so left enough time to make a good attempt at the question with the most marks and (b) candidates who made calculation errors still clearly explained their method and so were able to pick up significant marks for their working.

END OF EXAMINER'S REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and who are using past papers as a revision aid, and also those who have previously failed the subject. The Examiners are charged by Council with examining the published syllabus. Although Examiners have access to the Core Reading, which is designed to interpret the syllabus, the Examiners are not required to examine the content of Core Reading. Notwithstanding that, the questions set, and the following comments, will generally be based on Core Reading.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report. Other valid approaches are always given appropriate credit; where there is a commonly used alternative approach, this is also noted in the report. For essay-style questions, and particularly the open-ended questions in the later subjects, this report contains all the points for which the Examiners awarded marks. This is much more than a model solution – it would be impossible to write down all the points in the report in the time allowed for the question.

T J Birse
Chairman of the Board of Examiners

July 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2012 paper

The general performance was broadly similar to the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q2(iii), Q5(iii) and Q6(iv) were less well answered than those that just involved calculation. Marginal candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) Price, P, of £100 nominal stock is:

$$P = 3v_{y_1} + 3v_{y_2}^2 + 103v_{y_3}^3$$

where

$y_1 = 0.041903$

$y_2 = 0.043625$

$y_3 = 0.045184$

$\Rightarrow P = 95.845$

And gross redemption yield, i%, solves:

$$95.845 = 3a_{\overline{3}|} + 100v^3 \text{ at } i\%$$

Try 4% RHS = 97.225
5% RHS = 94.554

$$\Rightarrow i = 0.04 + 0.01 \times \frac{97.225 - 95.845}{97.225 - 94.554}$$

$= 0.0452$

i.e. 4.5% p.a.

(ii) y_1, y_2 and y_3 as above. $y_4 = 0.046594$

$$1 = (yc_4)\left(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + v_{Y_4}^4\right) + v_{y_4}^4$$

$$\Rightarrow 1 = yc_4 \times 3.587225 + 0.8334644$$

$$\Rightarrow yc_4 = 0.04642 \text{ i.e. } 4.642\% \text{ p.a.}$$

2 (i) TWRR, i, is given by:

$$\frac{2.9}{2.3} \times \frac{4.2}{2.9 + 1.5} = 1 + i \Rightarrow i = 0.204 \text{ or } 20.4\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$2.3 \times (1+i) + 1.5(1+i)^{8/12} = 4.2$$

Then, we have:

$$\left.\begin{array}{l} i = 12\% \quad \Rightarrow \quad LHS = 4.1937 \\ i = 13\% \quad \Rightarrow \quad LHS = 4.2263 \end{array}\right\} \Rightarrow i = 0.12 + (0.13 - 0.12) \times \left(\frac{4.2 - 4.1937}{4.2263 - 4.1937} \right)$$

= 0.122

or 12.2% p.a.

(iii) The MWRR is lower as fund performs better before the cash inflow than after. Then, as the fund is larger after the cash inflow on 1 May 2011, the effect of the poor investment performance after this date is more significant in the calculation of the MWRR.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. This type of explanation is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

3 (i) Let R = annual repayment

$$500,000 = R\, a_{\overline{10}|}^{9\%} = R \times 6.4177$$

$$\Rightarrow R = 77,910.04$$

and total interest $= 10 \times 77,910.04 - 500,000$

$$= 279,100$$

(ii) (a) Capital outstanding at beginning of 8th year is:

$$77910.04\, a_{\overline{3}|}^{9\%} = 77909.53 \times 2.5313$$

$$= 197,213.28$$

Let R' be new payment per annum then

$$R'\, a_{\overline{4}|}^{(4)} = R' \times 1.043938 \times 3.0373 = 197,213.28$$

$$\Rightarrow \quad R' = 62,196.62$$

and quarterly payment is £15,549.16

(b) Interest content of 2nd quarterly payment is:

$$15549.16 \times \left(1 - v_{12\%}^{3\frac{3}{4}}\right) = 5383.41$$

[Or Capital in 1st quarterly payment is

$$15549.16 - 197213.28 \times \left(1.12^{1/4} - 1\right) = 9,881.77$$

So capital outstanding after 1st quarterly payment

$$= 197213.28 - 9881.77 = 187331.51$$

$\Rightarrow$ Interest in next payment is

$$187331.51 \times \left(1.12^{\frac{1}{4}} - 1\right) = 5383.41]$$

Generally answered well but a number of candidates made errors in calculating the remaining term in part (ii)

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The current value of the forward price of the old contract is:

$$7.20 \times (1.025)^4 - 1.20\, a_{\overline{5}|}^{2\frac{1}{2}\%}$$

whereas the current value of the forward price of a new contract is:

$$10.45 - 1.20\, a_{\overline{5}|}^{2\frac{1}{2}\%}$$

Hence, current value of old forward contract is:

$$10.45 - 7.20 \times (1.025)^4 = £2.5025$$

(iii) The current value of the forward price of the old contract is:

$$7.20(1.025)^4(1.03)^{-9} = 6.0911$$

whereas the current value of the forward price of a new contract is:

$$10.45(1.03)^{-5} = 9.0143$$

$\Rightarrow$ current value of old forward contract is:

$$9.0143 - 6.0911 = £2.9232$$

This was the most poorly answered question on the paper but well-prepared candidates still scored full marks. Some candidates in part (ii) assumed that the dividend income was received during the lifetime of the forward contract. Whilst the examiners did not believe that such an approach was justified, candidates who assumed this alternative treatment of the income were not penalised. It was very clear that the poor performance on the question was not as result of this alternative interpretation.

5 (i) The equation of value is:

$$1309.5 = 100\left(a_{\overline{5}|}^{(4)} + (1.05)^5 v^5 a_{\overline{5}|}^{(4)} + \ldots + (1.05)^{20} v^{20} a_{\overline{5}|}^{(4)}\right) - 12a_{\overline{25}|}^{(4)}$$

Rearranging:

$$1309.5 = 100a_{\overline{5}|}^{(4)}\left(\frac{1-(1.05v)^{25}}{1-(1.05v)^5}\right) - 12a_{\overline{25}|}^{(4)}$$

At 9%, RHS is:

$$100 \times 1.033144 \times 3.8897 \times \left(\frac{1-\left(\frac{1.05}{1.09}\right)^{25}}{1-\left(\frac{1.05}{1.09}\right)^5}\right) - 12 \times 1.033144 \times 9.8226$$

$$= 401.8570 \times \frac{0.607292}{0.170505} - 121.7779$$

$= 1309.53 \Rightarrow$ IRR is 9% p.a.

(ii) For Project B the equation of value is

$$1000 = 85\ddot{a}_{\overline{20}|}^{(4)} + v^{20}\ 90\ddot{a}_{\overline{5}|}^{(4)}$$

Roughly $1000 \simeq 85a_{\overline{25}|} \Rightarrow i \simeq 7\%$

At 7% RHS is 1039.05
= 85 × 1.043380 × 10.5940 + 0.25842 × 1.043380 × 4.1002 × 90

8% RHS is 956.78
= 85 × 1.049519 × 9.8181 + 0.21455 × 1.049519 × 3.9927 × 90

$\Rightarrow i \simeq 7.5\%$ p.a.

(iii) Project A is more attractive since it has the higher IRR. However, the investor will also need to take into account other factors such as:

- the outlay is much higher for Project A than Project B
- the interest rate at which the investor might need to borrow at to finance a project since it will affect the net present values and discounted payback periods of the projects
- the risks for each project that the rents and expenses will not be those assumed in the calculations.

In part (i) candidates were asked to demonstrate that the internal rate of return was a given value. In such questions, candidates should set up the equation of value and clearly show each stage of their algebra and their calculations (including the evaluation of all factors that make up the equation). Many candidates claimed that they had shown the correct answer despite obvious errors and/or insufficient working. Candidates who tried to create a "proof" where the arguments didn't follow logically gained few marks. In this type of question, if you can't complete a proof, it is better to show how far you have got and be open about being unable to proceed further. This will generally gain more intermediate markst.

Part (ii) was answered well but in part (iii) few candidates came up with any of the other factors that should be considered.

6 (i) Price per £100 nominal is given by:

$$P = 5 \times a_{\overline{18}|}^{3.158\%} + 100v_{3.158\%}^{18} = 5 \times \left(\frac{1 - v_{3.158\%}^{18}}{0.03158} \right) + 100v_{3.158\%}^{18} = 125.00$$

(ii) As coupons are payable annually and the gross redemption yield is equal to the annual coupon rate, the new price per £100 nominal is £100.

i.e. $P = 5a_{\overline{13}|}^{5\%} + 100v_{5\%}^{13} = 5\left(\frac{1 - v_{5\%}^{13}}{0.05} \right) + 100v_{5\%}^{13} = 100.00$

(iii) Equation of value is:

$$125.00 = 5a_{\overline{5}|} + 100v^5 \Rightarrow i = 0\%$$

Thus, the investor makes a return of 0% per annum over the period.

(iv) Longer-dated bonds are more volatile.

Thus, as a result of the rise in gross redemption yields from 3.158% per annum on 1 March 2007 to 5% on 1 March 2012, the fall in the price of the bond would be greater.

Thus, as the income received over the period would be unchanged, the overall return achieved would be reduced (as a result of the greater fall in the capital value).
[In fact, the price on 1 March 2007 would have been £133.91 per £100 nominal falling to £100 per £100 nominal on 1 March 2012.

i.e. in this case, we need to find *i* such that $133.91 = 5a_{\overline{5}|} + 100V^5 \Rightarrow i < 0\%$.]

The first three parts were generally well-answered although relatively few candidates noticed that parts (ii) and (iii) could be answered quickly and consequently many candidates made avoidable calculation errors.

7 (i) $E(1+i) = e^{\mu + \frac{1}{2}\sigma^2}$

$= e^{0.05 + \frac{1}{2} \times 0.004}$

$= 1.0533757$

$\therefore E[i] = 0.0533757$ since $E(1+i) = 1 + E(i)$

Let A be the accumulation of £5000 at the end of 20 years

then $E[A] = 5000\, \ddot{s}_{\overline{20|}}$ at rate $j = 0.0533757$

$$= 5000 \frac{\left((1+j)^{20} - 1\right)}{j} \times (1+j)$$

$$= 5000 \frac{\left(1.0533757^{20} - 1\right)}{0.0533757} \times 1.0533757$$

$$= £180{,}499$$

(ii) Let the accumulation be S_{20}

S_{20} has a log-normal distribution with parameters 20μ and $20\sigma^2$

$$\therefore E[S_{20}] = e^{20\mu + \frac{1}{2}\left(20\sigma^2\right)} \qquad \left\{\text{or } (1+j)^{20}\right\}$$

$$= \exp(20 \times 0.05 + 10 \times 0.004)$$

$$= e^{1.04} \quad = 2.829217$$

$$\ln S_{20} \sim N\left(20\mu, 20\sigma^2\right)$$

i.e. $\ln S_{20} \sim N(1,\ 0.08)$

$$P\left(S_{20} > e^{1.04}\right) = P(\ln S_{20} > 1.04)$$

$$= P\left(Z > \frac{1.04 - 1}{\sqrt{0.08}}\right) \quad \text{where} \quad Z \sim N(0,1)$$

$$= P(Z > 0.14) = 1 - \Phi(0.14)$$

$$= 1 - 0.56 = 0.44$$

Questions regarding annual investments are comparatively rarely asked on this topic and students seemed to struggle with part (i). Part (ii) was answered better in general than equivalent questions in previous exams.

8 (i) for $t > 8$

$$v(t) = \exp -\left\{\int_0^5 0.04 + 0.003t^2 dt + \int_5^8 0.01 + 0.03t dt + \int_8^t 0.02 dt\right\}$$

$$= \exp -\left\{\left[0.04t + 0.001t^3\right]_0^5 + \left[0.01t + 0.015t^2\right]_5^8 + \left[0.02t\right]_8^t\right\}$$

$$= \exp -\left\{0.2 + 0.125 + 0.01 \times 3 + 0.015\left(8^2 - 5^2\right) + 0.02t - 0.02 \times 8\right\}$$

$$= \exp -\left\{0.325 + 0.615 + 0.02t - 0.16\right\}$$

$$= e^{-(0.78 + 0.02t)}$$

Hence PV of £1,000 due at t = 10 is:

$$1000 \times \exp -\left(0.78 + 0.02 \times 10\right) = £375.31$$

(ii) $$1000\left(1 - \frac{d^{(4)}}{4}\right)^{4 \times 10} = 375.31$$

$$\left(1 - \frac{d^{(4)}}{4}\right)^{40} = \frac{375.31}{1000}$$

$$d^{(4)} = 4\left(1 - \left(\frac{375.31}{1000}\right)^{1/40}\right)$$

$$\underline{= 0.09681}$$

(iii) $$PV = \int_{10}^{18} \rho(t) v(t) dt$$

$$= \int_{10}^{18} 100e^{0.01t} \times e^{-(0.78 + 0.02t)}$$

$$= 100e^{-0.78} \int_{10}^{18} e^{-0.01t} dt$$

$$= 100e^{-0.78} \left\{\left[\frac{e^{-0.01t}}{-0.01}\right]_{10}^{18}\right\}$$

$$= \frac{100}{0.01} e^{-0.78} \left(e^{-0.1} - e^{-0.18} \right)$$

$$= £318.90$$

Parts (i) and (ii) were answered well. Some candidates made errors in part (iii) by not discounting the payment stream back to time 0.

9

(i)

$$PV = 100 \times 0.35 \left(1.03v + 1.03 \times 1.05v^2 + 1.03 \times 1.05 \times 1.06v^3 + 1.03 \times 1.05 \times 1.06^2 v^4 + \cdots \right)$$

$$= 35 \left(1.03v + 1.03 \times 1.05v^2 + \frac{1.03 \times 1.05 \times 1.06v^3}{1 - 1.06v} \right) \quad @ \ 8\%$$

$$= 35 \left(\frac{1.03}{1.08} + \frac{1.03 \times 1.05}{1.08^2} + \frac{1.03 \times 1.05 \times 1.06}{1.08^3} \times \frac{1.08}{0.02} \right)$$

$$= 35 \left(0.95370 + 0.92721 + 49.14223 \right)$$

$$= £1785.81$$

(ii) Real rate of return is i such that:

$$1720 = 35 \left(1.03 \times \frac{110}{112.3} v + 1.03 \times 1.05 \times \frac{110}{113.2} v^2 + 1.03 \times 1.05 \times 1.06 \times \frac{110}{113.8} v^3 \right) + 1800 \times \frac{110}{113.8} v^3$$

$$= 35 \left(1.0089047v + 1.050928v^2 + 1.108110v^3 \right) + 1739.894552v^3$$

$$= 35.3116645v + 36.78248v^2 + 1778.678402v^3$$

For initial estimate, assume all income received at end of 3 years:

$1720 \approx 1850.77$v3

$\Rightarrow v \approx 0.9758696 \quad \Rightarrow i \approx 2.4727$

Try $i = 2.5\%$, RHS = 1721.14 ≈ 1720

so $i = 2.5\%$

Most candidates made a good attempt at part (i) although slight errors in setting up the equation and/or in the calculation were common. Many candidates struggled with setting up the required equation in part (ii).

10 (i) Working in 000's

$$PV_L = 200a_{\overline{20|}} + 300v^{15} \quad @ \ 8\%$$
$$= 200 \times 9.818147 + 300 \times 0.315242$$
$$= 2058.20199$$

i.e. £2,058,201.99

(ii)

$$DMT_L = \frac{200v + 200 \times 2v^2 + 200 \times 3v^3 + \cdots + 200 \times 20v^{20} + 300 \times 15v^{15}}{200a_{\overline{20|}} + 300v^{15}}$$
$$= \frac{200(Ia)_{\overline{20|}} + 300 \times 15v^{15}}{2058.20199} \quad @ \ 8\%$$
$$= \frac{200 \times 78.9079 + 300 \times 15 \times 0.31524}{2058.20199}$$
$$= \frac{17200.175}{2058.20199} = 8.3569 \text{ years}$$

(iii) Redington's first two conditions are:

$$\Rightarrow PV_L = PV_A$$
$$\Rightarrow DMT_L = DMT_A$$

Let the nominal amount in securities A and B be X and Y respectively.

$$PV_A = PV_L \Rightarrow X\left(0.09a_{\overline{12|}} + v^{12}\right) + Y\left(0.04a_{\overline{30|}} + v^{30}\right) = 2058201.99 \ @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 7.5361 + 0.39711\right) + Y\left(0.04 \times 11.2578 + 0.09938\right)$$
$$\Rightarrow 1.075361X + 0.549689Y = 2058201.99$$
$$\Rightarrow X = \frac{2058201.99 - 0.549689Y}{1.075361}$$

$$DMT_A = DMT_L \Rightarrow \frac{X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right)}{2058201.99} = 8.3569$$
$$\Rightarrow X\left(0.09(Ia)_{\overline{12|}} + 12v^{12}\right) + Y\left(0.04(Ia)_{\overline{30|}} + 30v^{30}\right) = 17200175 \quad @ \ 8\%$$
$$\Rightarrow X\left(0.09 \times 42.17 + 12 \times 0.39711\right) + Y\left(0.04 \times 114.7136 + 30 \times 0.09938\right) = 17200175$$
$$\Rightarrow 8.56066X + 7.56986Y = 17200175$$
$$\Rightarrow 8.56066 \times \frac{(2058201.99 - 0.549689Y)}{1.075361} + 7.56986Y = 17200175$$
$$\Rightarrow 3.19394Y = 815370.9$$
$$\Rightarrow Y = 255287, \ X = 1783470$$

Hence company should purchase £1,783,470 nominal of security A and £255,287 nominal of security B for Redington's first two conditions to be satisfied.

(iv) Redington's third condition is that the convexity of the asset cash flow series is greater than the convexity of the liability cash flow series. Therefore the convexities of the asset cash flows and the liability cash flows will need to be calculated and compared.

Generally well answered but candidates' workings in part (iii) were often unclear which made it difficult for examiners to award marks when calculation errors had been made.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2012 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

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For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

D C Bowie
Chairman of the Board of Examiners

December 2012

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2012 paper

The general performance was of a lower standard compared with the previous two exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q3(ii), Q4(ii) and Q9(iii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations. At the other end of the spectrum, there was a difficulty for many candidates when it came to answering questions involving introductory ideas.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 Let d be the annual simple rate of discount.

The discounted value of 100 in the deposit account would be x such that:

$$x = 100(1.04)^{-91/365} = 99.0269$$

$\therefore$ to provide the same effective rate of return a treasury bill that pays 100 must have a price of 99.0269 and $100\left(1 - \frac{91}{365} \times d\right) = 99.0269$

$$\therefore\ 1 - \frac{91}{365} \times d = \frac{99.0269}{100} = 0.990269$$

$$d = (1\text{-}0.990269) \times \frac{365}{91} = 0.03903$$

Many candidates scored full marks on this question but many others failed to score any marks at all. Some candidates incorrectly used (1-*nd*) *as an accumulation factor*

2 (i) $e^{-\delta/4} = 1 - \frac{0.08}{4} = 0.98 \quad \therefore \delta = 0.080811$

(ii) $(1+i)^{-1} = \left(1 - \frac{0.08}{4}\right)^4 = 0.92237 \therefore\ i = 0.084166$

(iii) $\left(1 - \frac{d^{(12)}}{12}\right)^{12} = \left(1 - \frac{0.08}{4}\right)^4 = 0.92237 \therefore d^{(12)} = 0.080539$

A lot of marginal candidates scored very badly on this question even though it was covering an introductory part of the syllabus.

3 (i) $\left(1+i\right)^{2.5} = \frac{140}{120} \times \frac{600}{140+200} = 2.05882$

$\therefore\ 1 + i = 1.33490$

$\therefore\ i = 33.49\%$ p.a. effective.

(ii) The money weighted rate of return weights performance according to the amount of money in the fund. The fund was performing better after it had been given the large injection of money on 1/1/2011.

Part (i) was answered well. The type of explanation asked for in part (ii) is commonly asked for in CT1 exams. To get full marks, candidates should address the specific situation given in the question rather than just repeat the bookwork.

4 (i) Present value of dividends, I, is:

$$I = \left(v^{1/4} + v^{1/2} + v^{3/4}\right)$$

Calculated at $i'\%$ when $(1+i') = (1.04)^2 = 1.0816$

So $I = 1.0816^{-1/4} + 1.0816^{-1/2} + 1.0816^{-3/4}$
$= 2.88499$

Hence, forward price, F, is:

$$F = (10 - 2.88499)(1+i')^{10/12} \text{ at } 8.16\%$$
$$= \left(10 - 2.88499\right) \times 1.0816^{10/12} = £7.5956$$

(ii) The price of the forward can be determined from the price of the share (for which it is a close substitute). The forward is like the share but with delayed settlement and without dividends.

5 (i) The characteristics of a Eurobond are:

- Medium- or long-term borrowing
- Unsecured
- Regular coupon payments
- Redeemed at par
- Issued and traded internationally/not in the jurisdiction of any one country
- Can be denominated in any currency (e.g. not the currency of issuer)
- Tend to be issued by large companies, governments or supra-national organisations
- Yields depend on issue size and issuer (or marketability and risk)
- Issue characteristics may vary – market free to allow innovation

(ii) (a) The characteristics of a certificate of deposit are:

- Tradable certificate issued by banks stating that money has been deposited
- Terms to maturity between one and six months
- Interest payable on maturity/issued at a discount
- Security and marketability will depend on issuing bank
- Active secondary market

(b) Answer is i such that $(1.01)^{12}\left(1+\frac{i^{(12)}}{12}\right)^{12} = (1.02)^{24}$ giving

$i^{(12)} = 36.119\%$

6 (i) Amount of loan is:

$100(Ia)_{\overline{10}|} + 100a_{\overline{10}|}$ at 6% p.a.

= 100 ×36.9624 + 100 × 7.3601
= 3696.24 + 736.01 = £4,432.25

(ii) (a) the o/s loan after sixth instalment is:

$100(Ia)_{\overline{4}|} + 700\,a_{\overline{4}|}$

=100 × 8.4106 + 700 × 3.4651 = 841.06 + 2425.57 = £3,266.64

The interest component is therefore:

0.06 × 3266.64 = £196.00

(b) The capital component =

800-196.00 = £604.00

(iii) The capital remaining after the seventh instalment is
3266.64 – 604.00 = 2662.64

Let the new instalment = X

$Xa_{\overline{8}|} = 2,662.64$ at 8%

$a_{\overline{8}|} = 5.7466;\ X = 2,662.64/5.7466 = £463.34$

7 (i) Expected annual interest rate in both ten-year periods =
0.04 ×0.3 + 0.06 × 0.7 = 0.054 or 5.4%

Amount of the investment would be X such that:

$X\,(1.054)^{20} = 200,000$

$\therefore X = £69,858.26$

(ii) Expected accumulation factors in both ten-year periods are:

$0.3\,(1.04)^{10} + 0.7(1.06)^{10} = 1.697667$

The accumulation factors in each ten-year period are independent.

Therefore the expected accumulation is:

$69{,}858.26 \times 1.697667 \times 1.697667$

$= £201{,}336.55$

Therefore the value of investment over and above £200,000 = £1,336.55.

(iii) The extreme outcomes for the investment are:

$69{,}858.26 \times 1.04^{20} = 153{,}068.06$
$69{,}858.26 \times 1.06^{20} = 224{,}044.91.$

Therefore the range is: £70,976.85

Many candidates struggled with this question and seemed to have difficulty particularly with part (ii). Part (iii) was also badly answered even though part (ii) was not needed to answer part (iii).

8 (i) $t \le 9$

$$\begin{aligned} v(t) &= e^{-\int_0^t (0.03+0.01s)\,ds} \\ &= e^{-\left[0.03s+\frac{0.01s^2}{2}\right]_0^t} \\ &= e^{-\left[0.03t+0.005t^2\right]} \end{aligned}$$

$t > 9$

$$\begin{aligned} V(t) &= e^{-\left[\int_0^9 \delta(s)\,ds+\int_9^t 0.06\,ds\right]} \\ &= V(9).e^{0.06(t-9)} \\ &= e^{-0.675}.e^{-0.06(t-9)} \\ &= e^{-(0.135+0.06t)} \end{aligned}$$

(ii) (a) $PV = 5{,}000\, e^{-(0.135+0.06\times 15)}$

$= 5{,}000\, e^{-1.035}$

$= £1{,}776.13$

(b) $1{,}776.13\, e^{\delta\times 15} = 5{,}000$

$e^{\delta\times 15} = 2.81511$

$15\delta = \ln 2.81511$

$\delta = \dfrac{\ln 2.81511}{15} = 0.0690$

(iii)

$$P.V. = \int_{11}^{15} e^{-(0.135+0.06t)} \times 100\, e^{-0.02t}\, dt$$

$$= \int_{11}^{15} 100\, e^{-0.135-0.08t}\, dt$$

$$= 100\, e^{-0.135} \left[\frac{e^{-0.08t}}{-0.08} \right]_{11}^{15}$$

$$= 100\, e^{-0.135}(5.18479 - 3.76493)$$

$$= 124.055$$

Generally answered well but some candidates lost marks in part (i) by not deriving the discount factor for t < 9.

9 (i) Expectations theory: yields on short and long-term bonds are determined by expectations of future interest rates as it is assumed that a long-term bond is a substitute for a series of short-term bonds.
[If interest rates are expected to rise (fall) long-term bonds will have higher (lower) yields that short-term bonds.]

Liquidity preference: it is assumed that investors have an inherent preference for short-term bonds because interest-rate sensitivity is lower. As such, (there is an upward bias on the expectations-based yield curve) and longer-term bonds will offer a higher expected return than implied by expectations theory on its own. *N.B. the part in brackets is not in core reading.*

Market segmentation: bonds of different terms to redemption are attractive to different investors with different liabilities.

The supply of bonds of different terms to redemption will depend on the strategy of the relevant issuer. The term structure is determined by the interaction of supply and demand in each term-to-redemption segment.

(ii) Duration = $\dfrac{\sum tC_t v^t}{\sum C_t v^t} = \dfrac{4\times(Ia)_{\overline{n}|}+100n v^n}{4\times a_{\overline{n}|}+100v^n}$

For n = 1 to 5. Clearly duration on one-year bond is one year.

Term	$(Ia)_{\overline{n}\rvert}$	$100v^n$	$a_{\overline{n}\rvert}$	$n100v^n$	$4(Ia)_{\overline{n}\rvert}$
3	5.3580	86.384	2.7232	259.152	21.432
5	12.5664	78.353	4.3295	391.765	50.2656

Duration of three-year bond:

$$\frac{21.432+259.152}{4\times 2.7232+86.384} = 2.884 \text{ years}$$

Duration of five-year bond:

$$\frac{50.2656+391.765}{4\times 4.3295+78.353} = 4.620 \text{ years}$$

(iii) The duration of a bond is the average time of the cashflows weighted by present value. The coupon payments of the 8% coupon bond will be a higher proportion of the total proceeds than for the 4% coupon bond. Thus, a greater proportion of the total proceeds of the 8% coupon bond will be received before the end of the term. The average time of the cashflows will be shorter and hence the duration will be lower.

(iv) **Option 1**

The equation of value would be:

$$95 = 4a_{\overline{4}|} + 79v^5$$

The rate of return is zero (incoming and outgoing cash flows are equal).

Option 2

The equation of value would be:

$$95 = 4a_{\overline{4}|} + v^4 a_{\overline{8}|} + 100v^{12}$$

$i = 2.5\%$

RHS = 4 × 3.762 + 0.90595 × 7.1701 + 100 × 0.74356
= 15.0479 + 6.4958 + 74.3556 = 95.8993

$i = 3\%$

RHS = 4 × 3.7171 + 0.88849 × 7.0197 + 100 ×0.70138
= 14.8684 + 6.2369 + 70.1380
= 91.2433

By interpolation:

$$i = 0.005 \times \left(\frac{95.8998 - 95}{95.8998 - 91.2433} \right) + 0.025$$

= 0.025966 or 2.6% per annum effective.

Hence Option 2 would provide the higher rate of return

(v) Two of the following:

- Option 2 creates a higher duration bond which might not be suitable for the investor
  ….e.g. alternative investments may be available in the longer term
- The credit risk over the longer duration may be greater
- The inflation risk over the longer duration may be greater
- There may be tax implications because of the differing capital and income combinations.
- the institution could reinvest the proceeds from option 1 at whatever rate of return prevails.

Part (i) was often poorly answered even though this was bookwork and candidates also struggled with part (ii). In part (ii) it is important to include the correct units for the duration (in this case, years). Most candidates made a good attempt at part (iv) even if some made calculation errors (e.g. in the calculation of the outstanding term of the bond under Option 2). Marginal candidates scored badly on parts (iii) and (v).

10 (i) The payback period simply looks at the time when the total incoming cash flows are greater than the total outgoing cash flows. It takes no account of interest at all.

Though the discounted payback period takes account of interest that would have to be paid on loans, it only looks at when loans used to finance outgoing cash flows would be repaid and not at the overall profitability of the projects.

(ii) (a) Outgoing cash flow = £3m

In £m, at time t, total incoming cash flows are £0.64t

We need t such that $3 = 0.64t$

$$t = {}^{3}\!/_{0.64} = 4.6875 \text{ years}$$

(b) Present value of incoming cash flows at time t is:

$$0.64\bar{a}_{\overline{t}|} = 0.64\left(\frac{1-v^t}{\delta}\right) \text{where } \delta = 0.039221$$

Require t such that:

$$0.64\left(\frac{1-v^t}{0.039221}\right) = 3$$

$1 - v^t = 0.183848$
$v^t \quad = 0.816152$
$t \ln v = \ln 0.816152$

$$t = \frac{\ln 0.816152}{\ln v}$$

$$= -\frac{-0.203155}{-0.039221} = 5.1798 \text{ years}$$

(iii) Crossover point is the rate of interest at which the n.p.v. of the two projects is equal. As the present value of the cash outflows for both projects is the same at all rates of interest, the crossover point is the rate of interest at which the present value of the cash inflows from both projects is equal.

P.V of cash inflows from Project B = $0.64\bar{a}_{\overline{6}|}$
P.V of cash inflows from Project A =

$$0.5\, v^{1/2} + 1.1 \times 0.5 \times v^{1^{1/2}} + \cdots + 1.1^5 \times 0.5 \times v^{5^{1/2}}$$
$$= 0.5v^{1/2}\left[\frac{1-1.1^6 \times v^6}{1-1.1\times v}\right]$$

Therefore require i such that:

$$0.64\ \bar{a}_{\overline{6}|} - 0.5\ v^{1/2}\left[\frac{1-1.1^6 v^6}{1-1.1v}\right] = 0$$

Let i = 4%

$a_{\overline{6}|} = 5.2421 \quad i/\delta = 1.019869$

$\nu^{1/2} = 0.98058 \quad \nu = 0.96154$

$\nu^6 = 0.79031$

$1.1^6 = 1.77156$

$$\text{LHS} = 0.64 \times 5.2421 \times 1.019869 - 0.5 \times 0.98058 \left[\frac{1 - 1.77156 \times 0.79031}{1 - 1.1 \times 0.96154} \right]$$

= 3.4216 – 0.49029 ×6.93490 = 3.4216 – 3.4001
= 0.0215

Let i = 0%

$$\text{LHS} = 0.64 \times 6 - 0.5 \times \left[\frac{1 - 1.77156}{1 - 1.1} \right] = 3.84 - 3.8578 = -0.0178$$

Given that NPV of Project A is greater than that of project B at 0% per annum effective and the reverse is true at 4% per annum effective, the NPV of the two projects must be equal at some point between 0% and 4%.

(iv) **Project A**

Duration is:

$$\frac{\nu^{1/2} 0.5(0.5 + 1.1 \times 1.5\nu + 1.1^2 \times 2.5\nu^2 + 1.1^3 \times 3.5\nu^3 + 1.1^4 \times 4.5\nu^4 + 1.1^5 \times 5.5 \times \nu^5)}{0.49029 \times 6.93490}$$

Term in brackets is

0.5 + 1.58654 + 2.79678 + 4.14139 + 5.63183 + 7.28047 = 21.93702.

$$\therefore \text{ Duration} = \frac{0.98059 \times 0.5 \times 21.93702}{0.49029 \times 6.93490} = 3.163 \text{ years}$$

Project B

$$\text{Duration is}: \frac{0.64\int_0^6 t\nu^t dt}{0.64\int_0^6 \nu^t dt} = \frac{(\bar{I}\bar{a})_{\overline{6}|}}{\bar{a}_{\overline{6}|}} = \frac{\left(i/\delta \, a_{\overline{6}|} - 6\nu^6 \right) / \delta}{i/\delta \, a_{\overline{6}|}}$$

$$= \frac{(1.019869 \times 5.2421 - 6 \times 0.79031)/0.039221}{1.019869 \times 5.2421}$$

$$\frac{15.41000}{5.3462} = 2.882 \text{ years}$$

(v) Project A has a longer duration and therefore the present value of its incoming cash flows is more sensitive to changes in the rate of interest. As such, when the interest rate rises, the present value of incoming cash flows falls more rapidly than for Project B.

Most candidates could calculate the discounted payback period but struggled with the undiscounted equivalent. As in Q9, the units should be included within the answer. The working of many candidates in part (iii) was often unclear even when the formulae were correctly derived. In part (iv) many candidates incorrectly thought the duration should be $\frac{(I\bar{a})_{\overline{6}|}}{\bar{a}_{\overline{6}|}}$ *for Project B.*

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

July 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2013 paper

This paper proved to be marginally more challenging than other recent papers and the general performance was of a slightly lower standard compared with the previous April exams. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(iii) and Q4(ii) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) TWRR, i, is given by:

$$\frac{1.9}{1.3} \times \frac{0.8}{1.9 - 0.9} = 1 + i \Rightarrow i = 0.169 \text{ or } 16.9\% \text{ p.a.}$$

(ii) MWRR, i, is given by:

$$1.3 \times (1+i) - 0.9 \times (1+i)^{\frac{3}{12}} = 0.8$$

Then, we have:

$$\left.\begin{array}{lll} i = 30\% & \Rightarrow & LHS = 0.729 \\ i = 40\% & \Rightarrow & LHS = 0.841 \end{array}\right\} \Rightarrow i \approx 0.3 + (0.4 - 0.3) \times \left(\frac{0.8 - 0.729}{0.841 - 0.729} \right) = 0.36$$

or 36% p.a.

(iii) MWRR is higher as fund performs much better before the cash outflow than after. As the fund is smaller after 1 October 2012, the effect of the poor investment performance is less significant.

The calculations were performed well but the quality of the explanations in part (iii) was often poor. A common error was to cite the large withdrawal itself as the reason for the superior MWRR.

2 (i) (a) Options – holder has the right but not the obligation to trade.

Futures – both parties have agreed to the trade and are obliged to do so.

(b) Call Option – right but not the obligation to BUY specified asset in the future at specified price.

Put Option – right but not the obligation to SELL specified asset in the future at specified price.

(ii) Assume no arbitrage.

The present value of the dividends, I, is:

$$I = 1.1v_{2.25\%} + 1.1v_{2.5\%}^2 = 1.1 \times (0.977995 + 0.951814)$$
$$= 2.12279$$

Hence, forward price $F = (10.50 - 2.12279) \times 1.025^2$
$= £8.8013$

3 (i) $97 = 6\,a_{\overline{3}|} + 103v^3$

Try 9% RHS = 94.723
Try 8% RHS = 97.227

Interpolation gives

$$0.08 + \frac{97.227 - 97}{97.227 - 94.723} \times 0.01$$

$= 0.08091$

i.e. 8.09% p.a. (exact answer is 8.089%)

(ii) Let i_n = spot rate for term n

Then $97 = 109 v_{i_1\%}$

$\Rightarrow i_1$ = 12.371% p.a.

$$97 = 6v_{i_1\%} + 109v_{i_2\%}^2$$

$$109(1+i_2)^{-2} = 97 - \frac{6}{1.12371}$$

$\Rightarrow i_2$ = 9.049% p.a.

Part (i) was generally well answered. Some candidates wasted time in (ii) through using linear interpolation to solve the yield for the one year bond.

4 (i) Maximum price payable by investor is given by:

$$P = 0.30 \times 1.05 \times v_{9\%} + 0.30 \times 1.05^2 \times v_{9\%}^2 + \ldots$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \left[1 + \left(\frac{1.05}{1.09}\right) + \left(\frac{1.05}{1.09}\right)^2 + \ldots\right]$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1}{1 - \frac{1.05}{1.09}}$$

$$= 0.30 \times \left(\frac{1.05}{1.09}\right) \times \frac{1.09}{0.09 - 0.05} = 0.30 \times \frac{1.05}{0.09 - 0.05}$$

$$= \pounds 7.875$$

(ii) (a) Increasing the expected rate of dividend growth, g , will increase the maximum price that the investor is prepared to pay to purchase the share since the dividend income is expected to be higher.

(b) An increase in the expected rate of future price inflation is likely to lead to an increase in both the expected rate of dividend growth (as nominal level of profits should increase in line with inflation) and the nominal return required from the investment (as the investor is likely to want to maintain the required real return).

Thus, the maximum price that the investor is prepared to pay will be (largely) unchanged – in fact, it will increase slightly due to (1 + g) term in numerator.

(c) If the investor is more uncertain about the rate of future dividend growth (whilst the expected dividend growth is unchanged), then the required return, i, is likely to be increased to compensate for the increased uncertainty.

Thus, the maximum price that the investor is prepared to pay will reduce.

Part (i) was generally well answered although common errors included adding an extra 30 pence dividend at the start or to assume that the first dividend was payable immediately.

The examiners expected candidates to find part (ii) challenging and this was indeed the case with very few candidates scoring full marks. In (ii)(b) full marks were awarded for a reasoned argument that led to a final answer of either an increase or no change in the price. In general, some credit was given for valid reasoning even if the final conclusion was incorrect.

5 (i) Accumulated value at time 10 is:

$$100 \times \exp\left(\int_0^{10} \delta(t)\, dt\right) + 50 \times \exp\left(\int_7^{10} \delta(t)\, dt\right)$$

$$= 100 \times \exp\left(\int_0^6 (0.1 - 0.005t)\, dt + \int_6^{10} 0.07 dt\right) + 50 \times \exp\left(\int_7^{10} 0.07 dt\right)$$

$$= 100 \times \exp\left(\left[0.1t - 0.0025t^2\right]_{t=0}^{t=6} + \left[0.07t\right]_{t=6}^{t=10}\right) + 50 \times \exp\left(\left[0.07t\right]_{t=7}^{t=10}\right)$$

$$= 100 \times \exp\left([0.6 - 0.09] + 0.28\right) + 50 \times \exp(0.21)$$

$$= 220.34 + 61.68$$

$$= £282.02$$

(ii) Present value at time 0 is:

$$= \int_{12}^{15} \rho(t) v(t)\, dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\int_0^t \delta(s)\, ds\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\left[\int_0^6 (0.1 - 0.005s)\, ds + \int_6^t 0.07 ds\right]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-\left[[0.1s - 0.0025s^2]_{s=0}^{s=6} + [0.07s]_{s=6}^{s=t}\right]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times \exp\left(-[0.51 + (0.07t - 0.42)]\right) dt$$

$$= \int_{12}^{15} 50e^{0.05t} \times e^{-0.09-0.07t} dt$$

$$= 50e^{-0.09} \times \int_{12}^{15} e^{-0.02t} dt$$

$$= 50e^{-0.09} \times \left[\frac{e^{-0.02t}}{-0.02}\right]_{t=12}^{t=15}$$

$$= 2{,}500e^{-0.09} \times \left(e^{-0.24} - e^{-0.30}\right)$$

$$= £104.67$$

6 (i) $j = 0.05 \times 0.2 + 0.07 \times 0.6 + 0.09 \times 0.2$

$= 0.07$

$\Rightarrow$ mean accumulation $= 10{,}000 \times (1+j)^{15}$

$= 10{,}000 \times (1.07)^{15}$

$= £27{,}590.32$

(ii) $s^2 = 0.05^2 \times 0.2 + 0.07^2 \times 0.6 + 0.09^2 \times 0.2 - 0.07^2$

$= 0.00506 - 0.00490$

$= 0.00016$

Var (accumulation) $= 10{,}000^2\{(1 + 2j + j^2 + s^2)^{15} - (1+j)^{30}\}$

$= 10{,}000^2\ \{1.14506^{15} - 1.07^{30}\}$

$= 1{,}597{,}283.16$

SD (accumulation) $= \sqrt{1597283.16} = £1{,}263.84$

(iii) (a) By symmetry $j = 0.07$ (as in (i))

Hence, mean (accumulation) will be the same as in (i) (i.e. £27,590.32).

The spread of the yields around the mean is lower than in (i). Hence, the standard deviation of the accumulation will be lower than £1,263.84.

(b) Mean (accumulation) < £27,590.32 since the investment is being accumulated over a shorter period.

SD (accumulation) < £1,263.84 since investing over a shorter term than in (i) will lead to a narrower spread of possible accumulated amounts.

In part (i) some candidates misread the question and assumed the yield was fixed for the whole ten years rather than varying each year.

7 (i) Need $V_A(i) = V_L(i)$ with $i = 0.08$

$$V_L(i) = 6v^8 + 11v^{15}$$
$$V_A(\text{i}) = Xv^5 + Yv^{20}$$

Need $V_A'(i) = V_L'(i)$ with $i = 0.08$

$$V_L' = -48v^9 - 165v^{16}$$
$$V_A' = -5Xv^6 - 20Y\,v^{21}$$

Thus we have to solve simultaneous equations:

(a) $6v^8 + 11v^{15} = Xv^5 + Yv^{20}$

(b) $-48v^9 - 165v^{16} = -5Xv^6 - 20Y\,v^{21}$

Taking 5 times (a) + (1+i) times (b) we get

$$-18v^8 - 110v^{15} = -15Yv^{20}$$

$$\Rightarrow Y = \frac{18(1+i)^{12} + 110(1+i)^5}{15}$$

$$\Rightarrow Y = 13.79688$$

Substitute back in (a) to get $X = 5.50877$

Hence the values of the zero-coupon bonds are £5.50877 million and £13.79688 million.

(ii) We need to check that the third condition is satisfied:

$$V_A' = -5Xv^6 - 20Yv^{21}$$
$$\Rightarrow V_A'' = 30Xv^7 + 420Yv^{22}$$
$$\Rightarrow V_A''(0.08) = 30 \times 5.50877 \times 1.08^{-7} + 420 \times 13.79688 \times 1.08^{-22}$$
$$= 1162.31$$

$$V_L' = -48v^9 - 165v^{16}$$
$$\Rightarrow V_L'' = 432v^{10} + 2640v^{17}$$
$$\Rightarrow V_L''(0.08) = 432 \times 1.08^{-10} + 2640 \times 1.08^{-17}$$
$$= 913.61$$

Therefore $V_A''(0.08) > V_L''(0.08)$

Thus the third condition is satisfied.

[Or note that since the assets have terms of 5 years and 20 years and the liabilities have terms of 8 years and 15 years, the spread of assets around the mean term is greater than that of the liabilities. Hence, the convexity of assets is greater than the convexity of liabilities].

The best answered question on the paper.

8 (i) Work in £ millions

Let Discounted Payback Period from 1 January 2014 be n.

Then, considering project at the end of year n but before the outgo at the start of year $n + 1$

$$\begin{aligned}&-\ 19-9v^{1/2}-5v\\ &-\ 6\times 9.5\left(v^2+1.04v^3+\ldots+(1.04)^{n-3}v^{n-1}\right)\\ &+\ 6\times 12.6\left(v^3+1.04v^4+\ldots+(1.04)^{n-3}v^{n}\right)\geq 0 \quad \text{at } 9\%\end{aligned}$$

Hence, $19+8.6204+4.5872\leq\left(75.6v^3-57v^2\right)\left(\dfrac{1-\left(\dfrac{1.04}{1.09}\right)^{n-2}}{1-\dfrac{1.04}{1.09}}\right)$

and RHS = $10.4013\times 21.8\times\left(1-\left(\dfrac{1.04}{1.09}\right)^{n-2}\right)$

Hence, $\dfrac{32.2076}{10.4013\times 21.8}\leq 1-\left(\dfrac{1.04}{1.09}\right)^{n-2}$

$$\Rightarrow\left(\frac{1.04}{1.09}\right)^{n-2}\geq 0.85796$$

$$\Rightarrow(n-2)\log\left(\frac{1.04}{1.09}\right)\geq\ \log 0.85796$$

$$\Rightarrow n-2\geq\frac{-0.06653}{-0.02039}=3.262$$

$$\Rightarrow n\geq 5.262$$

But sales are only made at the end of each calendar year.

$\Rightarrow$ DPP = 6 years

(ii) The DPP would be shorter using an effective rate of interest less than 9% p.a. This is because the income (in the form of car sales) does not commence until a few years have elapsed whereas the bulk of the outgo occurs in the early years. The effect of discounting means that using a lower rate of interest has a greater effect on the value of the income than on the value of the outgo (although both values increase). Hence the DPP becomes shorter.

In part (i), many candidates valued the total outgo for the whole production run and then attempted to find when the present value of income exceeded this. The working of many marginal candidates was difficult to follow and it was not clear to the examiners what the candidates were attempting to do.

9 (i) $\frac{D}{R}(1-t_1)=\frac{0.08}{1}\times 0.7=0.056< \underset{6\%}{i}^{(2)}=0.059126$

$\Rightarrow$ There is a capital gain and assume redeemed as late as possible.

Let $P=$ Price at 1/5/11per £100 nominal

$$P=\left[0.7\times 8\, a_{\overline{11}|}^{(2)}+100v^{11}-0.25(100-P)v^{11}\right]\times(1+i)^{4/12}$$

$$\Rightarrow P \;=5.6\times 1.014782\times 7.8869\times(1.06)^{4/12}+75v^{108/12}+0.25Pv^{108/12}$$

$$\Rightarrow P \;=\frac{45.6985+40.2839}{1-0.25v^{108/12}}$$

$$=£99.319$$

(ii) $\frac{D}{R}=0.08>i_{7\%}^{(2)}=0.068816$

$\Rightarrow$ Assume redeemed as soon as possible

Sale Price per £100 nominal $=\left(8\, a_{\overline{4}|}^{(2)}+100v^{4}\right)\times(1+i)^{3/12}$

$$=(8\times 1.017204\times 3.3872+100\times 0.76290)\times(1.07)^{3/12}$$

$$=£\,105.625$$

(iii) CGT is payable of $(105.625-99.319)\times 0.25$

= £1.5765

Equation of value:

$$99.319 = 0.7\times4\times v^{2/12} + 0.7\times4\times v^{8/12} + 0.7\times4\times v^{12/12} + 0.7\times4\times v^{18/12} + (105.625 - 1.5765)v^{11/12}$$

$$\Rightarrow 99.319 = (1+i)^{4/12}\times5.6\, a_{\overline{2}|}^{(2)} + 104.0485v^{11/12}$$

At 8%, RHS is $1.08^{4/12}\times5.6\times1.019615\times1.7833 + 104.0485v^{11/12}$
$= 100.226$

At 9% RHS is $(1.09)^{4/12}\times5.6\times1.022015\times1.7591 + 104.0485v^{11/12}$
$= 98.568$

and since 98.568 < 99.319 < 100.226, the net yield is between 8% and 9% p.a.

Many candidates struggled with the four month adjustment in part (i). Common errors included:

- *ignoring the adjustment completely.*
- *discounting the present value of payments by four months rather than accumulating.*
- *adjusting the price at the end of the calculations (which does not allow for CGT correctly).*

In part (iii) some candidates wasted time by trying to solve the yield exactly rather than just show that 8% was too low and 9% too high.

10 (i) Original amount of loan is:

$$\begin{aligned} L &= 5{,}000v + 4{,}800v^2 + 4{,}600v^3 + \ldots + 1{,}200v^{20} \\ &= 5{,}200\times\left(v + v^2 + \ldots + v^{20}\right) - 200\times\left(v + 2v^2 + \ldots + 20v^{20}\right) \\ &= 5{,}200a_{\overline{20}|} - 200(Ia)_{\overline{20}|} \\ &= 5{,}200\times13.5903 - 200\times125.1550 \\ &= £45{,}638.56 \end{aligned}$$

(ii) Amount of 12th instalment is £2,800.

Loan o/s after 11th instalment is given by PV of future repayments:

$$\begin{aligned} L_{11} &= 2{,}800v + 2{,}600v^2 + 2{,}400v^3 + \ldots + 1{,}200v^9 \\ &= 3{,}000a_{\overline{9}|} - 200(Ia)_{\overline{9}|} \\ &= 3{,}000\times7.4353 - 200\times35.2366 \\ &= £15{,}258.58 \end{aligned}$$

Then, interest component of 12th instalment is: $0.04\times15{,}258.58 = £610.34$.

Hence, capital repaid in 12th instalment is $2,800 - 610.34 = £2,189.66$.

(iii) (a) Then, after 12th instalment, loan o/s is

$15,258.58 - 2,189.66 = £13,068.92$.

This will be repaid by level instalments of £2,800.

Thus, remaining term of loan is n given by:

$$13,068.92 \le 2,800 \times a_{\overline{n}|}^{4\%} \Rightarrow a_{\overline{n}|}^{4\%} \ge 4.6675 \Rightarrow n = 6$$

i.e. remaining term is 6 years (i.e. loan is repaid by time 18)

(b) We need to find reduced final payment, R, such that:

$$13,068.92 = 2,800 \times a_{\overline{5}|}^{4\%} + Rv_{4\%}^{6} \Rightarrow 0.79031R$$
$$= 13,068.92 - 2,800 \times 4.4518 \Rightarrow R = £764.11$$

(c) Total amount of interest paid is given by:

$$5,000 + 4,800 + 4,600 + \ldots + 2,800 + 5 \times 2,800 + 764.11 - 45,638.56$$
$$= £15,925.55$$

In part (ii) the most common error was to not round n up, i.e. quoting a non-integer number of years for the revised loan. Part (iii) was answered poorly with candidates often not correctly allowing for the payments prior to the change in payment schedule.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2013 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

D C Bowie
Chairman of the Board of Examiners

December 2013

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2013 paper

This paper proved to have some questions where the vast majority of candidates scored well and others where many candidates found challenging. Well-prepared candidates scored well across the whole paper. As in previous diets, questions that required an element of explanation or analysis, such as Q1(ii), Q8(iv), Q10(iii) and Q11(v) were less well answered than those that just involved calculation. This is an area to which attention should be paid. Candidates should note that it is important to explain and show understanding of the concepts and not just mechanically go through calculations.

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

1 (i) (a) $d = \dfrac{0.045}{1.045} = 0.043062 = 4.3062\%$

(b) $\left(1 - d^{(12)}/_{12}\right) = (1.045)^{-1/12}$

$\therefore 1 - \dfrac{d^{(12)}}{12} = 0.99634$

$\therefore d^{(12)} = 0.043936$ or 4.3936%

(c) $\left(1 + \dfrac{i^{(4)}}{4}\right)^4 = 1.045$

$\therefore \left(1 + \dfrac{i^{(4)}}{4}\right) = 1.011065$

$\therefore i^{(4)} = 0.044260$ or 4.4260%

(d) $1.045^5 = 1.24618$

$\therefore$ five-yearly effective rate is 24.618%

(ii) The answer to (i)(b) is bigger than the answer to (i)(a) because the rate of discount convertible monthly is applied each month to a smaller (already discounted) sum of money. As such, in order to achieve the same total amount of discounting the rate has to be slightly more than one twelfth of the annual rate of discount. [An answer relating to the concept of interest payable in advance would also be acceptable].

The calculations were performed well but the quality of the explanations in part (ii) was often poor. A common error in (i)(d) was to state the answer as $i^{(1/5)}$ *rather than* $\dfrac{i^{(1/5)}}{1/5}$.

2 Present value of dividend = $0.1 \times 1.04^{-0.5} = 0.09806$

Value of forward is $(1.8 - 0.09806) \times 1.04^{0.75} = £1.75275$

3 (i) Work in half years.

$$P = 2a_{\overline{20|}} + 100v^{20} \text{ @ } 1\tfrac{1}{2}\%$$
$$= 2\times 17.1686+100\times 0.74247$$
$$= £108.584$$

(ii) $$P = (2\times 0.75a_{\overline{20|}} + 100v^{20})(1.015)^{91/182.5}$$
$$= (2\times0.75\times17.1686+100\times0.74247)\times (1.015)^{91/182.5}$$
$$= £100.7452$$

Part (i) was answered well although some candidates assumed an annual effective rate of 3%. In part (ii) many candidates did not deal with the 91 days elapsed duration – discounting instead of accumulating the 10-year bond price and/or assuming that 91 days equated to a quarter of a year.

4 (i) The investor pays a purchase price at outset.

The investor receives a series of coupon payments and a capital payment at maturity

The coupon and capital payments are linked to an index of prices (possibly with a time lag)

[Time lag does not have to be mentioned].

(ii) The investor pays a purchase price at outset

Shareholders are paid dividends. These are not fixed but declared out of profits.
Dividends may be expected to increase over time ….
….but may cease if the company fails.

There is a high degree of uncertainty with regard to future cash flows.

No maturity date
Would receive a sale price on the sale of the shares

Generally poorly answered with many candidates just writing down all characteristics they knew about these assets rather than concentrating on the cashflows. Many candidates omitted mention of the initial purchase price in each part.

5 (i) The return from the bonds issued by Country A is: $\frac{106}{101} - 1 = 0.049505$

The expected cash flows from the bonds from Country B are:

$$0.1 \times 0 + 0.2 \times 100 + 0.3 \times 50 + 0.4 \times 106 = 77.4$$

The price to provide the same expected return is P such that:

$$P = \frac{77.4}{1.049505} = €73.749$$

(ii) The gross redemption yield from the bond is such that:

$$73.749 \times (1+i) = 106$$
$$\therefore i = 43.731\%$$

(iii) The risk is higher for Country B's bond. Although the gross redemption yield is such that the expected returns are equal, the investor may want a higher expected return to compensate for the higher risk.

Many candidates had trouble with part (ii) not recognising that the gross redemption yield calculation will not include any allowance for default.

6 Divide the number of cars by 100 to obtain the share due to the pension fund

PV of income = $365 \times 400\, \bar{a}_{\overline{1}|} + 365 \times 500\, \bar{a}_{\overline{1}|}\, v \times 1.1 \times \left(1 + 1.01^2 v + 1.01^4 v^2 + \ldots\ldots + 1.01^{36} v^{18}\right)$

$$= 365 \times 400 \frac{i}{\delta} a_{\overline{1}|} + 365 \times 500 \frac{i}{\delta} a_{\overline{1}|}\, v \times 1.1 \times \left(\frac{1 - 1.01^{38} v^{19}}{1 - 1.01^2 v}\right)$$

$$= 365 \times 400 \times 1.039487 \times 0.92593$$
$$+ 365 \times 500 \times 1.039487 \times 0.92593 \times 0.92593 \times 1.1 \times \left(\frac{1 - 1.45953 \times 0.23171}{1 - 1.0201 \times 0.92593}\right)$$

= 140,523 + 178,907 × 11.93247

= 140,523 + 2,134,801 = 2,275,324 so NPV = £275,324

Candidates made a variety of errors in this question often ignoring one or more parts of the scenario (e.g. pension fund's 1% share of the project, the fact that daily vehicle numbers were given in the question, 1% increases in both vehicle numbers and tolls from the second year). Nevertheless, candidates who set out their workings clearly and logically often scored the majority of the available marks.

7 (i) $(1 + i_t) \sim \text{lognormal}\ (\mu, \sigma^2)$

$$\ln(1+i_t) \sim N(\mu, \sigma^2)$$

$$\ln \prod_{t=1}^{5}(1+i_t) = \ln(1+i_t) + \ln(1+i_t) + \ldots + \ln(1+i_t)$$

$$\sim N(5\mu, 5\sigma^2) \quad \text{by independence}$$

$$\therefore \prod_{t=1}^{5}(1+i_t) \sim \text{ lognormal } (5\mu, 5\sigma^2)$$

$$E(1+i_t) = \exp\left(\mu + \frac{\sigma^2}{2}\right) = 1.055$$

$$\text{Var}(1+i_t) = \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right] = 0.04^2$$

$$\frac{0.04^2}{1.055^2} = \left[\exp\left(\sigma^2\right) - 1\right] \therefore \sigma^2 = 0.0014365$$

$$\exp\left(\mu + \frac{0.0014360}{2}\right) = 1.055 \Rightarrow$$

$$\mu = \ln 1.055 - \frac{0.0014365}{2} = 0.052823$$

$5\mu = 0.264113$

$5\sigma^2 = 0.007182.$

Let S_5 be the accumulation of one unit after five years:

$$E(S_5) = \exp\left(5 \times \mu + \frac{5\sigma^2}{2}\right) = \exp\left(0.264113 + \frac{0.007182}{2}\right)$$

$$= 1.30696$$

$$\text{Var}(S_5) = \exp(2 \times 5\mu + 5\sigma^2)\left[\exp\left(5\sigma^2\right) - 1\right]$$

$$= \exp(2 \times 0.264113 + 0.007182).(\exp 0.007182 - 1)$$

= exp 0.53541 (exp 0.007182 – 1)

= 0.012313

Mean value of the accumulation of premiums is

7,850,000 × 1.30696 = £10,259,636.

Standard deviation of the accumulated value of the premiums is

$$7{,}850{,}000 \times \sqrt{0.012313} = £871{,}061$$

Alternatively:

Let i_t be the (random) rate of interest in year t. Let S_5 be the accumulation of a single investment of 1 unit after five years:

$$E(S_5) \;=\; E\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]$$

$$E(S_5) \;=\; E[1+i_1]E[1+i_2]\ldots E[1+i_5] \text{ as } \{i_t\} \text{ are independent}$$

$$E[i_t] \;=\; 0.055$$

$$\therefore \quad E(S_5) = (1.055)^5 = 1.30696$$

$$E\left(S_5^2\right) = E\left[\left[(1+i_1)(1+i_2)\ldots(1+i_5)\right]^2\right]$$

$$= E(1+i_1)^2 E(1+i_2)^2 \ldots E(1+i_5)^2 \text{ (using independence)}$$

$$= E\left(1+2i_1+i_1^2\right)E\left(1+2i_2+i_2^2\right)\ldots E\left(1+2i_5+i_5^2\right)$$

$$= \left(1+2\times 0.055+0.04^2+0.055^2\right)^5$$

as $E\left[i_i^2\right] = V[i_t] + E[i_t]^2 = 0.04^2 + 0.055^2$

$$\therefore \text{ Var}[S_5] = \left(1+2\times 0.055+0.04^2+0.055^2\right)^5 - (1.055)^{10}$$

$$\therefore = 1.114625^5 - (1.055)^{10} = 0.0123128$$

Mean value of the accumulation of premiums is

$$7{,}850{,}000 \times 1.30696 = £10{,}259{,}636.$$

Standard deviation of the accumulated value of the premiums is

$$7{,}850{,}000 \times \sqrt{0.012313} = £871{,}061$$

(ii) If the company invested in fixed-interest securities, it would obtain a guaranteed accumulation of £7,850,000 $(1.04)^5$ = £9,550,725. In one sense, there is a 100% probability that a loss will be made and therefore the policy is unwise. The "risky" investment strategy leads to an expected profit. On the other hand, the standard deviation of the accumulation from the risky investment strategy is £871,061. Whilst there is a chance of an even greater profit from this strategy, there is also a chance of a more considerable loss than from investing in fixed-interest securities.

A poorly answered questions with many candidates not including enough derivation of the required results in part (i). Some candidates mixed their answers between the two methods given above e.g. they calculated μ and σ^2 for the log normal route, then used these in the alternative method for the mean and variance of i_t. Other candidates just used 0.055 and $.04^2$ as their values of μ and σ^2 .

8 (i) Purchase price of the annuity (working in half-years)

$$5{,}000 a_{\overline{50}|}^{(6)} \text{ calculated at } i = 2\%$$

$$= 5{,}000 \frac{i}{i^{(6)}} a_{\overline{50}|}$$

$$i = 0.02$$

$$i^{(6)} = 0.019835$$

$$a_{\overline{50}|} = 31.4236$$

$$\text{Purchase price} = 5{,}000 \times \frac{0.02}{0.019835} \times 31.4236$$

$$= £158{,}422$$

Individual needs to invest X such that: (working in years)

$$X\,1.05^{20} = 158{,}422$$

$$1.05^{20} = 2.653297$$

$$\therefore X = \frac{158{,}422}{2.653297} = £59{,}708$$

(ii) Real return is j such that:

$$59{,}708 = \frac{158{,}422}{(1+j)^{20}} \times \frac{143}{340}$$

$$\therefore (i+j)^{20} = \frac{158{,}422}{59{,}708} \times \frac{143}{340}$$

$$=1.11595$$

$$\therefore j = 0.550\%$$

(iii) The amount of the capital gain is:

$$158{,}422 - 59{,}708 = 98{,}714$$

Tax = 0.25 × 98,714 = 24,679

Proceeds of investment = 133,744

Net real return is j' such that:

$$59{,}708 = \frac{133{,}744}{(1+j')^{20}} \times \frac{143}{340}$$

$$\therefore (1+j')^{20} = \frac{133{,}744}{59{,}708} \times \frac{143}{340}$$

$$= 0.942106$$

$$\therefore j' = -0.2977\%$$

(iv) The capital gains taxed has taxed the nominal gain, part of which is merely to compensate the investor for inflation. The tax has therefore reduced the real value of the investor's capital and led to a negative real return.

Parts (i) and (ii) were generally answered well but many candidates struggled with the calculation of the capital gain in part (iii) not recognising that this would be based on money values.

9 (i) PV is:

$$400v + 380v^2 + 360v^3 + \cdots + 120v^5$$

$$420\, a_{\overline{15|}} - 20\left(Ia\right)_{\overline{15|}} \qquad @\,4\%$$

$= 420 \times 11.1184 - 20 \times 80.8539$

$= 4{,}669.728 – 1{,}617.078 = £3{,}052.65.$

(ii) Interest component:

$= 0.04 \times 3{,}052.65 = £122.106$

Capital component $= 400 – 122.106$
$= £277.894$

(iii) Seven repayments remain and the PV of the remaining payments is:

$$240v + 220v^2 + \cdots + 120\, v^7$$

$$260\, a_{\overline{7|}} - 20\left(Ia\right)_{\overline{7|}} \qquad @\,4\%$$

$$= 260 \times 6.0021 - 20 \times 23.0678 = £1{,}099.19$$

The loan is written down to: $0.5 \times 1{,}099.19$
$= £549.595$

The present value of the new repayment is:

$$\left(X-2\right)a_{\overline{10|}} + 2(Ia)_{\overline{10|}} \qquad @\,8\%$$

$$\therefore 549.595 = \left(X-2\right)6.7101 + 2 \times 32.6869$$

$$\therefore X - 2 = \frac{549.595 - 2 \times 32.6869}{6.7101} = 72.163$$

$$\therefore X = £74.16$$

The best answered question on the paper although some candidates, when calculating the outstanding loan in part (iii), stated that the repayment in year 8 was £420. Some candidates also used the incorrect formula $Xa_{\overline{10|}} + 2(Ia)_{\overline{10|}}$ *for the repayment in part (iv).*

10 (i) (a) $e^{\int_0^7 0.05+0.002t\,dt}$

$= e^{\left[0.05t+\frac{0.002t^2}{2}\right]_0^7}$

$= \exp\left[0.05\times 7\right]+\dfrac{0.002\times 49}{2}$

$= \exp(0.399) = 1.490331$

(b) $e^{\int_0^6 0.05+0.002t\ \mathrm{dt}}$

$= e^{\left[0.05\times 6+\frac{0.002\times 36}{2}\right]}$

$= \exp(0.336) = 1.399339$

(c) $\dfrac{1.490331}{1.399339} = 1.06503$

(ii) (a) Let spot rate = i_7

$(1+i_7)^7 = 1.490334$

$\Rightarrow i_7 = 5.8656\%$ p.a. effective

(b) $(1+i_6)^6 = 1.39934$

$\therefore i_6 = 5.7598\%$ p.a. effective

(c) From (i) (c) 6.503% per annum effective.

(iii) The forward rate is the rate of interest in the seventh year. The spot rate, in effect, is the rate of interest per annum averaged over the seven years (a form of geometric average). As the force of interest is rising the rate of interest in the seventh year must be higher than the rate averaged over the full seven year period.

(iv) $v(t) = e^{-\int_0^t 0.05+0.002s \, ds}$

$$= e^{-\left[0.05s+\frac{0.002s^2}{2}\right]_0^t}$$

$$= e^{-0.05t-0.001t^2}$$

We require

$$\int_3^{10} \rho(t)v(t)dt$$

$$= \int_3^{10} \frac{30 \, \mathrm{e}^{-0.01t}}{e^{-0.001t^2}} . e^{-0.05t} . e^{-0.001t^2} dt$$

$$30\int_3^{10} e^{-0.06t} \ dt$$

$$\frac{30}{-0.06}\left[e^{-0.06t}\right]_3^{10}$$

$$= \frac{30}{-0.06}\left[e^{-0.6} - e^{-0.18}\right]$$

= –500(0.548812 – 0.83527)

=143.229

The calculations were well-done but only the best candidates clearly explained their reasoning in part (iii).

11 (i) Present value of liabilities annuity

$10\ a_{\overline{40}|}$ at 4% $\quad a_{\overline{40}|} = 19.7928$

= 10 × 19.7928 = £197.928m

(ii) Call 10 year security "security A" and five year security "security B".

We need to calculate the PV of £100 nominal for each of security A and security B

P.V of £100 nominal of A is:

$$5a_{\overline{10|}} + 100v^{10} \quad @4\%$$

$$a_{\overline{10|}} = 8.1109 \quad v^{10} = 0.67556$$

$\therefore$ PV = 5×8.1109 + 67.556 = 108.1105

P.V of £100 nominal of B is:

$$10\ a_{\overline{5|}} + 100v^{5} \quad @\ 4\%$$

$$a_{\overline{5|}} = 4.4518 \quad v^{5} = 0.82193$$

$\therefore$ PV = 44.518 + 82.193 = 126.711

£98.964m, is invested in each security.

$$\frac{98{,}964{,}000}{108.1105} \times 100 \text{ per £100 nominal of A is bought.}$$

= £91,539,674 nominal

$$\frac{98{,}964{,}000}{126.711} \times 100 \text{ per £100 nominal of B is bought}$$

= £78,102,138 nominal

[other ways of expressing units are okay, but marks will be deducted if units are not correct]

(iii) Duration of the liabilities

$$= \frac{\sum tc_t \ \mathrm{v}^t}{\sum c_t \ \mathrm{v}^t}$$

$$\text{Numerator } = \sum_{t=1}^{40} 10\ t\ v^t \quad \text{(in £ m)}$$

$$= 10(Ia)_{\overline{40|}} = 10 \times 306.3231 = 3063.231 \quad \text{at 4\% p.a. effective}$$

$\therefore$ Duration = 3063.231/197.928 = 15.48 years

(iv) Numerator of duration is:

$$(5(Ia)_{\overline{10|}} + 10 \times 100\, v^{10}) \times 915{,}396.74$$

$$+(10(Ia)_{\overline{5|}} + 5 \times 100\, v^{5}) \times 781{,}021.38$$

Following the same reasoning as for the calculation of the duration of the annuity payments, adding the capital repayment and multiplying by the number of units of £100 nominal bought.

$$(Ia)_{\overline{10|}} = 41.9922$$

$$v^{10} = 0.67556$$

$$(Ia)_{\overline{5|}} = 13.0065$$

$$v^{5} = 0.82193$$

$$= (5 \times 41.9922 + 10 \times 100 \times 0.67556) \times 915{,}396.74$$

$$+(10 \times 13.0065 + 5 \times 100 \times 0.82193) \times 781{,}021.38$$

$$= 810{,}603{,}000 + 422{,}554{,}000$$

$$= 1{,}233{,}157{,}000$$

$\therefore$ Duration = 1,233,157,000/197,928,000
= 6.23 years

(v) The duration (and therefore the volatility) is greater for the liabilities than for the assets. As a result, when interest rates fall, the present value of the liabilities will rise by more than the present value of the assets and so a loss will be made.

Many candidates wrongly assumed that the same nominal amounts were bought of each asset rather than each asset amount having the same present value. This assumption made the calculations in part (ii) somewhat easier and the marks awarded in this part took this into account. Part (iii) was answered well. The explanations in part (v) were often poorly stated although time pressures at the end of the paper may have contributed to this.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2014 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

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D C Bowie
Chairman of the Board of Examiners

June 2014

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2014 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates

1 We can ignore the fund values given at 30 June.

Working in £000s:

$$870(1+i)^3 + 26(1+i)^{2\frac{1}{2}} + 27(1+i)^{1\frac{1}{2}} + 33(1+i)^{\frac{1}{2}} = 990$$

Approximate i comes from:

$$(870+26+27+33)(1+i)^3 = 990$$

$$\Rightarrow i = 1.2\%$$

Try 1%, LHS = 983.587

Try 2%, LHS = 1011.713

So

$$i = 0.01 + (0.02 - 0.01) \times \frac{990 - 983.587}{1011.713 - 983.587}$$

$$= 0.0123$$

Answer = 1.2% p.a.

Well answered although many candidates ignored the instruction to give the answer to the nearest 0.1%, and were penalised accordingly.

2 (a) **Debentures**

Debentures are part of the loan capital of companies.
The term "loan capital" usually refers to long-term borrowings rather than short-term.
Payments consist of regular coupons…
…and a final redemption payment
The issuing company provides some form of security to holders of the debenture…
…e.g. via a fixed or floating charge on the company's assets
Debenture stocks are considered more risky than government bonds…
…and are considered less marketable than government bonds.
Accordingly the yield required by investors will be higher than for a comparable government bond.

(b) **Unsecured loan stocks**

Issued by various companies.
They are unsecured – holders rank alongside other unsecured creditors.
Yields will be higher than on comparable debentures issued by the same company…
…to reflect the higher default risk.

This question was poorly answered despite being completely based on bookwork.

The above shows the variety of points that could be made (and not all were required for full marks). Many marginal candidates either made no significant attempt at the question or did not make enough distinct points.

3 (i) $$900 \times \left(1 + \frac{i^{(2)}}{2}\right)^{2*\frac{4}{12}} = 925$$

$$\Rightarrow \left(1 + \frac{i^{(2)}}{2}\right)^{\frac{8}{12}} = \frac{925}{900} \Rightarrow 1 + \frac{i^{(2)}}{2} = \left(\frac{925}{900}\right)^{\frac{12}{8}} = 1.041954693$$

$$\Rightarrow i^{(2)} = 8.39\% \quad (8.3909385)$$

(ii) $$900 = 925 \times \left(1 - \frac{d^{(4)}}{4}\right)^{4*\frac{4}{12}}$$

$$\Rightarrow \left(1 - \frac{d^{(4)}}{4}\right)^{\frac{16}{12}} = \frac{900}{925} \Rightarrow 1 - \frac{d^{(4)}}{4} = \left(\frac{900}{925}\right)^{\frac{12}{16}} = 0.979660466$$

$$\Rightarrow d^{(4)} = 8.14\% \quad (8.1358136)$$

(iii) $$900 \times \left(1 + \frac{4}{12} i'\right) = 925 \Rightarrow i' = 8.33\% \left(8.\dot{3}\right)$$

Where i' is the simple rate of interest per annum.

This question was answered very well although some candidates calculated $i^{(4)}$ rather than $d^{(4)}$ for part (ii).

4 Firstly we must consider $i^{(2)}$ and $(1-t)\dfrac{D}{R}$

where $i^{(2)}$ is evaluated at the net yield rate (6% p.a.) = 5.9126%

$t = 0.30$, the income tax rate

$$\frac{D}{R} = \frac{8}{1.03} = 7.7670 \text{ p.a.} \quad \Rightarrow (1-t)\frac{D}{R} = 5.4369\%$$

We have $i^{(2)} > (1-t)\dfrac{D}{R}$

$\Rightarrow$ there is a capital gain and the stock will be redeemed at the last possible date if the minimum yield is received. i.e. at the end of 25 years.
Hence, let P be price per £100 nominal, then

$$P = (1-0.3)8\, a^{(2)}_{\overline{25|}} + \left(103-(103-P)\times 0.4\right)v^{25} \text{at 6\% p.a.}$$

$$= 5.6 a^{(2)}_{\overline{25|}} + (61.8 + 0.4P)v^{25}$$

$$\Rightarrow P = \frac{5.6\dfrac{i}{i^{(2)}}\, a_{\overline{25|}} + 61.8v^{25}}{1-0.4v^{25}}$$

$$= \frac{5.6\times 1.014782\times 12.7834 \; + 61.8\times 0.23300}{1-0.4\times 0.23300}$$

$$= \frac{72.6452 + 14.3994}{1-0.0932}$$

= £95.99

Generally well-answered although some candidates' arguments for choosing the latest possible date were unclear.

5 (i) The amounts of cash flows:

Coupon on 25/4/2013

$$= 10{,}000 \times \frac{0.03}{2} \times \frac{RPI_{4/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{171.4}{149.2} = £172.319$$

Coupon on 25/10/2013

$$= 10000 \times \frac{0.03}{2} \times \frac{RPI_{10/2013}}{RPI_{10/2008}}$$

$$= 10000 \times \frac{0.03}{2} \times \frac{173.8}{149.2} = £174.732$$

Redemption on 25/10/2013 $= 10000 \times \dfrac{173.8}{149.2}$

$$= £11{,}648.794$$

(ii) Purchase Price at 25/10/2012 = *PV* at real rate of $3\frac{1}{2}\%$ p.a. effective of future cash flows.

= *PV* at $3\frac{1}{2}\%$ p.a. effective of "25/10/2012 money values" of future cash flows.

Future cash flows expressed in 25/10/2012 money values

Coupon at 25/4/2013 $= 172.319 \times \dfrac{RPI_{10/2012}}{RPI_{4/2013}}$

$$= 172.319 \times \frac{169.4}{171.4} = £170.308$$

Coupon at 25/10/2013 $= 174.732 \times \dfrac{RPI_{10/2012}}{RPI_{10/2013}}$

$$= 174.732 \times \frac{169.4}{173.8} = £170.308$$

(same as 25/4/2013, as expected)

Redemption at 25/10/2013 = $11648.794 \times \dfrac{169.4}{173.8} \quad = \quad £11,353.888$

$$\left[\text{or } 10000 \times \frac{RPI_{10/2012}}{RPI_{10/2008}} = 10000 \times \frac{169.4}{149.2} = 11353.888\right]$$

Hence Price at 25/10/2012

$$= 170.308 \times \frac{1}{(1.035)^{\frac{1}{2}}} + \frac{170.308 + 11353.888}{(1.035)}$$

$$= £11,301.89$$

Many candidates had difficulty in recognising that the real yield would be based on using the inflation-adjusted cashflows as at the time of purchase. Some candidates made no adjustment at all whereas others incorrectly assumed that the inflation rate would be constant throughout the holding period.

6 (i) Redington's first condition states that the *pv* of the assets should equal the *pv* of the liabilities.

Working in £ million:

pv of assets
$$\begin{aligned} &= 7.404v^2 + 31.834v^{25} \text{ at } 7\% \\ &= 7.404 * 0.87344 + 31.834 * 0.18425 \\ &= 6.467 + 5.865 \\ &= 12.3323 \end{aligned}$$

pv of liabilities
$$\begin{aligned} &= 10v^{10} + 20v^{15} \text{ at } 7\% \\ &= 10 * 0.50835 + 20 * 0.36245 \\ &= 5.0835 + 7.249 \\ &= 12.3324 \end{aligned}$$

Allowing for rounding, Redington's first condition is satisfied.

Redington's second condition states that the DMT of the assets should equal the DMT of the liabilities. Given denominator of DMTs of assets and liabilities have been shown to be equal, we only need to consider the numerators.

Numerator of DMT of assets
$$\begin{aligned} &= 7.404 * 2 * v^2 + 31.834 * 25 * v^{25} \text{ at } 7\% \\ &= 6.467 * 2 + 5.865 * 25 \\ &= 159.569 \end{aligned}$$

Numerator of DMT of liabilities $= 10 * 10 * v^{10} + 20 * 15 * v^{15}$ at 7%
$= 5.0835 * 10 + 7.249 * 15$
$= 159.569$

Allowing for rounding, Redington's 2nd condition is satisfied.

(ii) Profit $= 7.404v^2 + 31.834v^{25} - 10v^{10} - 20v^{15}$ at 7.5%
$= 6.40692 + 5.22011 - 4.85194 - 6.75932$
$= 0.015772$ i.e. a profit of £15,772

(iii) It can be seen that the spread of the assets is greater than the spread of the liabilities. This will mean that Redington's third condition for immunization is also satisfied, and that therefore a profit will occur if there is a small change in the rate of interest. Hence we would have anticipated a profit in (ii).

Parts (i) was answered well. Equating volatilities instead of DMTs was perfectly acceptable in this part. Part (ii) was also generally answered well although some candidates estimated the answer by using an estimation based on volatility rather than calculating the answer directly as asked. Part (iii) was less well answered with some candidates ignoring this part completely and others stating that Redington's 3rd condition was satisfied without further explanation.

7 Let K_t and S_t denote the forward price of the contract at time t, and the stock price at time t respectively.

Let r be the risk-free rate per annum at time $t = \frac{1}{2}$

Then, $K_0 = S_0 e^{0.04}$

and $K_{\frac{1}{2}} = 0.98 \ S_0 \ e^{\frac{1}{2}r}$

The value of the contract $V_{\frac{1}{2}}$ is $\left(K_{\frac{1}{2}} - K_0\right) e^{-\frac{1}{2}r}$

Hence $V_{\frac{1}{2}} = \left(K_{\frac{1}{2}} - K_0\right) e^{-\frac{1}{2}r}$

$= S_0 \times \left(0.98 e^{\frac{1}{2}r} - e^{0.04}\right) e^{-\frac{1}{2}r}$

And

$V_{\frac{1}{2}} > 0$ when $0.98e^{\frac{1}{2}r} > e^{0.04}$

which is when $r > 2\ln\left(\frac{e^{0.04}}{0.98}\right) = 12.041\%$ p.a.

One of the worst answered questions on the paper. Some candidates, who did not complete the question, lost some of the marks that would have been available to them by not showing clear working e.g. writing down one half of a formula without explaining what the formula was supposed to represent.

8 (i) Let DPP be t. We want (all figures in £000s)

$$50{,}000 = 6{,}000a_{\overline{t}|}^{(2)} \quad \text{at 9\% p.a.}$$

$$= 6{,}000 \times \frac{i}{i^{(2)}} \times a_{\overline{t}|}$$

$$\Rightarrow a_{\overline{t}|} = \frac{50}{6 \times 1.022015}$$

$$= 8.1538268$$

$$\Rightarrow v^t = 1 - 8.1538268 \times 0.09$$

$$\Rightarrow t = \frac{\ln(1 - 8.1538268 \times 0.09)}{\ln 1.09^{-1}} = 15.360$$

$\therefore$ Take DPP as 15.5 years

(ii) Profit at the end of 20 years is

$$-50{,}000 \times (1.09)^{15.5} \times (1.07)^{4.5} + 6{,}000 \times s_{\overline{15.5}|}^{(2)} \times (1.07)^{4.5} + X$$

where

$$s_{\overline{15.5}|}^{(2)} = \frac{(1+i)^{15.5} - 1}{i^{(2)}} \quad \text{at 9\%}$$

$$= \frac{1.09^{15.5} - 1}{0.088061}$$

$$= 31.8285476$$

and to find X we work in half-years:

$$X = 3,000 s_{\overline{9}|} \text{ at } j\% \text{ where } (1+j)^2 = 1.07$$

$$= 3,000 \times \frac{(1+j)^9 - 1}{j}$$

$$= 3,000 \times \frac{(1.07)^{9/2} - 1}{(1.07)^{1/2} - 1}$$

$$= 31,030.35528$$

$$\therefore \text{ Profit} = -257,814.7272 + 258,937.5717 + 31,030.35528$$

$$= 32,153.20$$

$$(= £32,153,200)$$

Part (i) was answered well although candidates lost marks for not recognising that the DPP could only be at the time of income receipt i.e. at the end of a half-year. Part (ii) was answered badly with some candidates ignoring the initial profit obtained at the end of the DPP. A common error in the calculation of the profit arising after the DPP was to calculate the present value rather than the accumulated value.

9 (i) We can find the one-year forward rates $f_{1,1}$ and $f_{2,1}$ from the spot rates y_1, y_2 and y_3:

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$

$$\Rightarrow (1+0.037)^2 = (1+0.036)(1+f_{1,1})$$

$$\Rightarrow f_{1,1} = 3.800\% \text{ p.a.}$$

and

$$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1})$$

$$\Rightarrow (1.038)^3 = (1.037)^2(1+f_{2,1})$$

$$\Rightarrow f_{2,1} = 4.000\% \text{ p.a.}$$

(ii) (a) Price per £100 nominal

$$= 4\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2} + \underset{3.8\%}{v^3} \right) + 105 \underset{3.8\%}{v^3}$$

$$= 4 \times 2.78931 + 105 \times 0.89414$$

$$= £105.0425$$

(b) Let yc_2 = two-year par yield

$$1 = yc_2\left(\underset{3.6\%}{v} + \underset{3.7\%}{v^2} \right) + \underset{3.7\%}{v^2}$$

$$\Rightarrow yc_2 = 3.6982\ \%\ \text{p.a.}$$

Questions on the term structure of interest rates have caused significant problems for candidates in past years but this question was generally answered very well.

10 (i) Let X = initial payment

$$20000 = (X - 50)a_{\overline{25}|} + 50(Ia)_{\overline{25}|}$$

$$= (X - 50) \times 10.6748 + 50 \times 98.4789$$

$$= 10.6748X - 533.74 + 4923.95$$

$$\Rightarrow X = \frac{15609.80}{10.6748} = £1,462.31.$$

(ii) After 3 years, capital o/s is:

$$1562.31a_{\overline{22}|} + 50(Ia)_{\overline{22}|}$$

$$= 1562.31 \times 10.2007 + 50 \times 87.1264$$

$$= £\,20,293.01$$

(iii) The loan has actually increased from £20,000 to £20,293.01. The reason for this is that the loan is being repaid by an increasing annuity and, in the early years, the interest is not covered by the repayments (e.g. 1st year: Interest is 0.08 × 20000 = £1,600 but 1st instalment is £1462.31 and so interest is not covered).

(iv) Total of instalments paid

$$= 25 \times 1462.31 + \frac{24 \times 25}{2} \times 50 = 51557.66$$

$\Rightarrow$ Total interest = $51557.66 - 20000$ = £31557.66

Parts (i) and (ii) were answered well, although in part (ii) some candidates incorrectly calculated the instalment that would be paid in the fourth year. Part (iii) was also answered relatively better than similar explanation questions in previous years. Many candidates failed to include the effect of the increasing payments in the calculation of the total instalments in part (iv) despite having correctly allowed for this in earlier parts.

11 (i) $PV = \int_4^{10} 3{,}000 \nu(t)\, dt$

where $\nu(t)$ is as follows:

$0 \le t < 4$

$$\nu(t) = e^{-\int_{o}^{t}(0.03+0.01t)dt} \quad = e^{-\left[0.03t+\frac{1}{2}x0.01t^2\right]}$$

$4 \le t < 6$

$$\nu(t) \ = e^{-0.20}.e^{-\int_4^t 0.07dt} = e^{-0.20}.e^{(-0.07t+0.28)}$$

$$= e^{0.08-0.07t}$$

$t \ge 6$

$$\nu(t) \ = e^{-0.34}.e^{-\int_6^t 0.09dt} = e^{-0.34}.e^{(-0.09t+0.54)}$$

$$= e^{(0.20-0.09t)}$$

$$\Rightarrow PV \ = 3{,}000\int_4^6 \left(e^{0.08-0.07t}\right)dt + 3{,}000\int_6^{10} e^{(0.20-0.09t)}dt$$

$$= \frac{3{,}000e^{0.08}}{-0.07}\left[e^{-0.42} - e^{-0.28}\right] + \frac{3{,}000e^{0.20}}{-0.09}\left[e^{-0.90} - e^{-0.54}\right]$$

$$= 4584.02 + 7172.83 = \$11{,}756.85$$

(ii) $11.75685 = 3\left(\bar{a}_{\overline{10}|} - \bar{a}_{\overline{4}|}\right)$

at i = 6%, RHS = 3(1.029709)[7.3601 – 3.4651] = 12.03215

at i = 7%, RHS = 3(1.034605)[7.0236 – 3.3872] = 11.28671

by interpolation

$$\therefore i = 0.06 + \left(\frac{12.03215 - 11.75685}{12.03215 - 11.28671} \times 0.01\right) = 0.06369 \text{ i.e. } 6.4\%$$

(actual answer is 6.36%)

One of the worst answered questions on the paper with the different formulation of a question based on varying forces of interest causing problems for many candidates. It is also possible to answer part (i) as a combination of continuous deferred annuities. Part (ii) was poorly answered even by candidates who had made a good attempt to part (i).

12 (i) $1 + i_t \sim \text{LogNormal}\left(\mu, \sigma^2\right)$

$$S_{12} = \prod_1^{12} (1 + i_t)$$

$$\Rightarrow \ln S_{12} = \sum_1^{12} \ln(1 + i_t) \sim N\left(12\mu, 12\sigma^2\right)$$

$$E(1 + i_t) = 1.08 \quad = \exp\left\{\mu + \sigma^2 / 2\right\}$$

$$\text{Var}(1 + i_t) = 0.05^2 \quad = \exp\left\{2\mu + \sigma^2\right\}\left(\exp\left(\sigma^2\right) - 1\right)$$

$$= 1.08^2\left(\exp\left(\sigma^2\right) - 1\right)$$

$$\Rightarrow e^{\sigma^2} \quad = 1 + \frac{0.05^2}{1.08^2}$$

$$\Rightarrow \sigma^2 \quad = 0.002141053$$

$$\mu \quad = \ln 1.08 - \sigma^2 / 2$$

$$= 0.075890514$$

Hence S_{12} has LogNormal distribution with parameters 0.910686 and 0.025692636

(ii) *PV* of annuity at time 12:

$$PV = 4000\underbrace{\ddot{a}^{(12)}_{\overline{4}|}}_{7\%} + 5000\underbrace{\ddot{a}^{(4)}_{\overline{2}|}v^4}_{7\%} + 5000\underbrace{\ddot{a}^{(4)}_{\overline{2}|}}_{9\%}\underbrace{v^6}_{7\%} + 6000\underbrace{\bar{a}_{\overline{4}|}v^2}_{9\%}\underbrace{v^6}_{7\%}$$

$$\begin{aligned}
&= 1000\,(4\times 1.037525\times 3.3872 + 5\times 1.043380\times 1.8080\times 0.76290\\
&\qquad + 5\times 1.055644\times 1.7591\times 0.66634\\
&\qquad + 6\times 1.044354\times 3.2397\times 0.84168\times 0.66634)
\end{aligned}$$

$$\begin{aligned}
&= 1000\times(14.057219 + 7.195791 + 6.186911 + 11.385358)\\
&= 38{,}825.28
\end{aligned}$$

Hence

$$\begin{aligned}
\text{Prob}\,(18{,}000S_{12} \ge 38{,}825.28) &= \text{Prob}\,(S_{12} \ge 2.15696)\\
&= \text{Prob}\left(Z \ge \frac{\ln\left(2.15696\right) - 0.910686}{\sqrt{0.025692636}}\right)\\
&= \text{Prob}\,(Z \ge -0.8858)\\
&= \Phi\left(0.89\right)\\
&= 0.81\\
&\text{i.e. } 81\%
\end{aligned}$$

This question provided the greatest range of quality of answers. Many candidates scored well on part (i) although common errors included assuming that $E\left(1+i_t\right) = 0.08$ *and/or that* $\text{Var}\left(1+i_t\right) = 0.05$. *Few candidates calculated the correct value of the required present value in part (ii) and candidates who made errors in this part lost further marks by not showing clear working or sufficient intermediate steps (although the examiners recognise that some candidates might have been under time pressure by the time they attempted this question). The probability calculation was often answered well by candidates who attempted this part.*

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2014 examinations

Subject CT1 – Financial Mathematics
Core Technical

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F Layton
Chairman of the Board of Examiners

November 2014

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Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the September 2014 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates. In general the non-numerical questions were answered poorly by marginal candidates. This applied to bookwork questions such as Q1 and Q8(i) as well as questions requiring interpretation of answers such as Q4(iii), Q8(iv) and (v) and Q9(iv) and (v).

1 One party agrees to pay to the other a regular series of fixed amounts for a certain term. In exchange, the second party agrees to pay a series of variable amounts in the same currency based on the level of a short term interest rate. [2]

2 (i) Expected annual interest rate

$= 0.25 \times 4 + 0.75 \times 7 = 6.25\%$

Let premium = P

$P\,(1.0625)^{20} = 200{,}000$

$\therefore\ P = £59{,}490.99$ [2]

(ii) Expected accumulation is:

$59{,}490.99\,(0.25 \times 1.04^{20} + 0.75 \times 1.07^{20})$

$= 205{,}246.55$

$\therefore$ Expected profit = £5,246.55 [2]

[Total 4]

Many candidates struggled to distinguish between the use of an expected annual interest rate and the expected accumulation after 20 years.

3 (i) $98.83 = 100\,(1 + i)^{-91/365}$

$$\ln(1+i) = \left(-\frac{365}{91}\right) \times \ln(98.83/100) = 0.047205$$

Therefore $i = 0.04834$. [3]

(ii) The rate of interest over 91 days is

$(100 - 98.83) / 98.83 = 0.011839$

The simple rate per annum is:

$$0.011839 \times \frac{365}{91} = 0.04748$$ [2]

[Total 5]

4 (i) Let i be the TWRR per annum effective, then:

$$1+i=\frac{2.1}{2.0}\times\frac{4.2}{2.1+2.5}=0.95870$$

$\Rightarrow$ TWRR $= -4.130\%$p.a. [2]

(ii) Let i be the MWRR per annum effective, then:

$$2.0(1+i)+2.5(1+i)^{2/3}=4.2$$

Try: -8% LHS = 4.20482
-9% LHS = 4.16765

Then $$-i=0.08+0.01\times\left(\frac{4.20482-4.2}{4.20482-4.16765}\right)$$

$i\approx -8.130\% = -8.1\%$p.a. (Exact answer is -8.12985%) [3]

(iii) The MWRR is affected by the timing and amount of cashflows. The fund performs relatively worse when the size of the fund is largest and this will have a greater effect on the MWRR which is consequently lower than the TWRR. [2]

[Total 7]

Parts (i) and (ii) were well-answered. In part (iii), examiners were looking for specific comments regarding this scenario and not just a statement of the bookwork.

5 (i) $\frac{i}{i^{(12)}}a_{\overline{5}|}$ = 1.022715 × 4.3295 = 4.4278 [1]

(ii) $a_{\overline{19}|}-a_{\overline{4}|}$ = 12.0853 – 3.5460 [1]

= 8.5394

(iii) $$\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{\delta}$$

$$=\frac{1.05\times 7.7217-10\times 0.61391}{0.04879}$$

$= 40.3501$ [1]

(iv) $\dfrac{\bar{a}_{\overline{10|}} - 10v^{10}}{\delta} = \dfrac{1.024797 \times 7.7217 - 10 \times 0.61391}{0.04879}$

$= 36.3613.$ [1]

(v) Present value is:

$13\ddot{a}_{\overline{10|}} - (I\ddot{a})_{\overline{10|}}$

$= 1.05 \times (13 \times 7.7217 - 39.3738)$

$= 64.0592$ [2]

[Total 6]

Generally well-answered although some candidates were unable to distinguish between the increasing annuities in parts (iii) and (iv).

6 (i) $P = 0.8 \times 5\, a_{\overline{10|}} + 100v^{10}$ @ 4% per annum.

$a_{\overline{10|}} = 8.1109 \qquad v^{10} = 0.67556$

$P = 0.8 \times 5 \times 8.1109 + 100 \times 0.67556 = 100$

No loss of marks for general reasoning. [2]

(ii) DMT $= \dfrac{\sum tC_t\, v^t}{\sum C_t v^t}$

$= \dfrac{5(Ia)_{\overline{10|}} + 10 \times 100v^{10}}{5a_{\overline{10|}} + 100v^{10}}$

$= \dfrac{5 \times 41.9922 + 10 \times 100 \times 0.67556}{5 \times 8.1109 + 100 \times 0.67556}$

$= \dfrac{885.525}{108.1108} = 8.19$ years [3]

(iii) (a) On average the gross cash flows are earlier because of the higher coupon payments. Therefore the discounted mean term would be lower.

(b) Term of the bond
The gross redemption yield/interest rate at which payments are discounted.

[3]

(iv) All payments are 3 months closer. Therefore, purchase price would be

$$100(1.04)^{1/4} = 100.9853$$ [1]

[Total 9]

Parts (i) and (ii) were answered well. Many candidates' arguments in part (iii)(a) were unclear.

7 (i)

$$A(0,10) = e^{\int_0^{10} 0.03\,dt} = e^{[0.03t]_0^{10}} = e^{0.3}$$

$$A(10,20) = e^{\int_{10}^{20} 0.003t\,dt}$$

$$= e^{\left[\frac{0.003t^2}{2}\right]_{10}^{20}}$$

$$= e^{(0.6-0.15)} = e^{0.45}$$

$$A(20,28) = e^{\int_{20}^{28} 0.0001t^2\,dt}$$

$$= e^{\left[\frac{0.0001t^3}{3}\right]_{20}^{28}} = e^{0.73173-0.26666} = e^{0.46507}$$

Required PV

$$= \frac{1}{A(0,10)\,A(10,20)\,A(20,28)} = e^{-0.3-0.45-0.46507} = e^{-1.21507}$$

$$= 0.29669$$ [7]

(ii) (a) $0.29669 = e^{-28\delta}$

$$\frac{\ln 0.29669}{-28} = \delta = 0.04340 = 4.340\% \text{ per annum}$$

(b) $(1-d)^{28} = 0.29669$

$1 - d = 0.95753$

$d = 0.04247 = 4.247\%$ per annum [3]

(iii) $v(t) = e^{-\int_0^t 0.03ds} = e^{-0.03t}$

$\rho(t) = e^{-0.04t}$

We require:

$$\int_3^7 e^{-0.03t}e^{-0.04t}dt = \int_3^7 e^{-0.07t}dt$$

$$= \left[\frac{-e^{-0.07t}}{0.07}\right]_3^7$$

$= -8.75181 + 11.57977$

$= 2.82797$ [4]

[Total 14]

Answered well. The common mistake was to calculate the effective rate of interest rather than the effective rate of discount in part (ii)(b).

8 (i) (a) Bonds of different terms are attractive to different investors, who will choose assets that are appropriate for their liabilities. The shape of the yield curve is determined by supply and demand at different terms to redemption.

(b) Longer dated bonds are more sensitive to interest rate movements than short dated bonds. It is assumed that risk averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds.

[4]

(ii) Let i_t be the spot yield over t years.

One year: yield is 6% therefore $i_1 = 0.06$
Two years: $(1 + i_2)^2 = 1.06 \times 1.05$ therefore $i_2 = 0.054988$
Three years: $(1 + i_3)^3 = 1.06 \times 1.05 \times 1.04$ therefore $i_3 = 0.049968$.
Four years: $(1 + i_4)^4 = 1.06 \times 1.05 \times 1.04 \times 1.03$ therefore $i_4 = 0.04494$.

[4]

(iii) Price of bond is:

$4[(1.06)^{-1} + (1.054988)^{-2} + (1.049968)^{-3} + (1.04494)^{-4}]$
$\quad + 110 \times 1.04494^{-4}$
$= 4 \times 3.54454 + 92.26294$
$= 106.4411$

Find the gross redemption yield from:

$$106.4411 = 4a_{\overline{4|}} + 110v^4$$

Try 4%

$$a_{\overline{4|}} = 3.6299 \quad v^4 = 0.85480$$

RHS = 108.5476

Try 5%

$$a_{\overline{4|}} = 3.5460 \quad v^4 = 0.82270$$

RHS = 104.681

Interpolate between 4% and 5%:

$$i = 0.04 + 0.01 \times \frac{108.5476 - 106.4411}{108.5476 - 104.681}$$

$= 0.0454$

$= 4.54\%$

[5]

(iv) On average, the payments would be received earlier and discounted at higher spot rates. This means that the gross redemption yield (which is a weighted average of the interest rates used to discount the payments) would be higher.

[2]

(v) The earlier spot rates are likely to fall as a result of greater demand for the bonds with shorter terms to redemption. [2]

[Total 17]

Parts (i) and (iii) were generally answered well with correct approaches in part (iii) given full credit even if the calculations in part (ii) had been incorrect. In common with other similar questions on this paper, the reasoning questions in parts (iv) and (v) were poorly answered.

9 (i)

Date	*Nominal Cash Flow £m*	*Indexed Cash Flow £m*
1/12/2012	0.0075×3.5 = 0.02625	(116/112)×0.02625 = 0.0271875
1/6/2013	0.0075×3.5 = 0.02625	(117/112)×0.02625 = 0.0274219
1/12/2013	0.0075×3.5 = 0.02625	(120/112)×0.02625 = 0.028125
1/6/2014	(1 + 0.0075)×3.5 = 3.52625	(121/112)×3.52625 = 3.8096094

[5]

(ii)

Date	*Indexed Cash Flow £m*	*Index Ratio*	*Real Value of Cash flow £m*
1/12/2012	0.0271875	113/117	0.0262580
1/6/2013	0.0274219	113/118	0.0262599
1/12/2013	0.028125	113/121	0.0262655
1/6/2014	3.8096094	113/122	3.5285726

[4]

(iii) Value of £3.5m nominal is:

$0.0262580v^{½} + 0.0262599v + 0.0262655v^{1½} + 3.5285726v^2$

$= 0.0262580 \times 0.992583 + 0.0262599 \times 0.98522 + 0.0262655 \times 0.97791$

$+ 3.5285726 \times 0.97066$

= £3.502657m

Per £100 nominal $= \dfrac{3.502671}{3.5} \times 100$

= £100.0763 [3]

(iv) The expected rate of return at issue is likely to have been higher. Although the investor is compensated for the higher-than-expected inflation, the time lag used for indexation is likely to mean that he is not fully compensated. Therefore the actual real value of the cash flows is less than the expected real value of the cash flows at issue. [2]

(v) It is likely that the price will fall. The expected real value of the cash flows measured will be lower because the cash flows will be linked to an index expected to rise at a lower rate. [2]

[Total 16]

The most poorly answered question on the paper. Better candidates took advantage of the relatively large number of marks available in parts (i) and (ii) for straightforward calculation work. The important point in part (iii) is to note that the real redemption yield *equation uses inflation adjusted cashflows (in terms of 1 June 2012 prices in this case). In part (iv), the important point is that the time lag causes the investor not to be fully protected against inflation. If there had been no time lag, the actual increase in the retail price index would have no effect on the investor's real rate of return.*

10 (i) Present value of earning if university is not attended:

$$15,000\ a_{\overline{1}|}^{(12)} + 18,000\ a_{\overline{1}|}^{(12)} v + 20,000\ a_{\overline{1}|}^{(12)} v^2 + 20,000 a_{\overline{1}|}^{(12)}$$

$$\times 1.01 v^3 (1 + 1.01v + \ldots + 1.01^{36} v^{36})$$

$$= \frac{i}{i^{(12)}} a_{\overline{1}|} (15,000 + 18,000v + 20,000v^2)$$

$$+ \left(20,000 \frac{i}{i^{(12)}} a_{\overline{1}|} * 1.01 v^3\right)\left(\frac{1 - 1.01^{37} v^{37}}{1 - 1.01v}\right)$$

$$\frac{i}{i^{(12)}} = 1.031691;\ a_{\overline{1}|} = v = 0.93458$$

$v^2 = 0.87344$; $v^3 = 0.81630$; $1.01^{37} = 1.445076$

$v^{37} = 0.08181$.

1.031691 × 0.93458 (15,000 + 18,000 × 0.93458 + 20,000 × 0.8734)

+ (20,000 × 1.031691 × 0.93458

$$\times 1.01 \times 0.81630)\ \left(\frac{1 - 1.445076 \times 0.081809}{1 - 1.01 \times 0.9346}\right)$$

= 47,527.46 + 15,898.86 × 15.7252

= £297,537.30 [7]

(ii) The cost of going to university is:

$$15,000 \times \ddot{a}_{\overline{3}|} @ 7\% = 42,120.27$$

$$\ddot{a}_{\overline{3}|} = 2.6243 \times 1.07 = 2.8080$$

The PV of the salary from attending university at the time of leaving university is:

$$22{,}000\, a_{\overline{1}|}^{(12)} + 25{,}000\, {}_{\overline{1}|}^{(12)} v + 28{,}000\, a_{\overline{1}|}^{(12)} v^2$$

$$+28{,}000 \quad a_{\overline{1}|}^{(12)} \times 1.015 v^3 (1 + 1.015v + \ldots + 1.015^{33} v^{33})$$

$$= \frac{i}{i^{(12)}} a_{\overline{1}|} (22{,}000 + 25{,}000v + 28{,}000v^2)$$

$$+ \left(28{,}000 \frac{i}{i^{(12)}} a_{\overline{1}|} 1.015 v^3 \right) \left(\frac{1 - 1.015^{34} v^{34}}{1 - 1.015v} \right)$$

$1.015^{34} = 1.658996$

$v^{34} = 0.10022$

$= 1.031691 \times 0.93458(22{,}000 + 25{,}000 \times 0.93458$

$+ 28{,}000 \times 0.87344) + 28{,}000 \times 1.031691 \times 0.93458 \times 1.015$

$\times\ 0.81630 \left(\dfrac{1 - 1.658996 \times 0.10022}{1 - 1.015 \times 0.93458} \right)$

$= 67{,}321.02 + 22{,}368.60 \times 16.21996$

$= 430{,}138.80$

PV at time of decision $= 430{,}138.80 \times v^3$

$= 430{,}138.80 \times 0.81630 = £351{,}121.46$

There are various ways in which the answer can be rationalised.

NPV of benefit of going to university (net of earnings lost through the alternative course of action)

$= 351{,}121.46 - 42{,}120.27 - 297{,}537.30$

$= £11{,}463.89$ [9]

(iii) The costs of going to university are incurred earlier and the benefits received later. If the rate of interest is lower, then any loans taken out to finance attendance at university will be repaid more easily at a lower interest cost (answer could say that value of payments received later will rise by more when the interest rate falls). [2]

(iv) Tax is paid on income only at rate t.

Therefore, equation of value is:

$$351{,}121.46\,(1-t) = 42{,}120.27 + 297{,}537.30 \times (1-t)$$

$$351{,}121.46 - 42{,}120.27 - 297{,}537.30$$

$$= t\,(351{,}121.46 - 297{,}537.30)$$

$$\therefore\ 11{,}463.89 = 53{,}584.16t$$

$$\therefore\ t = 0.2139 \text{ or } 21.39\%$$ [2]

[Total 20]

Parts (i) and (ii) were often well-answered although marginal candidates would have benefited from setting out their working more clearly and some candidates failed to' determine whether attending university would be a more attractive option' despite having completed the requisite calculations.

Part (iii) was poorly answered by marginal candidates with few such candidates correctly considering the relative timing of the costs and benefits. Few such candidates attempted part (iv) perhaps because of time pressure. Stronger candidates, however, often obtained close to full marks on the question.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2015 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

June 2015

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2015 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates. In general, the non-numerical questions were answered poorly by marginal candidates.

1 Dividends usually increase annually whereas rents are reviewed less often.
Property is less marketable.
Expenses associated with property investment are higher.
Large, indivisible units of property are less flexible.
On average, dividends will tend to rise more rapidly than rents because dividends benefit from retention and reinvestment of profits in earlier years.

The worst answered question on the paper with over one-third of candidates scoring no marks.

2 (a) Let the answer be t days

$$3{,}000\left(1+0.04\times\frac{t}{365}\right)=3{,}800$$

$t = 2{,}433.33$ days

(b) Let the answer be t days:

$$3{,}000\ (1.04)^{\frac{t}{365}} = 3{,}800$$

$$\therefore (1.04)^{\frac{t}{365}} = \frac{3{,}800}{3{,}000}$$

$$\frac{t}{365}\ln 1.04 = \ln\left(\frac{3{,}800}{3{,}000}\right)$$

$\therefore t = 2{,}199.91$ days.

3 (i) $96.5(1.04)^t = 98$

$t \times \ln(1.04) = \ln(98/96.5)$

Therefore, $t = 0.3933$ years $= 143.54$ days (144 days)

(ii) The second investor held the bill for 182–144 = 38 days

Therefore $98\left(1+\frac{38}{365}i\right) = 100$

$$i = \left(\frac{100}{98}-1\right)\times\frac{365}{38} = 0.19603 \text{ or } 19.603\%$$

(iii) The actual rate of interest over 38 days was (100/98) – 1 = 0.020408

Annual effective rate over 1 year would be:

$$(1 + 0.020408)^{365/38} - 1 = 0.21416 \text{ or } 21.416\%$$

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The theoretical price per share of the forward contract is $£6e^{(0.09-0.035)\times\frac{9}{12}}$ $= £6.2527$

(iii) In this case the actual forward price is too expensive in relation to the stock.

The investor should borrow $£6e^{-0.035\times\frac{9}{12}}$ and use this to buy $e^{-0.035\times\frac{9}{12}}$ units of the stock. The investor will also go short in one forward contract. The continuous dividends are reinvested in the stock. (Mark given for general strategy, exact amounts not required).

[After nine months, the investor will have $e^{0.035\times\frac{9}{12}} \times e^{-0.035\times\frac{9}{12}} = 1$ unit of stock that can be sold under the terms of the forward contract for £6.30. The investor will also have to repay cash of $£6e^{-0.035\times\frac{9}{12}}e^{0.09\times\frac{9}{12}} = £6.2527$.]

Whilst it was not required for candidates to give a full mathematical explanation for part (iii), they were expected to recognise that the forward was overpriced and to determine the arbitrage strategy accordingly.

5 We will use the ½-year as the time unit because the interest rate is convertible half yearly. The effective rate of interest is 3% per half year.

Accumulated amount = $\dfrac{120}{\ddot{a}_{\overline{2}|}}\ddot{s}_{\overline{8}|}\times(1.03)^{16}+60\ddot{s}_{\overline{8}|}^{(2)}\times(1.03)^{8}+60\ddot{s}_{\overline{8}|}^{(6)}$ at 3%

We need $d^{(6)}$ from $\left(1-\dfrac{d^{(6)}}{6}\right)^{6}=1-d=\dfrac{1}{1+i}=\dfrac{1}{1.03}$

$$\Rightarrow 1-\frac{d^{(6)}}{6}=\left(\frac{1}{1.03}\right)^{1/6}\Rightarrow\quad d^{(6)}=\left(1-\left(\frac{1}{1.03}\right)^{1/6}\right)\times 6$$

$$=0\ .029486111$$

Thus accumulated amount =

$$\frac{120}{a_{\overline{2}|}}s_{\overline{8}|}\times(1.03)^{16}+60\frac{i}{d^{(2)}}s_{\overline{8}|}\times(1.03)^{8}+60\frac{i}{d^{(6)}}s_{\overline{8}|}\text{ at }3\%$$

$$=\frac{120}{1.9135}*8.8923*1.60471+60\times1.022445*8.8923*1.26677+60*\frac{0.03}{0.029486111}*8.8923$$

= 894.877 + 691.040 + 542.837

= £2,128.75

(above uses factors in formulae and tables book; if book not used then exact answer is £2,128.7**7)**.

Generally well-answered but marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity.

6 Let:

$i =$ nominal yield
$e =$ inflation rate
$i' =$ real rate

Then

$$1+i=(1+i')(1+e)$$

We first find i and then use above equation to find i':

$$175 = \frac{6}{(1+i)^{1/2}}+\frac{6\times 1.06}{(1+i)^{1\frac{1}{2}}}+\frac{6\times(1.06)^2}{(1+i)^{2\frac{1}{2}}}+\cdots$$

$$=\frac{6}{(1+i)^{1/2}}\left(1+\frac{1.06}{1+i}+\frac{(1.06)^2}{(1+i)^2}+\cdots\right)$$

$$=\frac{6}{(1+i)^{1/2}}\left(\frac{1}{1-\dfrac{1.06}{1+i}}\right)$$

$$=\frac{6(1+i)^{1/2}}{(1+i)\left(1-\dfrac{1.06}{1+i}\right)}$$

$$=\frac{6(1+i)^{1/2}}{i-0.06}$$

Let $i=10\%$, RHS = 157.32
$i=9\%$, RHS = 208.81

Hence

$$i \approx 0.09+0.01\times\left(\frac{208.81-175}{208.81-157.32}\right)$$

$= 0.09657$ (answer to nearest 0.1% is 9.6%)

Then we have

$$1.09657 = (1+i')1.04$$

$\Rightarrow \quad i' = 5.44\%$ p.a. real return (answer to nearest 0.1% is 5.4%)

An alternative method of formulating the equation in real terms to find i' directly was perfectly valid.

7 (i)

Time	*Spot rate of interest*
½	0.01
1½	0.03
2½	0.05
3½	0.07
4½	0.09

Value of liabilities (£m)

$$V = 1\left(v_{1\%}^{0.5} + v_{3\%}^{1.5} + v_{5\%}^{2.5} + v_{7\%}^{3.5}\right) + 2v_{9\%}^{4.5}$$

$_{1\%}v^{½} = 0.99504$

$_{3\%}v^{1.5} = 0.95663$

$_{5\%}v^{2.5} = 0.88517$

$_{7\%}v^{3.5} = 0.78914$

$_{9\%}v^{4.5} = 0.67855$

$V = 4.98308$

(ii) Because expectations of short-term interest rates rise with term and the yield curve is determined by expectations theory.

Because investors have a preference for liquidity which puts an upwards bias on the yield curve (e.g. because long-term bonds are more volatile). A rising curve would be compatible, for example, with constant expectations of interest rates.

Because the market segmentation theory holds and investors short-term bonds might be in demand by investors such as banks (or there is an undersupply of short-term bonds or less demand/more supply for long-term bonds).

(iii) Spot rate to time 4.5 is 9%. Spot rate to time 3.5 is 7%. Therefore:

$$1.09^{4.5}/(1.07)^{3.5} = \text{forward rate from 3.5 to 4.5} = 16.3\%$$

Common errors in part (i) were to assume payments at the end of the year and/or to assume that the payments should be valued with the end of year spot rate (2%, 4%, 6% etc.)

8 (i) $P = 9a_{\overline{10|}7\%} + 100v_{7\%}^{10}$

$v^{10} = 0.50835; \quad a_{\overline{10|}} = 7.02358$

$P \; = \; 9 \times 7.02358 + 100 \times 0.50835$

$= 114.047$

(ii) Discounted mean term

$$= \frac{\sum tC_t v^t}{\sum C_t v^t}$$

$$= \frac{9(Ia)_{\overline{10|}} + 10 \times 100v^{10}}{114.047}$$

$$= \frac{9 \times 34.7391 + 10 \times 100 \times 0.50835}{114.047}$$

= 7.199 years

(iii) Duration will be higher because the payments will be more weighted towards the end of the term.

(iv) (a) Effective duration = duration $/(1+i)$

$= 7.199 / 1.07 = 6.728$ years

(b) Effective duration would indicate the extent to which the value of the bond would change if there were a uniform change in interest rates. It is therefore an indication of the risk to which the investor is exposed if interest rates rise and the price of the security falls before it is sold.

Many of the explanations from candidates in part (iii) were very unclear.

9 Present value of outgoings = 4,000,000 + 900,000$v^{½}$

$$@12\% = 4,850,420$$

Present value of income =

$$360,000v\ddot{a}_{\overline{1}|}^{(4)} + 360,000(1+k)v^2\ \ddot{a}_{\overline{1}|}^{(4)}$$

$$+\ldots+360,000v^5(1+k)^4\ \ddot{a}_{\overline{1}|}^{(4)} + 6,800,000\ v^6$$

$$= 360,000\ \ddot{a}_{\overline{1}|}^{(4)}\ v\left(1+v_j+v_j^2+v_j^3+v_j^4\right) + 6,800,000\ v^6$$

where $j = \dfrac{1.12}{1+k} - 1$

$$\ddot{a}_{\overline{1}|}^{(4)} = 0.95887 @ 12\%$$

So, present value of income $= 360,000 \times 0.95887 \times \dfrac{1}{1.12} \times \ddot{a}_{\overline{5}|}^{j} + 6,800,000\ v^6$

$$= 308,209\ddot{a}_{\overline{5}|}^{j} + 3,445,092$$

Hence, for IRR = 12%, 4,850,420 = $308,209\ \ddot{a}_{\overline{5}|}^{j} + 3,445,092$

so $\ddot{a}_{\overline{5}|}^{j} = 4.55966$

At 4% $\ddot{a}_{\overline{5}|} = 4.62990$

5% $\ddot{a}_{\overline{5}|} = 4.54595$

$$j \approx 4 + \frac{4.62990 - 4.55966}{4.62990 - 4.54595} = 4.837\%$$

$$j = \frac{1.12}{1+k} - 1$$

$$0.04837 = \frac{1.12}{1+k} - 1$$

$$\therefore 1.04837(1+k) = 1.12$$

$$k = \frac{1.12}{1.04837} - 1 = 0.0683 = 6.83\% \text{ (exact answer is 6.84\%)}$$

Marginal candidates again would have benefited from showing more intermediate working. In project appraisal questions, it is good exam technique to show working and answers for each component of income and outgo separately so that partial marks can be given if any errors are made within a component.

10 (i) For $0 \le t \le 4$:

$$v(t) = \exp\left(-\int_0^t \delta(s)\,ds\right) = \exp\left(-\int_0^t 0.08ds\right)$$

$$= e^{-0.08t} \text{ and } v(4) = e^{-0.08\times 4} = e^{-0.32}$$

For $4 < t \le 9$:

$$v(t) = \exp\left(-\int_0^4 \delta(s)\,ds - \int_4^t \delta(s)\,ds\right) = e^{-0.32}\exp\left(-\int_4^t 0.12 - 0.01sds\right)$$

$$= e^{-0.32}\exp\left[-0.12s + 0.005s^2\right]_4^t$$

$$= e^{-0.32}\exp\left(0.48 - 0.08 - 0.12t + 0.005t^2\right)$$

$$= e^{0.08-0.12t+0.005t^2}$$

and $v(9) = e^{0.08-0.12\times 9+0.005\times 81} = e^{-0.595}$

For $t > 9$:

$$v(t) = \exp\left(-\int_0^9 \delta(s)\,ds - \int_9^t \delta(s)\,ds\right) = e^{-0.595}\exp\left(-\int_9^t 0.05ds\right)$$

$$= e^{-0.595}\exp\left[-0.05s\right]_9^t = e^{-0.595}\exp\left(0.45 - 0.05t\right) = e^{-0.145-0.05t}$$

(ii) Present value is

$$\int_{10}^{12} 100e^{0.03t}\left(\exp\left(-\int_0^t \delta(s)\,ds\right)\right)dt$$

$$=\int_{10}^{12} 100e^{0.03t}e^{-0.145-0.05t}dt$$

$$=\int_{10}^{12} 100e^{-0.145-0.02t}dt=\left[\frac{100e^{-0.145-0.02t}}{-0.02}\right]_{10}^{12}$$

$$=\left[-5,000e^{-0.145-0.02t}\right]_{10}^{12}=5,000\left(e^{-0.345}-e^{-0.385}\right)$$

$$=138.85$$

(iii) Present value $=1,000a_{\overline{3}|\delta=8\%}=1,000\dfrac{1-e^{-0.08\times 3}}{e^{0.08}-1}=£2,561.89$

11 (i) $(Ia)_{\overline{n}|}=v+2v^2+3v^3+\ldots+nv^n$ (1)

$(1+i)(Ia)_{\overline{n}|}=1+2v+3v^2+\ldots+nv^{n-1}$ (2)

$(2)-(1)\Rightarrow i(Ia)_{\overline{n}|}=1+v+v^2+\ldots+v^{n-1}-nv^n$

$$\Rightarrow (Ia)_{\overline{n}|}=\frac{\left(1+v+v^2+\ldots+v^{n-1}\right)-nv^n}{i}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i}$$

(ii) Work in months i.e. use a monthly interest rate of 1.25% per month effective:

$$30,000=Xv+2Xv^2+\ldots+60Xv^{60}=X\,(Ia)_{\overline{60}|@1.25\%}=X\left(\frac{\ddot{a}_{\overline{60}|}-60v^{60}}{i}\right)$$

$$=X\left(\frac{\dfrac{1-v^{60}}{d}-60v^{60}}{i}\right)=X\left(\frac{\dfrac{1-1.0125^{-60}}{0.0125/1.0125}-60\times 1.0125^{-60}}{0.0125}\right)$$

$$=1126.8774X\Rightarrow X=£26.62$$

(iii) Equation of value:

$$30,000 = v^{36} 958.32 a_{\overline{60|}}$$

$$\Rightarrow v^{36} a_{\overline{60|}} = 31.3048$$

Try i = 1%: LHS = 31.4202

Try i = 1.1%: LHS = 29.5098

Interpolate: $i = 1\% + 0.1\% \left(\dfrac{31.3048 - 31.4202}{29.5098 - 31.4202} \right) = 1.0060\%$

APR is $(1 + 0.010060)^{12} - 1 = 12.8\%$ to 1 d.p.

The bank is unlikely to be happy to accept the suggestion as it will be earning a lower rate of return compared with the original proposal of 15% per annum convertible monthly (=16.1% per annum effective).

(iv) The student's arrangement will lead to a greater total of payments (60 payments of 36X) when compared to the original arrangement (60 payments of 30.5X on average) but will incur a lower rate of interest. This is because under the student's arrangement no capital or interest will be paid for three years. The extra total of payments will not be sufficient to cover the deferred interest at the bank's preferred rate.

In part (i), candidates were expected to show the first and last terms of each series used to derive the result so that the proof is absolutely clear. In part (ii), candidates should show enough steps to demonstrate that they have performed the calculations required to actually prove the answer (e.g. show the numerical values for the factors used) . In part (iv), if interpolating on a monthly interest rate (as in the above solution) the guesses most be close enough together to ensure the estimated annual rate is close enough to the correct answer.

12 (i) Let S_n = Accumulated value at time n of £1 invested at time 0

$$S_n = (1+i_1)(1+i_2)....(1+i_n)$$

$$\Rightarrow E[S_n] = E[(1+i_1)(1+i_2)....(1+i_n)]$$

$$= E(1+i_1).\,E(1+i_2)....\,.E(1+i_n) \text{ by independence}$$

and $E(1+i_t) = 1 + E(i_t) = 1 + j$

Hence

$$E(S_n) = (1+j)^n$$

Now

$$\text{Var}[S_n] = E\left[S_n^2\right] - \left(E[S_n]\right)^2$$

$$E\left[S_n^2\right] = E\left[(1+i_1)^2 (1+i_2)^2 \ldots (1+i_n)^2\right]$$

$$= E\left[(1+i_1)^2\right].E\left[(1+i_2)^2\right]\ldots\quad .E\left[(1+i_n)^2\right]$$

by independence

and

$$E\left[(1+i_t)^2\right] = E\left[\left(1+2i_t+i_t^2\right)\right]$$

$$= 1+2\,E(i_t)+E\left(i_t^2\right)$$

and

$$\text{Var}[i_t] = s^2 = E\left(i_t^2\right) - \left[E(i_t)\right]^2$$

$$= E\left(i_t^2\right) - j^2$$

$$\Rightarrow E\left(i_t^2\right) = s^2 + j^2$$

Hence

$$E\left[S_n^2\right] = \left(1+2j+j^2+s^2\right)^n$$

and

$$\text{Var}[S_n] = \left(1+2j+j^2+s^2\right)^n - (1+j)^{2n}$$

(ii) (a) $E\left(1+i_t\right)=1+E\left(i_t\right)=1+j=1.04=e^{\left(\mu+\sigma^2/2\right)}$

$$\operatorname{Var}\left(1+i_t\right)=\operatorname{Var}\left(i_t\right)=s^2=0.12^2=e^{\left(2\mu+\sigma^2\right)}\times\left(e^{\sigma^2}-1\right)$$

$$\Rightarrow \frac{0.12^2}{(1.04)^2} = e^{\sigma^2}-1$$

$$\Rightarrow \sigma^2=Ln\left[1+\left(\frac{0.12}{1.04}\right)^2\right]$$

$$\Rightarrow \sigma^2 = 0.013226$$

$$1.04=e^{\left(\mu+\frac{0.013226}{2}\right)}$$

$$\Rightarrow \mu = Ln\,1.04-\frac{0.013226}{2}$$
$$= 0.032608$$

(b) $Ln\left(1+i_t\right)\sim N\left(0.032608, 0.013226\right)$

and we require probability $0.06<i_t<0.08$

$$=\Pr\left(1.06<1+i_t<1.08\right)$$

$$=\Pr\left(Ln\,1.06<Ln\left(1+i_t\right)<Ln\,1.08\right)$$

$$=\Pr\left(\frac{Ln\,1.06-0.032608}{\sqrt{0.013226}}<\frac{Ln\left(1+i_t\right)-\mu}{\sigma}<\frac{Ln\,1.08-0.032608}{\sqrt{0.013226}}\right)$$

$$=\Pr\left(0.22<Z<0.39\right) \qquad \text{where } Z\sim N\left(0,1\right)$$

$$=\Phi\left(0.39\right)-\Phi\left(0.22\right)$$

$$=0.65173-0.58706=0.0647$$

i.e. 6% probability (using exact Φ function gives probability of 6.2%)

(iii) The probability in (ii) (b) is small. This is reasonable since the expected return in any year is 4%, and we are being asked to calculate the probability that the return is between 6%, and 8% (i.e. a range which does not include the expected value).

In part (i), it is important to note when the assumption of independence is required for both proofs. Common mistakes in the calculation of μ and σ were to assume that s^2 was 0.12 (rather than s) and 1 + j was 0.04 (rather than 1.04).

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2015

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners
December 2015

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

2. Student performance was poorer than in previous years. The performance across all questions was of a lower standard which indicated that the lower performance was not as the result of some particularly difficult individual questions.

3. There were some ambiguities in the wording and mark scheme for Q8 but the marking process was adjusted to ensure that candidates were not disadvantaged.

4. There were also elements of non-numerical explanation and analysis required in several questions and, as in previous papers, students performed relatively badly on these sections.

5. Finally, it appeared that many students left themselves short of time for the last question, Q9, which was worth 26 marks.

C. Comparative pass rates for the past 3 years for this diet of examination

Year	*%*
September 2015	44
April 2015	55
September 2014	57
April 2014	60
September 2013	57
April 2013	60

Reasons for any significant change in pass rates in current diet to those in the past:

Historically, the papers have been set by experienced examiners and this has led to very stable pass marks and relatively stable pass rates. This paper was set by the same examining team.

As highlighted in Section B, the performance by candidates on this paper has been significantly poorer than for past exams in this subject. The examiners do not believe that the paper was significantly different this session and any potential ambiguities in the paper were fully allowed for within the marking process.

We have no definite cause for the lower marks. An analysis on a question by question basis shows that candidates on average were only scoring above 60% on average on two questions out of nine, one of which was the 3-mark Q1. This all suggests a relatively weak cohort.

Solutions

Q1 $P = 100(1.03)^{-91/365} = 0.99266$ or £99.266

Let annual simple rate of discount = d

$$\frac{99.266}{100} = 1 - \frac{91}{365}d$$

Therefore $\frac{91}{365}d = 0.00734$; $d = 0.02945$

Q2 (i) (a) $\delta = \ln(1+i) = \ln\left[\left(1-\frac{d^{(12)}}{12}\right)^{-12}\right] = \ln\left[\left(1-\frac{0.055}{12}\right)^{-12}\right] = 0.055126$

$= 5.513\%$

(b) $1+i = \left(1-\frac{d^{(12)}}{12}\right)^{-12} \Rightarrow i = \left(1-\frac{0.055}{12}\right)^{-12} - 1 = 0.056674 = 5.667\%$

(c) $1+\frac{i^{(12)}}{12} = \left(1-\frac{d^{(12)}}{12}\right)^{-1} \Rightarrow i^{(12)} = 12\left[\left(1-\frac{0.055}{12}\right)^{-1} - 1\right] = 0.055253$

$= 5.525\%$

(ii) When interest is paid monthly, the interest that is paid in earlier months itself earns interest. This means that to achieve the same effective rate over the year, the nominal rate must be lower.

(iii) $100 \;=\; 159(1-d)^8 \Rightarrow d = 1-\left(\dfrac{100}{159}\right)^{1/8} = 0.056319 = 5.632\%$

(iv) $1+i = 1.12^{1/2} \Rightarrow i = 0.058301 = 5.830\%$

> Parts (i) and (iii) were generally calculated well although many candidates chose not to give their final answer to the requested accuracy. Some candidates attempted to use their answers to parts (i)((a) and (i)(b) to find an answer to part (i)(c) – this was not accepted by the examiners . Part (iv) proved more challenging to many candidates. Amongst the marginal candidates, there were very few who gave a clear explanation for part (ii). It is a matter of concern that so many candidates were unable to articulate the relationship between nominal and effective interest rates.

Q3 (i) Duration of the annuity payment is $\dfrac{(Ia)_{\overline{12|}}}{a_{\overline{12|}}} = \dfrac{56.6328}{9.3851} =$ 6.0343 years

(ii) Duration of bond is:

$$\frac{5(Ia)_{\overline{8|}} + 800v^8}{5a_{\overline{8|}} + 100v^8}$$

$$= \frac{5\times 28.9133 + 800\times 0.73069}{5\times 6.7327 + 100\times 0.73069}$$

$$= \frac{729.119}{106.733} = 6.8313 \text{ years}$$

(iii) The duration of the assets (the bond) is greater than the duration of the liabilities (pension payments). If there is a rise in interest rates, the present value of the assets will fall by more than the present value of the liabilities and the insurance company will make a loss.

> Parts (i) and (ii) were answered well. Where a term is calculated, it is particularly important to include the units in the final answer. Part (iii) was very poorly answered with many candidates stating that the company must make a loss because the durations were not equal.

Q4 (i) Assuming no arbitrage

The present value of the dividends (in £), I, is:

$I = 0.5(e^{-0.05\times(1/12)} + e^{-0.05\times(7/12)}) = 0.5 \times (0.995842 + 0.971255) = 0.98355$

Hence, forward price $F = (10 - 0.98355)e^{0.05(9/12)} = £9.3610$

(ii) The expected price of the share does not have to be taken into account because, using the no-arbitrage assumption, the purchaser of the forward is simply able to use the current price of the share (and the value of the dividends) given that the forward is simply an alternative way of exposing the investor to the same set of cash flows.

[The expected future price of the share will be taken into account by investors when determining the price they wish to pay for the share and therefore the current share price.]

> Part (i) was often answered well although some candidates miscalculated the timing of the dividends and the statement of the arbitrage assumption was often missed. Part (ii) was poorly answered despite being similar to previous exam questions.

Q5 (i) Present value is

$$\int_3^6 \rho(t)v(t)dt = \int_3^6 5000v(t)dt$$

For $t \geq 3$

$$v(t) = \exp\left(-\int_0^t \delta(t)\,dt\right) = \exp\left(-\int_0^3 0.03 + 0.005t\,dt - \int_3^t 0.005\,dt\right)$$

$$= \exp\left[-0.03t - 0.0025t^2\right]_0^3 \exp\left[-0.005t\right]_3^t$$

$$= \exp\left[-0.1125\right]\exp\left[0.015 - 0.005t\right]$$

$$= \exp(-0.005t - 0.0975)$$

Hence present value is

$$\int_3^6 5,000\exp(-0.005t - 0.0975)dt$$

$$= 5,000e^{-0.0975}\int_3^6 e^{-0.005t}dt$$

$$= \frac{5,000e^{-0.0975}}{-0.005}\left[e^{-0.005t}\right]_3^6 = -1,000,000e^{-0.0975}(e^{-0.03} - e^{-0.015})$$

$$= -880.293.42 + 893,597.35 = £13,303.93$$

(ii) $5,000(\bar{a}_{\overline{6|}} - \bar{a}_{\overline{3|}}) = 13,303.93$

i = 2%: LHS = 13,723
i = 3%: LHS = 13,136

Interpolating

$$i \approx 0.02 + 0.01 \times \frac{13,304 - 13,723}{13,136 - 13,723} = 2.714\% \text{ say } 2.7\%$$

(iii) Accumulation =

$$= 300A(50) = 300\exp(0.005 \times 50 + 0.0975)$$

$$= 300e^{0.3475} = £424.66$$

The discount factor was usually calculated correctly although some candidates just calculated this factor for t = 6 and assumed that the value of a single payment at this time was required. Part (ii) was poorly answered. The important point is that the rate of interest is obtained by equating the amount initially invested as calculated in part (i) with the present value of the annuity.

Q6 (i) $101 = 3a_{\overline{3|}} + 100v^3$

i = 3%: RHS = 100
i = 2.5%: RHS = 101.428

Interpolating

$$i \approx 0.025 + 0.005 \times \frac{101 - 101.428}{100 - 101.428} = 2.65\%$$

(ii) Let i_n = spot rate for term n

One year bond gives

$$101 = 103v_{i_1}$$

$$v_{i_1} = \frac{101}{103} = 0.98058$$

$$\Rightarrow i_1 = \frac{103}{101} - 1 = 1.980\%$$

Two year bond gives

$$101 = 3v_{i_1} + 103v_{i_2}^2$$

$$\Rightarrow v_{i_2}^2 = \frac{101 - 3\frac{101}{103}}{103} = 0.95202$$

$$\Rightarrow i_2 = 2.489\%$$

Three year bond gives

$$101 = 3v_{i_1} + 3v_{i_2}^2 + 103v_{i_3}^3$$

$$\Rightarrow v_{i_3}^3 = \frac{101 - 3\times 0.98058 - 3\times 0.95202}{103} = 0.92429$$

$$\Rightarrow i_3 = 2.659\%$$

(iii) Forward rate is $f_{2,1}$ where

$$1 + f_{2,1} = \frac{(1+i_3)^3}{(1+i_2)^2} = \frac{1.02659^3}{1.02489^2} = 1.03000 \Rightarrow f_{2,1} = 3.000\%$$

(iv) Reasons could include:

Expectations theory suggests that if short-term interest rates are expected to rise then if yields are the same on both long- and short-term bonds, short-term bonds will be more attractive and longer term bonds less attractive and so the yields on short-term bonds will fall relative to those on long-term bonds.

[Expected higher inflation could be a reason for this but could be allowed as a distinct point]

Liquidity preference theory suggests that investors demand higher rates of return for less liquid/longer term-to-maturity investments which are more sensitive to interest rate movements.

Market segmentation with the supply of bonds being restricted at shorter terms or some factor that leads to the demand for bonds of longer terms to be lower

> Common errors on this question included assuming the price of each bond was 100 and that the gross redemption yield of the 3-year bond was equal to the three-year spot rate. In part (iv) many candidates just gave the names of theories of the yield curve without explaining how this applied in this particular scenario. Otherwise, this was the best answered question on the paper apart from Q1.

Q7 (i) Maximum rate of return after 20 years

$$100 = 0.75\left(7a_{\overline{20}|}^{(2)} - 3a_{\overline{10}|}^{(2)}\right) + 130v^{20}$$

Try i = 5%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.05^{-20}) - 3\times(1-1.05^{-10})}{2(1.05^{1/2}-1)}\right) + 130\times1.05^{-20}$$

$$= 48.6460 + 48.9956 = 97.6417$$

Try i = 4%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.04^{-20}) - 3\times(1-1.04^{-10})}{2(1.04^{1/2}-1)}\right) + 130\times1.04^{-20}$$

$$= 53.6255 + 59.3303 = 112.9558$$

Interpolating

$$i \approx 0.04 + 0.01\times\frac{100-112.9558}{97.6417-112.9558} = 4.846\% = 4.8\%$$

(Exact answer is 4.8338%.)

(ii) Minimum rate of return after 10 years

In this case, the investor invests 100, receives 100 back and receives a net income at a rate of 1.5 per half-year. The rate of return per half-year effective is therefore 1.5 per cent.

The annual effective rate of return is $1.015^2 - 1 = 3.0225\%$.

(iii) There is a 0.5 probability of both early redemption and of late redemption. The expected return is therefore $0.5(4.8338\% + 3.0225\%) \approx 3.928\%$

(iv) If the investor buys the whole loan, the present value of the cash flows from the loan is as follows (per €100 nominal):

$$= 0.75 \times 4 a_{\overline{10}|}^{(2)} + 0.5 \times 100 v^{10} + 0.5\left(0.75 \times 7 a_{\overline{10}|}^{(2)} v^{10} + 130 v^{20}\right)$$

At i = 3.928% this is

$$= 3\frac{1-1.03928^{-10}}{2(1.03928^{1/2}-1)} + 50 \times 1.03928^{-10}$$

$$+0.5\left(5.25\frac{1-1.03928^{-10}}{2(1.03928^{1/2}-1)}1.03928^{-10} + 130 \times 1.03928^{-20}\right)$$

$$= 24.6575 + 34.0125 + 0.5(29.3538 + 60.1562)$$

$$= 103.4264$$

This is greater than 100 and so the rate of return will be greater than 3.928% (exact return is 4.212%).

This was the worst answered question on the paper with many candidates not recognising that the cases where the bond is redeemed after 10 years and after 20 years have to be calculated separately for parts (i) and (ii). If candidates obtained answers for parts (i) and (ii) then part (iii) was usually done well. However, few candidates recognised that substituting the return from part (iii) into the required equation for part (iv) would lead to the required answer.

The question did not state specifically that the coupons in the second 10 years were semi-annual although most students assumed this. Candidates who assumed a different coupon frequency were given full credit.

Q8 (i)

- Generally issued by commercial undertakings and other bodies.
- Shares are held by the owners of a company who receive a share in the company's profits in the form of dividends
- Potential for high returns relative to other asset classes…
- …but high risk particularly risk of capital losses
- Dividends are not fixed or known in advance and…
- …the proportion of profits paid out as dividends will vary from time-to-time
- No fixed redemption date
- Lowest ranking finance issued by companies.
- Return made up of income return and capital gains.

- Initial running yield low but has potential to increase with dividend growth…
- …in line with inflation and real growth in company earnings.
- Marketability depends on the size of the issue.
- Ordinary shareholders receive voting rights in proportion to their holding.

(ii) Let

P = price investor is willing to pay
d = next expected dividend
g = expected annual dividend growth rate
r = annual required return
t = tax rate

Then

$$P = \frac{d(1-t)}{1+r} + \frac{d(1-t)(1+g)}{(1+r)^2} + \frac{d(1-t)(1+g)^2}{(1+r)^3} + \dots$$

$$= \frac{d(1-t)}{1+r}\left[1 + \frac{1+g}{1+r} + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \dots\right]$$

$$= \frac{d(1-t)}{1+r}\frac{1}{1-\frac{1+g}{1+r}} = \frac{d(1-t)}{r-g}$$

(iii) $d = 6p$
$g = 0.01$
$r = 0.06$
$t = 0.2$

$$P = \frac{d(1-t)}{r-g} = \frac{6(1-0.2)}{0.06-0.01} = 96p$$

(iv) If the share were regarded as more risky, then the required return, r, would increase. If r were to increase, this would reduce the value of the share as r is in the denominator (and is positive).

(v) Equation of value would be (working in money terms):

$$960 = 0.8 \times 60v + 1200v - 0.25(1200 - 960)v$$

$$\Rightarrow v = \frac{96}{118.8}$$

Therefore net money rate of return, i, is $\frac{118.8}{96} - 1 = 23.75\%$

Net real rate of return is $\frac{1.2375}{126/123} - 1 = 20.80\%$

> There was an error in the paragraph prior to part (v) of this question where the calculation was designed to be based on the purchase/sale of 1,000 shares but the question referred to the sale of "the share". Nearly all candidates based their calculation on the purchase/sale of the same number of shares (whether it be 1 share or 1,000) but candidates who made a different assumption were not penalised.
>
> There was no split of the marks between parts (ii) and part (iii) given on the paper. Candidates who just performed the calculation without the derivation of the formula were give appropriate credit but a formula derivation was required to obtain the full five marks for these parts. The question did not state specifically that the dividends were paid annually although almost all candidates assumed this. Candidates who assumed other payment frequencies were given full credit.

Q9 (i) (a) The payback period is the first time at which the total incoming cash flows are equal or greater in amount than the total outgoing cash flows.

Total incoming cash flows at the beginning of year t = 60,000t.

Determine t for which $60,000t \geq 1,000,000 \Rightarrow t \geq 16.67$

Therefore the payback period is 16 years.

(b) The net total income received in any year from project A is never greater than £60,000. As the costs are incurred at the beginning of the year, there is no point at which the total income from project A is greater than the total income from project B until the very end of the project when the properties are sold. The payback period for B must therefore be less than that for A.

(ii) (a) The discounted payback period occurs where the present value (or accumulated value) of incoming cash flows is equal to or greater than that of outgoing cash flows for the first time.

(b) Equation of value for project B is (in £000):

$$60\ddot{a}_{\overline{t}|} \geq 1,000$$

$$\Rightarrow a_{\overline{t}|} \geq \frac{1,000}{60(1+i)} = \frac{1,000}{60.6} = 16.5016$$

Need to solve for *t*. From inspection of tables, *t* = 19 and so the discounted payback period is 18 years.

(c) Again, given that the net income from project A is never greater in an individual year, than that from project B, at no rate of interest can the discounted value of the net income from project A be greater than that for the income from project B.

(iii) Internal rate of return from project B is the solution to the following equation of value (all figures in 000s):

$$1,000 = 60\ddot{a}_{\overline{20}|} + 1,000v^{20}$$

This can be solved by general reasoning.

As the investor invests 1,000 and receives an annual income of 60 in advance and receives his capital back at the end, the total rate of return, *d*, expressed as an effective rate of discount per annum is 6 per cent.

Internal rate of return is $i = \dfrac{d}{1-d} = \dfrac{0.06}{0.94} = 6.383\%$

(iv) If the IRR from project A is higher then it must have a net present value > zero at a rate of interest of 6.383 per cent.

Note that $v = 0.94$ at $i = 6.383\%$

Present value of costs for project A:

$$= 1,000 + 10(1 + 1.005v + 1.005^2v^2 + \ldots + 1.005^{19}v^{19})$$

$$= 1,000 + 10\left(\frac{1 - 1.005^{20}v^{20}}{1 - 1.005v}\right)$$

$$= 1,000 + 10\left(\frac{1 - (1.005 \times 0.94)^{20}}{1 - 1.005 \times 0.94}\right)$$

$$=1,000+10\frac{0.67946}{0.0553}=1,122.869$$

Present value of revenue for project A:

$$=2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}v^5(1.005+1.005^2v+\ldots+1.005^{15}v^{14})$$

$$=2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}1.005v^5(1+1.005v+\ldots+1.005^{14}v^{14})$$

$$=2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}1.005v^5\left(\frac{1-1.005^{15}v^{15}}{1-1.005\ \ v}\right)$$

$$=2,000\times 0.94^{20}+60\times 0.94\left(\frac{1-0.94^4}{12(1-(1-0.06)^{1/12})}\right)$$

$$+60\times 1.005\times 0.94^5\left(\frac{1-0.94}{12(1-(1-0.06)^{1/12})}\right)\left(\frac{1-(1.005\times 0.94)^{15}}{1-1.005\times 0.94}\right)$$

$$=580.212+200.365+446.577=1,227.154$$

NPV of project at IRR from project A is: 1,227.154 – 1,122.869 = 104.285 (= £104,285)

This is clearly positive so project A has a higher IRR.

(v) Project B would be preferred on the basis of both payback period and discounted payback period.

However, both these measures have shortcomings. The first does not take into account interest at all and the second does not take into account cash flows after the discounted payback period [or in the case of project A the occurrence of one large cash flow at the time of the discounted payback period]

Project A would be preferred on the basis of internal rate of return.

The internal rate of return measures the total return on the project and therefore is a better decision criterion than payback period or discounted payback period.

There may be other factors (comparison of NPVs at a particular rate of interest or the risk of the two projects) that should be taken into account.

Other factors could include, for example:

student’s need for return of original investment
reliability of estimates of future cashflows

It appeared that many candidates were under time pressure when attempting this question. Nearly all candidates failed to recognise how the payments being at the start of the year for project B would impact on the payback period and the discounted payback period. Many candidates’ general reasoning arguments for parts (i)(b) and (ii)(c) were unclear. In part (c), a common mistake was to miss out the return of the original investment in the calculation of the IRR. In part (iv), common errors were to miscount the number of terms (for both costs and revenue). As for similar long questions in previous years, marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity. There were a wide range of points that could be made to score marks in part (v) but few candidates scored well on this part.

END OF EXAMINERS’ REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2016 (with mark allocations)

Subject CT1 – Financial Mathematics Core Technical

Introduction

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F Layton
Chair of the Board of Examiners
June 2016

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

2. Performance was of a similar standard to most recent diets with the weaker performance in September 2015 being an exception to the general standard. As in previous diets, the non-numerical questions were often answered poorly by marginal candidates.

C. Pass Mark

The Pass Mark for this exam was 60%.

Solutions

Q1 Convertible Securities

- Generally unsecured loan stocks.
- Can be converted into ordinary shares of the issuing company.
- Pay interest/coupons until conversion.
- The date of conversion might be a single date or, at the option of the holder, one of a series of specified dates.
- Risk characteristics of convertible vary as the final date for convertibility approaches (behaviour will tend towards the security into which it is likely to convert)
- Generally less volatility than in the underlying share price before conversion.
- Combine lower risk of debt securities with the potential for gains from equity investment.
- Security and marketability depend upon issuer.
- Generally provide higher income than ordinary shares and lower income than conventional loan stock or preference shares.
- Option to convert will have time value which is reflected in price of the security.

[½ mark for each valid point]

[MAX 3]

> Despite being a bookwork question, this was answered poorly. This performance is consistent with questions in previous years on the same area of the syllabus.

Q2 (i) PV of asset proceeds is:

$$V_A(0.08) = 5.5088v_{8\%}^5 + 13.7969v_{8\%}^{20} = 6.7093$$

PV of liability outgo is:

$$V_L(0.08) = 6v_{8\%}^8 + 11v_{8\%}^{15} = 6.7093 = V_A(0.08)$$

Hence, condition (1) for immunisation is satisfied. [2]

Also, DMT of asset proceeds is:

$$\tau_A(0.08) = \frac{5 \times 5.5088v_{8\%}^5 + 20 \times 13.7969v_{8\%}^{20}}{6.7093} = 11.618$$

And, DMT of liability outgo is:

$$\tau_L(0.08) = \frac{8 \times 6v_{8\%}^8 + 15 \times 11v_{8\%}^{15}}{6.7093} = 11.618 = \tau_A(0.08)$$

Hence, condition (2) for immunisation is also satisfied. [3]

(ii) Yes, the insurance company is immunised.

As the asset proceeds are received at times 5 and 20, whereas the liability outgo is paid at times 8 and 15, the spread of the asset proceeds around the DMT is greater than the spread of the liability outgo around the same DMT. [2]

[TOTAL 7]

> Part (i) was generally answered well; however, candidates must include sufficient factors and workings to demonstrate that the respective asset and liability values are the same for each of the two conditions. Part (ii) was often answered poorly. To get full marks for this part, candidates were required to make reference to the actual data in the question rather than just repeating the theory (e.g. stating the actual figures for the spread of the assets and the liabilities around the DMT).

Q3 (i) Issue price (per £100 nominal) = $\frac{4}{1+i_i}+\frac{4}{(1+i_2)^2}+\frac{4}{(1+i_3)^3}+\frac{105}{(1+i_3)^3}$ [1½]

where i_t is the t-year zero coupon rate at time $t = 0$ and we have that:

$$\left(1+i_{t-1}\right)^{t-1} * (1+f_{t-1,1}) = (1+i_t)^t$$

where $f_{t-1,1}$ is the one-year forward rate at time $t - 1$

we have $1 + i_1 = 1.04$ (i_1 is given)

$$(1+i_2)^2 = (1+i_1)(1+f_{1,1})$$

$$= 1.04*1.05$$

$$(1+i_3)^3 = (1+i_2)^2(1+f_{2,1})$$

$$= 1.04*1.05*1.06$$ [1½]

$$\Rightarrow \text{Issue Price} = \frac{4}{1.04}+\frac{4}{1.04*1.05}+\frac{4+105}{1.04*1.05*1.06}$$

$$= £101.68$$ [1]

(ii) Let y_{c_3} be the 3 year par yield (%). Then y_{c_3} is given by

$$100 = y_{c_3}\left(\frac{1}{1+i_1}+\frac{1}{(1+i_2)^2}+\frac{1}{(1+i_3)^3}\right)+\frac{100}{(1+i_3)^3} \quad [1½]$$

$$= y_{c_3}\left(\frac{1}{1.04}+\frac{1}{1.04*1.05}+\frac{1}{1.04*1.05*1.06}\right)+\frac{100}{1.04*1.05*1.06}$$

$$= y_{c_3}*2.741205336+86.39159583$$

$$\Rightarrow y_{c_3} = 4.9644\% \quad [1½]$$

[TOTAL 7]

Generally well answered.

Q4 (i) $\left(1+\dfrac{i^{(4)}}{4}\right)^4 = 1.05 \Rightarrow i^{(4)} = 0.049089$

$$\frac{D}{R}(1-t_1) = \frac{0.07}{1.08}\times 0.75 = 0.04861$$

$$\Rightarrow i^{(4)} > (1-t_1)g$$

$\Rightarrow$ Capital gain on contract and we assume loan is redeemed as late as possible (i.e. after 20 years) to obtain minimum yield. [2]

Let price of stock = P

$$P = 0.07\times 100000\times 0.75\times a_{\overline{20|}}^{(4)} + (108000-0.40(108000-P))v^{20} \text{ at } 5\%$$

$$\Rightarrow P = \frac{5250\, a_{\overline{20|}}^{(4)} + 64800\, v^{20}}{1-0.40\, v^{20}}$$

$$= \frac{5250\times 1.018559\times 12.4622 + 64800\times 0.37689}{1-0.40\times 0.37689}$$

$$= £107,228.63 \quad [3]$$

(above uses factors from Formulae and Tables Book – exact answer is £107,228.67)

(ii) As the redemption date is at the option of the borrower, it is outside the investor's control when the stock will be redeemed. Hence the investor must assume a worst case scenario in pricing the loan. [2]

[TOTAL 7]

Part (i) was answered well. The reasoning of marginal candidates in part (ii) was often unclear. The key point is that the date of redemption is out of the control of the investor.

Q5 (i) Loan = $950\, a_{\overline{15|}} + 250(Ia)_{\overline{15|}}$ at 6%

$= 950 \times 9.7122 + 250 \times 67.2668$

$= £26{,}043.29$

[3]

(ii) Capital outstanding after 9 payments:

$$3200\, a_{\overline{6|}} + 250(Ia)_{\overline{6|}} = 3200 \times 4.9173 + 250 \times 16.3767 = £19{,}829.54$$

[2]

(iii) Capital outstanding after 14 payments $= 4700v$ at 6%

$= £4{,}433.96$

$=$ Capital in final payment

$\Rightarrow$ Interest in final payment $= 4700 - 4433.96$

$= £266.04$

[2]

(above uses factors from Formulae and Tables Book – exact answers are £26,043.34 for (i) and £19,829.61 for (ii))

[TOTAL 7]

The best answered question on the paper.

Q6 (i) $$pv = 10{,}000 \times \exp\left[-\int_7^{10}(0.01t - 0.04)\,dt\right] \times \exp\left[-\int_5^7(0.10 - 0.01t)\,dt\right]$$ [1]

$$= 10{,}000\exp \times \left(-\left[\frac{0.01t^2}{2} - 0.04t\right]_7^{10}\right) \times \exp\left(-\left[0.10t - \frac{0.01t^2}{2}\right]_5^7\right)$$

$$= 10{,}000 \times \exp\left(-\left[\frac{0.01*51}{2} - 0.04\times 3\right]\right) \times \exp\left(-\left[0.10*2 - \frac{0.01*24}{2}\right]\right)$$

$= 10{,}000\exp(-0.255 + 0.12 - 0.20 + 0.12)$
$= 10{,}000\exp(-0.215)$

$= £8{,}065.41$ [4]

(ii) Required discount rate p.a. convertible monthly is given by

$$10{,}000\left(1 - \frac{d^{(12)}}{12}\right)^{12\times 5} = 8{,}065.41$$

$d^{(12)} = 4.2923\%$ p.a. convertible monthly.

[2]
[TOTAL 7]

Generally well answered.

Q7 Forward price of the contract is:

$$K_0 = (S_0 - I)e^{\delta T} = (8.70 - I)e^{0.07}$$ [1]

where I is the present value of the income expected during the contract

$$\Rightarrow I = 1.10 \times e^{-0.07\times 8/12} = 1.049846$$ [1]

$$\Rightarrow K_0 = (8.70 - 1.049846)\times e^{0.07} = 8.204853$$ [½]

Forward price of contract set up at time r (where r = 5 months) is

$$K_r = (S_r - I_r)e^{\delta(T-r)}$$ [1]

where I_r is the value at time r of the income expected during the contract

$$= 1.10 \times e^{-0.065 \times 3/12} = 1.082269 \qquad [1]$$

$$\Rightarrow K_r = (9.90 - 1.082269)e^{0.065 \times 7/12} = 9.158489 \qquad [½]$$

Value of original forward contract

$$= (K_r - K_0)e^{-\delta(T-r)}$$

$$= (9.158489 - 8.204853)e^{-0.065 \times \frac{7}{12}}$$

$$= 0.918154$$

$$= £0.92 \qquad [2]$$

[TOTAL 7]

> Although this question was answered better than similar questions in past diets, the workings shown by marginal candidates were often unclear.

Q8 (i) Work in £000's

Let total accumulation at 1/6/20 be X, and i_y = investment return for the year starting from 1 June 2016 + y

$$E(X) = E\left[3(1+i_1)(1+i_2)(1+i_3) + 3(1+i_2)(1+i_3) + 3(1+i_3) + 3 + 110\right] \qquad [1½]$$

Due to independence:

$$E(X) = 3\left[E(1+i_1)E(1+i_2)E(1+i_3) + E(1+i_2)E(1+i_3) + E(1+i_3)\right] + 113$$

[1]

$$= 3\left[(1+E[i_1])(1+E[i_2])(1+E[i_3]) + (1+E[i_2])(1+E[i_3]) + (1+E[i_3])\right] + 113$$

[½]

where $E(i_y)$ $= 0.55 \times 6\% + 0.45 \times 5.5\%$

$= 5.775\%$ [1]

(ii) $E(X) = 3\,\ddot{s}_{\overline{3|}}^{5.775\%} + 113$

$$= 3\left(\frac{1.05775^3 - 1}{0.05775/1.05775}\right) + 113$$

$= 123.080$ (= £123,080 for £100,000 nominal)

[2]

[TOTAL 6]

> Whilst the calculations were often correct, relatively few candidates followed the instructions to derive the required formula for these calculations. For full marks, such derivation was required including identifying where the independence assumption is used.

Q9 (i) Cash Flows:

Issue Price: Jan 14 $-0.97 \times 100{,}000 = -£97{,}000$

Interest Payments: July 14 $0.03 \times 100{,}000 \times \frac{122.3}{120.0} = £3{,}057.50$

Jan 15 $0.03 \times 100{,}000 \times \frac{124.9}{120.0} = £3{,}122.50$

July 15 $0.03 \times 100{,}000 \times \frac{127.2}{120.0} = £3{,}180.00$

Jan 16 $0.03 \times 100{,}000 \times \frac{129.1}{120.0} = £3{,}227.50$

Capital redeemed: Jan 16 $100{,}000 \times \frac{129.1}{120.0} = £107{,}583.33$

[3]

(ii) Express all amounts in "January 2014 money", and we get:

$$97000 = 3057.50 \times \frac{122.3}{124.9} v^{\frac{1}{2}} + 3122.50 \times \frac{122.3}{127.2} v$$

$$+3180.00 \times \frac{122.3}{129.1} v^{1\frac{1}{2}} + \frac{122.3}{131.8} \times (107583.33 + 3227.50) v^2$$ [2]

$$\Rightarrow 97000 = 2993.85 v^{\frac{1}{2}} + 3002.22 v + 3012.50 v^{1\frac{1}{2}} + 102823.71 v^2$$ [1]

Try 7%, RHS = 98232.04
8%, RHS = 96499.48

$$i = 0.07 + \left(\frac{98232.04 - 97000}{98232.04 - 96499.48}\right) \times 0.01 \qquad [2]$$

= 7.7% p.a. effective real yield (exact answer is 7.708%).

[TOTAL 8]

This question seemed to strongly differentiate between stronger and weaker candidates. Common errors from the latter included not correctly allowing for the time lag in part (i) or not uplifting the nominal cashflows for inflation at all.

Q10 (i) TWRR is i such that:

$$(1+i)^3 = \frac{12,700}{12,000} \times \frac{13,000}{12,700 + 2,600} \times \frac{14,100}{13,000 - 3,700} \times \frac{17,200}{14,100 + 1,800}$$

= 1.474830 $\Rightarrow$ i = 13.8% [2 for formula, 1 for solution]

(ii) If the MWRR achieved by the fund were 13.8% p.a., then fund value at 31 December 2015 would be (in £000's):

$$12000 \times (1.138)^3 + 2600 \times (1.138)^{2\frac{1}{2}} - 3700 \times (1.138)^{1\frac{1}{2}} + 1800 \times (1.138)^{\frac{1}{2}}$$

=18,706 which is greater than 17,200. This means that the MWRR must be less than 13.8% p.a. [2]

(iii) The MWRR is lower because the fund performed badly immediately after receiving the large positive cash flow in July 2013 and also performed well immediately after the large negative cash flow in July 2014. [2]

(iv) The TWRR is not influenced by the amount and timing of the cash flows (which are generally considered to be outside of the control of the fund manager) and, thus, better reflects the manager's performance over the period. [2]

[TOTAL 9]

Parts (i) and (ii) were answered well. In part (ii), it is not necessary to calculate the MWRR.

As in previous diets, candidates had difficulty explaining the relative values for the MWRR and TWRR. For full marks in part (iii), candidates needed to make reference to the actual data in the question. In part (iv), the key point is that

the amount and timing of the cash flows are generally considered to be outside of the control of the fund manager.

Q11 (i) 10,000 shares give a total dividend on the next payment date of £650.

Then, working in half-year periods, we have:

$$V = 650 \times \left(v_{6\%} + 1.02 v_{6\%}^2 + 1.02^2 v_{6\%}^3 + \ldots \right) \quad [2]$$

$$= 650 v_{6\%} \times \left(1 + 1.02 v_{6\%} + \left(1.02 v_{6\%}\right)^2 + \ldots \right)$$

$$= 650 v_{6\%} \times \left(\frac{1}{1 - 1.02 v_{6\%}} \right)$$

$$= £16{,}250 \quad [2]$$

(ii) (a) We now have

$$v = 650 v_{6\%} \times \left(\frac{1}{1 - 1.025 v_{6\%}} \right) = £18{,}571.43 \quad [1]$$

(b) The higher rate of dividend growth means that expected future dividend income is increased and, thus, the investor is prepared to pay a higher price to purchase the shares. [1]

(iii) (a) The rumoured change in legislation might be thought of as increasing the uncertainty of the future growth prospects for the company (without necessarily either increasing or decreasing them).

Thus it is appropriate that the investor requires a higher return to compensate for this greater uncertainty. [1]

(b) We now have:

$$v = 650 v_{7\%} \times \left(\frac{1}{1 - 1.02 v_{7\%}} \right) = £13{,}000 \quad [1]$$

(c) The higher risk (as reflected by the higher effective rate of return required) means that the investor is now prepared to pay a lower maximum price to purchase the shares. [1]

(iv) (a) Lower inflation is likely to lead to lower (nominal) profits and, thus, lower (nominal) dividend payments.

Also, as many investors are more concerned with real returns (i.e. in excess of inflation), it is appropriate to reduce the effective rate of return to reflect the lower expected inflation. [2]

(b) We now have:

$$v = 650v_{5\%} \times \left(\frac{1}{1 - 1.01v_{5\%}} \right) = £16,250 \qquad [1]$$

(c) In this case, the maximum price that the investor is prepared to pay is unchanged. Lower expected inflation leads to lower nominal dividend payments, which are then discounted at a lower nominal interest rate. Thus, the price is unaffected (i.e. equities are a real asset).

[2]
[TOTAL 14]

The calculations in this question were relatively simple and generally done well. The explanatory parts of the questions were answered better than expected.

Q12 (i)

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n te^{-\delta t} dt = \left[t \times \frac{e^{-\delta t}}{-\delta} \right]_0^n - \int_0^n \frac{e^{-\delta t}}{-\delta} dt$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta} \int_0^n e^{-\delta t} \; dt$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta} \left[-\frac{e^{-\delta t}}{\delta} \right]_o^n$$

$$= -\frac{n.e^{-\delta n}}{\delta} + \frac{1}{\delta} \left[\frac{1 - e^{-\delta n}}{\delta} \right] = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

[4]

(ii) Project lasts for 33 years as follows:

Time take to reopen mine = 1 year

Time taken for net revenue to go from zero to \$3,600,000 is 12 years

$$\left(\text{from } \frac{3,600,000}{300,000} = 12 \right)$$

Time taken for net revenue to decline to \$600,000 is 20 years

$$\left(\text{from } \frac{3,600,000-600,000}{150,000} = 20\right)$$

[2]

(iii) PV of reopening costs and additional costs $= 700,000\ \bar{a}_{\overline{1|}} + 200,000\ \bar{a}_{\overline{33|}}$ at 25%

where

$$\bar{a}_{\overline{1|}}^{25\%} = \frac{i}{\delta} a_{\overline{1|}} = 1.120355 \times 0.8 = 0.896284$$

$$\bar{a}_{\overline{33|}}^{25\%} = \frac{i}{\delta}\ .\ a_{\overline{33|}} = 1.120355 \times 3.9975 = 4.478619$$

$\Rightarrow$ PV = 627,399 + 895,724 = 1,523,123 [3]

PV of net revenue

$$v.\ 300,000\left(\bar{I}\bar{a}\right)_{\overline{12|}} + v^{13}\left\{3,600,000\ \bar{a}_{\overline{20|}} - 150,000\left(\bar{I}\bar{a}\right)_{\overline{20|}}\right\}$$ [2]

where $\bar{a}_{\overline{12|}} = \frac{i}{\delta}.\ a_{\overline{12|}} = 1.120355 \times 3.7251 = 4.173434$ [1]

$$\left(\bar{I}\bar{a}\right)_{\overline{12|}} = \frac{\bar{a}_{\overline{12|}} - 12v^{12}}{\delta} = \frac{4.173434 - 12 \times 0.06872}{0.223144} = 15.0073$$ [1]

$$\bar{a}_{\overline{20|}} = \frac{i}{\delta}.\ a_{\overline{20|}} = 1.120355 \times 3.9539 = 4.429772$$ [1]

$$\left(\bar{I}\bar{a}\right)_{\overline{20|}} = \frac{4.429772 - 20v^{20}}{\delta} = 18.818324$$ [1]

$\Rightarrow$ PV of net revenue

$$= \frac{300,000}{1.25} \times 15.0073 + 0.05498\{3,600,000 \times 4.429772 - 150,000 \times 18.818324\}$$ [1]

= 3,601,752 + 721,581 = 4,323,333 [1]

$\Rightarrow$ Price to obtain IRR of 25% p.a. is:

4,323,333 – 1,523,123 = \$2,800,210. [1]

(above uses factors from Formulae and Tables Book – exact answer is 4,323,319 – 1,523,115 = \$2,800,204)

[TOTAL 18]

The proof in part (i) was answered very poorly. Also, since the result is given, candidates must provide enough steps in deriving the result to convince the examiners that they haven't just jumped to the result. In part (iii), the workings of many marginal candidates were very unclear. The examiners recommend that candidates set out their working clearly e.g. by calculating each component of the costs and benefits separately. This enables examiners to give full credit for correct working even if errors are made in the calculations.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2016

Subject CT1 – Financial Mathematics
Core Technical

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Luke Hatter
Chair of the Board of Examiners
December 2016

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.

2. Performance was very slightly weaker when compared with most recent diets. As in previous diets, the non-numerical questions were often answered poorly by marginal candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i) $e^{\delta/4} = 1.0125 \Rightarrow \delta = 4 \times \ln 1.0125 = 0.0496901 = 4.969\%$ [1]

(ii) $1+i = 1.0125^4 \Rightarrow i = 0.0509453 = 5.095\%$ [1]

(iii) From (i) (though could be done in other ways)

$$\left(1-\frac{d^{(12)}}{12}\right) = e^{-\delta/12} = e^{-0.0041408} = 0.9958677 \Rightarrow d^{(12)} = 0.049587 = 4.959\%$$

[2]
[Total 4]

> This was generally answered well although many candidates ignored the specific rounding instructions or rounded incorrectly.

Q2 Various approaches (e.g. effective interest period can be changed etc.).
Work in quarters. Interest rate per quarter = 0.5%. Rate of payment per quarter = 25.
Number of quarters = 48.

PV = $25\ddot{a}^{(3)}_{\overline{48|}} = 25\frac{i}{d^{(3)}}a_{\overline{48|}}$ [2]

$d^{(3)} = 3\left(1-1.005^{-1/3}\right) = 0.0049834,\ a_{\overline{48|}} = 42.5803$ [1]

Therefore, PV = 25 × (0.005/0.0049834) × 42.5803 = €1,068.05 [1]
[Total 4]

> Generally well answered

Q3 A loan repayable by a series of payments at fixed times set in advance.

Typically issued by banks and building societies

Typically long-term …
…e,g. used to fund house purchase
…and secured against the property

Each payment contains an element to pay interest on the loan with the remainder being used to repay capital

In its simplest form, the interest rate will be fixed …
….and the payments will be of fixed equal amounts.

The interest payment portion of the repayments will fall over time…
… and the capital payments will rise over time.

Risk that borrower defaults on loan

Complications might be added such as (i) allowing the loan to be repaid early or (ii) allowing the interest rate to vary

[½ mark for each point]
[Max 3]
[Total 3]

Despite being a bookwork question, this was the worst answered question on the paper. It was not necessary to make all the above points for full marks.

Q4 (i) $X = (112 + 23) \times 1.1 = £148.5\text{m}$ [1]

(ii) TWRR is found from

$$(1+i)^2 = \frac{148.5}{135}\frac{160}{148.5+43} = 0.91906 \Rightarrow i = -0.04132 = -4.132\%$$ [3]

[Total 4]

A significant number of marginal candidates failed to answer part (i) correctly.

Q5 Let original price of zero coupon bond = P

$P = 100v^{80}$ at 2.5%

$\Rightarrow P = 100 \times 0.13870 = 13.87$ [2]

Equation of value for the purchaser:

$13.87e^{\delta t} = 80$ [1]

$$t = \frac{\ln(80/13.67)}{\delta} \Rightarrow t = \frac{\ln 5.7678}{0.048} = 36.506 \text{ years}$$ [1]

0.506 years is 185 days. There are 181 days to the end of June. Default is therefore on 4 July 2011. [1]

[Total 5]

> This question was answered poorly with many candidates not able to formulate separate equations of value for:
>
> - the original terms to determine the issue price.
> - the revised terms to determine the date of default.

Q6 (i) Simple rate of return is $(100 - 96)/96 = 0.041666$ [1]

Expressed as an annual rate, this is: $0.041666 \times (365/182) = 8.3562\%$ [1]

(ii) Let the time in years = t

$(97.5 - 96)/96 = 0.035t$ [1]

$t = (97.5 - 96)/(0.035 \times 96) = 0.44643$ years = 163 days [1]

(iii) Equation of value for the second investor:

$97.5(1 + i)^{(182-163)/365} = 100$ [1]

$$(1+i)^{19/365} = \frac{100}{97.5} \Rightarrow i = \left(\frac{100}{97.5}\right)^{365/19} - 1 = 62.640\%$$ [1]

[Total 6]

> Parts (i) and (ii) were very well answered. In part (iii), some candidates continued to assume a simple rate of return was required. Alternative answers to part (iii) based on different rounding of the answer in part (ii) were given full credit.

Q7 (i) Present value of dividends, I, is:

$$0.1\left(v^{1/4} + v^{1/2} + v^{3/4}\right)$$

Calculated at $i'\%$ when $1 + i' = 1.04^2 = 1.0816$

So $I = 0.1\,(1.0816^{-0.25} + 1.0816^{-0.5} + 1.0816^{-0.75}) = 0.288499$ [2]

Hence, forward price, K, is:

$K = (1.1 - 0.288499) \times 1.0816^{0.75} = 0.86068 = 86.068p$ [2]

(ii) The price of the forward can be determined from the price of the share (for which it is a close substitute). The forward is like the share but with delayed settlement and without dividends. [Could also be said that the price of the share already takes into account expectations.] [2]

[Total 6]

> Part (i) was answered well although some candidates ignored the fact that the interest rate given was a convertible half-yearly rate. Part (ii) was answered less well with the arguments of many marginal candidates being very unclear.

Q8 (i) $96 = 4a_{\overline{3}|} + 100v^3$ [1]

Try 6% RHS = 94.654 [½]

Try 5% RHS = 97.2768 [½]

Interpolation gives

$i \approx 0.05 + 0.01 \times \dfrac{97.2768 - 96}{97.2768 - 94.6540} = 0.0549 \approx 5.5\%$ [1]

(ii) Let i_n = spot rate for term n

Then $96 = 104v$ at $i_1 \Rightarrow 1 + i_1 = \dfrac{104}{96} \Rightarrow i_1 = 8.333\%$ [1]

$96 = 4v_{i_1} + 104v_{i_2}^2 \Rightarrow (1+i_2)^2 = \dfrac{104}{96 - 4v_{i_1}} = \dfrac{104}{96 - 3.69231} \Rightarrow i_2 = 6.145\%$ [2]

(iii) Let the forward rate be $f_{1,1}$

$(1+i_2)^2 = (1+i_1)(1+f_{1,1})$ [1]

$\Rightarrow 1.06145^2 = 1.08333 \times (1+f_{1,1}) \Rightarrow f_{1,1} = 0.04000 = 4\%$ [1]

(iv) The three year gross redemption yield is a complex form of weighted average of the three spot rates. [1]

The one-year spot rate is over 8%, the two-year rate is over 6% and the gross redemption yield is 5.5%. Therefore, the three-year rate must be less than 5.5% if the weighted average is 5.5%. [1]

[Total 10]

> Part (i), (ii) and (iii) were generally answered well, although in part (iii) some candidates were not clear as to the forward rate required by the question. Part (iv) was very poorly answered. For this part no marks were available for calculation without explanation.

Q9 (i) Let S_n = Accumulated value at time n of £1 invested at time 0

$$S_n = (1+i_1)(1+i_2)....(1+i_n)$$

$$\Rightarrow E[S_n] = E[(1+i_1)(1+i_2)....(1+i_n)]$$

$$= E(1+i_1).E(1+i_2).... .E(1+i_n) \text{ by independence}$$

and $E(1+i_t) = 1+E(i_t) = 1+j$

Hence $E(S_n) = (1+j)^n$

Now

$$\text{Var}[S_n] = E\left[S_n^2\right] - (E[S_n])^2$$

$$E\left[S_n^2\right] = E\left[(1+i_1)^2(1+i_2)^2....(1+i_n)^2\right]$$

$$= E\left[(1+i_1)^2\right].E\left[(1+i_2)^2\right].... .E\left[(1+i_n)^2\right]$$ [1]

by independence [½]

and

$$E\left[(1+i_t)^2\right] = E\left[\left(1+2i_t+i_t^2\right)\right] = 1+2E(i_t)+E\left(i_t^2\right)$$

and $\text{Var}[i_t] = s^2 = E\left(i_t^2\right) - [E(i_t)]^2 = E\left(i_t^2\right) - j^2$

$$\Rightarrow E\left(i_t^2\right) = s^2 + j^2$$ [1]

Hence

$$E\left[S_n^2\right] = \left(1+2j+j^2+s^2\right)^n$$

And $\mathrm{Var}\left[S_n\right]=\left(1+2j+j^2+s^2\right)^n-(1+j)^{2n}$ [1]

Hence mean accumulation $=8,000,000E\left(S_5\right)$ [½]

$$=8,000,000(1.055)^5=£10,455,680$$ [1]

Standard deviation of accumulation $=8,000,000\sqrt{\mathrm{Var}\left(S_5\right)}$ [½]

$$=8,000,000\sqrt{\left(1+2\times0.055+0.055^2+0.04^2\right)^5-(1.055)^{10}}$$

$$=8,000,000\sqrt{1.7204573-1.7081445}=8,000,000\times\sqrt{0.01231284}$$

$$=£887,706$$ [2]

Alternative Solution

$(1+i_t) \sim$ lognormal (μ,σ^2)

$$\ln(1+i_t) \sim N(\mu,\sigma^2)$$

$$\ln(1+i_t)^5=\ln(1+i_t)+\ln(1+i_t)+\cdots+\ln(1+i_t) \sim N(5\mu,5\sigma^2)$$

Given assumption that they are independent and identically distributed
$\therefore (1+i_t)^5 \sim$ lognormal $(5\mu,5\sigma^2)$ [2]

$$E(1+i_t)=\exp\left(\mu+\frac{\sigma^2}{2}\right)=1.055$$ [1]

$$\mathrm{Var}(1+i_t)=\exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right]=0.04^2$$ [1]

$$\frac{0.04^2}{1.055^2}=\left[\exp\left(\sigma^2\right)-1\right]\Rightarrow\sigma^2=0.0014365$$ [1]

$$\exp\left(\mu+\frac{0.0014360}{2}\right)=1.055\Rightarrow$$

$$\mu = \ln 1.055 - \frac{0.0014365}{2} = 0.052823$$ [1]

$5\mu = 0.264113$

$5\sigma^2 = 0.0071825.$

Let S_5 be the accumulation of one unit after five years:

$$E(S_5) = \exp\left(5 \times \mu + \frac{5\sigma^2}{2}\right) = \exp\left(0.264113 + \frac{0.0071825}{2}\right)$$

$=1.30696$ [½]

$$\text{Var}(S_5) = \exp(2 \times 5\mu + 5\sigma^2)\left[\exp\left(5\sigma^2\right) - 1\right]$$

$= \exp(2 \times 0.264113 + 0.0071825) . (\exp 0.0071825 -1)$

$= \exp 0.53541 (\exp 0.0071825 - 1)$

$= 0.01231284$ [½]

Mean value of the accumulation of premiums is

$8{,}000{,}000 \times 1.30696 =$ £10,455,680. [1]

Standard deviation of the accumulated value of the premiums is

$8{,}000{,}000 \times 0.012312849^{0.5} =$ £887,706 [1]

(ii) If the company invested in fixed-interest securities, it would obtain a guaranteed accumulation of £8,000,000 * $(1.04)^5$ = £9,733,223. In one sense, there is a 100% probability that a loss will be made and therefore the policy is unwise. [1]

The "risky" investment strategy leads to an expected profit. [1]

On the other hand, the standard deviation of the accumulation from the risky investment strategy will be higher than investing in the fixed-interest securities. Whilst there is a chance of an even greater profit from this strategy, there is also a chance of a more considerable loss than from investing in fixed-interest securities. [1]

[Total 12]

> Many candidates either ignored the requirement in part (i) to derive the necessary formulae or had difficulty in performing the derivation often trying to combine the two methods above without success. Part (ii) was better answered than the other explanation questions on the paper.

Q10 (i) The payback period simply tells an investor when the total cash inflows from the investment have exceeded the total cash outflows. This tells the investor nothing about the overall profitability of the project. [2]

(ii) The present value of outgoing cash flows at a rate of return of 6% per annum effective is as follows (in £m):

$$\ddot{a}_{\overline{3}|} + 0.05\left(a_{\overline{13}|} - a_{\overline{3}|}\right)$$

$= 1.06\times2.6730 + 0.05(8.8527 - 2.6730) = 3.14238$ [2]

The present value of the incoming cash flows is as follows (in £m):

$$= 0.495v^3\ddot{a}_{\overline{1}|}^{(12)}\left(1+1.01v+1.01^2v^2+...+1.01^9v^9\right)$$

$$= 0.495v^3\frac{d}{d^{(12)}}\left(1+1.01v+1.01^2v^2+...+1.01^9v^9\right)$$

$= 0.495\times0.83962 \times 0.973784 \times (1 - 1.01^{10}/1.06^{10}) / (1 - 1.01/1.06)$

$= 0.404716 \times 8.12352 = 3.2877$ [3½]

NPV of cash flows at 6% = 3.2877 – 3.1424 = £0.1453m = £145,300 [1]

The project has a positive NPV at 6% and therefore an IRR higher than 6% and the first criteria is met. [½]

By the end of the 10^{th} year, the total outgoing cash flows will have been: £3,000,000 plus 7 × £50,000 or £3,350,000. [1]

Total incoming cash flows are:

$495{,}000\times (1 + 1.01 + 1.01^2 +...+ 1.01^6)$ (i.e. rate of payment of £495,000 rising by 1% per year for seven years). [1]

Geometric progression with common ratio 1.01 and seven terms

$= 495{,}000(1 - 1.01^7)/(1 - 1.01) = £3{,}570{,}700$ [1]

This is greater than total outgoing cash flows and therefore second criterion is met. [½]

There is clearly a positive cash flow in the fifth year as the incoming cash flows will be greater than £495,000 and the outgoing cash flows will be £50,000. [1]

Therefore final criterion is met. [½]

[Total 14]

> This was the worst answered of the longer questions. The examiners strongly recommend that candidates take a systematic approach to the question and e.g. derive the PVs of the outgo and income separately. Marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity, explaining all steps.
>
> Candidates who assumed that the repayments continued for 11 years, rather than 10, were not penalised.

Q11 (i) Duration $= \sum_t tC_t v^t \Big/ \sum_t C_t v^t$

$$= \frac{\left(\sum_{t=1}^{3} 4\times 0.04\times tv^t\right) + \left(\sum_{t=1}^{10} 5\times 0.04\times tv^t\right) + 4\times 1\times 3v^3 + 5\times 1\times 10v^{10}}{\left(\sum_{t=1}^{3} 4\times 0.04\times v^t\right) + \left(\sum_{t=1}^{10} 5\times 0.04\times v^t\right) + 4\times 1\times v^3 + 5\times 1\times v^{10}}$$

$$= \frac{0.16(Ia)_{\overline{3}|} + 0.20(Ia)_{\overline{10}|} + 12v^3 + 50v^{10}}{0.16a_{\overline{3}|} + 0.20a_{\overline{10}|} + 4v^3 + 5v^{10}}$$ [4]

Therefore, duration

$$= \frac{0.16\times 5.2422 + 0.20\times 36.9624 + 12\times 0.83962 + 50\times 0.55839}{0.16\times 2.6730 + 0.20\times 7.3601 + 4\times 0.83962 + 5\times 0.55839} = \frac{46.2264}{8.05015}$$

$= 5.742$ years [3]

(ii) Present value of new portfolio per unit nominal $= \dfrac{1}{i}$ [1]

Volatility of new portfolio per unit nominal $= -\dfrac{\frac{d}{di}\left(\frac{1}{i}\right)}{\frac{1}{i}} = \dfrac{1}{i}$ [1]

Duration of new portfolio, applying equation above is

$\text{volatility} \times (1+i) = \dfrac{1+i}{i} = \dfrac{1.06}{0.06}$

$= 17.666$ years [2]

(iii) Present value of existing bonds:

$= £8.05013\text{bn}$ [½]

Let the nominal amount of new bonds issued $= X$

Present value of new bonds $= 0.05Xa_{\overline{\infty}|} = \dfrac{0.05X}{0.06}$

$$\Rightarrow \frac{0.05X}{0.06} = 0.8 \times 8.05013 \Rightarrow X = £7.728\text{bn}$$

[1½]

[Total 13]

> In part (i), many candidates incorrectly calculated DMTs separately for the two bonds which simplified the question and did not produce the DMT of the whole portfolio. Part (ii) was poorly answered with many candidates not recognising the relationship between DMT and volatility. It is also important in such questions to state the time units in the final answer. Part (iii) did seem to act as a differentiator between candidates, with the strongest candidates able to proceed clearly through the question.

Q12 (i) $50 = 100v(20)$

where $v(20) = \exp\left(-\int_0^{10} 0.03dt\right)\exp\left(-\int_{10}^{20} at\,dt\right)$ [3]

$$= \exp[-0.03t]_0^{10} \exp\left[-\frac{at^2}{2}\right]_{10}^{20}$$

$$= e^{-0.3}e^{-150a} = e^{-0.3-150a} = 0.5 \qquad [1]$$

$$\Rightarrow a = \frac{\ln 2 - 0.3}{150} = 0.0026210 \qquad [1]$$

(ii) $40 = 100v(28)$

where $v(28) = v(20)\exp\left(-\int_{20}^{28} bt\,dt\right)$ [1]

$$= -0.5\exp\left[\frac{-bt^2}{2}\right]_{20}^{28}$$

$$= 0.5e^{-192b} = 0.4 \qquad [2]$$

$$\Rightarrow b = \frac{-\ln 0.8}{192} = 0.0011622 \qquad [1]$$

(iii) Equivalent annual effective rate of discount can be found from:

$$100(1-d)^{28} = 40 \Rightarrow (1-d)^{28} = 0.4 \qquad [1]$$

$$\Rightarrow d = 0.032195 = 3.220\% \qquad [1]$$

(iv) (a) We require: $\int_3^7 \rho(t)v(t)\,dt = \int_3^7 e^{-0.04t}e^{-0.03t}dt = \int_3^7 e^{-0.07t}dt$ [1½]

$$= \left[\frac{e^{-0.07t}}{-0.07}\right]_3^7 \qquad [1]$$

$$= -8.751806 + 11.579775 = 2.827969 \qquad [1½]$$

(b) The present value of the payment stream $2.827969 = X\left(\bar{a}_{\overline{7}|} - \bar{a}_{\overline{3}|}\right)$ where X is the continuous payment stream using $\delta = 0.03$. [2]

$$X = \frac{2.827969}{\bar{a}_{\overline{7}|} - \bar{a}_{\overline{3}|}} = \frac{2.827969}{\left(\frac{1-e^{-0.03\times 7}}{0.03}\right) - \left(\frac{1-e^{-0.03\times 3}}{0.03}\right)}$$

$$= \frac{2.827969}{6.31386 - 2.86896} = 0.82092 \qquad [2]$$

[Total 19]

This was well answered apart from part (iv)(b). It was pleasing to see many candidates using good exam technique to leave enough time for this question which proved to be more straightforward than the other longer questions.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2017

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
July 2017

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.

2. Performance was of a similar standard to that of most recent examinations. As in previous examinations, the non-numerical questions were often answered poorly by marginal candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i) $i^{(4)} = 4\%$

$$\left(1-\frac{d^{(12)}}{12}\right)^3 = v^{\frac{1}{4}}$$

$$= \left(1+\frac{i^{(4)}}{4}\right)^{-1}$$

$$\Rightarrow d^{(12)} = 12\left(1-\left(1+\frac{i^{(4)}}{4}\right)^{-\frac{1}{3}}\right) = 12(1-(1.01)^{-\frac{1}{3}})$$

$$\approx 3.9735\%$$ [2]

(ii) $$1-\frac{d^{(12)}}{12} = e^{-\delta\frac{1}{12}}$$

$$\Rightarrow d^{(12)} = 12\left(1-e^{-0.05\frac{1}{12}}\right)$$

$$\approx 4.9896\%$$ [2]

(iii) $$1-\frac{d^{(4)}}{4} = v^{\frac{1}{4}}$$

$$= \left(1-\frac{d^{(12)}}{12}\right)^3$$

$$\Rightarrow d^{(12)} = 12\left(1-\left(1-\frac{d^{(4)}}{4}\right)^{\frac{1}{3}}\right)$$

$$\approx 4.0134\%$$ [2]

[Total 6]

Generally well-answered.

Q2 Present value for 1st option:

$$7,800\, a_{\overline{16}|}^{(4)} = 7,800 \times 1.018559 \times 10.8378$$

$$= £86,103.52$$ [2]

Present value for 2nd option:

$$16,400 \times \left(v^2 + v^4 + ... + v^{16}\right)$$

$$16,400\, v^2 \left(\frac{1-v^{16}}{1-v^2}\right)$$

$$16,400 \times 0.90703 \times \left(\frac{(1-0.45811)}{(1-0.90703)}\right)$$

$$= £86,702.94$$ [2]

(above uses factors from Formulae and Tables Book – exact answer is £86.702.16)

Therefore, 1st option is better for the borrower as the total value of the repayments is less than with the 2nd option. [1]

[Total 5]

> For the second option, a common mistake was to assume the annual rate of payment was £8,200. Some students omitted to give a final conclusion (and so did not actually answer the question).

Q3 Forward price of the contract is $K_0 = (S_0 - I)e^{\delta T} = (78 - I)e^{0.14 \times 1}$ [1]

where I is the present value of income during the term of the contract.

$$\Rightarrow I = 3.2\left(e^{-0.14 \times \frac{3}{12}} + e^{-0.14 \times \frac{9}{12}}\right) = 5.97098$$ [1]

$$\Rightarrow K_0 = (78 - 5.97098)e^{0.14} = 82.85309$$ [½]

Forward price when new contract issued at time r (1 month) is

$$K_r = \left(S_r - I^*\right)e^{\delta(T-r)} = (80 - I^*)e^{0.11 \times \frac{11}{12}}$$ [1]

where I^* is the present value of income during the term of the contract.

$$\Rightarrow I^* = 3.2\left(e^{-0.11\times\frac{2}{12}} + e^{-0.11\times\frac{8}{12}}\right) = 6.115599$$ [1]

$$\Rightarrow K_{\frac{1}{12}} = \left(80 - 6.115599\right)e^{0.11\times\frac{11}{12}}$$

$$= 81.72297$$ [½]

Value of original contract $= \left(K_r - K_0\right)e^{-\delta(T-r)}$ [½]

$$= \left(81.72297 - 82.85309\right)e^{-0.11\times\frac{11}{12}}$$

$$= -1.02172 = -£1.02172$$ [1½]

(above uses the rounded forward prices shown – exact answer is −£1.02174)

[Total 7]

Reasonably well-answered although a common mistake was not to deal with the change in the interest rate correctly. There are quicker ways to answer the question but candidates who took a methodical approach such as that outlined above were able to maximise the marks for working even if they made calculation errors.

Q4 (i) $v(t) = e^{-\int_0^t \delta(s)ds}$

For $0 \le t \le 2$

$$v(t) = e^{-\int_0^t 0.04ds} = e^{-0.04t}$$ [2]

For $t > 2$

$$v(t) = v(2).e^{-\int_2^t (0.02+k.s)ds}$$

$$= e^{-0.08} \times e^{-\left[0.02s+\frac{1}{2}ks^2\right]_2^t}$$

$$= e^{-0.08} \times e^{-\left[\left(0.02t+\frac{1}{2}kt^2\right)-\left(0.04+2k\right)\right]}$$

$$= e^{-0.02t-\frac{1}{2}kt^2-0.04+2k}$$ [3]

(ii) To calculate maximum value of k:

Now PV of outlay (in £000s)

$= 500 + 250\left(e^{-0.02} + e^{-0.04}\right)$

$= 985.247$ [1]

At $t = 10$, PV of sale proceeds

$= 2000\, e^{-0.08}.e^{-(0.2+50k-0.04-2k)}$

$= 2000e^{-0.24-48k}$ [1]

So, for DPP = 10 years, we need PV of sale proceeds ≥ PV of outlay

$\Rightarrow 2000\, e^{-0.24-48k} \geq 985.247$

$\Rightarrow \ln\left(0.4926235\right) \leq -0.24 - 48k$

$\Rightarrow k \leq 0.00975$ so maximum value is 0.00975 [2]

[Total 9]

In part (i), it was important to give the required expression for $t \leq 2$ explicitly.

Unfortunately in part (ii), there was a typographical error in the question paper. The intention was to ask for the maximum value of *k* rather than the minimum and the solution above obtains this maximum value. The answer to the question actually on the paper is that the minimum value would be $k = -\infty$. Students who gave this answer were given full credit as were students who obtained the maximum value above. Marginal candidates did not appear to have been disadvantaged as such candidates had typically been unable to calculate the PV of the sale proceeds in terms of *k*.

Q5 (i) No, as with only a single asset, the spread of the asset proceeds would be less than the spread of the liability outgo (at times 7 and 11). [1]
Thus, the convexity of the assets would be less than the convexity of the liabilities and the third condition of immunisation could not be satisfied. [1]

(ii) Redington's first condition states that the PV of the assets should equal the PV of the liabilities $\left(\text{using } v = \dfrac{1}{1.055} = 0.94787\right)$ and working in £ millions:

$V_A = 15.363v^{7.5} + 3.787v^{14.25} = 12.048$ [1]

$V_L = 11v^7 + 8.084v^{11} = 12.048$ [1]

Allowing for rounding (using three decimal places), Redington's first condition applies. [½]

Redington's second condition states that the discounted mean term (DMT) of the assets should be equal to the DMT of the liabilities, which equivalently can be written as

$V'_A = V'_L$ (where in the calculations below the derivatives are with respect to the force of interest)

$V'_A = 15.363 \times 7.5v^{7.5} + 3.787 \times 14.25v^{14.25} = 102.28$. [1]

$V'_L = 11 \times 7v^7 + 8.084 \times 11v^{11} = 102.28$ [1]

Allowing for rounding (using 2 decimal places), Redington's second condition applies. [½]

Since the spread of asset proceeds exceeds the spread of liability outgo (as asset proceeds are received at times 7.5 and 14.25, whereas liability outgo is paid at times 7 and 11), the convexity of the assets is greater than the convexity of the liabilities.

Alternatively:

$V''_A = 15.363 \times 7.5^2 v^{7.5} + 3.787 \times 14.25^2 v^{14.25} = 936.94$.

$V''_L = 11 \times 7^2 v^7 + 8.084 \times 11^2 v^{11} = 913.32 < V''_A$.

[1½]

Thus, the third condition is also satisfied and the company is immunised against small changes in the rate of interest. [½]

[Total 9]

In part (i), candidates tended to score 0 or 2 marks depending on whether they recognised the problem with using a single asset. Part (ii) was answered well.

Q6 The investor's proceeds in £ millions at the time of purchase can be calculated as:

$$\text{PV}_{\text{in}} = 1.25 \times (1-0.35) a_{\overline{5|}}^{(12)} \left[1 + 1.042^5 v^5 + 1.042^{10} v^{10} + \ldots + 1.042^{30} v^{30}\right] v^{\frac{9}{12}}$$

$$+ 11.5 v^{35+\frac{9}{12}} \quad @ i = 8\% \text{ p.a.}$$

[3]

$$= 0.8125 \times a_{\overline{5|}}^{(12)} \left(\frac{1-r^7}{1-r}\right) v^{0.75} + 11.5 v^{35.75} \ @ i = 8\% \text{ p.a.}$$ [1½]

$= 0.8125 \times 4.1371$ [1]

$\times 4.35767$ [1]

$\times 0.94391$ [½]

$+ 11.5 \times 0.06384$ [1]

$= 14.56039$ [½]

where we have:

$$r = \left(\frac{1.042}{1.08}\right)^5 = 0.836026\,; \quad a_{\overline{5|}}^{(12)} = \frac{1-0.68058}{0.07721} = 4.1371$$

At the same time, the investor's costs (in millions) are:

$$\text{PV}_{\text{out}} = 5.8 + 0.85 v^{\frac{6}{12}} \ @ i = 8\% \text{ p.a.}$$

$$= 5.8 + 0.85 \times 0.96225 = 6.6179$$ [1½]

Thus, the investor's net proceeds (in millions) are given by:

$$\text{NPV} = \text{PV}_{\text{in}} - \text{PV}_{\text{out}} = 14.5604 - 6.6179 = £7.9425m$$ [1]

[Total 11]

> This was a question where it was beneficial to use a methodical approach. Common errors included not allowing for the three month period since purchase and assuming the rent increases were 4.2% every five years.

Q7 (i) Let i = money rate of return

Then

$$9{,}800 = 400\, a_{\overline{20|}}^{(2)} + 10{,}500 v^{20} - 0.30 \times 400 v^{5/12}\, a_{\overline{20|}} - 0.40\,(10{,}500 - 9{,}800) v^{20 5/12}$$

[4]

Try $i = 3\%$

RHS = 400 × 1.007445 × 14.8775 + 10,500 × 0.55368 – 120 × 0.98776 × 14.8775 – 280 × 0.546898 = 9,892.37 [1½]

Try i =4%

RHS = 400 ×1.009902 × 13.5903 + 10,500 ×0.45639 – 120 × 0.983791 × 13.5903 – 280 × 0.448989 = 8551.92 [1]

Since 8551.92 < 9800 < 9892.37

then 3% < i < 4% [½]

(ii) We can find i from:

$$i = 0.03 + \frac{9892.37 - 9800}{9892.37 - 8551.92} \times 0.01$$

= 0.0307 i.e. i = 3.07% p.a. [1½]

If inflation = 2% p.a. = e, then i' = net real yield can be found from

$$1 + i' = \frac{1+i}{1+e} = \frac{1.0307}{1.02}$$

$\Rightarrow i'$ = net real yield = 1.05% p.a. [1½]

(iii) If tax were collected on 1 April instead of 1 June each year then tax payments would be brought forward which would increase the present value of these payments. [1]

This would decrease both the net money yield and the net real yield. [1]

[Total 12]

In terms of average mark, this was the worst answered question on the paper. Many candidates simplified part (a) to assume that taxes were paid at the same time as the coupon/redemption payments (ignoring the 5-month time lag and/or assuming income tax was paid half-yearly) and they lost marks accordingly. It was possible to get full marks on part (b) even if part (a) was answered incorrectly. Part (c) was very poorly answered even though the points required were straightforward.

Q8 (i) The TWRR for fund A and B results from the annual rates achieved for 2015 and 2016:

$$\text{TWRR}_A: \quad (1+i)^2 = 1.42 \times 1.03 \quad => \quad i = 20.94\%$$

$$\text{TWRR}_B: \quad (1+i)^2 = 1.36 \times 1.02 \quad => \quad i = 17.78\%$$

[3]

(ii) In order to calculate the MWRR, first we need to calculate the values of the funds at the beginning and at the end of 2016. Working in £m, we have for fund A and B where $F_{t,A}$ and $F_{t,B}$ are the fund values at the end of 2014 + t for Funds A and B respectively:

$$F_{1,A} = (1.5 + 0.3) \times 1.42 = 2.556 \quad \text{and}$$

$$F_{2,A} = (F_{1,A} + 1.7) \times 1.03 = 4.38368$$

$$F_{1,B} = (2.3 + 2) \times 1.36 = 5.848 \quad \text{and}$$

$$F_{2,B} = (F_{1,B} + 0.2) \times 1.02 = 6.1689\,6$$

[2]

Then the MWRR result from the EV at the end of 2016:

$$\text{MWRR}_A : (1.5 + 0.3) \times (1+i)^2 + 1.7 \times (1 \times i) = 4.38368$$

$$\therefore 1.8x^2 + 1.7x - 4.38368 = 0$$

$$=> \quad x = 1.158229 \quad \Rightarrow \quad i_A = 15.823\%$$

[3]

$$\text{MWRR}_B : (2.3 + 2) \times (1+i)^2 + 0.2 \times (1+i) = 6.16896$$

$$\therefore 4.3x^2 + 0.2x - 6.16896 = 0$$

$$=> \quad x = 1.174735 \quad \Rightarrow \quad i_B = 17.474\%$$

[3]

(iii) The TWRR is a more reliable indicator of the manager's performance since it is independent of the size of the amounts and the time at which investments are made … [½]
…both of which are outside the manager's control. [½]

In this case, manager A performed better than manager B for both 2015 and 2016 by achieving a higher TWRR for each of those years (i.e. $42\% > 36\%$ and $3\% > 2\%$). [1]

It should be noted that manager A had a worse MWRR for the 2 year period than manager B because manager A had so few funds invested during the best period for investment which was 2015/manager A received a large cashflow just before a period of poor performance. [1]

[Total 14]

> Many candidates failed to notice the quick way that part (i) could be solved although much of the extra working that they undertook was needed for part (ii) anyway.
> Part (iii) was poorly answered although other approaches to those given above could be used to gain full credit. It is important in this type of question to refer to the actual results obtained and the actual data given. Unsubstantiated answers to this part were given no credit.

Q9 (i) Let $p(t)$ = Price of t-year bond

$g_1 = i_1 = 0.071 = 7.100\%$ p.a. [½]

$p(2) = 5a_{\overline{2|}} + 100v^2 \;@\; g_2 = 7.2\%$ p.a. [1]

$= 5 \times 1.8030185 + 100 \times 0.8701827$

$= 96.0334$ [1]

and $96.0334 = \dfrac{5}{1.071} + \dfrac{105}{(1+i_2)^2}$ [1]

$\Rightarrow \; i_2 = 7.203\%$ p.a. [1]

$p(3) = 5a_{\overline{3|}} + 100v^3 \;@\; g_3 = 7.3\%$ p.a.

$= 5 \times 2.609998 + 100 \times 0.8094701$

$= 93.9970$ [1]

$\Rightarrow \; 93.9970 = \dfrac{5}{1.071} + \dfrac{5}{(1.07203)^2} + \dfrac{105}{(1+i_3)^3}$ [½]

$\Rightarrow \; i_3 = 7.307\%$ p.a. [1]

(ii) $f_0 = i_1 = 7.1\% = 7.100\%$ p.a. [½]

$(1+f_0)(1+f_1) = (1+i_2)^2$ [1]

$$\Rightarrow 1+f_1 = \frac{(1.07203)^2}{1.071}$$

$\Rightarrow f_1 = 7.306\%$ p.a. [1]

(*Above answer is based on rounded answer for* i_2. *Exact answers is 7.305%*).

$$(1+i_2)^2(1+f_2)=(1+i_3)^3$$

$$\Rightarrow \quad 1+f_2 = \frac{(1.07307)^3}{(1.07203)^2}$$

$\Rightarrow \quad f_2 = 7.515\%$ p.a. [1½]

(*Above answer is based on rounded answers for* i_2 *and* i_3. *Exact answers is 7.516%*).

(iii) The spot rate for a term is the geometric average of the forward rates making up that term. [1]
Since the spot rates increase with term, the forward rates must increase at a faster rate than the spot rates to ensure that the geometric average of the forward rates is itself increasing with term. [1]
[Total 13]

Many marginal candidates answered part (i) as if the gross redemption yields given were actually spot yields. Others assumed the price of the bonds all to be par. Part (ii) was answered well even by candidates who had struggled with part (i). The examiners recognised that part (iii) would stretch many candidates and indeed this part was found to be challenging.

Q10 (i) Value of annuity = $20,000\, \ddot{a}_{\overline{1}|}^{(12)}\left(1+1.03v+1.03^2v^2+...+1.03^{19}v^{19}\right)$ [2]

$$= 20,000 \times 1.037525 \times 0.93458 \times \left(\frac{1-\left(\frac{1.03}{1.07}\right)^{20}}{1-\frac{1.03}{1.07}}\right)$$ [1]

= 19,393.4173×14.26488

= £276,645. [1]

(above uses factors from Formulae and Tables Book – exact answer is £276,639)

(ii) Let S_5 = Accumulation of £1 after 5 years and let i_t = investment return for year t.

$$[\,E(S_5) = E\left(\prod_{t=1}^{5} 1+i_t\right)$$

$$= \prod_{t=1}^{5} E(1+i_t) \text{ using independence}$$

$$= \prod_{t=1}^{5} (1+E(i_t))\,]$$

Now $E(i_t) = 0.6 \times 0.04 + 0.4 \times 0.07$

$= 0.052$ for t = 1, 2, … 5 [1]

$\Rightarrow$ Expected accumulation $= 200,000 E(S_5)$

$= 200,000 \times (1.052)^5$ [1]

$= 200,000 \times 1.288483$

$= £257,696.60$ [1]

(iii) The variance of the accumulation is

$$200,000^2 \times \left(E\left(S_5^2\right) - E\left(S_5\right)^2\right)$$ [1]

[where $E\left(S_5^2\right) = E\left(\prod_{t=1}^{5} (1+i_t)^2\right)$

$$= E\left(\prod_{t=1}^{5} \left(1+2i_t+i_t^2\right)\right)$$

$= \prod_{t=1}^{5} \left(1+2E(i_t)+E(i_t^2)\right)$ from independence]

Now $E(i_t^2) = 0.6 \times 0.04^2 + 0.4 \times 0.07^2$

$= 0.00292$ for t = 1, 2, …5 [1]

Hence, $E(S_5^2) = (1+2\times 0.052+0.00292)^5$

$= 1.661809$ [1]

$\Rightarrow$ Standard deviation of accumulated fund is

$$200,000 \times \left(1.661809 - 1.288483^2\right)^{\frac{1}{2}}$$

= £8,051.23 [1]

(above uses factors from Formulae and Tables Book – exact answer is £8,051.74)

(iv) Note that $200,000 \times (1.07)^4 (1.04)$ = 272,645.57 < 276,639
and $200,000 \times (1.07)^5$ = 280,510.35 > 276,639

Hence, the individual would require the annual return to be 7% p.a. for each of the 5 years in order to reach the required fund. [2]

The probability of this happening is

$(0.4)^5 = 0.01024$ [1]

[Total 14]

Parts (i) and (ii) were answered reasonably well although a few candidates appeared to be under time pressure if this was the last question to be attempted. Part (iii) was less well answered with many marginal candidates confusing $E(i_t^2)$ and $\mathrm{Var}(i_t)$. Part (iv) was generally only answered by the strongest candidates with many candidates incorrectly applying a lognormal distribution to the problem.

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2017

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2017

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i) (a) $6,000\left(1+\dfrac{0.03t}{365}\right)=7,600$ [1]

$t=\left(\dfrac{7,600}{6,000}-1\right)\times\dfrac{365}{0.03}$ = 3,244.4 days [1]

(b) $6,000(1+0.03)^{t/365}=7,600$ [1]

$t\times\dfrac{\ln 1.03}{365}=\ln\left(\dfrac{7,600}{6,000}\right)=\ln 1.26667=0.23639$

$\Rightarrow t=365\times\dfrac{0.23639}{\ln 1.03}$ = 2,919.0 days [1]

(c) $6,000e^{0.03t/365}=7,600$ [1]

$\Rightarrow t=\dfrac{365}{0.03}\ln\left(\dfrac{7,600}{6,000}\right)$ = 2,876.1 days [1]

(ii) Effective interest rate per half year is $\dfrac{i^{(2)}}{2}$ where

$\left(1+\dfrac{i^{(2)}}{2}\right)=e^{\delta/2}=e^{0.015}=1.0151131\Rightarrow\dfrac{i^{(2)}}{2}=1.51131\%$ [1]

[Total 7]

Well answered although some candidates gave $i^{(2)}$ as their final answer to part (ii).

Q2 One party agrees to pay to the other a regular series of fixed amounts… [½]
…for a certain/given term. [½]
In exchange, the second party agrees to pay a series of variable amounts [½]
…based on the level of a short-term interest rate. [½]
[Total 2]

The worst-answered question on the paper even though the above comes directly from the Core Reading.

Q3 Let *d* be the annual simple rate of discount.

The discounted value of 100 in the deposit account would be *X* such that:

$X = 100(1.03)^{-91/365} = 99.26576$ [1]

To provide the same effective rate of return a government bill that pays 100 must have a price of 99.26576 and so $100\left(1 - \frac{91d}{365}\right) = 99.26576$

$d = \frac{365}{91}(1 - 0.9926576) = 0.029450$ [2]

[Total 3]

> *There was a potential ambiguity with this question in that the term of the government bill was not separately stated. Most students assumed the term of the bill was also 91 days as the examiners intended but candidates who assumed another term were also given credit.*

Q4 Assuming no arbitrage: [1]

Present value of dividends

$= 0.10v_{5\%}^{0.5} + 0.10v_{6\%} = 0.1 \times 0.97590 + 0.1 \times 0.94340 = 0.19193$ [2]

Forward price $= (4 - 0.19193) \times 1.06 = \$4.03655$ [1]

[Total 4]

> *No comments.*

Q5 (i) Let $S_{20} =$ Accumulated value at time 20 of £1 invested at time 0

then $E[S_{20}] = (1+j)^{20}$

$E[100S_{20}] = 100E[S_{20}] = 200 \Rightarrow E[S_{20}] = 2$

$(1+j)^{20} = 2 \Rightarrow j = 0.035265$ [1]

(ii) Let s be the standard deviation of the annual effective rate of return.

$\text{Var}[100S_{20}] = 50^2$

$10{,}000\text{Var}[S_{20}] = 2{,}500 \Rightarrow \text{Var}[S_{20}] = 0.25$ [1]

$$\text{Var}\left[S_{20}\right]=\left(\left(1+j\right)^2+s^2\right)^{20}-E\left[S_{20}\right]^2$$ [1]

$$0.25=\left(2^{1/10}+s^2\right)^{20}-2^2$$

$$\Rightarrow s^2=\left(0.25+2^2\right)^{1/20}-2^{1/10}=0.00325372$$

$$\Rightarrow s=0.057041$$ [2]

[Total 5]

> *Part (i) was well answered although many candidates struggled with part (ii). The above solution uses the formulae developed in the core reading in the case where the returns in each year are assumed to be independent and identically distributed although these assumptions are not necessary for the calculation of the above answer.*

Q6 Accumulated amount from Fund A

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3\%}=1,200\frac{1.03^{15}-1}{12\left(1-1.03^{-1/12}\right)}$$

$$=\$22,679.74$$

[2]

Accumulated amount from Fund B

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3.7\%}-12\times 15\ddot{s}^{(12)}_{\overline{1}|3.7\%}\left(1.037\right)^{14}$$

$$=1,200\frac{1.037^{15}-1}{12\left(1-1.037^{-1/12}\right)}-180\frac{1.037-1}{12\left(1-1.037^{-1/12}\right)}\left(1.037\right)^{14}$$

$$=23,967.992-305.313=\$23,662.68$$

[3]

The percentage by which B is greater is found from

$$\frac{23,662.68-22,679.74}{22,679.74}-1=4.33\%$$ [1]

> *A comparatively straightforward question that was poorly done by marginal candidates.*

Q7 (i) Let P be the price per £100 nominal.

$P = 0.8 \times 5 a_{\overline{2}|}^{(2)} + 110 v^2$ with a gross redemption yield of 4% per annum. [1]

$$\Rightarrow P = 0.8 \times 5 \frac{1 - 1.04^{-2}}{2\left(1.04^{1/2} - 1\right)} + 110 \times 1.04^{-2}$$

$$\Rightarrow P = 4 \times 1.904771 + 110 \times 0.924556 = £109.320$$ [2]

(ii)

Time t	Government bond spot rate y_t + 1%	Present value factor	Payment	Present value of payment
0.5	0.0175	0.99136	2	1.9827
1	0.025	0.97561	2	1.9512
1.5	0.0325	0.95316	2	1.9063
2	0.04	0.92456	112	103.5503

Total present value = £109.391 per £100 nominal [3]

(iii) Forward rate $= \dfrac{(1+y_2)^2}{1+y_1} - 1 = \dfrac{1.03^2}{1.015} - 1 = 0.04522$ [2]

(iv) It may be because interest rates are expected to rise in the future and the yield curve is determined by expectations theory.

And/or because investors might expect inflation to rise leading to expectations of higher interest rates over the longer term.

And/or because investors have a preference for liquidity which puts an upwards bias on the yield curve. A rising curve would be compatible, for example, with constant expectations of interest rates.

And/or because the market segmentation theory holds and short-term bonds might be in demand by investors such as banks.

[1½ each point, maximum 3]
[Total 11]

Marginal candidates struggled with this question with a common error in part (ii) being to assume coupons were annual (which simplified the question considerably).
Part (iv) was poorly answered. Explanations of why the yield curve would be the given shape were required. It was not sufficient just to name the various theories of the yield curve.

Q8 (i) Amount of loan is $50(Ia)_{\overline{10|}} + 50a_{\overline{10|}}$ at 5% per annum effective [1]

$= 50 \times 39.3738 + 50 \times 7.7217$

= 1968.69 + 386.09 = £2,354.78 [1]

(ii) (a) The outstanding loan after fifth instalment is:

$50(Ia)_{\overline{5|}} + 300a_{\overline{5|}}$ [1]

$= 628.32 + 1,298.85$ = £1,927.17 [1]

The interest component is therefore $0.05 \times 1,927.17 = £96.36$ [1]

(b) The capital component = 350 - 96.36 = £253.64 [1]

(iii) The capital remaining after the sixth instalment is
1,927.17 – 253.64 = £1,673.53 [1]

Let the new instalment = *X*

$Xa_{\overline{4|}6\%} = 1,673.53$

$X = \frac{1,673.53}{3.4651} = £482.96$ [2]

[Total 9]

The best answered question on the paper (excluding Q1)

Q9 (i) $A(0,10) = \exp \int_0^{10} 0.09 - 0.003s\, ds$

$= \exp\left[0.09s - 0.0015s^2\right]_0^{10} = \exp(0.9 - 0.15) = e^{0.75} = 2.1170$ [3]

Require *i* where $(1+i)^{10} = 2.1170 \Rightarrow i = 0.077884$ [1]

(ii) $d^{(2)} = 2\left(1 - (1+i)^{-½}\right) = 0.073611$ [1]

(iii) $A(5,10) = \exp \int_5^{10} 0.09 - 0.003s\, ds$

$$= \exp\left[0.09s - 0.0015s^2\right]_5^{10} = \exp(0.75 - 0.45 + 0.0375) = e^{0.3375}$$

$$A(10,15) = e^{5\times 0.06} = e^{0.3}$$

$$A(5,15) = A(5,10)\,A(10,15) = e^{0.6375} = 1.89175$$

Accumulated amount = 1,500 $e^{0.6375}$ =£2,837.62 [3]

(iv) Equivalent annual effective rate of discount is d such that
$(1-d)^{-10} = e^{0.6375} \Rightarrow d = 0.061760$ [1]

(v) For t > 10,

$$v(t) = v(10)\exp\left[-\int_{10}^{t} 0.06\, ds\right]$$

$$= e^{-0.75}\exp\left[-0.06s\right]_{10}^{t}$$

$$= e^{-0.75}\exp\left[-0.06t + 0.6\right] = e^{-0.06t-0.15}$$

[3]

Present value $= \int_{11}^{15} \rho(t)v(t)\,dt = \int_{11}^{15} 10e^{0.01t}e^{-0.06t-0.15}\,dt = \int_{11}^{15} 10e^{-0.05t-0.15}\,dt$

$$= 10\left[\frac{e^{-0.05t-0.15}}{-0.05}\right]_{11}^{15} = -200\left(e^{-0.9} - e^{-0.7}\right)$$

$$= -81.314 + 99.317 = 18.003$$

[3]

Another standard question that was well-answered.

Q10 (i) (a) Work in £ millions

PV of liabilities = $100v^{10} + 200v^{20}$ at 3% per annum

$=100\times 0.74409+200\times 0.55368=185.145$ [1½]

(b) DMT of liabilities = $\dfrac{1,000\times 0.74409+4,000\times 0.55368}{185.145}=\dfrac{2,958.797}{185.145}$

=15.981 years [2½]

(ii) PV of assets = $144.054v^{15}+Xa_{\overline{t|}}$ where t is the term of the annuity and X is the annual payment.

So $Xa_{\overline{t|}}=185.145-144.054v^{15}=185.145-144.054\times 0.64186=92.682$ for first condition to be satisfied. [1]

DMT of assets = $\dfrac{144.054\times 15\times 0.64186+X\left(Ia\right)_{\overline{t|}}}{185.145}$ = 15.981 years [2½]

So $X\left(Ia\right)_{\overline{t|}}=2,958.797-144.054\times 15\times 0.64186=1,571.859$ for second condition to be satisfied. [1]

Thus $\dfrac{X\left(Ia\right)_{\overline{t|}}}{Xa_{\overline{t|}}}=\dfrac{1,571.859}{92.682}\Rightarrow\dfrac{\left(Ia\right)_{\overline{t|}}}{a_{\overline{t|}}}=16.960$

From inspection of tables, t = 41 years. [1½]

(iii) $Xa_{\overline{41|}}=92.682\Rightarrow X=£3.95865m$ [1]

(iv) Redington's third condition requires that the convexity or spread of the terms of the asset proceeds around the discounted mean term is greater than that for the liabilities.
It is likely that this is the case given that the asset proceeds consist in part of an annuity of term 41 years (though not certain). [2]

(v) If the insurance company sells the security and buys one with a shorter term, the discounted mean term of its assets will no longer be equal to that of its liabilities (it will be shorter). This will mean that, if interest rates were to fall, the insurance company would make a loss. [2]

[Total 15]

Part (i) was answered well. In a 'Show that…' question as in part (ii), it is important to show steps clearly. Many marginal candidates did not do this or, more seriously, appeared to claim that incorrect workings led to the required final answer.
Part (v) was answered very poorly with few candidates explaining the precise scenario where a loss would be made.

Q11 (i) $PV_A=10,000\ddot{a}^{(12)}_{\overline{1|}}+10,000\times 1.05v\times\ddot{a}^{(12)}_{\overline{1|}}+10,000\times 1.05^2v^2\times\ddot{a}^{(12)}_{\overline{1|}}$ [2]

$$= 10{,}000\ddot{a}_{\overline{1}|}^{(12)}\left(1 + 1.05v + (1.05v)^2\right)$$

$$= 10{,}000\ddot{a}_{\overline{1}|}^{(12)}\frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10{,}000\frac{1-v}{d^{(12)}}\frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10{,}000 \times 0.986579 \times 3.058629$$

$$= £30{,}176$$

[2]

[or from 2nd line in 1 above:

$$= 10{,}000\frac{1-v}{d^{(12)}} \times (1 + 1.019417 + 1.039212)$$

$$= 10{,}000 \times 0.986579 \times 3.058629$$

$$= £30{,}176]$$

(ii) $PV_B(1+i)^6 = 1{,}300\ddot{a}_{\overline{45}|}^{(4)} + 200\left(\ddot{a}_{\overline{45}|}^{(4)} - \ddot{a}_{\overline{15}|}^{(4)}\right) + 300\left(\ddot{a}_{\overline{45}|}^{(4)} - \ddot{a}_{\overline{30}|}^{(4)}\right)$ [2]

$$PV_B = v^6\left[1{,}800\ddot{a}_{\overline{45}|}^{(4)} - 200\ddot{a}_{\overline{15}|}^{(4)} - 300\ddot{a}_{\overline{30}|}^{(4)}\right]$$

$$\Rightarrow PV_B = 1.03^{-6}\frac{\left[1{,}800\left(1-v^{45}\right) - 200\left(1-v^{15}\right) - 300\left(1-v^{30}\right)\right]}{4\left(1-1.03^{-1/4}\right)}$$

$$\Rightarrow PV_B = 0.837484 \times \frac{1{,}800 \times 0.735561 - 200 \times 0.358138 - 300 \times 0.588013}{0.0294499} = £30{,}598$$

[3]

[or

$$PV_B(1+i)^6 = 1{,}300\ddot{a}_{\overline{15}|}^{(4)} + 1{,}500v^{15}\ddot{a}_{\overline{15}|}^{(4)} + 1{,}800v^{30}\ddot{a}_{\overline{15}|}^{(4)}$$

$$= \ddot{a}_{\overline{15}|}^{(4)}\left(1{,}300 + 1{,}500v^{15} + 1{,}800v^{30}\right)$$

$$= \frac{\left(1-v^{15}\right)}{4\left(1-1.03^{-1/4}\right)} \times \left(1{,}300 + 1{,}500 \times 0.641862 + 1{,}800 \times 0.411987\right)$$

$$= \frac{0.358138}{0.0294499} \times 3{,}004.3696$$

$$\Rightarrow PV_B = 0.837484 \times 36{,}535.91 = £30{,}598\,]$$

(iii) Option A has the lower present value out of A and B. Therefore, the student has to calculate the salary level so that $PV_C = 30{,}176$ [1]

Let the initial salary level in relation to option C be S_C

$$30{,}176 = 0.03S_C v^3\left(v + 1.03v^2 + \ldots + 1.03^9 v^{10}\right) + 0.03S_C 1.03^{10} v^{18}\left(v + 1.01v^2 + \ldots + 1.01^{29} v^{30}\right)$$

[3]

$$= 0.03S_C v^4\left(10 + 1.03^{10} v^{15}\left(1 + 1.01v \ + \ldots + 1.01^{29} v^{29}\right)\right)$$

$$= 0.03S_C v^4\left(10 + v^5 \frac{1 - 1.01^{30} v^{30}}{1 - 1.01v}\right)$$

$$= 0.03S_C 1.03^{-4}\left(10 + 0.862609 \times 22.90226\right) = 0.79313S_C$$

$\Rightarrow S_C = £38{,}047$ [3]

Therefore, the starting salary has to be less than £38,047 for option C to have the lowest net present value. [1]

(iv) The risks to the students of the three options are very different. For example, the payments under option C vary with salaries and probably with general inflation and the time spent out of the labour market, whereas under options A and B payments are fixed. Therefore, it does not seem reasonable to use the same interest rate (and therefore risk premium) to evaluate all three options. [2]

(v) Possible risks could be:

Student defaults on loan payments (for those that choose option B)

Student salaries are less than the university expects (for those that choose option C) – this could include lower than expected general inflation

Salary earning periods being shorter than expected (for those that choose option C) e.g. because of periods of maternity/paternity leave.

Mortality risk: e.g. under options B and C, if mortality were higher than expected, payments received would be lower than expected.

Students select against the university with those expecting low salaries or poor employment prospects choosing C and those expecting high salaries choosing options A or B.

Students choosing option C artificially restrict official salary (e.g. 'cash-in-hand', payment via dividends, working abroad). [1½ each point, maximum 4]

[Total 23]

There was an ambiguity in part (iii) where the examiners intended for the maximum initial level of salary to be given as the answer. All marginal candidates appeared to read this part as the examiners had intended.

Parts (i) and (ii) were answered well but the later parts were answered poorly, possibly as a result of time pressure. Parts (iv) and (v) did not *require reference to the earlier calculations but were still not answered well by marginal candidates.*

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2018

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
June 2018

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.
2. Student performance was similar to that in recent diets with the average mark being very close to the average of the previous six diets although lower than that in September 2017. Students seemed to have difficulty with the early part of the paper with the four worst answered questions all in the first five questions.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 The characteristics of a Eurobond are:

- Medium- or long-term borrowing
- Usually unsecured
- Regular interest payments
- Redeemed at par
- Issued and traded internationally/not in the jurisdiction of any one country
- Can be denominated in any currency (e.g. not the currency of issuer)
- Tend to be issued by large companies, governments or supra-national
- organisations
- Yields depend on issue size and issuer (or marketability and risk)…
- …(although typically yields will be higher than those on gilts and lower than those on equities)
- Issue characteristics may vary – market free to allow innovation

[½ mark for each point, max 4]

> *This was a bookwork question similar to the type asked in most diets. This was generally answered poorly particularly by marginal candidates.*

Q2 (i) An equity which is offered for sale without the next dividend is called ex-dividend [1]

(ii) Value of dividends to investor =

$$0.07\times 10{,}000\times\left(v^{7/12}+1.02\left(v^{13/12}+v^{19/12}\right)+1.02^2\left(v^{25/12}+v^{31/12}\right)+....\right) \quad [1]$$

$$=700v^{1/12}\left(v^{6/12}+1.02\left(v+v^{1½}\right)+1.02^2\left(v^2+v^{2½}\right)+....\right)$$

$$=700v^{7/12}+700v^{1/12}\ 1.02\left(v+v^{1½}\right)\times\left[1+1.02v+1.02^2v^2+...\right] @\ 7\%$$

$$=700v^{7/12}+700v^{1/12}\ 1.02\left(v+v^{1½}\right)\times\left(\frac{1}{1-(1.02/1.07)}\right) \quad [2]$$

$$=672.91+709.99\times 1.83807\times\left(\frac{1}{1-(1.02/1.07)}\right)$$

= \$28,600 [2]

Candidates who scored well on this question tended to score very well overall but this was poorly answered by marginal candidates. Very few got the timing right, with many failing to include the extra one month offset. Many also struggled with simplifying the long equation into a format which could be more easily calculated.

Q3 Effective rate of interest per month for first 10 years, i_1, comes from:

$$1+i_1=(1.03)^{1/6} \Rightarrow i_1=0.49386\% \text{ per month} \qquad [1]$$

and effective rate of interest per month for last 15 years, i_2, comes from:

$$1+i_2=e^{0.06/12} \Rightarrow i_2=0.50125\% \text{ per month} \qquad [1]$$

$\Rightarrow$ Accumulation after 25 years $=80\ \ddot{s}_{\overline{120}|}^{0.49386\%}\times\left(1.0050125\right)^{180}+80\ \ddot{s}_{\overline{180}|}^{0.50125\%}$

where $\ddot{s}_{\overline{120}|}^{0.49386\%}=1.0049386\times\dfrac{(1.0049386^{120}-1)}{0.0049386}$

$= 164.0318$ [1½]

and $\ddot{s}_{\overline{180}|}^{0.50125\%}=1.0050125\times\dfrac{\left(1.0050125^{180}-1\right)}{0.0050125}$ $= 292.6504$ [1½]

$\Rightarrow$ Accumulation $= 80 \times 164.0318 \times 1.0050125^{180} + 80 \times 292.6504$

$= 32276.13 + 23412.03 = £55,688.16$ (exact answer is £55,688.38)
[1]

[or working in years:

$$1+i_1=(1.03)^2 \Rightarrow i_1=6.09\% \text{ per year}$$
$$1+i_2=e^{0.06} \Rightarrow i_2=6.1837\% \text{ per year}$$

$\Rightarrow$ Accumulation after 25 years $=960\ \ddot{s}_{\overline{10}|}^{(12)@6.09\%}\times\left(1.061837\right)^{15}+960\ \ddot{s}_{\overline{15}|}^{(12)@6.1837\%}$

where $\ddot{s}_{\overline{10}|}^{(12)@6.09\%}=\dfrac{(1.0609^{10}-1)}{12\times\left(1-\left(1-\dfrac{0.0609}{1.0609}\right)^{1/12}\right)}=13.6693$

and $\ddot{s}_{\overline{15}|}^{(12)@6.1837\%} = \dfrac{(1.061837^{15} - 1)}{12 \times \left(1 - \left(1 - \dfrac{0.061837}{1.061837}\right)^{1/12}\right)} = 24.3877$

$\Rightarrow$ Accumulation $= 960 \times 13.6693 \times 1.061837^{15} + 960 \times 24.3877$

$= 32276.42 + 23412.17 = £55{,}688.59]$

Many of the comments on Q2 also apply here although the performance on this question was better. Common errors included those in the calculation of the appropriate interest rate and in the calculation of the accumulation factors.

Q4 (i) Let S_n denote the accumulation at time n of an initial investment of 1 at time 0.

Then, the accumulation at time 10 is:

$$S_{10} = \prod_{t=1}^{10}(1+i_t) \Rightarrow \ln(S_{10}) = \sum_{t=1}^{10}\ln(1+i_t) \sim N(10\mu, 10\sigma^2) \qquad [1]$$

Also, an initial investment of X at time 0 will accumulate to XS_{10} at time 10.

Then, we require to find the value of X such that:

$$P(XS_{10} \geq 800{,}000) = 0.95 \qquad [1]$$

$$\Rightarrow P\left(S_{10} \geq \frac{800{,}000}{X}\right) = 0.95$$

$$\Rightarrow P\left[\ln(S_{10}) \geq \ln\left(\frac{800{,}000}{X}\right)\right] = 0.95$$

$$\Rightarrow P\left[Z \geq \frac{\ln\left(\frac{800{,}000}{X}\right) - 10\mu}{\sqrt{10\sigma^2}}\right] = 0.95$$

$$\Rightarrow \frac{\ln\left(\frac{800{,}000}{X}\right) - 10\mu}{\sqrt{10\sigma^2}} = -1.645$$

$$\Rightarrow \ln\left(\frac{800{,}000}{X}\right) = 0.019708$$

$$\Rightarrow X = 784{,}388 \approx €784{,}000$$

(exact answer is £784,333) [3]

(ii) (a) Increasing the value of μ will increase the expected annual investment return and so the amount required at time 0 (to meet the liability with probability 95%) will decrease. [1]

(b) Increasing the value of σ will increase the volatility of the annual investment return $\Rightarrow$ the amount required at time 0 (to meet the liability with probability 95%) will increase. [1]

(c) If the probability of meeting the liability is increased from 95% to 99%, then the risk of not meeting the liability has been reduced and so the amount required to be invested now must be increased so that, with a greater initial investment, there is more certainty that the target figure of £800.000 after 10 years will be reached. [1]

This was the worst answered question on the paper. Some candidates tried to calculate the parameters of the distribution for the 10-year accumulation from first principles and others made method/calculation errors when manipulating the Normal distribution.

In part (ii), many candidates stated conclusions with no supporting reasoning. No credit was awarded in such cases.

Q5 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss; nor

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit. [2]

(ii) (a) The current value of the forward price of the old contract is:

$$7.10 \times (1.02)^3 - 1.1 a_{\overline{6|}}^{2\%}$$

whereas the current value of the forward price of a new contract is:

$$10.20 - 1.1\, a_{\overline{6|}}^{2\%}$$

Hence, current value of old forward contract is:

$$10.20 - 7.10 \times (1.02)^3 = £2.6654 \qquad [3]$$

Alternative solution

$$K_0 = (7.1 - 1.1v^3 a_{\overline{6|}}) \times 1.02^9 = 1.5462$$
$$K_3 = (10.2 - 1.1a_{\overline{6|}}) \times 1.02^6 = 4.5479$$
$$V_4 = (4.5479 - 1.5462) \times 1.02^{-6} = £2.6654$$

Solution if it is assumed that dividends are paid from the start:

$$K_0 = (7.1 - 1.1a_{\overline{9|}}) \times 1.02^9 = -2.22449$$
$$K_3 = (10.2 - 1.1a_{\overline{6|}}) \times 1.02^6 = 4.5479$$
$$V_3 = (4.5479 + 2.2449) \times 1.02^{-6} = £6.0319$$

(ii) (b) The current value of the forward price of the old contract is:

$$7.10(1.02)^3 \ (1.025)^{-9} = 6.0331$$

Whereas the current value of the forward price of a new contract is

$$10.20\left(1.025\right)^{-6} = 8.7954$$

$\Rightarrow$ current value of old forward contract is

8.7954 – 6.0331 = £2.7623 [3]

Alternative Solution

$$K_0 = 7.1 \times 1.025^{-9} \times 1.02^9 = 6.7943$$
$$K_3 = 10.2 \times 1.025^{-6} \times 1.02^6 = 9.9051$$
$$V_3 = (9.9051 - 6.7943) \times 1.02^{-6} = £2.7623$$

In part (i), many candidates seemed confused between 'arbitrage' and 'no arbitrage'. In part (ii)(a), candidates who assumed the dividends were payable from outset also received full credit.

Q6 (i) Let i = money yield per annum.

Consider £100 nominal stock purchased on 1/4/2018.

$$102 = 0.75 \times 3 \times a^{(2)}_{\overline{10|}} + \left(105 - 0.35 \times (105 - 102)\right)v^{10}$$

$$\Rightarrow 102 = 2.25\ a^{(2)}_{\overline{10|}} + 103.95v^{10}$$ [2]

Try 2%, RHS = 2.25 × 1.004975 × 8.9826 + 85.2752

= 105.59

Try 3%, RHS = 2.25 × 1.007445 ×8.5302 + 77.3486

= 96.68

$$i = 0.02 + \frac{105.59 - 102}{105.59 - 96.68} \times 0.01$$

= 0.0240 (exact answer is 0.0239)

[3]

and (1 + *i*) = (1 + *i′*)(1 + *e*)

$\Rightarrow i' = \frac{1.0239}{1.02} - 1 = 0.00382$ i.e. Real yield = 0.4% per annum [1]

(ii) If inflation had been less than 2% per annum throughout the term then the real rate of return would have been higher. This is because one would be stripping out a lower rate of inflation from the money yield to obtain the real yield. [2]

Generally well-attempted although in part (b) some candidates, as in Q4, gave a conclusion without supporting reasoning.

Q7 (i) Present value of initial outlay $= 2 + 0.5\, v^{1/2} = 2.4767$ [1]

PV of 1st year's net revenue = $0.2\, v\, \bar{a}_{\overline{1|}} = 0.2\, v^2\, i/\delta$

= 0.2 × 0.82645 × 1.049206

= 0.1734 [2]

PV of 2nd to 14th year of net revenue

$$= 0.25\, v^2\, \bar{a}_{\overline{1|}} + 0.25 \times 1.04\, v^3\, \bar{a}_{\overline{1|}} + \ldots\ldots + 0.25 \times 1.04^{12} v^{14}\, \bar{a}_{\overline{1|}}$$

$$= 0.25 v^2\, \bar{a}_{\overline{1|}} \left(1 + 1.04\, v + \ldots + 1.04^{12}\, v^{12}\right)$$

$$= 0.25 v^3\, \frac{i}{\delta} \left[\frac{1 - \left(1.04/1.10\right)^{13}}{1 - 1.04/1.10} \right]$$

$= 0.25 \times 0.75131 \times 1.049206 \times 9.49094$

$= 1.8704$ [3]

PV of refit = $0.8v^8 = 0.3732$ [½]

PV of sale proceeds = $6.4v^{15}$

$= 1.5321$ [½]

$\Rightarrow$ NPV = 0.1734 + 1.8704 + 1.5321 – 2.4767 – 0.3732

$=$ £0.726m [1]

(ii) If the net revenue had been received mid-year rather than continuously then we would be replacing $\bar{a}_{\overline{1}|}$ with $v^{1/2}$ in the formulae for the PV of the net revenue.

Since we can observe that $\bar{a}_{\overline{1}|} = \frac{i}{\delta}v > v^{1/2}$ we can see that the PV of the net revenue would decrease. Therefore, the NPV of the profit would decrease. [2]

Generally well-attempted. In questions like part (i), candidates are advised to show their working for each element separately as this provides a clear 'audit trail' for markers to follow and appropriate partial credit can be awarded for correct elements.

Q8 (i) Let X and Y be the maturity proceeds from the amounts invested in the 7-year and 14-year zero-coupon bonds respectively.

Redington's first condition states that the PV of the assets should equal the PV of the liabilities (using $= \frac{1}{1.045} = 0.95694$ and working in £million):

$$V_L = 20v^8 + 15v^{12} = 22.9087$$
$$V_A = Xv^7 + Yv^{14} = 22.9087 \ (1)$$ [2]

Redington's second condition states that the discounted mean term (DMT) of the assets should be equal to the DMT of the liabilities. The denominators for the DMTs will be the respective PVs, which are assumed to be equal from the first condition above, so we can just consider the numerators:

For the liabilities: $= 20 \times 8v^8 + 15 \times 12v^{12} = 218.6491$

For the assets: $= 7X\text{v}^7 + 14Y\text{v}^{14} = 218.6491$ (2) [2]

Taking (2) – 7 × (1):

$$7Y\text{v}^{14} = 218.6491 - 7 \times 22.9087 = 58.2882$$ [1]

$$Y = \frac{58.2882}{7 \times 1.045^{-14}} = £15.421m$$

with an amount invested of $Y\text{v}^{14} = £8.327m$ [1]

Sub back in (1):

$$X = \frac{22.9087 - 15.421 \times 1.045^{-14}}{1.045^{-7}} = £19.844m$$

with an amount invested of $X\text{v}^7 = £14.582m$ [1]

Since the spread of asset proceeds exceeds the spread of liability outgo (as asset proceeds are received at times 7 and 14, whereas liability outgo is paid at times 8 and 12), the convexity of the assets is greater than the convexity of the liabilities. Thus, the third condition is also satisfied and the company is immunised against small changes in the rate of interest. [2]

(ii) The small increase in interest rates will mean that the present value of both assets and liabilities will fall. The greater convexity of the assets mean that the assets will fall by a smaller amount. There is a greater positive contribution from the convexity term in the present value of the assets than that of the present value of the liabilities. [2]

Part (i) was answered well although, for full credit, the amounts invested needed to be given rather than the maturity values. Part (ii) was less well answered with many marginal candidates not appreciating how the greater asset convexity would influence the change in relative values.

Q9 (i) Let the 1-year and 2-year zero-coupon yields (spot rates) be i_1 and i_2 respectively.

$$\frac{106}{1+i_1} = 106v @ 5.2\%$$

$$\therefore i_1 = 0.052 \text{ (=5.200\% to 3 dp)}$$ [1]

For the 2-year spot rate:

$$\frac{6}{1+i_1}+\frac{106}{\left(1+i_2\right)^2}=6a_{\overline{2}|6.1\%}+100v_{6.1\%}^2$$ [1]

$$\frac{6}{1.052}+\frac{106}{\left(1+i_2\right)^2}=6\frac{\left(1-\frac{1}{1.061^2}\right)}{0.061}+\frac{100}{1.061^2}$$

= 10.984960 + 88.831957

= 99.816917

$$\frac{106}{\left(1+i_2\right)^2}=99.816917-\frac{6}{1.052}$$

$$\Rightarrow (1+i_2)^2=\frac{106}{94.113495}$$

$\Rightarrow i_2 = 6.1273\%$ p.a.(= 6.127% to 3 dp) [3]

For the 3- year spot rate:

The 3-year par yield is 6.6% p.a.

$$\Rightarrow 1=0.066\left(\frac{1}{1+i_1}+\frac{1}{(1+i_2)^2}+\frac{1}{(1+i_3)^3}\right)+\frac{1}{(1+i_3)^3}$$ [1]

$$\Rightarrow \frac{1.066}{(1+i_3)^3}=1-\frac{0.066}{1.052}-\frac{0.066}{(1.061273)^2}$$

$$\Rightarrow (1+i_3)^3=\frac{1.066}{0.878663}$$

$\Rightarrow i_3 = 6.6543\%$ p.a. (= 6.654% to 3 dp) [2]

(ii) 1-year forward rates:

$f_0 = 1_i = 5.2\%$ p.a. [1]

$$(1+i_1)(1+f_1)=(1+i_2)^2$$

$$\Rightarrow 1+f_1=\frac{1.061273^2}{1.052}$$

$\Rightarrow f_1 = 7.0628\%$ p.a (=7.063% to 3dp). [1½]

(answer is 7.062% if rounded spot rates used)

$$(1+i_2)^2(1+f_2) = (1+i_3)^3$$

$$\Rightarrow 1+f_2 = \frac{(1.066543)^3}{(1.061273)^2}$$

$\Rightarrow f_2 = 7.7162\%$ p.a. (= 7.716% to 3 dp) [1½]

(answer unchanged if rounded spot rates used)

> *Candidates who made errors in part (i) often scored full marks in part (ii) after allowance was made for the effects of the earlier errors.*

Q10 (i) We make use of: $v(t) = \exp\left(-\int_0^t \delta(s)\,ds\right)$.

For $0 < t \le 6$ $\quad v(t) = \exp\left(-\int_0^t (0.24 - 0.02s)\,ds\right)$

$$= \exp\left(-0.24s + 0.01s^2\Big|_0^t\right) = \exp\left(-0.24t + 0.01t^2\right)$$ [2]

For $t > 6$ $\quad v(t) = v(6) \times \exp\left(-\int_6^t 0.12\,ds\right)$

$$= \exp\left(-0.24 \times 6 + 0.01 \times 36\right) \times \exp\left(-0.12s\Big|_6^t\right)$$
$$= \exp\left(-0.36 - 0.12t\right)$$ [3]

(ii) Discounted value

$$= 1{,}000 \times v(4,10) = 1{,}000\frac{v(10)}{v(4)} = 1{,}000\frac{\exp(-0.36 - 0.12 \times 10)}{\exp\left(-0.24 \times 4 + 0.01 \times 4^2\right)}$$

$$= 1{,}000e^{-1.56-(-0.8)} = 1{,}000e^{-0.76} = 467.67$$ [2]

(iii) $$\left(1+\frac{i^{(12)}}{12}\right)^{12\times(10-4)} = \frac{1,000}{1,000e^{-0.76}} = e^{0.76}$$

$$=> \; i^{(12)} = \left(\sqrt[72]{e^{0.76}} - 1\right)\times 12 = 0.12734$$ [2]

(iv) $$PV = \int_6^{10} \rho(t)v(t)\,dt = \int_6^{10} 20\exp(0.36+0.32t)\times\exp(-0.36-0.12t)\,dt$$

$$= \int_6^{10} 20e^{0.2t}\,dt = \frac{20}{0.2}e^{0.2t}\Big|_6^{10} = 100\times\left(e^2 - e^{1.2}\right) = 406.89$$ [4]

This calculation question was the best-answered on the page.

Q11 (i) Denote PV of annuity by:

$$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \ldots + 2v^{n-1} + v^n$$ [1]

$$\Rightarrow (1+i)\times(Da)_{\overline{n}|} = n + (n-1)v + (n-2)v^2 + \ldots + 2v^{n-2} + v^{n-1}$$

$$\Rightarrow i\times(Da)_{\overline{n}|} = n - \left(v + v^2 + \ldots + v^n\right)$$

$$\Rightarrow (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

[2]

(ii) Initial amount of loan, L, is given by:

$$L = 8,000v_{5.5\%} + 7,800v_{5.5\%}^2 + 7,600v_{5.5\%}^3 + \ldots + 3,200v_{5.5\%}^{25}$$

$$= 3,000\times a_{\overline{25}|}^{5.5\%} + 200\times(Da)_{\overline{25}|}^{5.5\%}$$

[1]

where $a_{\overline{25}|}^{5.5\%} = \dfrac{1-v_{5.5\%}^{25}}{0.055} = 13.4139$, and

$$(Da)_{\overline{25}|}^{5.5\%} = \frac{25 - a_{\overline{25}|}^{5.5\%}}{0.055} = 210.6558$$ [1½]

Thus, initial amount of loan is:

$$L = 3,000\times 13.4139 + 200\times 210.6558 = £82,372.95$$ [½]

[or can use

$$L = 8,200 \times a_{\overline{25}|}^{5.5\%} - 200 \times (Ia)_{\overline{25}|}^{5.5\%}$$
$$= 8,200 \times 13.4139 - 200 \times 138.1065$$
$$= £82,372.95$$

where $(Ia)_{\overline{25}|}^{5.5\%} = \dfrac{\ddot{a}_{\overline{25}|}^{5.5\%} - 25v_{5.5\%}^{25}}{0.055} = \dfrac{1.055 \times 13.4139 - 25 \times 1.055^{-25}}{0.055} = 138.1065$]

(iii) Need loan outstanding immediately after 9th instalment (i.e. PV of future repayments).

Amount of tenth instalment is £6,200.

Loan outstanding is PV of future repayments, given by:

$$L^* = 6,200v_{5.5\%} + 6,000v_{5.5\%}^2 + 5,800v_{5.5\%}^3 + \ldots + 3,200v_{5.5\%}^{16}$$

$$= 3,000 \times a_{\overline{16}|}^{5.5\%} + 200 \times (Da)_{\overline{16}|}^{5.5\%}$$

where

$$a_{\overline{16}|}^{5.5\%} = \frac{1 - v_{5.5\%}^{16}}{0.055} = 10.4622 \text{ and } (Da)_{\overline{16}|}^{5.5\%} = \frac{16 - a_{\overline{16}|}^{5.5\%}}{0.055} = 100.6880$$

Thus, amount of loan outstanding is:

$$L^* = 3,000 \times 10.4622 + 200 \times 100.6880 = £51,524.08$$ [4]

[or can use

$$L^* = 6,400 \times a_{\overline{16}|}^{5.5\%} - 200 \times (Ia)_{\overline{16}|}^{5.5\%}$$
$$= 6,400 \times 10.4622 - 200 \times 77.1688$$
$$= £51,524.08$$

where $(Ia)_{\overline{16}|}^{5.5\%} = \dfrac{\ddot{a}_{\overline{16}|}^{5.5\%} - 16v_{5.5\%}^{16}}{0.055} = \dfrac{1.055 \times 10.4622 - 16 \times 1.055^{-16}}{0.055} = 77.1688$]

Then, we have:

- interest component of 10th instalment is $0.055 \times 51,524.08 = £2,833.83$, and
- capital component of 10th instalment is $6,200 - 2,833.83 = £3,366.17$

[2]

(iv) Total amount repaid is:

$$3,200 + 3,400 + 3,600 + \ldots + 7,800 + 8,000$$

$= \left(3,000+3,000+\ldots+3,000\right)+\left(200+400+\ldots+5,000\right)$

$= 3,000\times 25+200\times 0.5\times 25\times 26$

$= 140,000$ [1½]

Thus, total interest paid is:

$$140,000-82,372.95=\text{£}57,627.05$$ [½]

Many attempts at proofs in part (i) were unclear. Part (ii) was generally answered well although a common error was to miscalculate the amount of the level annuity component of the loan outstanding. Some candidates also deducted the decreasing annuity component (or equivalently added the increasing component).

END OF EXAMINERS' REPORT

INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2018

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Mike Hammer
Chair of the Board of Examiners
December 2018

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.
2. The number of candidates taking this exam was much lower than in previous diets. This was not surprising given that non-members were not permitted to take this exam due to the fact CT1 without CT5 will not translate to a pass in any subject under the Curriculum 2019 structure.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1

(i) Let d be the annual simple rate of discount.

Assume the bank bond also pays out €100.

The present value of the amount invested in the bank bond would be X such that:

$X = 100(1.04)^{-91/365} = 99.0269$ (99.0276 if 365.25 days in a year used) [1]

To provide the same effective rate of return a treasury bill that pays 100 must have a price of 99.0269 and so $100\left(1-\dfrac{91d}{365}\right) = 99.0269$ [1]

$d = \dfrac{365}{91}(1-0.990269) = 0.03903$ (unchanged if 365.25 days in a year used) [1]

(ii) An additional factor could be the risk of the investments [1]

[Total 4]

Part (i) was well answered although some candidates did not explicitly give the price of treasury bill as asked for in the question. In part (ii), answers referring to present value (which is directly related to the rate of return) or term (which was the same for both investments) were not given credit. Credit was given for answers mentioning marketability or liquidity.

Q2

(i) $\delta = \ln(1+i) = -\ln(1-d) = -\ln 0.95 = 0.051293$ [1]

(ii) $\left(1+\dfrac{i^{(12)}}{12}\right)^{12} = 1+i = \dfrac{1}{1-d} = 0.95^{-1} \Rightarrow i^{(12)} = 12\left(0.95^{-1/12}-1\right) = 0.051403$ [2]

(iii) $\left(1-\dfrac{d^{(12)}}{12}\right)^{12} = 1-d = 0.95 \Rightarrow d^{(12)} = 12\left(1-0.95^{1/12}\right) = 0.051184$ [1]

[Total 4]

The best answered question on the paper.

Q3 (i) Time-weighted rate of return is i where:

$$(1+i)^{2.5} = \frac{70}{60}\frac{300}{70+100} = 2.05882$$ [2]

$$\Rightarrow 1+i = 1.33489 \Rightarrow i = 0.3349$$ [1]

(ii) The money weighted rate of return gives a greater weighting to performance when there is more money in the fund. [½]
The fund was performing better after it had been given the large injection of money on 1 January 2017. [1½]
[Total 5]

Part (i) was answered well. As with similar questions in previous diets, part (ii) was poorly answered. It is important in this type of question to refer to the actual results obtained and the actual data given and the majority of marks in this part were awarded for this.

Q4 (i) (a) Price $= 3a_{\overline{50}|} + 103v^{50}$ at 1.5% working in half-years [1]

$$= 3 \times 34.9997 + 103 \times 0.47500 = 153.925$$ [1]

(b) Three months later the price will be $= 153.925(1.015)^{1/2} = 155.075$ [1]

(ii) Is there a Capital gain?

$$i^{(2)} = 2\left(1.1^{1/2} - 1\right) = 9.762\%$$ [½]

$$\frac{D}{R}(1-t_1) = \frac{6}{1.03} \times 0.7 = 4.078\%$$ [½]

$$i^{(2)} \le g(1-t) \Rightarrow \text{Capital gain}$$ [½]

Price paid per £100 nominal = P where

$$P = 0.7 \times 6a_{\overline{25}|}^{(2)} + 103v^{25} - 0.4(103 - P)v^{25} \text{at } 10\%$$

$$P = \frac{0.7 \times 6a_{\overline{25}|}^{(2)} + 0.6 \times 103v^{25}}{1 - 0.4v^{25}}$$
$$= \frac{0.7 \times 6 \times 1.024404 \times 9.0770 + 0.6 \times 103 \times 0.09230}{1 - 0.4 \times 0.09230}$$
$$= \frac{39.05395 + 5.70389}{0.96308} = 46.474$$

Price = £46.474 per £100 nominal [2½]

[Total 7]

The convertible half-yearly interest rate seemed to confuse some candidates but otherwise the questions was generally answered well.

Q5 (i) (a) Options – holder has the right but not the obligation to trade. [1]
Futures – both parties have agreed to the trade and are obliged to do so. [1]

(b) Call Option – right but not the obligation to BUY specified asset in the future at specified price. [1]
Put Option – right but not the obligation to SELL specified asset in the future at specified price. [1]

(ii) Present value of dividends

$$= 0.1\left(1.025^{-1/2} + 1.025^{-1} + 1.025^{-3/2}\right) = 0.29270$$ [2]

Forward price $= (1.1 - 0.29270) \times 1.025^2 = £0.84817$ [2]

[Total 8]

Well answered although some candidates in part (i) seemed to write down everything that they knew about options whereas the answer required was quite specific. In part (i)(a), partial credit was given for stating the option involved the payment of an initial premium by the holder.

Q6 (i) $$(1+i_t) \sim \log N\left(\mu, \sigma^2\right) \Rightarrow S_{10} = \prod_{t=1}^{10} (1+i_t) \sim \log N\left(10\mu, 10\sigma^2\right)$$ [1]

$$E(1+i_t) = 1 + E(i_t) = 1 + j = 1.08 = e^{\left(\mu + \sigma^2/2\right)}$$

$$\mathrm{Var}(1+i_t)=\mathrm{Var}(i_t)=s^2=0.07^2=e^{(2\mu+\sigma^2)}\times\left(e^{\sigma^2}-1\right)$$ [1]

$$\Rightarrow \quad \frac{0.07^2}{(1.08)^2} \quad = \quad e^{\sigma^2}-1$$

$$\Rightarrow \quad \sigma^2=\ln\left[1+\left(\frac{0.07}{1.08}\right)^2\right]= \ 0.0041922$$

[1]

$$1.08=e^{\left(\mu+\frac{0.0041922}{2}\right)}$$

$$\Rightarrow \mu = \ \ln 1.08-\frac{0.0041922}{2}=0.074865$$ [1]

$$\Rightarrow S_{10}\sim \log N(0.74865, 0.041922)$$ [1]

(ii) $\ln S_{10}\sim N(0.74865, 0.041922)$

and we require X such that $P(6,000S_{10}>X)=0.975$

$$\Rightarrow P\left(\ln S_{10}>\ln\frac{X}{6,000}\right)=0.975$$ [1]

$$\Rightarrow 1-\Phi\left(\frac{\ln\frac{X}{6,000}-0.74865}{\sqrt{0.041922}}\right)=0.975$$ [1]

$$\Rightarrow \frac{\ln\frac{X}{6,000}-0.74865}{\sqrt{0.041922}}=-1.96$$

$$\Rightarrow X=6,000\exp\left(-1.96\times\sqrt{0.041922}+0.74865\right)=\pounds 8,492$$

[1]
[Total 8]

This question proved to be a good differentiator with strong candidates scoring well on both parts but many weaker candidates scoring very little. Full marks could still be scored in part (ii) even if candidates made errors.

Q7 (i) Present value is $v(20) = e^{-\int_0^{20} \delta(s)ds}$ [1]

$$\int_0^{20} \delta(s)\,ds = \int_0^{10} 0.03ds + \int_{10}^{20} 0.003sds$$
$$= [0.03s]_0^{10} + \left[0.0015s^2\right]_{10}^{20}$$
$$= 0.3 + 0.0015(400 - 100) = 0.75$$

[2]

$v(20) = e^{-0.75} = 0.47237$ [1]

(ii) Require δ such that $e^{-20\delta} = e^{-0.75} \Rightarrow \delta = 0.0375$ [2]

(iii) Present value $\int_4^8 \rho(t)v(t)\,dt = \int_4^8 e^{-0.06t}e^{-0.03t}dt = \int_4^8 e^{-0.09t}dt$ [2]

$= \left[\dfrac{e^{-0.09t}}{-0.09}\right]_4^8$ [1]

= -5.40836 + 7.75196 = 2.34360 [1]

[Total 10]

Well answered.

Q8 (i) (a) The payback period is the first point at which the total revenues from a project exceed the total cost, with no allowance made for interest. [1½]

(b) The payback period takes no account of interest at all. It is therefore inappropriate for assessing an investment project which should provide the investor with a return or be paid for from borrowings. [1]

The payback period takes no account of what happens after the payback period. In this particular case, it is known that the revenue from the project might be weighted towards the end and the payback period will make no allowance for this. [1½]

(ii) Work in £2017 millions at 6% per annum

PV of initial costs = $15a_{\overline{5}|} = 15 \times 4.2124 = 63.1855$ [1]

PV of running costs = $3\bar{a}_{\overline{30}|} = 3 \times \dfrac{1 - 1.06^{-30}}{\ln(1.06)} = 3 \times 14.1738 = 42.5213$ [1½]

PV of revenue in first ten years =

$$3.1\bar{a}_{\overline{10|}} = 3.1 \times \frac{1-1.06^{-10}}{\ln(1.06)} = 3.1 \times 7.57875 = 23.4941$$ [1½]

PV of revenue in years 11 to 30 =

$$v^{10}\left(3.2\bar{a}_{\overline{1|}} + 3.2 \times 1.05v\bar{a}_{\overline{1|}} + \ldots 3.2 \times 1.05^{19} v^{19}\bar{a}_{\overline{1|}}\right)$$ [1½]

$$= 3.2v^{10}\bar{a}_{\overline{1|}}\left(1 + 1.05v + \ldots (1.05v)^{19}\right)$$

$$= 3.2v^{10}\bar{a}_{\overline{1|}}\frac{1-(1.05v)^{20}}{1-1.05v}$$

[1]

$$= 3.2 \times 1.06^{-10} \times \frac{1-1.06^{-1}}{\ln 1.06}\frac{1-(1.05/1.06)^{20}}{1-1.05/1.06}$$

$$= 3.2 \times 0.55839 \times 0.97142 \times 18.30506 = 31.7739$$

[1]

Sales proceeds are P such that

$$Pv^{30} = 63.1855 + 42.5213 - 23.4941 - 31.7739 = 50.4388$$

$$P = 50.4388 \times 1.06^{30} = £289.69m$$ [1½]

(iii) Probabilities could be assigned to the cash flows… [1]
… or a higher discount rate could be used to account for risk. [1]
[Total 15]

The calculations in part (ii) were generally done well but parts (i) and (iii) were poorly answered. Part (i) has been asked in previous diets and generally answered better by candidates. Whilst part (iii) has not often been asked, the answer comes directly from the Core Reading.

Q9 (i) The investor pays a purchase price at outset. [½]
The investor receives a series of coupon payments and a capital payment at maturity. [1]
The coupon and capital payments are linked to an index of prices (possibly with a time lag). [½]
[Time lag does not have to be mentioned].

(ii) Let the RPI three months before issue (end 9/2015) = 100
Relevant RPI values are three months before first coupon payment (end 3/2016), three months before second coupon payment (end 9/2016) etc.

Cash payments from the bond are in the following table:

Nominal payment per £100 nominal (*1*)	Base index (*2*)	Index three months before payment (*3*)	(*3*) divided by (*2*) (*4*)	Cash payment (*4*) ×1%×£1m
1	100	$100(1.02)^{0.5}$	1.00995	£10,100
1	100	100(1.02)	1.02	£10,200
1	100	$100(1.02)^{1.5}$	1.0301	£10,301
1	100	$100(1.02)^{2}$	1.0404	£10,404

There is also the sale value of £1,010,000 [3]

(iii) Equation of value is:

$$1{,}000{,}000 = 10{,}100v^{0.5} + 10{,}200v + 10{,}301v^{1.5} + 10{,}404v^2 + 1{,}010{,}000v^2 \quad [2½]$$

Try 2.5%.
RHS of equation of value becomes 1,001,089

Try 3%
RHS of equation of value becomes 991, 538 [1½]

Interpolating:

$$i = 0.025 + (0.03 - 0.025) \times \frac{1{,}000{,}000 - 1{,}001{,}089}{991{,}538 - 1{,}001{,}089} \approx 0.0256 \quad [1]$$

(iv) The equation of value for the real cash flows is as follows (working in half years):

$$\begin{aligned}1{,}000{,}000 = {} & 10{,}000\frac{RPI(\text{March 2016})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{June 2016})}v \\ & +10{,}000\frac{RPI(\text{September 2016})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{December 2016})}v^2 \\ & +10{,}000\frac{RPI(\text{March 2017})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{June 2017})}v^3 \\ & +10{,}000\frac{RPI(\text{September 2017})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{December 2017})}v^4 \\ & +1{,}010{,}000\frac{RPI(\text{December 2015})}{RPI(\text{December 2017})}v^4\end{aligned}$$

All the RPI factors cancel out except the last two because each is the ratio of RPI at three-month intervals multiplied by the inverse of that ratio. [4]

The equation of value therefore becomes:

$$1,000,000 = 10,000v + 10,000v^2 + 10,000v^3$$
$$+ (10,404 + 1,010,000) \times \frac{RPI\left(\text{December 2015}\right)}{RPI\left(\text{December 2017}\right)} v^4$$

Let rate of inflation per annum in the last three months = x

Equation of value becomes:

$$1,000,000 = 10,000a_{\overline{3|}} + \frac{1,020,404}{1.02^{1.75}\left(1+x\right)^{0.25}} v^4 \text{ at } 0.5\%$$

$$\Rightarrow \left(1+x\right)^{0.25} = \frac{1,020,404 \times 1.005^{-4}}{\left(1,000,000 - 10,000a_{\overline{3|}}\right) \times 1.02^{1.75}} = 0.995755$$

$$\Rightarrow \left(1+x\right) = 0.98313 \Rightarrow x = -0.01687$$ [3]

[Total 17

This proved to be the most difficult question on the paper by some margin. Whilst the examiners expected the calculations in part (iii) and especially part (iv) to be challenging, it was more surprising to see candidates having considerable difficulty with part (ii).

Q10 (i) A loan repayable by a series of payments at fixed times set in advance. [½]

Typically issued by banks and building societies [½]

Typically long-term … [½]
…e,g. used to fund house purchase
…and secured against the property [½]

Each payment contains an element to pay interest on the loan with the remainder being used to repay capital [½]

In its simplest form, the interest rate will be fixed … [½]
….and the payments will be of fixed equal amounts. [½]

The interest payment portion of the repayments will fall over time… [½]
… and the capital payments will rise over time. [½]

Risk that borrower defaults on loan [½]

Complications might be added such as allowing the loan to be repaid early or allowing the interest rate to vary. [½]
[Max 3]

(ii) (a) Annual repayment is X where

$10,000 = Xa_{\overline{10|}}$ at 4%

$$X = \frac{10,000}{8.1109} = \$1,232.91$$ [2]

(b) DMT of repayments =

$$\frac{\sum_{t=1}^{10} 1,232.91 \times tv^t}{\sum_{t=1}^{10} 1,232.91 \times v^t} = \frac{1,232.91 \times (Ia)_{\overline{10|}}}{10,000}$$ [1½]

$$= \frac{1,232.91 \times 41.9922}{10,000} = 5.1773 \text{ years}$$

[1½ including ½ for units]

(iii) Let the nominal amount invested in the second zero coupon bond = X and let term = n

If the present values are to be equal:

$$10,000 = 5,000v^2 + Xv^n \quad (1)$$ [1]

If the discounted mean terms are to be equal:

$$5.1773 = \frac{2 \times 5,000v^2 + nXv^n}{10,000}$$

$$\Rightarrow 51,773 = 10,000v^2 + nXv^n \quad (2)$$

[1½]

$$\text{Sub } (1) \text{ into} (2) \Rightarrow 51,773 = 10,000v^2 + n\left(10,000 - 5,000v^2\right)$$

$$\Rightarrow n = \frac{51,773 - 10,000v^2}{10,000 - 5,000v^2} = \frac{42,527}{5,377} = 7.909 \text{ years}$$

[2½]

$$\text{Sub back in } (1) \Rightarrow X = \frac{10,000 - 5,000v^2}{v^{7.909}} = \$7,332.81$$ [1]

(iv) (a) Monthly instalment is M where

$$10,000 = 12Ma_{\overline{25|}}^{(12)}$$ [1]

$$10,000 = 12M \frac{\left(1 - 1.04^{-25}\right)}{12\left(1.04^{1/12} - 1\right)}$$

$$\Rightarrow M = \frac{10,000 \times 0.0032737}{0.62488} = \$52.39$$

[1½]

(b) The assets now have a much longer duration than the liabilities. [1]
Therefore, if interest rates rise, the assets will fall in value by more than the liabilities and the bank will make a loss. [1]

(c) After ten years of payments, the capital outstanding is $12 \times 52.39 a_{\overline{15}|}^{(12)}$

[1]

$= 12 \times 52.39 \dfrac{1-1.04^{-15}}{0.039289} = 7{,}117.15$ [1]

Interest component of 121st payment $= 7{,}117.15 \times \dfrac{0.039289}{12} = \$23.30$

[1]

Capital component of 121st payment $= 52.39 - 23.30 = \$29.09$ [½]

[Total 22]

It was pleasing to see that many candidates scored well on this question even if they had struggled with the previous question. Parts (ii), (iii), (iv)(a) and (iv)(c) were also generally answered well. Part (i) was answered less well despite the wide range of mark-scoring points available to be made.

END OF EXAMINERS' REPORT