0 Introduction

In this chapter we will consider the implication of specifying the order in which the two lives die, which leads to the two types of benefit:

- *contingent assurances* – these are payable on the death of one life, contingent upon another life being in a specified state (alive or dead); and
- *reversionary annuities* – these are payable to one life following the death of another life.

We will also look at multiple life functions that depend on duration in a variety of ways, and annuities based on two lives that are payable more frequently than annually.

1 Contingent probabilities of death

So far we have examined events which depend on the joint lifetime xy or last survivor lifetime $\overline{xy}$ of two lives. We now look at events that depend on the order in which the deaths occur.

We will study two events:

- **the event that (*x*) is the first to die of two lives (*x*) and (*y*):** $\overset{1}{x}y$
- **the event that (*x*) is the second to die of two lives (*x*) and (*y*):** $\overset{2}{x}y$

Events that depend upon the order in which the lives die are called *contingent* events.

Then we use ${}_nq^{1}_{xy}$ and ${}_nq^{2}_{xy}$ to denote the probabilities that each of these two events occurs in the next n years.

Let's consider the meaning of ${}_nq^{1}_{xy}$. The way to read this is by first considering *only* the probability as it would relate to the life *with the number above it* ((*x*) in this case), *ie* initially ignoring the other life (*y*). So we first read this as we would ${}_nq_x$:

- the probability that (*x*) dies within *n* years.

We then bring in the second life, and notice that the number superscript over the *x* is a '1', so this tells us that (*x*) has to die first. However, we have already said something about (*x*), so we need to express this in terms of what happens to (*y*); that is, in addition to the above, we have:

- (*y*) must die after (*x*).

So the whole thing reads:

${}_nq^{1}_{xy} =$ probability that (*x*) dies within n years, and (*y*) dies (any time) after (*x*).

(In this probability, *y* may or may not die within the *n*-year period – *y* can die as long as *x* is already dead.)

Similarly:

${}_nq^{2}_{xy} =$ probability that (*x*) dies within n years, and (*y*) dies before (*x*)

This is because the superscript over the *x* is a '2', so (*x*) has to die second.

These probabilities can be evaluated by an appropriate integration of the density functions of the random variables T_x and T_y.

For example, we can write:

$${}_nq^{\,1}_{xy} = \int_{t=0}^{t=n} \int_{s=t}^{s=\infty} {}_tp_x\,\mu_{x+t}\;{}_sp_y\,\mu_{y+s}\,ds\,dt$$

which corresponds to the event $T_x \le T_y$, $T_x \le n$.

If we rewrite this as:

$${}_nq^{\,1}_{xy} = \int_{t=0}^{t=n} {}_tp_x\,\mu_{x+t}\left\{\int_{s=t}^{s=\infty} {}_sp_y\,\mu_{y+s}\,ds\right\}dt$$

$$= \int_{t=0}^{t=n} {}_tp_x\,\mu_{x+t}\;{}_tp_y\,dt$$

then we are calculating the product of:

- the probability that (x) dies at exact moment t ($\approx {}_tp_x\,\mu_{x+t}\,dt$)
- the probability that (y) is still living at exact moment t ($= {}_tp_y$), *ie* that (y) dies after time t and therefore after (x) dies

and then summing (*ie* integrating) over all possible times at which (x) could die over the next n years.

Finally we can neaten this up slightly by writing:

$${}_nq^{\,1}_{xy} = \int_{t=0}^{t=n} {}_tp_{xy}\,\mu_{x+t}\,dt$$

The ability to express probabilities of life and death as integrals for single and joint lives is important.

The key considerations are:

- what event should be modelled as occurring at time t (*eg* should it be a death or the survival of a particular life)?
- who is dying at time t (if anyone)?
- who has to be alive/dead at time t?
- over what range of times t should the integral be evaluated?

So, in expressing ${}_nq^{\,1}_{xy}$ as an integral, we could proceed as follows:

- we want to model (*x*)'s death as occurring at time *t* (we choose (*x*) rather than (*y*) on this occasion because we know that (*x*) has to die within *n* years, so when we integrate over 0 to *n* we will have covered all the possible times that this can happen);
- at time *t*, (*x*) must have survived to that point in order then to die, and then die instantly (so we want the factor ${}_tp_x\,\mu_{x+t}\,dt$);
- at time *t*, (*y*) needs to be alive (so we want the factor ${}_tp_y$).

So the integral must be: $\int_{t=0}^{n} {}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt$.

Question

Express the probability ${}_nq^{\,2}_{xy}$ as an integral, using the above approach.

Solution

In the probability ${}_nq^{\,2}_{xy}$, (*x*) has to die within the n-year period. So we model (*x*) as dying at time t , meaning that we need a factor of:

$${}_tp_x\,\mu_{x+t}\,dt$$

This can be thought of as the probability that (*x*) dies at the exact future time *t*.

However, since (*x*) has to be the second life to die, we also need to include the probability that (*y*) has already died by this time, *ie* the probability that (*y*) dies before time *t*. This is:

$${}_tq_y$$

Since the time of (*x*)'s death must be between time 0 and time *n*, the integral is:

$${}_nq^{\,2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\,{}_tq_y\,dt$$

Using the solution above and writing ${}_tq_y = \int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds$, we see that:

$${}_nq^{\,2}_{xy} = \int_0^n {}_tp_x\,\mu_{x+t}\left(\int_{s=0}^{s=t} {}_sp_y\,\mu_{y+s}\,ds\right)dt \qquad (1)$$

This can be rewritten as:

$$ {}_nq^{2}_{xy} = \int_{t=0}^{t=n} \int_{s=0}^{s=t} {}_tp_x\, \mu_{x+t}\, {}_sp_y\, \mu_{y+s}\, ds\, dt $$

This corresponds to the event $T_y < T_x \le n$.

If we substitute $\left(1 - {}_tp_y\right)$ for ${}_tq_y$ in the solution to the previous question, then we have:

$$ {}_nq^{2}_{xy} = \int_{t=0}^{t=n} (1 - {}_tp_y)\, {}_tp_x\, \mu_{x+t}\, dt $$

$$ = \int_0^n {}_tp_x\, \mu_{x+t}\, dt - \int_0^n {}_tp_{xy}\, \mu_{x+t}\, dt $$

$$ = {}_nq_x - {}_nq^{1}_{xy} $$

This result implies that 'second death' probabilities can always be expressed in terms of 'single death' and 'first death' probabilities. This provides a method of evaluating 'second death' probabilities.

The truth of this expression can be argued by general reasoning if it is rewritten as:

$$ {}_nq_x = {}_nq^{1}_{xy} + {}_nq^{2}_{xy} $$

The right-hand side is the probability that a life aged x dies in an n-year period either before or after a life aged y. As these are the only two possibilities for (x)'s death in relation to (y), this is equal to the probability that (x) dies at some point in the n-year period.

By changing the order of integration in expression (1) for ${}_nq^{2}_{xy}$ we can show that:

$$ {}_nq^{2}_{xy} = {}_nq_{x\overset{1}{y}} - {}_np_x\, {}_nq_y $$

Question

Prove this result.

Solution

From expression (1), we have:

$$ {}_nq^{2}_{xy} = \int_0^n {}_tp_x\, \mu_{x+t} \left(\int_0^t {}_sp_y\, \mu_{y+s}\, ds \right) dt $$

Changing the order of integration, we have:

$$ {}_nq^{2}_{xy} = \int_0^n {}_sp_y\,\mu_{y+s}\left(\int_s^n {}_tp_x\,\mu_{x+t}\,dt\right)ds $$

In the original double integral we have $0 \le s \le t \le n$. So, when we change the order, we have s going from 0 to n and t going from s to n.

Now $\int_s^n {}_tp_x\,\mu_{x+t}\,dt$ is the probability that (*x*) dies between time s and time n, and so:

$$ \int_s^n {}_tp_x\,\mu_{x+t}\,dt = {}_sp_x - {}_np_x $$

Substituting this expression into the equation above gives:

$$ \begin{aligned} {}_nq^{2}_{xy} &= \int_0^n {}_sp_y\,\mu_{y+s}\left({}_sp_x - {}_np_x\right)ds \\ &= \int_0^n {}_sp_y\,\mu_{y+s}\,{}_sp_x\,ds - {}_np_x\int_0^n {}_sp_y\,\mu_{y+s}\,ds \\ &= {}_nq^{1}_{xy} - {}_np_x\,{}_nq_y \end{aligned} $$

as required.

This relationship can also be argued by considering the events defined by each of the functions.

So, the events defined by ${}_nq^{1}_{xy}$ are:

(A) (*y*) dies followed by (*x*) dying, all within the *n*-year period, or

(B) (*y*) dies within the *n*-year period and (*x*) dies after the *n*-year period has elapsed.

On the other hand, only event (A) is covered by ${}_nq^{2}_{xy}$. So ${}_nq^{1}_{xy} - {}_nq^{2}_{xy}$ must be the probability of event (B) only, which is ${}_nq_y\,{}_np_x$, as above.

The expressions become simple and easy to evaluate when $x = y$. We can write:

$$ \mathbf{{}_nq_x = {}_nq^{1}_{xx} + {}_nq^{2}_{xx}} $$

$$ \mathbf{{}_nq^{2}_{xx} = {}_nq^{1}_{xx} - {}_np_x\,{}_nq_x} $$

by substituting x in place of y in the previous two results. When we use these formulae, we are assuming not only that the two lives are the same age, but that they are subject to identical mortality, and that mortality operates independently between the two lives.

Recall that:

$$ \mathbf{{}_nq_{xx} = 1 - {}_np_{xx} = 1 - {}_np_x\,{}_np_x} $$

So we have:

$$
\begin{aligned}
{}_nq^2_{xx} &= {}_nq^1_{xx} - {}_np_x(1-{}_np_x) \\
&= {}_nq^1_{xx} - {}_np_x + {}_np_{xx} \\
&= {}_nq^1_{xx} - (1-{}_nq_x) + (1-{}_nq_{xx}) \\
&= {}_nq^1_{xx} + {}_nq_x - {}_nq_{xx} \\
&= {}_nq^1_{xx} + \left({}_nq^1_{xx} + {}_nq^2_{xx}\right) - {}_nq_{xx}
\end{aligned}
$$

Cancelling the ${}_nq^2_{xx}$ terms, we obtain:

$$0 = 2\,{}_nq^1_{xx} - {}_nq_{xx}$$

to give:

$$\boldsymbol{{}_nq^1_{xx} = \tfrac{1}{2}\,{}_nq_{xx}}$$

If $n=\infty$ then ${}_\infty q_{xx} = 1$, leading to:

$$\boldsymbol{{}_\infty q^1_{xx} = {}_\infty q^2_{xx} = \tfrac{1}{2}}$$

Note also that:

$${}_nq^2_{xx} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} = \tfrac{1}{2}\left({}_nq_x\right)^2$$

Arguments of symmetry can often lead to simplifications when we have joint life problems involving lives of equal ages. However, it is often useful to start off by considering the more general case with unequal ages.

Question

Simplify the sum ${}_nq^2_{xy} + {}_nq_{x\overset{2}{y}}$, and use the result to prove that ${}_nq^2_{xx} = \frac{1}{2}\,{}_nq_{\overline{xx}}$.

Solution

${}_nq^2_{xy} + {}_nq_{x\overset{2}{y}}$ is the probability that either (x) is the second of the two lives to die within the n-year period, or (y) is the second of the two lives to die within the n-year period. So, overall, this is the probability that both lives die within the n-year period, *ie* ${}_nq_{\overline{xy}}$.

So:

$${}_nq^2_{xy} + {}_nq_{x\overset{2}{y}} = {}_nq_{\overline{xy}}$$

Now if we have two lives of equal age who both experience the same mortality, we have:

$$ {}_nq^{2}_{x\,x} = {}_nq_{x\,\overset{2}{x}} $$

so the expression above becomes:

$$ {}_nq^{2}_{x\,x} + {}_nq_{x\,\overset{2}{x}} = {}_nq_{\overline{xx}} \quad \Rightarrow \quad 2\,{}_nq^{2}_{x\,x} = {}_nq_{\overline{xx}} \quad \Rightarrow \quad {}_nq^{2}_{x\,x} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} $$

The formulae:

$$ {}_nq^{1}_{xx} = \tfrac{1}{2}\,{}_nq_{xx} \quad \text{and} \quad {}_nq^{2}_{x\,x} = \tfrac{1}{2}\,{}_nq_{\overline{xx}} $$

enable us to calculate order of death probabilities in terms of single life probabilities.

Question

Calculate:

(i) ${}_5q^{1}_{40:40}$

(ii) ${}_5q^{2}_{40:40}$

using AM92 mortality.

Solution

(i) We have:

$$ {}_5q^{1}_{40:40} = \frac{1}{2}\,{}_5q_{40:40} = \frac{1}{2}\left(1 - {}_5p_{40:40}\right) = \frac{1}{2}\left(1 - \left({}_5p_{40}\right)^2\right) = \frac{1}{2}\left(1 - \left(\frac{l_{45}}{l_{40}}\right)^2\right) $$

$$ = \frac{1}{2}\left(1 - \left(\frac{9,801.3123}{9,856.2863}\right)^2\right) = 0.005562 $$

(ii) We have:

$$ {}_5q^{2}_{40:40} = \frac{1}{2}\,{}_5q_{\overline{40:40}} = \frac{1}{2}\left({}_5q_{40}\right)^2 = \frac{1}{2}\left(1 - \frac{l_{45}}{l_{40}}\right)^2 = \frac{1}{2}\left(1 - \frac{9,801.3123}{9,856.2863}\right)^2 = 0.000016 $$

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2 Contingent assurances

In Section 1 we saw that contingent events depending on the future lifetime of two lives (x) and (y) can be written in terms of the random variables T_x and T_y. The random variables representing the present value of contingent assurances can be expressed as functions of these two random variables.

For example, the present value of a sum assured of 1 paid immediately on the death of (x) provided that (y) is still alive can be expressed as:

$$\bar{Z} = \begin{cases} v_i^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

where i is the valuation rate of interest.

It is important to realise that the sum assured is not necessarily paid *to* (y) when (x) dies.

Using similar methods to those used in Section 1 the mean of $\bar{Z}$ is:

$$E\left[\bar{Z}\right] = \bar{A}^{1}_{xy} = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \, \mu_{x+t} \, dt$$

The actuarial notation is consistent with the term assurance notation $\bar{A}^{1}_{x:\overline{n|}}$ that we defined earlier in the course, but here it involves a second life status *y* instead of the duration status.

So, $\bar{A}^{1}_{xy}$ represents the EPV of a sum assured of 1 paid immediately on the death of life (x), provided that life (x) dies before life (y).

A point to note is that the *positioning* of the number (over the *x*) tells us *on whose death* the benefit will be paid. Then the *value* of that number (1 in this case) tells us the required *order* in which the deaths must occur.

Using a stochastic approach, we obtain the expectation as shown above. We can alternatively derive this integral using general reasoning. As we saw when expressing probabilities as integrals, we consider an integral based on some event happening at time t and what conditions have to hold at time t. In addition, we need to discount the benefit payment back to time 0. It is therefore normally necessary for the event that we are modelling at time t to be the event that triggers payment.

So for this assurance we have:

- a benefit payable on (*x*)'s death provided (*y*) is still alive, so we model (*x*)'s death at time t
- at time t, we require (*x*) to have survived to t and then to die, giving ${}_tp_x\mu_{x+t}$
- at time t, we require (*y*) to be alive, giving ${}_tp_y$
- we discount to time 0 from the moment of benefit payment at time t, giving v^t
- and we integrate over all possible values of t (from 0 to ∞) as the term is not limited.

So the integral is:

$$\int_{t=0}^{\infty} v^t\,{}_tp_x\,{}_tp_y\,\mu_{x+t}\,dt = \int_{t=0}^{\infty} v^t\,{}_tp_{xy}\mu_{x+t}\,dt$$

The variance of $\bar{Z}$ is:

$$\mathbf{var(\bar{Z}) = {}^2\bar{A}^1_{xy} - \left(\bar{A}^1_{xy}\right)^2}$$

where ${}^2\bar{A}$ is evaluated at a valuation rate of interest i^2+2i.

To obtain this formula we start by noting that if:

$$\bar{Z} = \begin{cases} v_i^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

then:

$$\bar{Z}^2 = \begin{cases} \left(v_i^{T_x}\right)^2 = \left(v_i^2\right)^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases}$$

We see that $\bar{Z}^2$ has the same form as $\bar{Z}$ with v_i replaced by v_i^2, and so we can replace v^t by $\left(v^2\right)^t$ in the expression for $E\left[\bar{Z}\right]$ to obtain:

$$E\left[\bar{Z}^2\right] = \int_{t=0}^{\infty} \left(v^2\right)^t\,{}_tp_{xy}\mu_{x+t}\,dt$$

This is equal to ${}^2\bar{A}^1_{xy}$ at an interest rate of i^2+2i (or force of interest 2δ). Hence:

$$\text{var}\left(\bar{Z}\right) = E\left[\bar{Z}^2\right] - \left(E\left[\bar{Z}\right]\right)^2 = {}^2\bar{A}^1_{xy} - \left(\bar{A}^1_{xy}\right)^2$$

These functions are usually evaluated by using numerical methods, such as Simpson's rule to determine the values of the integrals.

Simpson's rule is similar to the trapezium rule except that, instead of joining pairs of consecutive points with straight-line segments, we take groups of three consecutive points and fit quadratics to them.

In some cases, joint and single life values obtained from tables can be useful in conjunction with the following and similar relationships:

(i) $\boldsymbol{\bar{A}_{xy} = \bar{A}^{1}_{x\,y} + \bar{A}_{x\,\overset{1}{y}}}$

We can prove this result as follows:

$$\bar{A}^{1}_{x\,y} + \bar{A}_{x\,\overset{1}{y}} = \int_{t=0}^{\infty} v^t\,{}_tp_{xy}\mu_{x+t}\,dt + \int_{t=0}^{\infty} v^t\,{}_tp_{xy}\mu_{y+t}\,dt$$

$$= \int_{t=0}^{\infty} v^t\,{}_tp_{xy}\left(\mu_{x+t} + \mu_{y+t}\right)dt$$

$$= \int_{t=0}^{\infty} v^t\,{}_tp_{xy}\,\mu_{x+t:y+t}\,dt$$

$$= \bar{A}_{xy}$$

$\bar{A}_{xy}$ is the expected present value of 1, paid immediately when the first of the two lives dies. The first life to die can either be (*x*) or (*y*), so $\bar{A}_{xy}$ is equal to the sum of an assurance that makes a payment immediately if (*x*) is the first life to die ($\bar{A}^{1}_{x\,y}$) and an assurance that makes a payment immediately if (*y*) is the first life to die ($\bar{A}_{x\,\overset{1}{y}}$).

(ii) $\boldsymbol{\bar{A}_{x} = \bar{A}^{1}_{x\,y} + \bar{A}^{2}_{x\,y}}$

We can see that this is true by general reasoning.

$\bar{A}_x$ is the expected present value of 1 paid immediately on the death of (*x*). In relation to (*y*), (*x*) must either die first or second. Since these two possibilities are mutually exclusive and exhaustive, $\bar{A}_x$ is equal to the sum of an assurance that makes a payment immediately if (*x*) is the first life to die ($\bar{A}^{1}_{x\,y}$) and an assurance that makes a payment immediately if (*x*) is the second life to die ($\bar{A}^{2}_{x\,y}$).

We can also prove this algebraically as follows:

$$\begin{aligned}\bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y} &= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_tp_y \, dt + \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, {}_tq_y \, dt \\ &= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \left({}_tp_y + {}_tq_y \right) dt \\ &= \int_0^\infty v^t \, {}_tp_x \mu_{x+t} \, dt \\ &= \bar{A}_x\end{aligned}$$

(iii) $\boldsymbol{\bar{A}_{\overset{1}{x}x} = \tfrac{1}{2}\bar{A}_{xx}}$

We can prove this result by making use of the formula:

$$\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$$

Making both lives the same age gives:

$$\bar{A}_{xx} = \bar{A}_{\overset{1}{x}x} + \bar{A}_{x\overset{1}{x}}$$

By symmetry:

$$\bar{A}_{\overset{1}{x}x} = \bar{A}_{x\overset{1}{x}}$$

So:

$$\bar{A}_{\overset{1}{x}x} = \tfrac{1}{2}\bar{A}_{xx}$$

(iv) $\boldsymbol{\bar{A}_{\overset{2}{x}x} = \tfrac{1}{2}\bar{A}_{\overline{xx}}}$

To derive this result, we start by obtaining another relationship.

Question

Prove that $\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$.

Solution

By rearranging relationship (ii) above we have:

$$\bar{A}_{\overset{2}{x}y} = \bar{A}_x - \bar{A}_{\overset{1}{x}y}$$

Similarly:

$$\bar{A}_{x\overset{2}{y}} = \bar{A}_y - \bar{A}_{x\overset{1}{y}}$$

So:

$$\begin{aligned}\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} &= \bar{A}_x - \bar{A}_{\overset{1}{x}y} + \bar{A}_y - \bar{A}_{x\overset{1}{y}} \\ &= \bar{A}_x + \bar{A}_y - \left(\bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}\right) \\ &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\ &= \bar{A}_{\overline{xy}}\end{aligned}$$

where the last line above uses the 'last survivor (L) = single (S) + single (S) − joint (J)' result that we met in the previous chapter.

Now, using $\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$ with $y = x$:

$$\bar{A}_{\overset{2}{x}x} + \bar{A}_{x\overset{2}{x}} = \bar{A}_{\overline{xx}}$$

By symmetry:

$$\bar{A}_{\overset{2}{x}x} = \bar{A}_{x\overset{2}{x}}$$

So:

$$\bar{A}_{\overset{2}{x}x} = \tfrac{1}{2}\bar{A}_{\overline{xx}}$$

Question

Two lives are aged 50 and 60. The 50-year-old is subject to a constant force of mortality of 0.02 *pa*, and the 60-year-old is subject to a constant force of mortality of 0.025 *pa*.

Assuming that the constant force of interest is 6% *pa*, calculate:

(i) $\bar{A}_{\overset{1}{50}:60}$

(ii) $\bar{A}_{\overset{2}{50}:60}$

Solution

Since both lives experience a constant force of mortality, we have:

$$\mu_{50+t} = 0.02 \quad \text{and} \quad {}_tp_{50} = e^{-0.02t}$$

and: $$\mu_{60+t} = 0.025 \quad \text{and} \quad {}_tp_{60} = e^{-0.025t}$$

for all t. In addition, $v^t = e^{-0.06t}$.

(i) We have:

$$\bar{A}^{\,1}_{50:60} = \int_0^\infty v^t \, {}_tp_{50:60} \, \mu_{50+t} \, dt = \int_0^\infty v^t \, {}_tp_{50} \, {}_tp_{60} \, \mu_{50+t} \, dt$$

$$= \int_0^\infty e^{-0.06t} \times e^{-0.02t} \times e^{-0.025t} \times 0.02 \, dt$$

$$= 0.02 \int_0^\infty e^{-0.105t} \, dt = 0.02 \left[\frac{e^{-0.105t}}{-0.105} \right]_0^\infty = \frac{0.02}{0.105} = 0.19048$$

(ii) We have:

$$\bar{A}_{50} = \int_0^\infty v^t \, {}_tp_{50} \, \mu_{50+t} \, dt$$

$$= \int_0^\infty e^{-0.06t} \times e^{-0.02t} \times 0.02 \, dt$$

$$= 0.02 \int_0^\infty e^{-0.08t} \, dt = 0.02 \left[\frac{e^{-0.08t}}{-0.08} \right]_0^\infty = \frac{0.02}{0.08} = 0.25$$

So, using relationship (ii):

$$\bar{A}^{\,2}_{50:60} = \bar{A}_{50} - \bar{A}^{\,1}_{50:60} = 0.25 - \frac{0.02}{0.105} = 0.05952$$

Alternatively, we could calculate this as:

$$\bar{A}^{\,2}_{50:60} = \int_0^\infty v^t \, {}_tp_{50} \, \mu_{50+t} \, {}_tq_{60} \, dt$$

The four relationships we have met above all relate to payments made immediately on death.

If the benefit is payable at the end of the policy year in which the contingent event occurs, then we can show that:

$$A^{1}_{xy} = \sum_{t=0}^{t=\infty} v^{t+1}\, {}_{t}p_{xy}\, q^{1}_{x+t:y+t}$$

with analogous expressions for the variance to those derived for assurances payable immediately on the occurrence of the contingent event.

Such expressions are usually evaluated by using the approximate relationship:

$$A^{1}_{xy} \approx (1+i)^{-\frac{1}{2}}\, \bar{A}^{1}_{xy}$$

and similar expressions.

3 Reversionary annuities

The simplest form of a reversionary annuity is one that begins on the death of (x), if (y) is then alive, and continues during the lifetime of (y). The life (x) is called the counter or failing life, and the life (y) is called the annuitant. The random variable $\bar{Z}$ representing the present value of this reversionary annuity if it is payable continuously can be written as a function of the random variables T_x and T_y, where:

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

So, if $T_y > T_x$, $\bar{Z}$ is the present value of a continuously payable annuity of 1 *pa* beginning in exactly T_x years' time (when (*x*) dies) and ending exactly T_y years from now (when (*y*) dies). So we can write, for $T_y > T_x$:

$$\bar{Z} = \int_{T_x}^{T_y} v^t \, dt = \int_0^{T_y} v^t \, dt - \int_0^{T_x} v^t \, dt = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$$

as before.

Also, we can see that, by substituting $r = t - T_x$:

$$\bar{Z} = \int_0^{T_y - T_x} v^{T_x + r} \, dr = v^{T_x} \int_0^{T_y - T_x} v^r \, dr = v^{T_x} \, \bar{a}_{\overline{T_y - T_x}|}$$

(remembering that this is only for $T_y > T_x$).

So the reversionary annuity is paid for a total of $T_y - T_x$ years, beginning (and therefore discounted from) T_x years from now.

As an alternative to the above expressions for the present value of the reversionary annuity, we can write:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{\min\{T_x, T_y\}}|}$$

so that when $T_x < T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}$, and when $T_x > T_y$ the value is $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_y}|} = 0$. Then, because $\min\{T_x, T_y\} = T_{xy}$, we can conveniently write:

$$\bar{Z} = \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}$$

This is a very useful form for the present value of a reversionary annuity.

Using similar methods to those used for contingent assurances, we can show that:

$$E\left[\bar{Z}\right] = \bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

$$= \int_{t=0}^{t=\infty} v^t\, \bar{a}_{y+t}\ {}_tp_{xy}\, \mu_{x+t}\, dt$$

The variance can also be expressed as an integral in this way.

The actuarial notation for the EPV of this reversionary annuity is $\bar{a}_{x|y}$. The vertical bar in the subscript represents a period of deferment, as before. Here, it specifically means that the annuity is paid to (y) but is deferred for the lifetime of (x).

$\bar{a}_y - \bar{a}_{xy}$ is generally the most useful result for calculating expected present values of reversionary annuities. This follows easily from our final expression for $\bar{Z}$ above:

$$E\left[\bar{Z}\right] = E\left[\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{xy}}|}\right] = E\left[\bar{a}_{\overline{T_y}|}\right] - E\left[\bar{a}_{\overline{T_{xy}}|}\right] = \bar{a}_y - \bar{a}_{xy}$$

Also, since:

$$\bar{a}_y = \frac{1-\bar{A}_y}{\delta} \quad \text{and} \quad \bar{a}_{xy} = \frac{1-\bar{A}_{xy}}{\delta}$$

by premium conversion, we have:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{1-\bar{A}_y}{\delta} - \frac{1-\bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

Question

Explain the integral expression:

$$\bar{a}_{x|y} = \int_{t=0}^{t=\infty} v^t\, \bar{a}_{y+t}\ {}_tp_{xy}\, \mu_{x+t}\, dt$$

by general reasoning.

Solution

Under this reversionary annuity, payments will begin at exact time t if (*x*) dies at that moment (with probability density ${}_tp_x\, \mu_{x+t}$) and (*y*) is still alive (with probability ${}_tp_y$). This gives a factor of ${}_tp_x\, \mu_{x+t} \times {}_tp_y = {}_tp_{xy}\, \mu_{x+t}$.

From time t onwards, an annuity of 1 *pa* is payable continuously until the subsequent death of (*y*). As (*y*) is aged $y+t$ at the time the annuity starts, the expected present value of the annuity as at time t is $\bar{a}_{y+t}$. Discounting to time 0 requires further multiplication by v^t.

Finally, integrating over all possible values of t covers all possible start times for the annuity payments (*ie* all the moments at which (x) could die with (y) still living).

A simpler alternative integral expression is:

$$\bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_y \, {}_tq_x \, dt$$

The logic here is that payments are made at (and hence discounted from) each time t provided (y) is alive at that moment and (x) is dead by that time.

Using this integral expression, we can again obtain the result $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$:

$$\begin{aligned}\bar{a}_{x|y} &= \int_0^\infty v^t \, {}_tp_y \, {}_tq_x \, dt = \int_0^\infty v^t \, {}_tp_y \left(1 - {}_tp_x\right) dt \\ &= \int_0^\infty v^t \, {}_tp_y \, dt - \int_0^\infty v^t \, {}_tp_{xy} \, dt \\ &= \bar{a}_y - \bar{a}_{xy}\end{aligned}$$

If the annuity begins at the end of the year of death of (x) and is then paid annually in arrears during the lifetime of (y), the random variable Z representing the present value of the payments can be written as a function of K_x and K_y, where:

$$Z = \begin{cases} a_{\overline{K_y|}} - a_{\overline{K_x|}} & \text{if } K_y > K_x \\ 0 & \text{if } K_y \le K_x \end{cases}$$

$$= a_{\overline{K_y|}} - a_{\overline{K_{xy}|}}$$

We can show that:

$$E[Z] = a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}$$

Question

Calculate $a_{65|60}$, assuming:

(i) (65) is male and (60) is female

(ii) (65) is female and (60) is male.

Basis: Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective

Solution

(i) $a^{m\ f}_{65|60} = a^{f}_{60} - a^{m\ f}_{65:60} = 15.652 - 11.682 = 3.970$

(ii) $a^{f\ m}_{65|60} = a^{m}_{60} - a^{m\ f}_{60:65} = 14.632 - 12.101 = 2.531$

To find the values of reversionary annuities in arrears, we take the tabulated values of the annuities-due in the *Tables*, and subtract one from both. The two –1 terms cancel out, so that we can say:

$$a_{x|y} = a_y - a_{xy} = (\ddot{a}_y - 1) - (\ddot{a}_{xy} - 1) = \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}_{x|y}$$

4 Joint life functions dependent on term

All of the theory involving assurance and annuity benefits considered above can be easily modified to allow for policies with a specified term.

4.1 Expected present values of joint life assurances and annuities which also depend upon term

Joint life assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values, if they are paid immediately on death, can be expressed as:

$$\bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t:y+t} \, dt$$

$$\bar{A}_{xy:\overline{n|}} = \bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}} + \bar{A}_{xy:\overset{\;\;1}{\overline{n|}}}$$

where:

$$\bar{A}_{xy:\overset{\;\;1}{\overline{n|}}} = {}_np_{xy} v^n$$

The bracket used in the notation for the term assurance $\bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}}$ indicates that the joint life status must end within the fixed term of n years.

We want to indicate that the joint life status has to fail before the *n*-year term runs out, but we don't want to specify that a particular life has to die. So we draw a bracket over the joint life status to denote that we are treating the joint lives as a single entity.

Question

Describe precisely the meaning of each of the three assurance functions used above.

Solution

$\bar{A}^{\;1}_{\overbrace{xy}:\overline{n|}}$ = EPV of 1 paid immediately on the first death out of (x) and (y), provided that death occurs within n years.

$\bar{A}_{xy:\overline{n|}}$ = EPV of 1 paid in n years' time, or immediately on the first death of (x) and (y) if this occurs sooner.

$\bar{A}_{xy:\overset{\;\;1}{\overline{n|}}}$ = EPV of 1 paid in n years' time, provided neither (x) nor (y) has died by then.

The expected present value of the temporary joint life annuity payable continuously can be written as:

$$\bar{a}_{xy:\overline{n}|} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, dt$$

Similar expressions involving summation operators can be developed if assurances are paid at the end of the year of death or if annuities are payable annually in advance or in arrears.

4.2 Expected present values of last survivor assurances and annuities that also depend upon term

Last survivor assurances that are dependent on a fixed term of n years can be term assurances or endowment assurances. Their expected present values can be expressed in terms of single and joint life functions by making use of the results set out in Chapter 21, which can be generalised to any two statuses u and v. So we write:

$$T_{\overline{uv}} = T_u + T_v - T_{uv}$$

$$K_{\overline{uv}} = K_u + K_v - K_{uv}$$

The resulting expressions for assurances payable immediately on death are:

$$\bar{A}_{\overline{xy}:\overline{n}|} = \bar{A}_{x:\overline{n}|} + \bar{A}_{y:\overline{n}|} - \bar{A}_{xy:\overline{n}|}$$

$$\bar{A}^{\,1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + \bar{A}^{\,1}_{y:\overline{n}|} - \bar{A}^{\,1}_{xy:\overline{n}|}$$

The expected present value of the temporary last survivor annuity payable continuously can be written as:

$$\bar{a}_{\overline{xy}:\overline{n}|} = \bar{a}_{x:\overline{n}|} + \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Similar expressions involving summation operators can be developed if assurances are paid out at the end of the year of death or if annuities are payable annually in advance or arrears.

Question

Express $A_{\overline{60:60}:\overline{5}|}$ in terms of single and joint life functions (assuming the two lives have identical, and independent, mortality).

Solution

$$A_{\overline{60:60}:\overline{5}|} = 2A_{60:\overline{5}|} - A_{60:60:\overline{5}|}$$

With last survivor assurances and annuities we must be very careful when allowing for duration. For example:

$$\ddot{a}_{\overline{xy}:\overline{n}|} \neq \ddot{a}_{\overline{xy}} - v^n \, {}_np_{\overline{xy}} \, \ddot{a}_{\overline{x+n:y+n}}$$

Question

Explain why.

Solution

In this kind of formula, we are considering the temporary annuity as being equal to a whole life annuity minus another whole life annuity from higher ages, reflecting the payments that will not be received after the n-year term has elapsed.

In this case, after n years, the status $\overline{xy}$ is still active if:

- (x) only is alive
- (y) only is alive
- both (x) and (y) are alive.

However, the annuity value $\ddot{a}_{\overline{x+n:y+n}}$ is conditional on *both* lives being alive at time n.

To correct the formula, we have to allow for all three possibilities separately. That is:

$$\ddot{a}_{\overline{xy}:\overline{n}|} = \ddot{a}_{\overline{xy}} - v^n \left[{}_np_x \, {}_nq_y \, \ddot{a}_{x+n} + {}_nq_x \, {}_np_y \, \ddot{a}_{y+n} + {}_np_{xy} \, \ddot{a}_{\overline{x+n:y+n}} \right]$$

The formulae shown in the Core Reading are therefore simpler.

For similar reasons:

$$A^{\;1}_{\overline{xy}:\overline{n}|} \neq A_{\overline{xy}} - v^n \, {}_np_{\overline{xy}} \, A_{\overline{x+n:y+n}}$$

Question

Using PA92C20 mortality and 4% *pa* interest, calculate $\ddot{a}_{\overline{50:50}:\overline{20}|}$, assuming that one life is male and the other is female.

Solution

The last survivor temporary annuity can be thought of as the sum of the two single life temporary annuities less the temporary joint life annuity:

$$\ddot{a}_{\overline{50:50}:\overline{20}|} = \ddot{a}_{50:\overline{20}|} + \ddot{a}_{50:\overline{20}|} - \ddot{a}_{50:50:\overline{20}|}$$

Now:

$$\ddot{a}_{50:\overline{20|}} = \ddot{a}_{50} - v^{20}\frac{l_{70}}{l_{50}}\ddot{a}_{70}$$

which for males gives:

$$\ddot{a}_{50:\overline{20|}} = 18.843 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times 11.562 = 13.940$$

and for females gives:

$$\ddot{a}_{50:\overline{20|}} = 19.539 - 1.04^{-20} \times \frac{9,392.621}{9,952.697} \times 12.934 = 13.968$$

Also:

$$\ddot{a}_{50:50:\overline{20|}} = \ddot{a}_{50:50} - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times \ddot{a}_{70:70}$$

$$= 17.688 - 1.04^{-20} \times \frac{9,238.134}{9,941.923} \times \frac{9,392.621}{9,952.697} \times 9.766$$

$$= 13.780$$

So:

$$\ddot{a}_{\overline{50:50}:\overline{20|}} = 13.940 + 13.968 - 13.780 = 14.128$$

4.3 More complex conditions

We frequently employ a technique of describing probabilities and expected present values with appropriately constructed integrals. We can generally use this technique to deal with more complex conditions, and this is often the most efficient way to solve such problems. The technique can be summarised as follows:

(1) identify the critical dates / times in the problem,

(2) express as an integral,

(3) simplify (often by making some appropriate substitution), and

(4) re-express in terms of simpler (non-integral) functions.

For discrete benefits we would use a summation expression rather than an integral.

Question

Two sisters, Xanthe and Yolanda, are aged x and y, respectively.

(i) Derive an expression for the probability that Xanthe will die more than 5 years after the death of Yolanda, giving the answer in terms of single life probabilities and probabilities based on the first death.

Xanthe wishes to take out an assurance with a sum assured of £50,000 payable immediately on her death under the conditions described in part (i).

(ii) Derive an expression for the expected present value of this benefit, giving the answer in terms of single life assurances and assurances payable on the first death.

Solution

(i) ***Probability***

When deriving probabilities for two lives, we can usually condition on either death. In this case, if Yolanda dies at time t, we require Xanthe to be alive 5 years later. This can be expressed in integral form as:

$$\int_0^\infty {}_{t+5}p_x \; {}_tp_y \; \mu_{y+t} \, dt$$

Since ${}_{t+5}p_x = {}_5p_x \; {}_tp_{x+5}$, the integral can be written:

$${}_5p_x \int_0^\infty {}_tp_{x+5} \; {}_tp_y \; \mu_{y+t} \, dt = {}_5p_x \; {}_\infty q^{\;\;1}_{x+5:y}$$

(ii) ***Expected present value***

When deriving the expected present value of assurances, it is easier to condition on the assured life. If Xanthe dies at time t, the sum assured will be paid provided Yolanda died before time $t-5$, for $t \geq 5$. The expected present value of the assurance per £1 sum assured is therefore:

$$\int_5^\infty v^t \, {}_tp_x \; \mu_{x+t} \; {}_{t-5}q_y \, dt = \int_5^\infty v^t \, {}_tp_x \; \mu_{x+t} \, (1 - {}_{t-5}p_y) \, dt$$

Substituting $u = t-5$, the integral becomes:

$$\int_0^\infty v^{u+5} \, {}_{u+5}p_x \; \mu_{x+u+5} \, (1 - {}_up_y) \, du = v^5 \, {}_5p_x \int_0^\infty v^u \, {}_up_{x+5} \; \mu_{x+u+5} (1 - {}_up_y) \, du$$

since ${}_{u+5}p_x = {}_5p_x \; {}_up_{x+5}$.

Multiplying out the bracket, we have:

$$v^5 {}_5p_x \left[\int_0^\infty v^u {}_up_{x+5}\, \mu_{x+u+5}\, du - \int_0^\infty v^u {}_up_{x+5}\, \mu_{x+u+5}\, {}_up_y\, du \right]$$

Expressing this in terms of assurance functions and multiplying by the sum assured of £50,000 gives an expected present value of:

$$50{,}000 v^5 {}_5p_x (\bar{A}_{x+5} - \bar{A}^{\ 1}_{x+5:y})$$

We will use this kind of approach when developing formulae in the next section.

4.4 Expected present values of reversionary annuities that depend upon term

There are several different types of reversionary annuities that depend on term, and some of the possibilities are listed below.

Type 1 – an annuity payable after a fixed term has elapsed

A reversionary annuity in which the counter or failing status is a fixed term of n years is exactly equivalent to a deferred life annuity. The expected present value of an annuity that is paid continuously can be written:

$$\bar{a}_{\overline{n}|\,|y} = {}_{n|}\bar{a}_y = \bar{a}_y - \bar{a}_{y:\overline{n}|}$$

However, for calculation purposes it is much quicker to use:

$${}_{n|}\bar{a}_y = v^n {}_np_y\, \bar{a}_{y+n}$$

as we have done before.

Type 2 – an annuity payable to (y) on the death of (x), but ceasing at time n

If a reversionary annuity ceases in any event after n years, *ie* is payable to (y) after the death of (x) with no payment being made after n years, the expected present value can be expressed as:

$$\bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

We can obtain this expression using integrals, by considering the payment made at time t. A payment will be made at time t if:

- (y) is alive at time t
- (x) is dead by time t
- $t < n$.

So the expected present value is:

$$\int_0^n v^t \, {}_tp_y \, {}_tq_x \, dt = \int_0^n v^t \, {}_tp_y \left(1 - {}_tp_x\right) dt = \int_0^n v^t \, {}_tp_y \, dt - \int_0^n v^t \, {}_tp_{xy} \, dt = \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

Question

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death. There will be no payments on or beyond Ralph's 80th birthday in any circumstances.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

This is a reversionary annuity payable to Ralph, after Betty's death, but ceasing after 15 years.

We have:

$$\ddot{a}^{m}_{65:\overline{15}|} - \ddot{a}^{m\ f}_{65:65:\overline{15}|} = \ddot{a}_{65} - v^{15} \, {}_{15}p_{65} \, \ddot{a}_{80} - \left[\ddot{a}_{65:65} - v^{15} \, {}_{15}p_{65:65} \, \ddot{a}_{80:80}\right]$$

$$= 13.666 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times 7.506$$

$$- \left[11.958 - 1.04^{-15} \times \frac{6,953.536}{9,647.797} \times \frac{7,724.737}{9,703.708} \times 5.857\right]$$

$$= 0.5700$$

So the EPV to Ralph is:

$$£20,000 \times 0.5700 = £11,401$$

Type 3 – an annuity payable to (*y*) on the death of (*x*) provided that (*x*) dies within *n* years

If the payment commences on the death of (x) within n years and then continues until the death of (y), the expected present value can be expressed as:

$$\int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt = \bar{a}_y - \bar{a}_{xy} - v^n \, {}_np_{xy} \left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n}\right)$$

$$= \bar{a}_{x|y} - v^n \, {}_np_{xy} \, \bar{a}_{x+n|y+n}$$

This is what we might most accurately describe as a 'temporary reversionary annuity'.

The rationale for the integral expression:

$$\int_{t=0}^{n} v^t \, {}_t p_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt$$

is that the only restriction compared to the normal whole life version is that payments made as a result of the death of (x) after n years will not be made. We can therefore calculate the expected present value by integrating from $t=0$ to n rather than to infinity.

Question

Prove that $\int_{t=0}^{n} v^t \, {}_t p_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, dt = \bar{a}_{x|y} - v^n \, {}_n p_{xy} \, \bar{a}_{x+n|y+n}$.

Solution

We already have, from Section 3, the result:

$$\int_{t=0}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt = a_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

So we can write:

$$\int_{t=0}^{n} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt = \int_{t=0}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt - \int_{t=n}^{\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt$$

$$= \bar{a}_y - \bar{a}_{xy} - \int_{s=0}^{\infty} v^{s+n} \, \bar{a}_{y+s+n} \, {}_{s+n} p_{xy} \, \mu_{x+s+n} \, ds$$

using the substitution $s = t - n$.

Pulling a factor of $v^n \, {}_n p_{xy}$ outside the integral gives:

$$\int_{t=0}^{n} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt = \bar{a}_y - \bar{a}_{xy} - v^n \, {}_n p_{xy} \int_{s=0}^{\infty} v^s \, \bar{a}_{y+s+n} \, {}_s p_{x+n:y+n} \, \mu_{x+s+n} \, ds$$

$$= \bar{a}_y - \bar{a}_{xy} - v^n \, {}_n p_{xy} \left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n} \right)$$

$$= \bar{a}_{x|y} - v^n \, {}_n p_{xy} \, \bar{a}_{x+n|y+n}$$

The expression:

$$\bar{a}_{x|y} - v^n {}_np_{xy}\, \bar{a}_{x+n|y+n}$$

can be justified by noting that, compared to the normal whole life reversionary annuity $\bar{a}_{x|y}$, we need to deduct all annuity payments that relate to the death of (x) after n years with (y) alive at the time of (x)'s death. (Note that the death of (x) after n years, with (y) dead at the time of (x)'s death, is already excluded from $\bar{a}_{x|y}$). The required deduction therefore equates to a reversionary annuity from time n, with both (x) and (y) alive at that point, appropriately discounted to the start of the contract.

Question

Ralph and Betty are both aged 65 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life starting from the end of the year of Betty's death, provided that Betty dies in the next 10 years.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}_{65}^{m} - \ddot{a}_{65:65} - v^{10}{}_{10}p_{65:65}(\ddot{a}_{75}^{m} - \ddot{a}_{75:75})$$

$$= 13.666 - 11.958 - 1.04^{-10} \times \frac{8,405.160}{9,647.797} \times \frac{8,784.955}{9,703.708} \times (9.456 - 7.679)$$

$$= 0.76117$$

So, the EPV to Ralph is:

$$£20,000 \times 0.76117 = £15,223$$

Type 4 – an annuity payable to (*y*) on the death of (*x*) for a maximum of *n* years

If the conditions of payment say that the payment will:

- **begin on the death of (x) and**
- **cease on the death of (y) or n years after the death of (x) (whichever event occurs first),**

then the expected present value can be expressed as:

$$\int_{t=0}^{t=\infty} v^t \,{}_tp_{xy}\, \mu_{x+t}\, \bar{a}_{y+t:\overline{n}|}\, dt = \bar{a}_{y:\overline{n}|} + v^n \,{}_np_y\, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

Similar expressions involving summation operators can be developed if the annuity payments are made at annual intervals from the date on which the counter status fails.

Question

Explain in words the integral formula on the left-hand side of the equation above.

Solution

The annuity begins at time *t*, which is the instant of (*x*)'s death. This gives a factor of ${}_tp_x\, \mu_{x+t}\, dt$.

The annuity is payable to (*y*) for life thereafter, but for a maximum of *n* years. Hence:

- (*y*) has to be alive at time *t* (with probability ${}_tp_y$), and
- the expected present value of the temporary annuity from that time is $\bar{a}_{y+t:\overline{n}|}$.

We then need to discount this value to time zero, using v^t, and integrate over all times t at which (*x*) can die.

To obtain the expression on the right-hand side above:

$$\bar{a}_{y:\overline{n}|} + v^n \,{}_np_y\, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

it is easiest to consider an alternative integral expression for this annuity.

First, consider an annuity that starts *n* years after the death of (*x*) and is payable to (*y*). This annuity will be in payment at time $t+n$ if:

- (*x*) has died before time *t* and
- (*y*) is alive at time $t+n$.

So, the expected present value of this annuity is:

$$\int_0^\infty v^{t+n}\, {}_tq_x\, {}_{t+n}p_y\, dt$$

Now, the Type 4 annuity is equal to a reversionary annuity less this one above. So, the expected present value of the Type 4 annuity is:

$$\bar{a}_{x|y} - \int_0^\infty v^{t+n}\, {}_tq_x\, {}_{t+n}p_y\, dt = \bar{a}_{x|y} - v^n\, {}_np_y \int_0^\infty v^t\, {}_tq_x\, {}_tp_{y+n}\, dt$$

$$= \bar{a}_{x|y} - v^n\, {}_np_y\, \bar{a}_{x|y+n}$$

Finally, using the formulae for a reversionary annuity, we can write:

$$\bar{a}_{x|y} - v^n\, {}_np_y\, \bar{a}_{x|y+n} = \bar{a}_y - \bar{a}_{xy} - v^n\, {}_np_y \left(\bar{a}_{y+n} - \bar{a}_{x:y+n}\right)$$

$$= \bar{a}_y - v^n\, {}_np_y\, \bar{a}_{y+n} + v^n\, {}_np_y\, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

$$= \bar{a}_{y:\overline{n}|} + v^n\, {}_np_y\, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

Question

Ralph and Betty are both aged 70 exact. Upon Betty's death, Ralph will receive £20,000 *pa* payable annually in advance starting from the end of the year of Betty's death and ceasing on Ralph's death. Ralph will receive a maximum of 20 payments.

Ralph's mortality follows PMA92C20, Betty's mortality follows PFA92C20 and the interest rate for all future years is 4% *pa*.

Calculate the EPV of this benefit to Ralph.

Solution

We have:

$$\ddot{a}^m_{70:\overline{20}|} + v^{20}\, {}_{20}p^m_{70}\, \ddot{a}^{f\,m}_{70:90} - \ddot{a}_{70:70}$$

$$= 11.562 - 1.04^{-20} \times \frac{2,675.203}{9,238.134} \times 4.527 + 1.04^{-20} \times \frac{2,675.203}{9,238.134} \times 4.339 - 9.766$$

$$= 1.771154$$

So, the EPV to Ralph is:

$$£20,000 \times 1.77115 = £35,423$$

Type 5 – an annuity payable to (*y*) on the death of (*x*) and guaranteed for *n* years

The expected present value of this benefit is:

$$\bar{A}^{1}_{x:y}\,\bar{a}_{\overline{n}|}+v^n\,{}_np_y\,\bar{a}_{x|y+n}$$

Question

Prove this result.

Solution

We can write the expected present value of this benefit as an integral by considering time t to be the instant of (*x*)'s death, which is when the annuity begins. This gives a factor of ${}_tp_x\,\mu_{x+t}\,dt$.

The annuity is payable to (*y*) for life thereafter, but for a minimum of *n* years. Hence:

- (*y*) has to be alive at time *t* (with probability ${}_tp_y$), and
- the expected present value of the guaranteed annuity from that time is $\bar{a}_{\overline{y+t:\overline{n}|}}$.

We then need to discount the payments to time zero, using v^t, and integrate over all times t at which (*x*) can die, to give:

$$\int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_y\,\bar{a}_{\overline{y+t:\overline{n}|}}\,dt=\int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_y\left(\bar{a}_{\overline{n}|}+v^n\,{}_np_{y+t}\,\bar{a}_{y+t+n}\right)dt$$

If we multiply out the brackets, then the first term is:

$$\bar{a}_{\overline{n}|}\int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt=\bar{a}_{\overline{n}|}\,\bar{A}^{1}_{x:y}$$

The second term can be written as:

$$\int_0^\infty v^{t+n}\,{}_tp_x\,\mu_{x+t}\,{}_{t+n}p_y\,\bar{a}_{y+t+n}\,dt=v^n\,{}_np_y\int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_{y+n}\,\bar{a}_{y+t+n}\,dt$$

$$=v^n\,{}_np_y\,\bar{a}_{x|y+n}$$

So the overall EPV is:

$$\bar{A}^{1}_{x:y}\,\bar{a}_{\overline{n}|}+v^n\,{}_np_y\,\bar{a}_{x|y+n}$$

The first term in the above expression is the expected present value of the guaranteed benefit, which is paid to (y) following the death of (x). The second term is the expected present value of the benefit paid to (y) once the n-year guarantee period has elapsed.

Question

Ralph and Ted are both aged 60 exact. Upon Ted's death, Ralph will receive £20,000 *pa* for the rest of his life payable annually in advance starting from the end of the year of Ted's death. The payments to Ralph are guaranteed for 5 years.

Ralph's mortality and Ted's mortality both follow PMA92C20 and the interest rate for all future years is 4% *pa*.

Given that $A_{60:60} = 0.47585$ and $A_{60:65} = 0.51084$, where both lives follow PMA92C20, calculate the EPV of this benefit to Ralph.

Solution

We need to calculate:

$$A^{1}_{60:60}\,\ddot{a}_{\overline{5|}} + v^5\,{}_5p_{60}\,\ddot{a}_{60|65}$$

The first term in the expression above relates to the 5 guaranteed annuity payments, the first of which is made at the end of the year of Ted's death. We can think of the value of these as an assurance that pays a benefit of $\ddot{a}_{\overline{5|}}$ at the end of the year of Ted's death, provided Ted dies first.

The second term relates to the annual annuity payments made after the 5-year guarantee period if Ralph is alive.

Since, by symmetry:

$$A^{1}_{60:60} = 0.5A_{60:60}$$

we have, using premium conversion:

$$0.5A_{60:60}\,\ddot{a}_{\overline{5|}} + v^5\,{}_5p_{60}\left(\ddot{a}_{65} - \ddot{a}_{60:65}\right)$$

$$= 0.5A_{60:60}\,\ddot{a}_{\overline{5|}} + v^5\,{}_5p_{60}\left(\ddot{a}_{65} - \frac{1 - A_{60:65}}{d}\right)$$

$$= 0.5 \times 0.47585 \times \frac{1 - 1.04^{-5}}{0.04/1.04} + 1.04^{-5} \times \frac{9{,}647.797}{9{,}826.131} \times \left(13.666 - \frac{1 - 0.51084}{0.04/1.04}\right)$$

$$= 1.86648$$

So, the EPV to Ralph is:

$$£20{,}000 \times 1.86648 = £37{,}330$$

Type 6 – an annuity payable to (*y*) on the death of (*x*) and continuing for *n* years after (*y*)'s death

The expected present value of this benefit is:

$$\bar{a}_{x|y} + \bar{A}^{\;2}_{x:y}\,\bar{a}_{\overline{n}|}$$

The first term is the expected present value of the benefit payable after the death of (x) while (y) is still alive.

The second term is the expected present value of the annuity paid for n years following the death of (y), provided that (y) dies after x. We can think of this as an assurance that provides a lump sum payment of $\bar{a}_{\overline{n}|}$ immediately on the death of (y), provided (y) dies second.

Question

Ralph and Ted are both aged 60 exact. Upon Ted's death, Ralph will receive £20,000 *pa* payable annually in advance for the rest of his life, starting from the end of the year of Ted's death. The payments to Ralph will continue for 12 years after Ralph has died. No payments are made if Ralph dies first.

Ralph's mortality and Ted's mortality both follow PMA92C20 and the interest rate for all future years is 4% *pa*.

Given that $A_{60:60} = 0.47585$ where both lives follow PMA92C20, calculate the EPV of this benefit to Ralph.

Solution

We need to calculate:

$$\ddot{a}_{60|60} + A^{\;2}_{60:60}\,\ddot{a}_{\overline{12}|}$$

Here we use a factor of $A^{\;2}_{60:60}$ *as the first annuity payment is made at the end of the year of Ted's death.*

Since, by symmetry:

$$A^{\;2}_{60:60} = 0.5A_{\overline{60:60}}$$

we have, using premium conversion:

$$\ddot{a}_{60} - \ddot{a}_{60:60} + 0.5A_{\overline{60:60}}\,\ddot{a}_{\overline{12}|} = \ddot{a}_{60} - \frac{1-A_{60:60}}{d} + 0.5A_{\overline{60:60}}\,\ddot{a}_{\overline{12}|}$$

Now:

$$A_{\overline{60:60}} = A_{60} + A_{60} - A_{60:60}$$

$$= 2\left(1 - \frac{0.04}{1.04} \times 15.632\right) - 0.47585$$

$$= 0.32169$$

So:

$$\ddot{a}_{60} - \frac{1 - A_{60:60}}{d} + 0.5 A_{\overline{60:60}}\, \ddot{a}_{\overline{12}|} = 15.632 - \frac{1 - 0.47585}{0.04/1.04} + 0.5 \times 0.32169 \times \frac{1 - 1.04^{-12}}{0.04/1.04}$$

$$= 3.57402$$

The EPV to Ralph is therefore:

$$£20{,}000 \times 3.57402 = £71{,}480$$

The different types of reversionary annuity above are not exhaustive. In unusual cases, where it is not obvious what formula to use, we have to think carefully to develop the correct expressions.

Question

Jack, aged 60, wants to buy a reversionary annuity. If he dies before age 65 and before his wife Vera, who is also now aged 60, she will receive an income of £10,000 *pa*. The income will be paid annually in arrears (from the end of the year of Jack's death) until Vera's 75th birthday or until her earlier death.

Calculate the single premium payable assuming PMA92C20 mortality for Jack, PFA92C20 mortality for Vera and 4% *pa* interest.

Solution

Here, we should consider the period before Jack and Vera are aged 65, and the period after they are aged 65, separately. This is because before age 65 the annuity payments can commence at any time (due to Jack's death), but after age 65, if the annuity payments have not already started, they never will.

The payments made to Vera before she is aged 65 are a Type 2 annuity, *ie* they are payments to Vera, after the death of Jack, but ceasing after 5 years. The expected present value of these payments is:

$$10{,}000\left(a^{\;f}_{60:\overline{5}|} - a^{m\;f}_{60:60:\overline{5}|}\right)$$

Payments will only be made in the period from age 65 to age 75 if Jack dies before age 65 (with probability ${}_5q_{60}^{m}$) and Vera is still alive at age 65 (with probability ${}_5p_{60}^{f}$). The EPV at age 65 of the payments made to Vera in the period from age 65 to age 75 is $10{,}000a_{65:\overline{10}|}^{f}$ and this will need to be discounted back to age 60. So the EPV of this part of the benefit is:

$$10{,}000v^5 \, {}_5q_{60}^{m} \, {}_5p_{60}^{f} \, a_{65:\overline{10}|}^{f}$$

Overall, the single premium is given by:

$$P = 10{,}000\left(a_{60:\overline{5}|}^{f} - a_{60:60:\overline{5}|}^{m\ f} + v^5 \, {}_5q_{60}^{m} \, {}_5p_{60}^{f} \, a_{65:\overline{10}|}^{f}\right)$$

Now:

$$a_{60:\overline{5}|}^{f} = a_{60}^{f} - v^5 \, {}_5p_{60}^{f} \, a_{65}^{f}$$

$$= 15.652 - 1.04^{-5} \times \frac{9{,}703.708}{9{,}848.431} \times 13.871$$

$$= 4.419$$

$$a_{60:60:\overline{5}|}^{m\ f} = a_{60:60}^{m\ f} - v^5 \, {}_5p_{60}^{m} \, {}_5p_{60}^{f} \, a_{65:65}^{m\ f}$$

$$= 13.090 - 1.04^{-5} \times \frac{9{,}647.797}{9{,}826.131} \times \frac{9{,}703.708}{9{,}848.431} \times 10.958$$

$$= 4.377$$

$$a_{65:\overline{10}|}^{f} = a_{65}^{f} - v^{10} \, {}_{10}p_{65}^{f} \, a_{75}^{f}$$

$$= 13.871 - 1.04^{-10} \times \frac{8{,}784.955}{9{,}703.708} \times 9.933$$

$$= 7.796$$

So:

$$P = 10{,}000\left[4.419 - 4.377 + 1.04^{-5} \times \left(1 - \frac{9{,}647.797}{9{,}826.131}\right) \times \frac{9{,}703.708}{9{,}848.431} \times 7.796\right]$$

$$= £1{,}566$$

4.5 Expected present values of contingent assurances that depend upon term

Only term assurances are meaningful in this context. The expected present value of an assurance payable immediately on the death of (x) within n years provided (y) is then alive can be written:

$$\bar{A}^{\;1}_{xy:\overline{n}|} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t} \, dt$$

with a similar expression involving summation operators if the sum assured is payable at the end of the year of death.

If the sum assured is payable at the end of the year of (x)'s death, then the expected present value is:

$$A^{\;1}_{xy:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \, {}_kp_{xy} \, q^{\;1}_{x+k:y+k}$$

5 Expected present value of annuities payable *m* times a year

In Chapter 17 we determined (for a single life status x) the approximations:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

and:

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

The first of these formulae appears on page 36 of the *Tables*.

It is important to note that the nature of the above approximation means that the single life status x can equally be replaced by any life status, 'u' say.

In particular, with $u = xy$, we obtain the joint life annuity approximation:

$$a_{xy}^{(m)} \approx a_{xy} + \frac{m-1}{2m}$$

For a last survivor annuity we write:

$$a_{\overline{xy}}^{(m)} = a_x^{(m)} + a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses x, y and xy to obtain:

$$a_{\overline{xy}}^{(m)} \approx a_x + a_y - a_{xy} + \frac{m-1}{2m}$$

in its simplest form.

For a reversionary annuity we write:

$$a_{x|y}^{(m)} = a_y^{(m)} - a_{xy}^{(m)}$$

and use the result with statuses y and xy to obtain:

$$a_{x|y}^{(m)} \approx a_y - a_{xy}$$

in its simplest form. Notice that there is no 'correction term' in this case because they are cancelled out.

Similar expressions may be developed if the m thly annuities are temporary.

We can write, for the generalised status u:

$$a_{u:\overline{n}|}^{(m)} = a_u^{(m)} - {}_{n|}a_u^{(m)}$$

We know from above that $a_u^{(m)} \approx a_u + \dfrac{m-1}{2m}$.

Also:

$$
\begin{aligned}
{}_{n|}a_u^{(m)} &= v^n \frac{l_{u+n}}{l_u} a_{u+n}^{(m)} \\
&\approx v^n \frac{l_{u+n}}{l_u}\left(a_{u+n} + \frac{m-1}{2m}\right) \\
&= {}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}
\end{aligned}
$$

Hence:

$$a_{u:\overline{n}|}^{(m)} \approx a_u + \frac{m-1}{2m} - \left({}_{n|}a_u + \frac{m-1}{2m} v^n \frac{l_{u+n}}{l_u}\right)$$

or:

$$a_{u:\overline{n}|}^{(m)} \approx a_{u:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{u+n}}{l_u}\right)$$

Hence we obtain the following expressions:

$$a_{xy:\overline{n}|}^{(m)} \approx a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)$$

and:

$$
\begin{aligned}
a_{\overline{xy}:\overline{n}|}^{(m)} &= a_{x:\overline{n}|}^{(m)} + a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \\
&\approx a_{x:\overline{n}|} + a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n \frac{l_{x+n}}{l_x} - v^n \frac{l_{y+n}}{l_y} + v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y}\right)
\end{aligned}
$$

For a reversionary annuity which ceases in any event after n years we can write:

$$a_{y:\overline{n}|}^{(m)} - a_{xy:\overline{n}|}^{(m)} \approx a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(v^n \frac{l_{x+n}\, l_{y+n}}{l_x\, l_y} - v^n \frac{l_{y+n}}{l_y}\right)$$

Similar expressions can be developed for annuities payable in advance and, letting $m \to \infty$, continuous annuities.

Question

Jim and Dot, both aged 60, buy an annuity payable monthly in advance for at most 20 years, where payments are made while at least one of them is alive.

Calculate the expected present value of the annuity assuming PMA92C20 mortality for Jim, PFA92C20 mortality for Dot, and interest of 4% *pa*.

Solution

The expected present value of this temporary last survivor annuity is:

$$\ddot{a}^{(12)}_{\overline{60:60}:\overline{20|}} = \ddot{a}^{(12)m}_{60:\overline{20|}} + \ddot{a}^{(12)f}_{60:\overline{20|}} - \ddot{a}^{(12)}_{60:60:\overline{20|}}$$

Now, using the approximation on page 36 of the *Tables*:

$$\begin{aligned}\ddot{a}^{(12)m}_{60:\overline{20|}} &= \ddot{a}^{(12)m}_{60} - v^{20}\,{}_{20}p^m_{60}\,\ddot{a}^{(12)m}_{80}\\ &\approx \ddot{a}^m_{60} - \frac{11}{24} - v^{20}\,{}_{20}p^m_{60}\left(\ddot{a}^m_{80} - \frac{11}{24}\right)\\ &= 15.632 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \left(7.506 - \frac{11}{24}\right)\\ &= 12.898\end{aligned}$$

Similarly:

$$\begin{aligned}\ddot{a}^{(12)f}_{60:\overline{20|}} &\approx \ddot{a}^f_{60} - \frac{11}{24} - v^{20}\,{}_{20}p^f_{60}\left(\ddot{a}^f_{80} - \frac{11}{24}\right)\\ &= 16.652 - \frac{11}{24} - 1.04^{-20} \times \frac{7,724.737}{9,848.431} \times \left(8.989 - \frac{11}{24}\right)\\ &= 13.140\end{aligned}$$

Finally, the joint life annuity is given by:

$$\begin{aligned}\ddot{a}^{(12)}_{60:60:\overline{20|}} &= \ddot{a}^{(12)}_{60:60} - v^{20}\,{}_{20}p_{60:60}\,\ddot{a}^{(12)}_{80:80}\\ &\approx \ddot{a}_{60:60} - \frac{11}{24} - v^{20}\,{}_{20}p_{60:60}\left(\ddot{a}_{80:80} - \frac{11}{24}\right)\\ &= 14.090 - \frac{11}{24} - 1.04^{-20} \times \frac{6,953.536}{9,826.131} \times \frac{7,724.737}{9,848.431} \times \left(5.857 - \frac{11}{24}\right)\\ &= 12.264\end{aligned}$$

So:

$$\ddot{a}^{(12)}_{\overline{60:60}:\overline{20|}} \approx 12.898 + 13.140 - 12.264 = 13.77$$

6 Further aspects

We round off this chapter with consideration of:

- premium conversion relationships,
- the premium payment term.

6.1 Premium conversion relationships

Earlier in the course we met the premium conversion relationships for single life policies:

$$A_x = 1 - d\ddot{a}_x$$

and: $$A_{x:\overline{n|}} = 1 - d\ddot{a}_{x:\overline{n|}}$$

As we have seen, these have analogous equivalents for joint life and last survivor statuses. These relationships are particularly useful for calculating the value of joint life and last survivor assurances, as the *Tables* provide annuity values only.

Question

Marge and Homer, both aged exactly 55, take out a policy that provides a lump sum of £50,000 payable immediately when the second of them dies. Premiums are payable annually in advance while at least one of Marge and Homer is alive.

Calculate the annual premium for the policy assuming PMA92C20 mortality for Homer, PFA92C20 mortality for Marge, 4% *pa* interest, and no expenses.

Solution

The payment of the premiums and the payment of the benefits both depend on the last survivor status. So, the equation of value is:

$$P\ddot{a}_{\overline{55:55}} = 50,000\bar{A}_{\overline{55:55}}$$

The annuity function is:

$$\ddot{a}_{\overline{55:55}} = \ddot{a}_{55}^{m} + \ddot{a}_{55}^{f} - \ddot{a}_{55:55} = 17.364 + 18.210 - 16.016 = 19.558$$

By premium conversion, we have:

$$A_{\overline{55:55}} = 1 - d\ddot{a}_{\overline{55:55}} = 1 - \frac{0.04}{1.04} \times 19.558 = 0.24777$$

So:

$$\bar{A}_{\overline{55:55}} \approx 1.04^{0.5} \times 0.24777 = 0.25268$$

Therefore:

$$P = \frac{50,000 \times 0.25268}{19.558} = £645.97$$

6.2 Premium payment term

One of the principal uses of the theory developed in this chapter is to determine the premium suitable for any given assurance or annuity benefit involving two lives. However, we may then find a complication in that it is possible for the joint life function defining the premium payment annuity to be different from that defining the desired benefit.

For example, if a man took out a reversionary annuity contract to provide a pension for his wife (specified by name) after his death, it would be unreasonable to expect him to continue paying premiums if his wife died before him. So premium payment would continue only up to the first death of the pair, *ie* failure of their joint life status.

Normally for contracts involving two lives, premiums will be payable until one of the following events occurs:

- the benefit is paid,
- the term of the contract expires,
- the person paying the premium dies, or
- it becomes impossible for the benefit to be paid at any time in the future.

Question

The following assurance functions represent the factors used for valuing the benefits from annual premium insurance contracts:

(a) A_{xy}

(b) $A_{\overline{xy}}$

(c) A^{1}_{xy}

(d) $A_{xy}^{\;\;2}$

Specify the appropriate annuity factor to be used when valuing premiums.

Solution

(a) A_{xy} represents a benefit payable when the first death of the two lives occurs. No premiums will be paid after the first death has occurred (as the benefit will already have been paid).

So premiums will be payable until the first death occurs, *ie* while both lives are alive, and the appropriate annuity factor is $\ddot{a}_{xy}$.

(b) $A_{\overline{xy}}$ represents a benefit payable when the second death of the two lives occurs. No premiums will be paid after the second death has occurred (as the benefit has already been paid, and both policyholders are dead).

So premiums will be payable until the second death occurs, and the appropriate annuity factor is $\ddot{a}_{\overline{xy}}$.

(c) A^{1}_{xy} represents a benefit payable if (x) dies first. If (x) dies first, the benefit is paid, so no premiums will be paid after (x)'s death. If (y) dies first, the benefit can never be paid, so no premiums will be paid after (y)'s death.

Therefore premiums will be payable until the first death occurs, and the appropriate annuity factor is $\ddot{a}_{xy}$.

(d) $A_{x\overset{2}{y}}$ represents a benefit payable if (y) dies second. When (y) dies, either the benefit will be paid (if (x) is already dead), or the benefit will never be paid (if (x) is still alive).

So (y)'s death is the deciding factor and premiums should stop when (y) dies. The appropriate annuity factor is therefore $\ddot{a}_{y}$.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 22 Summary

Contingent probabilities

${}_tq^{\,1}_{xy}$ represents the probability that (x) dies within the next t years, with (y) still alive at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$$ {}_tq^{\,1}_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sp_y\,ds $$

${}_tq^{\,2}_{xy}$ represents the probability that (x) dies within the next t years, with (y) already dead at the time of (x)'s death. It can be expressed in terms of an integral as follows:

$$ {}_tq^{\,2}_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sq_y\,ds $$

By writing ${}_sq_y = 1 - {}_sp_y$ in the integral above, we see that:

$$ {}_tq^{\,2}_{xy} = \int_0^t {}_sp_x\,\mu_{x+s}\left(1-{}_sp_y\right)ds = \int_0^t {}_sp_x\,\mu_{x+s}\,ds - \int_0^t {}_sp_x\,\mu_{x+s}\,{}_sp_y\,ds = {}_tq_x - {}_tq^{\,1}_{xy} $$

When we have two lives of the same age (and we assume that their mortality is identical), we can use a symmetry argument to write:

$$ {}_tq^{\,1}_{xx} = \tfrac{1}{2}\,{}_tq_{xx} \quad\text{and}\quad {}_tq^{\,2}_{xx} = \tfrac{1}{2}\,{}_tq_{\overline{xx}} $$

Contingent assurances

The present value of a benefit of 1 payable immediately on the death of (x) provided that (y) is still alive is:

$$ \bar{Z} = \begin{cases} v^{T_x} & \text{if } T_x \le T_y \\ 0 & \text{if } T_x > T_y \end{cases} $$

The expected present value of this benefit is denoted by $\bar{A}^{\,1}_{xy}$ and can be expressed in integral form as follows:

$$ E\left(\bar{Z}\right) = \bar{A}^{\,1}_{xy} = \int_0^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_tp_y\,dt $$

The variance of the present value random variable is:

$$ \operatorname{var}\left(\bar{Z}\right) = {}^2\bar{A}^{\,1}_{xy} - \left(\bar{A}^{\,1}_{xy}\right)^2 $$

Similar expressions involving summation operators can be developed if the assurance is payable at the end of the year of death.

Reversionary annuities

The present value of an annuity of 1 *pa* payable continuously throughout life to (y) following the death of (x) is:

$$\bar{Y} = \begin{cases} \bar{a}_{\overline{T_y|}} - \bar{a}_{\overline{T_x|}} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \le T_x \end{cases}$$

The expected present value of this benefit is denoted by $\bar{a}_{x|y}$ and can be expressed in integral form as:

$$E(\bar{Y}) = \bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, {}_tp_y \, \bar{a}_{y+t} \, dt$$

We can also write:

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$$

Similar expressions involving summation operators can be developed if the reversionary annuity is payable at discrete intervals.

Functions with specified terms

We can adapt the relationships we have observed for single life functions to help us to calculate the corresponding joint life values. For example:

$$\ddot{a}_{xy:\overline{n|}} = \ddot{a}_{xy} - {}_np_x \times {}_np_y \times v^n \times \ddot{a}_{x+n:y+n}$$

$$A_{xy:\overline{n|}} = A^{\;1}_{xy:\overline{n|}} + A_{xy:\overset{\;\;1}{\overline{n|}}}$$

Premium conversion formulae

$$A_{xy} = 1 - d\ddot{a}_{xy} \qquad A_{xy:\overline{n|}} = 1 - d\ddot{a}_{xy:\overline{n|}}$$

$$\bar{A}_{xy} = 1 - \delta\bar{a}_{xy} \qquad \bar{A}_{xy:\overline{n|}} = 1 - \delta\bar{a}_{xy:\overline{n|}}$$

Similar results hold for last survivor annuities and assurances.

Chapter 22 Practice Questions

22.1 Express each of the following symbols:

- in terms of the random variables T_x and T_y, and
- as an integral.

(i) ${}_{\infty}q_{xy}^{1}$

(ii) $\bar{A}_{\overset{1}{x}y}$

(iii) $\bar{A}_{xy}^{\;\;2}$

22.2 Two lives, each aged x, are subject to the same mortality table. According to this mortality table, and at a certain rate of interest, $A_x = 0.4$ and $A_{xx} = 0.6$.

Calculate the value of $A_{xx}^{\;\;2}$ based on this mortality table and interest rate.

22.3 Given that:

$$\mu_x = \frac{1}{100-x} \quad \text{for} \quad 0 \le x < 100$$

calculate the value of ${}_{30}q_{50:60}^{\;\;\;\;2}$.

22.4 Calculate:

(i) ${}_{10}q_{xy}^{1}$

(ii) $\bar{a}_{y|x}$

assuming that:

- (x) is subject to a constant force of mortality of 0.01 *pa*
- (y) is subject to a constant force of mortality of 0.02 *pa*
- the force of interest is 0.04 *pa*.

22.5 Calculate $a_{68:58}^{(4)}$, assuming PMA92C20 mortality for the life aged 68, PFA92C20 mortality for the life aged 58, and 4% *pa* interest.

22.6 Calculate $a^{(12)}_{70|60}$ on the following basis:

Mortality: 70-year-old: PMA92C20

60-year-old: PFA92C20

Interest: 4% *pa*

22.7 Two lives, aged 40 and 44, purchase a policy from an insurance company that pays 50,000 in 20 years' time, if at least one of them is still alive at that time.

Assuming that the mortality of each life follows the AM92 Select table, and the annual effective interest rate is 6%, calculate the expected present value of the benefits from this policy.

22.8 Two lives aged x and y take out a policy that will pay £15,000 immediately on the death of (x) provided that (y) has died at least 5 years earlier and no more than 15 years earlier.

Exam style

(i) Express the present value of this benefit in terms of the random variables denoting the future lifetimes of (x) and (y). [2]

(ii) Write down an integral expression (in terms of single integrals only) for the expected present value of the benefit. [3]

(iii) Prove that the expected present value is equal to:

$$15{,}000\left[v^5\,{}_5p_x\,\bar{A}^{\,2}_{x+5:y} - v^{15}\,{}_{15}p_x\,\bar{A}^{\,2}_{x+15:y}\right] \quad [3]$$

(iv) Describe the appropriate premium payment term for this policy, assuming premiums are to be paid annually in advance. [2]

[Total 10]

22.9 A 65-year-old male and a 62-year-old female take out a joint whole life policy with a sum assured of £10,000 that is payable immediately on the first death. Premiums are payable monthly in advance while the policy is in force for at most 5 years.

Exam style

(i) Show that the monthly premium is £100, using the basis given below. [6]

(ii) Calculate the prospective reserve at the end of the third policy year, using the basis given below. [4]

Basis: Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective

Expenses: None

[Total 10]

22.10 A male and a female, aged 60 and 64 respectively, take out a policy under which the benefits are:

Exam style

- A lump sum of £50,000 payable at the end of the year of the first death provided this occurs within 10 years.
- An annuity payable annually in advance with the first payment due to be made 10 years from the date of issue. The annuity payments will be £10,000 *pa* for as long as both lives are still alive or £5,000 while only one of them is alive.

Level premiums are payable annually in advance for at most 10 years and will cease on the first death if this occurs earlier.

Calculate the amount of the annual premium on the following basis:

Interest:	4% *pa*	
Mortality:	PMA92C20 for the male life and PFA92C20 for the female life	
Expenses:	Initial:	£750
	Renewal:	3% of each premium excluding the first

[8]

22.11 A pension scheme provides the following benefit to the spouse of a member, following the death of the member in retirement:

Exam style

A pension of £25,000 *pa* payable during the lifetime of the spouse, but ceasing 20 years after the death of the member if that is earlier. All payments are made on the anniversary of the member's retirement.

Calculate the expected present value of the spouse's benefit in the case of a female member retiring now on her 65th birthday, who has a husband aged exactly 55.

Basis:	Mortality:	PMA92C20 for the male life, PFA92C20 for the female life
	Interest:	4% *pa* effective
	Expenses:	None

[7]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 22 Solutions

22.1 (i) ${}_{\infty}q^{1}_{xy}$ represents the probability that (x) dies before (y).

In terms of T_x and T_y:

$${}_{\infty}q^{1}_{xy} = P(T_x < T_y)$$

As an integral:

$${}_{\infty}q^{1}_{xy} = \int_0^{\infty} {}_tp_x\, \mu_{x+t}\, {}_tp_y\, dt = \int_0^{\infty} {}_tp_{xy}\, \mu_{x+t}\, dt$$

(ii) $\bar{A}^{1}_{xy}$ represents the expected present value of a payment of 1 unit made immediately on the death of (x), provided (x) dies before (y).

In terms of T_x and T_y:

$$\bar{A}^{1}_{xy} = E\left[g(T_x, T_y)\right] \quad \text{where} \quad g(T_x, T_y) = \begin{cases} v^{T_x} & \text{if } T_x < T_y \\ 0 & \text{if } T_x \geq T_y \end{cases}$$

As an integral:

$$\bar{A}^{1}_{xy} = \int_0^{\infty} v^t\, {}_tp_x\, \mu_{x+t}\, {}_tp_y\, dt = \int_0^{\infty} v^t\, {}_tp_{xy}\, \mu_{x+t}\, dt$$

(iii) $\bar{A}_{xy}^{\;\;2}$ represents the expected present value of a payment of 1 unit made immediately on the death of (y), provided (y) dies after (x).

In terms of T_x and T_y:

$$\bar{A}_{xy}^{\;\;2} = E\left[g(T_x, T_y)\right] \quad \text{where} \quad g(T_x, T_y) = \begin{cases} v^{T_y} & \text{if } T_y > T_x \\ 0 & \text{if } T_y \leq T_x \end{cases}$$

As an integral:

$$\bar{A}_{xy}^{\;\;2} = \int_0^{\infty} v^t\, {}_tq_x\, {}_tp_y\, \mu_{y+t}\, dt$$

22.2 Since the two lives are the same age and have the same mortality, by symmetry we have:

$$A_{\overset{2}{x}x} = \tfrac{1}{2}A_{\overline{xx}} = \tfrac{1}{2}\left(A_x + A_x - A_{xx}\right) = \tfrac{1}{2}\left(0.4 + 0.4 - 0.6\right) = 0.1$$

Alternatively:

$$A_{\overset{2}{x}x} = A_x - A_{\overset{1}{x}x} = A_x - \tfrac{1}{2}A_{xx} = 0.4 - \tfrac{1}{2}\times 0.6 = 0.1$$

22.3 The probability can be expressed in integral form as follows:

$${}_{30}q_{50:\overset{2}{60}} = \int_0^{30} {}_tp_{50}\,\mu_{50+t}\,{}_tq_{60}\,dt$$

Now:

$$\mu_{50+t} = \frac{1}{50-t}$$

and:

$${}_tp_x = \exp\left(-\int_0^t \mu_{x+s}\,ds\right) = \exp\left(-\int_0^t \frac{1}{100-x-s}\,ds\right)$$

$$= \exp\left[\ln\left(100-x-s\right)\right]_0^t = \exp\left[\ln\left(\frac{100-x-t}{100-x}\right)\right]$$

$$= \frac{100-x-t}{100-x}$$

So:

$${}_tp_{50} = \frac{50-t}{50} \quad \text{and} \quad {}_tq_{60} = 1 - {}_tp_{60} = 1 - \frac{40-t}{40} = \frac{t}{40}$$

Substituting these factors into the integral gives:

$${}_{30}q_{50:\overset{2}{60}} = \int_0^{30}\left(\frac{50-t}{50}\times\frac{1}{50-t}\times\frac{t}{40}\right)dt = \int_0^{30}\frac{t}{2{,}000}\,dt = \left[\frac{t^2}{4{,}000}\right]_0^{30} = 0.225$$

22.4 (i) In terms of an integral, this probability is:

$$_{10}q^{\;1}_{xy} = \int_0^{10} {}_tp_x \mu_{x+t}\, {}_tp_y \, dt$$

Evaluating this gives:

$$_{10}q^{\;1}_{xy} = \int_0^{10} e^{-0.01t} \times 0.01 \times e^{-0.02t}\, dt$$

$$= 0.01\int_0^{10} e^{-0.03t}\, dt$$

$$= 0.01\left[\frac{e^{-0.03t}}{-0.03}\right]_0^{10}$$

$$= \frac{0.01}{0.03}\left(1-e^{-0.3}\right) = 0.08639$$

(ii) In terms of an integral, the expected present value of this reversionary annuity is:

$$\bar{a}_{y|x} = \int_0^{\infty} v^t\, {}_tp_x\, {}_tq_y\, dt = \int_0^{\infty} v^t\, {}_tp_x (1-{}_tp_y)\, dt$$

Evaluating this gives:

$$\bar{a}_{y|x} = \int_0^{\infty} e^{-0.04t} e^{-0.01t} (1-e^{-0.02t})\, dt$$

$$= \int_0^{\infty} e^{-0.05t} - e^{-0.07t}\, dt$$

$$= \left[\frac{e^{-0.05t}}{-0.05} - \frac{e^{-0.07t}}{-0.07}\right]_0^{\infty}$$

$$= \frac{1}{0.05} - \frac{1}{0.07} = 5.7143$$

22.5 We can calculate this using the formula $a^{(m)}_{xy} \approx a_{xy} + \frac{m-1}{2m}$ with $m=4$:

$$a^{(4)}_{68:58} \approx a_{68:58} + \frac{3}{8} = \ddot{a}_{68:58} - 1 + \frac{3}{8} = 11.849 - 1 + \frac{3}{8} = 11.224$$

Alternatively:

$$a^{(4)}_{68:58} = \ddot{a}^{(4)}_{68:58} - \frac{1}{4} \approx \ddot{a}_{68:58} - \frac{3}{8} - \frac{1}{4} = 11.849 - \frac{3}{8} - \frac{1}{4} = 11.224$$

The relationship $a^{(m)}_{xy} = \ddot{a}^{(m)}_{xy} - \frac{1}{m}$ holds because the only difference between the annuity payable in arrears and the annuity payable in advance is the very first payment of $\frac{1}{m}$ made at time 0.

22.6 We have:

$$a_{70|60}^{(12)} = a_{60}^{(12)} - a_{70:60}^{(12)} \approx a_{60} + \frac{11}{24} - \left(a_{70:60} + \frac{11}{24} \right)$$

$$= a_{60} - a_{70:60} = \ddot{a}_{60} - 1 - \left(\ddot{a}_{70:60} - 1 \right) = \ddot{a}_{60} - \ddot{a}_{70:60}$$

where the 60-year-old experiences female mortality and the 70-year-old experiences male mortality.

Taking values from the *Tables*, we find:

$$a_{70|60}^{(12)} = 16.652 - 10.978 = 5.674$$

22.7 This is a 20-year pure endowment based on the last survivor status. The EPV of the benefits from this policy can be written as:

$$50{,}000 A_{\overline{[40]:[44]}:\overset{\;\;1}{\overline{20|}}}$$

This can be evaluated as follows:

$$50{,}000 A_{\overline{[40]:[44]}:\overset{\;\;1}{\overline{20|}}} = 50{,}000 v^{20} \; {}_{20}p_{\overline{[40]:[44]}}$$

$$= 50{,}000 v^{20} \left(1 - {}_{20}q_{[40]} \, {}_{20}q_{[44]} \right)$$

$$= 50{,}000 v^{20} \left(1 - \left(1 - \frac{l_{60}}{l_{[40]}} \right) \left(1 - \frac{l_{64}}{l_{[44]}} \right) \right)$$

$$= 50{,}000 (1.06)^{-20} \left(1 - \left(1 - \frac{9{,}287.2164}{9{,}854.3036} \right) \left(1 - \frac{8{,}934.8771}{9{,}811.4473} \right) \right)$$

$$= 15{,}510$$

22.8 (i) ***Present value random variable***

For the benefit to be paid, (x) must die between 5 years and 15 years after (y).

Letting Z denote the present value of the benefit, we have:

$$Z = \begin{cases} 15{,}000 v^{T_x} & \text{if } T_y + 5 < T_x < T_y + 15 \\ 0 & \text{otherwise} \end{cases}$$

[Total 2]

(ii) ***Integral expression for expected present value***

We can start by considering the expected present value of a benefit paid immediately on the death of (x) provided that (y) died at least 5 years earlier (*ie* ignoring the 15-year condition for the time being). This is:

$$15{,}000\int_5^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt$$

The lower limit on this integral is 5, as the benefit cannot be paid during the first 5 years.

Similarly, the expected present value of a benefit paid immediately on the death of (x) provided that (y) died at least 15 years earlier is:

$$15{,}000\int_{15}^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-15}q_y\,dt$$

The benefit we require is the difference between these two (*ie* it is paid out provided that (y) died at least 5 years earlier but no more than 15 years earlier). So:

$$E[Z]=15{,}000\left\{\int_5^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt-\int_{15}^\infty v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-15}q_y\,dt\right\}$$ [3]

Alternative solution

If (x) dies between time 0 and time 5, no benefit will be paid (as (y) cannot have died at least 5 years earlier).

If (x) dies between time 5 and time 15, the benefit is payable on the death of (x) at time t provided (y) is dead by time $t-5$. (In this case, (y) cannot have died more than 15 years before (x)). The expected present value of the benefit paid in this time interval is then:

$$15{,}000\int_5^{15} v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt$$

If (x) dies after time 15, the benefit is payable on the death of (x) at time t provided (y) died between time $t-15$ and time $t-5$. The expected present value of the benefit paid in this time interval is then:

$$15{,}000\int_{15}^\infty v^t\,{}_tp_x\mu_{x+t}\left({}_{t-5}q_y-{}_{t-15}q_y\right)dt$$

The overall expected present value of the benefit can therefore be written as:

$$E[Z]=15{,}000\left\{\int_5^{15} v^t\,{}_tp_x\,\mu_{x+t}\,{}_{t-5}q_y\,dt+\int_{15}^\infty v^t\,{}_tp_x\mu_{x+t}\left({}_{t-5}q_y-{}_{t-15}q_y\right)dt\right\}$$

This simplifies to the same expression as before as follows:

$$E[Z]=15{,}000\left\{\int_5^{15} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-5}q_y\,dt+\int_{15}^{\infty} v^t\,{}_tp_x\mu_{x+t}\;{}_{t-5}q_y\,dt-\int_{15}^{\infty} v^t\,{}_tp_x\mu_{x+t}\;{}_{t-15}q_y\,dt\right\}$$

$$=15{,}000\left\{\int_5^{\infty} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-5}q_y\,dt-\int_{15}^{\infty} v^t\,{}_tp_x\mu_{x+t}\;{}_{t-15}q_y\,dt\right\}$$

[Total 3]

(iii) ***Proof***

Considering in the first integral in the expression:

$$E[Z]=15{,}000\left\{\int_5^{\infty} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-5}q_y\,dt-\int_{15}^{\infty} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-15}q_y\,dt\right\}$$

and making the substitution $t=s+5$, we see that:

$$\int_5^{\infty} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-5}q_y\,dt=\int_0^{\infty} v^{s+5}\,{}_{s+5}p_x\,\mu_{x+s+5}\;{}_sq_y\,ds$$

$$=v^5\,{}_5p_x\int_0^{\infty} v^s\,{}_sp_{x+5}\,\mu_{x+s+5}\;{}_sq_y\,ds$$

$$=v^5\,{}_5p_x\,\bar{A}^{2}_{x+5:y}$$ [1½]

Similarly:

$$\int_{15}^{\infty} v^t\,{}_tp_x\,\mu_{x+t}\;{}_{t-15}q_y\,dt=v^{15}\,{}_{15}p_x\,\bar{A}^{2}_{x+15:y}$$ [½]

Hence:

$$E[Z]=15{,}000\left[v^5\,{}_5p_x\,\bar{A}^{2}_{x+5:y}-v^{15}\,{}_{15}p_x\,\bar{A}^{2}_{x+15:y}\right]$$ [1]

[Total 3]

(iv) ***Premium payment term***

For premiums to be payable for this policy, we require:

- (x) to be alive (since the benefit is paid on (x)'s death), and [1]
- either (y) to be alive, or (y) to have died no more than 15 years ago. [1]

[Total 2]

22.9 (i) ***Monthly premium***

Let P denote the monthly premium. Then the equation of value is:

$$12P\ddot{a}^{(12)}_{65:62:\overline{5}|} = 10{,}000\,\bar{A}_{65:62} \quad [1]$$

Now:

$$\ddot{a}^{(12)}_{65:62:\overline{5}|} = \ddot{a}^{(12)}_{65:62} - v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} \times \ddot{a}^{(12)}_{70:67} \quad [1]$$

$$\ddot{a}^{(12)}_{65:62} \approx \ddot{a}_{65:62} - \frac{11}{24} = 12.427 - \frac{11}{24} = 11.969 \quad [½]$$

$$\ddot{a}^{(12)}_{70:67} \approx \ddot{a}_{70:67} - \frac{11}{24} = 10.233 - \frac{11}{24} = 9.775 \quad [½]$$

and:

$$v^5 \times \frac{l_{70}}{l_{65}} \times \frac{l_{67}}{l_{62}} = 1.04^{-5} \times \frac{9{,}238.134}{9{,}647.797} \times \frac{9{,}605.483}{9{,}804.173} = 0.77108 \quad [½]$$

So:

$$\ddot{a}^{(12)}_{65:62:\overline{5}|} \approx 11.969 - 0.77108 \times 9.775 = 4.432 \quad [½]$$

We need to use the premium conversion relationship to determine the value of the benefit:

$$\bar{A}_{65:62} \approx (1+i)^{½} A_{65:62} = 1.04^{½}\left(1 - d\,\ddot{a}_{65:62}\right) = 1.04^{½}\left(1 - \frac{0.04}{1.04} \times 12.427\right) = 0.53238 \quad [1]$$

So the monthly premium is:

$$P = \frac{10{,}000 \times 0.53238}{12 \times 4.432} = £100 \quad [1]$$

[Total 6]

(ii) ***Prospective reserve after 3 years***

The reserve after three years will be:

$${}_3V = 10{,}000\bar{A}_{68:65} - 12P\ddot{a}^{(12)}_{68:65:\overline{2}|} \quad [1]$$

where:

$$\ddot{a}^{(12)}_{68:65:\overline{2}|} = \ddot{a}^{(12)}_{68:65} - v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} \times \ddot{a}^{(12)}_{70:67} \quad [½]$$

Now:

$$\ddot{a}^{(12)}_{68:65} \approx \ddot{a}_{68:65} - \frac{11}{24} = 11.112 - \frac{11}{24} = 10.654$$ [½]

$$\ddot{a}^{(12)}_{70:67} \approx 9.775 \text{ from (i)}$$

and:

$$v^2 \times \frac{l_{70}}{l_{68}} \times \frac{l_{67}}{l_{65}} = 1.04^{-2} \times \frac{9,238.134}{9,440.717} \times \frac{9,605.483}{9,703.708} = 0.89556$$ [½]

So:

$$\ddot{a}^{(12)}_{68:65:\overline{2|}} \approx 10.654 - 0.89556 \times 9.775 = 1.900$$ [½]

Also:

$$\bar{A}_{68:65} \approx (1+i)^{½} A_{68:65} = (1+i)^{½}\left(1 - d\,\ddot{a}_{68:65}\right) = 1.04^{½}\left(1 - \frac{0.04}{1.04} \times 11.112\right) = 0.58396$$ [½]

So the reserve at time 3 is:

$$10,000 \times 0.58396 - 1,200 \times 1.900 = £3,560$$ [½]

[Total 4]

22.10 Let P denote the annual premium. The expected present value of the premiums is:

$$P\ddot{a}_{60:64:\overline{10|}} = P\left(\ddot{a}_{60:64} - v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\,\ddot{a}_{70:74}\right)$$ [1]

where the 60-year-old is subject to male mortality and the 64-year-old is subject to female mortality.

So, using the *Tables*:

$$P\ddot{a}_{60:64:\overline{10|}} = \left(13.325 - 1.04^{-10} \times \frac{9,238.134}{9,826.131} \times \frac{8,937.791}{9,742.640} \times 9.005\right)P = 8.078P$$ [1]

The expected present value of the expenses is:

$$750 + 0.03P\left(\ddot{a}_{60:64:\overline{10|}} - 1\right) = 750 + 0.03P \times 7.078 = 750 + 0.2123P$$ [1]

The expected present value of the benefit payable on death is $50,000A^{\;1}_{\overline{60:64}:\overline{10|}}$. We can evaluate this as follows, using premium conversion to find the endowment assurance value:

$$50{,}000A_{\overset{1}{60:64}:\overline{10|}} = 50{,}000\left(A_{60:64:\overline{10|}} - A_{60:64:\overset{1}{\overline{10|}}}\right)$$

$$= 50{,}000\left(1 - d\ddot{a}_{60:64:\overline{10|}} - v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\right) \qquad [1]$$

Using the *Tables*, and the value of the temporary annuity factor calculated earlier, the EPV of the death benefit is:

$$50{,}000\left(1 - \frac{0.04}{1.04}\times 8.078 - 1.04^{-10}\times\frac{9{,}238.134}{9{,}826.131}\times\frac{8{,}937.791}{9{,}742.640}\right) = 5{,}331.81 \qquad [1]$$

Alternatively, the EPV of the death benefit can be calculated as:

$$50{,}000A_{\overset{1}{60:64}:\overline{10|}} = 50{,}000\left(A_{60:64} - v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}A_{70:74}\right)$$

$$= 50{,}000\left(1 - d\ddot{a}_{60:64} - v^{10}\,{}_{10}p_{60}\,{}_{10}p_{64}\left(1 - d\ddot{a}_{70:74}\right)\right)$$

to obtain the same answer.

The deferred annuity is £10,000 *pa* while both lives are alive or £5,000 *pa* if only one life is alive. This is equivalent to each life receiving a single life annuity of £5,000 (since £10,000 will be paid in total if both are alive). [1]

So the expected present value of this benefit is:

$$5{,}000\left(v^{10}\,{}_{10}p_{60}\,\ddot{a}_{70} + v^{10}\,{}_{10}p_{64}\,\ddot{a}_{74}\right)$$

$$= 5{,}000\times 1.04^{-10}\left(\frac{9{,}238.134}{9{,}826.131}\times 11.562 + \frac{8{,}937.791}{9{,}742.640}\times 11.333\right)$$

$$= 71{,}835.77 \qquad [1]$$

Setting the EPV of premiums equal to the EPV of the benefits and expenses gives:

$$8.078P = 5{,}331.81 + 71{,}835.77 + 750 + 0.2123P$$

$$\Rightarrow \quad 7.866P = 77{,}917.58$$

$$\Rightarrow \quad P = £9{,}906 \qquad [1]$$

[Total 8]

22.11 *This is an example of a 'Type 4' reversionary annuity, as described in Section 4.4.*

The expected present value of the spouse's benefit is:

$$25{,}000\left(a_{65|55} - v^{20}\,{}_{20}p_{55}\,a_{65|75}\right) \qquad [3]$$

The terms in this expression are:

$$a_{65|55} = a_{55}^{m} - a_{55:65}^{m\ f} = (\ddot{a}_{55}^{m} - 1) - (\ddot{a}_{55:65}^{m\ f} - 1) = \ddot{a}_{55}^{m} - \ddot{a}_{55:65}^{m\ f} = 17.364 - 13.880 = 3.484 \qquad [1]$$

$$a_{65|75} = a_{75}^{m} - a_{75:65}^{m\ f} = (\ddot{a}_{75}^{m} - 1) - (\ddot{a}_{75:65}^{m\ f} - 1) = \ddot{a}_{75}^{m} - \ddot{a}_{75:65}^{m\ f} = 9.456 - 8.833 = 0.623 \ [1]$$

and:

$$v^{20}\,{}_{20}p_{55} = 1.04^{-20} \times \frac{8{,}405.160}{9{,}904.805} = 0.38729 \qquad [1]$$

So the expected present value of the benefit is:

$$25{,}000(3.484 - 0.38729 \times 0.623) = £81{,}068 \qquad [1]$$

[Total 7]

To work out where to start with a question like this, it can help to imagine that the annuity benefit is payable continuously, and then write down and simplify an integral expression, as follows:

$$\int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t:\overline{20}|}\,dt$$

$$= \int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\left(\bar{a}_{55+t} - v^{20}\,{}_{20}p_{55+t}\,\bar{a}_{75+t}\right)dt$$

$$= \int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{55}\,v^t\,\bar{a}_{55+t}\,dt - v^{20}\,{}_{20}p_{55}\int_0^\infty {}_tp_{65}\,\mu_{65+t}\,{}_tp_{75}\,v^t\,\bar{a}_{75+t}\,dt$$

$$= \bar{a}_{65|55} - v^{20}\,{}_{20}p_{55}\,\bar{a}_{65|75}$$

Converting the continuous annuities to annual annuities gives us the required formula:

$$a_{65|55} - v^{20}\,{}_{20}p_{55}\,a_{65|75}$$

End of Part 4

What next?

1. Briefly **review** the key areas of Part 4 and/or re-read the **summaries** at the end of Chapters 19 to 22.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 4. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X4**.
4. Attempt the questions relating to Chapters 19 to 22 of the **Paper B Online Resources (PBOR)**.

23

Mortality profit

Syllabus objectives

6.3 Define and calculate, for a single policy or a portfolio of policies (as appropriate):

- death strain at risk
- expected death strain
- actual death strain
- mortality profit

for policies with death benefits payable immediately on death or at the end of the year of death, policies paying annuity benefits at the start of the year or on survival to the end of the year, and policies where single or non-single premiums are payable.

0 Introduction

In Section 5 of Chapter 20 we described the recursive relation between reserves and how an expression for the profit earned over a particular year could be derived from this.

In this chapter we look at that part of the profit earned during the year that is due to mortality, referred to as *mortality profit*.

In Chapter 20 it was shown that, if the experience exactly follows the reserve basis, then, on average, the income and outgo in each policy year are equal.

In this case 'outgo' includes the increase in reserves. When talking about outgo from the insurance company's point of view, we consider reserves as money for policyholders. Hence increase in reserves is a form of outgo.

If the experience does not follow the assumptions, there will either be an excess of income over outgo (a profit, or surplus) or an excess of outgo over income (a loss or negative profit). Profits and losses may arise from any element of the reserve basis. For example:

1. **If the interest earned is greater than that assumed in the reserve, then the income will accumulate to more than the sum required to cover the cost of the benefits and the year-end reserve, giving an interest surplus.**

2. **If the policyholder decides to surrender his or her policy (that is, to cease paying premiums, and take some lump sum in respect of the future benefits already paid for) then the year-end outgo is not as assumed. If the lump sum is less than the reserve there will be a surrender profit. If no surrender benefit is paid, the profit will be equal to the reserve.**

3. **If the experienced mortality is heavier than that assumed in the basis, then there will be a profit or loss from mortality, depending on the nature of the contract. Where benefits are paid out on death, such as under a term assurance, lighter mortality than assumed will give rise to a profit. Where benefits are paid out on survival, such as under an annuity, then lighter mortality will give rise to a loss.**

So, if experience is not as assumed, profits or losses will arise. Exactly the same principle applies in pension schemes; surpluses and deficits arise because experience is not in line with the actuary's view of future experience.

Here we consider mortality profit only. We assume, therefore, that in all elements other than mortality, experienced rates follow the assumed rates exactly.

In practice, they do not; and each of the above will give rise to profits or losses. The impact of each element may be quantified. This procedure is known as analysis of surplus.

1 Mortality profit on a single policy

1.1 Death strain at risk (DSAR)

Consider a policy issued t years ago to a life then aged x, with sum assured S payable at the end of the year of death. Also, assume that no survival benefit is due if the life survives to $t+1$. (We will extend these ideas to death benefits payable immediately on death, and to survival benefits, in later sections). Let ${}_tV$ be the reserve at time t. Then we define the *death strain* in the policy year t to $t+1$ to be the random variable, DS, say,

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ (S - {}_{t+1}V) & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

The maximum death strain, $(S - {}_{t+1}V)$ is called the *death strain at risk* or *DSAR*.

If we think of the reserve as money already set aside for the policyholder, the death strain at risk for the current policy year is the amount of extra money that the company would need to pay if the policyholder died during that policy year. The death strain at risk is sometimes also called the sum at risk.

The word *strain* is used loosely to mean a cost to the company. The reasoning behind the DSAR definition is seen more clearly if we rearrange the recursive relationship between ${}_tV$ and ${}_{t+1}V$.

Recall from Chapter 20 the general recursive relationship between gross premium reserves ${}_tV'$ and ${}_{t+1}V'$:

$$\left({}_tV' + G - e\right)(1+i) = q_{x+t}\left(S+f\right) + p_{x+t}\,{}_{t+1}V'$$

There is a similar recursive relationship that applies to net premium reserves, which involves the net premium P and ignores all expenses (e and f in the above formula).

Assuming level net premiums for simplicity:

$$\begin{aligned} \left({}_tV + P\right)(1+i) &= q_{x+t}\,S + p_{x+t}\,{}_{t+1}V \\ &= q_{x+t}\,S + (1-q_{x+t})\,{}_{t+1}V \\ &= {}_{t+1}V + q_{x+t}(S - {}_{t+1}V) \end{aligned}$$

In words, the reasoning is that for each policy we must pay out at least ${}_{t+1}V$ at the end of the year. In addition, if the policy becomes a claim during the year, with probability q_{x+t}, then we must pay out an extra sum of $(S - {}_{t+1}V)$ which is the DSAR. Note that q_{x+t} is the probability of dying in the year t to $t+1$, and therefore $x+t$ is the age at the *start* of the year.

1.2 Expected death strain (EDS)

The expected amount of the death strain is called the *expected death strain* (*EDS*). This is the amount that the life insurance company expects to pay in addition to the year-end reserve for the policy. The probability of claiming in the policy year t to $t+1$ is q_{x+t} so that:

$$\text{EDS} = q_{x+t}\left(S - {}_{t+1}V\right)$$

1.3 Actual death strain (ADS)

The *actual death strain* is simply the observed value at $t+1$ of the death strain random variable, that is:

$$\text{ADS} = \begin{cases} 0 & \text{if the life survived to } t+1 \\ \left(S - {}_{t+1}V\right) & \text{if the life died in the year } [t,\ t+1) \end{cases}$$

1.4 Mortality profit

The *mortality profit* is defined as:

Mortality profit = Expected Death Strain – Actual Death Strain

The EDS is the amount the company *expects* to pay out, in addition to the year-end reserve for a policy. The ADS is the amount it *actually* pays out, in addition to the year-end reserve. If it actually pays out less than it expected to pay, there will be a profit. If the actual strain is greater than the expected strain, there will be a loss.

2 Mortality profit on a portfolio of policies

We are often interested in analysing the experience of a group of similar policies. We use the term *portfolio of policies* to mean any group of policies. In this case we simply sum the EDS and the ADS over all the relevant policies. If all lives are the same age, and subject to the same mortality table, this gives:

$$\textbf{Total DSAR} = \sum_{\text{all policies}} (S - {}_{t+1}V)$$

$$\textbf{Total EDS} = \sum_{\text{all policies}} q_{x+t}(S - {}_{t+1}V)$$

$$= q_{x+t}\left(\sum_{\text{all policies}} (S - {}_{t+1}V)\right)$$

$$= q_{x+t}(\text{total DSAR})$$

$$\textbf{Total ADS} = \sum_{\text{death claims}} (S - {}_{t+1}V)$$

$$\textbf{Mortality Profit} = \text{total EDS} - \text{total ADS}$$

If the policies are identical, then:

Total EDS = expected number of deaths × DSAR

Total ADS = actual number of deaths × DSAR

In many situations the DSAR of each individual policy is not known, but the total DSAR is simply the total sum assured less the total year-end reserve, and the total EDS is:

$$q_{x+t}(\text{total DSAR})$$

In the above equations:

- the summation is over all policies that are in force at the *start* of the year
- the mortality rate q_{x+t} relates to the age of the policyholder(s) at the *start* of the year
- the reserves ${}_{t+1}V$ are calculated as at the *end* of the year.

Question

A life insurance company has a portfolio of 10,000 single premium one-year term assurances. For each policy, there is a sum assured of $50,000 payable at the end of the year if the policyholder dies during the year. The company assumes that mortality will be 1% *pa*.

(i) Calculate the expected death strain for this portfolio.

(ii) Given that 89 people die during the year, calculate the actual death strain and hence the mortality profit or loss for this portfolio.

Solution

Since this is a one-year policy, no reserves will be required at the end of the year. The death strain at risk for each policy is therefore \$50,000.

(i) The expected number of deaths is $10,000 \times 0.01 = 100$ and the expected death strain is $100 \times 50,000 = \$5m$.

(ii) The actual death strain is $89 \times 50,000 = \$4.45m$.

The mortality profit is the difference between these, *ie* \$550,000.

The insurance company has made a profit here because the actual number of deaths (89) was less than the expected number (100).

3 Allowing for death benefits payable immediately

Where death benefits are payable immediately on death, in the calculation of the death strain we allow for interest between the time of payment and the end of the year of death. In this case, the death strain defined in Section 1.1 would become:

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S(1+i)^{1/2} - {}_{t+1}V\right) & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

The death strain formula requires the value of the death benefit payment as at the *end* of the year of death. As the sum assured is paid out during the year, the value of this payment will increase with interest between the date of death and the end of the year of death. The above formula therefore assumes that death occurs half way through the year, on average.

Question

Explain how the formula for the death strain should be adjusted if the sum assured is paid two months after the actual date of death.

Solution

The death strain formula would change to:

$$DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ S(1+i)^{4/12} - {}_{t+1}V & \text{if the life dies in the year } [t,\ t+1) \end{cases}$$

This is because (assuming deaths occur half way through the year on average) the payment of the sum assured occurs 8/12 through the year, on average. So, in order to revalue the payment to the end of the year of death, we need to accumulate the payment with 4 months' interest.

Similar adjustments would be applied to the formulae in Sections 1.2, 1.3 and 2.

Question

Consider the following group of whole life assurance policies:

- year of issue: 2011
- number in force at the policy anniversary in 2016: 1,900
- number in force at the policy anniversary in 2017: 1,867
- exact age at the policy anniversary in 2016: 70
- sum assured: 60,000 per policy, payable immediately on death
- level premiums are payable annually in advance for the whole of life.

Calculate the mortality profit for this group of policies for the policy year commencing at the policy anniversary in 2016, assuming death is the only cause of policy termination, and that the insurer holds net premium reserves for these contracts calculated assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The death strain at risk for a single policy is calculated as:

$$DSAR = 60{,}000 \times (1+i)^{\frac{1}{2}} - {}_6V$$

where ${}_6V$ is the reserve held at the policy anniversary in 2017 (which is at exact duration 6).

For the net premium reserve, we first need to calculate the net premium. This is found by solving the premium equation of value at policy outset (when the policyholder was aged 65) using the reserving basis assumptions. This gives:

$$P = \frac{60{,}000\bar{A}_{65}}{\ddot{a}_{65}} \approx \frac{60{,}000 \times 1.04^{\frac{1}{2}} \times 0.52786}{12.276} = 2{,}631.05$$

using $\bar{A}_x \approx (1+i)^{\frac{1}{2}} A_x$.

Then:

$${}_6V = 60{,}000\bar{A}_{71} - P\ddot{a}_{71}$$

$$\approx 60{,}000 \times 1.04^{\frac{1}{2}} \times A_{71} - 2{,}631.05\ddot{a}_{71}$$

$$= 60{,}000 \times 1.04^{\frac{1}{2}} \times 0.61548 - 2{,}631.05 \times 9.998 = 11{,}354.90$$

and the death strain at risk is:

$$DSAR = 60{,}000 \times 1.04^{\frac{1}{2}} - 11{,}354.90 = 49{,}833.33$$

The expected death strain for this group of policies is:

$$EDS = 1{,}900 \times q_{70} \times DSAR = 1{,}900 \times 0.024783 \times 49{,}833.33$$

$$= 2{,}346{,}537$$

During the policy year, 33 people died, so the actual death strain for the group of policies is:

$$ADS = 33 \times DSAR = 33 \times 49{,}833.33 = 1{,}644{,}500$$

This gives a mortality profit of:

$$EDS - ADS = £702{,}037$$

4 Allowing for survival benefits

Suppose the contract provides for a benefit at the end of policy year t to $t+1$. By convention, the expected present value of this will have been included in ${}_tV$ but will fall outside the computation of ${}_{t+1}V$. So, the survival benefit needs to be allowed for as an additional payment.

Let R be the benefit payable at the end of the policy year t to $t+1$ contingent on the survival of the policyholder. Assuming death benefits are paid at the end of the year of death, the recursive relationship between successive reserves is now:

$$\left({}_tV+P\right)(1+i)=q_{x+t}S+p_{x+t}({}_{t+1}V+R)$$

$$=q_{x+t}S+(1-q_{x+t})({}_{t+1}V+R)$$

$$={}_{t+1}V+R+q_{x+t}(S-({}_{t+1}V+R))$$

The DS is now:

$$\text{DS}=\begin{cases}0 & \text{if the life survives to } t+1\\ \left(S-({}_{t+1}V+R)\right) & \text{if the life dies in the year } [t,\ t+1)\end{cases}$$

as the office must pay out:

- $({}_{t+1}V+R)$ **for all policyholders, and an additional**
- $S-({}_{t+1}V+R)$ **for policies becoming claims by death.**

So the DSAR for a single policy is $S-\left({}_{t+1}V+R\right)$.

The expected death strain is then $q_{x+t}\left[S-({}_{t+1}V+R)\right]$; the actual death strain is 0 if the life survived the year and $S-({}_{t+1}V+R)$ if the life died during the year; the mortality profit is EDS – ADS, as before.

Question

Repeat the question at the end of Section 2, assuming that the survivors are paid a lump sum benefit of \$20,000 at the end of the year.

Solution

For policyholders who die during the year we need the sum assured of \$50,000 at the end of the year. So in the above formula $S=50,000$.

For policyholders who survive we need the endowment payment of \$20,000. So $R=20,000$.

As the reserve at the end of the year is zero, the death strain at risk is now:

$$50{,}000 - 20{,}000 = \$30{,}000$$

The expected death strain is:

$$0.01 \times 10{,}000 \times 30{,}000 = 3{,}000{,}000$$

and the actual death strain is:

$$89 \times 30{,}000 = 2{,}670{,}000$$

Therefore the mortality profit is:

$$3{,}000{,}000 - 2{,}670{,}000 = \$330{,}000 .$$

In the next question, the only benefit payable is on survival to the end of the term of the contract.

Question

On 1 January 2009 a life insurance company issued a number of 30-year pure endowment contracts to lives then aged 35. Level premiums are payable annually in advance throughout the term of the contract or until earlier death. In each case, the only benefit is a sum assured of £20,000, payable on survival to the end of the term.

On 1 January 2017, 600 policies were still in force. During 2017, 3 policyholders died. Assuming that the company holds net premium policy reserves, calculate the profit or loss from mortality for calendar year 2017 in respect of this group of policies.

Basis:	Mortality:	AM92 Ultimate
	Interest:	4% *pa*

Solution

The reserve at 31 December 2017, *ie* at time 9, is:

$$_9V = \text{EPV future benefits} - \text{EPV future premiums}$$

$$= 20{,}000 \frac{D_{65}}{D_{44}} - P\ddot{a}_{44:\overline{21|}}$$

$$= 20{,}000 \times \frac{689.23}{1{,}747.41} - P \times 14.233$$

$$= 7{,}888.59 - 14.233P$$

In the above formula:

$$P = \frac{20{,}000\dfrac{D_{65}}{D_{35}}}{\ddot{a}_{35:\overline{30|}}} = \frac{20{,}000 \times \dfrac{689.23}{2{,}507.40}}{17.629} = £311.85$$

So:

$${}_9V = £3{,}450.06$$

Since there is no benefit payable on death during the year, or on survival to the end of the year:

$$DSAR = -{}_9V = -£3{,}450.06$$

The expected death strain is:

$$EDS = 600q_{43} \times DSAR = -600 \times 0.001208 \times 3{,}450.06 = -£2{,}500.60$$

The actual death strain is:

$$ADS = 3 \times DSAR = -£10{,}350.17$$

So the mortality profit is:

$$-2{,}500.60 - (-10{,}350.17) = £7{,}850$$

A mortality profit has been made this year because there were more deaths during the year than expected (ie 3 actual deaths as opposed to the expected number of $600 \times 0.001208 = 0.7248$ *deaths). This has caused a profit because a policyholder who survives is more costly to the insurance company than a person who dies, as a surviving policyholder presents a non-zero probability of a claim being eventually paid at maturity whereas a death causes no claim cost, either now or in the future.*

4.1 Annuities

In the case of an annuity of R *pa*, payable annually in arrears, with no death benefit, the DSAR would be $-\left({}_{t+1}V + R\right)$. In this case each death causes a negative strain or *release of reserves*.

Question

At the start of a particular year a life insurance company had a portfolio of 5,000 female pensioners, all aged exactly 60, who each receive an income of £10,000 per annum, paid annually in arrears.

The company holds net premium reserves, calculated using PFA92C20 mortality and 4% *pa* interest.

During that year, 9 pensioners died. Calculate the mortality profit or loss for that year.

Solution

For policyholders who die during the year, no funds are required at the end of the year.

The reserve required at the end of the year for each surviving policy plus the annuity payment due at that time is:

$$10{,}000a_{61} + 10{,}000 = 10{,}000\ddot{a}_{61} = 10{,}000 \times 16.311 = £163{,}110$$

So the death strain at risk is $0 - 163{,}110 = -£163{,}110$.

From the *Tables*, $q_{60} = 0.002058$. So the expected number of deaths during the year is $5{,}000q_{60} = 10.29$.

So the EDS is $-£163{,}110 \times 10.29 = -£1{,}678{,}402$.

The ADS is $-£163{,}110 \times 9 = -£1{,}467{,}990$.

So the mortality profit is:

$$EDS - ADS = -1{,}678{,}402 - (-1{,}467{,}990) = -£210{,}412$$

ie a loss of about £210,400. (The loss arises because fewer people died than expected.)

In the case of an annuity of R *pa*, payable annually in advance, with no death benefit, the DSAR would be $-{}_{t+1}V$ only. The annuity payment is made by all policies in force at the start of the year, and is not affected by whether or not the policyholder survives the year.

The annuity payment due at time $t+1$ is paid at the start of the *next* year, and is therefore included in ${}_{t+1}V$. If we took the DSAR to be $-\left({}_{t+1}V + R\right)$ for the current year we would be double-counting the annuity payment.

5 Allowing for different premium or annuity payment frequencies

The above formulae for the death strain and mortality profit are appropriate where premiums are either paid annually in advance or as a single payment at the outset of the policy.

Where premiums (or annuities) are paid more frequently than annually, the formulae for the death strain and mortality profit will be different, because the death or survival of the policyholder during the year will affect how many premiums are actually received, or how many annuity payments are actually made. These variations are beyond the scope of the CM1 syllabus.

6 Calculation of mortality profit for policies involving two lives

We shall now show how we can calculate the mortality profit for policies that each involve two lives. Essentially we can use the same methods as before (although the calculations can sometimes be more complex).

For life assurance policies where the death benefit is payable on the first death of two people, the calculations are fairly simple, and almost identical to those used for a single life policy. Let's look at an example.

Question

A company sells joint life whole of life assurances to male lives aged 60 exact and female lives aged 58 exact. The sum assured is £80,000, payable at the end of the year of death of the first life to die. Level premiums are payable annually in advance whilst the policy is in force.

(i) Calculate the annual premium for these policies.

Basis: Mortality: PMA92C20 for the male life and PFA92C20 for the female life

Interest: 4% *pa* effective

Expenses: Nil

A group of these policies was sold on 1 January 2010. On 1 January 2018, 200 of these policies were still in force. Of these policies, two made a claim during 2018.

(ii) Calculate the profit or loss from mortality for calendar year 2018, using the premium basis given above.

Solution

(i) ***Premium***

The premium equation is:

$$P\ddot{a}_{60:58} = 80{,}000 A_{60:58}$$

The annuity is tabulated ($\ddot{a}_{60:58} = 14.393$), and the assurance function is calculated using premium conversion:

$$A_{60:58} = 1 - d\,\ddot{a}_{60:58} = 1 - \frac{0.04}{1.04} \times 14.393 = 0.44642$$

So the premium is:

$$P = \frac{80{,}000 \times 0.44642}{14.393} = £2{,}481.33$$

(ii) ***Mortality profit or loss***

We first need to calculate the reserve needed for each policy on 31 December 2018, *ie* at time 9. At this point the lives will be aged 69 and 67, so the prospective reserve is:

$$_9V = 80{,}000 A_{69:67} - 2{,}481.33\ddot{a}_{69:67}$$

The joint life annuity is in the *Tables* ($\ddot{a}_{69:67} = 10.526$), and the assurance function is calculated using premium conversion:

$$A_{69:67} = 1 - d\,\ddot{a}_{69:67} = 1 - \frac{0.04}{1.04} \times 10.526 = 0.59515$$

So the reserve is:

$$_9V = 80{,}000 \times 0.59515 - 2{,}481.33 \times 10.526 = 21{,}493.83$$

So the death strain at risk is:

$$S - {_9V} = 80{,}000 - 21{,}493.83 = 58{,}506.17$$

The probability that a policy will become a claim during 2018 is the probability that at least one of the policyholders dies, which is one minus the probability that they both survive. Using the policyholders' ages at the *start* of 2018, the probability of a claim is:

$$1 - p_{68}^{m} \times p_{66}^{f} = 1 - (1 - 0.009930)(1 - 0.005467) = 0.015343$$

So the expected number of claims during the year from this cohort of policies is:

$$200 \times 0.015343 = 3.06854$$

and the actual number of claims is 2. So we can find the expected death strain (EDS) and the actual death strain (ADS):

$$EDS = 3.06854 \times 58{,}506.17 = 179{,}529$$

$$ADS = 2 \times 58{,}506.17 = 117{,}012$$

The mortality profit is the difference between these:

$$= 179{,}529 - 117{,}012 = £62{,}516$$

For policies where the death benefit is payable on the second death, the situation is more complex.

Consider a last survivor assurance where both lives are still alive at the start of the year. If both lives die during the year, the death strain at risk is, as usual, the sum assured less the reserve that would have been held at the end of the year if both lives had survived.

Suppose, however, that only one of the lives dies. There is still a mortality cost even though the sum assured has not been paid out, because the death of one of the policyholders means that a higher reserve will be needed at the end of the year compared to what it would have been if both people had survived. The death strain at risk in this case is therefore calculated as the amount by which the end-year reserve increases as a result of the particular death that occurs.

An example should help to make this clear.

Question

A company sells last survivor whole of life assurances to male lives aged 60 exact and female lives aged 58 exact. The sum assured is £80,000, payable at the end of the year of death of the second life to die. Level premiums (calculated using the basis below) of £1,237.61 are paid annually in advance whilst the policy is in force.

Basis: Mortality: PMA92C20 for the male life and PFA92C20 for the female life

Interest: 4% *pa* effective

Expenses: Nil

A group of these policies was sold on 1 January 2010. On 1 January 2018, there were 1,000 policies still in force where both lives were still alive. Of these policies, during 2018:

- there was one policy for which both lives died
- there were two policies where the female life died (but the male life did not)
- there were three policies where the male life died (but the female life did not).

Calculate the profit or loss from mortality for calendar year 2018, using the premium basis.

Solution

We first calculate the reserve at the end of 2018, assuming that both lives are still alive. Working prospectively, this is:

$$_9V = 80{,}000 A_{\overline{69:67}} - 1{,}237.61\,\ddot{a}_{\overline{69:67}}$$

The last survivor annuity is calculated from the single and joint life annuities:

$$\ddot{a}_{\overline{69:67}} = \ddot{a}_{69} + \ddot{a}_{67} - \ddot{a}_{69:67} = 11.988 + 14.111 - 10.526 = 15.573$$

The last survivor assurance can be calculated using premium conversion:

$$A_{\overline{69:67}} = 1 - d\,\ddot{a}_{\overline{69:67}} = 1 - \frac{0.04}{1.04} \times 15.573 = 0.40104$$

So the reserve is:

$$_9V = 80{,}000 \times 0.40104 - 1{,}237.61 \times 15.573 = £12{,}809.78$$

We now consider separately the different events that occurred.

Both lives die

Consider a policy for which both lives die during 2018. If neither life had died, the reserve needed would have been £12,809.78. If both die, we need to pay the sum assured of £80,000. So the death strain at risk (DSAR) is the difference between these two figures:

$$DSAR = 80{,}000 - 12{,}809.78 = 67{,}190.22$$

The expected number of policies for which both lives die (using the ages at the start of 2018) is:

$$1{,}000 \times q_{68}^{m} \times q_{66}^{f} = 1{,}000 \times 0.009930 \times 0.005467 = 0.05429$$

So the expected death strain (EDS) for these policies is:

$$EDS = 0.05429 \times 67{,}190.22 = 3{,}647.58$$

The actual number of policies for which both lives died is one, so the actual death strain (ADS) is just 67,190.22. So the mortality profit (MP) for this group of policies is:

$$MP = EDS - ADS = 3{,}647.58 - 67{,}190.22 = -£63{,}542.65$$

(These figures are quite sensitive to rounding.)

Male life dies, female survives

If this event had not happened, so that neither life had died during the year, we would again have needed the last survivor reserve of £12,809.78. If the event does happen, we will need the single life reserve for the female life only:

$$\begin{aligned} {}_9V^{\textit{female alive}} &= 80{,}000 A_{67}^{f} - 1{,}237.61\, \ddot{a}_{67}^{f} \\ &= 80{,}000 \times \left(1 - \frac{0.04}{1.04} \times 14.111\right) - 1{,}237.61 \times 14.111 = 19{,}117.62 \end{aligned}$$

So the DSAR for these policies is the difference between these two reserves:

$$DSAR = 19{,}117.62 - 12{,}809.78 = 6{,}307.85$$

The expected number of policies for which only the male dies is:

$$1{,}000 \times q_{68}^{m} \times p_{66}^{f} = 1{,}000 \times 0.009930 \times (1 - 0.005467) = 9.87571$$

So the EDS for these policies is:

$$EDS = 9.87571 \times 6{,}307.85 = 62{,}294.49$$

The actual number of policies for which only the male dies is three, so the ADS is:

$$ADS = 3 \times 6{,}307.85 = 18{,}923.54$$

So the MP for this group of policies is:

$$MP = EDS - ADS = 62{,}294.49 - 18{,}923.54 = £43{,}370.95$$

Female life dies, male life survives

Again, if this event had not happened during the year, we would have needed the last survivor reserve of £12,809.78. If the event does happen, we will need the single life reserve for the male life only:

$$\begin{aligned} {}_9V^{\text{male alive}} &= 80{,}000 A_{69}^m - 1{,}237.61\, \ddot{a}_{69}^m \\ &= 80{,}000 \times \left(1 - \frac{0.04}{1.04} \times 11.988\right) - 1{,}237.61 \times 11.988 = 28{,}277.38 \end{aligned}$$

The DSAR for these policies is the difference between these two reserves:

$$DSAR = 28{,}277.38 - 12{,}809.78 = 15{,}467.60$$

The expected number of policies for which only the female dies is:

$$1{,}000 \times p_{68}^m \times q_{66}^f = 1{,}000 \times (1 - 0.009930) \times 0.005467 = 5.41271$$

So the EDS for these policies is:

$$EDS = 5.41271 \times 15{,}467.60 = 83{,}721.68$$

The actual number of policies for which only the female dies is two, so the ADS is:

$$ADS = 2 \times 15{,}467.60 = 30{,}935.20$$

So the MP for this group of policies is:

$$MP = EDS - ADS = 83{,}721.68 - 30{,}935.20 = £52{,}786.48$$

Adding together the mortality profit from the three groups of lives:

$$MP = -63{,}542.65 + 43{,}370.95 + 52{,}786.48 = £32{,}615$$

We can see that the calculation of the mortality profit for policies of this type can be quite time-consuming.

In real life, as well as the class of policy we saw in the calculation above (where both lives were alive at the start of 2018), we could have had two further classes of policy – those for which the male life had died before the start of 2018, and those for which the female life had died before the start of 2018. However, both these classes of policy are effectively single life policies by now, and so the mortality profit is calculated in the usual way for policies of this type.

Question

In the previous question, the company paid out more claims than expected (1 actual claim as opposed to the 0.05 policies that are expected to claim during the year).

Explain briefly why the insurance company has still made an overall profit from mortality during the year.

Solution

For the policies where both lives die, there was a higher claim rate than expected, so the mortality here is heavier than expected.

However, for the policies where only one life died, mortality was lighter than expected (we expected 9.88 policies with the male life only dying, but observed only 3; we expected 5.41 policies with the female life only dying, but observed only 2).

So this lighter than expected mortality is sufficient to enable us to make an overall mortality profit.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes

Chapter 23 Summary

Single life and joint life (first death) policies

In general:

$$\text{DSAR} = S - R - {}_{t+1}V$$

$$\text{Total EDS} = \sum_{\text{all policies}} q_{x+t} \times \text{DSAR}$$

$$\text{Total ADS} = \sum_{\text{all deaths}} (S - R - {}_{t+1}V)$$

$$\text{Mortality profit} = \text{Total EDS} - \text{Total ADS}$$

where:

S is the sum assured paid on death during the year (revalued to the end of the year)

R is the sum paid on survival to the end of the year

${}_{t+1}V$ is the reserve at the end of the year

$x+t$ is the age at the start of the year

the EDS is calculated by summing over all policies in force at the *start* of the year.

Last survivor policies

To calculate the mortality profit for a last survivor policy, we need the death strain at risk for the different classes of lives subject to the different possible death events:

Death event	***Death strain at risk***
Both lives die	$S - {}_{t+1}V^{both\ alive}$
x dies, y survives	${}_{t+1}V^{y\ alive} - {}_{t+1}V^{both\ alive}$
y dies, x survives	${}_{t+1}V^{x\ alive} - {}_{t+1}V^{both\ alive}$

The EDS then uses the relevant probability for the death event, and the ADS is worked out using the actual numbers of each event that have occurred.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes

Chapter 23 Practice Questions

Exam style All of the following questions are exam style.

23.1 On 1 January 2018 a life insurance company sold a number of 10-year pure endowment policies, each with a benefit amount of £40,000, to lives then aged 30. Level premiums are payable annually in advance.

(i) Calculate the annual premium. [3]

(ii) On 1 January 2019, there were 50 of these policies still in force. During 2019, one policyholder died. Calculate the company's mortality profit for 2019. [5]

Basis: Mortality: AM92 Select

Interest: 4% *pa* effective

Expenses: None [Total 8]

23.2 On 31 December 2018 a pension scheme had 100 members aged 75 exact, each eligible for a pension of £10,000 *pa*, payable annually on each 1 January. In addition, the members were entitled to a death benefit of £20,000 payable at the end of the year of death. No premiums were being paid in respect of these contracts after December 2018. Given that 4 of the lives died during 2019, calculate the mortality profit for these contracts for calendar year 2019 using the following basis:

Mortality: PFA92C20

Interest: 4% *pa*

Expenses: none [5]

23.3 (i) Express in the form of symbols, and also explain in words, the expressions 'death strain at risk' as it applies to a single policy, and the 'expected death strain' and 'actual death strain' for a group of policies. [3]

(ii) On 1 January 2009 a life insurance company issued a number of annual premium policies to a group of lives, each of whom was then aged exactly 45. All policies were for a term of 20 years and were of the following types:

(a) endowment assurances under which the sum assured was payable on survival to the end of the term or at the end of the year of earlier death

(b) term assurances under which the sum assured was payable only at the end of the year of death within the policy term

(c) pure endowments under which the only benefit payable is the sum assured on survival to the end of the policy term.

Assuming that there is no source of decrement other than death, calculate the profit or loss from mortality for the calendar year 2018 in respect of the policies issued to this group of lives, given the following information:

Type of policy	*Sums assured in force on 1 January 2018*	*Sums assured discontinued by death during 2018*
Endowment assurance	£600,000	£4,000
Term assurance	£200,000	£2,000
Pure endowment	£80,000	£500

Basis: Mortality: AM92 Ultimate

Interest: 4% *pa* effective

Expenses: None

[11]
[Total 14]

23.4 An insurance company issues a special single premium annuity contract, which pays £10,000 *pa* in arrears for 10 years. If the policyholder dies within the 10-year term the annuity payments cease, and a lump sum benefit is paid out immediately on death. The amount of the death benefit is calculated as:

$$100{,}000-10{,}000k$$

where k is the curtate duration of the policy at the time of death.

The policy cannot be terminated for any reason other than through death.

1,500 of these policies were issued during a particular year to lives who were all aged exactly 55 when they took out the policy. It is now more than four years since the most recent of these policies was issued.

The mortality experience to date of this group of policyholders is given as follows:

- number of policyholders receiving exactly 2 or fewer annuity payments = 8
- number of policyholders receiving exactly 3 annuity payments = 4
- number of policyholders receiving exactly 4 or more annuity payments = 1,488

Calculate the mortality profit earned for the insurance company in the fourth policy year of this block of business, on the following basis:

Mortality: AM92 Ultimate

Interest: 4% *pa*

Expenses: None [10]

23.5 Under a 10-year 'double endowment' assurance policy issued to a group of lives aged 50, a sum assured of £10,000 is payable at the end of the year of death and £20,000 is paid if the life survives to the maturity date. Premiums are payable annually in advance.

You are given the following information:

reserve at the start of the 8th year (per policy in force):	£12,951
number of policies in force at the start of the 8th year:	200
number of deaths during the 8th year:	3
annual premium (per policy)	£1,591

(i) Assuming that reserves are calculated according to the basis specified below, calculate the profit or loss arising from mortality in the 8th year. [5]

(ii) Comment on your results. [1]

Basis: Mortality: ELT15 (Males)

Interest: 4% *pa* effective

Expenses: None [Total 6]

23.6 On 1 January 2012, a life insurance company issued joint life whole life assurance policies. Each policy was issued to a male life aged 65 exact and a female life aged 60 exact. A sum assured of 75,000 is payable immediately on the death of the second of the lives to die.

Premiums of 1,395.11 are payable annually in advance for each policy while at least one of the lives is alive.

At the beginning of 2014, there were 5,997 policies in force. For all of these policies, both lives were still alive. During 2014, the following experience was observed:

- for 2 policies, both lives died
- for 12 policies, only the male life died
- for 8 policies, only the female life died.

Calculate, showing all your workings, the mortality profit or loss for the group of policies for the calendar year 2014.

Basis: Mortality: PMA92C20 for the male
PFA92C20 for the female

Interest: 4% per annum

Expenses: Ignore [10]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 23 Solutions

23.1 (i) ***Annual premium***

Let P denote the annual premium. Then:

$$\text{EPV premiums} = P\ddot{a}_{[30]:\overline{10}|} = P\left(\ddot{a}_{[30]} - \frac{D_{40}}{D_{[30]}}\ddot{a}_{40}\right)$$ [½]

From the *Tables*:

$$\frac{D_{40}}{D_{[30]}} = \frac{2,052.96}{3,059.68} = 0.67097$$ [½]

$$\ddot{a}_{[30]} = 21.837 \text{ and } \ddot{a}_{40} = 20.005$$ [½]

So:

$$\text{EPV premiums} = P(21.837 - 0.67097 \times 20.005) = 8.414P$$ [½]

Also:

$$\text{EPV benefits} = 40,000\frac{D_{40}}{D_{[30]}} = 26,838.89$$ [½]

So the annual premium is:

$$P = \frac{26,838.89}{8.414} = £3,189.71$$ [½]

[Total 3]

(ii) ***Mortality profit***

The reserve per policy in force at the end of 2019 is:

$${}_2V = 40,000\frac{D_{40}}{D_{32}} - 3,189.71\ddot{a}_{32:\overline{8}|}$$ [½]

From the *Tables*:

$$\frac{D_{40}}{D_{32}} = \frac{2,052.96}{2,825.89} = 0.72648$$ [½]

and:

$$\ddot{a}_{32:\overline{8}|} = \ddot{a}_{32} - \frac{D_{40}}{D_{32}}\ddot{a}_{40} = 21.520 - 0.72648 \times 20.005 = 6.987$$ [1]

So:

$${}_{2}V = 40{,}000 \times 0.72648 - 3{,}189.71 \times 6.987 = 6{,}773.71$$ [½]

There is no death benefit, so the death strain at risk (DSAR) for calendar year 2019 is:

$$DSAR = S - {}_{2}V = 0 - 6{,}773.71 = -6{,}773.71$$ [½]

The expected death strain (EDS) is:

$$EDS = 50q_{[30]+1} \times DSAR = 50 \times 0.000569 \times (-6{,}773.71) = -192.71$$ [1]

and the actual death strain (ADS) is:

$$ADS = 1 \times DSAR = -6{,}773.71$$ [½]

So the company's mortality profit for calendar year 2019 is:

$$EDS - ADS = -192.71 - (-6{,}773.71) = £6{,}581$$ [½]

[Total 5]

23.2 The death strain at risk (DSAR) for a single contract for calendar year 2019 is:

$$20{,}000 - {}_{31.12.19}V$$ [½]

where:

$$\begin{aligned} {}_{31.12.19}V &= 10{,}000\ddot{a}_{76} + 20{,}000A_{76} \\ &= 10{,}000\ddot{a}_{76} + 20{,}000(1 - d\ddot{a}_{76}) \\ &= 10{,}000 \times 10.536 + 20{,}000\left(1 - \frac{0.04}{1.04} \times 10.536\right) \\ &= 117{,}255.38 \end{aligned}$$ [2]

So the DSAR is:

$$20{,}000 - 117{,}255.38 = -97{,}255.38$$ [½]

The expected number of deaths during 2019 is:

$$100q_{75} = 100 \times 0.019478 = 1.9478$$ [½]

So the expected death strain is:

$$EDS = 100q_{75} \times DSAR = -189{,}434.04$$ [½]

The actual number of deaths during 2019 is 4. So the actual death strain is:

$$ADS = 4 \times DSAR = -389{,}021.54$$ [½]

Hence the mortality profit for the group of policies for 2019 is:

$$EDS - ADS = £199,587$$ [½]

[Total 5]

23.3 (i) ***Definitions***

The 'death strain at risk' for a policy for year $t+1$ (*ie* the year beginning at time t and ending at time $t+1$, $t=0,1,2,\ldots$) is the excess of the sum assured (*ie* the value at time $t+1$ of all benefits payable on death during year $t+1$) over the end-of-year reserve. [1]

The 'expected death strain' for a group of policies for year $t+1$, is the total death strain that would be incurred in respect of all policies in force at the start of year $t+1$ if deaths conformed to the numbers expected.

$$\text{EDS for year } t+1 = \sum_{\substack{\text{policies in force}\\ \text{at start of year}}} q(S - {}_{t+1}V)$$ [1]

The 'actual death strain' for a group of policies for year $t+1$ is the total death strain incurred in respect of all claims actually arising during year $t+1$.

$$\text{ADS for year } t+1 = \sum_{\text{claims during year}} (S - {}_{t+1}V)$$ [1]

[Total 3]

(ii) ***Mortality profit***

The premiums per unit sum assured for the three types of policies can be found as follows:

$$P_a \ddot{a}_{45:\overline{20}|} = A_{45:\overline{20}|} \Rightarrow P_a = 0.46998/13.780 = 0.03411$$ [1]

$$P_b \ddot{a}_{45:\overline{20}|} = A^{1}_{45:\overline{20}|}$$

where:

$$A^{1}_{45:\overline{20}|} = A_{45:\overline{20}|} - \frac{D_{65}}{D_{45}} = 0.46998 - \frac{689.23}{1,677.97} = 0.05923$$

$$\Rightarrow P_b = 0.05923/13.780 = 0.00430$$ [2]

$$P_c = P_a - P_b = 0.02981$$ [½]

The reserves at the end of the year per unit sum assured are:

$${}_{10}V_a = A_{55:\overline{10|}} - P_a \ddot{a}_{55:\overline{10|}} = 0.68388 - 0.03411 \times 8.219 = 0.4036$$ [1]

$${}_{10}V_b = A^{1}_{55:\overline{10|}} - P_b \ddot{a}_{55:\overline{10|}} = 0.06037 - 0.00430 \times 8.219 = 0.02505$$ [1]

$${}_{10}V_c = \frac{D_{65}}{D_{55}} - P_c \ddot{a}_{55:\overline{10|}} = 0.62351 - 0.02981 \times 8.219 = 0.3785$$ [1]

The total expected death strain is:

$$\begin{aligned} EDS &= EDS_a + EDS_b + EDS_c \\ &= q_{54}[600{,}000(1 - {}_{10}V_a) + 200{,}000(1 - {}_{10}V_b) + 80{,}000(0 - {}_{10}V_c)] \\ &= 0.003976[600{,}000(1 - 0.4036) + 200{,}000(1 - 0.02505) + 80{,}000(-0.3785)] \\ &= 2{,}078 \end{aligned}$$ [2]

The total actual death strain is:

$$\begin{aligned} ADS &= ADS_a + ADS_b + ADS_c \\ &= 4{,}000(1 - 0.4036) + 2{,}000(1 - 0.02505) + 500(-0.3785) \\ &= 4{,}146 \end{aligned}$$ [2]

So there is a profit of $2{,}078 - 4{,}146 = -2{,}069$, *ie* a loss of £2,069. [½]

[Total 11]

23.4 If the policyholder dies in the fourth policy year, the curtate duration will equal 3. So the sum assured payable immediately on death during this year is:

$$100{,}000 - 3 \times 10{,}000 = 70{,}000$$

In the death strain at risk, this needs accumulating to the end of the year. So, assuming deaths occur on average half way through the year, we need to multiply this by $1.04^{½}$. From this we need to deduct the payment made on survival at the end of the year (which is the annuity payment of 10,000) and the reserve at the end of the year. So the death strain at risk in the fourth policy year is:

$$DSAR = 70{,}000 \times 1.04^{½} - 10{,}000 - {}_4V$$ [2]

where ${}_4V$ is the reserve at the end of year 4.

There are no future premiums or expenses, so the reserve is equal to the expected present value (EPV) at time 4 (when the policyholder is aged 59 exact) of the future death benefits and annuity benefits.

The death benefit in year 5 is 60,000; in year 6, it is 50,000, and so on, decreasing by 10,000 each year until by year 10 it has reduced to 10,000. We can calculate the EPV of these benefits by deducting an increasing term assurance with 10,000 *pa* increases from a level term assurance with a sum assured of 70,000. So, including the EPV of the remaining annuity payments, the reserve becomes:

$$_4V = 70{,}000\bar{A}^{\,1}_{59:\overline{6}|} - 10{,}000(I\bar{A})^{\,1}_{59:\overline{6}|} + 10{,}000a_{59:\overline{6}|}$$ [1]

To work out these factors, we will need:

$$\frac{D_{65}}{D_{59}} = \frac{689.23}{924.76} = 0.74531$$

Now:

$$\bar{A}^{\,1}_{59:\overline{6}|} \approx (1+i)^{½}\left(A_{59} - \frac{D_{65}}{D_{59}}A_{65}\right)$$

$$= 1.04^{½} \times (0.44258 - 0.74531 \times 0.52786)$$

$$= 0.050136$$ [1]

$$(I\bar{A})^{\,1}_{59:\overline{6}|} \approx (1+i)^{½}\left\{(IA)_{59} - \frac{D_{65}}{D_{59}}\left[(IA)_{65} + 6A_{65}\right]\right\}$$

$$= 1.04^{½} \times \left(8.42588 - 0.74531 \times \left[7.89442 + 6 \times 0.52786\right]\right)$$

$$= 0.185205$$ [1½]

$$a_{59:\overline{6}|} = (\ddot{a}_{59} - 1) - \frac{D_{65}}{D_{59}}(\ddot{a}_{65} - 1) = 13.493 - 0.74531 \times 11.276 = 5.089$$ [1]

So:

$$_4V = 70{,}000 \times 0.050136 - 10{,}000 \times 0.185205 + 10{,}000 \times 5.089 = 52{,}546.66$$ [½]

Hence:

$$DSAR = 70{,}000 \times 1.04^{½} - 10{,}000 - 52{,}546.66 = 8{,}839.61$$ [½]

8 of the initial 1,500 policyholders received two or fewer annuity payments, which means that 8 policyholders died during the first three policy years. So there were 1,492 policies still in force at the start of the fourth policy year, when the policyholders were all aged 58. [½]

So the expected death strain was:

$$EDS = 1{,}492 \times q_{58} \times DSAR = 1{,}492 \times 0.006352 \times 8{,}839.61 = 83{,}775$$ [1]

4 policyholders received exactly 3 annuity payments, which means they must have died during the fourth policy year. So the actual death strain was:

$$ADS = 4 \times 8,839.61 = 35,358 \quad [½]$$

and the mortality profit was:

$$EDS - ADS = 83,775 - 35,358 = £48,416 \quad [½]$$

[Total 10]

23.5 (i) ***Mortality profit***

The reserve required (per policy) at the end of the 8th year can be found from the equation of equilibrium:

$$1.04 \times \left({}_7V + P \right) = q_{57} \times 10,000 + p_{57} \times {}_8V \quad [1]$$

Inserting the values gives:

$$1.04 \times (12,951 + 1,591) = 0.00995 \times 10,000 + 0.99005 \times {}_8V$$

So:

$${}_8V = 15,024.18 / 0.99005 = 15,175.17 \quad [1]$$

The expected death strain is:

$$200 q_{57} (10,000 - {}_8V) = 1.99(10,000 - 15,175.17) = -10,298.59 \quad [1]$$

The actual death strain is:

$$3(10,000 - {}_8V) = 3(10,000 - 15,175.17) = -15,525.52 \quad [1]$$

So the mortality profit for the year is:

$$EDS - ADS = -10,298.59 - (-15,525.52) = £5,227 \quad [1]$$

[Total 5]

(ii) ***Comment***

In this case the reserve exceeds the death benefit, so the company makes a profit when people die. More people than expected died, so the result is a mortality profit. [1]

23.6 *This question is Subject CT5, April 2016, Question 11.*

We need the mortality profit for calendar year 2014, at the start of which all male and female policyholders were aged exactly 67 and 62 respectively.

End-year reserve required for the death strain at risk

Let $V_{\overline{(xy)}}$ denote the reserve at the end of 2014 (*ie* at time 3) if both policyholders are then still alive. This is the reserve required where no policyholder dies during the year, and is calculated as:

$$V_{\overline{(xy)}} = 75{,}000\,\bar{A}_{\overline{68:63}} - 1{,}395.11\,\ddot{a}_{\overline{68:63}} \qquad [½]$$

where (68) and (63) indicate male and female lives respectively.

Now:

$$\ddot{a}_{\overline{68:63}} = \ddot{a}_{68} + \ddot{a}_{63} - \ddot{a}_{68:63} = 12.412 + 15.606 - 11.372 = 16.646 \qquad [½]$$

So:

$$\bar{A}_{\overline{68:63}} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{\overline{68:63}}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 16.646\right) = 0.366894 \qquad [½]$$

$$\Rightarrow \quad V_{\overline{(xy)}} \approx 75{,}000 \times 0.366894 - 1{,}395.11 \times 16.646 = 4{,}294.05$$

Mortality profit

Policies where both policyholders die during the year

The cost to the insurer of both policyholders dying during the year, accumulated to the end of the year, is:

$$75{,}000 \times 1.04^{½}$$

So the death strain at risk for this event is:

$$DSAR_{(\text{both})} = 75{,}000 \times 1.04^{½} - 4{,}294.05 = 72{,}191.24 \qquad [½]$$

The expected death strain, totalled over all 5,997 policies in force at the start of the year, is:

$$\begin{aligned} EDS_{(\text{both})} &= 5{,}997 \times q_{67} \times q_{62} \times DSAR_{(\text{both})} \\ &= 5{,}997 \times 0.008439 \times 0.002885 \times 72{,}191.24 \\ &= 10{,}540.36 \qquad [½] \end{aligned}$$

where (67) and (62) denote male and female lives respectively.

The actual death strain is:

$$ADS_{(\text{both})} = 2 \times DSAR_{(\text{both})} = 2 \times 72{,}191.24 = 144{,}382.48 \qquad [½]$$

The mortality profit is then:

$$MP_{(\text{both})} = EDS_{(\text{both})} - ADS_{(\text{both})} = 10,540.36 - 144,382.48$$

$$= -133,842.12$$ [½]

Policies where only the male dies during the year

The cost to the insurer of just the male life dying during the year, is the reserve at the end of the year where only the female is alive. Denoting $V_{(y)}$ to be this reserve, we have:

$$V_{(y)} = 75,000\,\bar{A}_{63} - 1,395.11\ddot{a}_{63}$$ [½]

Now:

$$\bar{A}_{63} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{63}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 15.606\right) = 0.407686$$ [½]

So:

$$V_{(y)} \approx 75,000 \times 0.407686 - 1,395.11 \times 15.606 = 8,804.38$$

So the death strain at risk for this event is:

$$DSAR_{(\text{male only})} = 8,804.38 - 4,294.05 = 4,510.33$$ [½]

The expected death strain is:

$$EDS_{(\text{male only})} = 5,997 \times q_{67} \times \left(1 - q_{62}\right) \times DSAR_{(\text{male only})}$$

$$= 5,997 \times 0.008439 \times \left(1 - 0.002885\right) \times 4,510.33$$

$$= 227,603.13$$ [½]

The actual death strain is:

$$ADS_{(\text{male only})} = 12 \times DSAR_{(\text{male only})} = 12 \times 4,510.33 = 54,123.91$$ [½]

The mortality profit is then:

$$MP_{(\text{male only})} = EDS_{(\text{male only})} - ADS_{(\text{male only})}$$

$$= 227,603.13 - 54,123.91 = 173,479.22$$ [½]

Policies where only the female dies during the year

The cost to the insurer of just the female life dying during the year, is the reserve at the end of the year where only the male is alive. Denoting $V_{(x)}$ to be this reserve, we have:

$$V_{(x)} = 75{,}000\,\bar{A}_{68} - 1{,}395.11\,\ddot{a}_{68} \qquad [½]$$

where:

$$\bar{A}_{68} \approx 1.04^{½} \times \left(1 - d\,\ddot{a}_{68}\right) = 1.04^{½} \times \left(1 - \frac{0.04}{1.04} \times 12.412\right) = 0.532965 \qquad [½]$$

So:

$$V_{(x)} \approx 75{,}000 \times 0.532965 - 1{,}395.11 \times 12.412 = 22{,}656.29$$

So the death strain at risk for this event is:

$$DSAR_{\text{(female only)}} = 22{,}656.29 - 4{,}294.05 = 18{,}362.23 \qquad [½]$$

The expected death strain is:

$$EDS_{\text{(female only)}} = 5{,}997 \times \left(1 - q_{67}\right) \times q_{62} \times DSAR_{\text{(female only)}}$$

$$= 5{,}997 \times \left(1 - 0.008439\right) \times 0.002885 \times 18{,}362.23$$

$$= 315{,}010.30 \qquad [½]$$

The actual death strain is:

$$ADS_{\text{(female only)}} = 8 \times DSAR_{\text{(female only)}} = 8 \times 18{,}362.23 = 146{,}897.85 \qquad [½]$$

The mortality profit is then:

$$MP_{\text{(female only)}} = EDS_{\text{(female only)}} - ADS_{\text{(female only)}}$$

$$= 315{,}010.30 - 146{,}897.85 = 168{,}112.45 \qquad [½]$$

So the total mortality profit for all possible death events is:

$$MP_{\text{(both)}} + MP_{\text{(male only)}} + MP_{\text{(female only)}}$$

$$= -133{,}842.12 + 173{,}479.22 + 168{,}112.45 = 207{,}750 \qquad [½]$$

[Total 10]

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Competing risks

Syllabus objectives

5.2 Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events.

- 5.2.1 Define health insurance, and describe simple health insurance premium and benefit structures.
- 5.2.2 Explain how a cashflow, contingent upon multiple transition events, may be valued using a multiple-state Markov model, in terms of the forces and probabilities of transition.
- 5.2.3 Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

5.3 Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events.

5.3.1 Describe the construction and use of multiple decrement tables.

5.3.2 Define a multiple decrement model as a special case of a multiple state Markov model.

5.3.3 Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

5.3.4 Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

0 Introduction

So far we have considered contingencies where a life is exposed to death only. If we suppose that a life is subject to more than one transition, then the transitions are referred to as a set of *competing risks*. For example, a member of a pension scheme can, in order for the associated pension scheme benefits to be valued, be regarded as exposed to the competing risks of retirement and death.

In a similar way, a person with a health insurance policy who is in good health, can be considered as exposed to the competing risks of becoming sick and dying.

Some of the ideas in this chapter involve concepts that are also covered in Subject CS2.

1 Health insurance contracts

In the same way as insurance contracts exist that pay benefits contingent upon death or survival, so contracts also exist that pay benefits contingent upon the state of health of a person. In this case a policyholder can be considered to be subject to the competing risks of death and of becoming sick.

An *income protection* insurance contract pays an income to the policyholder while that policyholder is deemed as being 'sick' (with the definition of sickness being carefully specified in the policy conditions). If the policyholder recovers, the cover under the policy usually continues, so that subsequent bouts of qualifying sickness would merit further benefit payments.

Such policies are usually subject to a *deferred period* (*eg* 3 months) of continuous sickness that has to have elapsed before any benefits start to be paid, and during which no benefit is payable.

Premiums for these policies would normally be regular (*eg* monthly) and would typically be *waived* during periods of qualifying sickness. This means that premiums would not be paid at the same time as benefits are payable.

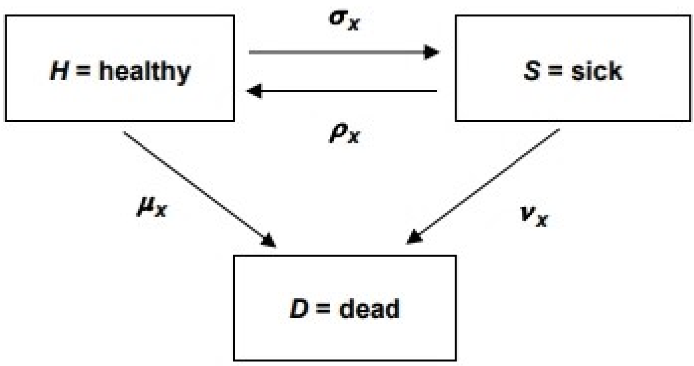
There are many varieties of insurance contracts that can be based on health-dependent contingencies. Other examples (covered in Subject SP1) include:

- ***critical illness* insurance, which normally pays a lump sum on diagnosis of a defined 'critical' illness (such as cancer); and**
- ***long-term care* insurance, which pays an income contingent upon the policyholder requiring long-term care, and hence supports the costs of receiving that care.**

2 Multiple state models

Multiple state models are well suited to valuing cashflows that are dependent on multiple transitions, such as of health insurance contracts. The model will be chosen to include the relevant states, and transitions between states, that are necessary to replicate the required cashflows for the contract concerned.

For example, for the simple income protection policy described in the previous section (with no deferred period), the general three-state healthy-sick-dead model would be suitable.



The labels on the arrows relate to the *transition intensities* from one state to another. A transition intensity can also be referred to as a *transition rate* or a *force of transition*, and so is the same type of quantity as the force of mortality that we met in earlier chapters.

Here, we use μ to refer specifically to the force of transition from the healthy state to the dead state (rather than the force of transition from 'alive' to 'dead', as is it used earlier in this course). The transition intensity from the *sick* state to the *dead* state is represented by ν (nu, the 13th letter of the Greek alphabet), not v (the 22nd letter of the English alphabet).

2.1 Notation

Let i and j denote any two different states. Define μ_x^{ij} to be the transition intensity from state i to state j at age x (so, for example, $\mu_x^{HS} = \sigma_x$ in the above model). Also define the related transition probability:

$$ {}_tp_x^{ij} = P\big[\text{in state } j \text{ at age } x+t \mid \text{ in state } i \text{ at age } x\big]$$

where now i and j need not be different.

For example, for the model shown above:

${}_tp_x^{HD}$ represents the probability that a life in the *healthy* state at age x will be in the *dead* state at age $x+t$. This probability encompasses all possible routes from *healthy* to *dead*, which may or may not include one or more visits to the *sick* state.

${}_tp_x^{SS}$ represents the probability that a life in the *sick* state at age x will be in the *sick* state at age $x+t$. This includes the probability that the life remained in the *sick* state throughout the period from x to $x+t$ and the probability that the life made one or more visits to the *healthy* state, returning on each occasion to the *sick* state.

The event whose probability is defined by the expression:

$${}_tp_x^{ij} = P\left[\text{in state } j \text{ at age } x+t \mid \text{in state } i \text{ at age } x\right]$$

does not specify what must happen between age x and age $x+t$, however. In particular, if $i=j$, it does not require that the life remains in state i between these ages. So for any state i, also define the related transition probability:

$${}_tp_x^{\overline{ii}} = P\left[\text{in state } i \text{ from age } x \text{ to } x+t \mid \text{in state } i \text{ at age } x\right]$$

This is sometimes referred to as the *occupancy probability*, as it relates to the probability of staying in (or occupying) state i from age x to age $x+t$.

If return to state i is impossible, then ${}_tp_x^{\overline{ii}} = {}_tp_x^{ii}$, but this is not true (for example) in the case of states H and S in the healthy-sick-dead model above.

Question

A life insurance company uses the three-state healthy-sick-dead model described above to calculate premiums for a 3-year sickness policy issued to healthy policyholders aged 60.

Let S_t denote the state occupied by the policyholder at age $60+t$, so that $S_0 = H$ and $S_t = H, S \text{ or } D$ for $t = 1,2,3$.

The transition probabilities used by the insurer are defined in the following way:

$$p_{60+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right)$$

For $t = 0,1,2$, it is assumed that:

$$p_{60+t}^{HH} = 0.9 \qquad p_{60+t}^{HS} = 0.08$$

$$p_{60+t}^{SH} = 0.7 \qquad p_{60+t}^{SS} = 0.25$$

Calculate the probability that a new policyholder is:

(a) sick at exact age 62.

(b) dead at exact age 62.

Solution

(a) Since the policyholder is healthy at age 60, the probability that he is sick at age 62 is:

$$
\begin{aligned}
{}_2p_{60}^{HS} &= \left(p_{60}^{HH} \times p_{61}^{HS}\right) + \left(p_{60}^{HS} \times p_{61}^{SS}\right) \\
&= (0.9 \times 0.08) + (0.08 \times 0.25) \\
&= 0.092
\end{aligned}
$$

(b) The probability that the policyholder is dead at age 62 is:

$$
\begin{aligned}
{}_2p_{60}^{HD} &= p_{60}^{HD} + \left(p_{60}^{HH} \times p_{61}^{HD}\right) + \left(p_{60}^{HS} \times p_{61}^{SD}\right) \\
&= 0.02 + (0.9 \times 0.02) + (0.08 \times 0.05) \\
&= 0.042
\end{aligned}
$$

Note that $p_{60}^{HH} + p_{60}^{HS} + p_{60}^{HD} = 1$, and a similar equation holds for lives starting from the sick state.

In CM1, we assume that:

- **the above probabilities, as well as the transition intensities, are available**
- **the differential equation for ${}_tp_x^{\overline{ii}}$ has the closed form solution:**

$$
{}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} \, ds\right)
$$

This result is particularly important in the construction of multiple decrement models.

Multiple decrement models are described in Section 3 of this chapter.

The differential equation for ${}_tp_x^{\overline{ii}}$ referred to above is derived in Subject CS2, but it is not required here. In this course, we will just use the solution provided.

In the formula for ${}_tp_x^{\overline{ii}}$, the term $\sum_{j \neq i} \mu_{x+s}^{ij}$ relates to the *total* force of transition out of state i at age $x+s$. If the forces of transition are constant over the period, and written as μ^{ij}, the formula for the occupancy probability simplifies to:

$$
{}_tp_x^{\overline{ii}} = \exp\left(-t \times \sum_{j \neq i} \mu^{ij}\right)
$$

Question

Using the three-state healthy-sick-dead model described above with transition intensities:

$$\sigma_{45+t} = 0.001t \qquad \mu_{45+t} = 0.002t$$

$$\rho_{45+t} = 0.002t^2 \qquad \nu_{45+t} = 0.01$$

calculate the probability that a life remains healthy from age 45 to age 50.

Solution

Using the form of the occupancy probability given above:

$$_5p_{45}^{\overline{HH}} = \exp\left(-\int_0^5 (\sigma_{45+t} + \mu_{45+t})\,dt\right) = \exp\left(-\int_0^5 0.003t\,dt\right)$$

Carrying out the integration gives:

$$_5p_{45}^{\overline{HH}} = \exp\left(-\left[0.0015t^2\right]_0^5\right) = e^{-0.0015\times 25} = e^{-0.0375} = 0.96319$$

2.2 Valuing continuous cashflows using multiple state models

Consider, for example, a healthy life who is subject to the competing risks of sickness and death. A multiple state model can be used to construct integral expressions for the EPVs of the following types of cashflows:

- **a lump sum paid immediately on transition from one state to another** (Type 1)
- **an income payable while occupying a particular state** (Type 2).

Examples

The healthy-sick-dead model described in the previous section will be used.

1. **The EPV of a lump sum of 1 payable on death (whether directly from healthy or from having first become sick) of a healthy life currently aged x is:**

$$\int_0^\infty e^{-\delta t} ({}_tp_x^{HH}\mu_{x+t} + {}_tp_x^{HS}\nu_{x+t})\,dt$$

(assuming a constant force of interest δ).

This is a Type 1 cashflow. We can think about the integral as being built up as follows:

- The benefit of 1 is payable at time t if the policyholder dies at time t. This may be from the healthy state (in which case the policyholder is healthy at time t, and dies from this state at age $x+t$) or from the sick state (in which case the policyholder is sick at time t and dies from the sick state at age $x+t$).
- The benefit is then discounted back to time 0.
- The expression is then integrated over all possible points in time when death might occur.

2. The EPV of an annuity of 1 per annum payable continuously during sickness of a healthy life currently aged x is:

$$\int_0^\infty e^{-\delta t}\,{}_tp_x^{HS}\,dt$$

This is a Type 2 cashflow. Here we can think about a benefit being paid at time t provided that the life is sick at time t. So we need the probability that the currently healthy life is sick at time t. The benefit amount of 1 is then discounted back to time 0, and we integrate over all points in time at which a benefit could be paid.

3. The EPV of a premium of 1 per annum payable continuously, but waived during periods of sickness, by a healthy life currently aged x is:

$$\int_0^\infty e^{-\delta t}\,{}_tp_x^{HH}\,dt$$

This is also a Type 2 cashflow. The premium is payable at time t only by someone who is healthy at time t. So we need the probability that the currently healthy life is also healthy at time t, discounted back to time 0, and then integrated over all points in time during which the premium could be paid.

Question

Using the three-state healthy-sick-dead model and the notation defined in Section 2.1, write down an expression for the expected present value of each of the following sickness benefits for a healthy life aged 30.

(i) £3,000 *pa* payable continuously while sick, but ceasing at age 60.

(ii) £3,000 *pa* payable continuously throughout the first period of sickness only, but ceasing at age 60.

(iii) £3,000 *pa* payable continuously while sick provided that the life has been sick for at least one year. Again, any benefit ceases to be paid at age 60.

Solution

In all of these solutions v^t could be replaced by $e^{-\delta t}$.

(i) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{30} v^t\, {}_tp_{30}^{HS}\, dt$$

This is very similar to Example 2 above, except that the upper limit on the integral is 30, as the payments continue for at most 30 years.

(ii) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{30} v^t\, {}_tp_{30}^{\overline{HH}}\, \sigma_{30+t}\, \bar{a}^S_{30+t:\overline{30-t}|}\, dt$$

$$=3{,}000\int_0^{30} v^t\, {}_tp_{30}^{\overline{HH}}\, \sigma_{30+t}\left(\int_0^{30-t} v^u\, {}_up_{30+t}^{\overline{SS}}\, du\right)dt$$

*This can be reasoned as follows. The 'probability' that the policyholder falls sick for the first time at age $30+t$ is ${}_tp_{30}^{\overline{HH}}\,\sigma_{30+t}$ (where the use of the occupancy probability ensures that this is the **first** transition to the sick state).*

Given that this has occurred, sickness benefits will be received continuously from age $30+t$ until the policyholder recovers or reaches age 60, whichever is earlier. The expected value at time t of this benefit can be written as $3{,}000\bar{a}^S_{30+t:\overline{30-t}|}$ where the S superscript indicates that this annuity will be payable whilst the life remains in the sick state. The annuity can then be expressed in integral form to give $3{,}000\int_0^{30-t} v^u\, {}_up_{30+t}^{\overline{SS}}\, du$.

Discounting back to time 0 and integrating over all possible values of t gives the required result.

(iii) The expression for the EPV of this benefit is:

$$3{,}000\int_0^{29} v^t\, {}_tp_{30}^{HH}\, \sigma_{30+t}\left(v\, p_{30+t}^{\overline{SS}}\, \bar{a}^S_{31+t:\overline{29-t}|}\right)dt$$

$$=3{,}000\int_0^{29} v^t\, {}_tp_{30}^{HH}\, \sigma_{30+t}\left(\int_1^{30-t} v^u\, {}_up_{30+t}^{\overline{SS}}\, du\right)dt$$

*This can be reasoned in a similar way. The 'probability' that the policyholder falls sick at time t (ie age $30+t$) is ${}_tp_{30}^{HH}\,\sigma_{30+t}$ (where we do not use an occupancy probability in this case, as the payments are made during **all** bouts of sickness, not just the first).*

Given that this occurs, if the policyholder stays sick for one year (ie until age $30+t+1$ or age $31+t$), then sickness benefits will be received continuously until the policyholder recovers or reaches age 60, whichever is earlier.

The expected value at age $31+t$ of this benefit can be written as $3{,}000\,\bar{a}^{S}_{31+t:\overline{29-t|}}$, and the expected value at age $30+t$ (or time t) is $3{,}000\,v\,p^{\overline{SS}}_{30+t}\,\bar{a}^{S}_{31+t:\overline{29-t|}}$. Writing the annuity in integral form gives:

$$3{,}000\,v\,p^{\overline{SS}}_{30+t}\int_0^{29-t} v^w\,{}_wp^{\overline{SS}}_{31+t}\,dw = 3{,}000\int_0^{29-t} v^{w+1}\,{}_{w+1}p^{\overline{SS}}_{30+t}\,dw$$

$$= 3{,}000\int_1^{30-t} v^u\,{}_up^{\overline{SS}}_{30+t}\,du$$

using the substitution $u=w+1$. The final integral here reflects the fact that benefits will be received if the policyholder is sick at age $30+t+u$, where $1\le u<30-t$.

Discounting back to time 0 and integrating over all possible values of t gives the required result. If the policyholder falls sick after age 59, there will be no benefit payable, so the upper limit for t is 29.

The actual evaluation of the above integrals may, in general, be done numerically.

For example, we could use a numerical technique such as the trapezium rule to approximate the value of an integral. The trapezium rule says that if we want to integrate a function $f(t)$ between the limits of $t=a$ and $t=b$, we divide the interval $[a,b]$ into n strips of equal length $h=\dfrac{b-a}{n}$ and use the formula:

$$\int_a^b f(t)\,dt \approx \frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+\cdots+2f(b-h)+f(b)\right]$$

Question

We consider a simple income protection insurance contract that pays at a rate of 20,000 per annum continuously while a policyholder is sick. The policy is issued to a healthy life aged 50 exact for a term of three years. Calculate an approximate EPV of the sickness benefit at outset on the following assumptions:

$${}_1p^{HS}_{50}=0.02 \qquad {}_2p^{HS}_{50}=0.04 \qquad {}_3p^{HS}_{50}=0.07$$

Interest of 3% per annum

(with symbols as defined in Section 2.1).

Solution

As an integral the EPV is:

$$\text{EPV} = \int_0^3 20{,}000\,e^{-\delta t}\,{}_tp^{HS}_{50}\,dt = 20{,}000\int_0^3 v^t\,{}_tp^{HS}_{50}\,dt$$

If we assume the function $K_x = k$ varies linearly over each policy year, we can approximate the integral using the trapezium rule:

$$\text{EPV} \approx 20{,}000\left(\tfrac{1}{2}v^0\,{}_0p_{50}^{HS} + v^1\,{}_1p_{50}^{HS} + v^2\,{}_2p_{50}^{HS} + \tfrac{1}{2}v^3\,{}_3p_{50}^{HS}\right)$$

$$= 20{,}000\left(0 + \frac{0.02}{1.03} + \frac{0.04}{1.03^2} + \tfrac{1}{2}\times\frac{0.07}{1.03^3}\right)$$

$$= 1{,}783$$

(Other suitable approximations for the integral could be used).

The integral $\int_0^3 v^t\,{}_tp_{50}^{HS}\,dt$ has been evaluated using the form of the trapezium rule stated before the question with $h=1$, $a=0$, $b=3$ and $f(t)=v^t\,{}_tp_{50}^{HS}$.

2.3 Designing the multiple state model

An important actuarial skill is to be able to choose, or design, an appropriate model for a particular purpose or application. Here, we need to be able to design or select a multiple state model for the purpose of valuing cashflows, where the cashflows depend on life and/or health-dependent events.

Defining the model

A multiple state Markov model, of the type we have been using in this chapter, is fully defined by the following:

- the possible states that can be occupied
- the possible transitions that can be made between the states (*ie* all the arrows in the transition diagram)
- the values of the forces of transition between the states at each age.

Essentially, defining a multiple state Markov model involves drawing the transition diagram, indicating (and defining as necessary) the forces of transition that will apply on each arrow. So, for example, the transition diagram shown at the start of Section 2 above fully defines the multiple state model that is needed for valuing the cashflows covered in this chapter so far. In practice, of course, we will also need numerical values for all the transition rates in the model.

How to design a model

In order to decide on an appropriate model, we need to consider *which transitions affect the expected present value of the cashflows involved*. So, for example, the expected present value of the income benefits paid under a typical income protection policy (described in Section 1 of this chapter) will fairly obviously be affected by any:

- transition from healthy to sick (because this will cause payments to start)
- transition from sick to healthy (because this will cause payments to stop)
- transition from sick to dead (because this will cause payments to stop).

Question

State two other transitions that, in real life, might affect the expected present value of the sickness benefits for a new policyholder who is currently healthy. Assume that the policyholder is to pay regular premiums.

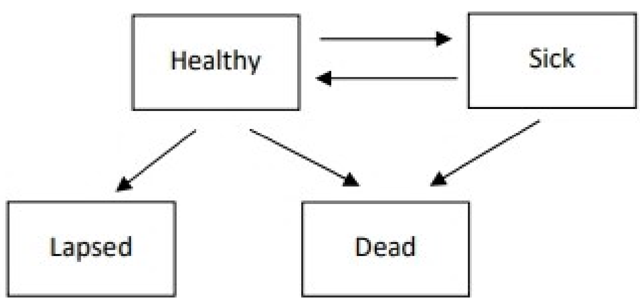
Solution

The expected present value of the sickness payments will also be affected by any:

- transition from healthy to dead
- transition from healthy to lapsed (*ie* where cover under the contract ceases as the policyholder has stopped paying premiums).

This is because both of these transitions would mean that the policyholder could not receive a sickness benefit in the future, reducing the expected present value of the sickness benefit payments.

So, in order to include all of the necessary transitions, the model will require four states (Healthy, Sick, Dead, Lapsed), with transitions between the states as shown in the following diagram:



To complete the model, we would also need to define:

$$\mu_x^{ij} = \text{force of transition from state } i \text{ to state } j \ (i \neq j) \text{ at exact age } x$$

Question

(i) Explain what would happen to the expected present value of the sickness payments if lapses were ignored in the model (and we just used the standard healthy-sick-dead model).

(ii) Give an example of a situation in which we could we justify ignoring lapses in practice.

Solution

(i) If we did not include lapses in the model, then the expected present value of the future benefit payments would be increased. This is because more policies would be assumed to stay in force to each future time point, which increases the chance of a sickness payment being made.

(ii) We might be justified in doing this if, for example, we wished to take a conservative (*ie* prudent) view of the expected present value of these benefit payments.

So, while it might be permissible to exclude lapses from the model in certain situations, *including* them allows us to value the sickness payments more accurately. That is, the model as defined above will enable us to perform a more *realistic* valuation of the sickness benefits.

Question

Explain whether adding a transition from lapsed to dead in the above model would increase the accuracy of the expected present value of the sickness benefit payments calculated.

Solution

Adding the extra transition from lapsed to dead into the model would make no difference to the expected present value of the sickness benefits calculated using the model. This is because the death of a lapsed policyholder does not affect the sickness benefit payments.

So introducing a transition from lapsed to dead would mean that the model was more complex than it needed to be for the desired purpose. Unnecessary complexity should always be avoided when designing a model, because, for example, the model may be slower and take longer to build, it may be harder to understand, be more prone to errors, require estimation of additional parameters – and so on. A model that contains no unnecessary features for its required purpose is described as *parsimonious*, and therefore parsimony is a desired feature of any model design.

In summary, an optimal design for a particular multiple state model will be achieved if we:

- include all transitions that materially affect the output from the model (*eg* the expected present value of some cashflows, a premium, or a reserve) so that we obtain the degree of accuracy we require
- exclude all transitions that do not materially affect the output from the model.

An example of this approach will be found in Section 3.1 below.

3 Multiple decrement models

A multiple decrement model is a multiple state model which has:

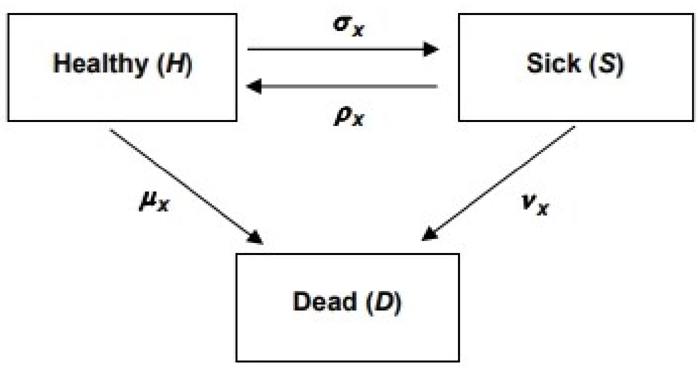
- **one active state, and**
- **one or more absorbing exit states.**

Many practical situations involving competing risks can be modelled adequately using this simplified model structure.

3.1 A simple example

We will use a simple example to illustrate the operation of multiple decrement models.

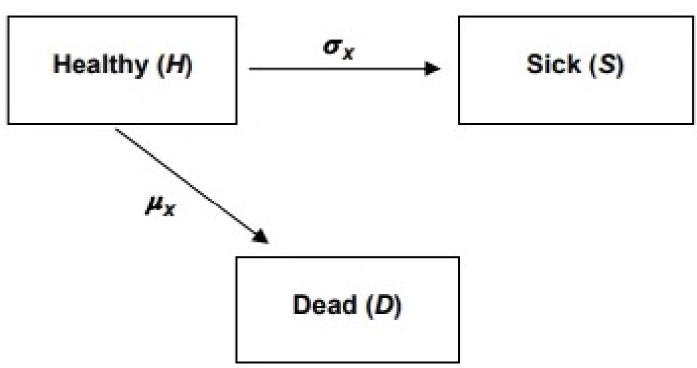
We begin with the general healthy-sick-dead multiple state model described in Section 2:



Suppose we have an insurance contract, which only pays:

- **a once-only payment of X on transition from healthy to sick, or**
- **a once-only payment of Y on transition from healthy to dead.**

In this case, the following multiple decrement model should be sufficient to enable us to value these cashflows adequately:



In this multiple decrement model, H is the active state, and there are two absorbing exit states, S and D. The life H is subject to the competing risks of S and D.

3.2 Multiple decrement probabilities

In a multiple decrement model, we only need to define two types of probability.

The first is ${}_t(aq)_x^r$, which is defined as the *dependent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By 'dependent', we mean *in the presence of all other risks of decrement in the population*). When $t=1$ this is written as $(aq)_x^r$.

The second is ${}_t(ap)_x$, which is defined as the dependent probability that an individual aged x in the active state will still be in the active state at age $x+t$. When $t=1$ this is written as $(ap)_x$.

For comparison, using multiple state notation, the dependent probabilities for our example would be written:

$$(aq)_x^s = {}_1p_x^{HS}$$

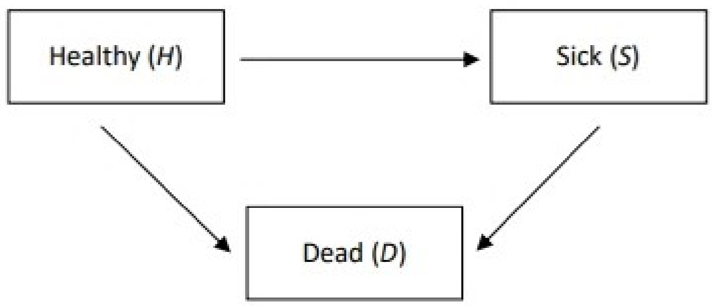
$$(aq)_x^d = {}_1p_x^{HD}$$

$$(ap)_x = {}_1p_x^{HH}$$

These equalities are *only* true in the case of the revised healthy-sick-dead model above. They are *not* true for the general healthy-sick-dead multiple state model described at the start of Section 3.1.

Question

Consider the following 3-state model:



Assuming that H is the active state, explain whether or not each of the following is true, and if not, state with reasons which of the two probabilities is the larger.

(a) $(aq)_x^s = {}_1p_x^{HS}$

(b) $(aq)_x^d = {}_1p_x^{HD}$

(c) $(ap)_x = {}_1p_x^{HH}$

Solution

(a) $(aq)_x^s = {}_1p_x^{HS}$

$(aq)_x^s$ is the probability that a healthy life aged x will leave the healthy state due to sickness during the next year. ${}_1p_x^{HS}$ is the probability that a life who is healthy at age x, will be sick at age $x+1$.

${}_1p_x^{HS}$ will be smaller than $(aq)_x^s$, because some of the lives who become sick during the year go on to die during the same year, and so are not present in State S at age $x+1$.

(b) $(aq)_x^d = {}_1p_x^{HD}$

$(aq)_x^d$ is the probability that a healthy life aged x will leave the healthy state due to death during the next year. ${}_1p_x^{HD}$ is the probability that a life who is healthy at age x, will be dead by age $x+1$.

${}_1p_x^{HD}$ will be larger than $(aq)_x^d$, because some of those who start the year healthy and who end the year dead, will have become sick and then died from sick during the same year. So ${}_1p_x^{HD}$ will include some lives who leave the healthy state through sickness, as well as all those who leave the healthy state directly through death.

(c) $(ap)_x = {}_1p_x^{HH}$

$(ap)_x$ is the probability that a healthy life aged x stays healthy until at least age $x+1$.

${}_1p_x^{HH}$ is the probability that a life who is healthy at age x, is also healthy at age $x+1$.

Because (in *this* model) it is impossible to return to State H once the state has been left, ${}_1p_x^{HH}$ also implies that the life remains in H for at least one year, and so these two probabilities are the same.

So, the relationship between the dependent probability notation (*eg* $(aq)_x^d$) and multiple state model probability notation (*eg* ${}_1p_x^{HD}$) depends on the precise form the model takes.

Question

Explain whether $(ap)_x = {}_1p_x^{HH}$ for the general healthy-sick-dead model defined at the start of Section 3.1.

Solution

As before, $(ap)_x$ means the probability of staying (continuously) healthy for at least one year, and ${}_1p_x^{HH}$ means the probability of a life, who is healthy at age x, being also healthy one year later. These probabilities are not the same in the general healthy-sick-dead model, as lives are able to leave healthy and go back again during the same year, which means that:

$$(ap)_x < {}_1p_x^{HH}$$

In this case, $(ap)_x$ is equal to the occupancy probability, ${}_1p_x^{\overline{HH}}$.

We also note that:

$$(ap)_x + (aq)_x^s + (aq)_x^d = 1$$

and that we also write:

$$(aq)_x = (aq)_x^s + (aq)_x^d$$

so that:

$$(ap)_x + (aq)_x = 1$$

Question

Active members of a pension scheme are subject to the following probabilities of decrement at the given ages (where r and d stand for retirement and death, respectively).

Age x	$(aq)_x^r$	$(aq)_x^d$
60	0.1	0.03
61	0.2	0.04

Calculate the following probabilities, all relating to an active member who is currently exactly aged 60.

(i) The probability of retiring during the year of age 61 to 62.

(ii) The probability of dying as an active member before age 62 (*ie* without retiring first).

(iii) The probability of still being an active member at age 62.

Solution

(i) This is $(ap)_{60} \times (aq)^r_{61}$, where:

$$(ap)_{60} = 1 - (aq)^r_{60} - (aq)^d_{60} = 1 - 0.1 - 0.03 = 0.87$$

So the required probability is $0.87 \times 0.2 = 0.174$

This probability can also be written as ${}_{1|}(aq)^r_{60}$.

(ii) This probability is:

$$\begin{aligned} {}_2(aq)^d_{60} &= (aq)^d_{60} + (ap)_{60} \times (aq)^d_{61} \\ &= 0.03 + 0.87 \times 0.04 \\ &= 0.0648 \end{aligned}$$

(iii) This is:

$${}_2(ap)_{60} = (ap)_{60} \times (ap)_{61} = 0.87 \times [1 - 0.2 - 0.04] = 0.6612$$

It is also useful to consider the special case of a single decrement model, which only has one cause of decrement. For this we define ${}_tq^r_x$ to be the *independent* probability that an individual aged x in the active state will be removed from that state between ages x and $x+t$ by the decrement r. (By 'independent', we mean *when r is the only risk of decrement acting on the population*). When $t=1$ this is written as q^r_x.

Independent probabilities of decrement assume that there are no other decrements operating on the population of interest. Dependent probabilities of decrement take into account the competing forces of decrement that operate on the population.

Question

A population is subject to two causes of decrement, death (d) and withdrawal (w). Explain whether the value of q^d_x would be larger or smaller than the value of $(aq)^d_x$.

Solution

q_x^d is the probability of a life aged x dying between ages x and $x+1$, where death is the only cause of decrement occurring.

$(aq)_x^d$ is the probability of a life aged x leaving the population directly through death, when withdrawals are also taking place. This probability is lower than it would be if death were the only decrement, because lives may withdraw before they die during the same year, *ie* $(aq)_x^d < q_x^d$.

3.3 Deriving probabilities from transition intensities

We can use the Kolmogorov forward differential equations to derive transition probabilities, as in the case of multiple state models. We note from Section 2.1 that, in the multiple state model, this produces the following general result:

$$ {}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij}\,ds\right) $$

The Kolmogorov differential equations are derived in Subject CS2.

In the case of the multiple decrement model, in which return to the active state is not possible, we have:

$$ {}_t(ap)_x = {}_tp_x^{HH} = {}_tp_x^{\overline{HH}} $$

Since our double decrement model has decrements of sickness and death, we have:

$$ {}_t(ap)_x = {}_tp_x^{\overline{HH}} = \exp\left[-\int_{s=0}^t \left(\mu_{x+s}^{HS} + \mu_{x+s}^{HD}\right)ds\right] = \exp\left[-\int_{s=0}^t \left(\sigma_{x+s} + \mu_{x+s}\right)ds\right] $$

Therefore, assuming constant transition intensities:

$$ {}_t(ap)_x = e^{-(\mu+\sigma)t} \qquad \textbf{(1)} $$

For the other probabilities, the differential equations (again assuming constant transition intensities) are:

$$ \frac{\partial}{\partial t}\,{}_t(aq)_x^s = \sigma\,{}_t(ap)_x = \sigma e^{-(\mu+\sigma)t} $$

and

$$ \frac{\partial}{\partial t}\,{}_t(aq)_x^d = \mu\,{}_t(ap)_x = \mu e^{-(\mu+\sigma)t} $$

These differential equations have the closed form solutions (with $t = 1$):

$$(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) \qquad (2)$$

and

$$(aq)_x^d = \frac{\mu}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) \qquad (3)$$

These solutions are obtained by integrating the differential equations with respect to t between the limits of $t=0$ and $t=1$. For example, integrating the first differential equation gives:

$$\left[{}_t(aq)_x^s\right]_0^1 = \int_0^1 \sigma e^{-(\mu+\sigma)t}\,dt$$

$$\Rightarrow (aq)_x^s - {}_0(aq)_x^s = \frac{-\sigma}{\mu+\sigma}\left[e^{-(\mu+\sigma)t}\right]_0^1$$

$$\Rightarrow (aq)_x^s = \frac{\sigma}{\mu+\sigma}\left[1-e^{-(\mu+\sigma)}\right]$$

since ${}_0(aq)_x^s = 0$.

Note that $(aq)_x^s$ is the product of:

- $1-e^{-(\mu+\sigma)} = 1-(ap)_x$, *ie* the probability that a life who is in the active state at the start of the year, is *not* in the active state at the end of the year, and
- $\frac{\sigma}{\mu+\sigma}$, *ie* the proportion of the total force acting on the life in the active state that relates to transitions to the sick state. This represents the conditional probability that the transition from the active state takes the life into the sick state, given that there is a transition out of the active state.

We now have formulae for the dependent probabilities in terms of transition intensities. So, if we can estimate the transition intensities, it is very easy to estimate the dependent probabilities using these formulae. The estimation of transition intensities is covered in CS2 – in CM1 we will be told what values or functions to use.

For the independent (single decrement) probabilities we obtain:

$$q_x^s = 1-e^{-\sigma} \qquad (4)$$

and:

$$q_x^d = 1-e^{-\mu}$$

The first of these is obtained by setting $\mu=0$ in (2) above (so that sickness is the *only* decrement operating, *ie* sickness is operating *independently*). The second of these is obtained by setting $\sigma=0$ in (3) above.

When $t \neq 1$, we have, for example:

$$ {}_t(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-t(\mu+\sigma)}\right) $$

provided the transition intensities are constant over $[x, x+t]$ (and similar for other cases).

Question

A population of healthy people over the year of age 50 to 51 is subject to a constant force of decrement due to sickness of 0.08 per annum, and a constant force of mortality of 0.002 per annum.

Assuming that a double decrement model is used, calculate:

(i) the probability that a healthy person aged exactly 50 will still be healthy at exact age 51

(ii) the probability that a healthy person aged exactly 50 will leave the healthy population through death before exact age 51

(iii) the independent probability of a life aged exactly 50 dying before exact age 51.

Solution

(i) The probability of staying healthy is:

$$ (ap)_{50} = e^{-(0.08+0.002)} = 0.921272 $$

(ii) The dependent probability of leaving the population through death is:

$$ (aq)_{50}^d = \frac{0.002}{0.082}\left(1-e^{-0.082}\right) = 0.001920 $$

(iii) The independent probability of dying is:

$$ q_x^d = 1-e^{-0.002} = 0.001998 $$

Note that $(aq)_x^d < q_x^d$, as we would expect.

4 Multiple decrement tables

A multiple decrement table is a computational tool for dealing with a population subject to multiple decrements.

We introduce the following notation as an extension of the (single decrement) life table approach:

$$(al)_x = \text{active population at age } x$$

and $\alpha, \beta, \gamma, \ldots$ the labels for the types of independent decrements to which the population is subject.

Then the multiple decrement table is a numerical representation of the development of the population, such that:

$$\begin{aligned}(al)_{x+1} = (al)_x &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \alpha \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \beta \\ &- \text{number of lives removed between ages } x \text{ and } x+1 \\ &\quad \text{due to decrement } \gamma \\ &- \ldots\end{aligned}$$

In general, as the decrements are assumed to operate independently, the number of lives removed due to decrement 'k' will depend on the preceding population $(al)_x$ *as well as* the numbers removed by every other decrement other than k.

We need to be clear about what the Core Reading means by 'independently' here. It means that the presence of multiple causes of decrement in a population does not affect the *forces* of decrement by each cause – *ie* that the forces of decrement are independent of each other. However, the *numbers* of decrements by any cause will certainly be affected by how many decrements from other causes occur.

We define the number of lives removed over the year of age due to decrement k as $(ad)_x^k$. Hence, we have:

$$EPV_S = 3{,}000\left\{\frac{0.095045}{1.03^{1/2}} + \frac{2\times 0.902568\times 0.048703}{1.03^{1\frac{1}{2}}} + \frac{3\times 0.856137\times 0.048694}{1.03^{2\frac{1}{2}}}\right\}\times 0.75$$

$$= 661.30$$

$${}_n(aq)_x^k = \frac{(ad)_x^k + (ad)_{x+1}^k + \dots + (ad)_{x+n-1}^k}{(al)_x}$$

$(al)_x$

$${}_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

for $n = 0,1,\dots$.

Question

You are given the following extract from a double decrement table:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
50	100,000	175	2,490
51	97,335	180	2,160
52	94,995		

where d and w refer to death and withdrawal respectively.

Calculate:

(a) $(ap)_{51}$

(b) $(aq)_{51}^d$

(c) ${}_2(aq)_{50}^w$

(d) ${}_2(ap)_{50}$

Solution

(a) $(ap)_{51} = \dfrac{(al)_{52}}{(al)_{51}} = \dfrac{94{,}995}{97{,}335} = 0.975959$

(b) $(aq)_{51}^{d} = \dfrac{(ad)_{51}^{d}}{(al)_{51}} = \dfrac{180}{97{,}335} = 0.001849$

(c) ${}_2(aq)_{50}^{w} = \dfrac{(ad)_{50}^{w} + (ad)_{51}^{w}}{(al)_{50}} = \dfrac{2{,}490 + 2{,}160}{100{,}000} = 0.0465$

(d) ${}_2(ap)_{50} = \dfrac{(al)_{52}}{(al)_{50}} = \dfrac{94{,}995}{100{,}000} = 0.94995$

We can also use the table to calculate *deferred* dependent probabilities of the form:

$$_{n|}(aq)_x^k = \frac{(ad)_{x+n}^k}{(al)_x}$$

This is the probability that an active life, currently aged x, leaves the population by cause k in the year of age $x+n$ to $x+n+1$.

The expression can be constructed from:

$$_{n|}(aq)_x^k = {}_n(ap)_x \times (aq)_{x+n}^k = \frac{(al)_{x+n}}{(al)_x} \times \frac{(ad)_{x+n}^k}{(al)_{x+n}} = \frac{(ad)_{x+n}^k}{(al)_x}$$

Question

Using the multiple decrement table in the preceding question, calculate $_{1|}(aq)_{50}^{d}$, and state what this probability means in words.

Solution

The probability is:

$$_{1|}(aq)_{50}^{d} = \frac{(ad)_{51}^{d}}{(al)_{50}} = \frac{180}{100{,}000} = 0.0018$$

This is the probability that a person aged exactly 50 leaves the population through death between exact ages 51 and 52.

It is also conventional to write the transition intensity (or *force of decrement*) due to cause k at age .. in the multiple decrement model as $(a\mu)_x^k$.

4.1 Associated single decrement tables

For each of the causes in a multiple decrement table, it is possible to define a single decrement table which only involves a particular cause of decrement. In effect the other modes of decrement are assumed not to operate.

A life table, such as AM92 or ELT15, is an example of a single decrement table, where the only decrement is death. We can similarly construct other single decrement tables (at least in theory) where the decrement is just withdrawal, for example.

We define functions as in any single decrement table except that each symbol has a superscript indicating the mode of decrement that is being modelled.

The notation used is $l_x^j, d_x^j, q_x^j, p_x^j, \mu_x^j$ etc for mode of decrement j.

We've already met the independent probability q_x^j. The notation p_x^j $(=1-q_x^j)$ represents the probability that a life aged exactly x remains in the population for at least one year, when cause j is the only way in which people can exit from the population.

The importance of single decrement tables in practice is that they are often used, at least as a starting point, in the construction of a multiple decrement table.

For example, we might wish to construct a double decrement table for endowment assurance policyholders incorporating decrements of death and withdrawal, in which the underlying (or 'independent') mortality basis is represented by a standard mortality table, such as AM92 Select. The probabilities q_x^d and p_x^d can be read off from the table, and then used to help construct the relevant dependent probabilities, as described below.

4.2 Relationships between single and multiple decrement tables

The linking assumption between the tables is:

$$(a\mu)_x^j = \mu_x^j \text{ for all } j \text{ and all } x$$

Normally we would expect the independent decrement probabilities and the dependent decrement probabilities not to be equal. For instance, $(aq)_x^d$ and q_x^d might not be equal because the dependent probability will be affected by the fact that some people will withdraw, or retire (if these are the other decrements operating) 'before they could die' in our population. So the fact that a lot of things could happen over the course of a year causes a problem.

If, however, we look at the transition intensities μ, then we are in effect looking at some infinitesimally small time interval. In that very small time interval there is only time for one decrement – *eg* death – and so the number of deaths occurring in that time interval will not be reduced by people withdrawing. Thus it is reasonable to assume that the independent intensity of death μ_x^d will be equal to the dependent intensity of death $(a\mu)_x^d$. The same reasoning will hold for any decrement.

The assumption is often called the 'independence of decrements'. One of its consequences is that if we observe the forces of decrement from cause j in two groups, one which includes all those who have not left the population as a result of decrement from cause i and the other which includes all those who have left the population as a result of decrement from cause i, then the observed forces of decrement from cause j in the two groups are the same. So the force of decrement from cause j is independent of the force from cause i.

For instance, if we have a population of pension scheme members where cause j is death and cause i is withdrawal, then 'independence of decrements' implies that:

'the mortality of active pension scheme members is equal to that of pension scheme members who have withdrawn.'

This means that any change in $(a\mu)_x^w$ would not affect $(a\mu)_x^d$.

So a consequence of the independence of decrements is that the decrements are non-selective – they do not alter the decrement experience of those 'left behind'. In practice this may not be true, especially if we are considering populations of life assurance policyholders, for example, where it is very likely that people who lapse or surrender their policies would have lower than average mortality. However, the independence assumption is required to make the theory more tractable.

There are a number of theoretical shortcomings of these assumptions but they are beyond the scope of this course and application of these assumptions provides us with a viable working model for actuaries in practice.

The sum of all the (dependent) forces of mortality is denoted by:

$$(a\mu)_x = \sum_{all\ j} (a\mu)_x^j$$

4.3 Constructing a multiple decrement table

To construct a multiple decrement table, we need to obtain the relevant dependent probabilities $(aq)_x^k$ at each age and for each cause of decrement k (as described in Section 4.4 below).

Once these are obtained we then choose a suitable starting age x for the table, and a suitable value for the radix $(al)_\alpha$.

Then construct:

$$(ad)_x^k = (al)_x \, (aq)_x^k$$

for all k, and

$$(al)_{x+1} = (al)_x - \sum_k (ad)_x^k$$

recursively for all $x = \alpha, \alpha+1, \ldots$

4.4 Obtaining dependent probabilities

The formulae in this section all require the independence of decrements assumption, defined in Section 4.2, to hold.

From the forces of decrement

The most logical starting point is to begin with the relevant forces of decrement, and assume these are constant over single years of age. Formulae such as (1), (2) and (3) of Section 3.3 can then be used to calculate the dependent probabilities directly.

Question

In a certain population, forces of decrement are assumed to be constant over individual years of age.

The following independent forces of decrement will be assumed for this population between the exact ages of 50 and 52:

Force of decrement for year of age commencing from exact age x

Age x	due to mortality	due to sickness
50	0.011	0.075
51	0.012	0.081

Construct a double decrement table including the two decrements of mortality and sickness, for this population between exact ages 50 and 52, assuming a radix of $(al)_{50} = 100{,}000$.

Solution

First we need the dependent probabilities of decrement by each cause at each age.

We will assume decrements are independent, so that the given forces can be assumed to apply when the two decrements occur together in the same population.

We will use μ_x^j to be the constant force of decrement due to cause j operating over the year of age x to $x+1$, where $j = d$ (death), s (sickness).

From Section 3.3 we have:

$$(aq)_x^j = \frac{\mu_x^j}{\mu_x^d + \mu_x^s}\left(1 - e^{-(\mu_x^d + \mu_x^s)}\right)$$

We then obtain:

$$(aq)_{50}^{d} = \frac{0.011}{0.086}\left(1-e^{-0.086}\right) = 0.010540$$

$$(aq)_{50}^{s} = \frac{0.075}{0.086}\left(1-e^{-0.086}\right) = 0.071865$$

$$(aq)_{51}^{d} = \frac{0.012}{0.093}\left(1-e^{-0.093}\right) = 0.011459$$

$$(aq)_{51}^{s} = \frac{0.081}{0.093}\left(1-e^{-0.093}\right) = 0.077348$$

To construct the table, we use the radix of $(al)_{50} = 100{,}000$ and the formulae in Section 4.3, which give us:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	1,054.03	7,186.55
51	91,759.42	1,051.46	7,097.37
52	83,610.59		

From an existing multiple decrement table

Here we will need to calculate the implied (constant) forces of decrement underlying the existing table. Taking formula (2) from Section 3.3 as an example (where we have decrements of sickness and mortality):

$$\boldsymbol{(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right)}$$

$$= \frac{\sigma}{\mu+\sigma}\left(1-(ap)_x\right)$$

$$\boldsymbol{= \frac{\sigma}{\mu+\sigma}(aq)_x}$$

Rearranging:

$$\boldsymbol{\Rightarrow \sigma = \frac{(aq)_x^s}{(aq)_x}(\mu+\sigma) = \frac{(aq)_x^s}{(aq)_x}\left(-\ln(ap)_x\right)}$$

where:

$$(aq)_x = (aq)_x^s + (aq)_x^d = \sum_{\text{all } j}(aq)_x^j$$

Question

We wish to extend the multiple decrement table constructed in the previous question (incorporating the decrements of death (d) and sickness (s)) to include the decrement of withdrawal (w).

It is believed that the independent forces of withdrawal will conform to those underlying the withdrawal decrement in the multiple decrement table below:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
50	100,000	175	2,490
51	97,335	180	2,160
52	94,995		

Calculate the first two lines of the triple decrement table (*ie* between ages 50 and 52) incorporating the three decrements d , s and w , assuming that the forces of sickness and mortality are unchanged.

Solution

We first need to calculate the independent forces of withdrawal from the table provided, using:

$$\mu_x^w = \frac{(aq)_x^w}{(aq)_x}\left[-\ln(ap)_x\right] = \frac{(ad)_x^w}{(ad)_x^d + (ad)_x^w}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x}\right)\right]$$

So, for $x = 50,51$:

$$\mu_{50}^w = \frac{2,490}{175+2,490}\left[-\ln\left(\frac{97,335}{100,000}\right)\right] = 0.025238$$

$$\mu_{51}^w = \frac{2,160}{180+2,160}\left[-\ln\left(\frac{94,995}{97,335}\right)\right] = 0.022463$$

Indicating the new triple-decrement functions with '*b*' rather than '*a*' prefixes, the new dependent probabilities for age x are:

$$(bq)_x^j = \frac{\mu_x^j}{\mu_x^d + \mu_x^s + \mu_x^w}\left(1 - e^{-(\mu_x^d + \mu_x^s + \mu_x^w)}\right) \qquad \text{for } j = d, s, w$$

So for $x = 50$, we have:

$$\mu_{50}^d + \mu_{50}^s + \mu_{50}^w = 0.011 + 0.075 + 0.025238 = 0.111238$$

$$1 - e^{-0.111238} = 0.105274$$

Then:

$$(bq)^d_{50} = \frac{0.011}{0.111238} \times 0.105274 = 0.010410$$

$$(bq)^s_{50} = \frac{0.075}{0.111238} \times 0.105274 = 0.070979$$

$$(bq)^w_{50} = \frac{0.025238}{0.111238} \times 0.105274 = 0.023885$$

For $x = 51$, we have:

$$\mu^d_{51} + \mu^s_{51} + \mu^w_{51} = 0.012 + 0.081 + 0.022463 = 0.115463$$

$$1 - e^{-0.115463} = 0.109046$$

Then:

$$(bq)^d_{51} = \frac{0.012}{0.115463} \times 0.109046 = 0.011333$$

$$(bq)^s_{51} = \frac{0.081}{0.115463} \times 0.109046 = 0.076499$$

$$(bq)^w_{51} = \frac{0.022463}{0.115463} \times 0.109046 = 0.021214$$

The first two lines of the triple decrement table are:

Age x	$(bl)_x$	$(bd)^d_x$	$(bd)^s_x$	$(bd)^w_x$
50	100,000	1,041.03	7,097.90	2,388.47
51	89,472.60	1,014.01	6,844.53	1,898.09
52	79,715.97			

From existing single decrement tables

Here we will need to calculate the implied (constant) forces of decrement underlying the existing single decrement tables. Taking formula (4) from Section 3.3 as an example:

$$q^s_x = 1 - e^{-\sigma}$$

$$\Rightarrow \sigma = -\ln\left(1 - q^s_x\right)$$

In all cases, the forces of decrement obtained may need to be adjusted before being applied to construct the dependent probabilities for any particular application.

Question

We wish to construct a double decrement table for mortality and sickness only.

The independent force of sickness for the year of age from 50 to 51 is 0.075 and the independent force of mortality for the same year of age is 80% of the force of mortality according to the ELT15 (Males) mortality table.

Assuming that all forces of decrement are constant over individual years of age, calculate the first line of this double decrement table, using a radix of $(al)_{50} = 100,000$.

Solution

The required force of mortality is found from:

$$\mu^d_{50} = 0.8\left[-\ln\left(1-q^d_{50}\right)\right]$$

where q^d_{50} is the relevant probability of death in the ELT15 (Males) table. So:

$$\mu^d_{50} = 0.8\left[-\ln\left(1-0.00464\right)\right] = 0.003721$$

Calculating the dependent probabilities, we have:

$$(aq)^d_{50} = \frac{0.003721}{0.003721+0.075}\left(1-e^{-(0.003721+0.075)}\right) = 0.003578$$

$$(aq)^s_{50} = \frac{0.075}{0.078721}\times\left(1-e^{-0.078721}\right) = 0.072124$$

So the first line of this double decrement table is:

Age x	$(al)_x$	$(ad)^d_x$	$(ad)^s_x$
50	100,000	357.80	7,212.39
51	92,429.81		

These formulae can also be used to construct single decrement tables from existing multiple decrement tables, if desired. The key is always to define any existing tables in terms of their underlying forces of decrement, and then apply these forces to calculate the dependent or independent probabilities as required.

There is a direct relationship between the independent and dependent probabilities, which is sometimes useful.

If we have n decrements, labelled $j = 1, 2, \ldots n$:

$$
\begin{aligned}
{}_t(ap)_x &= \exp\left[-\int_{s=0}^{t} (a\mu)_{x+s}\, ds\right] \\
&= \exp\left[-\int_{s=0}^{t} \sum_{j=1}^{n} (a\mu)^j_{x+s}\, ds\right] \\
&= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} (a\mu)^j_{x+s}\, ds\right]
\end{aligned}
$$

Assuming $(a\mu)^j_{x+s} = \mu^j_{x+s}$ for all j and s, as usual, then:

$$
\begin{aligned}
{}_t(ap)_x &= \prod_{j=1}^{n} \exp\left[-\int_{s=0}^{t} \mu^j_{x+s}\, ds\right] \\
&= \prod_{j=1}^{n} {}_tp^j_x
\end{aligned}
$$

So we can obtain the overall dependent probability of remaining in the active state, by multiplying the independent 'not-leaving' probabilities for all causes together.

4.5 Integral formulae for multiple decrement probabilities

We can also obtain expressions for multiple decrement probabilities without making the assumption that forces of decrement are constant over each year of age.

For example, with a time-varying sickness intensity, the differential equation for ${}_t(aq)^s_x$ in Section 3.3 would become:

$$\frac{\partial}{\partial t}\, {}_t(aq)^s_x = {}_t(ap)_x\, \sigma_{x+t} = {}_t(ap)_x\, (a\mu)^s_{x+t}$$

Integrating over $t = 0$ to $t = 1$ we obtain:

$${}_1(aq)^s_x - {}_0(aq)^s_x = \int_{t=0}^{1} {}_t(ap)_x\, (a\mu)^s_{x+t}\, dt$$

As ${}_0(aq)^s_x = 0$:

$$(aq)^s_x = \int_{t=0}^{1} {}_t(ap)_x\, (a\mu)^s_{x+t}\, dt$$

In general, we have:

$$(aq)_x^j = \int_{t=0}^{1} {}_t(ap)_x \, (a\mu)_{x+t}^j \, dt = \int_{t=0}^{1} {}_t(ap)_x \, \mu_{x+t}^j \, dt$$

assuming independence of decrements.

Such integrals for dependent probabilities are equivalent to the integral expression we met for the probability of death in Chapter 14:

$${}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds$$

5 Using multiple decrement tables to evaluate expected present values of cashflows

The approach to calculating EPVs of cashflows that are contingent on multiple decrements is similar to the single decrement case described in earlier chapters.

5.1 Continuous approach

Integrals can be used to express the expected present value of cashflows when these are continuously contingent on multiple decrements.

Question

An employee benefits package pays 50,000 immediately on the death of an employee while in employment. The employee population is assumed to be subject to multiple decrements, of which death in employment is one. All benefits (including death benefits) under the package cease on an employee's 70th birthday.

Write down an integral expression to represent the expected present value of the death benefits to an employee who is currently aged exactly 48.

Solution

The payment of 50,000 needs to be discounted back from the moment of death. Letting t denote the time of death measured from the current time (time 0), we have:

$$EPV = \int_{t=0}^{22} 50{,}000 v^t \, {}_t(ap)_{48} \, (a\mu)^d_{48+t} \, dt$$

We integrate from $t=0$ to $t=22$ because no benefit is payable after 22 years (which is when the employee attains age 70).

5.2 Discrete approach

As usual, the basic approach is summarised as:

$$EPV = \sum \{\text{cashflow}\} \times \{\text{discount factor}\} \times \{\text{probability}\}$$

where the summation is over all future payment periods. In the case of multiple decrements, the probability will be the relevant dependent probability that the cashflow will occur.

Question

An endowment assurance pays 10,000 in three years' time or immediately on the earlier death of a life aged 50 at entry. On surrender at any time, a surrender value equal to 75% of the premiums paid by the date of surrender (without interest) is payable. Surrender payments are assumed to occur immediately at the time of surrender. A level annual premium of 3,000 is paid at the start of each year.

Calculate the expected present value of the death, maturity and surrender benefits for a single policy at outset, using the following assumptions:

- mortality: AM92 Ultimate
- annual force of decrement due to surrender: 0.1 in year 1, and 0.05 in each of years 2 and 3
- interest: 3% per annum compound

State any further assumptions you make.

Solution

We will use a multiple decrement model with decrements of surrender and death, represented by s and d respectively.

Assuming that decrements occur half way through each year of age, on average, the EPV of the death and maturity benefits combined is:

$$EPV_{D,M} = 10{,}000\left\{v^{½}\,{}_{0|}(aq)^{d}_{50} + v^{1½}\,{}_{1|}(aq)^{d}_{50} + v^{2½}\,{}_{2|}(aq)^{d}_{50} + v^{3}\,{}_{3}(ap)_{50}\right\} \qquad (*)$$

$$= 10{,}000\left\{\frac{v^{½}(ad)^{d}_{50} + v^{1½}(ad)^{d}_{51} + v^{2½}(ad)^{d}_{52} + v^{3}(al)_{53}}{(al)_{50}}\right\} \qquad (**)$$

The EPV of the surrender benefits is:

$$EPV_{S} = 3{,}000 \times 0.75\left\{v^{½}\,{}_{0|}(aq)^{s}_{50} + 2v^{1½}\,{}_{1|}(aq)^{s}_{50} + 3v^{2½}\,{}_{2|}(aq)^{s}_{50}\right\} \qquad (*)$$

$$= 3{,}000 \times 0.75\left\{\frac{v^{½}(ad)^{s}_{50} + 2v^{1½}(ad)^{s}_{51} + 3v^{2½}(ad)^{s}_{52}}{(al)_{50}}\right\} \qquad (**)$$

We can either use the dependent probabilities directly, using (*), or we can construct a multiple decrement table and then use (**).

Calculating EPVs using probabilities directly

The required values are shown in the following table:

Age x Year t	50 1	51 2	52 3
μ^d	0.002511	0.002813	0.003157
μ^s	0.1	0.05	0.05
$(aq)_x$	0.097432	0.051443	0.051769
$(aq)_x^d$	0.002387	0.002740	0.003075
$(aq)_x^s$	0.095045	0.048703	0.048694
$(ap)_x$	0.902568	0.948557	0.948231
${}_t(ap)_x$	0.902568	0.856137	0.811816

where:

$$\mu^d = -\ln\left(1-q_x^d\right)$$

$$(aq)_x = 1-e^{-(\mu^d+\mu^s)}$$

$$(aq)_x^d = \frac{\mu^d}{\mu^d+\mu^s}(aq)_x$$

$$(aq)_x^s = (aq)_x - (aq)_x^d$$

$$(ap)_x = 1-(aq)_x$$

$${}_t(ap)_x = \prod_{r=0}^{t-1}(ap)_{x+r}$$

Using (*):

$$EPV_{D,M} = 10{,}000\left\{\frac{0.002387}{1.03^{½}} + \frac{0.902568 \times 0.002740}{1.03^{1½}} + \frac{0.856137 \times 0.003075}{1.03^{2½}} + \frac{0.811816}{1.03^{3}}\right\}$$

$$= 7{,}500.89$$

$$EPV_{S} = 3{,}000 \times 0.75\left\{\frac{0.095045}{1.03^{½}} + \frac{2 \times 0.902568 \times 0.048703}{1.03^{1½}} + \frac{3 \times 0.856137 \times 0.048694}{1.03^{2½}}\right\}$$

$$= 661.30$$

Calculating EPVs using a multiple decrement table

Using (**) we first need to construct the multiple decrement table. We choose an arbitrary radix of $(al)_{50} = 100{,}000$:

Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^s$
50	100,000	238.7	9,504.5
51	90,256.8	247.3	4,395.8
52	85,613.7	263.3	4,168.9
53	81,181.5		

using:

$$(ad)_x^d = (al)_x \times (aq)_x^d$$

$$(ad)_x^s = (al)_x \times (aq)_x^s$$

$$(al)_{x+1} = (al)_x - (ad)_x^d - (ad)_x^s$$

Then:

$$EPV_{D,M} = \frac{10{,}000}{100{,}000}\left\{1.03^{-½} \times 238.7 + 1.03^{-1½} \times 247.3 + 1.03^{-2½} \times 263.3 + 1.03^{-3} \times 81{,}181.5\right\}$$

$$= 7{,}500.89$$

$$EPV_{S} = \frac{3{,}000 \times 0.75}{100{,}000}\left\{1.03^{-½} \times 9{,}504.5 + 2 \times 1.03^{-1½} \times 4{,}395.8 + 3 \times 1.03^{-2½} \times 4{,}168.9\right\}$$

$$= 661.30$$

as before.

This page has been left blank so that you can keep the chapter summaries together for revision purposes.

Chapter 24 Summary

Multiple state models

Definitions

$\mu_x^{ij} =$ force of transition (transition intensity) from state i to state j at exact age x

${}_tp_x^{ij} =$ probability of being in state j at age $x+t$ given in state i at age x

${}_tp_x^{\overline{ii}} =$ probability of staying continuously in state i between ages x and $x+t$ given in state i at age x

$$= \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds \right)$$

Valuing cashflows

A lump sum benefit of S payable immediately on transition from state i to state j, for a life currently in state a, has expected present value:

$$S\int_0^\infty v^t {}_tp_x^{ai} \mu_{x+t}^{ij} dt$$

An income benefit of B *pa* payable continuously while in state j, for a life currently in state a, has expected present value:

$$B\int_0^\infty v^t {}_tp_x^{aj} dt$$

Integral expressions can be evaluated using numerical techniques, such as the trapezium rule.

Multiple decrement models

A multiple decrement model is appropriate if there is one active state, out of which transitions occur into one or more absorbing exit states only.

Dependent and independent probabilities

The dependent probability $(aq)_x^{\alpha}$ is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, in the presence of all other decrements in the population.

The independent probability q_x^{α} is the probability that a life aged x in a particular state will be removed from that state within a year by the decrement α, where α is the only decrement acting on the population.

Results for the sickness-death (2 decrement) model

If the active state A is subject to decrements to exit states D and S, with constant forces of transition μ and σ over integer ages, then the independent probabilities are given by:

$$q_x^d = 1 - e^{-\mu} \quad \text{and} \quad q_x^s = 1 - e^{-\sigma}$$

The dependent probabilities are given by:

$$(aq)_x^d = p_x^{AD} = \frac{\mu}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right) \quad \text{and} \quad (aq)_x^s = p_x^{AS} = \frac{\sigma}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right)$$

The probability of remaining in the active state is:

$$(ap)_x = 1 - (aq)_x^d - (aq)_x^s = e^{-(\mu+\sigma)}$$

Given dependent probabilities, the forces of transition can be obtained as:

$$\mu = \frac{(aq)_x^d}{(aq)_x}\left(-\ln(ap)_x\right) \quad \text{and} \quad \sigma = \frac{(aq)_x^s}{(aq)_x}\left(-\ln(ap)_x\right)$$

where $(aq)_x = (aq)_x^d + (aq)_x^s$.

Independence of decrements

The 'independence of decrements' equation is $(a\mu)_x^j = \mu_x^j$ from which it follows that:

$$(ap)_x = p_x^1 p_x^2 p_x^3 \dots p_x^m$$

where there are m causes of decrement $j = 1, 2, \dots m$, and $p_x^j = 1 - q_x^j$.

The dependent probability expressed in integral form is:

$$(aq)_x^j = \int_0^1 {}_t(ap)_x \, (a\mu)_{x+t}^j \, dt = \int_0^1 {}_t(ap)_x \, \mu_{x+t}^j \, dt$$

(The second expression is true only if the independence of decrements assumption holds.)

Multiple decrement tables

Definitions

$(al)_x =$ expected number of active lives at exact age x

$(ad)_x^j =$ expected number of decrements due to cause j over the year of age x to $x+1$, given a radix of $(al)_\alpha$ lives active at age α.

Calculating probabilities

$$(aq)_x^j = \frac{(ad)_x^j}{(al)_x} \qquad {}_n(ap)_x = \frac{(al)_{x+n}}{(al)_x}$$

$$ {}_n(aq)_x^j = \frac{\sum_{r=0}^{n-1} (ad)_{x+r}^j}{(al)_x} \qquad {}_{n|}(aq)_x^j = \frac{(ad)_{x+n}^j}{(al)_x}$$

Construction of multiple decrement tables

$$(ad)_x^j = (al)_x (aq)_x^j \qquad (al)_{x+1} = (al)_x - \sum_{j=1}^{m} (ad)_x^j$$

where there are m causes of decrement $j = 1, 2, \dots m$.

Valuing cashflows

Cashflows can be evaluated either using integrals (as for multiple state models), or using a discrete, annual summation approach:

$$EPV = \sum \{\text{cashflow}\} \times \{\text{discount factor}\} \times \{\text{probability}\}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 24 Practice Questions

24.1 Determine which of the following assertions relating to a multiple decrement table are correct:

I The dependent probabilities of decrement can never exceed the corresponding independent probabilities.

II Forces of decrement can never exceed 1 in value.

III The total of all the decrement numbers summed over all ages equals the initial radix of the table.

24.2 The active male membership of a large pension scheme follows the experience of the multiple decrement table below.

Age x	Active lives $(al)_x$	Age retirements $(ad)_x^r$	Ill-health retirements $(ad)_x^i$	Withdrawals $(ad)_x^w$	Deaths $(ad)_x^d$
16	10,000	0	0	750	5
17	9,245	0	0	600	5
18	8,640	0	0	500	5
…	…	…	…	…	…
39	3,150	0	2	30	7
40	3,111	0	3	20	8
…	…	…	…	…	…
63	2,200	100	25	0	20
64	2,055	200	30	0	25
65	1,800	1,800	-	-	-

Assume that decrements occur continuously, except for age retirements at age 65, which all occur on the 65th birthday. Stating any other assumptions that you make, calculate the following:

(i) The probability that a man who joins the scheme on his 18th birthday, will retire for any reason after his 63rd birthday.

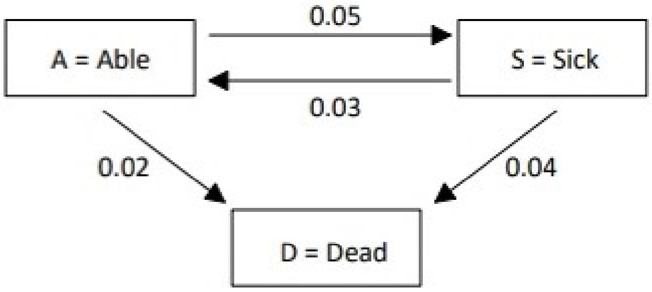
(ii) The independent probability of withdrawal between the ages of 17 and 18.

(iii) The expected present value of a lump sum retirement benefit of £10,000 payable on retirement at age 65, for a member now aged exactly 18, calculated using 4% *pa* interest.

(iv) The expected present value of a lump sum death benefit of £20,000 payable immediately on death while an active member if this occurs after the 63rd birthday, for a member now aged exactly 40, using 4% *pa* interest.

24.3 An insurance company prices its sickness contracts using the three-state model and transition intensities shown below:

Exam style



Level premiums of 2,500 *pa* are payable continuously while the policyholder is in the able state.

An able life aged 50 takes out a 15-year sickness contract that provides a 'no claim' benefit equal to 50% of the total premiums paid (without interest) if the life remains able for the full duration of the contract.

Calculate the expected present value of this 'no claim' benefit at outset, assuming that the force of interest is 6% *pa*. [3]

24.4 (i) Explain what is meant by a dependent probability of decrement and by an independent probability of decrement.

(ii) The following is an extract from a multiple decrement table subject to 3 modes of decrement, α, β and γ:

Age, x	$(al)_x$	$(ad)_x^\alpha$	$(ad)_x^\beta$	$(ad)_x^\gamma$
50	5,000	86	52	14
51	4,848	80	56	20

(a) Calculate the probability that a 50-year-old leaves the population through decrement γ between the ages of 51 and 52.

(b) Assuming that forces of decrement are constant between integer ages, calculate the independent probabilities q_{50}^α and ${}_{1|}q_{50}^\alpha$.

24.5 A multiple decrement model involves three decrements a, b and c. Decrements a and b occur continuously over the year of age $(x, x+1)$, but decrement c occurs only at age $x + ¼$. Also:

Exam style

- the forces of decrement due to causes a and b are constant over the year of age $(x, x+1)$ and are equal to 0.03 and 0.01 per annum respectively
- the probability of decrement by cause c at exact age $x + ¼$ is 0.06.

Calculate the value of $(aq)_x^a$. [6]

24.6 A life insurance company issues sickness insurance policies to healthy individuals. Each policy pays the following benefits:

Exam style

- an income of 6,000 per year payable continuously during all periods of temporary sickness (*ie* from which it is possible to recover), which is doubled during periods of permanent sickness (*ie* from which it is not possible to recover), with all sickness benefits ceasing at age 65
- on death at any time before age 65, a return of all premiums payable to date (including any waived premiums), without interest, payable immediately on death.

Level annual premiums are payable continuously to age 65 or to earlier death, except that they are waived (*ie* paid for by the insurer) during any period of sickness during that time.

(i) Draw and label a transition diagram that would be suitable for modelling this process for pricing purposes. [4]

(ii) Using the transition rates defined in your diagram, and probabilities of the form:

$${}_tp_x^{ij} = P\left(\text{person is in state } j \text{ at age } x+t \,\middle|\, \text{person is in state } i \text{ at age } x\right)$$

construct a formula, using integrals, for the annual premium for this contract (ignoring expenses), for a life aged exactly 45 at entry. [4]

[Total 8]

24.7 The decrement table extract below is based on the historical experience of a very large multinational company's workforce:

Exam style

	Number of employees	Deaths	Withdrawals
Age x	$(al)_x$	$(ad)_x^d$	$(ad)_x^w$
40	10,000	25	120
41	9,855	27	144
42	9,684		

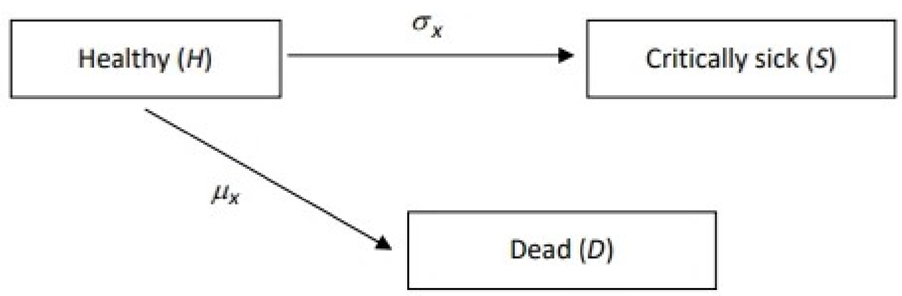
Recent changes in working conditions have resulted in an estimate that the annual independent force of withdrawal is now 75% of that previously used.

Calculate a revised table assuming no changes to the independent forces of mortality, stating your results to one decimal place. [7]

24.8 An insurance company is considering the sale of a 'critical illness extra' term assurance policy. The critical illness benefit is £25,000, payable immediately on diagnosis of a critical illness within the 25-year term. The death benefit is £75,000, payable immediately on death during the term. Only one benefit is payable under any one policy and once the benefit has been paid, both the premiums and the cover cease. Annual premiums of £P *pa* are payable continuously while the policy is in force.

Exam style

The company assesses the profitability of the policy using the following multiple state model:



${}_tp_x^{ab}$ is defined as the probability that a life who is in state a at age x ($a = H, S, D$) is in state b at age $x+t$ ($t \geq 0$ and $b = H, S, D$).

(i) Suggest, with reasons, one group of customers the insurance company may wish to target in their marketing of this policy. [1]

(ii) Express in integral form, using the probabilities and the forces of transition, the expected present value of the profit from one such policy with an annual premium of £1,200 that has just been sold to a life aged exactly 50. [2]

After careful consideration, the company modifies the policy by changing both the death benefit and the critical illness benefit to be £50,000.

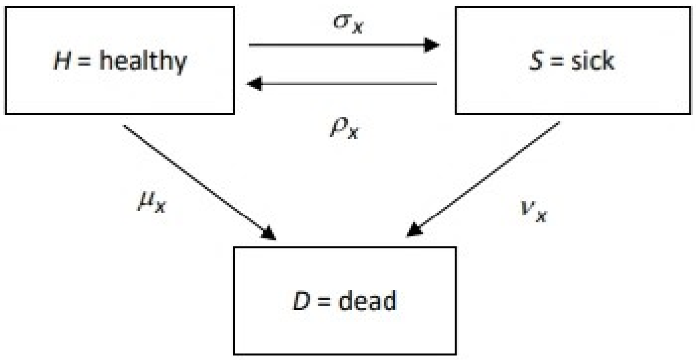
(iii) Explain how this modification could considerably reduce the cost of assessing claims. [2]

(iv) Given that $\mu_x = 0.0006$ and $\sigma_x = 0.0014$ for all $45 \leq x \leq 70$, and that the force of interest is 4% *pa*, calculate the expected present value of the benefits for the modified policy sold to a life aged exactly 45. [3]

[Total 8]

24.9 A life insurance company uses a three-state model as shown below to calculate premiums for a two-year combined sickness and endowment assurance policy issued to healthy policyholders aged 58.

Exam style



Premiums are payable at the start of each policy year and are waived if the policyholder is sick at the time the premium is due.

At the end of each policy year a benefit of £10,000 is payable if the life is then sick.

A sum assured of £15,000 is payable at the end of the year of death if this occurs during the term of the policy, or at the end of the term if the life is alive and has never claimed any sickness benefit.

S_t denotes the state occupied by the policyholder at age $58+t$, so that $S_0 = H$ (healthy) and $S_t = H, S$ or D (healthy, sick or dead) for $t = 1, 2$.

The transition probabilities used by the insurer are defined in the following way:

$$p_{58+t}^{jk} = P\left(S_{t+1} = k \mid S_t = j\right)$$

where for $t = 0, 1$:

$p_{58+t}^{HD} = 0.02$ $\qquad p_{58+t}^{HS} = 0.1$ $\qquad p_{58+t}^{SD} = 0.05$ $\qquad p_{58+t}^{SS} = 0.09$

Calculate the annual premium for this policy using the equivalence principle, based on the above transition probabilities and the following additional assumptions:

Interest:	3% *pa* effective
Initial expenses:	£200 incurred at outset
Renewal expenses:	£40 at time 1, whether the policyholder is healthy or sick
Claim expense:	£30 at the date of payment of any benefit

[8]

24.10 A company provides the following benefits for its employees:

Exam style

- immediately on death in service, a lump sum of £20,000
- immediately on withdrawal from service (other than on death or in ill-health), a lump sum equal to £1,000 for each completed year of service
- immediately on leaving due to ill-health, a benefit of £5,000 *pa* payable monthly in advance for 5 years certain and then ceasing, and
- on survival in service to age 65, a pension of £2,000 *pa* for each complete year of service, payable monthly in advance from age 65 for 5 years certain and life thereafter.

The forces of decrement for the employees at each age, assumed to be constant over each year of age, are as follows:

Age x	μ_x^d	μ_x^i	μ_x^w
62	0.018	0.10	0.020
63	0.020	0.15	0.015
64	0.023	0.20	0.010

where μ_x^j is the (assumed constant) force of decrement by cause j over $(x, x+1)$, d represents death, i represents ill-health retirement and w represents withdrawal.

(i) Construct a multiple decrement table with radix $(al)_{62} = 100{,}000$ to show the numbers of deaths, ill-health retirements and withdrawals at ages 62, 63 and 64, and the number remaining in employment until age 65. [5]

(ii) Calculate the expected present value of each of the above benefits for a new entrant aged exactly 62. Assume that interest is 6% *pa* effective before retirement and 4% *pa* effective thereafter, and that mortality after retirement follows the PMA92C20 table. [10]

[Total 15]

Chapter 24 Solutions

24.1 I is true. For a dependent probability, the other decrements operate to reduce the exposure to the decrement of interest, so that a smaller proportion of lives are expected to leave by a particular cause than if the decrement was operating on its own.

II is false. Forces of decrement are rates of transition per unit time, and there is no restriction on their size (other than they cannot be negative). For example, the force of mortality exceeds 1 at high ages in the AM92 mortality table.

III is true. All the active lives at any age will eventually become a decrement by one cause or another at some future age.

24.2 (i) The probability that an 18-year-old will retire (voluntarily or through ill-health) after attaining age 63 is:

$$\frac{[(ad)_{63}^r+(ad)_{64}^r+(ad)_{65}^r]+[(ad)_{63}^i+(ad)_{64}^i]}{(al)_{18}}=\frac{(100+200+1{,}800)+(25+30)}{8{,}640}$$

$$=0.249421$$

(ii) Assuming the force of decrement due to withdrawal is constant over the year of age 17 to 18, the independent probability of withdrawal at age 17 can be calculated from:

$$q_{17}^w=1-e^{-\mu_{17}^w}$$

where μ_x^j is defined to be the (assumed constant) force of decrement due to cause j over the year of age $[x,x+1]$.

At age 17, there are two decrements operating (withdrawal and death). Again assuming constant forces of decrement over the year of age, we can find the value of $\bar{\mu}_{17}^w$ from:

$$\mu_{17}^w=\frac{(aq)_{17}^w}{(aq)_{17}}\left(-\ln[(ap)_{17}]\right)$$

$$=\frac{(ad)_{17}^w}{(ad)_{17}}\left(-\ln\left[\frac{(al)_{18}}{(al)_{17}}\right]\right)$$

$$=\frac{600}{605}\left(-\ln\left[\frac{8{,}640}{9{,}245}\right]\right)$$

$$=0.067121$$

Therefore:

$$q_{17}^w=1-e^{-0.067121}=0.064918$$

(iii) The expected present value is:

$$10{,}000 \times \frac{1{,}800}{8{,}640} \times v^{65-18} = £329.76$$

(iv) The expected present value (assuming deaths occur mid-year) is:

$$20{,}000 \times \left(\frac{20}{3{,}111} v^{63\frac{1}{2}-40} + \frac{25}{3{,}111} v^{64\frac{1}{2}-40} \right) = £112.64$$

24.3 If the policyholder remains in the able state for the full duration of the contract, 15 years of premiums will be paid, so the 'no claim' benefit will be:

$$0.5 \times 15 \times 2{,}500 = 18{,}750$$ [1]

The probability that the policyholder remains able for the full 15 years is:

$${}_{15}p_{50}^{\overline{AA}} = e^{-(0.02+0.05)\times 15} = e^{-1.05}$$ [1]

So, the EPV of the 'no claim' benefit is:

$$18{,}750 \times e^{-0.06\times 15} \times {}_{15}p_{50}^{\overline{AA}} = 18{,}750 \times e^{-0.9} \times e^{-1.05} = 2{,}667.64$$ [1]

[Total 3]

24.4 (i) ***Dependent and independent probabilities***

A dependent probability of decrement takes into account the action of other decrements operating on the population. For example, the dependent probability $(aq)_x^\alpha$ is the probability that a life aged x will leave the active population through decrement α before age $x+1$, while all other decrements are operating.

An independent probability of decrement is a purely theoretical quantity that assumes there are no other decrements operating. For example, q_x^α is the probability that a life aged x will leave the active population through decrement α before age $x+1$, where α is the only decrement operating.

(ii)(a) ***Probability***

The probability that a 50-year-old member of the population leaves through decrement γ between the ages of 51 and 52 is:

$$\frac{(ad)_{51}^\gamma}{(al)_{50}} = \frac{20}{5{,}000} = 0.004$$

(ii)(b) ***Calculation of independent probabilities***

Since the forces of decrement are constant over each year of age, we have:

$$q_x^\alpha = 1 - e^{-\mu_x^\alpha}$$

where μ_x^j is defined to be the (assumed constant) force of decrement due to cause *j* over the year of age $[x, x+1]$.

We can find the value of μ_x^α from:

$$\mu_x^\alpha = \frac{(aq)_x^\alpha}{(aq)_x}\left(-\ln\left[(ap)_x\right]\right) = \frac{(ad)_x^\alpha}{(ad)_x}\left(-\ln\left[\frac{(al)_{x+1}}{(al)_x}\right]\right)$$

So:

$$\mu_{50}^\alpha = \frac{86}{86+52+14}\left(-\ln\left[\frac{4,848}{5,000}\right]\right) = 0.017467$$

and:

$$q_{50}^\alpha = 1 - e^{-0.017467} = 0.017315$$

Also:

$$\mu_{51}^\alpha = \frac{80}{80+56+20}\left(-\ln\left[\frac{4,848-(80+56+20)}{4,848}\right]\right) = 0.016773$$

so:

$$q_{51}^\alpha = 1 - e^{-0.016773} = 0.016633$$

From these we can then calculate:

$${}_{1|}q_{50}^\alpha = \left(1 - q_{50}^\alpha\right) \times q_{51}^\alpha = (1 - 0.017315) \times 0.016633 = 0.016345$$

24.5 We will use notation such as ${}_t(ap)_x{}'$ and ${}_t(aq)_x{}'$ to represent dependent probabilities that *ignore* decrement c. We can then write:

$$(aq)_x^a = {}_{\frac{1}{4}}(aq)_x^{a\,\prime} + {}_{\frac{1}{4}}(ap)_x{}'\left(1 - q_{x+\frac{1}{4}}^c\right){}_{\frac{3}{4}}(aq)_{x+\frac{1}{4}}^{a\,\prime} \qquad [2]$$

In order for the life to leave the population due to decrement a in the year of age $(x, x+1)$, it could either leave due to decrement a in the first quarter of the year, or, if it remains in the population until age $x + \frac{1}{4}$, and does not leave due to decrement c at that age, it could leave due to decrement a in the last three quarters of the year.

Now for $0 \le r < 1$ and $0 < t \le 1-r$:

$${}_t(ap)_{x+r}{}' = \exp\left[-\left(\mu^a + \mu^b\right)t\right] = e^{-0.04t} \qquad [1]$$

and:

$${}_t(aq)^a_{x+r}{}' = \frac{\mu^a}{\mu^a + \mu^b}\left(1 - {}_t(ap)_{x+r}{}'\right) = \frac{0.03}{0.04} \times \left(1 - e^{-0.04t}\right) \qquad [1]$$

So:

$${}_{\frac{1}{4}}(aq)^{a}_x{}' = 0.75\left(1 - e^{-0.04 \times 0.25}\right) \qquad [½]$$

$${}_{\frac{1}{4}}(ap)_x{}' = e^{-0.04 \times 0.25} \qquad [½]$$

$${}_{\frac{3}{4}}(aq)^a_{x+\frac{1}{4}}{}' = 0.75\left(1 - e^{-0.04 \times 0.75}\right) \qquad [½]$$

Therefore:

$$(aq)^a_x = 0.75 \times \left(1 - e^{-0.01}\right) + e^{-0.01}(1 - 0.06) \times 0.75 \times \left(1 - e^{-0.03}\right) = 0.028091 \qquad [½]$$

[Total 6]

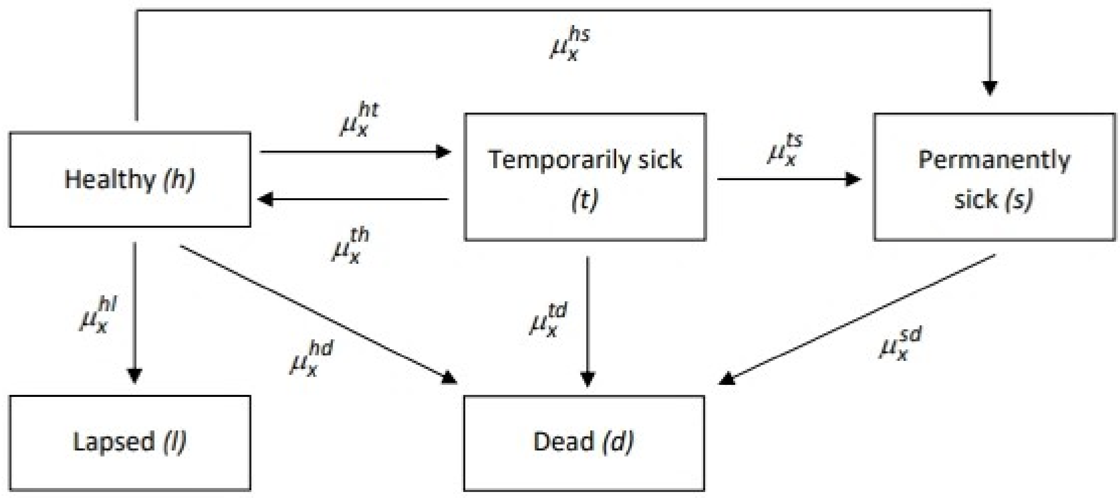
24.6 (i) ***Transition diagram***

Three key states to include will be healthy, sick and dead. However, we will need two separate sick states: a temporary (recoverable) sick state and a permanent (non-recoverable) sick state, as different benefit levels are paid in each. It would also be sensible to include lapses, as these policies are paid for by annual premiums so they can be lapsed if policyholders stop paying their premiums early.

We will also need to include the following transitions, each of which cause changes in payments:

- *healthy to temporary sickness and permanent sickness*
- *temporary sickness to healthy*
- *temporary sickness to permanent sickness*
- *healthy to lapsed*
- *healthy, temporary sickness, and permanent sickness, to dead.*

The following multiple state model would be appropriate:



[1½ *for including the correct states*]
[1½ *for including the correct transitions*]
[1 *for suitable labelling of transition rates*]
[Total 4]

(ii) ***Premium formula***

We will need to construct an equation of value:

$$EPV(\text{premiums}) = EPV(\text{benefits})$$

where P is the annual premium required (ignoring expenses).

(1) ***EPV of temporary sickness benefit***

This is:

$$6{,}000 \int_{r=0}^{20} v^r \, {}_r p_{45}^{ht} \, dr \qquad [1]$$

(2) ***EPV of permanent sickness benefit***

While permanently sick, 12,000 *pa* is paid. So the expected present value is:

$$12{,}000 \int_{r=0}^{20} v^r \, {}_r p_{45}^{hs} \, dr \qquad [½]$$

(3) ***EPV of death benefit***

On death at time r, a benefit of $P \times r$ would be payable, regardless of which state is then occupied (because the waived premiums are included in the benefit amount). As death can occur while in any of States *h*, *t*, or *s*, the expected present value is:

$$P\int_{r=0}^{20} r v^r \left[{}_r p_{45}^{hh} \mu_{45+r}^{hd} + {}_r p_{45}^{ht} \mu_{45+r}^{td} + {}_r p_{45}^{hs} \mu_{45+r}^{sd} \right] dr \qquad [1½]$$

(4) ***EPV of premiums***

Premiums are only payable by healthy lives. So the expected present value is:

$$P\int_{r=0}^{20} v^r {}_r p_{45}^{hh}\, dr \qquad [½]$$

So by setting $(1)+(2)+(3)=(4)$, we obtain:

$$P = \frac{6{,}000 \int_{r=0}^{20} v^r {}_r p_{45}^{ht}\, dr + 12{,}000 \int_{r=0}^{20} v^r {}_r p_{45}^{hs}\, dr}{\int_{r=0}^{20} v^r {}_r p_{45}^{hh}\, dr - \int_{r=0}^{20} r v^r \left[{}_r p_{45}^{hh} \mu_{45+r}^{hd} + {}_r p_{45}^{ht} \mu_{45+r}^{td} + {}_r p_{45}^{hs} \mu_{45+r}^{sd} \right] dr} \qquad [½]$$

[Total 4]

Alternatively, we could use $e^{-\delta r}$ *in place of* v^r.

24.7 *This question is Subject CT5, April 2010, Question 10 (revised to make it consistent with the current syllabus).*

We first need to calculate the (dependent) forces of decrement from the existing multiple decrement table. Assuming forces of decrement are constant over individual years of age, and that independent and dependent forces are equal, we can use:

$$\mu_x^j = (a\mu)_x^j = \frac{(ad)_x^j}{(ad)_x^d + (ad)_x^w}\left[-\ln\left(\frac{(al)_{x+1}}{(al)_x} \right) \right]$$

Using this formula gives the following values:

Age x	μ_x^d	μ_x^w
40	0.002518	0.012088
41	0.002764	0.014740

[2]

Reducing the forces of withdrawal by 25% gives:

Age x	$^*\mu_x^d$	$^*\mu_x^w$
40	0.002518	0.009066
41	0.002764	0.011055

[1]

We now need to obtain the revised dependent probabilities, using:

$$^*(aq)_x^j = \frac{^*\mu_x^j}{^*\mu_x^d + ^*\mu_x^w}\left(1 - e^{-(^*\mu_x^d + ^*\mu_x^w)}\right)$$

This gives:

Age x	$^*(aq)_x^d$	$^*(aq)_x^w$
40	0.002504	0.009014
41	0.002745	0.010979

[2]

Finally using:

$$^*(al)_{40} = 10,000, \qquad ^*(ad)_x^j = ^*(al)_x \times ^*(aq)_x^j$$

$$^*(al)_{x+1} = ^*(al)_x - ^*(ad)_x^d - ^*(ad)_x^w$$

we obtain the new table as:

Age x	$^*(al)_x$	$^*(ad)_x^d$	$^*(ad)_x^w$
40	10,000	25.0	90.1
41	9,884.8	27.1	108.5
42	9,749.2		

[2]
[Total 7]

24.8 (i) ***Possible customers***

This policy would be suitable for anybody with a family who has recently taken out a 25-year mortgage. [½]

If the policyholder dies or becomes critically ill, the benefit payment could be used to reduce the outstanding balance on the mortgage at a time of financial distress. [½]
[Total 1]

(ii) ***Expected present value of the profit***

The expected present value is:

$$1{,}200\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}dt - 75{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\,\mu_{50+t}\,dt - 25{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\,\sigma_{50+t}\,dt$$

$$=\int_0^{25} e^{-\delta t}\,{}_tp_{50}^{HH}\left(1{,}200-75{,}000\mu_{50+t}-25{,}000\sigma_{50+t}\right)dt$$ [Total 2]

In this model, since it is impossible to return to state H, ${}_tp_{50}^{HH} = {}_tp_{50}^{\overline{HH}}$.

(iii) ***Reducing the cost of underwriting claims***

The main problem with this policy in its original form is that the death benefit is three times larger than the critical illness benefit. Families of policyholders who become critically ill and then die may try (possibly fraudulently) to claim the death benefit instead of the critical illness benefit. [1]

The company would need to ensure that death claims are genuine in that the policyholder was not already critically ill when they died. This would involve running certain checks as part of the claims underwriting procedure and would cost money. [½]

If the death benefit is equal to the critical illness benefit then there is no need for these extra checks and hence costs. [½]

[Total 2]

(iv) ***Expected present value of the benefits***

The probability that a 45-year-old remains healthy for t years is:

$${}_tp_{45}^{HH} = \exp\left\{-\int_0^t\left(\mu_{45+s}+\sigma_{45+s}\right)ds\right\} = \exp\left\{-\int_0^t 0.002ds\right\} = \exp\left(-0.002t\right)$$ [1]

The expected present value of the benefits is:

$$50{,}000\int_0^{25} e^{-\delta t}\,{}_tp_{45}^{HH}(\mu_{45+t}+\sigma_{45+t})dt = 50{,}000\int_0^{25} e^{-0.04t}e^{-0.002t}(0.0006+0.0014)dt$$

$$=50{,}000\int_0^{25} 0.002e^{-0.042t}dt$$

$$=\frac{100}{0.042}\times\left[-e^{-0.042t}\right]_0^{25}$$

$$=£1{,}548$$ [2]

[Total 3]

24.9 *Expected present value of death benefit and death claim expenses*

This is:

$$15,030\times\left[p_{58}^{HD}v+\left(p_{58}^{HH}p_{59}^{HD}+p_{58}^{HS}p_{59}^{SD}\right)v^2\right]=15,030\times\left[\frac{0.02}{1.03}+\frac{0.88\times0.02+0.1\times0.05}{1.03^2}\right]$$

$$=612.02 \quad [2]$$

where:

$$p_{58}^{HH}=1-p_{58}^{HS}-p_{58}^{HD}=1-0.1-0.02=0.88$$

Expected present value of sickness benefit and sickness claim expenses

This is:

$$10,030\times\left[p_{58}^{HS}v+\left(p_{58}^{HH}p_{59}^{HS}+p_{58}^{HS}p_{59}^{SS}\right)v^2\right]=10,030\times\left[\frac{0.1}{1.03}+\frac{0.88\times0.1+0.1\times0.09}{1.03^2}\right]$$

$$=1,890.85 \quad [2]$$

Expected present value of maturity benefit and maturity claim expenses

This is:

$$15,030\ p_{58}^{HH}\ p_{59}^{HH}\ v^2=\frac{15,030\times0.88\times0.88}{1.03^2}=10,971.09 \quad [1]$$

since $p_{58}^{HH}=p_{59}^{HH}$.

Expected present value of other expenses

This is:

$$200+40\left(p_{58}^{HH}+p_{58}^{HS}\right)v=200+\frac{40(0.88+0.1)}{1.03}=238.06 \quad [1]$$

Expected present value of premiums

Using P for the annual premium, this is:

$$P\left[1+p_{58}^{HH}v\right]=P\left[1+\frac{0.88}{1.03}\right]=1.85437P \quad [1]$$

Equating the EPV of the premiums with the EPV of the benefits and expenses, and solving for P, we obtain:

$$P=\frac{612.02+1,890.85+10,971.09+238.06}{1.85437}=£7,394.44 \quad [1]$$

[Total 8]

24.10 (i) ***Multiple decrement table***

Since the forces of decrement are constant over each year of age, we can calculate the dependent probabilities of decrement using formulae of the form:

$$(aq)_x^d = \frac{\mu_x^d}{\mu_x^d + \mu_x^i + \mu_x^w}\left[1 - e^{-(\mu_x^d + \mu_x^i + \mu_x^w)}\right]$$

The dependent probabilities of decrement are then:

x	$(aq)_x^d$	$(aq)_x^i$	$(aq)_x^w$
62	0.01681	0.09341	0.01868
63	0.01826	0.13694	0.01369
64	0.02052	0.17841	0.00892

[3]

A multiple decrement table with radix $(al)_{62} = 100,000$ can then be created:

x	$(al)_x$	$(ad)_x^d$	$(ad)_x^i$	$(ad)_x^w$
62	100,000	1,681	9,341	1,868
63	87,110	1,591	11,929	1,193
64	72,397	1,486	12,916	646
65	57,349			

[2]
[Total 5]

(ii) ***Expected present value of benefits***

Death benefit

Assuming that death occurs, on average, halfway through the year, the expected present value of the death benefit is:

$$\frac{20,000}{(al)_{62}}\left(v^{½}(ad)_{62}^d + v^{1½}(ad)_{63}^d + v^{2½}(ad)_{64}^d\right)$$ [1½]

Evaluating this gives:

$$\frac{20,000}{100,000}\left(\frac{1,681}{1.06^{½}} + \frac{1,591}{1.06^{1½}} + \frac{1,486}{1.06^{2½}}\right) = £875$$ [½]

Ill-health benefit

Assuming that ill-health retirement occurs, on average, halfway through the year, the expected present value of the ill-health retirement benefit is:

$$\frac{5{,}000}{(al)_{62}}\ddot{a}^{(12)}_{\overline{5}|}\left(v^{½}(ad)^{i}_{62}+v^{1½}(ad)^{i}_{63}+v^{2½}(ad)^{i}_{64}\right) \qquad [1½]$$

Evaluating this, using $\ddot{a}^{(12)}_{\overline{5}|}=a_{\overline{5}|}\times\dfrac{i}{d^{(12)}}$ and values from the *Tables* at 4% (as this annuity is received after retirement), gives:

$$\frac{5{,}000\times 4.4518\times 1.021537}{100{,}000}\left(\frac{9{,}341}{1.06^{½}}+\frac{11{,}929}{1.06^{1½}}+\frac{12{,}916}{1.06^{2½}}\right)=£7{,}087 \qquad [1]$$

Withdrawal benefit

Assuming that withdrawal occurs, on average, halfway through the year, the expected present value of the withdrawal benefit is:

$$1{,}000v^{1½}\frac{(ad)^{w}_{63}}{(al)_{62}}+2{,}000v^{2½}\frac{(ad)^{w}_{64}}{(al)_{62}}=\frac{1{,}000}{100{,}000}\left(\frac{1{,}193}{1.06^{1½}}+\frac{2\times 646}{1.06^{2½}}\right)=£22 \qquad [2]$$

Normal retirement benefit

The expected present value of the normal retirement benefit is:

$$3\times 2{,}000\times v^{3}\times\frac{(al)_{65}}{(al)_{62}}\times\left(\ddot{a}^{(12)}_{\overline{5}|}+v^{5}\frac{l_{70}}{l_{65}}\ddot{a}^{(12)}_{70}\right) \qquad [2]$$

Evaluating this gives:

$$\frac{6{,}000}{1.06^{3}}\times\frac{57{,}349}{100{,}000}\times\left[4.4518\times 1.021537+\frac{1}{1.04^{5}}\times\frac{9{,}238.134}{9{,}647.797}\times\left(11.562-\frac{11}{24}\right)\right]$$

$$=£38{,}386 \qquad [1½]$$

[Total 10]

25

Unit-linked and accumulating with-profits contracts

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.3 Describe the operation of conventional unit-linked contracts, in which death benefits are expressed as combination of absolute amount and relative to a unit fund.

4.1.4 Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:

- the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
- the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus (unitised with-profits).

In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary) or (b) allocations of additional units. In either case a guaranteed minimum monetary death benefit may be applied.

0 Introduction

In this chapter we describe policies for which the benefit takes the form of an accumulating fund of premiums, *ie* unit-linked (UL) and accumulating with-profits (AWP) contracts. Under UL contracts the insurance company has no discretion about the policy benefits that are paid but, under AWP, significant components of the benefit amounts are discretionary, following a similar rationale to conventional with-profits contracts, which were described in an earlier chapter. You should make sure you are familiar with this content before reading on.

The techniques required for valuing the cashflows for both these contract types are very different from the techniques used for the insurance contracts that we have looked at so far. These techniques, which involve the projection and discounting of future cashflows and profit flows, are described in later chapters.

(b) ***Maturity value***

At policy maturity the policyholder essentially exchanges their unit holdings for cash. The amount of cash received will be the total value of all the units that are cashed in. So the maturity value is:

$$927 \times 5.08 = £4,709.16$$

Once this policy has matured, the total fund value will reduce to:

$$£50,756 - £4,709.16 = £46,046.84$$

and the number of units in issue has reduced to:

$$10,000 - 927 = 9,073$$

So each unit's share of the fund remains equal to the current unit price, *ie*:

$$\frac{£46,046.84}{9,073} = £5.08$$

The reason for dividing the funds into units of equal value, in this way, is so that the returns credited to individual policies are as fair as possible. (As can be seen from our example, the fact that one policy has taken cash out of the fund does not affect the value of the remaining units, and so all the other policyholders who own units in the fund do not gain or lose any money as a result of the cash transaction.)

The calculation of the unit price from day to day is more complicated than this in practice. For example, there are usually *two* prices of each unit at the same time, the *bid price* (which is the cash-in value of each unit) and the *offer price* (which is the price that has to paid to purchase a unit in the fund). The difference between the two (with the offer price being greater than the bid price) is called the *bid-offer spread*, and this difference is money that the insurance company makes from each unit purchased and which helps cover its costs and enable it to make profits. Different approaches are also used for pricing units when the fund is expanding or contracting in size, but these details are beyond the scope of this course.

When each investment allocation is made, the number of units purchased by the policyholder is recorded. The value at the date of death or maturity of the cumulative number of units purchased is the sum assured under the policy.

The date of death or maturity would be the claim date of the policy and, in the absence of any guarantees or additional benefits, the benefit amount is simply the current value of the policyholders' allocated units at that point (usually just referred to as the current *unit fund* value).

Unit-linked policies may offer some guaranteed benefits. For example:

(1) on death during the policy term, the higher of a fixed sum assured or the value of units might be paid. This ensures that a significant benefit is paid out should the policyholder die early in the term.

(2) on survival to the maturity date of the policy, a minimum guaranteed sum assured, or a minimum average unit growth rate, may be applied. In either case, the maturity benefit is the higher of the guaranteed amounts (of sum assured or unit fund) and the actual unit fund value at the maturity date. This guarantee is to ensure that the policyholder avoids any difficulties arising from a particularly poor investment performance.

Death benefit guarantees are generally more common than maturity guarantees. In fact, policies with investment guarantees have proved very costly in the UK in the past.

In order to price and value unit-linked contracts, details of allocation percentages (usually specified in the policy) and an assumption about the future growth in the price of the units purchased are needed. The calculations involve projecting the expected profit flows on a year-on-year basis, and discounting these to obtain a measure of the expected profitability of the contract. The details of this are described in subsequent chapters.

So, with unit-linked contracts, the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds run by the life insurer. The value of these units moves in line with the performance of the fund.

Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;
- a single premium product offering a return of premium on death to give a pure savings bond; or
- a regular premium contract offering an annuity payment during periods of disability or unemployment.

Important terminology that we will use when discussing unit-linked contracts includes:

- unit account, or unit fund – this is the total value of the units in respect of the policy at any time.
- bid and offer price – as described above. The bid price is lower than the offer price (a 5% difference would be typical).
- charges – the company deducts money, either by cancelling units or by reducing the unit growth credited to the unit account on a periodic basis, for instance monthly. The company normally reserves the right to vary charges in the light of experience.

To illustrate these, consider a unit-linked endowment assurance policy, which has the following features and charges:

- Premiums are paid annually for ten years. 97% of every premium paid is used to buy units. The policyholder can choose between five different funds, *eg* the company's European Equities fund. There is a 5% bid/offer spread.
- There is a minimum guaranteed sum assured on death of £40,000 but no guaranteed minimum benefit on maturity.
- Every year the company deducts 1% from the bid value of the fund. This charge is made monthly in arrears (*ie* at the end of each month $1.01^{1/12}-1$ of the bid value of the units is deducted). This charge is called a fund management charge and is to pay for administration expenses. The company also deducts £50 *pa* from the policy's unit account. Again, this charge is made monthly in arrears. These charges may be varied in line with the company's experience.

Now suppose that in the first year the policyholder pays a premium of £1,500. Of this, 97% (= £1,455) is used to buy units. However, from this moment on, the fund is only worth 95% of £1,455 (*ie* £1,382.25) to the policyholder because of the 5% bid-offer spread. So the company only has to hold £1,382.25 worth of units, effectively taking the bid-offer spread from the policyholder at the moment the premium is paid.

Now let's imagine that the policyholder chooses to buy units entirely in the company's Mixed Fund, which is a mix of local and international bonds and equities. Then, over the next year, the fund grows by 12.3%.

So, by the end of the year, the unit account has (approximately) become:

$$1{,}382.25 \times 1.123 \times 0.99 - 50$$

(Actually a calculation of this type would have been done at least monthly, and possibly daily, so the above is slightly inaccurate.)

Contracts that are non-unit-linked are normally called *conventional*.

1.1 Unit funds and non-unit funds

The most important thing to bear in mind with unit-linked contracts is that we have two 'worlds' to keep track of: the *unit* world, and a *cash* (or *non-unit*) world. The policyholder pays premiums to acquire units, and the eventual benefit is normally denominated in these units, so we will need to keep track of the number of units bought by a policyholder, how they are growing, and what charges we are deducting from them.

However, the policyholder pays the life insurance company in real money. So we need to keep track of the cash not used to buy units, because that cash is a source of profit to the life insurance company. Conversely, if the policyholder dies there might be a cash denominated sum insured, and so we need to keep track of the cash outgo on claims. Another very significant cash outgo to consider is the company's expenses. These include expenses incurred in underwriting and maintaining the policy, as well as commission payments to whoever sold it.

Question

For a typical unit-linked contract, state:

(a) the different charges that might be used in the product, and

(b) the different non-unit cash outgoes that the company might have to pay.

Solution

(a) The charges include:

- money saved from the premium as a result of having an allocation rate of less than 100%
- bid/offer spread
- fund management charge
- expense charge (policy fee).

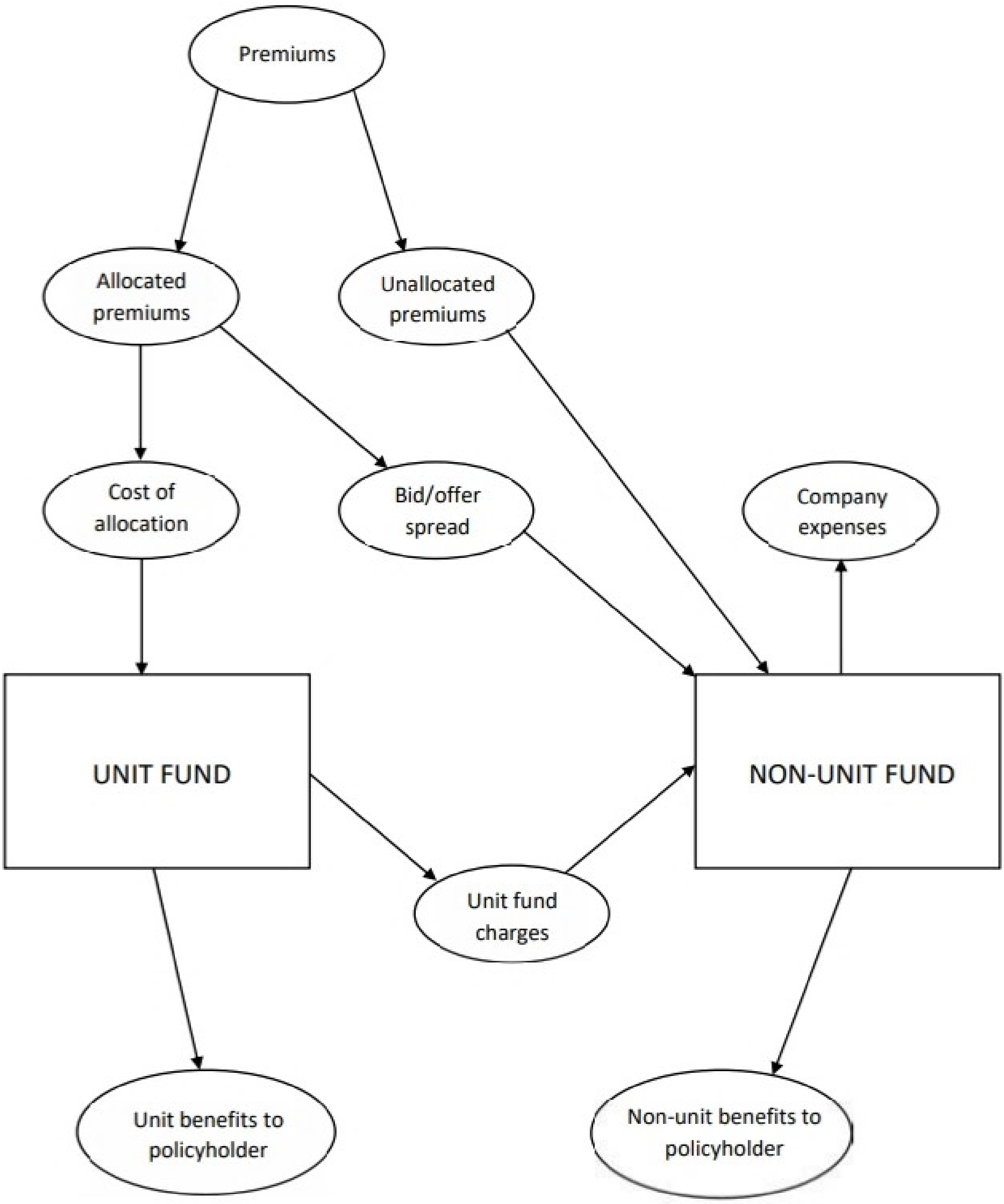
(b) The cash outgoes include:

- expenses
- maturity benefit (excess of any cash-denominated guaranteed maturity value over value of units)
- death benefit (excess of any cash-denominated guaranteed sum assured over value of units).

To clarify the situation, it is useful to consider things diagrammatically. Here, and at most other times in the course, we do not worry about splitting the policyholder's units into different funds, *eg* American Equities, Peruvian Goldmines, *etc*, but consider a generic unit fund.

It is also common to think of the cash world as a specific cash fund. We will refer to this as the *non-unit* fund.

The inter-relationship between the policyholder, the company, units and cash can be encapsulated in the following diagram.


Premiums
Allocated premiums
Unallocated premiums
Cost of allocation
Bid/offer spread
Company expenses
UNIT FUND
NON-UNIT FUND
Unit fund charges
Unit benefits to policyholder
Non-unit benefits to policyholder

Unit benefits are payable on surrender, death claim or maturity.

Non-unit benefits include, for instance, any sum insured payable on death in excess of the value of the units, or any guaranteed maturity value in excess of the value of the units.

Unit fund charges comprise:

- a fund management charge, for instance 0.5% *pa* of the unit fund in respect of fund management expenses,
- a policy fee (paid by cancellation of units) to cover other administration expenses, and
- a charge to cover the cost of providing any additional non-unit benefits, *eg* for any extra money paid out (over and above the unit fund value) when there is a guaranteed minimum death benefit.

One very important influence on the eventual experience of the policy is not shown in the diagram, because it occurs entirely 'within' the boxes. This is the fund growth experienced by the units, *eg* if the units are all invested in equities, it will be the combination of any capital appreciation of the equities, plus dividends received.

The insurance company is also likely to earn interest on any money it *holds* in the non-unit fund from time to time.

There are a number of important things to highlight in this diagram:

- The unit fund is worth only the *bid* value of the *allocated* premium – the rest of the premium goes to the non-unit fund.
- The unit fund charges are made in order to cover the expenses of fund management. However, charges and expenses are different cashflows (charges are part of the company's income while the expenses are part of its outgo) and are unlikely to be of the same amount.
- The profit or loss to the life company in each year is calculated as the difference between income (charges, unallocated premium, bid/offer spread) and outgo (expenses, non-unit benefits).
- The unit fund is what the policyholder sees – for instance, unit growth and all charges are communicated to the policyholder. The non-unit fund is what goes on within the company, and the policyholder does not see anything at this level.

Question

For the unit-linked endowment assurance shown on page 6 (details repeated below), calculate the unit fund value at the end of the first month of the policy (deducting all charges at the end of the month) for a 40-year-old policyholder, and calculate the total charges arising for the same period. Assume the unit fund has been growing at an annual rate of 12.3% over the first month (before the charges are deducted).

Policy details:

- Premiums of £1,500 are paid annually in advance.
- 97% of every premium paid is used to buy units.
- There is a 5% bid/offer spread.
- There is a minimum guaranteed sum assured on death of £40,000 but no guaranteed minimum benefit on maturity.
- A fund management charge payable at the rate of 1% *pa* is deducted at the end of each month from the bid value of the units.
- A policy fee payable at the rate of £50 *pa* is deducted at the end of each month by cancellation of units.

Solution

Premiums allocated are $97\% \times £1{,}500 = £1{,}455$.

We then deduct the bid/offer spread from this, so the value of units allocated is:

$$(1-5\%) \times £1{,}455 = £1{,}382.25$$

Growth over the month takes this up to $£1{,}382.25 \times 1.123^{\frac{1}{12}} = £1{,}395.68$.

We have the following charges:

- cash policy fee of: $\dfrac{£50}{12} = £4.17$
- management charge of $£1{,}395.68 \times \left(1.01^{\frac{1}{12}} - 1\right) = £1.16$.

So charges total £5.33 and the fund at the end of the month reduces to £1,390.35.

In the next question we consider what might happen to the *non*-unit fund in the same time period.

Question

The company's expenses in respect of this policy in the first month were 55% of the annual premium plus £178, and on average the mortality experience of all such policyholders was 58% of AM92 Ultimate.

Death claim payments are made at the end of the month, after all charges have been deducted.

Calculate the profit or loss to the company for the first month (ignoring any interest in the non-unit fund and assuming that the proportion of policyholders dying during the first month is one-twelfth of the annual proportion).

Solution

The policyholder pays a premium of £1,500 at the start of the first year. We saw in the solution to the previous question that the value of the allocated units is £1,382.25. The remainder of the premium, *ie* £117.75, goes into the non-unit fund.

Expenses are $0.55 \times 1{,}500 + 178 = £1{,}003$. Interest on the non-unit fund is to be ignored.

Since the value of the unit fund at the end of the first month is only £1,390.35, the guaranteed minimum benefit of £40,000 is paid out on death. The unit fund value will go towards paying the death benefit, but if the policyholder dies during the month an additional amount of $£40{,}000 - £1{,}390.35$ needs to be paid out of the non-unit fund to cover the shortfall.

To calculate the average actual cost of the death benefit we need to multiply the non-unit amount by the proportion of the policyholders dying over the first month. This is one-twelfth of 58% of the value of q_{40} in the AM92 Ultimate table (0.00937), and so the average actual death cost is therefore:

$$(40{,}000 - 1{,}390.35) \times \tfrac{1}{12} \times 0.58 \times 0.000937 = £1.75 .$$

From the solution to the previous question, the charges from the unit fund at the end of the month total £5.33.

So the profit is $117.75 - 1{,}003 - 1.75 + 5.33 = -£881.67$.

2 Accumulating with-profits contracts

These contracts originated in the UK, and now form almost all of the new with-profits business sold by UK insurers at the present time.

2.1 Definition

Under an accumulating with-profits (AWP) contract, the basic benefit takes the form of an accumulating fund of premiums (like a unit-linked policy, described in Section 1). If the accumulating fund at time t is denoted by F_t, the simplest form of an AWP contract follows the following recursive formula:

$$F_t = \left(F_{t-1} + P\right)(1+b_t)$$

This example assumes that annual premiums of P are payable at the start of each year. b_t is the annual *bonus interest* declared for year t.

In this particular example, the 'bonus' interest is the *only* interest that is credited to the policy during the year.

As in the case of the regular reversionary bonus described for conventional with-profits contracts, this is a discretionary amount determined by the insurance company each year.

So the word 'bonus' is used here to refer to this interest being a discretionary payment, rather than as some addition to any other interest that might have been earned.

The bonus will reflect both the returns achieved on the underlying assets over the period plus any additional profits made on the contract in this time. As it is discretionary, it does not exactly reflect these amounts, and in practice the insurer tends to smooth out the variations in returns and profits achieved from year to year to produce a bonus interest rate that is more stable over time than the underlying asset returns, for example. A key feature of the regular bonus interest is that it cannot be negative, whereas for certain asset types (*eg* equity portfolios) actual returns can be negative.

Question

A man pays a premium of £7,000 at the start of each year under an accumulating with-profits contract. Calculate the fund value of the policy after 3 years if the insurer declares annual bonus interest rates as follows:

year 1: 2.3%

year 2: 2.6%

year 3: 2.5%

Solution

The fund value after 3 years is calculated recursively as follows:

$$F_1 = 7{,}000 \times 1.023 = £7{,}161$$

$$F_2 = (7{,}161 + 7{,}000) \times 1.026 = £14{,}529.19$$

$$F_3 = (14{,}529.19 + 7{,}000) \times 1.025 = £22{,}067.42$$

Sometimes, as was often the case in the UK in the past, part of the bonus interest would be guaranteed. One way of including a guaranteed bonus interest rate of g per annum is shown in the following recursive formula:

$$F_t = \left(F_{t-1} + P\right)(1+g)(1+b_t)$$

An alternative approach is just to guarantee that the value for b_t in any given year cannot be lower than g.

It is unusual for any guaranteed rate to be applied to AWP in modern conditions (other than the degenerate case where $g = 0$).

This is because investment returns in general are currently too low to enable insurance companies to offer non-zero guaranteed rates without risking significant losses. However, interest rates have been much higher historically, and many of the early AWP contracts included significant guarantees. For example, in the 1980s guarantees of 3% and 4% *pa* were not uncommon.

As with conventional with-profits, the regular bonuses under AWP can be reduced so as to retain profit for subsequent deferred payment as a terminal bonus. The contractual benefit under an AWP policy (payable on death or maturity as appropriate) could then be defined as:

$$B_t = F_t + T_t$$

where T_t is the amount of terminal bonus payable on a claim at time t. The purpose and rationale for paying terminal bonus is the same under AWP as it is for conventional with-profits.

Question

State the underlying rationale that would underpin the calculation of the terminal bonus payable at the maturity date of an accumulating with-profits contract.

Solution

The terminal bonus should be such that the total policy maturity value is broadly equal to the asset share of the policy at that time. The aim is to distribute all the surplus available to the policy by the time the policy has terminated.

Apart from the terminal bonus component, these simple AWP contracts operate in a very similar way to a deposit account administered by a bank.

Question

A man currently aged exactly 42 wishes to provide himself with a pension of around £25,000 *pa* on his retirement at age 67. He intends to purchase an accumulating with-profits endowment policy that will mature on his 67th birthday, and which he hopes will provide enough funds at retirement to purchase the required pension.

Calculate the annual premium that would provide the required expected amount of pension based on the following assumptions:

- premiums are level and are paid at the start of each year throughout the duration of the AWP contract, which has no explicit charges
- the insurance company declares an annual bonus interest rate of 3.5% *pa* throughout the duration of the AWP contract
- terminal bonus is ignored
- the annuity is to be paid monthly in advance for the whole of life from age 67, without guarantee
- the insurance company projects that it will use the following annuity basis to convert cash into annuity payments at the time of retirement:

Interest:	4% *pa*
Mortality:	PMA92C20 with a 7-year deduction from the age
Expenses:	Ignored

Solution

The fund required at age 67 to produce an annuity of £25,000 *pa*, payable monthly for the whole of life, is:

$$25{,}000\ddot{a}_{67-7}^{(12)} \approx 25{,}000\times\left(\ddot{a}_{60}-\tfrac{11}{24}\right)$$

$$=25{,}000\times\left(15.632-\tfrac{11}{24}\right)$$

$$=379{,}342$$

This fund needs to equal the accumulated amount of the premiums, which are paid annually in advance over the preceding 25 years and which will be accumulated at the bonus interest rate of 3.5% *pa* over this period. Hence the premium P required to provide the fund under the AWP policy, using the given assumptions, satisfies:

$$P\ddot{s}_{\overline{25}|}^{@3.5\%} = 379,342$$

where:

$$\ddot{s}_{\overline{25}|}^{@3.5\%} = \frac{1.035^{25} - 1}{0.035/1.035} = 40.31310$$

Hence:

$$P = \frac{379,342}{40.31310} = £9,410 \ pa$$

2.2 Unitised (accumulating) with-profits contracts

Many companies that sell AWP administer the contract in unitised form (called *unitised with-profits* (UWP)). The policyholder is allocated units, and the fund value at any time for any policy is equal to the number of units held multiplied by the current price (or value) of each unit at that time.

In this way, UWP operates in a very similar way to unit-linked contracts. A key difference is the way the unit price is calculated. Two example possibilities are:

Method (1) **the unit price allows for guaranteed bonus interest increases only; the discretionary bonus is credited to the policy by awarding additional (bonus) units from time to time**

Method (2) **the unit price allows for both guaranteed and bonus interest increases.**

In both cases, it would be normal for unit prices to be changing on an effectively continuous basis (*eg* daily). The company would declare its regular bonus interest rate in advance, so that interest would accrue to policies at the equivalent daily rate.

2.3 Charges and benefits under UWP

The unitised nature of UWP means that it is easy to make allocations or deductions from the policyholder's fund at any time. As a result, much more complex product designs have been developed (often mirroring the unit-linked products that might be issued by the same insurance company). For UWP, insurers typically make explicit deductions for expense (and other) charges, as appropriate, in the same way as for unit-linked policies.

As for unit-linked policies, there will often be a minimum monetary benefit paid on death. For example, if a minimum death benefit of S was to be applied, then the death benefit payable on death at time t would be calculated as:

$$\max\left[S,\ B_t\right] = \max\left[S,\ F_t + T_t\right]$$

A minimum sum assured could also be applied at maturity in theory, but this is uncommon in practice.

A regular charge would then typically be deducted from the policyholder's fund to pay for the cost of providing the additional death (and/or maturity) benefit.

The charge for the minimum death benefit would be taken regularly, possibly annually but more likely on a monthly basis, and it would be proportional to the expected cost of providing the extra death benefit over the year (or month).

So the charge at policy duration t might be calculated as:

$$q'_{x+t} \times \max\{S - F_t, 0\}$$

where q'_{x+t} is the probability of the policyholder (aged $x+t$ at duration t) dying over the next year (or month), S is the guaranteed sum assured payable on death, and F_t is the fund value at time t. This is also how the charge would be calculated for a unit-linked contract.

Question

Two UWP contracts (*A* and *B*) were issued on the same day, both for the same annual premium of £5,000 and with the same term to maturity of 20 years. Policy *A* had a minimum death benefit of £50,000, while Policy *B* had no minimum death benefit. The policies were otherwise identical.

Explain the differences that would be expected between the two policies in terms of the likely amounts of their:

(a) death benefits in 5 years' time

(b) death benefits in 15 years' time

(c) maturity benefits.

Solution

(a) ***Death benefits in 5 years' time***

For Policy *A*, in 5 years' time the fund value will have accumulated due to the addition of 5 premiums of £5,000 each, plus some bonus interest less some charges. There might also be a small terminal bonus component at this stage. The total is therefore likely to be somewhat higher than $5 \times £5{,}000 = £25{,}000$, but almost certainly nowhere near as high as the minimum sum assured of £50,000. The death benefit under Policy *A* will therefore be £50,000 at this time.

As Policy *B* pays out the fund plus any terminal bonus at the time of death, the death benefit under this policy will be significantly smaller than for Policy *A* after 5 years.

(b) ***Death benefits in 15 years' time***

After 15 years, 15 annual premiums of £5,000 will have been paid under each policy – amounting to £75,000 (ignoring interest and charges) – so the fund value itself will be greater than the minimum death benefit. The death benefit under Policy *A* should therefore be somewhat higher than £75,000, after bonus interest has been added, charges deducted, and any terminal bonus added at the time of claim.

At first sight the death benefit under policy *B* should be the same, as the minimum death benefit does not apply. However, it is probable that the insurer will have made appropriate charges on Policy *A* to cover the additional cost of providing the minimum death benefit during the early years. These additional charges will have caused the fund value to grow slightly more slowly in the case of Policy *A*, and so by time 15 the fund value (and therefore the death benefit) for Policy *B* is likely to be slightly higher than for Policy *A*.

(c) ***Maturity benefits***

The policies mature after 20 years. There is no minimum maturity value, so both will equal their respective fund values plus any terminal bonuses paid. The situation will therefore be similar to (b), *ie* if Policy *A* has incurred charges for the minimum death benefit, then the maturity value for Policy *B* will be slightly greater than that for Policy *A*.

2.4 Comparison between UWP and the simple AWP designs

With the simple AWP design (described in Section 2.1), the bonus interest would distribute profits net of all expenses and other costs incurred. In this way it is similar to the with-profits approach that is embodied in conventional with-profits contracts, described in a previous chapter.

In the case of UWP designs (described in Sections 2.2 and 2.3), explicit charges are made to cover the various expense and other costs incurred for the policy. The bonus rates declared would then be closely related to the rates of return obtained on the underlying assets only, smoothed (and possibly deferred) over time as usual. These contracts then fit in well with the unit-linked products that the same insurers might be offering.

It should be noted that AWP (and UWP) essentially provides an accumulating fund approach to with-profits. Many individual variations on the basic design are possible and it is, therefore, impossible to document them all in this course. Students should be aware of the basic approach and main variations described above.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 25 Summary

Unit-linked contracts

With unit-linked contracts the policyholder's basic entitlement is expressed in terms of units, which represent a portion of a fund or funds of investments managed by the life insurer. This entitlement is referred to as the *unit fund* of the policy. The unit fund value moves in line with the performance of the backing investments. Given this basic concept, the unit-linked idea can be used to provide many different types of product, for instance:

- a regular premium product offering a guaranteed sum assured on death, to give an endowment assurance;
- a single premium product offering a return of premium on death to give a pure savings bond; or
- a regular premium contract offering an annuity payment during periods of disability or unemployment.

The *non-unit fund* represents the accumulation of all cashflows paid in that are not used to buy units, less all cashflows paid out that have not arisen from the cancellation of units. As such it represents the accumulation of the company's profits from the policy at any time.

Accumulating with-profits (AWP)

The basic benefit is an accumulating fund of premiums with discretionary interest rates. The accumulation follows the recursive relation:

$$F_t = \left(F_{t-1} + p\right)(1+g)(1+b_t)$$

where g is the guaranteed annual interest and b_t is the bonus annual interest for year t.

On death or survival, the benefits can be further increased by a terminal bonus.

AWP contracts can be unitised (UWP) or non-unitised. For UWP, guaranteed interest is factored into the unit price. The bonus interest can be factored into the unit price also, or can be allocated by creating new units. Policies may include a guaranteed minimum death benefit. UWP is usually subject to explicit charges, to cover expenses and any additional death benefit cost.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 25 Practice Questions

25.1 Describe the main features of a unit-linked policy.

25.2 Explain the terms 'unit fund' and 'non-unit fund' in the context of a unit-linked life assurance contract, listing the various items that make up the non-unit fund.

25.3 A woman now aged exactly 64 has paid £20,000 a year into an accumulating with-profits contract at the start of each of the last four years.

Exam style

The policy has incurred the following charges:

- £1,000 deducted at the start of year 1
- £100 deducted at the start of each subsequent year.

The following rates of regular bonus interest have been applied:

Year t	1	2	3	4
Bonus interest b_t	2.9%	3.1%	3.2%	3.4%

Additionally, there is a terminal bonus on contractual claim, currently payable at the rate of $0.015\times(t-1)$ of the fund value, where t is the number of years the policy has been in force at the time of claim.

The policy is now maturing, and the woman is using all of the maturity proceeds to buy a level annuity from the insurance company. The annuity will be payable monthly in advance for a minimum of 5 years and for the whole of life thereafter.

Calculate the monthly amount of annuity that the woman will receive, if the insurance company uses the following basis in its annuity pricing:

Mortality: PFA92C20 with a 3-year age deduction

Interest: 4% *pa*

Expenses: £400 initial plus 0.35% of each annuity payment. [7]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 25 Solutions

25.1 Features of unit-linked policies:

- Benefits are directly linked to the value of the underlying investment.
- The benefit payable in respect of each policy depends on the value of the units allocated to that policy.
- Every time the policyholder pays a premium, part of it (the allocated premium) is invested on the policyholder's behalf in a fund chosen by the policyholder. The remainder goes into the company's non-unit fund.
- The investment fund is divided into units, which are priced continuously.
- Most companies use a bid/offer spread to help cover expenses and contribute to profit. The policyholder buys units at the offer price and sells them back to the company at the bid price. The bid price is usually about 5% lower than the offer price.
- Every time the policyholder pays a premium, the number of units purchased is recorded. When the policy matures or a claim is made, the value of the cumulative number of units purchased is available to be paid out as the benefit payment.
- The company will deduct money from the unit account on a periodic basis, *eg* monthly. This is to cover expenses and the cost of providing insurance. The charges are usually variable, and can be modified in the light of the company's experience.
- There may be a minimum guaranteed sum assured to protect the policyholder against poor investment performance, or to provide some benefit in the event of an early death.
- The most common types of unit-linked assurance are whole life and endowment assurances.

25.2 The unit fund is the amount held in units on behalf of the policyholder at any time.

It may not necessarily be the amount that the policyholder is entitled to at that time. For example, if the policy is surrendered, the policyholder may receive only a proportion of the full bid value of the units, and on death there may be a guaranteed minimum sum assured which means that more than the unit fund value might be paid.

On death, maturity or surrender, the units held will be used to pay the benefit. Any excess/ shortfall in the unit fund will give rise to a positive/negative cashflow in the non-unit fund.

The amount of money in the non-unit fund is the net result of the life office's (non-unit) cashflows.

These cashflows arise from the following sources:

- premium less cost of allocation, *ie* the difference between the premium paid by the policyholder and the amount invested in the unit fund on the policyholder's behalf
- expenses incurred by the life office
- interest earned/charged on the non-unit fund

- management charges taken from the unit fund
- extra death or maturity costs (if the benefit payable on death or maturity is greater than the value of the units held at the time of death or maturity)
- profit on surrender (if the benefit payable on surrender is less than the value of the units held at the time of surrender).

25.3 First we need to calculate the maturity benefit of the accumulating with-profits policy. The fund at maturity can be calculated recursively using the formula:

$$F_0 = 0$$

$$F_t = \left(F_{t-1} + P - E_t\right)\left(1 + b_t\right) \qquad t = 1, 2, 3, 4$$

where:

F_t = fund value at time t

P = premium

E_t = expense charge at start of year t

b_t = bonus interest for year t

This produces the fund values shown in the following table:

Year	Premium	Expense charge	Bonus interest rate	Fund at end of year
1	20,000	1,000	2.9%	19,551.00
2	20,000	100	3.1%	40,673.98
3	20,000	100	3.2%	62,512.35
4	20,000	100	3.4%	85,214.37

[2]

The terminal bonus rate at maturity is $0.015 \times 3 = 0.045$, so the total maturity value is:

$$85{,}214.37 \times 1.045 = 89{,}049.01$$ [1]

Using the equivalence principle, the monthly annuity payment (of X) can be obtained from the following equation, where the annuity factor is calculated for a life aged 61 in the *Tables* (being the age at maturity of 64 less the assumed three-year age reduction):

$$89{,}049.01 = 400 + (12X) \times \ddot{a}^{(12)}_{61:\overline{5}|} + (12X) \times 0.0035 \times \ddot{a}^{(12)}_{61:\overline{5}|}$$

$$\Rightarrow X = \frac{89{,}049.01 - 400}{12 \times 1.0035 \times \ddot{a}^{(12)}_{61:\overline{5}|}}$$ [1½]

Now:

$$\ddot{a}^{(12)}_{61:\overline{5}|} = \ddot{a}^{(12)}_{\overline{5}|} + v^5 \, {}_5p_{61} \, \ddot{a}^{(12)}_{66}$$

$$\approx a_{\overline{5}|} \frac{i}{d^{(12)}} + v^5 \frac{l_{66}}{l_{61}} \left(\ddot{a}_{66} - \frac{11}{24} \right)$$

$$= 4.4518 \times 1.021537 + 1.04^{-5} \times \frac{9,658.285}{9,828.163} \times \left(14.494 - \frac{11}{24} \right)$$

$$= 15.88457$$ [2]

where the values used to calculate the annuity-certain are taken from page 56 of the *Tables*.

So the monthly amount of the annuity is:

$$X = \frac{89,049.01 - 400}{12 \times 1.0035 \times 15.88457} = £463.45$$ [½]

[Total 7]

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Profit testing

Syllabus objectives

6.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate.

6.4.1 Profit test life insurance contracts of the types listed above and determine the profit vector, the profit signature, the net present value and the profit margin.

6.4.2 Show how a profit test can be used to price a product, and use a profit test to calculate a premium for life insurance contracts of the types listed above.

0 Introduction

In this chapter we introduce the technique of profit testing. This is the process of projecting the income and outgo emerging from a policy, and discounting the results. The results can then be used for various different purposes, such as setting the premium for a life policy that will give us our required level of profitability.

We shall see in the next chapter how we can also use profit tests to set reserves, and various other applications.

1 Evaluating expected cashflows for various contract types

Profit testing begins with the projection of the expected cashflows of a hypothetical policy. We first need to decide what units of time to use – for instance, whether we are going to consider every month of the policy's lifetime, or only every year.

The standard approach is to divide the total duration of a contract into a series of consecutive non-overlapping time periods. The length of each time period is chosen so that it is reasonable to make simple assumptions about the cashflows within each period, *eg* funds earn a constant rate of interest during the period, a particular cashflow accrues uniformly during the period. These assumptions allow the expected cashflows during the period to be evaluated.

We consider in great detail below exactly what these assumptions are.

The arithmetic of these calculations is usually most straightforward when the expected cashflows per contract in force at the start of the time period are calculated.

This turns out to be simpler than calculating the expected cashflows per contract in force at policy inception.

In practice, time periods will be short where there are many rapidly changing cashflows, *eg* the start of a contract, and longer where there are fewer cashflows.

So for instance we might project monthly for the first year of a policy's lifetime, and then yearly thereafter. In fact, given the extent of computational power now easily available, it may be simpler to just carry on with a monthly breakdown for the duration of the policy.

Question

Explain why cashflows are often changing rapidly in the first year of a contract, to the extent that a yearly breakdown could be very inaccurate.

Solution

We would expect the pattern of expenses to be very uneven over the course of the first year: big initial expenses on the first day, then small admin expenses spread over the year.

The expected cashflows, both positive and negative, are used to construct a projected revenue account (per contract in force at the start of the period) for each time period. The balancing item in the projected revenue account is the profit emerging at the end of the time period.

The revenue account for a life company is:

(+) premiums

(–) expenses

(+) investment income

(–) benefit payouts (claims, maturity, surrender values)

(–) increase in reserves

= *profit gross of tax*

(–) tax

= *profit net of tax*

Here the investment income is the income on reserves, plus interest on the balance of the cashflow itself; we exclude the investment income on the life company's other assets unconnected with the policies under consideration.

All that we are trying to do is to construct such a revenue account in respect of an individual (and probably hypothetical) policy for some given contract type. If we are doing this for a unit-linked or unitised with-profits (UWP) contract, then our projections get more complicated, and include separate projections of the unit fund and the non-unit fund. We shall consider an example of this later.

For some contracts there is only one in-force status and so only one projected revenue account is needed. For example, for a life assurance contract, it is in force when the policyholder is alive. For other contracts there is more than one in-force status. For example, for a disability insurance contract the policyholder can be alive and not receiving disability benefits or alive and receiving disability benefits. Projections for these contracts will require separate revenue accounts for each in-force status. Details of this are beyond the scope of this subject.

To calculate the individual elements of the projected revenue account for a policy, we require a basis, *ie* a set of assumptions (as described in Chapter 19).

Very often, the basis for these projected cashflows is a realistic estimate of expected future experience (as opposed to the more prudent estimates we might use for reserving). We consider the question of different bases in more detail in the next chapter.

In order to calculate the expected cashflows the following information is needed:

- **premiums received and their times of payment**
- **expected expenses (from the basis) and their times of payment**
- **contingent benefits payable under the contract, *eg* death benefit, annuity payment, survival benefit for endowment, difference between guaranteed sum assured and value of unit fund for unit-linked endowment**
- **other benefits payable under the contract, *eg* surrender values**
- **other expected cash payments, *eg* taxes**
- **other expected cash receipts, *eg* management charges levied on a unit fund, and**
- **the reserves required for a contract, usually at the beginning and end of the time period, calculated using the valuation basis**

together with the different probabilities of the various events leading to the payment of particular cash amounts. Any balance on the expected revenue account during the time period will be invested, and an assumption about the rate of investment return is needed. This allows the expected investment income during the period to be calculated and credited at the end of the period.

1.1 Example 1: Conventional whole life assurance

The contract is issued to a select life aged x and has a sum assured of S secured by level annual premiums of P payable in advance. The premium basis assumes initial expenses of I and renewal expenses of e. The valuation basis requires reserves of $S.{}_tV$ for an in force policy with sum assured S at policy duration t. The basis assumes that invested funds earn an effective rate i. The surrender value basis determines that a surrender value $(SV)_t$ will be paid to policies surrendered at policy duration t.

Note the difference between $(SV)_t$, where 'S' stands for surrender and $S.{}_tV$, where 'S' stands for the sum assured.

The probabilities of events are determined from a multiple decrement table with decrements of death, d, and surrender, w, having dependent probabilities at age x, of $(aq)_x^d$ and $(aq)_x^w$.

The following shows the expected profit calculation for all policy years except the first year. In this formula time t is the beginning of the policy year, and time $t+1$ is the end of that year. This profit is the what we would expect each year in relation to a policy that is *in force* at the start of that year, *ie* for a policy that is in force at exact time t.

Income		
Premiums	P **(from data)**	
Interest on reserves	$i.S._tV$	
Interest on balances	$(P-e)i$	(*)
Expenditure		
Expenses	e **(from data)**	(*)
Expected surrender value	$(aq)^w_{[x]+t}.(SV)_{t+1}$	
Expected death claims	$(aq)^d_{[x]+t}.S$	
Transfer to reserves	$(ap)_{[x]+t}.S._{t+1}V - S._tV$	
Profit	**Balancing item**	

This assumes that expenses occur at the start of each time period, and death claims and surrender values are paid at the end of each time period.

The 'transfer to reserves' item needs some explanation. At the start of the year the insurance company holds reserves of amount $S._tV$. This is money that is currently 'in the bank' for a policy that is in force at this point. When we reach the end of the year, we need to account for the fact that the company is *required* to hold reserves of $S._{t+1}V$ for each policy that is then in force. The increase in reserve between the start and the end of the year represents additional money that the company will have to set aside (*ie* transfer) from other income, *ie* it will be a deduction from the profit earned during the year. However, not all of the policies that start out the year will still be in force at the end of the year, and we will only need to hold reserves for policies that are still in force. So, we multiply the reserve at the end of the year by the probability of the policy staying in force during the year (*ie* $(ap)_{[x]+t}$) and so the item in the profit calculation is the *expected* cost of the profit transfer per policy in force at the start of the year.

Combining the items together and rearranging them slightly, we can show the expected profit arising for the time period $[t,t+1]$, for $t=1,2,\dots$, as:

$$P-e+(P-e)i-(aq)^d_{[x]+t}.S-(aq)^w_{[x]+t}.(SV)_{t+1}+S._tV+i.S._tV-(ap)_{[x]+t}.S._{t+1}V$$

This is the formula we are going to use later when we start doing some example calculations.

In the first policy year (*ie* for $t=0$), the renewal expenses e are replaced by the initial expenses I at (*) above, and in the above formula.

Question

In the first policy year, the reserve at the start of the year requires the value of ${}_0V$. Explain what the value of ${}_0V$ would be, and hence write down a formula for the expected profit arising in the first policy year.

Solution

According to the reserve notation used in this section of Core Reading, ${}_0V$ is the reserve per unit of sum assured for a policy in force at the very start of the policy, *ie* just before the first premium is paid. The aim of the calculation is to work out the expected profit in the first policy year, and so it is sensible to assume that the insurance company has assets of zero (*ie* has no money) in relation to this policy at this point. Hence we assume ${}_0V = 0$.

The expected profit in the first policy year is therefore:

$$P - I + (P - I)i - (aq)^d_{[x]}.S - (aq)^w_{[x]}.(SV)_1 - (ap)_{[x]}.S.{}_1V$$

Temporary assurances and endowment assurances follow a similar approach.

1.2 Example 2: Conventional endowment assurance

Suppose a life insurance company sells a 5-year regular-premium endowment assurance policy to a 55-year old male. The sum insured is £10,000 payable at the end of year of death. Initial expenses are 50% of annual premium, renewal expenses are 5% of subsequent premiums. Premiums are payable annually in advance.

There is a surrender benefit payable equal to a return of premiums paid, with no interest. This is paid at the end of the year of withdrawal.

The company is required to hold net premium reserves, calculated ignoring surrenders.

We shall now calculate the projected yearly cashflows per policy in force at the start of each year, using the following bases.

For pricing:	AM92 Ultimate mortality, 4% *pa* interest, expenses as above and ignoring surrenders, using the equivalence principle
For reserving:	Interest and mortality as per pricing

For future cashflow projection: Interest and expenses as per pricing, dependent surrender and mortality probabilities as in the table below.

Age x	$(aq)_x^d$	$(aq)_x^w$
55	0.005	0.1
56	0.006	0.05
57	0.007	0.05
58	0.008	0.01
59	0.009	0

We first need to calculate the annual premium payable, P. This satisfies the equation

$$P\ddot{a}_{55:\overline{5}|} = 10{,}000A_{55:\overline{5}|} + 0.5P + 0.05P\left(\ddot{a}_{55:\overline{5}|} - 1\right)$$

So:

$$P = \frac{10{,}000A_{55:\overline{5}|}}{0.95\ddot{a}_{55:\overline{5}|} - 0.45} = \frac{10{,}000 \times 0.82365}{0.95 \times 4.585 - 0.45} = £2{,}108.81$$

We now need to work out the reserves that the company must hold over the term of the policy. We are told the company holds net premium reserves, and for this policy type we can use the formula given on page 37 of the *Tables*, remembering to include a factor of 10,000 for the sum assured. So the reserve at time t is:

$${}_tV = 10{,}000\left(1 - \frac{\ddot{a}_{55+t:\overline{5-t}|}}{\ddot{a}_{55:\overline{5}|}}\right)$$

So we have:

$${}_0V = 0$$

$${}_1V = 10{,}000\left(1 - \frac{\ddot{a}_{56:\overline{4}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 10{,}000\left(1 - \frac{3.745}{4.585}\right) = 1{,}832.06$$

$${}_2V = 10{,}000\left(1 - \frac{\ddot{a}_{57:\overline{3}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 10{,}000\left(1 - \frac{2.870}{4.585}\right) = 3{,}740.46$$

and so on.

This gives the following development of reserves.

Year t	Reserve at start of year, ${}_{t-1}V$
1	0
2	1,832.06
3	3,740.46
4	5,736.10
5	7,818.97

We now need to calculate the expected cashflows for a policy in force at the start of each year. For instance, for the first year we have:

Premium paid in of	2,108.81	
Expenses	–1,054.41	from $0.5 \times 2{,}108.81$
Interest on other cashflow	42.18	from $(P-e)i$
Expected death claims	–50	from $10{,}000(aq)^d_{55}$
Expected surrenders	–210.88	from $P(aq)^w_{55}$
Expected maturities	0	
Expected Increase in reserves	–1,639.69	

The expected increase in reserve (or 'transfer to reserve' as it is called in Section 1.1) is calculated as follows. At the start of the year we have no reserve. This also means there is no interest earned on the reserve. At the end of the year we need 1,832.06 per policy still in force. The probability that a policy in force at the start of Year 1 is still in force at the end of Year 1 is:

$$(ap)_{55} = 1-(aq)^d_{55}-(aq)^w_{55} = 0.895$$

So the expected cost of increasing the reserve is:

$$1{,}832.06 \times 0.895 - 0 \times 1.04 = 1{,}639.69$$

Altogether this gives an expected profit at the end of Year 1 of:

$$2{,}108.81 - 1{,}054.41 + 42.18 - 50 - 210.88 - 1{,}639.69 = -803.99$$

We can carry on and do the same over the remaining years of the policy's term, as shown in the table below (where outgoing cashflows are shown as negative values):

Year	Premium	Expense	Interest	Expected claim cost	Expected surrender cost	Expected cost of increase in reserves	Expected profit per policy in force at start of year
1	2,108.81	−1,054.41	42.18	−50	−210.88	−1,639.69	−803.99
2	2,108.81	−105.44	80.13	−60	−210.88	−1,625.65	186.97
3	2,108.81	−105.44	80.13	−70	−316.32	−1,519.06	178.12
4	2,108.81	−105.44	80.13	−80	−84.35	−1,712.68	206.47
5	2,108.81	−105.44	80.13	−10,000	0	8,131.73	215.23

where the expected claim cost includes the cost of maturing policies.

Question

(i) Verify the entries for Year 3 of this policy.

(ii) For Year 5, verify the entry for the expected cost of increase in reserves.

Solution

(i) ***Year 3 calculations***

The premium is £2,108.81.

Expenses are 5% of premium = −105.44.

Interest is $4\% \times (2{,}108.81 - 105.44) = 80.13$.

Expected claim cost is $-10{,}000(aq)^d_{57} = -10{,}000 \times 0.007 = -70$.

Expected surrender cost is $-3P(aq)^w_{57} = -3 \times 2{,}108.81 \times 0.05 = -316.32$.

The reserve at the start of Year 3 is 3,740.46. This earns interest at the rate of 4% *pa* to become 3,890.08 by the end of Year 3. At the end of Year 3, we need a reserve per policy in force of:

$$ {}_3V = 10{,}000\left(1 - \frac{\ddot{a}_{58:\overline{2}|}}{\ddot{a}_{55:\overline{5}|}}\right) = 10{,}000 \times \left(1 - \frac{1.955}{4.585}\right) = 5{,}736.10 $$

The probability that a policy in force at the start of Year 3 is still in force at the end of Year 3 is $(ap)_{57} = 1-(aq)^d_{57}-(aq)^w_{57} = 0.943$. So the expected cost of the increase in reserves is $5,736.10 \times 0.943 - 3,890.08 = 1,519.06$. (This is a cost to the company so it is shown as a negative entry in the table of cashflows.)

The expected profit emerging at the end of Year 3 per policy in force at the start of Year 3 is then:

$$2,108.81-105.44+80.13-70-316.32-1,519.06=£178.12$$

(ii) ***Year 5 calculations***

At the start of Year 5 we have reserves of 7,818.97 per policy in force. This earns interest at the rate of 4% *pa* to become 8,131.73 by the end of Year 5. We don't need to hold any reserve at time 5 (once we have paid the benefits) because the policies are finished. So there is a release of reserves of £8,131.73, which is shown as a positive cashflow in the table.

The entries in the table for other policy years are calculated similarly.

Question

Explain why there is a big loss in the first year, despite the fact that the expenses 'experienced' were the same as those used to price the policy.

Solution

It's not how we *priced* the contract that matters here, but how we calculated the *reserves* that we have used. If we had used a gross premium reserve, we would have deducted the future gross premiums in the reserve calculations, which would have made the reserve at the end of Year 1 much smaller than 1,832, for example. This would have offset the large loss in Year 1 caused by the large expenses in Year 1.

However, instead we have a used a net premium reserve. In a net premium reserve we only deduct the future *net* premiums in the reserve calculation. Net premiums are significantly smaller than gross premiums, which makes the reserve larger. This bigger reserve, combined with the high initial expenses that occur in Year 1, leads to the large loss that we have seen in that year.

1.3 Example 3: Unit-linked endowment assurance

The contract is issued to a life aged x and has a sum assured equal to the bid value, at the time of death, of the units purchased, subject to a minimum guaranteed sum assured of S. It is secured by level annual premiums of P, of which a_t% is allocated to the unit fund at the start of policy year t at the offer price.

Recall that the bid price is the price at which units are bought back from the policyholder or, in other words, the price at which the insured redeems units; the bid value is the number of units multiplied by the bid price. The offer price is the price at which units are sold to the policyholder or, in other words, the price at which the insured purchases units.

The $a_t\%$ is an arbitrary number, not to be confused with an annuity value

The premium basis assumes initial expenses of I and renewal expenses of e per annum incurred at the beginning of each of the second and subsequent policy years. Unit reserves are held in the unit fund, and no allowance is made for reserves in the non-unit (cash) fund.

By unit reserve, here we mean the bid value of units. We shall see in the next chapter that it is sometimes necessary to hold not only unit reserves but also non-unit reserves, *ie* a cash-denominated reserve as a contingency against future negative (non-unit) cashflows. In this example no such reserves are required.

Investments in the non-unit fund are assumed to grow at $i\%$ *pa*. Suppose the unit fund projections show a fund value of F_t at policy duration t after management charges at a rate $k_t\%$ *pa* have been paid to the non-unit fund. The bid price (sale price) of units is $(1-\lambda)\times$ offer price, *ie* there is a bid-offer spread of $100\lambda\%$. F_t is evaluated at the bid price.

The annual management charge paid from the unit fund to the non-unit fund together with the bid-offer spread (*ie* units are sold to policyholders at a price greater than their underlying value) are analogous to charges covering the level annual expenses for a traditional assurance policy.

They are analogous in that the main way the company covers level annual expenses with unit-linked contracts is via these two charges. However, the fund management charge has a very different 'shape' to what we see with conventional contracts because, as the fund grows, the fund management charge will grow. This growth is very significant for regular premium contracts as the fund (and hence the charge) increases significantly every time a premium is paid.

The charge of $(1-a_t\%)$, which results from the allocation percentage imposed by the company, can be used to cover both initial expenses (with a very low a_1) and renewal expenses (with a sufficiently low a_1 for $t>1$).

On death, the bid value of the unit fund, after management charges, is paid at the end of the year of death, subject to the minimum guaranteed sum assured of S.

The following formulae look complex, but applying them to numerical examples is comparatively intuitive. So don't worry too much about the details of this on first reading; instead, study carefully the numerical example and questions that follow.

The expected cashflows to and from the non-unit fund in policy year t can be evaluated, where $t=1$ denotes the first policy year.

Income

Premiums not allocated to unit fund	$\left(1-\frac{a_t}{100}\right)P$	
Bid-offer spread	$\lambda\frac{a_t}{100}P$	
Management charge on the unit fund (taken at the year-end)	$\frac{F_{t+1}}{\left(1-\frac{k_t}{100}\right)}\left(\frac{k_t}{100}\right)$ (from unit fund projections)	
Investment income on balances	$\left\{\left(1-\frac{a_t}{100}+\frac{\lambda a_t}{100}\right)P-I\right\}i$	$t=1$
	$\left\{\left(1-\frac{a_t}{100}+\frac{\lambda a_t}{100}\right)P-e\right\}i$	$t>1$

Expenditure

Expenses	**e (from data) (or I when $t=1$)**
Expected cost of death claims	**$(S-F_{t+1})q_{x+t}$ if positive**
Profit	**Balancing item**

The fact that no non-unit reserves were deemed necessary means that there is no 'interest on reserves' element in the income section.

It is important to realise that the above gives us only the cashflows in the non-unit fund. It will normally be necessary to calculate the projected unit fund first, so as to calculate the F_{t+1} values needed for several of the above items.

We shall see how this works in the following numerical example.

A life insurance company is studying the profitability of a 5-year unit-linked endowment assurance contract. Details are as follows:

Age at issue	50	
Annual premium	£2,000	
Benefit	The greater of the bid value of units and £5,000 (paid at maturity or at the end of the policy year of earlier death)	
Allocation rate	First year:	60%
	Other years:	98%
Bid/offer spread	5%	

Management charge	1% *pa* (deducted at end of year)		
Unit growth rate	6% *pa*		
Interest for non-unit fund	6% *pa*		
Mortality	AM92 Ultimate		
Expenses	Initial	£1,150	
	Renewal	£75 at the start of the second year, subsequently inflating at 4% *pa*	

We shall calculate the expected profit or loss on the non-unit fund in each year, per policy in force at the start of each year.

We first need to project the size of the unit fund over the term of the policy.

Question

Calculate the value of the unit fund at the end of the first year.

Solution

At the start of Year 1 the allocated premium is $2,000 \times 0.6 = 1,200$. The bid value of the allocated premium is then $1,200 \times 0.95 = 1,140$. (Alternatively we could just calculate this as $2,000 \times 0.6 \times 0.95 = 1,140$.) This amount is referred to as the 'cost of allocation'.

The fund at end of year, before deduction of the management charge, is $1,140 \times 1.06 = 1,208.40$.

The management charge is $-1\% \times 1,208.40 = -12.08$.

The fund after deduction of management charge is $1,208.40 - 12.08 = 1,196.32$ (this can alternatively be calculated as 99% of 1,208.40).

We can repeat the logic described above to project the value of the unit fund to the end of the five-year term (where outgoing cashflows are shown as negative entries):

Year	Prem rec'd	Prem all'd	Cost of all'n	Fund after all'n	Fund before mgt charge	Mgt charge	Fund at year end
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	2,000	1,200	1,140	1,140.00	1,208.40	−12.08	1,196.32
2	2,000	1,960	1,862	3,058.32	3,241.81	−32.42	3,209.40
3	2,000	1,960	1,862	5,071.40	5,375.68	−53.76	5,321.92
4	2,000	1,960	1,862	7,183.92	7,614.96	−76.15	7,538.81
5	2,000	1,960	1,862	9,400.81	9,964.86	−99.65	9,865.21

The entries in each column are calculated as follows:

(3) premium allocated = premium received × allocation percentage

(4) cost of allocation = premium allocated (3) × (1 – bid/offer spread)

(5) fund after allocation = fund at end of previous year (8) + cost of allocation (4)

(6) fund at the end of the year before deduction of management charge = fund after allocation (5) × (1 + unit growth rate)

(7) fund management charge = fund on day 364 (6) × management charge

(8) fund at year end after deduction of management charge = fund before deduction (6) – management charge (7).

Question

Verify the entries for Year 3.

Solution

At the start of the year the allocated premium is $2,000 \times 0.98 = 1,960$.

The bid value of the allocated premium is $1,960 \times 0.95 = 1,862$.

The fund after allocation is $1,862 + 3,209.40 = 5,071.40$.

The fund at end of year, before management charge, is $5,071.40 \times 1.06 = 5,375.68$.

The management charge is 1% of this , *ie* – 53.76.

The fund after deduction of this management charge is:

$$5,375.68 - 53.76 = 5,321.92$$

Having done this, we can start to calculate the non-unit cashflows. For instance, in the first year we have the following elements of outgo:

Expenses $-1,150$

Expected death cost $-q_{50}(\textit{sum assured} - \textit{unit fund}) = 0.002508 \times (5,000 - 1,196.32)$

$= -9.54$

Question

Calculate the other expected cashflows in the non-unit fund in this first year, remembering to allow for interest and the fund management charge. Hence calculate the expected profit in the non-unit fund in the first policy year.

Solution

We have the following items of income in the non-unit fund:

- at the start of the year we have the remainder of the premium, after the cost of allocating the premium has been deducted, *ie* $2{,}000-1{,}140=860$, and
- at the end of the year the fund management charge of 12.08 (from the unit fund calculations).

Interest on the non-unit fund is negative in the first year due to the effect of expenses:

$$(860-1{,}150)\times 6\% = -17.40$$

The total of all these elements is:

$$860+12.08-17.40=854.68$$

The expected profit in the non-unit fund at the end of the first year is then:

$$854.68-1{,}150-9.54=-£304.86$$

We can do the same for every year of the contract, to determine the profit at the end of each year per policy in force at the start of that year.

One area where we need to be slightly careful is in calculating the expected death cost, because if the value of the units goes above the guaranteed sum insured then the death cost will be zero.

We find the following development of non-unit cashflows. It helps to think chronologically through the company's cashflows when constructing tables like this.

Year t	Premium less cost of all'n	Expenses	Interest	Expected death cost	Mgt charge	Expected profit
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	860	−1,150.00	−17.40	−9.54	12.08	−304.86
2	138	−75.00	3.78	−5.03	32.42	94.17
3	138	−78.00	3.60	0.00	53.76	117.36
4	138	−81.12	3.41	0.00	76.15	136.44
5	138	−84.36	3.22	0.00	99.65	156.51

The entries in each column are calculated as follows:

(2) premium less cost of allocation is (2) – (4) in unit fund table (also referred to as the 'profit on allocation')

(3) expenses from the set of assumptions