4.2 Accumulations

Similarly, we define $s_{\overline{n}|}^{(p)}$ and $\ddot{s}_{\overline{n}|}^{(p)}$ to be the accumulated amounts of the corresponding *p*thly immediate annuity and annuity-due respectively. Thus:

$$s_{\overline{n}|}^{(p)} = (1+i)^n a_{\overline{n}|}^{(p)} = (1+i)^n \frac{i}{i^{(p)}} a_{\overline{n}|} \qquad \text{(by (4.1))}$$

$$= \frac{i}{i^{(p)}} s_{\overline{n}|}$$

Using the formula for $s_{\overline{n}|}$ gives:

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}}$$

Also:

$$\ddot{s}_{\overline{n}|}^{(p)} = (1+i)^n \ddot{a}_{\overline{n}|}^{(p)} = (1+i)^n \frac{i}{d^{(p)}} a_{\overline{n}|} \qquad \text{(by (4.3))}$$

$$= \frac{i}{d^{(p)}} s_{\overline{n}|}$$

Again, using the formula for $s_{\overline{n}|}$ gives:

$$\ddot{s}_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}}$$

As previously, notice the similarity between the formulae for the present and accumulated values of annuities:

$$a_{\overline{n}|}^{(p)} = \frac{1-v^n}{i^{(p)}} \qquad \ddot{a}_{\overline{n}|}^{(p)} = \frac{1-v^n}{d^{(p)}}$$

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}} \qquad \ddot{s}_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{d^{(p)}}$$

The numerators are always consistent – the only difference between the formulae is in the denominators.

Question

An investment manager has invested £1,000 in a fixed-interest security that pays interest of £40 at the end of each half-year. In addition, the full £1,000 is returned on redemption in 12 years' time. The investment manager deposits all of the proceeds from the security in a bank account that pays an effective rate of interest of 8% *pa*.

Calculate the amount of money in the bank account after 10 years.

Solution

The interest payments are £80 *pa* payable half-yearly in arrears. After 10 years, the accumulated value of these will be:

$$80s_{\overline{10}|}^{(2)@8\%}=80\frac{i}{i^{(2)}}s_{\overline{10}|}=80\times 1.019615\times 14.4866=£1{,}182$$

using values from the *Tables*.

No payments received after time 10 years are included in this calculation.

The above proportional arguments may be applied to other varying series of payments. Consider, for example, an annuity payable annually in arrears for *n* years, the payment in the *t* th year being x_t. The present value of this annuity is obviously:

$$a=\sum_{t=1}^{n}x_t v^t \tag{4.6}$$

Consider now a second annuity, also payable for *n* years with the payment in the *t* th year, again of amount x_t, being made in *p* equal instalments in arrears over that year. If $a^{(p)}$ denotes the present value of this second annuity, by replacing the *p* payments for year *t* (each of amount x_t/p) by a single equivalent payment at the end of the year of amount $x_t[i/i^{(p)}]$, we immediately obtain:

$$a^{(p)}=\frac{i}{i^{(p)}}a$$

where *a* is given by equation (4.6) above.

Question

A series of payments of \$1 at time 1, \$4 at time 2, \$9 at time 3, and so on up to \$100 at time 10 has a present value of \$245.32 when evaluated at an effective interest rate of 6% *pa*.

Calculate the present value of the series of payments if, instead of being paid at the end of each year, each annual amount is split into three equal instalments paid at the end of each third of a year.

Solution

The present value will be a factor of $i/i^{(3)}$ greater. Therefore:

$$PV=245.32\times\frac{i}{i^{(3)}}=245.32\times\frac{0.06}{0.058838}=\$250.16$$

4.3 Annuities payable *p*thly where $p < 1$

If $p<1$, then we are considering an annuity payable less frequently than annually. For example, if $p=0.5$, the annuity is payable every two years. The formulae for $a_{\overline{n}|}^{(p)}$ and $\ddot{a}_{\overline{n}|}^{(p)}$ are still valid for $p<1$.

In Section 4.1 the symbol $a_{\overline{n}|}^{(p)}$ was introduced. Intuitively, with this notation one considers *p* to be an integer greater than 1 and assumes that the product $n.p$ is also an integer. (This, of course, will be true when *n* itself is an integer, but one might, for example, have $p=4$ and $n=5.75$ so that $np=23$.) Then $a_{\overline{n}|}^{(p)}$ denotes the value at time 0 of $n.p$ payments, each of amount $1/p$, at times $1/p, 2/p, \ldots, (np)/p$.

Non-integer values of *n* will be looked at more closely in the next section.

From a theoretical viewpoint it is perhaps worth noting that when *p* is the reciprocal of an integer and $n.p$ is also an integer (*eg* when $p=0.25$ and $n=28$), $a_{\overline{n}|}^{(p)}$ still gives the value at time 0 of $n.p$ payments, each of amount $1/p$, at times $1/p, 2/p, \ldots, (np)/p$.

For example, the value at time 0 of a series of seven payments, each of amount 4, at times 4, 8, 12 ,..., 28 may be denoted by $a_{\overline{28}|}^{(0.25)}$.

It follows that this value equals:

$$\frac{1-v^{28}}{(0.25)\left[(1+i)^4-1\right]}$$

This last expression may be written in the form:

$$\left[\frac{4}{\frac{(1+i)^4-1}{i}}\right].\frac{1-v^{28}}{i}=\frac{4}{s_{\overline{4}|}}.a_{\overline{28}|}$$

Although $a_{\overline{28}|}^{(0.25)}$ *can* be written in the form immediately above, it is unlikely that we would ever choose to do so.

5 Non-integer values of *n*

Let p be a positive integer. Until now the symbol $a_{\overline{n}|}^{(p)}$ has been defined only when n is a positive integer. For certain non-integral values of n the symbol $a_{\overline{n}|}^{(p)}$ has an intuitively obvious interpretation. For example, it is not clear what meaning, if any, may be given to $a_{\overline{23.5}|}$, but the symbol $a_{\overline{23.5}|}^{(4)}$ ought to represent the present value of an immediate annuity of 1 per annum payable quarterly in arrears for 23.5 years (*ie* a total of 94 quarterly payments, each of amount 0.25). On the other hand, $a_{\overline{23.25}|}^{(2)}$ has no obvious meaning.

Suppose that n is an integer multiple of $1/p$, say $n = r/p$, where r is an integer. In this case we define $a_{\overline{n}|}^{(p)}$ to be the value at time 0 of a series of r payments, each of amount $1/p$, at times $1/p, 2/p, 3/p, \ldots, r/p = n$. If $i = 0$, then clearly $a_{\overline{n}|}^{(p)} = n$. If $i > 0$, then:

$$
\begin{aligned}
a_{\overline{n}|}^{(p)} &= \frac{1}{p}\left(v^{1/p} + v^{2/p} + v^{3/p} + \cdots + v^{r/p}\right) \\
&= \frac{1}{p} v^{1/p} \left(\frac{1 - v^{r/p}}{1 - v^{1/p}}\right) \\
&= \frac{1}{p} \left[\frac{1 - v^{r/p}}{(1+i)^{1/p} - 1}\right]
\end{aligned}
$$

The derivation above uses the formula for the sum of a geometric progression of r terms, with first term $v^{1/p}$ and common ratio $v^{1/p}$.

Thus:

$$
a_{\overline{n}|}^{(p)} = \begin{cases} \dfrac{1 - v^n}{i^{(p)}} & \text{if } i > 0 \\ n & \text{if } i = 0 \end{cases} \tag{5.1}
$$

The standard formula for $a_{\overline{n}|}^{(p)}$ therefore applies for non-integer values of n when n is an integer multiple of $\frac{1}{p}$.

Note that, by working in terms of a new time unit equal to $\frac{1}{p}$ times the original time unit and with the equivalent effective interest rate of $\frac{i^{(p)}}{p}$ per new time unit, we see that:

$$a^{(p)}_{\overline{n|}} \text{ (at rate } i\text{)} = \frac{1}{p} a_{\overline{np|}} \text{ (at rate } \frac{i^{(p)}}{p}\text{)}$$

This formula is useful when $\frac{i^{(p)}}{p}$ is a tabulated rate of interest.

Question

Calculate $a^{(12)}_{\overline{3.5|}}$ given that $i = 19.5618\%$.

Solution

We could evaluate this directly using (5.1):

$$a^{(12)}_{\overline{3.5|}} = \frac{1-v^{3.5}}{i^{(12)}} = \frac{1-1.195618^{-3.5}}{12(1.195618^{1/12}-1)} = 2.5828$$

Alternatively, we could spot that when $i = 19.5618\%$, $i^{(12)} = 12(1.195618^{1/12}-1) = 18.00\%$ and $\frac{i^{(12)}}{12} = 1.5\%$. So, using values from the *Tables*:

$$a^{(12)}_{\overline{3.5|}} = \frac{1}{12} a^{@1.5\%}_{\overline{42|}} = \frac{1}{12} \times 30.9941 = 2.5828$$

Note that the definition of $a^{(p)}_{\overline{n|}}$ given by equation (5.1) is mathematically meaningful for *all* non-negative values of *n*. For our present purpose, therefore, it is convenient to adopt equation (5.1) as a definition of $a^{(p)}_{\overline{n|}}$ for all *n*.

This is only a mathematical definition. It is not easily translated into the present value of a series of payments.

If *n* is not an integer multiple of $\frac{1}{p}$, there is no universally recognised definition of $a^{(p)}_{\overline{n|}}$. For example, if $n = n_1 + f$, where n_1 is an integer multiple of $1/p$ and $0 < f < 1/p$, some writers define $a^{(p)}_{\overline{n|}}$ as:

$$a^{(p)}_{\overline{n_1|}} + fv^n$$

With this alternative definition:

$$a^{(2)}_{\overline{23.75}|} = a^{(2)}_{\overline{23.5}|} + 1/4\, v^{23.75}$$

which is the present value of an annuity of 1 per annum, payable half-yearly in arrears for 23.5 years, together with a final payment of 0.25 after 23.75 years. Note that this is *not* equal to the value obtained from definition (5.1).

For example, using $i = 0.03$, definition (5.1) gives:

$$a^{(2)}_{\overline{23.75}|} = \frac{1 - v^{23.75}}{i^{(2)}} = \frac{1 - 1.03^{-23.75}}{2(1.03^{1/2} - 1)} = 16.9391$$

but using the alternative definition instead gives:

$$a^{(2)}_{\overline{23.75}|} = a^{(2)}_{\overline{23.5}|} + \tfrac{1}{4} v^{23.75} = \frac{1 - 1.03^{-23.5}}{2(1.03^{1/2} - 1)} + \tfrac{1}{4} \times 1.03^{-23.75} = 16.9395$$

We can extend the above results to develop formulae for $\ddot{a}^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$ and $\ddot{s}^{(p)}_{\overline{n}|}$ for all non-negative n.

If $i > 0$, we define for all non-negative n:

$$\left.\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|} &= (1+i)^{1/p} a^{(p)}_{\overline{n}|} = \frac{(1 - v^n)}{d^{(p)}} \\
s^{(p)}_{\overline{n}|} &= (1+i)^n a^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}} \\
\ddot{s}^{(p)}_{\overline{n}|} &= (1+i)^n \ddot{a}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}}
\end{aligned}\right\} \qquad (5.2)$$

If $i = 0$, each of these last three functions is defined to equal *n*.

Whenever *n* is an integer multiple of $1/p$, say $n = r/p$, then $\ddot{a}^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, are values at different times of an annuity-certain of *r* payments, each of amount $1/p$, at intervals of $1/p$ time unit.

As before, we use the simpler notations $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ to denote $a^{(1)}_{\overline{n}|}$, $\ddot{a}^{(1)}_{\overline{n}|}$, $s^{(1)}_{\overline{n}|}$ and $\ddot{s}^{(1)}_{\overline{n}|}$ respectively, thus extending the definition of $a_{\overline{n}|}$ *etc*, to all non-negative values of *n*.

It is a trivial consequence of our definitions that the formulae:

$$\left.\begin{aligned}
a_{\overline{n}|}^{(p)} &= \frac{i}{i^{(p)}} a_{\overline{n}|} \\
\ddot{a}_{\overline{n}|}^{(p)} &= \frac{i}{d^{(p)}} a_{\overline{n}|} \\
s_{\overline{n}|}^{(p)} &= \frac{i}{i^{(p)}} s_{\overline{n}|} \\
\ddot{s}_{\overline{n}|}^{(p)} &= \frac{i}{d^{(p)}} s_{\overline{n}|}
\end{aligned}\right\} \qquad (5.3)$$

(valid when $i > 0$) now hold for *all* values of *n*.

6 Perpetuities

We can also consider an annuity that is payable forever. This is called a *perpetuity*. For example, consider an equity that pays a dividend of £10 at the end of each year. An investor who purchases the equity pays an amount equal to the present value of the dividends. The present value of the dividends is:

$$10v + 10v^2 + 10v^3 + \cdots$$

This can be summed using the formula for an infinite geometric progression:

$$10v + 10v^2 + 10v^3 + \cdots = \frac{10v}{1-v} = \frac{10}{i}$$

The formula for the sum to infinity of a geometric progression with first term a and common ratio r is $\frac{a}{1-r}$ provided $|r| < 1$.

Recall the formula for the present value of an annuity of £10 *pa* that continues for *n* years:

$$10a_{\overline{n}|} = \frac{10(1-v^n)}{i}$$

We have let $n \to \infty$ in this expression in order to arrive at the formula $\frac{10}{i}$.

Note that this formula only holds when i is positive.

Question

Calculate the present value of an annuity that pays £150 *pa* annually in arrears forever using an annual effective rate of interest of 8%.

Solution

The present value is:

$$150a_{\overline{\infty}|} = \frac{150}{i} = \frac{150}{0.08} = £1,875$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $150v$ and common ratio v :

$$150v + 150v^2 + 150v^3 + \cdots = \frac{150v}{1-v} = £1,875$$

In general, the present value of payments of 1 *pa* payable at the start and end of each year forever is $\frac{1}{d}$ and $\frac{1}{i}$, respectively. That is, $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$ and $a_{\overline{\infty}|} = \frac{1}{i}$.

Question

Calculate the present value of payments of £2,000 at times $0,1,2,\ldots$ using $i = 7.6\%$ *pa* effective.

Solution

The present value is:

$$2{,}000\ddot{a}_{\overline{\infty|}} = \frac{2{,}000}{d} = \frac{2{,}000}{0.076/1.076} = £28{,}315.79$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $2{,}000$ and common ratio v :

$$2{,}000 + 2{,}000v + 2{,}000v^2 + 2{,}000v^3 + \cdots = \frac{2{,}000}{1-v} = £28{,}315.79$$

Perpetuities payable pthly

The present value of payments of 1 *pa* payable in instalments of $\frac{1}{p}$ at the end of each *p*thly time period forever is:

$$a^{(p)}_{\overline{\infty|}} = \frac{1}{i^{(p)}}$$

The present value of payments of 1 *pa* payable in instalments of $\frac{1}{p}$ at the start of each *p*thly time period forever is:

$$\ddot{a}^{(p)}_{\overline{\infty|}} = \frac{1}{d^{(p)}}$$

Question

Calculate the present value of an annuity that pays £300 *pa* monthly in arrears forever using an annual effective rate of interest of 6%.

Solution

The present value is:

$$300a^{(12)}_{\overline{\infty|}} = \frac{300}{i^{(12)}} = \frac{300}{0.0584106} = £5{,}136.05$$

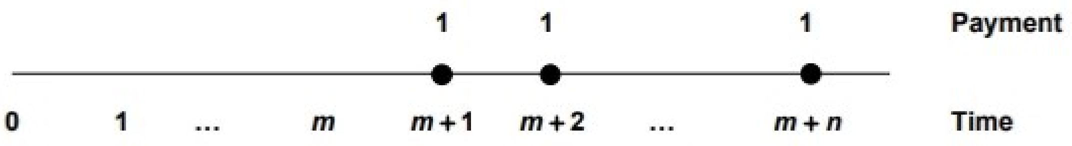
We could alternatively calculate this using the sum to infinity of a geometric progression with first term $\frac{300}{12}v^{\frac{1}{12}}$ and common ratio $v^{\frac{1}{12}}$:

$$\frac{300}{12}v^{\frac{1}{12}}+\frac{300}{12}v^{\frac{2}{12}}+\frac{300}{12}v^{\frac{3}{12}}+\cdots=\frac{300}{12}\times\frac{v^{\frac{1}{12}}}{1-v^{\frac{1}{12}}}=£5,136.05$$

7 Deferred annuities

7.1 Annual payments

Suppose that m and n are non-negative integers. The value at time 0 of a series of n payments, each of amount 1, due at times $(m+1), (m+2), \ldots, (m+n)$ is denoted by ${}_{m|}a_{\overline{n}|}$ (see the figure below).



Such a series of payments may be considered as an immediate annuity, deferred for m time units. When $n > 0$:

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + v^{m+3} + \cdots + v^{m+n} \\ &= (v + v^2 + v^3 + \cdots + v^{m+n}) - (v + v^2 + v^3 + \cdots + v^m) \\ &= v^m (v + v^2 + v^3 + \cdots + v^n) \end{aligned}$$

The last two equations show that:

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} \tag{7.1}$$

and:

$${}_{m|}a_{\overline{n}|} = v^m a_{\overline{n}|} \tag{7.2}$$

Either of these two equations may be used to determine the value of a deferred immediate annuity. Together they imply that:

$$a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|}$$

This formula could easily be deduced using general reasoning. The present value of a series of $(n+m)$ payments of one unit payable at the end of each time period is equal to the sum of:

(a) the present value of m payments of one unit payable at the end of each time period, and

(b) the present value of n payments of one unit payable at the end of each time period, deferred for m years.

Question

Using both equations (7.1) and (7.2), calculate the value of ${}_{8|}a_{\overline{12}|}$ using an interest rate of 6.2% *pa* convertible half-yearly.

Solution

An interest rate of 6.2% *pa* convertible half-yearly is equivalent to an effective annual interest rate of:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = 1.031^2 - 1 = 6.2961\%$$

Using Equation (7.1), we have:

$$_{8|}a_{\overline{12|}} = a_{\overline{20|}} - a_{\overline{8|}} = \frac{1-v^{20}}{i} - \frac{1-v^8}{i} = 11.19923 - 6.13767 = 5.06156$$

Using Equation (7.2), we have:

$$_{8|}a_{\overline{12|}} = v^8 a_{\overline{12|}} = 0.613566 \times 8.249406 = 5.06156$$

We may define the corresponding deferred annuity-due as:

$$_{m|}\ddot{a}_{\overline{n|}} = v^m \ddot{a}_{\overline{n|}}$$

Since $\ddot{a}_{\overline{n|}} = (1+i)a_{\overline{n|}}$ (from equation 1.3), we can also write:

$$_{m|}\ddot{a}_{\overline{n|}} = v^m (1+i) a_{\overline{n|}} = v^{m-1} a_{\overline{n|}}$$

7.2 Continuously payable annuities

If *m* is a non-negative number, we use the symbol $_{m|}\bar{a}_{\overline{n|}}$ to denote the present value of a continuously payable annuity of 1 per unit for *n* time units, deferred for *m* time units. Thus:

$$\begin{aligned} _{m|}\bar{a}_{\overline{n|}} &= \int_m^{m+n} e^{-\delta t}\,dt \\ &= e^{-\delta m}\int_0^n e^{-\delta s}\,ds \\ &= \int_0^{m+n} e^{-\delta t}\,dt - \int_0^m e^{-\delta t}\,dt \end{aligned}$$

Hence:

$$\begin{aligned} _{m|}\bar{a}_{\overline{n|}} &= \bar{a}_{\overline{m+n|}} - \bar{a}_{\overline{m|}} \\ &= v^m \bar{a}_{\overline{n|}} \end{aligned}$$

Question

An alien on the planet Xi is currently aged exactly 48 years old. When the alien retires at exact age 60, it will receive an income of 500 per year, payable continuously.

Given that all aliens on the planet Xi die at exact age 76, calculate the present value of the retirement benefits using an interest rate of 10% *pa* effective.

Solution

The present value of the retirement benefits is:

$$500 \times {}_{12|}\bar{a}_{\overline{16}|} = 500v^{12}\left(\frac{1-v^{16}}{\delta}\right) = 500 \times 1.1^{-12} \times \frac{1-1.1^{-16}}{\ln 1.1} = 1{,}308$$

7.3 Annuities payable *p*thly

The present values of an immediate annuity and an annuity-due, payable *p*thly at the rate of 1 per unit time for *n* time units and deferred for *m* time units, are denoted by:

$$\left.\begin{aligned} &{}_{m|}a^{(p)}_{\overline{n}|} = v^m a^{(p)}_{\overline{n}|} \\ \text{and}\quad &{}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = v^m \ddot{a}^{(p)}_{\overline{n}|} \end{aligned}\right\} \qquad (7.3)$$

respectively.

Question

Calculate ${}_{5|}a^{(2)}_{\overline{10}|}$ and ${}_{6|}\ddot{a}^{(2)}_{\overline{10}|}$ using an interest rate of 5% *pa* effective, and explain why the value of ${}_{5|}a^{(2)}_{\overline{10}|}$ is higher.

Solution

Evaluating the functions using $i = 0.05$, we have:

$${}_{5|}a^{(2)}_{\overline{10}|} = v^5 a^{(2)}_{\overline{10}|} = 1.05^{-5} \times \frac{1-1.05^{-10}}{2(1.05^{1/2}-1)} = 6.125$$

and
$${}_{6|}\ddot{a}^{(2)}_{\overline{10}|} = v^6 \ddot{a}^{(2)}_{\overline{10}|} = 1.05^{-6} \times \frac{1-1.05^{-10}}{2(1-1.05^{-1/2})} = 5.977$$

Both annuity functions value a ten-year annuity payable half-yearly. The first payment of ${}_{5|}a^{(2)}_{\overline{10}|}$ is at time 5½, whereas the first payment of ${}_{6|}\ddot{a}^{(2)}_{\overline{10}|}$ is at time 6. Therefore since the payments are made sooner, ${}_{5|}a^{(2)}_{\overline{10}|}$ has the greater value.

In fact, since the first payment of ${}_{5|}a^{(2)}_{\overline{10}|}$ is made half a year sooner than the first payment of ${}_{6|}\ddot{a}^{(2)}_{\overline{10}|}$, we have:

$${}_{5|}a^{(2)}_{\overline{10}|} = (1+i)^{0.5}\ {}_{6|}\ddot{a}^{(2)}_{\overline{10}|}$$

7.4 Non-integer values of *n*

We may also extend the definitions of ${}_{m|}a^{(p)}_{\overline{n}|}$ and ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ to all values of *n* by the formulae:

$$\left.\begin{aligned} {}_{m|}a^{(p)}_{\overline{n}|} &= v^m a^{(p)}_{\overline{n}|} \\ {}_{m|}\ddot{a}^{(p)}_{\overline{n}|} &= v^m \ddot{a}^{(p)}_{\overline{n}|} \end{aligned}\right\} \tag{7.4}$$

and so:

$$\left.\begin{aligned} {}_{m|}a^{(p)}_{\overline{n}|} &= a^{(p)}_{\overline{n+m}|} - a^{(p)}_{\overline{m}|} \\ {}_{m|}\ddot{a}^{(p)}_{\overline{n}|} &= \ddot{a}^{(p)}_{\overline{n+m}|} - \ddot{a}^{(p)}_{\overline{m}|} \end{aligned}\right\} \tag{7.5}$$

This is easily proved:

$$v^m a^{(p)}_{\overline{n}|} = v^m \left(\frac{1-v^n}{i^{(p)}}\right) = \frac{v^m - v^{n+m}}{i^{(p)}} = \frac{(1-v^{n+m})-(1-v^m)}{i^{(p)}} = a^{(p)}_{\overline{n+m}|} - a^{(p)}_{\overline{m}|}$$

and similarly for $v^m \ddot{a}^{(p)}_{\overline{n}|}$.

7.5 Sudden changes in interest rates

We considered sudden changes in interest rates for single payments earlier in the course. We now consider the impact of changes in interest rates on a series of payments. This involves the use of deferred annuities.

Question

Calculate the present value as at 1 January 2019 of payments of £100 on the first day of each quarter during calendar years 2021 to 2030 inclusive.

Assume the effective rate of interest is 8% per annum until 31 December 2025 and 6% per annum thereafter.

Solution

Here we must value a ten-year annuity payable quarterly in advance, deferred for two years. We have to split the ten-year annuity into two five-year annuities to allow for the change in interest rates.

The present value here is:

$$400v^{2@8\%}(\ddot{a}_{\overline{5}|}^{(4)@8\%} + v^{5@8\%}\ \ddot{a}_{\overline{5}|}^{(4)@6\%})$$

$$= 400 \times 0.85734\left(\frac{1-0.68058}{0.076225} + 0.68058 \times \frac{1-0.74726}{0.057847}\right)$$

$$= 400 \times 6.1420 = £2,457$$

Chapter 7 Summary

An annuity consists of a regular series of payments. Payments continue for a specified period. The amounts of the payments may be level, increasing or decreasing. Continuous annuities involve continuous payments paid at specified rates.

The symbol $a_{\overline{n}|}$ ($\ddot{a}_{\overline{n}|}$) represents the PV of an annuity consisting of n payments of 1 unit made at the end (start) of each of the next n time periods. This is called an annuity paid in arrears (advance). The formulae for the present values are:

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{i}{d} a_{\overline{n}|} = (1+i) a_{\overline{n}|} = a_{\overline{n-1}|} + 1$$

$s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are the accumulated values at the end of any period of length n of a series of n payments, each of amount 1, made at unit time intervals over the period, where the payments are made in arrears and in advance respectively. The formulae for the accumulated values are:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \frac{i}{d} s_{\overline{n}|} = (1+i) s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

The value at time 0 of an annuity payable continuously between time 0 and time n, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$. The formulae for continuous payments are:

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\overline{n}|} \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} s_{\overline{n}|} \qquad PV = \int_0^n \rho(t) v^t \, dt$$

If p and n are positive integers, the notation $a^{(p)}_{\overline{n}|}$ is used to denote the value at time 0 of a level annuity payable $p$thly in arrears at the rate of 1 per unit time over the time interval $[0,n]$. For this annuity the payments are made at times $1/p, 2/p, 3/p, \ldots, n$ and the amount of each payment is $1/p$. The required formulae are:

$$a^{(p)}_{\overline{n}|} = \frac{1-v^n}{i^{(p)}} = \frac{i}{i^{(p)}} a_{\overline{n}|} \qquad s^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(p)}} = \frac{i}{i^{(p)}} s_{\overline{n}|}$$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1-v^n}{d^{(p)}} = \frac{i}{d^{(p)}} a_{\overline{n}|} \qquad \ddot{s}^{(p)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(p)}} = \frac{i}{d^{(p)}} s_{\overline{n}|}$$

There is more than one definition of $a^{(p)}_{\overline{n}|}$ when n is not an integer multiple of $1/p$.

An annuity that is payable forever is called a perpetuity. The required formulae are:

$$a_{\overline{\infty}|} = \frac{1}{i} \quad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad a^{(p)}_{\overline{\infty}|} = \frac{1}{i^{(p)}} \qquad \ddot{a}^{(p)}_{\overline{\infty}|} = \frac{1}{d^{(p)}}$$

Deferred annuities are annuities where no payment is made during the first time period.

The value at time 0 of a series of n payments, each of amount 1, due at times $(m+1), (m+2), \ldots, (m+n)$ is denoted by ${}_{m|}a_{\overline{n}|}$:

$${}_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|} = v^m a_{\overline{n}|}$$

Other functions exist for annuities payable in advance, continuously and *p*thly:

$${}_{m|}\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|} = v^m \ddot{a}_{\overline{n}|} \qquad {}_{m|}\bar{a}_{\overline{n}|} = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} = v^m \bar{a}_{\overline{n}|}$$

$${}_{m|}a^{(p)}_{\overline{n}|} = v^m a^{(p)}_{\overline{n}|} \qquad {}_{m|}\ddot{a}^{(p)}_{\overline{n}|} = v^m \ddot{a}^{(p)}_{\overline{n}|}$$

Chapter 7 Practice Questions

7.1 (i) Calculate the following functions at $i = 9\%$:

(a) $a_{\overline{3}|}^{(4)}$

(b) $\ddot{a}_{\overline{4}|}^{(4)}$

(c) $\bar{s}_{\overline{10}|}$

(ii) Calculate the following functions at $i = 25\%$:

(a) $\ddot{a}_{\overline{10}|}^{(12)}$

(b) $a_{\overline{6.5}|}^{(12)}$

7.2 Calculate the accumulated value as at 1 January 2020 of payments of £100 paid every two years from 1 January 1980 to 1 January 2018 inclusive, using an interest rate of 12% *pa* effective.

7.3 Calculate the present value on 1 June 2019 of payments of £1,000 payable on the first day of each month from July 2019 to December 2019 inclusive, assuming a rate of interest of 8% per annum convertible quarterly.

7.4 An annuity provides payments of $40 at the end of each month forever. If the interest rate is 10% *pa* convertible quarterly, calculate the present value of the annuity.

7.5 An annuity-certain is payable monthly in advance for 40 years. The annuity is to be paid at the rate of £100 *pa* for the first 20 years, £120 *pa* for the next 5 years and £200 *pa* for the last 15 years.

Determine whether each of the following expressions gives the correct present value of the payments as at the commencement date of the annuity, assuming an annual effective interest rate of i .

(i) $(100a_{\overline{20}|} + 120v^{20}a_{\overline{5}|} + 200v^{25}a_{\overline{15}|})\dfrac{i}{i^{(12)}}$

(ii) $200\ddot{a}_{\overline{40}|}^{(12)} - 80\ddot{a}_{\overline{25}|}^{(12)} - 20\ddot{a}_{\overline{20}|}^{(12)}$

(iii) $100(1 + a_{\overline{39}|}^{(12)}) + 20v^{20}(1 + a_{\overline{19}|}^{(12)}) + 80v^{25}(1 + a_{\overline{14}|}^{(12)})$

7.6 Calculate the present value as at 1 June 2018 of 41 monthly payments each of £100 commencing on 1 January 2019, assuming a rate of interest of 10% *pa* convertible half-yearly.

7.7 Payments of £1,000 *pa* are payable quarterly in arrears from 1/1/20 to 31/12/25.

The annual effective rate of interest is 3.4% for calendar years 2020-2023 (inclusive) and 4.2% thereafter. Calculate:

(i) the present value of the payments at 1/1/20

(ii) the accumulated value of the payments at 1/1/27.

7.8 X denotes the present value of an annuity consisting of payments of £2,000 payable at the end of each of the next 8 years, valued using an interest rate of 8% *pa* convertible quarterly.

Y denotes the present value of an annuity consisting of payments of £4,000 payable at the end of every fourth year for the next 16 years, valued using an interest rate of 8% *pa* convertible half-yearly.

Calculate the ratio X/Y.

7.9 Using an interest rate of 12% *pa* convertible monthly, calculate:

Exam style

(i) the combined present value of an immediate annuity payable monthly in arrears such that payments are £1,000 *pa* for the first 6 years and £400 *pa* for the next 4 years, together with a lump sum of £2,000 at the end of the 10 years. [3]

(ii) the amount of the level annuity payable continuously for 10 years having the same present value as the payments in (i). [3]

(iii) the accumulated value of the first 7 years' payments at the end of the 7th year for the payments in (i) and (ii). [3]

[Total 9]

Chapter 7 Solutions

7.1 (i)(a) $a^{(4)}_{\overline{3|}} = \dfrac{1-v^3}{i^{(4)}} = \dfrac{1-1.09^{-3}}{0.087113} = 2.6152$

(i)(b) $\ddot{a}^{(4)}_{\overline{4|}} = \dfrac{1-v^4}{d^{(4)}} = \dfrac{1-1.09^{-4}}{0.085256} = 3.4200$

(i)(c) $\bar{s}_{\overline{10|}} = \dfrac{(1+i)^{10}-1}{\delta} = \dfrac{1.09^{10}-1}{\ln(1.09)} = 15.8668$

(ii)(a) $\ddot{a}^{(12)}_{\overline{10|}} = \dfrac{1-v^{10}}{d^{(12)}} = \dfrac{1-1.25^{-10}}{0.221082} = 4.0375$

(ii)(b) $a^{(12)}_{\overline{6.5|}} = \dfrac{1-v^{6.5}}{i^{(12)}} = \dfrac{1-1.25^{-6.5}}{0.225231} = 3.3989$

7.2 Working in two-yearly time periods with an effective two-yearly interest rate of:

$$i = 1.12^2 - 1 = 25.44\%$$

we can calculate the present value as:

$$100\ddot{s}_{\overline{20|}} = 100 \times \frac{1.2544^{20}-1}{0.2544/1.2544} = 100 \times 453.89 = £45,389$$

Alternatively, we could work in years and calculate $50\ddot{s}^{(½)}_{\overline{40|}}$ @12%.

7.3 As at 1 June 2019, the payments of £1,000 are made at the end of each of the next 6 months. We are given $i^{(4)} = 8\%$, so the effective annual interest rate is:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = 1.02^4 - 1 = 8.2432\%$$

and the effective monthly interest rate is:

$$1.082432^{1/12} - 1 = 0.66227\%$$

So, working in months, the present value of the payments on 1 June 2019 is:

$$1,000a_{\overline{6|}}^{@0.66227\%} = 1,000 \times \frac{1-1.0066227^{-6}}{0.0066227} = £5,863$$

Alternatively, the present value can be expressed as $1,000v\ddot{a}_{\overline{6|}}^{@0.66227\%}$ *or* $12,000a^{(12)\ @8.2432\%}_{\overline{0.5|}}$.

7.4 We are given that $i^{(4)} = 0.1$, so the effective annual interest rate is:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = 10.381\%$$

Using an interest rate of 10.381%, the present value is:

$$40 \times 12 a_{\overline{\infty|}}^{(12)} = \frac{40 \times 12}{i^{(12)}} = \frac{40}{(1+i)^{\frac{1}{12}} - 1} = \$4,839.78$$

We could alternatively calculate this using the sum to infinity of a geometric progression with first term $40v^{\frac{1}{12}}$ and common ratio $v^{\frac{1}{12}}$:

$$40v^{\frac{1}{12}} + 40v^{\frac{2}{12}} + 40v^{\frac{3}{12}} + \cdots = \frac{40v^{\frac{1}{12}}}{1 - v^{\frac{1}{12}}} = \$4,839.78$$

7.5 (i) This expression is incorrect. It is the present value of the same series of payments made monthly in arrears, not monthly in advance. As the payments are made monthly in advance, the ratio outside the bracket should be $\frac{i}{d^{(12)}}$ not $\frac{i}{i^{(12)}}$.

(ii) This expression is correct.

(iii) This expression is incorrect. A correct expression is:

$$100\ddot{a}_{\overline{40|}}^{(12)} + 20v^{20}\ddot{a}_{\overline{20|}}^{(12)} + 80v^{25}\ddot{a}_{\overline{15|}}^{(12)}$$

but note that $1 + a_{\overline{39|}}^{(12)} \neq \ddot{a}_{\overline{40|}}^{(12)}$, for example.

7.6 10% *pa* convertible half-yearly is equivalent to an effective annual interest rate of:

$$i = \left(1 + \frac{0.1}{2}\right)^2 - 1 = 10.25\%$$

So, working in years, the present value of the annuity is:

$$1,200v^{7/12}\ddot{a}_{\overline{41/12|}}^{(12)} @10.25\% = 1,200 \times 1.1025^{-7/12} \times 2.91729 = £3,307$$

Alternatively, working in months, with an effective monthly interest rate of:

$$1.1025^{1/12} - 1 = 0.81648\%$$

the present value of the payments is:

$$100v^7\ddot{a}_{\overline{41|}} @0.81648\% = 100 \times 1.0081648^{-7} \times 35.00751 = £3,307$$

We could alternatively use either of the following expressions to evaluate this present value:

- $1{,}200v^{6/12}a^{(12)}_{\overline{41/12}|}$ @10.25% *(working in years)*
- $100v^{6}a_{\overline{41}|}$ @0.81648% *(working in months)*

7.7 (i) The present value is:

$$1{,}000a^{(4)}_{\overline{4}|} @3.4\% + 1{,}000v^{4}_{@3.4\%}a^{(4)}_{\overline{2}|} @4.2\%$$

Evaluating these:

$$i^{(4)}_{@3.4\%} = 4\left(1.034^{\frac{1}{4}} - 1\right) = 0.03357$$

$$i^{(4)}_{@4.2\%} = 4\left(1.042^{\frac{1}{4}} - 1\right) = 0.04135$$

The present value is then:

$$1{,}000 \times \frac{1-1.034^{-4}}{0.03357} + 1{,}000 \times 1.034^{-4} \times \frac{1-1.042^{-2}}{0.04135} = £5{,}399.40$$

(ii) We need to accumulate the answer to part (i) by 7 years. Four of these years have an interest rate of 3.4% *pa* and the remainder have an interest rate of 4.2% *pa*:

$$5{,}399.40 \times 1.034^{4} \times 1.042^{3} = £6{,}982.81$$

7.8 Consider X first. An interest rate of 8% *pa* convertible quarterly corresponds to an effective interest rate of 2% per quarter.

Working in years, using an effective annual interest rate of $1.02^{4} - 1 = 8.2432\%$, gives:

$$X = 2{,}000a^{@8.2432\%}_{\overline{8}|} = 2{,}000 \times \frac{1-(1.02^{4})^{-8}}{1.02^{4}-1} = £11{,}388$$

Now consider Y. An interest rate of 8% *pa* convertible half-yearly corresponds to an effective rate of 4% per half-year. Working in 4-year time periods, using an effective 4-yearly interest rate of $1.04^{8} - 1 = 36.8569\%$, gives:

$$Y = 4{,}000a^{@36.8569\%}_{\overline{4}|} = 4{,}000 \times \frac{1-(1.04^{8})^{-4}}{1.04^{8}-1} = £7{,}759$$

So $\dfrac{X}{Y} = \dfrac{11{,}388}{7{,}759} = 1.468$.

7.9 (i) An interest rate of 12% *pa* convertible monthly is equivalent to a monthly effective interest rate of 1%.

So, working in months, the present value is:

$$PV = \frac{1,000}{12} a_{\overline{72}|} + \frac{400}{12} v^{72} a_{\overline{48}|} + 2,000 v^{120} \quad @1\% \qquad [1½]$$

$$= \frac{1,000}{12} \times 51.1504 + \frac{400}{12} \times 1.01^{-72} \times 37.9740 + 2,000 \times 1.01^{-120}$$

$$= 4,262.53 + 618.34 + 605.99 = £5,486.86 \qquad [1½]$$

[Total 3]

(ii) If the annual rate of payment is X, then, still working in months:

$$5,486.86 = \frac{X}{12} \bar{a}_{\overline{120}|} = \frac{X}{12} \left(\frac{1 - v^{120}}{\delta} \right) = \frac{X}{12} \times 70.0484 \qquad [2]$$

So:

$$X = 5,486.86 \times \frac{12}{70.0484} = £939.95 \qquad [1]$$

[Total 3]

(iii) Working in months, the accumulated value at time 7 years of the first 7 years' (*ie* 84 months') payments in (i) is:

$$AV = (1+i)^{84} \left(\frac{1,000}{12} a_{\overline{72}|} + \frac{400}{12} v^{72} a_{\overline{12}|} \right)$$

$$= (1.01)^{84} \left(\frac{1,000}{12} \times 51.1504 + \frac{400}{12} \times 1.01^{-72} \times 11.2551 \right) = £10,255.23 \qquad [2]$$

Using the value of X calculated in (ii) rounded to the nearest pence, the accumulated value for (ii) is:

$$\frac{939.95}{12} \bar{s}_{\overline{84}|} = \frac{939.95}{12} \times 131.325 = £10,286.54 \qquad [1]$$

[Total 3]

8

Increasing annuities

Syllabus objectives

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i , v , n , d , δ , $i^{(p)}$ and $d^{(p)}$:

2.5.3 $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\bar{a})_{\overline{n}|}$, $(\overline{Ia})_{\overline{n}|}$ and the respective deferred annuities.

0 Introduction

In this chapter we will derive formulae for calculating the present values of simple increasing annuities. A simple increasing annuity is an annuity where the payments increase each time by a fixed amount. For example, a payment of £3 at time 1, £6 at time 2, £9 at time 3 *etc*.

At the end of the chapter we give some special cases that use the techniques that have been developed so far in the course.

1 Varying annuities

1.1 Annual payments

For an annuity in which the payments are not all of an equal amount, it is a simple matter to find the present (or accumulated) value from first principles. Thus, for example, the present value of such an annuity may always be evaluated as:

$$\sum_{i=1}^{n} X_i v^{t_i}$$

where the *i*th payment, of amount X_i, is made at time t_i.

If there are also continuous payments, then the present value may be calculated as:

$$\sum_{i=1}^{n} X_i v^{t_i} + \int_{-\infty}^{\infty} \rho(t) v^t dt$$

where $\rho(t)$ is the rate of payment per time unit at time *t*.

In the particular case when $X_i = t_i = i$, the annuity is known as an 'increasing annuity' and its present value is denoted by $(Ia)_{\overline{n|}}$.

$(Ia)_{\overline{n|}}$, therefore, represents the present value of payments of 1 at the end of the first time period, 2 at the end of the second time period, …, *n* at the end of the *n*th time period.

Thus:

$$(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \cdots + nv^n \tag{1.1}$$

Hence: $(1+i)(Ia)_{\overline{n|}} = 1 + 2v + 3v^2 + \cdots + nv^{n-1}$

By subtraction, we obtain:

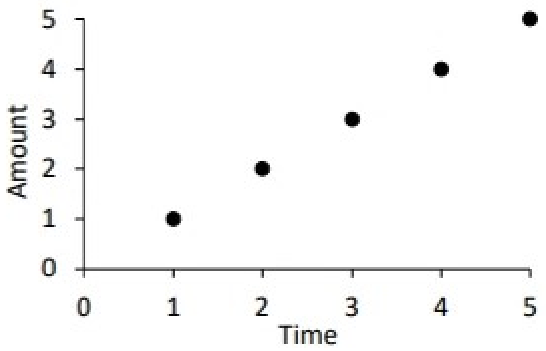
$$i(Ia)_{\overline{n|}} = 1 + v + v^2 + \cdots + v^{n-1} - nv^n$$
$$= \ddot{a}_{\overline{n|}} - nv^n$$

So:

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \tag{1.2}$$

This formula can be found in the *Tables* on page 31.

The graph below shows the pattern of the payments of $(Ia)_{\overline{5}|}$.



The present value of any annuity payable in arrears for *n* time units for which the amounts of successive payments form an arithmetic progression can be expressed in terms of $a_{\overline{n}|}$ and $(Ia)_{\overline{n}|}$. If the first payment of such an annuity is *P* and the second payment is $(P+Q)$, the *t*th payment is $(P-Q)+Qt$, then the present value of the annuity is

$$(P-Q)a_{\overline{n}|}+Q(Ia)_{\overline{n}|} \tag{1.3}$$

Alternatively, the present value of the annuity can be derived from first principles.

On a timeline we can show the payments as:

	P	$P+Q$	$P+2Q$	$P+3Q$	...	$P+(n-1)Q$	Payments
0	1	2	3	4	...	n	Time

Alternatively, these payments can be thought of as the sum of the following two sets of payments:

	$P-Q$	$P-Q$	$P-Q$	$P-Q$	...	$P-Q$	Payments
0	1	2	3	4	...	n	Time

	Q	$2Q$	$3Q$	$4Q$	...	nQ	Payments
0	1	2	3	4	...	n	Time

ie we have a level annuity with payments of $P-Q$ and an increasing annuity that increases by Q each time.

Question

Calculate the present value of a series of 10 annual payments, where the first payment is £500 made in one year's time, and each payment is £100 higher than the previous one. Assume an effective rate of interest of 8% *pa*.

Solution

We can think of this series of payments as a combination of:

- a level annuity of £400 payable annually in arrears and
- an increasing annuity of £100 payable annually in arrears.

So the present value of the payments is:

$$400a_{\overline{10}|}+100(Ia)_{\overline{10}|}=400a_{\overline{10}|}+100\left(\frac{\ddot{a}_{\overline{10}|}-10v^{10}}{i}\right)$$

$$=400\times 6.7101+100\times\frac{7.2469-10\times 1.08^{-10}}{0.08}$$

$$=£5{,}953$$

or this can be calculated using values from the *Tables*.

The notation $(I\ddot{a})_{\overline{n}|}$ is used to denote the present value of an increasing annuity-due payable for n time units, the tth payment (of amount t) being made at time $t-1$. Thus:

$$(I\ddot{a})_{\overline{n}|}=1+2v+3v^2+\cdots+nv^{n-1}$$

$$=(1+i)(Ia)_{\overline{n}|} \qquad \textbf{(1.4)}$$

$$=1+a_{\overline{n-1}|}+(Ia)_{\overline{n-1}|} \qquad \textbf{(1.5)}$$

While the third of the expressions above is true, it is not especially useful for calculation purposes.

The second line above follows by general reasoning by noting that the payments for $(I\ddot{a})_{\overline{n}|}$ are the same as for $(Ia)_{\overline{n}|}$, but advanced by 1 year. This gives us the formula:

$$(I\ddot{a})_{\overline{n}|}=(1+i)(Ia)_{\overline{n}|}=(1+i)\frac{\ddot{a}_{\overline{n}|}-nv^n}{i}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i/(1+i)}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{d}$$

We can alternatively derive this formula for $(I\ddot{a})_{\overline{n}|}$ using the same approach that we used to obtain a formula for $(Ia)_{\overline{n}|}$.

Starting from:

$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1}$$

we can multiply through by v :

$$v(I\ddot{a})_{\overline{n}|} = v + 2v^2 + \cdots + (n-2)v^{n-2} + (n-1)v^{n-1} + nv^n$$

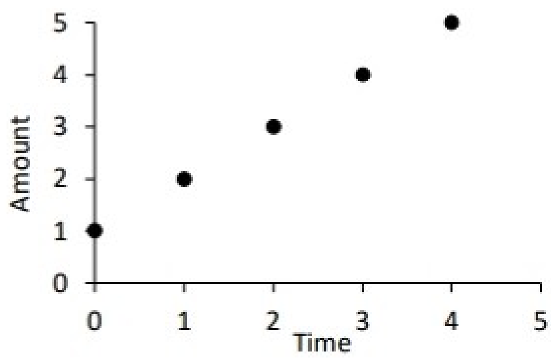
and subtract the second equation from the first to give:

$$(1-v)(I\ddot{a})_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-2} + v^{n-1} - nv^n$$

Since $1-v=d$ and $1+v+v^2+\cdots+v^{n-2}+v^{n-1}=\ddot{a}_{\overline{n}|}$, this gives:

$$d(I\ddot{a})_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n \quad \Rightarrow \quad (I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}$$

The graph below shows the pattern of the payments of $(I\ddot{a})_{\overline{5}|}$.



Question

Calculate the present value of payments of £50 at time 0, £60 at time 1 year, £70 at time 2 years and so on. The last payment is at time 10 years. Assume that the annual effective rate of interest is 4.2%.

Solution

First note that 11 payments are made in total. These can be shown on a timeline as follows:

We can think of this series of payments as a level annuity-due with payments of 40, and an increasing annuity-due with increases of 10 each time.

The present value of these payments is therefore:

$$40\ddot{a}_{\overline{11}|} + 10(I\ddot{a})_{\overline{11}|}$$

We have:

$$\ddot{a}_{\overline{11}|} = \frac{1-1.042^{-11}}{0.042/1.042} = 9.03074$$

$$(I\ddot{a})_{\overline{11}|} = \frac{\ddot{a}_{\overline{11}|} - 11v^{11}}{d} = \frac{9.03074 - 11\times 1.042^{-11}}{0.042/1.042} = 50.48174$$

So the present value is:

$$40\times 9.03074 + 10\times 50.48174 = £866.05$$

1.2 Continuously payable annuities

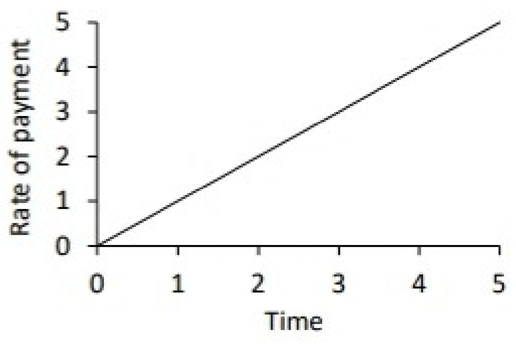
For increasing annuities which are payable continuously, it is important to distinguish between an annuity which has a constant rate of payment *r* (per unit time) throughout the *r*th period and an annuity which has a rate of payment t at time t. For the former the rate of payment is a step function taking the discrete values 1, 2,… . For the latter the rate of payment itself increases continuously. If the annuities are payable for *n* time units, their present values are denoted by $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}a)_{\overline{n}|}$ respectively.

A bar over the *a* indicates that the payments are made continuously and a bar over the *I* indicates that increases occur continuously, rather than at the end of the year.

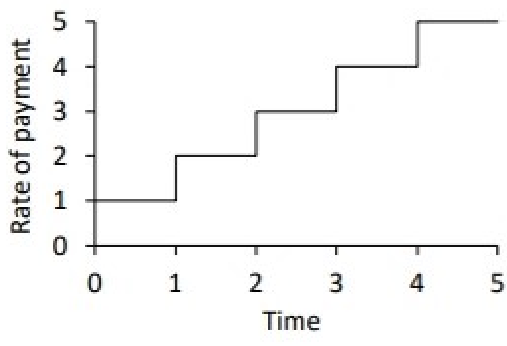
$(\bar{I}a)_{\overline{n}|}$ may also be written as $(\bar{I}\bar{a})_{\overline{n}|}$ – the two are equivalent.

The graphs below show the profiles of the payments.

$(\bar{I}a)_{\overline{5}|}$



$(I\bar{a})_{\overline{5}|}$



Clearly:

$$(I\bar{a})_{\overline{n}|} = \sum_{r=1}^{n}\left(\int_{r-1}^{r} r v^t \, dt\right)$$

and:

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t v^t \, dt$$

and it can be shown that:

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{\delta} \qquad \textbf{(1.6)}$$

and:

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta} \qquad \textbf{(1.7)}$$

The formula for $(\bar{I}\bar{a})_{\overline{n}|}$ is derived by integrating by parts. Using the formula on page 3 of the *Tables*, with $u = t$ and $\frac{dv}{dt} = e^{-\delta t}$:

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t v^t \, dt = \int_0^n t e^{-\delta t} \, dt$$

$$= \left[t\left(-\frac{1}{\delta}e^{-\delta t}\right)\right]_0^n - \int_0^n -\frac{1}{\delta}e^{-\delta t} \, dt$$

$$= -\frac{n v^n}{\delta} + \frac{1}{\delta}\int_0^n v^t \, dt = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$$

Question

A man agrees to make investments continuously for the next 10 years. He decides that he can afford to invest £$20t$ at time t , $0 \le t \le 10$. Calculate the:

(i) present value of the investment at time 0

(ii) accumulated value of the investment at time 10.

Assume that the annual effective rate of interest is 3.7% throughout the 10 years.

Solution

(i) The present value of these payments is:

$$20(I\bar{a})_{\overline{10|}} = 20\times\frac{\bar{a}_{\overline{10|}}-10v^{10}}{\delta} = 20\times\frac{\left(\frac{1-1.037^{-10}}{\ln 1.037}\right)-10\times 1.037^{-10}}{\ln 1.037} = 787.82$$

(ii) The accumulated value is the present value accumulated for 10 years:

$$787.82\times 1.037^{10} = 1{,}132.96$$

The formula given above for $(I\bar{a})_{\overline{n|}}$ can be derived by breaking it up into individual years:

$$(I\bar{a})_{\overline{n|}} = \int_0^1 v^t\,dt + \int_1^2 2v^t\,dt + \cdots + \int_{n-1}^n nv^t\,dt$$

$$= \bar{a}_{\overline{1|}} + 2v\bar{a}_{\overline{1|}} + \cdots + nv^{n-1}\bar{a}_{\overline{1|}}$$

$$= \bar{a}_{\overline{1|}}\left(1+2v+\cdots+nv^{n-1}\right)$$

$$= \bar{a}_{\overline{1|}}\times(I\ddot{a})_{\overline{n|}} = \frac{1-v}{\delta}\times\frac{\ddot{a}_{\overline{n|}}-nv^n}{d} = \frac{\ddot{a}_{\overline{n|}}-nv^n}{\delta}$$

since $1-v=d$.

Alternatively, by general reasoning, the payments made in each year under $(I\bar{a})_{\overline{n|}}$ are the same as for $(Ia)_{\overline{n|}}$, but they are paid continuously throughout the year, rather than at the end of the year. So, by proportioning:

$$(I\bar{a})_{\overline{n|}} = (Ia)_{\overline{n|}}\times\frac{\bar{a}_{\overline{1|}}}{a_{\overline{1|}}} = (Ia)_{\overline{n|}}\times\frac{\frac{1-v}{\delta}}{\frac{1-v}{i}} = (Ia)_{\overline{n|}}\times\frac{i}{\delta}$$

which simplifies to the formula given.

The formula $(I\bar{a})_{\overline{n|}} = (Ia)_{\overline{n|}}\times\frac{i}{\delta}$ may be useful for calculations since values of both $(Ia)_{\overline{n|}}$ and $\frac{i}{\delta}$ are given in the *Tables* at various rates of interest.

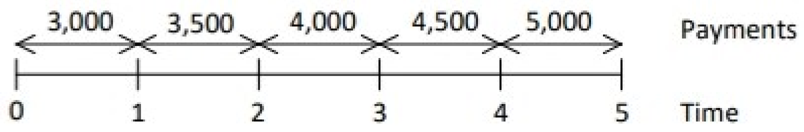
Question

Rent on a property is payable continuously for 5 years. The rent in the first year is £3,000, thereafter the annual rent increases by £500 *pa*.

Calculate the present value of the rent at the start of the 5 years, using an annual effective rate of interest of 6%.

Solution

Shown on a timeline, the rental payments are as follows:



The present value of these payments is:

$$2,500\bar{a}_{\overline{5}|} + 500(I\bar{a})_{\overline{5}|}$$

Evaluating these annuities:

$$\bar{a}_{\overline{5}|} = \frac{1-1.06^{-5}}{\ln 1.06} = 4.3375$$

$$\ddot{a}_{\overline{5}|} = \frac{1-1.06^{-5}}{0.06/1.06} = 4.4651$$

$$(I\bar{a})_{\overline{5}|} = \frac{4.4651 - 5 \times 1.06^{-5}}{\ln 1.06} = 12.5078$$

So the present value is:

$$2,500 \times 4.3375 + 500 \times 12.5078 = £17,097.66$$

Alternatively, we can use values from the Tables as follows:

$$\bar{a}_{\overline{5}|} = \frac{i}{\delta} \times a_{\overline{5}|} = 1.029709 \times 4.2124 = 4.3375$$

$$(I\bar{a})_{\overline{5}|} = \frac{i}{\delta} \times (Ia)_{\overline{5}|} = 1.029709 \times 12.1469 = 12.5078$$

The present values of deferred increasing annuities are defined in the obvious manner, for example:

$$_{m|}(Ia)_{\overline{n}|} = v^m (Ia)_{\overline{n}|}$$

1.3 Decreasing payments

We can also use increasing annuities to calculate the present values of annuities where the payments decrease by a fixed amount each time.

Question

A man makes payments into an investment account of \$200 at time 5, \$190 at time 6, \$180 at time 7, and so on until a payment of \$100 at time 15. Assuming an annual effective rate of interest of 3.5%, calculate:

(i) the present value of the payments at time 4,

(ii) the present value of the payments at time 0,

(iii) the accumulated value of the payments at time 15.

Solution

There are 11 payments in total. The payments can be thought of as:

210 at time 5, 210 at time 6, 210 at time 7, ..., 210 at time 15

LESS the following payments:

10 at time 5, 20 at time 6, 30 at time 7, ..., 110 at time 15

(i) The present value of these payments at time 4 (which is one time period before the first payment is made) is:

$$210a_{\overline{11|}} - 10(Ia)_{\overline{11|}}$$

Evaluating these, we have:

$$a_{\overline{11|}} = \frac{1-1.035^{-11}}{0.035} = 9.0016$$

$$\ddot{a}_{\overline{11|}} = \frac{1-1.035^{-11}}{0.035/1.035} = 9.3166 \quad (\text{or } \ddot{a}_{\overline{11|}} = 1.035 \times a_{\overline{11|}} = 9.3166)$$

$$(Ia)_{\overline{11|}} = \frac{9.3166 - 11\times 1.035^{-11}}{0.035} = 50.9201$$

So the present value is:

$$210\times 9.0016 - 10\times 50.9201 = 1,381.13$$

(ii) To find the present value at time 0, we need to discount the answer to part (i) by 4 years:

$$1,381.13\times 1.035^{-4} = 1,203.57$$

(iii) To find the accumulated value at time 15, we need to accumulate the answer to (i) by 11 years:

$$1,381.13 \times 1.035^{11} = 2,016.40$$

2 Special cases

2.1 Irregular payments

When the interest rate is constant, we can use the approach illustrated in the following example. This involves converting the payments into a simpler series of payments with the same present value.

Question

Write down an expression in terms of annuity functions for the present value as at 1 January 2019 of the following payments under the operation of a constant rate of interest:

£100 on 1 January, 1 April, 1 July and 1 October 2019

£200 on 1 January, 1 April, 1 July and 1 October 2020

£300 on 1 January, 1 April, 1 July and 1 October 2021

£400 on 1 January, 1 April, 1 July and 1 October 2022

£500 on 1 January, 1 April, 1 July and 1 October 2023

Solution

We can convert the payments for each calendar year to an equivalent single payment with the same present value payable on 1 January that year. For example, the payments in 2021 are equivalent to a single payment of $1,200\ddot{a}_{\overline{1}|}^{(4)}$ payable on 1 January 2021.

So, the payments are equivalent (in terms of present value) to the following five payments:

- $1 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2019,
- $2 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2020,
- *etc*
- $5 \times 400\ddot{a}_{\overline{1}|}^{(4)}$ on 1 January 2023.

This is a simple increasing annuity (payable annually in advance) where the payment amounts increase by $400\ddot{a}_{\overline{1}|}^{(4)}$ each year. So the present value is $(I\ddot{a})_{\overline{5}|} \times 400\ddot{a}_{\overline{1}|}^{(4)}$.

2.2 Compound increasing annuities

We have already looked at *simple* increasing annuities, such as $(Ia)_{\overline{n}|}$, where the payments increase by a constant *amount* each time. We also need to be able to value *compound* increasing annuities where the payments increase by a constant *factor* each time.

Question

Calculate the present value of an annuity payable annually in arrears for 15 years, where the first payment is 500 and subsequent payments increase by 3% per annum compound.

The effective annual rate of interest is 10%.

Solution – Method 1

The payment at the end of the first year is 500; the payment at the end of the second year is 500×1.03; the payment at the end of the third year is 500×1.03^2, and so on until the final payment of 500×1.03^{14} is made at the end of the fifteenth year.

From first principles, we can write down an expression for the present value of this annuity as follows:

$$PV = 500v + 500\times 1.03v^2 + 500\times 1.03^2 v^3 + \cdots + 500\times 1.03^{14} v^{15} \qquad (*)$$

The terms being summed in this present value expression form a geometric progression of 15 terms, with first term $500v$ and common ratio $1.03v = \frac{1.03}{1.1}$. So the present value is equal to:

$$PV = 500v\times\frac{1-\left(\frac{1.03}{1.1}\right)^{15}}{1-\frac{1.03}{1.1}} = \frac{500}{1.1}\times 9.853407 = 4,479$$

Method 2

We'll now consider a slightly different way of solving this problem. The equation (*) could be rearranged and written as:

$$PV = \frac{500}{1.03}\left(1.03v + 1.03^2 v^2 + \cdots 1.03^{15} v^{15}\right)$$

$$= \frac{500}{1.03}\left(v' + v'^2 + \cdots v'^{15}\right)$$

where $v' = 1.03v$.

Remember that:

$$a_{\overline{n}|} = v + v^2 + \cdots + v^n \qquad @ i\% \qquad \text{where } v = \frac{1}{1+i}$$

If we introduce a new interest rate i', we can define:

$$a_{\overline{n}|}' = v' + v'^2 + \cdots + v'^n \quad @i'\% \quad \text{where} \quad v' = \frac{1}{1+i'} = 1.03v \qquad (**)$$

This is the value, $a_{\overline{n}|}'$, of an annuity-certain, calculated at interest rate i'. From (**) we get:

$$\frac{1}{1+i'} = \frac{1.03}{1.1} \quad \Rightarrow \quad i' = \frac{1.1}{1.03} - 1 = 6.7961\%$$

So we can now write the present value as:

$$PV = \frac{500}{1.03} a_{\overline{15}|}' \quad @6.7961\%$$

$$= \frac{500}{1.03}\left(\frac{1-\left(\frac{1}{1.067961}\right)^{15}}{0.067961}\right) = \frac{500}{1.03} \times 9.2264 = 4,479$$

This example shows how a compound increasing annuity can be valued as a level annuity at a different interest rate.

Method 3

Finally, we'll consider a slight adaptation of Method 2, using an annuity-due instead of an annuity in arrears. Here, we rewrite the equation (*) as:

$$PV = 500v\left(1 + 1.03v + \cdots 1.03^{14} v^{14}\right) = 500v\left(1 + v' + v'^2 + \cdots v'^{14}\right)$$

where $v' = 1.03v$.

This time the expression in brackets is an annuity-due and so we can write:

$$PV = 500v\ddot{a}_{\overline{15}|}'$$

We must be slightly careful when evaluating this expression. The v in the expression must be calculated at 10%, whereas the annuity must be calculated at our new interest rate $i' = 6.7961\%$ as before. So:

$$PV = 500v\left(\frac{1-v'^{15}}{d'}\right) = \frac{500}{1.1} \times \left(\frac{1-\left(\frac{1}{1.067961}\right)^{15}}{0.067961/1.067961}\right) = \frac{500}{1.1} \times 9.8534 = 4,479$$

We have considered three slightly different methods here. There is no 'best method' to use. While Methods 2 and 3 may appear to be more complicated, it's worth trying them out as they're not quite as complex as they initially appear, and we will use a similar approach when considering life assurances and life annuities with compound increasing benefits later in the course.

Let's take another look at Method 2 in a more general context.

Consider an annuity under which the payment at time t is $(1+e)^t$, where e is any constant and $t=1,2,\ldots,n$. Then the present value of the single payment paid at time t is $\frac{(1+e)^t}{(1+i)^t}=\left(\frac{1+e}{1+i}\right)^t$. If we introduce a new interest rate i' defined by the equation $\frac{1}{1+i'}=\frac{1+e}{1+i}$, we find that the present value of the payment at time t can be expressed as $\left(\frac{1}{1+i'}\right)^t$, *ie* it is $(v')^t$, calculated using the interest rate i'.

Similarly, the present value of this compound increasing annuity is:

$$v'+(v')^2+\cdots+(v')^n$$

which is the value of an annuity-certain, $a_{\overline{n}|}'$, calculated at the interest rate i'.

We can rearrange the equation for i' above to find an expression for i' in terms of i and e:

$$\frac{1}{1+i'}=\frac{1+e}{1+i} \quad \Rightarrow \quad i'=\frac{1+i}{1+e}-1=\frac{(1+i)-(1+e)}{1+e}=\frac{i-e}{1+e}$$

Question

Calculate the present value (at time $t=0$) of payments of £$20,000\times1.0381^{t-1}$ payable at times $t=1,2,3,\ldots,10$, where time is measured in years, assuming a constant annual effective rate of interest of 9%.

Solution

The payment at time 1 is £20,000; the payment at time 2 is £$20,000\times1.0381$; the payment at time 3 is £$20,000\times1.0381^2$, and so on.

The present value of these payments is:

$$PV=20,000\left(v+1.0381v^2+\cdots+1.0381^9v^{10}\right)$$

The terms in brackets form a geometric progression of 10 terms, with first term v and common ratio $1.0381v=\frac{1.0381}{1.09}$. So the present value is equal to:

$$PV=20,000v\times\frac{1-\left(\frac{1.0381}{1.09}\right)^{10}}{1-\frac{1.0381}{1.09}}=\frac{20,000}{1.09}\times8.107974=£148,770$$

Alternatively, we can write the present value as:

$$PV = 20{,}000\left(v + 1.0381v^2 + \cdots + 1.0381^9 v^{10}\right)$$

$$= \frac{20{,}000}{1.0381}\left(1.0381v + 1.0381^2 v^2 + \cdots + 1.0381^{10} v^{10}\right)$$

$$= \frac{20{,}000}{1.0381} a_{\overline{10|}}'$$

In this case we have to divide by 1.0381 to ensure that the power of 1.0381 matches the power of v in each term of the expression.

The interest rate to use for the annuity is:

$$i' = \frac{i-e}{1+e} = \frac{0.09-0.0381}{1+0.0381} = 0.05000$$

ie almost exactly 5%. So the present value of the payments is:

$$\frac{20{,}000}{1.0381} a_{\overline{10|}@5\%} = \frac{20{,}000}{1.0381} \times 7.7217 = £148{,}770$$

As another alternative, we can write the present value as:

$$PV = 20{,}000\left(v + 1.0381v^2 + \cdots + 1.0381^9 v^{10}\right)$$

$$= 20{,}000v\left(1 + 1.0381v + \cdots + 1.0381^9 v^9\right)$$

$$= 20{,}000 \times \frac{1}{1.09} \times \ddot{a}_{\overline{10|}@5\%}$$

Evaluating this expression gives the same answer as before.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 8 Summary

$(Ia)_{\overline{n}|}$ represents the present value of payments of 1 at time 1, 2 at time 2, …, n at time n:

$$(Ia)_{\overline{n}|} = \sum_{t=1}^{n} tv^t = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

An increasing annuity but with payments in advance is given by:

$$(I\ddot{a})_{\overline{n}|} = \sum_{t=0}^{n-1} (t+1)v^t = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = (1+i)(Ia)_{\overline{n}|}$$

For the continuous annuity $(I\bar{a})_{\overline{n}|}$, the rate of payment is a step function taking the discrete values 1, 2, …, n. For $(\overline{Ia})_{\overline{n}|}$, the rate of payment itself increases continuously. The rate of payment at time t is t. The formulae are:

$$(I\bar{a})_{\overline{n}|} = \sum_{r=1}^{n} \left(\int_{r-1}^{r} rv^t dt \right) = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta} = \frac{i}{\delta}(Ia)_{\overline{n}|} \qquad (\overline{Ia})_{\overline{n}|} = \int_0^n tv^t \, dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

The present value of a compound increasing annuity can be found either by using the formula for the sum of terms in a geometric progression, or by writing it as a level annuity at an adjusted rate of interest.

The sum of the first n terms of a geometric progression with first term a and common ratio r is:

$$\frac{a(1-r^n)}{1-r}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 8 Practice Questions

8.1 (i) Calculate the following at $i = 9\%$:

(a) $(Ia)_{\overline{60|}}$

(b) $(I\ddot{a})_{\overline{60|}}$

(c) $(I\bar{a})_{\overline{100|}}$

(d) $(\overline{Ia})_{\overline{100|}}$

(ii) Calculate the following at $i = 7\%$:

(a) $(Is)_{\overline{1|}}$

(b) $(I\ddot{s})_{\overline{10|}}$

8.2 A series of payments is to be received annually in advance. The first payment is £10. Thereafter, payments increase by £2 per annum. The last payment is made at the beginning of the tenth year.

Determine whether each of the following is a correct expression for the present value of the annuity.

(i) $\sum_{t=0}^{9} 8v^t + 2\sum_{t=0}^{9} tv^t$

(ii) $10\ddot{a}_{\overline{10|}} + 2(Ia)_{\overline{9|}}$

(iii) $8\ddot{a}_{\overline{10|}} + 2(I\ddot{a})_{\overline{10|}}$

8.3 Given that $\delta(t) = 0.01t$ for $0 \le t \le 10$, calculate the value of $(\overline{Ia})_{\overline{10|}}$.

8.4 An annuity is payable annually in advance for a term of 20 years. The payment is £500 in year 1, £550 in year 2, and so on, increasing by £50 each year.

Exam style

Calculate the present value of this annuity, assuming that the effective rate of interest is 5% *pa* for the first twelve years and 7% *pa* thereafter. [5]

8.5 A continuous payment stream is such that the level rate of payment in year t is $100 \times 1.05^{t-1}$, for $t = 1, 2, \ldots, 10$. Calculate the present value of this payment stream as at its commencement date, assuming a rate of interest of 10% *pa.*

8.6 A series of 10 payments is received at times 5, 6, 7,..., 14. The first payment is \$200. Each of the next five payments is 5.7692% greater than the previous one, and thereafter each payment is 6.7961% greater than the previous one.

Calculate the present value of these payments at time 0 using an interest rate of 10% *pa* effective.

8.7 Exam style

An investor in property expects to receive rental payments for the next 50 years in line with the following assumptions:

- The current level of rental payments is £20,000 per annum, paid quarterly in advance.
- Payments will remain fixed for 5-year periods. At the end of each 5-year period the payments will rise in line with total inflationary growth over the previous five years.
- Inflation is assumed to be constant at 3% per annum.
- The interest rate is 12% per annum effective.

Calculate the present value of the rental income the investor expects to receive. [6]

8.8 Exam style

(i) Assuming a rate of interest of 6% *pa* effective, calculate the present value as at 1 January 2020 of the following annuities, each with a term of 25 years:

(a) an annuity payable annually, where the first payment is £3,000 made on 1 January 2021, and payments increase by £500 on each subsequent 1 January.

(b) an annuity as in (i), but only 10 increases are to be made, the annuity then remaining level for the remainder of the term. [5]

(ii) An investor is to receive a special annual annuity for a term of 10 years in which payments are increased by 5% compound each year to allow for inflation. The first payment is to be £1,000 on 1 November 2021. Calculate the accumulated value of the annuity payments as at 31 October 2038 if the investor achieves an effective rate of return of 4% per half-year. [4]

[Total 9]

Chapter 8 Solutions

8.1 (i)(a) Using the formula for the present value of an increasing annuity:

$$(Ia)_{\overline{60}|} = \frac{\ddot{a}_{\overline{60}|} - 60v^{60}}{i} = \frac{12.0423 - 60 \times 1.09^{-60}}{0.09} = 130.02$$

The value of $(Ia)_{\overline{60}|}$ can also be looked up directly in the Tables.

(i)(b) This can be calculated from the previous answer:

$$(I\ddot{a})_{\overline{60}|} = 1.09(Ia)_{\overline{60}|} = 1.09 \times 130.02 = 141.72$$

(i)(c) Using the formula:

$$(I\bar{a})_{\overline{100}|} = \frac{\ddot{a}_{\overline{100}|} - 100v^{100}}{\delta} = \frac{12.1089 - 100 \times 1.09^{-100}}{\ln(1.09)} = 140.30$$

Alternatively, using values from the Tables:

$$(I\bar{a})_{\overline{100}|} = \frac{i}{\delta}(Ia)_{\overline{100}|} = 1.044354 \times 134.3426 = 140.30$$

(i)(d) Using the formula:

$$(\overline{Ia})_{\overline{100}|} = \frac{\bar{a}_{\overline{100}|} - 100v^{100}}{\delta} = \frac{11.6018 - 100 \times 1.09^{-100}}{\ln(1.09)} = 134.42$$

(ii)(a) This is the accumulated value at time 1 of a payment of 1 unit made at time 1:

$$(Is)_{\overline{1}|} = 1$$

(ii)(b) This is the accumulated value of a 10-year increasing annuity due:

$$(I\ddot{s})_{\overline{10}|} = 1.07^{10}(I\ddot{a})_{\overline{10}|}$$

$$= 1.07^{10}\left(\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{0.07/1.07}\right) = 1.07^{10}\left(\frac{7.515232 - 10 \times 1.07^{-10}}{0.07/1.07}\right) = 73.12$$

Alternatively, using values from the Tables:

$$(I\ddot{s})_{\overline{10}|} = 1.07^{10}(I\ddot{a})_{\overline{10}|} = 1.07^{10} \times 1.07(Ia)_{\overline{10}|} = 1.07^{11} \times 34.7391 = 73.12$$

8.2 The present value of the annuity is:

$$PV = 10 + 12v + 14v^2 + \cdots + 26v^8 + 28v^9$$

Expression (i) is not correct since the first term (obtained when $t=0$) is 8, not 10.

Expression (ii) is correct since it can be rearranged to give the expression above:

$$10\ddot{a}_{\overline{10}|} + 2(Ia)_{\overline{9}|} = 10(1 + v + \cdots + v^9) + 2(v + 2v^2 + \cdots + 9v^9) = 10 + 12v + \cdots + 28v^9$$

Expression (iii) is also correct since it can be rearranged to give the expression above:

$$8\ddot{a}_{\overline{10}|} + 2(I\ddot{a})_{\overline{10}|} = 8(1 + v + \cdots + v^9) + 2(1 + 2v + \cdots + 10v^9) = 10 + 12v + \cdots + 28v^9$$

8.3 The discount factor for a payment at time t is:

$$v(t) = \exp\left(-\int_0^t 0.01s\, ds\right) = e^{-0.005t^2}$$

This continuously-increasing, continuously-payable annuity has a rate of payment of t at time t, so its present value can be expressed as an integral as:

$$(\bar{I}\bar{a})_{\overline{10}|} = \int_0^{10} t\, v(t)\, dt = \int_0^{10} te^{-0.005t^2} dt$$

Now, using the general result:

$$\int_a^b f'(t)e^{f(t)} dt = \left[e^{f(t)}\right]_a^b$$

we know that:

$$\int_0^{10} (-0.01t)e^{-0.005t^2}\, dt = \left[e^{-0.005t^2}\right]_0^{10}$$

So, we have:

$$(\bar{I}\bar{a})_{\overline{10}|} = \int_0^{10} te^{-0.005t^2} dt = -100\int_0^{10} (-0.01t)e^{-0.005t^2} dt$$

$$= -100\left[e^{-0.005t^2}\right]_0^{10} = -100\left(e^{-0.5} - 1\right) = 39.35$$

Alternatively, we can use the substitution $u=-0.005t^2$, so that:

$$\frac{du}{dt}=-0.01t \;\Rightarrow\; du=-0.01t\,dt \;\Rightarrow\; -100\,du=t\,dt$$

When $t=0$, $u=0$, and when $t=10$, $u=-0.5$, so we have:

$$(I\bar{a})_{\overline{10}|}=\int_0^{10} te^{-0.005t^2}dt=\int_0^{-0.5}-100e^u\,du=-100\left[e^u\right]_0^{-0.5}=-100\left(e^{-0.5}-1\right)=39.35$$

8.4 The present value of the payments can be expressed as:

$$450\ddot{a}_{\overline{12}|}^{@5\%}+50(I\ddot{a})_{\overline{12}|}^{@5\%}+v^{12@5\%}(1{,}050\ddot{a}_{\overline{8}|}^{@7\%}+50(I\ddot{a})_{\overline{8}|}^{@7\%}) \qquad [2]$$

To understand how this expression arises, first consider the initial 12 years, over which period the interest rate is 5% pa. As the payments start at 500 at time 0 and increase by 50 each year, we can view this as an increasing annuity $50(I\ddot{a})_{\overline{12}|}$ (which pays 50 at time 0, 100 at time 1, and so on, up to 600 at time 11), plus a level annuity $450\ddot{a}_{\overline{12}|}$ (which pays 450 at all times from 0 to 11).

Next, consider the final 8 payments. The first of these is for amount 1,100 at time 12, the second is for amount 1,150 at time 13, and so on. We can express the present value of these payments ***at time 12*** *using an increasing annuity $50(I\ddot{a})_{\overline{8}|}$, plus a level annuity $1{,}050\ddot{a}_{\overline{8}|}$. These annuities need to be evaluated at an interest rate of 7% pa, as we are considering the period from time 12 onwards. To obtain the present value of these payments* ***at time 0****, we need to discount the value at time 12 back to time 0, using an interest rate of 5% pa.*

The numerical values of the components are:

$$\ddot{a}_{\overline{12}|}^{@5\%}=\frac{1-v^{12}}{d}=\frac{1-0.55684}{0.05/1.05}=9.3064 \qquad [½]$$

$$(I\ddot{a})_{\overline{12}|}^{@5\%}=\frac{\ddot{a}_{\overline{12}|}-12v^{12}}{d}=\frac{9.3065-6.6821}{0.05/1.05}=55.1117 \qquad [½]$$

$$\ddot{a}_{\overline{8}|}^{@7\%}=\frac{1-v^{8}}{d}=\frac{1-0.58201}{0.07/1.07}=6.3893 \qquad [½]$$

$$(I\ddot{a})_{\overline{8}|}^{@7\%}=\frac{\ddot{a}_{\overline{8}|}-8v^{8}}{d}=\frac{6.3893-4.6561}{0.07/1.07}=26.4935 \qquad [½]$$

Substituting in these values gives a present value of:

$$450\times9.3064+50\times55.1117+0.55684(1{,}050\times6.3893+50\times26.4935)=£11{,}417 \qquad [1]$$

[Total 5]

8.5 The rate of payment is 100 in the first year, 100×1.05 in the second year, 100×1.05^2 in the third year, and so on. The present value of the payment stream is therefore:

$$100\bar{a}_{\overline{1|}}+100\times 1.05v\,\bar{a}_{\overline{1|}}+100\times 1.05^2v^2\bar{a}_{\overline{1|}}+\cdots+100\times 1.05^9v^9\bar{a}_{\overline{1|}}$$

$$=100\bar{a}_{\overline{1|}}\left(1+1.05v+1.05^2v^2+\cdots+1.05^9v^9\right)$$

The terms in brackets form a geometric progression of 10 terms with first term 1 and common ratio $1.05v$, so the present value is:

$$100\times\frac{1-v}{\delta}\times\frac{1-(1.05v)^{10}}{1-1.05v}=100\times 0.95382\times 8.1838=780.59$$

Alternatively, this could be evaluated as:

$$100\bar{a}_{\overline{1|}}^{@10\%}\left(1+1.05v+1.05^2v^2+\cdots+1.05^9v^9\right)=100\bar{a}_{\overline{1|}}^{@10\%}\left(1+V+V^2+\cdots+V^9\right)$$

$$=100\bar{a}_{\overline{1|}}^{@10\%}\ddot{a}_{\overline{10|}}^{@4.7619\%}$$

where $V=\dfrac{1}{1+I}=\dfrac{1.05}{1.1}\;\Rightarrow\; I=\dfrac{1.1}{1.05}-1=4.7619\%$.

8.6 Let $e=5.7692\%$ and $f=6.7961\%$. The present value of the payments is:

$$200v^5+200(1+e)v^6+200(1+e)^2v^7+\cdots+200(1+e)^5v^{10}$$

$$+200(1+e)^5(1+f)v^{11}+200(1+e)^5(1+f)^2v^{12}+\cdots+200(1+e)^5(1+f)^4v^{14}$$

This splits naturally into two geometric progressions. The first relates to the first line above – it contains 6 terms, has first term $200v^5$ and common ratio $(1+e)v$. The second relates to the second line above – it contains 4 terms, has first term $200(1+e)^5(1+f)v^{11}$ and common ratio $(1+f)v$. So the present value can be written as:

$$\frac{200v^5\left(1-((1+e)v)^6\right)}{1-(1+e)v}+\frac{200(1+e)^5(1+f)v^{11}\left(1-((1+f)v)^4\right)}{1-(1+f)v}$$

Evaluating this using $v=1.1^{-1}$, the present value is:

$$677.030+379.405=\$1,056.44$$

Alternatively, we can split up the series and pull out common factors:

$$\frac{200v^4}{(1+e)}\left((1+e)v+(1+e)^2v^2+\cdots+(1+e)^6v^6\right)$$

$$+200(1+e)^5v^{10}\left((1+f)v+(1+f)^2v^2+\cdots+(1+f)^4v^4\right)$$

$$=\frac{200v^4}{(1+e)}a'_{\overline{6}|}+200(1+e)^5v^{10}a''_{\overline{4}|}$$

where one dash represents an interest rate of $\frac{0.1-0.057692}{1.057692}=4\%$ *pa, and two dashes represents an interest rate of* $\frac{0.1-0.067961}{1.067961}=3\%$ *pa.*

The present value can then be evaluated as:

$$\frac{200\times1.1^{-4}}{1.057692}\times\frac{1-1.04^{-6}}{0.04}+200(1.057692)^5\times1.1^{-10}\times\frac{1-1.03^{-4}}{0.03}$$

$$=129.15167\times5.24214+102.07026\times3.71710$$

$$=\$1{,}056.44$$

8.7 The present value of the first five years' worth of payments (working in thousands of pounds) is $20\ddot{a}^{(4)}_{\overline{5}|}$ (calculated at $i=12\%$).

The present value of the next 5 years' worth of payments is:

$$20\times1.03^5\times\ddot{a}^{(4)}_{\overline{5}|}\times v^5$$

We must increase the annual payment by 1.03^5 *because we are given an annual rate of inflation and we are told that 'the payments will rise in line with* ***total*** *inflationary growth over the previous five years'.*

The next 5 years' worth of payments will be worth:

$$20\times1.03^{10}\times\ddot{a}^{(4)}_{\overline{5}|}\times v^{10}$$

and so on. So the total present value will be:

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\left[1+1.03^5v^5+1.03^{10}v^{10}+\cdots+1.03^{45}v^{45}\right]$$ [2]

The terms in brackets form a geometric progression of 10 terms, with first term 1 and common ratio $1.03^5v^5=\frac{1.03^5}{1.12^5}$.

So:

$$1+1.03^5v^5+1.03^{10}v^{10}+\dots+1.03^{45}v^{45}=\frac{1-\left(\frac{1.03^5}{1.12^5}\right)^{10}}{1-\frac{1.03^5}{1.12^5}}=2.877967 \qquad [2]$$

Also:

$$\ddot{a}^{(4)}_{\overline{5}|@12\%}=\frac{1-v^5}{d^{(4)}}=\frac{1-1.12^{-5}}{4\left(1-1.12^{-1/4}\right)}=3.871305 \qquad [1]$$

So the present value of the rental income is:

$$20\times 3.871305\times 2.877967=222.830$$

ie £222,830. [1]

[Total 6]

Alternatively, the present value expression can be written as:

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\left[1+1.03^5v^5+1.03^{10}v^{10}+\dots+1.03^{45}v^{45}\right]=20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\,\ddot{a}_{\overline{10}|@j\%}$$

where the last annuity term above is calculated at the interest rate j for which $v_j=\frac{1}{1+j}=\left(\frac{1.03}{1.12}\right)^5$. This gives $j=52.021\%$. So the total present value is:

$$20\,\ddot{a}^{(4)}_{\overline{5}|@12\%}\,\ddot{a}_{\overline{10}|@j\%}=20\times\frac{1-v^5}{d^{(4)}}\times\frac{1-v_j^{10}}{d_j}=20\times 3.871305\times 2.877967=222.830$$

ie £222,830, as before.

8.8 (i)(a) The present value of the payments can be expressed as:

$$PV=2{,}500a_{\overline{25}|}+500(Ia)_{\overline{25}|} \qquad [1]$$

Using annuity values from the *Tables*:

$$PV=2{,}500\times 12.7834+500\times 128.7565=£96{,}337 \qquad [1]$$

(i)(b) The present value of the payments is:

$$PV = 2{,}500a_{\overline{25}|} + 500(Ia)_{\overline{11}|} + 5{,}500v^{11}a_{\overline{14}|} \quad [2]$$

To understand how this expression arises, it can help to work from first principles. The present value of these payments is:

$$PV = 3{,}000v + 3{,}500v^2 + \cdots + 8{,}000v^{11} + 8{,}000v^{12} + 8{,}000v^{13} + \cdots + 8{,}000v^{25}$$

$$= 2{,}500v + 2{,}500v^2 + \cdots + 2{,}500v^{11} + 2{,}500v^{12} + 2{,}500v^{13} + \cdots + 2{,}500v^{25}$$

$$+ \left(500v + 1{,}000v^2 + \cdots + 5{,}500v^{11}\right) + \left(5{,}500v^{12} + 5{,}500v^{13} + \cdots + 5{,}500v^{25}\right)$$

$$= 2{,}500a_{\overline{25}|} + 500(Ia)_{\overline{11}|} + 5{,}500v^{11}\left(v + v^2 + \cdots + v^{14}\right)$$

$$= 2{,}500a_{\overline{25}|} + 500(Ia)_{\overline{11}|} + 5{,}500v^{11}a_{\overline{14}|}$$

So, using annuity values from the *Tables*:

$$PV = 2{,}500 \times 12.7834 + 500 \times 42.7571 + 5{,}500 \times 1.06^{-11} \times 9.2950 = £80{,}268 \quad [1]$$

[Total 5]

Alternatively, we could use the expression:

$$PV = 2{,}500a_{\overline{11}|} + 500(Ia)_{\overline{11}|} + 8{,}000v^{11}a_{\overline{14}|}$$

(ii) The first payment (of £1,000) needs to be accumulated for 17 years (or 34 half-years). The second payment (of $£1{,}000 \times 1.05$) needs to be accumulated for 16 years (or 32 half-years), and so on, until the tenth payment (of $£1{,}000 \times 1.05^9$), which needs to be accumulated for 8 years (or 16 half-years).

The accumulated value of the payments is therefore:

$$AV = 1{,}000 \times 1.04^{34} + 1{,}000 \times 1.05 \times 1.04^{32} + \cdots + 1{,}000 \times 1.05^9 \times 1.04^{16} \quad [2]$$

This is a geometric progression of 10 terms, with first term $1{,}000 \times 1.04^{34}$ and common ratio 1.05×1.04^{-2} , so the accumulated value is:

$$AV = 1{,}000 \times 1.04^{34} \times \frac{1-(1.05 \times 1.04^{-2})^{10}}{1-1.05 \times 1.04^{-2}} = £33{,}324 \quad [2]$$

[Total 4]

Alternatively, the accumulated value can be calculated using:

$$AV = 1{,}000 \times 1.04^{34}\left(1 + \frac{1.05}{1.04^2} + \frac{1.05^2}{1.04^4} + \cdots + \frac{1.05^9}{1.04^{18}}\right)$$

$$= 1{,}000 \times 1.04^{34}\left(1 + V + V^2 + \cdots + V^9\right)$$

$$= 1{,}000 \times 1.04^{34}\, \ddot{a}_{\overline{10}|}^{@3.0095\%}$$

where $V = \dfrac{1}{1+I} = \dfrac{1.05}{1.04^2} \;\Rightarrow\; I = \dfrac{1.04^2}{1.05} - 1 = 3.0095\%$.

9

Equations of value

Syllabus objectives

3.1 Define an equation of value.

- 3.1.1 Define an equation of value, where payment or receipt is certain.
- 3.1.2 Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.
- 3.1.3 Understand the two conditions required for there to be an exact solution to an equation of value.

0 Introduction

An *equation of value* equates the present value of money received to the present value of money paid out:

'PV income = PV outgo'

or equivalently:

'PV income – PV outgo = 0'

Equations of value are used throughout actuarial work. For example:

- the 'fair price' to pay for an investment such as a fixed-interest security or an equity (*ie PV outgo*) equals the present value of the proceeds from the investment, discounted at the rate of interest required by the investor.
- the premium for an insurance policy is calculated by equating the present value of the expected amounts received in premiums to the present value of the expected benefits and other outgo.

1 The equation of value and the yield on a transaction

1.1 The theory

Consider a transaction that provides that, in return for outlays of amount $a_{t_1}, a_{t_2}, \ldots, a_{t_n}$ at time $t_1, t_2, \ldots, t_n$, an investor will receive payments of $b_{t_1}, b_{t_2}, \ldots, b_{t_n}$ at these times respectively. (In most situations only one of a_{t_r} and b_{t_r} will be non-zero.) At what force or rate of interest does the series of outlays have the same value as the series of receipts? At force of interest δ the two series are of equal value if and only if:

$$\sum_{r=1}^{n} a_{t_r} e^{-\delta t_r} = \sum_{r=1}^{n} b_{t_r} e^{-\delta t_r}$$

This equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} = 0 \qquad (1.1)$$

where $c_{t_r} = b_{t_r} - a_{t_r}$ is the amount of the *net cashflow* at time t_r. (We adopt the convention that a negative cashflow corresponds to a payment *by* the investor and a positive cashflow represents a payment *to* the investor.)

Equation (1.1), which expresses algebraically the condition that, at force of interest δ, the total value of the net cashflows is 0, is called the equation of value for the force of interest implied by the transaction. If we let $e^{\delta} = 1+i$, the equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} = 0 \qquad (1.2)$$

The latter form is known as the equation of value for the rate of interest or the 'yield equation'. Alternatively, the equation may be written as:

$$\sum_{r=1}^{n} c_{t_r} v^{t_r} = 0$$

In relation to continuous payment streams, if we let $\rho_1(t)$ and $\rho_2(t)$ be the rates of paying and receiving money at time *t* respectively, we call $\rho(t) = \rho_2(t) - \rho_1(t)$ the net rate of cashflow at time *t*. The equation of value (corresponding to Equation (1.1)) for the force of interest is:

$$\int_0^{\infty} \rho(t) e^{-\delta t} dt = 0$$

When both discrete and continuous cashflows are present, the equation of value is:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0 \tag{1.3}$$

and the equivalent yield equation is:

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0 \tag{1.4}$$

For any given transaction, Equation (1.3) may have no roots, a unique root, or several roots. If there is a unique root, δ_0 say, it is known as the force of interest implied by the transaction, and the corresponding rate of interest $i_0 = e^{\delta_0} - 1$ is called the 'yield' per unit time. (Alternative terms for the yield are the 'internal rate of return' and the 'money-weighted rate of return' for the transaction.) Thus the yield is defined if and only if Equation (1.4) has precisely one root greater than –1 and, when such a root exists, it is the yield.

The yield must be greater than –1, since $e^{\delta_0} > 0$, so $i_0 = e^{\delta_0} - 1 > -1$. A yield of –1 corresponds to a return of –100%, *ie* losing all the money originally invested.

Question

An investor pays £100 now in order to receive £60 in 5 years' time and £60 in 10 years' time.

Calculate the annual effective rate of interest earned on this investment.

Solution

The equation of value is:

$$100 = 60v^5 + 60v^{10}$$

This is a quadratic in v^5, which can be solved to give:

$$v^5 = (1+i)^{-5} = \frac{-60 \pm \sqrt{60^2 + 4 \times 60 \times 100}}{120} = 0.8844 \text{ or } -1.884$$

Rearranging, this gives i to be 2.49% or –188%. Since i must be greater than –1, the annual effective rate of interest is 2.49%.

The analysis of the equation of value for a given transaction may be somewhat complex depending on the shape of the function $f(i)$ denoting the left-hand side of Equation (1.4). However, when the equation $f(i) = 0$ is such that f is a monotonic function, its analysis is particularly simple. The equation has a root if and only if we can find i_1 and i_2 with $f(i_1)$ and $f(i_2)$ of opposite sign. In this case, the root is unique and lies between i_1 and i_2. By choosing i_1 and i_2 to be 'tabulated' rates sufficiently close to each other, we may determine the yield to any desired degree of accuracy.

A monotonic function is one without any turning points, so it either always increases, or always decreases.

Having identified suitable values of i_1 and i_2, linear interpolation could be used to obtain the yield i_0 from these. Examples of this method are given a little later in this chapter.

It should be noted that, after multiplication by $(1+i)^{t_0}$, Equation (1.2) takes the equivalent form:

$$\sum_{r=1}^{n} c_{t_r}(1+i)^{t_0-t_r} = 0$$

This slightly more general form may be called the *equation of value at time* t_0. It is of course directly equivalent to the original equation (which is now seen to be the equation of value at time 0). In certain problems a particular choice of t_0 may simplify the solution.

1.2 Solving for an unknown quantity

Many problems in actuarial work can be reduced to solving an equation of value for an unknown quantity. We will look at how to do this using examples based on a hypothetical fixed-interest security which operates as follows:

Security *S*

A price P is paid (by the investor) in return for a series of interest payments of D payable at the end of each of the next n years and a final redemption payment of R payable at the end of the n years. The investor earns an annual effective rate of return of i.

					R
Cashflows	$-P$	D	...	D	D
Time	0	1	...	$n-1$	n

The equation of value for this investment is:

$$P = Da_{\overline{n|}} + Rv^n \quad \text{calculated at interest rate } i$$

In the remainder of this section we will consider how to solve this equation when each of the quantities P, D or R, n and i is unknown.

Solving for the present value (*P*)

The present value (which, in this case, represents the price) can be found using formulae we derived earlier in the course.

Question

Calculate P, given that $D=5$, $R=125$, $i=10\%$ and $n=10$.

Solution

The price P can be calculated directly (using 10% interest):

$$P = 5a_{\overline{10}|} + 125v^{10} = 5 \times 6.1446 + 125 \times 0.38554 = £78.92$$

Note from this example that the equation of value holds for the values $P = 78.92$, $D = 5$, $R = 125$, $i = 10\%$ and $n = 10$. We will treat these values as our reference values.

The result shown in the following question is sometimes useful.

Question

For Security *S*, show algebraically that if $D = iR$, then $P = R$.

Solution

The price P is given by the equation of value:

$$P = iRa_{\overline{n}|} + Rv^n \quad \text{(calculated at interest rate } i\text{)}$$

Using the formula for the annuity and simplifying gives:

$$P = iR\left(\frac{1-v^n}{i}\right) + Rv^n = R(1-v^n) + Rv^n = R$$

We can also obtain this result by general reasoning.

Suppose an investor deposits a sum of money R into a bank account that pays an effective annual interest rate i. The investor leaves the money in the account for n years. If interest is paid at the end of each year, and is withdrawn as soon as it is paid, the investor will receive interest payments of iR at the end of each year and the initial deposit of R will be repaid at the end of n years.

Under this arrangement the cashflows the investor receives exactly match the cashflows received by investing an amount P in Security S. Also, the rate of return obtained from the bank account will be i (by definition), which is the same as the interest rate i required from Security S.

So, investing R in the bank account or P in Security S leads to the same cashflows and gives the same rate of return. So P and R must be equal.

Question

Calculate P, given that $D=10$, $R=125$, $i=8\%$ and $n=10$.

Solution

The value of P is:

$$P=10a_{\overline{10}|}+125v^{10}=10\times 6.7101+125\times 0.46319=£125.00$$

This calculation verifies the result just proved, since here $D=10=0.08\times 125=iR$ *and we find that* $P=125=R$.

Solving for the amount of a payment (*D* or *R*)

Solving the equation of value for D or R is straightforward.

Question

Calculate D, given that $P=127.12$, $R=125$, $i=7.75\%$ and $n=10$.

Solution

The equation of value is:

$$127.12=Da_{\overline{10}|}+125v^{10}$$

So: $$127.12=D\times\frac{1-1.0775^{-10}}{0.0775}+125\times 1.0775^{-10}$$

ie $$127.12=D\times 6.7864+125\times 0.47405$$

This can be rearranged to find D:

$$D=\frac{127.12-125\times 0.47405}{6.7864}=10.00$$

Solving for the timing of a payment (*n*)

We can solve the equation of value for n by expressing the annuity function in terms of v .

Question

Calculate n, given that $P=78.92$, $D=5$, $R=125$ and $i=0.10$.

Solution

The equation of value is:

$$78.92=5a_{\overline{n|}}+125v^n$$

Substituting the formula for $a_{\overline{n|}}$ gives:

$$78.92=5\times\frac{1-v^n}{0.10}+125v^n$$

ie $$78.92=50(1-v^n)+125v^n=50+75v^n$$

This can be rearranged to find v^n :

$$v^n=\frac{78.92-50}{75}=0.38560$$

ie $$1.10^{-n}=0.38560$$

Taking logs, and using the result $\log a^b=b\log a$, we find:

$$-n\log 1.10=\log 0.38560 \quad ie \quad n=-\frac{\log 0.38560}{\log 1.10}=10.00$$

Solving for the interest rate (*i*)

Finding the interest rate is the hardest type of calculation, since the equation of value cannot usually be solved explicitly. If the equation of value cannot be solved explicitly, we could use trial and error, based on a rough initial guess.

To obtain an initial guess, we can approximate the interest rate by combining the cashflows into a single payment, payable on an average payment date. This is illustrated in the following question.

Question

Given that $P = 78.92$, $D = 5$, $R = 125$ and $n = 10$, determine a rough estimate for the value of i.

Solution

The equation of value is:

$$78.92 = 5a_{\overline{10|}} + 125v^{10} \quad \text{(calculated at interest rate } i\text{)}$$

Here, a payment of 5 is received at the end of each of years 1 to 10 (roughly equivalent to a total of 50 paid on average at time 5½), and in addition a payment of 125 is received at the end of year 10. Combining these (and weighting the timings by amounts) gives a single payment of 175 (*ie* $50+125$) at time 8.7 (*ie* $(50 \times 5\tfrac{1}{2} + 125 \times 10)/175$). This gives an equation we can solve more easily:

$$78.92 \approx 175v^{8.7} \text{ (calculated at rate } i\text{)}$$

So:

$$1+i \approx \left(\frac{78.92}{175}\right)^{-\frac{1}{8.7}} = 1.096 \quad \textit{ie} \quad i \approx 9.6\%$$

This rough estimate is quite close to the exact value of 10% (which we know from earlier questions).

An alternative method for finding a first guess is to use a first-order binomial expansion, replacing $(1+i)^n$ by $(1+ni)$. However, this is better suited to equations of value that contain no annuities. For example, using this method here we would have attained a first guess of 7.7%:

$$78.92 = 5\left(\frac{1-(1+i)^{-10}}{i}\right) + 125(1+i)^{-10}$$

$$\Rightarrow \quad 78.92 \approx 5\left(\frac{1-(1-10i)}{i}\right) + 125(1-10i)$$

$$\Rightarrow \quad 78.92 \approx 50 + 125(1-10i)$$

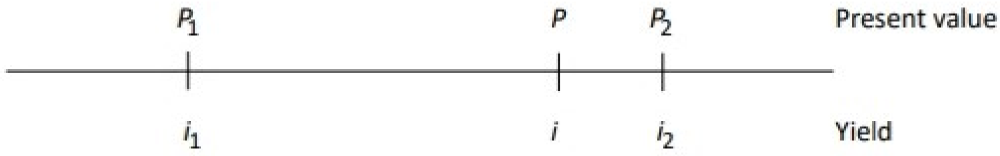
$$\Rightarrow \quad i \approx \frac{96.08}{1,250} = 7.7\%$$

Once an initial estimate has been obtained, a more accurate solution can then be found from the exact equation by linear interpolation, using the initial estimate as a starting point.

Estimating an unknown interest rate using linear interpolation

Suppose that the present values, calculated at interest rates i_1 and i_2, are P_1 and P_2 respectively, and we wish to work out the approximate interest rate corresponding to a present value of P.

This situation is illustrated on the following diagram



If the present value is a linear function of the interest rate, then the proportionate change in the interest rates will equal the proportionate change in the present values:

$$\frac{i-i_1}{i_2-i_1}=\frac{P-P_1}{P_2-P_1}$$

Rearranging this relationship gives the approximate value of i:

$$i \approx i_1+\frac{P-P_1}{P_2-P_1}\times(i_2-i_1)$$

This approximation works best if the trial values are close to the true value, *eg* values that are 1% apart. This formula also works even if the true value does not lie *between* the two trial values, but we would not recommend this approach (*ie* extrapolation) in the exam. We recommend that you interpolate between values that are either side of the true value and are a maximum of 1% apart. It is even better if the two values are 0.5% apart.

Question

Given that $P=75$, $D=5$, $R=125$ and $n=10$, calculate the value of i.

Solution

The equation of value is:

$$75=5a_{\overline{10|}}+125v^{10}$$

We know that when $i=10\%$, the right-hand side of this equation is equal to 78.92.

The price paid (75) is lower than this. So the value of i must be greater than 10%. Using $i=11\%$, the right-hand side is:

$$5a_{\overline{10|}}+125v^{10}=5\left(\frac{1-1.11^{-10}}{0.11}\right)+125(1.11)^{-10}=73.47$$

Interpolating linearly using these two values gives:

$$i \approx 10\% + \frac{75 - 78.92}{73.47 - 78.92} \times (11\% - 10\%) = 10.7\%$$

1.3 Example applications

Later in the course we will use equations of value in the context of the most common types of investment: fixed-interest bonds, index-linked bonds, equities (*ie* shares) and property.

Here, as an example, we will look at a question involving a property investment.

Question

A company has just bought an office block for £5m, which it will rent out to a number of small businesses. The total rent for the first year will be £100,000, and this is expected to increase by 4% *pa* compound in each future year. The office block is expected to be sold after 20 years for £7.5m.

Assuming that rent is paid in the middle of each year, calculate the yield the company will obtain on this investment.

Solution

Working in £000s, the equation of value here is:

$$5,000 = 100(v^{½} + 1.04v^{1½} + 1.04^2 v^{2½} + \cdots + 1.04^{19} v^{19.5}) + 7,500v^{20}$$

The terms in brackets form a geometric progression of 20 terms, with first term $v^{½}$ and common ratio $1.04v$. So the equation of value can be written:

$$5,000 = \frac{100v^{½}\left(1 - (1.04v)^{20}\right)}{1 - 1.04v} + 7,500v^{20} \qquad \text{provided } v \neq \frac{1}{1.04}$$

We can solve this by trial and error. At 4%, the right-hand side (using the first expression) is:

$$100\left(1.04^{-½} + 1.04^{-½} + \cdots + 1.04^{-½}\right) + \frac{7,500}{1.04^{20}} = 100 \times 20 \times 1.04^{-½} + \frac{7,500}{1.04^{20}} = 5,384.06$$

At 5%, the right-hand side (using the second expression) is 4,611.57. Interpolating between these two values, we obtain:

$$i \approx 4\% + \frac{5,000 - 5,384.06}{4,611.57 - 5,384.06} \times (5\% - 4\%) = 4.5\%$$

2 Uncertain payment or receipt

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- **apply a probability of payment/receipt to the cashflow at each time, or**
- **use a higher rate of discount.**

2.1 Probability of cashflow

The probability of payment/receipt can be allowed for by adapting the earlier equations. For example, Equation (1.4) can be revised to produce:

$$\sum_{r=1}^{n} p_{t_r} c_{t_r} (1+i)^{-t_r} + \int_0^{\infty} p(t)\rho(t)(1+i)^{-t}\,dt = 0 \qquad (2.1)$$

where p_t and $p(t)$ represent the probability of a cashflow at time t.

Where the force of interest is constant, and we can say that the probability is itself in the form of a discounting function, then Equation (1.3) can be generalised as:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} e^{-\mu t_r} + \int_0^{\infty} \rho(t) e^{-\delta t} e^{-\mu t}\,dt = 0 \qquad (2.2)$$

where μ is a constant force, rather than rate, of the probability of a cashflow at time t.

These probabilities of cashflows may often be estimated by consideration of the experience of similar cashflows. For example, this approach is used to assess the probabilities of cashflows that are dependent on the survival of a life – this is the theme of later chapters.

In other cases, there may be lack of data from which to determine an accurate probability for a cashflow. Instead a more approximate probability, or likelihood, may be determined after careful consideration of the risks.

In some cases, it may be spurious to attempt to determine the probability of each cashflow and so more approximate methods may be justified.

Wherever the uncertainty about the probability of the amount or timing of a cashflow could have significant financial effect, a sensitivity analysis may be performed. This involves calculations performed using different possible values for the likelihood and the amounts of the cashflows. Alternatively, a stochastic approach could be used to indicate possible outcomes.

Stochastic models were introduced in Chapter 1, and involve setting up some of the key assumptions as random variables. The model is then run many times to produce a distribution of outputs.

Question

A lottery ticket costs £1. The following table shows the different prizes available together with the chance of winning and the delay before receiving the prize money.

Prize	Probability of winning	Time before payment
£20	1 in 50	1 day
£200	1 in 1,000	1 day
£2,000	1 in 50,000	1 week
£200,000	1 in 2 million	2 weeks
£2 million	1 in 14 million	4 weeks

Calculate the expectation of the present value of the prize money assuming an effective rate of interest of 0.016% per day.

Solution

To calculate the expectation of the present value of the prize money, we take each possible present value, multiply it by the probability that it occurs, and then sum over all possibilities.

So, working in days, the expectation of the present value of the prize money is:

$$\frac{1}{50}\times 20v+\frac{1}{1,000}\times 200v+\frac{1}{50,000}\times 2,000v^7$$

$$+\frac{1}{2,000,000}\times 200,000v^{14}+\frac{1}{14,000,000}\times 2,000,000v^{28}$$

$$=£0.88$$

2.2 Higher discount rate

As the discounting functions and the probability functions in Equations (2.1) and (2.2) are both dependent on time, they can be combined into a single time-dependent function. In cases where there is insufficient information to objectively produce the probability functions, this combined function can be viewed as an adjusted discounting function that makes an implicit allowance for the probability of the cashflow.

Where the probability of the cashflow is a function that is of similar form to the discounting function, the combination can be treated as if a different discount rate were being used. For example, Equation (2.2) becomes:

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta' t_r} + \int_0^\infty \rho(t) e^{-\delta' t} dt = 0$$

where $\delta' = \delta + \mu$. The revised force of discount is therefore greater than the actual force of discount, as μ must be positive in order to give a probability between 0 and 1. It can therefore be shown that the rate of discount that is effectively used is greater than the actual rate of discount before the implicit allowance for the probability of the cashflow.

Question

A woman has invested some money in a company run by some ex-criminals. In return for the investment she expects to receive £100 at the end of each of the next ten years. The annual effective interest rate is 5%.

Calculate the present value of her investment by:

(i) ignoring the possibility that the payments might not be made.

(ii) assuming the probability of receiving the first payment is 0.95, the second payment is 0.9, the third payment is 0.85 *etc*.

(iii) increasing the force of interest by 0.04652.

Solution

(i) $PV = 100 a_{\overline{10|}}^{@5\%} = 100 \times 7.7217 = £772.17$

(ii) The present value allowing for the probabilities of payment is:

$$PV = 100v \times 0.95 + 100v^2 \times 0.9 + \cdots + 100v^{10} \times 0.5$$
$$= 95v + 90v^2 + \cdots + 50v^{10}$$

This is equivalent to an annuity in arrears with decreases of 5 each year. So the present value is:

$$PV = 100 a_{\overline{10|}} - 5(Ia)_{\overline{10|}} = 100 \times 7.7217 - 5 \times 39.3738 = £575.30$$

(iii) The new force of interest is $\ln(1.05) + 0.04652 = \ln(1.1)$. Therefore we can use an effective rate of interest of 10% *pa*:

$$PV = 100 a_{\overline{10|}}^{@10\%} = 100 \times 6.1446 = £614.46$$

Chapter 9 Summary

An equation of value equates the present value of money received to the present value of money paid out:

$$PV\ income - PV\ outgo = 0$$

$$\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^{\infty} \rho(t) e^{-\delta t} dt = 0$$

$$\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^{\infty} \rho(t)(1+i)^{-t} dt = 0$$

To calculate the yield on a transaction from an equation of value of the form $f(i)=0$, we need:

- to find i_1 and i_2 such that $f(i_1)$ and $f(i_2)$ are of opposite sign, and
- the final value obtained for i to be greater than -1.

Some equations of value cannot be solved algebraically. In such cases, we might calculate the yield using a trial and error approach, in conjunction with linear interpolation. The formula for linear interpolation is:

$$i \approx i_1 + \frac{P - P_1}{P_2 - P_1} \times (i_2 - i_1)$$

If there is uncertainty about the payment or receipt of a cashflow at a particular time, allowance can be made in one of two ways:

- apply a probability of payment/receipt to the cashflow at each time
- use a higher rate of discount such as a new force of interest δ', where $\delta' = \delta + \mu$.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 9 Practice Questions

Questions 9.1 to 9.3 relate to financial security *S*, which operates as follows:

A price P is paid (by the investor) in return for a series of interest payments of D payable at the end of each of the next n years, and a final redemption payment of R payable at the end of the n years. The investor earns an annual effective rate of return of i.

9.1 Calculate P, given that $D=5$, $R=125$, $i=10\%$ and $n=20$.

9.2 Calculate n, given that $P=83.73$, $D=4$, $R=101$ and $i=6\%$.

9.3 Calculate i, given that $P=75$, $D=5$, $R=120$ and $n=10$.

9.4 An investor is to pay £80,000 for a property. The investor will then be entitled to receive rental payments at the end of each year for 99 years. The rental payment will be fixed for the first 33 years, increasing to double the original amount for the next 33 years, and three times the original amount for the remaining 33 years. The value of the property at the end of the 99 years is expected to be £1,500,000.

Calculate the amount of the rent payable in the first year, if the investor expects to obtain a rate of return of 8% *pa* effective on the purchase.

9.5 An investor deposits £2,000 into an account, then withdraws level annual payments starting one year after the deposit is made. Immediately after the 11th annual withdrawal, the investor has £400 left in the account. Calculate the amount of each withdrawal, given that the effective annual rate of interest is 8%.

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 9 Solutions

9.1 The price P can be calculated directly as follows:

$$P = 5a_{\overline{20|}} + 125v^{20} = 5 \times 8.5136 + 125 \times 0.14864 = 61.15$$

9.2 The term n satisfies the equation:

$$83.73 = 4a_{\overline{n|}} + 101v^n = 4\left(\frac{1-1.06^{-n}}{0.06}\right) + 101 \times 1.06^{-n}$$

Rearranging gives:

$$\left(101 - \frac{4}{0.06}\right)1.06^{-n} = 83.73 - \frac{4}{0.06} \quad \Rightarrow \quad 1.06^{-n} = 0.49699$$

Taking logs, we find:

$$-n\ln 1.06 = \ln 0.49699 \quad \Rightarrow \quad n = -\frac{\ln 0.49699}{\ln 1.06} = 12.00 \text{ years}$$

9.3 To find the yield, we must solve the equation of value:

$$75 = 5a_{\overline{10|}} + 120v^{10}$$

At 10%, RHS = 76.99.

At 11%, RHS = 71.71.

Interpolating, we find that $i \approx 0.10 + \frac{76.99-75}{76.99-71.71}(0.11-0.10) = 0.1038$.

So i is approximately 10.4% *pa*.

9.4 If the amount of the rent payable in the first year is X, the equation of value is:

$$80,000 = Xa_{\overline{33|}} + 2Xv^{33}a_{\overline{33|}} + 3Xv^{66}a_{\overline{33|}} + 1,500,000v^{99}$$

ie $$80,000 = X(1 + 2v^{33} + 3v^{66})a_{\overline{33|}} + 1,500,000v^{99}$$

So:

$$80,000 = X(1 + 2 \times 1.08^{-33} + 3 \times 1.08^{-66}) \times 11.5139 + 1,500,000 \times 1.08^{-99}$$

This can be rearranged to find X:

$$X = \frac{80,000 - 736.444}{13.545} = £5,852$$

9.5 If the amount of the annual withdrawal is X, then we need to solve the equation:

$$2{,}000 = Xa_{\overline{11|}} + 400v^{11}$$

So:

$$2{,}000 = 7.1390X + 400 \times 1.08^{-11}$$

Rearranging to find X gives:

$$X = \frac{1{,}828.45}{7.1390} = £256.12$$

End of Part 1

What next?

1. Briefly **review** the key areas of Part 1 and/or re-read the **summaries** at the end of Chapters 1 to 9.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 1. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X1**.
4. Attempt the questions relating to Chapters 1 to 9 of the **Paper B Online Resources (PBOR)**.

10

Loan schedules

Syllabus objectives

3.2 Use the concept of equation of value to solve various practical problems.

3.2.1 Apply the equation of value to loans repaid by regular instalments of interest and capital. Obtain repayments, interest and capital components, the effective interest rate (APR) and construct a schedule of repayments.

0 Introduction

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Loans are mostly used by companies or individuals to raise funds, usually to buy buildings or equipment.

Most loans operate like a repayment mortgage, where the initial capital is repaid during the term of the loan. This is done by making repayments that are greater than the amount of interest due. The remainder of each repayment is used to repay part of the capital.

Some loans operate like interest-only mortgages, where the repayments represent interest only. This means that at the end of the term of the loan, the borrower will need to repay the capital using money from elsewhere.

Repayment loans and interest-only loans were first introduced in Chapter 2.

1 An example

This topic is best introduced through an example. We will consider a more general case afterwards.

Consider a very simple example. Assume a bank lends an individual £1,000 for three years, in return for three payments of £*X*, say, one at the end of each year. The bank will charge an effective rate of interest of 7% per annum.

The equation of value for the transaction gives:

$$1{,}000 = Xa_{\overline{3}|} \quad \Rightarrow \quad X = 381.05$$

Question

Verify the value of X obtained above.

Solution

Using the *Tables*, we see that $a_{\overline{3}|} = 2.6243$ at 7%. Therefore:

$$X = \frac{1{,}000}{2.6243} = 381.05$$

So, the borrower pays £381.05 at times $t = 1,2$ and 3 in return for the loan of £1,000 at time 0. These three payments cover both the interest due and the £1,000 capital.

It is helpful to see how this works in detail.

Each payment is first used to pay any interest due since the last payment, and then to reduce the amount of capital outstanding.

The initial amount of capital outstanding is £1,000 and the first payment is at time 1. Interest accrues before the first payment at 7% *pa*.

At time 1 the interest due on the loan of £1,000 is £70. The total payment made is £381.05. This leaves £311.05 that is available to repay some of the capital. The capital outstanding after this payment is then $£(1{,}000 - 311.05) = £688.95$.

At time 2 the interest due is now only 7% of $£688.95 = £48.22$, as the borrower does not pay interest on the capital that is already repaid, only on the amount outstanding. This leaves $£(381.05 - 48.22) = £332.83$ available to repay capital. The capital outstanding after this payment is then $£(688.95 - 332.83) = £356.12$.

Finally, at time 3 the interest due is 7% of $£356.12 = £24.93$, leaving $£(381.05 - 24.93) = £356.12$ available to pay the outstanding sum of £356.12, and the capital is precisely repaid.

One important point is that each repayment must pay first for interest due on the outstanding capital. The balance is then used to repay some of the capital outstanding. Each payment, therefore, comprises both interest and capital repayment. It may be necessary to identify the separate elements of the payments – for example if the tax treatment of interest and capital differs. Notice also that, where repayments are level, the interest component of the repayment instalments will decrease as capital is repaid, with the consequence that the capital payment will increase.

In this example, the interest payments reduce from £70 to £24.93 and the capital payments increase from £311.05 to £356.12. All payments, including the first payment and the last payment, include an interest element and a capital element.

Question

A bank lends a company £5,000 at a fixed rate of interest of 10% *pa* effective. The loan is to be repaid by five level annual payments.

Calculate the interest and capital payments of each repayment.

Solution

First, we calculate the amount of each repayment, Y.

$$Ya_{\overline{5}|} = 5{,}000 \quad \Rightarrow \quad Y = \frac{5{,}000}{3.7908} = £1{,}318.98$$

The following table shows how each repayment of £1,318.98 is split between interest and capital payments.

Year	Loan outstanding at start of the year (L)	Interest due at the end of the year (I = 10% of L)	Capital repaid at the end of the year ($C = 1{,}318.98 - I$)	Loan outstanding at end of the year ($L–C$)
1	5,000	500	818.98	4,181.02
2	4,181.02	418.10	900.88	3,280.14
3	3,280.14	328.01	990.97	2,289.17
4	2,289.17	228.92	1,090.06	1,199.11
5	1,199.11	119.91	1,199.07	0.04*

* This is non-zero due to rounding.

2 Calculating the capital outstanding

2.1 Introduction

In the example in the previous section, we calculated the capital outstanding just after each payment by rolling forward the loan contract year by year. This was fine for a short-term loan but if the term of the loan had been, say, 20 years, then it would have been very time-consuming. There are two much quicker ways to calculate the capital outstanding immediately after a repayment has been made:

(a) by calculating the accumulated value of the original loan less the accumulated value of the repayments made to date – called the *retrospective* method

(b) by calculating the present value of future repayments – called the *prospective* method.

Applying these two methods to the example in Section 1, the capital outstanding just after the first payment is:

(a) $1{,}000(1.07)-381.05=688.95$

(b) $381.05a_{\overline{2|}}=381.05\times 1.80802=688.95$

These values agree with those found earlier.

Question

Use the retrospective and prospective methods to verify that the capital outstanding just after the second payment is £356.12.

Solution

(a) $1{,}000(1.07)^2-381.05s_{\overline{2|}}=1{,}000(1.07)^2-381.05\times 2.0700=356.13$

(b) $381.05v=\dfrac{381.05}{1.07}=356.12$

The small difference is due to using a rounded repayment amount.

These results will now be proved for the more general case when the loan payments are not necessarily level.

2.2 The theory

Let L_t be the amount of the loan outstanding at time $t = 0, 1, \dots, n$, immediately after the repayment at *t*. The repayments are assumed to be in regular instalments, of amount X_t at time *t*, $t = 1, 2, 3, \dots, n$. (Note that we are not assuming all instalments are the same amount.) Let *i* be the effective rate of interest, per time unit, charged on the loan. Let f_t be the capital repaid at *t*, and let b_t be the interest paid at *t*, so that $X_t = f_t + b_t$.

The equation of value for the loan at time 0 is:

$$L_0 = X_1 v + X_2 v^2 + \cdots + X_n v^n \qquad (2.1)$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all *t*, the above equation simplifies to:

$$L_0 = X a_{\overline{n}|}$$

We can find the loan outstanding at *t prospectively* or *retrospectively.*

Prospective loan calculation

Calculating the loan prospectively involves looking forward and calculating the present value at the current point in time of future cashflows.

Consider the loan transactions at time *n*, which is the end of the contract term. After the final instalment of capital and interest the loan is exactly repaid. So the final instalment X_n must exactly cover the capital that remains outstanding after the instalment paid at $n-1$, together with the interest due on that capital. That is:

$$b_n = iL_{n-1} \; ; \quad f_n = L_{n-1}$$

so that:

$$X_n = iL_{n-1} + L_{n-1} = (1+i)L_{n-1} \quad \Rightarrow \quad L_{n-1} = X_n v$$

So the capital outstanding at time $n-1$, L_{n-1}, is equal to the present value at time $n-1$ of the future payment at time *n*.

Similarly, at any time $t+1$, $t \le n-2$, we know that the capital repaid is $L_t - L_{t+1}$, so that the instalment X_{t+1} is:

$$X_{t+1} = iL_t + (L_t - L_{t+1}) \quad \Rightarrow \quad L_t = (L_{t+1} + X_{t+1})v$$

Similarly, $L_{t+1} = (L_{t+2} + X_{t+2})v$, and working forward, successively substituting for L_{t+r} until we get to $L_n = 0$, we get:

$$\begin{aligned}L_t &= (L_{t+1} + X_{t+1})v \\ &= ((L_{t+2} + X_{t+2})v + X_{t+1})v = X_{t+1}v + X_{t+2}v^2 + L_{t+2}v^2 \\ &= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + L_{t+3}v^3 \\ &= \vdots \\ &= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + \cdots + X_n v^{n-t}\end{aligned}$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all *t*, the loan outstanding at time *t* is:

$$L_t = X(v + v^2 + v^3 + \cdots + v^{n-t}) = Xa_{\overline{n-t|}}$$

This gives the 'prospective method' for calculating the loan outstanding. What this equation tells us is that, for calculating the loan outstanding immediately after the repayment at *t*, say, we have:

Prospective Method: The loan outstanding at time *t* is the present (or discounted) value at time *t* of the future repayment instalments.

Note the condition for this method – the present value must be calculated at a repayment date.

Retrospective loan calculation

Calculating the loan retrospectively involves looking backwards and calculating the accumulated value of past cashflows.

At $t = 1$ the interest due is $b_1 = iL_0$, so the capital repaid is $f_1 = X_1 - iL_0$, leaving a loan outstanding of:

$$L_1 = L_0 - (X_1 - iL_0) = L_0(1+i) - X_1$$

In general, at time $t \geq 1$ the interest due is $b_t = iL_{t-1}$, leaving capital repaid at t of $X_t - iL_{t-1}$, giving:

$$L_t = L_{t-1}(1+i) - X_t$$

Similarly, $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$ and, working back from t to 0 we have:

$$\begin{aligned}L_t &= L_{t-1}(1+i) - X_t \\ &= (L_{t-2}(1+i) - X_{t-1})(1+i) - X_t = L_{t-2}(1+i)^2 - X_{t-1}(1+i) - X_t \\ &= L_0(1+i)^t - (X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t)\end{aligned}$$

If the loan is repaid by level regular instalments so that $X_t = X$ for all t, the loan outstanding at time t is:

$$L_t = L_0(1+i)^t - X((1+i)^{t-1} + (1+i)^{t-2} + \cdots + (1+i) + 1) = L_0(1+i)^t - X s_{\overline{t|}}$$

This gives the 'retrospective method' of calculating the outstanding loan. This may be described in words as:

Retrospective Method: The loan outstanding at time *t* is the accumulated value at time *t* of the original loan less the accumulated value at time *t* of the repayments to date.

Both approaches are very useful in calculating the capital outstanding at any time. Neither result depends on the interest rate being constant. It may be useful to work through the equations assuming the interest charged on the loan in year $r-1$ to r is i_r, say.

Since both methods calculate the loan outstanding at time t, they must both give the same result. This is fairly easy to show algebraically.

Consider Equation (2.1) and multiply it by $(1+i)^t$ giving:

$$L_0(1+i)^t = X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t + X_{t+1}v + \cdots + X_n v^{n-t}$$

Rearranging gives:

$$L_0(1+i)^t - (X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t)$$

$$= X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + \cdots + X_n v^{n-t}$$

which shows that the retrospective result equals the prospective result.

It is not really necessary to memorise the formulae given above. When looking at questions about loans, it is usually easier to apply the principles given here, rather than trying to use the general formulae.

Question

A loan of $50,000 is repayable by level annual payments at the end of each of the next 5 years. Interest is 8% *pa* effective for the first three years and 12% *pa* effective thereafter.

Calculate the loan outstanding immediately after the second repayment.

Solution

Let the amount of each repayment be X so that:

$$50{,}000 = X(a_{\overline{3}|8\%} + v_{8\%}^{3}\, a_{\overline{2}|12\%})$$

$$\Rightarrow \quad X = \frac{50{,}000}{2.5771 + 1.08^{-3} \times 1.6901}$$

$$\Rightarrow \quad X = \$12{,}759.15$$

Using the prospective approach, the loan outstanding immediately after the second payment is:

$$12{,}759.15 v_{8\%}(1 + a_{\overline{2}|12\%}) = 12{,}759.15 \times 1.08^{-1} \times (1 + 1.6901) = \$31{,}781$$

Alternatively, using the retrospective approach:

$$50{,}000(1.08)^2 - 12{,}759.15((1.08) + 1) = \$31{,}781$$

Note that, unless specifically mentioned, exam questions do not require the calculation to be performed both ways, but it is a good way to check an answer.

3 Calculating the interest and capital elements

Given the outstanding capital at any time, we can calculate the interest and capital element of any instalment.

For example, consider the instalment X_t at time *t*. We can calculate the interest element contained in this payment by calculating the loan outstanding immediately after the previous instalment, at $t-1$, L_{t-1}. The interest due on capital of L_{t-1} for one unit of time at effective rate *i* per time unit is iL_{t-1}, and this is the interest paid at *t*. The capital repaid may be found using $X_t - iL_{t-1}$, or by $L_{t-1} - L_t$.

In the example in Section 1 of this chapter, the capital repaid was calculated by deducting the interest due from each instalment. Alternatively, the capital repaid can be calculated by taking the difference between the capital outstanding before and after the instalment.

Similarly, it is a simple matter to calculate the interest paid and capital repaid over several instalments. For example, consider the five instalments from $t+1$ to $t+5$, inclusive. The loan outstanding immediately before the first instalment is L_t. The loan outstanding after the fifth instalment is L_{t+5}. The total capital repaid is therefore $L_t - L_{t+5}$. The total capital and interest paid is $X_{t+1} + X_{t+2} + \cdots + X_{t+5}$. Hence, the total interest paid is:

$$\sum_{k=t+1}^{t+5} b_k = (X_{t+1} + X_{t+2} + \cdots + X_{t+5}) - (L_t - L_{t+5})$$

Note the key differences between working out the interest paid and capital repaid in a single payment or in a series of payments.

For a single payment:

- calculate the loan outstanding after the previous payment
- calculate the interest paid by multiplying by the effective interest rate
- calculate the capital repaid by subtracting the interest paid from the repayment.

For a series of payments:

- calculate the capital repaid by subtracting the loan outstanding after the payments from the loan outstanding before the payments
- calculate the interest paid by subtracting the capital repaid from the total of the payments made.

Question

A loan of 16,000 is repayable by ten level payments, made annually in arrears. The annual effective rate of interest is 4%. Calculate:

(i) the interest element of the 4th payment

(ii) the capital element of the 7th payment

(iii) the capital repaid in the last five years of the loan

(iv) the total interest paid over the whole loan.

Solution

If the annual repayment is X, then:

$$Xa_{\overline{10}|} = 16{,}000 \quad \Rightarrow \quad X = \frac{16{,}000}{a_{\overline{10}|}} = \frac{16{,}000}{8.1109} = 1{,}972.66$$

(i) The capital outstanding after the 3rd payment (working prospectively) is:

$$1{,}972.66a_{\overline{7}|} = 1{,}972.66 \times 6.0021 = 11{,}840.01$$

The interest element of the 4th payment is:

$$0.04 \times 11{,}840.01 = 473.60$$

(ii) The capital outstanding after the 6th payment (working prospectively) is:

$$1{,}972.66a_{\overline{4}|} = 1{,}972.66 \times 3.6299 = 7{,}160.55$$

The interest element of the 7th payment is:

$$0.04 \times 7{,}160.55 = 286.42$$

Therefore the capital element of the 7th payment is:

$$1{,}972.66 - 286.42 = 1{,}686.24$$

(iii) The capital repaid over the last five years of the loan must be the capital outstanding after the 5th payment, *ie*:

$$1{,}972.66a_{\overline{5}|} = 1{,}972.66 \times 4.4518 = 8{,}781.93$$

(iv) The total interest payable over the whole loan is the total payment made less the capital borrowed, *ie*:

$$10 \times 1{,}972.66 - 16{,}000 = 3{,}726.60$$

4 The loan schedule

In the solution to one of the questions in Section 1, we set out the loan outstanding and the capital and interest part of each payment in a table. This type of table is called a *loan schedule*. It will now be defined in a more general context.

The loan payments can be expressed in the form of a table, or 'schedule', as follows.

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	L_0	X_1	iL_0	$X_1 - iL_0$	L_1 $= L_0 - (X_1 - iL_0)$
⋮	⋮	⋮	⋮	⋮	⋮
$t \to t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	L_{t+1} $= L_t - (X_{t+1} - iL_t)$
⋮	⋮	⋮	⋮	⋮	⋮
$n-1 \to n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

With spreadsheet software, it is a simple matter to construct the entire schedule for any loan.

Without a spreadsheet, it is quite time-consuming to have to construct a complete schedule for a loan with more than four or five repayments.

Question

A loan of amount L is to be repaid by level annual payments at the end of each of the next n years. The annual effective interest rate is i, and the annual repayment $P = \dfrac{L}{a_{\overline{n}|}}$.

Set out the loan schedule for this loan, simplifying expressions where possible.

Solution

Using the loan schedule structure outlined above:

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	$L = Pa_{\overline{n}\vert}$	$P = \dfrac{L}{a_{\overline{n}\vert}}$	$iL = iPa_{\overline{n}\vert}$ $= P(1-v^n)$	Pv^n	$P(a_{\overline{n}\vert} - v^n) = Pa_{\overline{n-1}\vert}$
$1 \to 2$	$Pa_{\overline{n-1}\vert}$	$P = \dfrac{L}{a_{\overline{n}\vert}}$	$iPa_{\overline{n-1}\vert}$ $= P(1-v^{n-1})$	Pv^{n-1}	$Pa_{\overline{n-2}\vert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	$Pa_{\overline{n-t}\vert}$	$P = \dfrac{L}{a_{\overline{n}\vert}}$	$P(1-v^{n-t})$	Pv^{n-t}	$Pa_{\overline{n-t-1}\vert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	$Pa_{\overline{1}\vert} = Pv$	$P = \dfrac{L}{a_{\overline{n}\vert}}$	$P(1-v)$	Pv	0

Here:

- the interest due is calculated as the annual effective interest rate i multiplied by the loan outstanding at the start of the year,
- the capital repaid is the repayment amount P minus the interest due, and
- the loan outstanding at the end of the year is calculated as the loan outstanding at the start of the year minus the capital repaid.

We see that the capital repaid increases by a factor of $(1+i)$ each year.

5 Instalments payable more frequently than annually

Most loans will be repaid in quarterly, monthly or weekly instalments. No new principles are involved where payments are made more frequently than annually, but care needs to be taken in calculating the interest due at any instalment date.

If the rate of interest used is effective over the same time unit as the frequency of the repayment instalments, then the calculations proceed exactly as above, with the time unit redefined appropriately.

For the case where the interest is expressed as an effective annual rate, with repayment instalments payable *p*thly, we have the equation of value for the loan, given repayments of X_t at time $t = \frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, n$:

$$L_0 = X_{1/p}v^{1/p} + X_{2/p}v^{2/p} + X_{3/p}v^{3/p} + \cdots + X_n v^n$$

In the case where the loan is repaid by level instalments of amount *X* payable *p*thly (so that the total repayment each year is pX), the loan equation simplifies to:

$$L_0 = X(v^{1/p} + v^{2/p} + v^{3/p} + \cdots + v^n) = pXa_{\overline{n}|}^{(p)}$$

Question

A loan of £900 is repayable by level monthly payments for 3 years, with interest payable at 18.5% *pa* effective. Calculate the amount of each monthly payment.

Solution

Let *M* equal the monthly payment. Then:

$$900 = 12Ma_{\overline{3}|}^{(12)} = 12M\left(\frac{1-v^3}{i^{(12)}}\right)$$

$$\Rightarrow \quad M = \frac{900i^{(12)}}{12(1-v^3)} = \frac{900 \times 12(1.185^{1/12} - 1)}{12(1 - 1.185^{-3})} = £32.13$$

It is easy to show that the two basic principles for calculating the loan outstanding hold when repayments are more frequent than annual. That is, the loan outstanding at any repayment date, immediately after an instalment has been paid, may still be calculated as the present value of the remaining repayment instalments, or as the accumulated value of the original loan less the repayments made to date.

Prospectively:

$$L_t = X_{t+1/p}v^{1/p} + X_{t+2/p}v^{2/p} + \cdots + X_n v^{n-t}$$

Retrospectively:

$$L_t = L_0(1+i)^t - (X_{1/p}(1+i)^{t-1/p} + X_{2/p}(1+i)^{t-2/p} + \dots + X_{t-1/p}(1+i)^{1/p} + X_t)$$

If the loan is repaid by level instalments of amount *X* payable *p*thly, these two expressions for the loan outstanding at time t simplify to:

Prospectively: $L_t = pXa_{\overline{n-t}|}^{(p)}$

Retrospectively: $L_t = L_0(1+i)^t - pXs_{\overline{t}|}^{(p)}$

Capital and interest elements

Given an annual effective rate of interest of *i*, the effective rate of interest over a period $\frac{1}{p}$ is $(1+i)^{1/p} - 1$, which is equal to $i^{(p)}/p$. The interest due at $t+\frac{1}{p}$, given capital outstanding of L_t at some repayment date *t*, is therefore $b_{t+1/p} = ((1+i)^{1/p} - 1)L_t$. The capital repaid at $t+\frac{1}{p}$ is then:

$$f_{t+1/p} = X_{t+1/p} - ((1+i)^{1/p} - 1)L_t = X_{t+1/p} - \frac{i^{(p)}}{p}L_t$$

This is just the total payment at time $t+\frac{1}{p}$ less the interest due at time $t+\frac{1}{p}$.

The capital repaid at $t+\frac{1}{p}$ is also equal to the capital outstanding at time *t* less the capital outstanding at time $t+\frac{1}{p}$, *ie* $L_t - L_{t+1/p}$.

Question

Calculate the interest and capital portions of the thirteenth repayment of the loan introduced in the previous question.

Solution

We calculated the monthly repayment to be £32.13. Working prospectively, the loan outstanding immediately after the twelfth payment is:

$$12 \times 32.13a_{\overline{2}|}^{(12)} = 385.56 \times \frac{1-v^2}{i^{(12)}} = 385.56 \times 1.6839 = £649.25$$

The interest portion of the thirteenth payment is therefore:

$$649.25 \times \frac{i^{(12)}}{12} = 649.25 \times (1.185^{1/12} - 1) = £9.25$$

The capital portion is:

$$32.13-9.25=£22.88$$

We can also use the same approach to calculate the capital and interest elements of a series of repayments that we used when the repayments were made annually.

Question

A loan of £4,000 is repayable by equal monthly payments for 5 years. Interest is payable at a rate of 7% *pa* effective.

Calculate the interest paid and the capital repaid in the 4th year.

Solution

Calculating the monthly payment, X:

$$12Xa_{\overline{5}|}^{(12)}=4{,}000 \quad \Rightarrow \quad X=\frac{4{,}000}{12\times 4.2301}=78.80$$

Working prospectively, the capital outstanding at the start of the 4th year (*ie* at time 3) is:

$$12\times 78.80\times a_{\overline{2}|}^{(12)}=12\times 78.80\times 1.8653=1{,}763.84$$

The capital outstanding at the end of the 4th year is:

$$12\times 78.80\times a_{\overline{1}|}^{(12)}=12\times 78.80\times 0.9642=911.75$$

So the capital repaid in the 4th year is:

$$1{,}763.84-911.75=852.09$$

The interest paid is the difference between the total payment and the capital repaid, *ie*:

$$12\times 78.80-852.09=945.60-852.09=93.51$$

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6 Consumer credit: APR

As we have seen in earlier chapters, there are various ways to display an interest rate. For example, it could be nominal but compounded monthly, or effective. We also noticed that 10% compounded monthly has an effective rate greater than 10% compounded quarterly.

Question

Calculate the annual effective interest rate that is equivalent to:

(i) a nominal rate of interest of 10% per annum convertible monthly

(ii) a nominal rate of interest of 10% per annum convertible quarterly.

Solution

(i) If $i^{(12)} = 10\%$:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.1}{12}\right)^{12} - 1 = 10.4713\%$$

(ii) If $i^{(4)} = 10\%$:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^{4} - 1 = \left(1 + \frac{0.1}{4}\right)^{4} - 1 = 10.3813\%$$

So, two interest rates that appear, at a glance, to be the same, might actually relate to different annual effective rates.

This affects the interest rate being advertised for a loan. Moreover, loans may have additional costs, such as opening costs or display simpler rates (for example, 1% daily for a pay-day loan). Since such an advertised rate ignores the effects of compounding or other costs, it will be considerably lower than the true effective rate of interest charged on the loan.

A 'pay-day' loan is a loan that is designed to be short-term in nature, to give the borrower sufficient funds to pay essential bills. The intention is that the loan will be repaid in full from the borrower's next pay packet. Such loans often charge very high rates of interest, to reflect the risk that the borrower may be unable to repay the loan.

Question

A loan provider quotes an interest rate of 1% per day effective.

(i) Calculate the annual effective interest rate on this loan.

(ii) Comment on why the loan provider has chosen to quote the interest rate as a daily rate.

Solution

(i) Assuming there are 365 days in a year, the annual effective interest rate equivalent to 1% per day effective is:

$$i = 1.01^{365} - 1 = 36.7834 \quad ie \;\; 3{,}678.34\%$$

(ii) An interest rate of 1% per day effective does not appear to be very high, and so is unlikely to put potential borrowers off taking out the loan. With the effect of compounding, the interest charged on an annual basis is very high indeed, and, if disclosed, this would cause most borrowers to look elsewhere.

To ensure that consumers can make informed judgements about the interest rates charged, lenders are required (in most circumstances) to give information about the effective rate of interest charged. In the UK, this is in the form of the Annual Percentage Rate of charge, or APR, which is defined as the effective annual rate of interest, rounded to the nearer 1/10th of 1%.

If all loan providers are required to quote the interest rate charged in the same format, it is easy for consumers to compare the rates and avoid being misled.

The APR is the rate of interest at which the present value of the amount borrowed equals the present value of the repayments (including all other charges), rounded to the nearer 0.1%.

Question

A motorist borrows £5,000 to buy a car. The loan is repaid by level payments of £458.33 at the end of each of the next 12 months. Calculate the APR paid by the motorist.

Solution

To determine the annual effective rate of interest on the loan, we need to solve the equation of value:

$$12 \times 458.33 a_{\overline{1}|}^{(12)} = 5{,}000 \quad ie \quad a_{\overline{1}|}^{(12)} = 0.90910$$

At 20%, $a_{\overline{1}|}^{(12)} = 0.90721$. This is too low. In order to increase the present value, we need to decrease the interest rate.

At 19.5%, $a_{\overline{1}|}^{(12)} = 0.90921$, and at 19.6%, $a_{\overline{1}|}^{(12)} = 0.90881$.

Since the 19.5% value is closer to 0.90910 than the 19.6% value, the APR is 19.5%.

To double check the rounding, we see that at 19.55%, $a_{\overline{1}|}^{(12)} = 0.90901$, *so the true value lies between 19.5% and 19.55%, ie 19.5% to 1 decimal place.*

In the question above, we see that the APR is effectively calculated by trial and error. To ease the calculation of the APR, we can consider a different measure of the interest charged on a loan, to give us an indication of where to start the trial and error process.

The *flat rate of interest* is calculated as:

$$\text{flat rate} = \frac{\text{total interest}}{\text{original loan} \times \text{term in years}} = \frac{\text{total repayment} - \text{original loan}}{\text{original loan} \times \text{term in years}}$$

This is quite a straightforward calculation to perform.

Question

Calculate the flat rate on the loan taken out by the motorist in the previous question.

Solution

The original loan amount is £5,000 and the term of the loan is 1 year. The total repayment made is 12×458.33. So, the flat rate is:

$$\frac{12 \times 458.33 - 5{,}000}{5{,}000 \times 1} = 10.0\%$$

We see that the APR (of 19.5%) is roughly equal to twice the flat rate (of 10%). Since this is often the case, when calculating an APR, we could start by calculating the flat rate, and then double it to provide a first guess for the APR.

To see why the APR is roughly twice the flat rate, we can rewrite the formula for the flat rate as:

$$\text{flat rate} = \frac{\text{total interest}}{\text{original loan} \times \text{term in years}} = \frac{\text{average annual interest}}{\text{original loan}}$$

This is because the total interest paid over the whole term of the loan divided by the term in years is equal to the average annual interest paid. However, to approximate the annual effective interest rate on the loan (*ie* the APR), we should divide the average annual interest by the average loan amount, rather than the original loan amount.

The loan outstanding varies from the original loan amount at outset to 0 at the end of the term, so the average loan amount is about one half of the original loan amount. Therefore, the APR is approximately:

$$\text{APR} \approx \frac{\text{average annual interest}}{\text{average loan amount}} = \frac{\text{average annual interest}}{\frac{1}{2} \times \text{original loan}} = 2 \times \frac{\text{average annual interest}}{\text{original loan}}$$

which is twice the flat rate.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 10 Summary

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

Each repayment must first pay the interest due on the outstanding capital; the balance is then used to repay some of the capital outstanding.

We can find the loan outstanding prospectively or retrospectively.

A prospective method involves finding the present value of future repayments.

A retrospective method involves calculating the accumulated value of the initial loan less the accumulated value of the repayments to date.

We can calculate the interest element contained in a single payment by calculating the loan outstanding immediately after the previous instalment and multiplying it by the rate of interest. The capital element is the total payment less the interest payment.

We can calculate the capital repaid in a period where there is more than one payment by subtracting the capital outstanding at the end of the period from the capital outstanding at the start of the period. The interest paid in this period is then the total payment less the capital repaid.

The interest and capital components in the repayments for a loan can be set out in the form of a loan schedule.

No new principles are involved where payments are made more frequently than annually, but care needs to be taken when calculating the interest due at any instalment date, to ensure the correct interest rate is used.

Lenders are usually required to give information about the effective rate of interest charged.

In the UK, this is in the form of the Annual Percentage Rate of charge, or APR, which is defined as the effective annual rate of interest, rounded to the nearest 1/10th of 1%.

The practice questions start on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 10 Practice Questions

10.1 A loan of £120,000 is repayable by equal quarterly payments for 25 years. The effective rate of interest is 6% *pa*.

Calculate the interest portion of the first payment.

10.2 A loan of £1,000 is to be repaid by level monthly instalments over 10 years using an interest rate of 10% *pa* effective.

Calculate the capital repaid in the sixth year.

10.3 A bank issues a 10-year loan for £100,000 to a businessman. The loan is to be repaid by annual repayments, payable in arrears, calculated using an interest rate of 8% *pa* effective. The repayment schedule has been designed so that half the capital will have been repaid by the end of the term. The remaining £50,000 will be repaid at the end of the term using funds from other sources.

Calculate the annual repayment.

10.4 A loan of £80,000 is repayable by eight annual payments, with the first payment being made in one year's time. The first three payments are half as much as the remaining five payments, and the annual effective interest rate is 4.5%.

Calculate the loan outstanding one year before the loan is completely repaid.

10.5 A customer borrows £4,000 under a consumer credit loan. Repayments are calculated based on an APR of 15.4%, and are paid monthly in arrears for 5 years.

Calculate the amount of each monthly repayment.

10.6 A man borrows £7,500 to buy a car. He repays the loan by 24 monthly instalments of £368.75, payable in arrears.

Exam style

Calculate the APR on this transaction. [4]

10.7 A loan of £50,000 is repaid over a period of 10 years by a series of level monthly instalments. Interest is charged on the loan at the rate of interest of 8% *pa* effective.

Exam style

(i) Calculate the monthly repayment. [2]

(ii) Calculate the amount of interest paid in the first year. [3]

After the payment at the end of 7 years, the borrower takes a 2-month payment break, *ie* the borrower does not pay the next 2 monthly instalments.

(iii) Calculate the extra amount that needs to be paid each month in order to fully repay the debt by the end of the 10th year. [4]

[Total 9]

10.8 A loan is to be repaid by payments at the end of each of the next 15 years. The first payment is £100 and the payments increase by £20 *pa* thereafter. Repayments are calculated using a rate of interest of 5% *pa* effective.

Exam style

(i) Calculate the amount of the loan. [3]

(ii) Set out a loan schedule showing the capital and interest elements in, and the amount of loan outstanding after, the 6th and 7th payments. [5]

(iii) Calculate the capital and interest element of the last instalment. [2]

[Total 10]

10.9 An actuarial student takes out a mortgage for £250,000 with a term of 25 years. The mortgage is repayable by level instalments made monthly in arrears. Interest is charged at a rate of 6% *pa* effective.

Exam style

(i) Calculate the monthly repayment. [2]

(ii) (a) Calculate the capital repaid in the fourth year.

(b) Calculate the interest element of the 49th repayment. [4]

After completing her exams, six years after taking out the mortgage, the newly-qualified actuary reviews her finances and realises that she can afford to make repayments at twice the rate calculated in (i).

(iii) Calculate the length of time by which this course of action reduces the remaining term of the loan. [4]

(iv) Calculate the amount of the final repayment, and hence the interest saved by the actuary, if she follows this course of action. [5]

[Total 15]

Chapter 10 Solutions

10.1 The interest portion of the first payment is:

$$120{,}000(1.06^{1/4}-1)=£1{,}760.86$$

The amount of each instalment is not needed here because we know the loan outstanding at the start.

10.2 Let X be the monthly payment, then:

$$12Xa_{\overline{10}|}^{(12)}=1{,}000 \quad\Rightarrow\quad X=\frac{1{,}000}{12\times 6.4213}=£12.98$$

The capital outstanding at the start of the sixth year (*ie* with 5 years still to run) is:

$$12\times 12.98a_{\overline{5}|}^{(12)}=12\times 12.98\times 3.9615=617.05$$

Similarly, the capital outstanding at the end of the sixth year (*ie* with 4 years still to run) is:

$$12\times 12.98a_{\overline{4}|}^{(12)}=12\times 12.98\times 3.3127=515.98$$

So the capital repaid during the sixth year is $617.05-515.98=£101.07$.

10.3 The annual repayment R can be found by setting the initial loan amount equal to the present value of all the repayments (*ie* the annual repayments plus the lump sum of £50,000):

$$100{,}000=Ra_{\overline{10}|}+50{,}000v^{10} \quad @8\%$$

ie:

$$100{,}000=6.7101R+50{,}000\times 1.08^{-10}$$

So:

$$R=\frac{76{,}840}{6.7101}=£11{,}451$$

10.4 Let X equal the amount of the first instalment, then:

$$80{,}000=Xa_{\overline{8}|}+Xv^3a_{\overline{5}|} \quad\Rightarrow\quad 80{,}000=X\left(\frac{1-1.045^{-8}}{0.045}+1.045^{-3}\times\frac{1-1.045^{-5}}{0.045}\right)$$

Therefore:

$$X=\frac{80{,}000}{6.5959+0.8763\times 4.3900}=7{,}660.77$$

Alternatively, the equation for X can be written:

$$80{,}000 = Xa_{\overline{3}|} + 2Xv^3 a_{\overline{5}|}$$

Working prospectively, the loan outstanding one year before the end of the term is equal to the present value at that time (*ie* time 7) of the final repayment:

$$2 \times 7{,}660.77v = £14{,}662$$

10.5 We can calculate the amount of each monthly repayment, M, from the equation of value:

$$4{,}000 = 12M a^{(12)}_{\overline{5}|@15.4\%} = 12M \frac{1-1.154^{-5}}{12(1.154^{1/12}-1)} = 42.5877M$$

$$\Rightarrow \quad M = £93.92$$

10.6 The APR is the annual effective rate of interest that solves the equation of value:

$$12 \times 368.75 a^{(12)}_{\overline{2}|} = 7{,}500 \quad ie \quad a^{(12)}_{\overline{2}|} = 1.6949$$ [1]

To obtain a first guess for the APR, we can calculate the flat rate, as follows:

$$\frac{24 \times 368.75 - 7{,}500}{7{,}500 \times 2} = 9\%$$

The APR is roughly twice the flat rate, so as a first guess we can try 18%:

$$a^{(12)}_{\overline{2}|@18\%} = 1.6909$$

Then trying 17%:

$$a^{(12)}_{\overline{2}|@17\%} = 1.7052$$

Interpolating between these two values gives an interest rate of 17.7%. Now trying 17.7% and 17.8%:

$$a^{(12)}_{\overline{2}|@17.7\%} = 1.6952$$ [1]

$$a^{(12)}_{\overline{2}|@17.8\%} = 1.6938$$ [1]

Since the 17.7% value is closer to 1.6949 than the 17.8% value, the APR is 17.7%. [1]

[Total 4]

To double check the rounding, we see that at 17.75%, $a^{(12)}_{\overline{2}|} = 1.6945$*, so the true value lies between 17.7% and 17.75%, ie 17.7% to 1 decimal place.*

10.7 (i) Let P denote the monthly repayment. Then, working in years:

$$12Pa_{\overline{10}|}^{(12)} @8\% = 50,000$$ [1]

Solving for P:

$$P = \frac{50,000}{12 \times 6.9527} = £599.29$$ [1]

[Total 2]

Alternatively, working in months, the equation of value is:

$$Pa_{\overline{120}|} = 50,000$$

where the annuity is calculated using the effective monthly interest rate:

$$i = 1.08^{1/12} - 1 = 0.00643403$$

(ii) Working prospectively, the amount outstanding at the end of the first year is given by:

$$(12 \times 599.29)a_{\overline{9}|}^{(12)} @8\% = 12 \times 599.29 \times 6.4728 = £46,548.71$$ [1]

The capital repaid in the first year is then:

$$50,000 - 46,548.71 = £3,451.29$$ [1]

and the interest paid in the first year is:

$$(12 \times 599.29) - 3,451.29 = £3,740.19$$ [1]

[Total 3]

Alternatively, working in months, the amount outstanding at the end of the first year is given by:

$$599.29a_{\overline{108}|} @0.643403\%$$

(iii) Working prospectively, the amount outstanding after 7 years (just before the payment break) is:

$$(12 \times 599.29)a_{\overline{3}|}^{(12)} @8\% = 12 \times 599.29 \times 2.6703 = £19,203.25$$ [1]

As no payments are made for the next 2 months, the capital outstanding after 7 years and 2 months is:

$$19,203.25 \times 1.08^{2/12} = 19,451.15$$ [1]

This is to be repaid in equal monthly instalments of Q over the next 2 years and 10 months. So:

$$12Qa^{(12)}_{\overline{2\frac{10}{12}}|} @8\% = 19,451.15 \quad \Rightarrow \quad Q = \frac{19,451.15}{12 \times 2.5375} = £638.78$$ [1]

The extra monthly payment is therefore $638.78 - 599.29 = £39.49$ [1]

[Total 4]

Alternatively, working in months, the amount outstanding after 7 years is given by:

$$599.29a_{\overline{36}|} @0.643403\%$$

and the equation to solve for Q is:

$$Qa_{\overline{34}|} @0.643403\% = 19,451.15$$

10.8 (i) The amount of the loan, *L*, is the present value of all the repayments:

$$L = 80a_{\overline{15}|} + 20(Ia)_{\overline{15}|}$$ [2]

Using values from the *Tables*:

$$L = 80 \times 10.3797 + 20 \times 73.6677 = £2,303.73$$ [1]

[Total 3]

(ii) We first calculate the capital outstanding immediately after the 5th payment has been made. The 6th payment is £200; the 7th payment is £220 and so on. Again, using values from the *Tables*:

$$180a_{\overline{10}|} + 20(Ia)_{\overline{10}|} = 180 \times 7.7217 + 20 \times 39.3738 = £2,177.38$$

The interest element of the 6th payment is:

$$0.05 \times 2,177.38 = £108.87$$

Hence, the capital element is $200 - 108.87 = £91.13$, and the capital outstanding after the 6th payment is $2,177.38 - 91.13 = £2,086.25$.

Similarly, the interest element of the 7th payment is:

$$0.05 \times 2,086.25 = £104.31$$

and the capital element is $220 - 104.31 = 115.69$. The capital outstanding after the 7th payment is therefore $2,086.25 - 115.69 = £1,970.56$.

Expressing these amounts in a loan schedule gives:

Payment	***Interest element (£)***	***Capital element (£)***	***Capital outstanding after payment (£)***
5			2,177.38
6	108.87	91.13	2,086.25
7	104.31	115.69	1,970.56

[1 mark for row 5, 2 marks each for rows 6 and 7]
[Total 5]

(iii) As there are 14 increases of £20 after the first repayment of £100, the last payment is £380. Working prospectively, the capital outstanding immediately after the penultimate payment is therefore:

$$380v = £361.90 \qquad [1]$$

This must also be the capital element of the last payment if that payment is to pay off the loan. Thus the interest element is:

$$380 - 361.90 = £18.10 \qquad [1]$$

[Total 2]

10.9 (i) ***Monthly repayment***

Let M be the monthly repayment. The equation of value is:

$$250{,}000 = 12Ma_{\overline{25}|}^{(12)} \qquad [1]$$

Using $a_{\overline{25}|}^{(12)} = 13.1312$, this gives:

$$M = £1{,}586.55 \qquad [1]$$

[Total 2]

(ii)(a) ***Capital repaid in the fourth year***

The capital outstanding at the start of the fourth year is calculated (prospectively) as:

$$12Ma_{\overline{22}|}^{(12)} = 12 \times 1{,}586.55 \times 12.36924 = £235{,}493.04 \qquad [1]$$

The capital outstanding at the end of the fourth year is calculated (prospectively) as:

$$12Ma_{\overline{21}|}^{(12)} = 12 \times 1{,}586.55 \times 12.08419 = £230{,}065.97 \qquad [1]$$

The capital repaid in the fourth year is therefore:

$$235{,}493.04 - 230{,}065.97 = £5{,}427 \qquad [1]$$

(ii)(b) ***Interest element in the 49th repayment***

To calculate the interest element in the 49th repayment, the capital outstanding immediately after the previous (*ie* 48th) repayment is needed. The 48th repayment is made at the end of four years, so the capital outstanding at that time is £230,065.97 from (ii)(a).

So the interest element in the 49th repayment is:

$$230{,}065.97 \times (1.06^{1/12} - 1) = £1{,}120$$ [1]

[Total 4]

(iii) ***Reduction in payment term***

After six years, when the student has qualified, the remaining term is 19 years. The capital outstanding at this point is:

$$12Ma_{\overline{19}|}^{(12)} = 12 \times 1{,}586.55 \times 11.46174 = £218{,}215.42$$ [1]

If the actuary makes monthly repayments at twice the original rate, the equation of value is:

$$218{,}215.42 = 12 \times 2 \times 1{,}586.55 \times a_{\overline{n}|}^{(12)}$$

where *n* is the reduced payment term. So:

$$a_{\overline{n}|}^{(12)} = \frac{1 - v^n}{i^{(12)}} = 5.73087$$ [1]

$$\Rightarrow 1 - v^n = 0.33474$$

$$\Rightarrow v^n = 0.66526$$

Taking logs of both sides:

$$-n\ln(1.06) = \ln(0.66526)$$

$$\Rightarrow n = 6.9949$$ [1]

Therefore, the final repayment will be made 7 years after the increased payments commence.

This means the payment term is shortened by 12 years. [1]

[Total 4]

(iv) ***Final repayment amount and total interest saved***

Now assume that the actuary makes twice the original monthly repayments, and let P be the amount of the final repayment made.

P can be found by solving the equation of value:

$$218{,}215.42 = 12 \times 2 \times 1{,}586.55 \times a_{\overline{6\frac{11}{12}}|}^{(12)} + Pv^7$$ [1]

Using $a^{(12)}_{\overline{6\frac{11}{12}|}} = 5.67886$, gives:

$$Pv^7 = 1{,}980.313 \quad \Rightarrow \quad P = £2{,}977.66 \qquad [1]$$

The total interest paid is equal to the difference between the total repayments made and the total capital repaid.

Hence, if the actuary is making twice the original monthly repayments, the total interest paid after the end of the sixth year is:

$$12 \times 2 \times 1{,}586.55 \times 6\tfrac{11}{12} + 2{,}977.66 - 218{,}215.42 = £48{,}129.54 \qquad [1]$$

If the actuary continues making only the original repayments, the total interest paid after the end of the sixth year is:

$$12 \times 1{,}586.55 \times 19 - 218{,}215.42 = £143{,}517.98 \qquad [1]$$

Hence, the total interest saved by following the new course of action is:

$$143{,}517.98 - 48{,}129.54 = £95{,}388 \qquad [1]$$

[Total 5]

Alternatively, we could calculate the total interest paid over the whole term of the loan, under each of the repayment schedules.

Where only the original repayments are made, the total interest is:

$$12 \times 1{,}586.55 \times 25 - 250{,}000 = £225{,}965.00$$

Where twice the original repayments are made after six years, the total interest is:

$$12 \times 1{,}586.55 \times 6 + 12 \times 2 \times 1{,}586.55 \times 6\tfrac{11}{12} + 2{,}977.66 - 250{,}000 = £130{,}576.56$$

Subtracting these gives the total interest saved as £95,388 as above.

11

Project appraisal

Syllabus objectives

3.3 Show how discounted cashflow and equation of value techniques can be used in project appraisals.

- 3.3.1 Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.
- 3.3.2 Calculate the internal rate of return, payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

0 Introduction

This chapter looks at methods that can be used to decide between alternative investment projects. We consider the following criteria:

- net present value
- accumulated profit
- internal rate of return
- payback period
- discounted payback period.

Estimating cashflows

Suppose an investor is considering the merits of an investment or business project. The investment or project will normally require an initial outlay and possibly other outlays in future, which will be followed by receipts, although in some cases the pattern of income and outgo is more complicated. The cashflows associated with the investment or business venture may be completely fixed (as in the case of a secure fixed-interest security maturing at a given date) or they may have to be estimated.

For an organisation issuing a fixed-interest security, there will be an initial positive cashflow, a single known negative cashflow on a specified future date, and a series of smaller known negative cashflows on a regular set of specified future dates.

The estimation of the cash inflows and outflows associated with a business project usually requires considerable experience and judgement. All the relevant factors (such as taxation and investment grants) and risks (such as construction delays) should be considered by the actuary, with assistance from experts in the relevant field (*eg* civil engineering for building projects). The identification and assessment of the risks may be conducted using the Risk Analysis and Management for Projects (RAMP) approach for risk analysis and management that has been developed by, and published on behalf of, the actuarial and civil engineering professions.

There's no need to know any more about RAMP than this for the Subject CM1 exam. However, more information about RAMP and initiatives between the actuarial and civil engineering professions can be found on the Institute and Faculty of Actuaries' website.

Considerable uncertainty will exist in the assessment of many of the risks, so it is prudent to perform calculations on more than one set of assumptions, *eg* on the basis of 'optimistic', 'average', and 'pessimistic' forecasts respectively.

A set of optimistic assumptions is often called a 'weak' basis and a pessimistic set of assumptions is often called a 'strong', 'prudent' or 'cautious' basis. Average assumptions are also called 'best-estimate' or 'realistic' assumptions.

More complicated techniques (using statistical theory) are available to deal with this kind of uncertainty. Precision is not attainable in the estimation of cashflows for many business projects and hence extreme accuracy is out of place in many calculations.

For example, there is no point in quoting a final answer to eight decimal places if the figures used in the calculation have only been estimated to the nearest million.

Net cashflow c_t at time t (measured in suitable time units) is:

$$c_t = \text{cash inflow at time } t - \text{cash outflow at time } t$$

If any payments may be regarded as continuous then $\rho(t)$, the net rate of cashflow per unit time at time t, is defined as:

$$\rho(t) = \rho_1(t) - \rho_2(t)$$

where $\rho_1(t)$ and $\rho_2(t)$ denote the rates of inflow and outflow at time t respectively.

We will now start looking at methods that can be used to decide between alternative investment projects. We will use the following two hypothetical projects as examples. Both relate to a small software company that has been asked to set up a new computer system for a major client.

Project R

Project R delegates all the development work to outside companies. The estimated cashflows for Project R are (where brackets indicate expenditure):

Beginning of Year 1	(£150,000)	(contractors' fees)
Beginning of Year 2	(£250,000)	(contractors' fees)
Beginning of Year 3	(£250,000)	(contractors' fees)
End of Year 3	£1,000,000	(sales)

Project S

Project S carries out all the development work in-house by purchasing the necessary equipment and using the company's own staff. The estimated cashflows for Project S are:

Beginning of Year 1	(£325,000)	(new equipment)
Throughout Year 1	(£75,000)	(staff costs)
Throughout Year 2	(£90,000)	(staff costs)
Throughout Year 3	(£120,000)	(staff costs)
End of Year 3	£1,000,000	(sales)

The staff costs can be assumed to be paid continuously throughout each year.

The next few sections will refer back to these two examples.

1 Fixed interest rates

Having ascertained or estimated the net cashflows of the investment or project under scrutiny, the investor will wish to measure its profitability in relation to other possible investments or projects. In particular, the investor may wish to determine whether or not it is prudent to borrow money to finance the venture.

Assume for the moment that the investor may borrow or lend money at a fixed rate of interest *i* per unit time. The investor could accumulate the net cashflows connected with the project in a separate account in which interest is payable or credited at this fixed rate. By the time the project ends (at time *T*, say), the balance in this account will be:

$$\sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}\,dt \tag{1.1}$$

where the summation extends over all t such that $c_t \neq 0$.

1.1 Accumulated value

One criterion that can be used to assess an investment project involves calculating the 'accumulated profit' at the end of the project. This is the accumulated value of the net cashflows (as at the time of the last payment).

The accumulated value, at time *T*, of a cashflow can be expressed as:

$$A(T) = \sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t}\,dt$$

The accumulated profit for a project has the intuitive appeal that it represents the final amount 'left over' if all the payments for the project were transacted through a bank account that earned interest at a rate *i*.

However, accumulated profit calculations suffer from the disadvantage that they can only be used in situations where there is a definite fixed time horizon for the project. This will not be the case if the time horizon (*ie* the time until the last cashflow payment) is:

- unlimited, or
- the timing of the payments is uncertain.

An example relating to the first bullet point is where an investor is considering purchasing a fixed-interest security with coupon payments that continue forever (known as an undated or irredeemable security). In order to determine a value for the accumulated profit, the investor would have to pick an arbitrary date and assume that the holding would be sold on that date (otherwise the accumulated profit would be infinite). The value calculated for the accumulated profit will then depend crucially on which date is selected.

An example relating to the second bullet point is where a retired man is considering using a lump sum to buy a pension payable for the rest of his life. Since he does not know when he will die, he cannot know the date to which he should accumulate the payments.

Question

Calculate the accumulated profit after 20 years of a project in which \$20,000 is paid out at time 0 and \$5,000 is received at times 5 to 15, inclusive. Assume an annual effective rate of interest of 3%.

Solution

There are 11 payments of \$5,000 and the accumulated value of these at the time the last payment is received (*ie* at time 15) is $5{,}000 s_{\overline{11|}}$. The accumulated profit is therefore:

$$-20{,}000 \times 1.03^{20} + 5{,}000 s_{\overline{11|}} \times 1.03^{5} = \$38{,}117$$

where $s_{\overline{11|}} = 12.8078$ when $i = 0.03$.

A further problem associated with accumulated profit calculations is that the accumulated profits for two different projects cannot be compared directly if they have different time horizons, since the calculated values will relate to different dates. However, this problem can be avoided by accumulating all the profits to the date of the last payment for the longer project.

These problems can be avoided by calculating the 'net present value' instead.

1.2 Net present values

The present value at rate of interest *i* of the net cashflows is called the 'net present value' at rate of interest *i* of the investment or business project, and is usually denoted by $NPV(i)$. Hence:

$$NPV(i) = \sum c_t (1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}\,dt \qquad (1.2)$$

The rate of interest *i* used to calculate the net present value is often referred to as the *risk discount rate*. Note that the risk discount rate is a rate of interest (i) not a rate of discount (d).

The net present value is similar to the accumulated profit, the only difference being that we are now looking at the value at the outset (which, by definition, is a fixed date), rather than the value at the end of the project. A higher net present value indicates a more profitable project.

(If the project continues indefinitely, the accumulation (1.1) is not defined, but the net present value may be defined by Equation (1.2) with $T = \infty$.) If $\rho(t) = 0$, we obtain the simpler formula:

$$NPV(i) = \sum c_t v^t$$

where $v = (1+i)^{-1}$.

Since the equation:

$$NPV(i) = 0$$

is the equation of value for the project at the present time, the yield i_0 on the transaction is the solution of this equation, provided that a unique solution exists.

It may readily be shown that $NPV(i)$ is a smooth function of the rate of interest i and that $NPV(i) \to c_0$ as $i \to \infty$.

Question

Calculate the net present value for both Project R and Project S (outlined in the introduction to this chapter) using a risk discount rate of 20% *pa*.

Based on the net present values obtained, comment on which project is preferable.

Solution

For Project R, the net present value (in £000s) is:

$$NPV_R = -150 - 250v - 250v^2 + 1{,}000v^3 \quad @20\%$$

$$= -150 - 250 \times 1.2^{-1} - 250 \times 1.2^{-2} + 1{,}000 \times 1.2^{-3} = 46.759$$

For Project S, the net present value (in £000s) is:

$$NPV_S = -325 - 75\bar{a}_{\overline{1}|} - 90v\bar{a}_{\overline{1}|} - 120v^2\bar{a}_{\overline{1}|} + 1{,}000v^3 \quad @20\%$$

$$= -325 - (75 + 90v + 120v^2)\bar{a}_{\overline{1}|} + 1{,}000v^3$$

$$= -325 - (75 + 90 \times 1.2^{-1} + 120 \times 1.2^{-2}) \times 0.9141 + 1{,}000 \times 1.2^{-3}$$

$$= 40.405$$

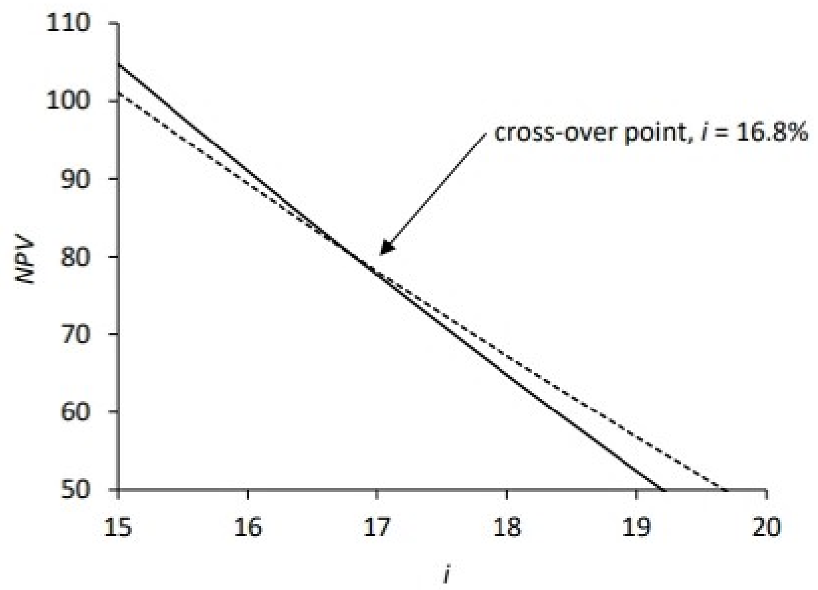
The net present values are £46,759 for Project R and £40,405 for Project S.

Using a risk discount rate of 20%, Project R has the higher net present value and so appears more favourable.

The net present value will depend on the risk discount rate used.

The following graph shows the net present values for Project R and Project S for different risk discount rates. There is a crossover point corresponding to about 16.8%. If the risk discount rate used exceeds 16.8%, the net present value is greater for Project R than for Project S. If the risk discount rate is less than 16.8%, the net present value is greater for Project S.

In the graph, the dotted line represents Project R and the solid line represents Project S.



In the graph above, as is the case for most projects in the real world, the net present value decreases as the risk discount rate (i) increases. The reason for this is that in Project R and Project S the income occurs later in time than the outgo, meaning it is discounted for a longer period, so its present value is more affected by the change in the risk discount rate. Therefore, as the risk discount rate increases, the present value of the project's income reduces by more than the present value of the project's outgo, and the net present value falls.

Question

Let the cashflow at time t be denoted C_t (where time t is measured in years and the amounts are in £000). The cashflows for two business ventures are as follows:

Venture 1: $C_0 = -100$, $C_1 = -40$, $C_2 = +50$, $C_3 = +120$

Venture 2: $C_1 = -45$, $C_3 = +25$, $C_4 = +25$, $C_5 = +25$

Calculate the accumulated profit at time 5 and the net present value for each of these ventures using a risk discount rate of 15% *pa*.

Solution

The accumulated profit for Venture 1 at time 5 is:

$$AP_1 = -100(1+i)^5 - 40(1+i)^4 + 50(1+i)^3 + 120(1+i)^2 \quad @15\% = -36.352$$

The accumulated profit for Venture 2 at time 5 is:

$$AP_2 = -45(1+i)^4 + 25(1+i)^2 + 25(1+i) + 25 \quad @15\% = 8.107$$

So the accumulated profits are –£36,352 (*ie* a loss) for Venture 1 and £8,107 for Venture 2.

The net present values are:

$$NPV_1 = -100 - 40v + 50v^2 + 120v^3 \quad @15\% = -18.073$$

$$NPV_2 = -45v + 25v^3 + 25v^4 + 25v^5 \quad @15\% = 4.031$$

So the NPVs are –£18,073 for Venture 1 and £4,031 for Venture 2.

Alternatively, the net present values could be calculated using the relationship:

$$NPV_i = AP_i \times 1.15^{-5} \quad \text{for } i = 1,2$$

1.3 Internal rate of return

In economics and accountancy the yield per annum is often referred to as the 'internal rate of return' (IRR) or the 'yield to redemption'. The latter term is frequently used when dealing with fixed-interest securities, for which the 'running' (or 'flat') yield is also considered.

We will leave the definition of the running yield until later in the course.

The internal rate of return for an investment project is the effective rate of interest that equates the present value of income and outgo, *ie* it makes the net present value of the cashflows equal to zero.

If all the payments for the project were transacted through a bank account that earned interest at the same rate as the internal rate of return, the net proceeds at the end of the project (*ie* the accumulated profit) would be zero.

For most projects, there is a unique solution to the equation defining the internal rate of return, since the quantity 'PV payments in − PV payments out' generally decreases as i increases, as we saw in the graph on the previous page.

The internal rate of return need not be positive. A zero return implies that the investor receives no return on the investment and if the yield is negative then the investor loses money on the investment. It is difficult, however, to find a practical interpretation for a yield less than –1, and so if there is not a solution to the equation greater than –1, the yield is undefined.

In some cases, it is possible for there to be more than one solution. In such cases the smallest positive solution is usually used. Also, if there are only inflows of cash (*ie* no outflow), the internal rate of return will be infinite.

Usually, the equation of value cannot be solved directly to find the interest rate. In these cases, an approximate solution can be found using the methods we looked at in Chapter 9. A more accurate value can then be found using linear interpolation by calculating the net present value for interest rates close to the initial estimate.

Question

Calculate the internal rate of return for both Project R and Project S (outlined in the introduction to this chapter) and, based on the values obtained, comment on which project is preferable.

Solution

Project R

For Project R, we need to calculate the interest rate i that satisfies the equation of value:

$$NPV_R(i) = -150 - 250v - 250v^2 + 1{,}000v^3 = 0$$

From an earlier question, we already know that $NPV_R(0.2) = 46.759$.

As this is too high, we need to try a higher rate. We find that $NPV_R(0.25) = 2.000$ and $NPV_R(0.26) = -5.977$.

We can approximate i by linearly interpolating using these two values:

$$i \approx 25\% + \frac{0 - 2.000}{-5.977 - 2.000} \times (26\% - 25\%) = 25.25\%$$

Now, $NPV_R(0.2525) = -0.022$, which is very close to 0, so the internal rate of return for Project R is approximately 25.25% *pa*.

Project S

For Project S, we need to calculate the interest rate that satisfies the equation of value:

$$NPV_S(i) = -325 - (75 + 90v + 120v^2)\bar{a}_{\overline{1}|} + 1{,}000v^3 = 0$$

From an earlier question, we already know that $NPV_S(0.2) = 40.405$.

As this is too high, we need to try a higher rate. We find that $NPV_S(0.23) = 6.898$ and $NPV_S(0.24) = -3.520$.

We can approximate i by linearly interpolating using these two values:

$$i \approx 23\% + \frac{0 - 6.898}{-3.520 - 6.898} \times (24\% - 23\%) = 23.66\%$$

Now $NPV_S(0.2366) = -0.018$, which is very close to 0, so the internal rate of return for Project S is approximately 23.66% *pa*.

Since the internal rate of return (or yield) for Project R exceeds that for Project S, Project R appears more favourable based on this criterion.

The practical interpretation of the net present value function $NPV(i)$ and the yield is as follows. Suppose that the investor may lend or borrow money at a fixed rate of interest i_1. Since, from Equation (1.2), $NPV(i_1)$ is the present value at rate of interest i_1 of the net cashflows associated with the project, we conclude that the project will be profitable if and only if:

$$NPV(i_1) > 0$$

Also, if the project ends at time T, then the profit (or, if negative, loss) at that time is (by Expression (1.1)):

$$NPV(i_1)(1+i_1)^T$$

Let us now assume that, as is usually the case in practice, the yield i_0 exists and $NPV(i)$ changes from positive to negative when $i = i_0$. Under these conditions it is clear that the project is profitable if and only if:

$$i_1 < i_0$$

***ie* the yield exceeds that rate of interest at which the investor may lend or borrow money.**

Many projects will need to provide a return to shareholders and so there will not be a specific fixed rate of interest that has to be exceeded. Instead a target, or hurdle, rate of return may be set for assessing whether a project is likely to be sufficiently profitable.

1.4 The comparison of two investment projects

Suppose now that an investor is comparing the merits of two possible investments or business ventures, which we call projects *A* and *B* respectively. We assume that the borrowing powers of the investor are not limited.

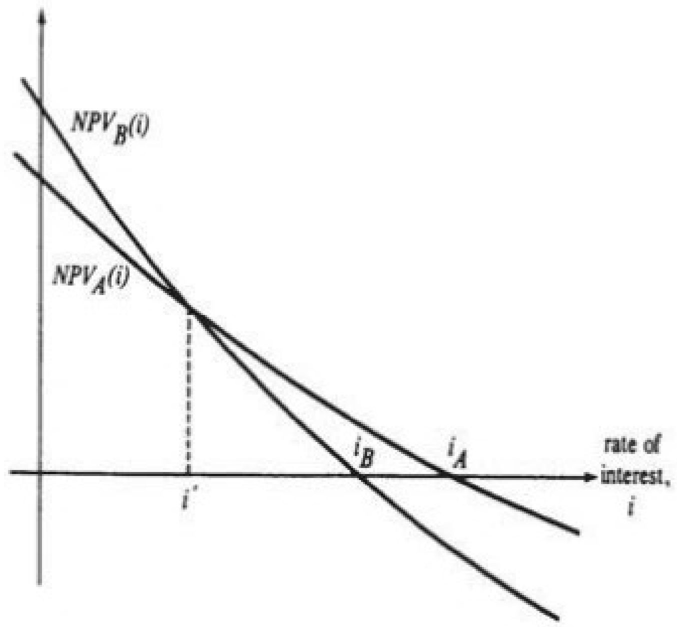
There are therefore no restrictions on how much money the investor can borrow.

Let $NPV_A(i)$ and $NPV_B(i)$ denote the respective net present value functions and let i_A and i_B denote the yields (which we shall assume to exist). It might be thought that the investor should always select the project with the higher yield, but this is not invariably the best policy. A better criterion to use is the *profit at time T* (the date when the later of the two projects ends) or, equivalently, the *net present value*, calculated at the rate of interest i_1 at which the investor may lend or borrow money. This is because *A* is the more profitable venture if:

$$NPV_A(i_1) > NPV_B(i_1)$$

The fact that $i_A > i_B$ may not imply that $NPV_A(i_1) > NPV_B(i_1)$ is illustrated in the following diagram. Although i_A is larger than i_B, the $NPV(i)$ functions 'cross-over' at i'. It follows that $NPV_B(i_1) > NPV_A(i_1)$ for any $i_1 < i'$, where i' is the *cross-over rate*. There may even be more than one cross-over point, in which case the range of interest rates for which project *A* is more profitable than project *B* is more complicated.

The following graph is similar to the graph in Section 1.2, although it has been extended to show the yields for the two projects.



We now give a final example for this section.

Example

An investor is considering whether to invest in either or both of the following loans:

Loan X: **For a purchase price of £10,000, the investor will receive £1,000 per annum payable quarterly in arrears for 15 years.**

Loan Y: **For a purchase price of £11,000, the investor will receive an income of £605 per annum, payable annually in arrears for 18 years, and a return of his outlay at the end of this period.**

The investor may lend or borrow money at 4% per annum. Would you advise the investor to invest in either loan, and, if so, which would be the more profitable?

Solution

We first consider loan *X*:

$$NPV_X(i) = -10,000 + 1,000a_{\overline{15|}}^{(4)}$$

and the yield is found by solving the equation $NPV_X(i) = 0$, or $a_{\overline{15|}}^{(4)} = 10$, which gives $i_X \approx 5.88\%$.

This value for i_X can be found using trial and error, and is easily checked by calculating $a_{\overline{15|}}^{(4)}$ using an interest rate of 5.88% *pa* effective:

$$a_{\overline{15|}}^{(4)} = \frac{1-v^{15}}{i^{(4)}} = \frac{1-1.0588^{-15}}{4(1.0588^{1/4}-1)} = 10.00$$

For loan *Y* we have:

$$NPV_Y(i) = -11{,}000 + 605a_{\overline{18}|} + 11{,}000v^{18}$$

and the yield (*ie* the solution of $NPV_Y(i) = 0$) is $i_Y = 5.5\%$.

The equation $NPV_Y(i) = 0$ can be rearranged and solved for i as follows:

$$-11{,}000 + 605a_{\overline{18}|} + 11{,}000v^{18} = 0$$

$$\Rightarrow \quad 605a_{\overline{18}|} = 11{,}000(1 - v^{18})$$

$$\Rightarrow \quad 605\left(\frac{1 - v^{18}}{i}\right) = 11{,}000(1 - v^{18})$$

$$\Rightarrow \quad i = \frac{605}{11{,}000} = 5.5\%$$

The rate of interest at which the investor may lend or borrow money is 4% per annum, which is less than both i_X and i_Y, so we compare $NPV_X(0.04)$ and $NPV_Y(0.04)$.

Now $NPV_X(0.04) = £1{,}284$ and $NPV_Y(0.04) = £2{,}089$, so it follows that, although the yield on loan *Y* is less than on loan *X*, the investor will make a larger profit from loan *Y*. We should, therefore, advise him that an investment in either loan would be profitable, but that, if only one of them is to be chosen, then loan *Y* will give the higher profit.

The above example illustrates the fact that the choice of investment depends very much on the rate of interest i_1 at which the investor may lend or borrow money. If this rate of interest were 5¾%, say, then loan *Y* would produce a loss to the investor, while loan *X* would give a profit.

2 Different interest rates for lending and borrowing

We have assumed so far that the investor may borrow or lend money at the same rate of interest i_1. In practice, however, the investor will probably have to pay a higher rate of interest (j_1, say) on borrowings than the rate (j_2, say) he receives on investments.

This is because banks make profits by borrowing money from savers at one rate of interest and lending it out for mortgages, business loans, *etc* at a higher rate.

The difference $j_1 - j_2$ between these rates of interest depends on various factors, including the credit-worthiness of the investor and the expense of raising a loan.

The concepts of net present value and yield are in general no longer meaningful in these circumstances. We must calculate the accumulation of net cashflows from first principles, the rate of interest depending on whether the investor's account is in credit. In many practical problems the balance in the investor's account (*ie* the accumulation of net cashflows) will be negative until a certain time t_1 and positive afterwards, except perhaps when the project ends.

In some cases the investor must finance his investment or business project by means of a fixed-term loan without an early repayment option. In these circumstances the investor cannot use a positive cashflow to repay the loan gradually, but must accumulate this money at the rate of interest applicable on lending, *ie* j_2.

Question

A company must choose between Project C and Project D, both of which would be financed by a loan, repayable only at the end of the project. The company must pay interest at a rate of 6.25% *pa* effective on money borrowed, but can only earn interest at a rate of 4% *pa* effective on money invested in its deposit account.

The cashflows for Project C, which has a term of 5 years, are:

Outgo		Income	
£100,000	(start of year 1)	£140,000	(end of year 5)

The cashflows for Project D, which has a term of 3 years, are:

Outgo		Income	
£80,000	(start of year 1)	£10,000	(end of year 1)
£20,000	(start of year 2)	£30,000	(end of year 2)
£5,000	(start of year 3)	£87,000	(end of year 3)

Calculate the accumulated profit at the end of 5 years for each project.

Solution

Project C

Since Project C does not generate any income, the company will be relying on the loan throughout the 5-year term. So only the borrowing rate of 6.25% will be relevant here.

The accumulated profit at time 5 years will be:

$$140{,}000 - 100{,}000 \times 1.0625^5 = £4{,}592$$

Project D

Since Project D does generate income during the project's term, we need to consider the company's net assets at the end of each year to see whether there are any excess funds available to invest.

At the end of year 1, there is an income payment of £10,000. However, at that time the company needs to pay interest of $80{,}000 \times 0.0625 = £5{,}000$ on the initial loan and it also has further outgo of £20,000 at the start of year 2. The net *outgo* at this time is therefore:

$$5{,}000 + 20{,}000 - 10{,}000 = £15{,}000$$

So further borrowing of £15,000 is required. This takes the total borrowing to:

$$80{,}000 + 15{,}000 = £95{,}000$$

At the end of year 2, there is an income payment of £30,000. However, at that time the company needs to pay interest of $95{,}000 \times 0.0625 = £5{,}937.5$ on its borrowing and it also has further outgo of £5,000 at the start of year 3. The net *income* at this time is therefore:

$$30{,}000 - 5{,}937.5 - 5{,}000 = £19{,}062.5$$

Since the loan can only be repaid at the end of the project, this money cannot be used to reduce the loan outstanding. Instead, the company has £19,062.5 available for investment.

So, just after the outgo of £5,000 at the start of year 3, the company has:

- loans totalling £95,000
- investments of £19,062.5.

At the end of the project (*ie* time 3 years), the company receives income of £87,000. The money invested at the start of year 3 has grown to:

$$19{,}062.5 \times 1.04 = £19{,}825$$

So the total amount at the company's disposal is $87{,}000 + 19{,}825 = £106{,}825$. From this, the company must repay the loan of £95,000 plus the interest accrued on this loan over the year. This leaves an amount at time 3 years of:

$$106{,}825 - 95{,}000 \times 1.0625 = £5{,}887.5$$

To obtain the accumulated profit at time 5, we take this money at time 3 and accumulate it for 2 years to time 5 (using the investment rate of interest of 4%):

$$5,887.5 \times 1.04^2 = £6,368$$

We now also consider the accumulated profit from Project C and Project D assuming that it is possible to use spare funds to repay part of the loan at any time.

Project C

The accumulated profit for Project C is unchanged, since there is no opportunity for repaying the loan early, as no income is received before the end of the project.

Project D

The calculation for the first year is unchanged, since there are no excess funds in the first year.

However, at the end of year 2, the net income of £19,062.5 could be used to repay part of the loan. This reduces the loan outstanding to $95,000 - 19,062.5 = £75,937.5$.

At the end of year 3, income of £87,000 is received, which can be used to repay the loan of £75,937.5 plus the interest accrued on this loan over the year. This leaves an amount at time 3 years of:

$$87,000 - 75,937.5 \times 1.0625 = £6,316.41$$

The accumulated profit at the end of year 5 is then:

$$6,316.41 \times 1.04^2 = £6,832$$

We see that the accumulated profit for Project D is greater in the case where the loan can be repaid early. This is because the cost of interest payments will be reduced, and the excess funds can be invested for a longer period.

2.1 Payback periods

Another quantity that is useful to calculate when an investment project is financed by outside borrowing is the discounted payback period (DPP). The DPP tells us how long it takes for the project to move into a position of profit.

In many practical problems the net cashflow changes sign only once, this change being from negative to positive. In these circumstances the balance in the investor's account will change from negative to positive at a unique time t_1, or it will always be negative, in which case the project is not viable. If this time t_1 exists, it is referred to as the 'discounted payback period' (DPP). It is the smallest value of *t* such that $A(t) \geq 0$, where:

$$A(t) = \sum_{s \leq t} c_s (1+j_1)^{t-s} + \int_0^t \rho(s)(1+j_1)^{t-s}\, ds \qquad (2.1)$$

Note that t_1 does not depend on j_2 but only on j_1, the rate of interest applicable to the investor's borrowings. Suppose that the project ends at time *T*. If $A(T) < 0$ (or, equivalently, if $NPV(j_1) < 0$) the project has no discounted payback period and is not profitable. If the project is viable (*ie* there is a discounted payback period t_1) the *accumulated profit* when the project ends at time *T* is:

$$P = A(t_1)(1+j_2)^{T-t_1} + \sum_{t>t_1} c_t(1+j_2)^{T-t} + \int_{t_1}^{T} \rho(t)(1+j_2)^{T-t}\,dt$$

This follows since the net cashflow is accumulated at rate j_2 after the discounted payback period has elapsed.

Other things being equal, a project with a shorter discounted payback period is preferable to a project with a longer discounted payback period because it will start producing profits earlier.

Question

Derive a formula for the accumulated value of an investment project that has cashflows at times $t_1, t_2, \dots, t_{10}$ (where $t_1 < t_2 < \cdots < t_{10}$), given that the first 3 cashflows are negative while the remainder are positive, and that the discounted payback period is t_7. Assume that the project is financed by borrowing at rate j that can be repaid at any time, and that excess funds can be invested at rate i ($i < j$).

Solution

Up to time t_7 (*ie* during the DPP), the project has to be funded by borrowing (at rate j).

So the accumulated value at this time is:

$$AV(t_7) = \sum_{k=1}^{7} C_{t_k}(1+j)^{t_7-t_k}$$

Thereafter, there are excess funds that can be invested at rate i up to time t_{10}.

So the accumulated value at the end of the project is:

$$AV(t_{10}) = AV(t_7)(1+i)^{t_{10}-t_7} + \sum_{k=8}^{10} C_{t_k}(1+i)^{t_{10}-t_k}$$

If interest is ignored in formula (2.1) (*ie* if we put $j_1 = 0$), the resulting period is called the 'payback period'. However, its use instead of the discounted payback period often leads to erroneous results and is therefore not to be recommended.

So, the payback period is the earliest time at which the total value of the income received to date is greater than or equal to the total value of the outgo to date. Since interest is ignored in the calculation of the payback period, it takes no account of the time value of money.

Question

The business plan for a new company that has obtained a 5-year lease for operating a local bus service is shown in the table below. Items marked with an asterisk represent continuous cashflows.

Cashflow item	*Timing*	*Amount (£000)*
Initial set up costs	Immediate	–250
Fees from advertising contracts	1 month	+200
Purchase of vehicles	3 months	–2,000
Fares from passengers*	From 3 months onwards	+1,000 *pa*
Staff costs & other operating costs*	From 3 months onwards	– 400 *pa*
Resale value of assets	5 years	+500

Determine the discounted payback period for this project assuming that it will be financed by a loan based on an effective annual interest rate of 10%, and that the loan can be repaid at any time.

Solution

If the DPP exists for this project, it will be at some point after 3 months, as the income from fares is needed to recoup the initial expenditure.

Remembering that the starred cashflows are continuous, the accumulated value of cashflows up to time t (where $3 \text{ months} < t < 5 \text{ years}$) is:

$$AV(t) = -250\times 1.10^{t} + 200\times 1.10^{t-1/12} - 2{,}000\times 1.10^{t-3/12} + (1{,}000-400)\bar{s}_{\overline{t-3/12}|}$$

$$= (-250\times 1.10^{3/12} + 200\times 1.10^{2/12} - 2{,}000)\times 1.10^{t-3/12} + 600\bar{s}_{\overline{t-3/12}|}$$

$$= -2{,}052.83\times 1.10^{t-3/12} + 600\bar{s}_{\overline{t-3/12}|}$$

To determine the DPP, we need to find the value of t for which the accumulated value of the cashflows up to that time is 0. This occurs when:

$$2{,}052.83\times 1.10^{t-3/12} = 600\bar{s}_{\overline{t-3/12}|}$$

Since $\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|}$, we can divide through by $1.10^{t-3/12}$ and rearrange to obtain:

$$\bar{a}_{\overline{t-3/12}|} = \frac{2{,}052.83}{600} = 3.42138$$

Now:

$$\frac{1-1.10^{-(t-3/12)}}{\ln 1.10}=3.42138 \quad \Rightarrow \quad 1.10^{-(t-3/12)}=0.67391$$

Taking logs and solving, we find:

$$-(t-3/12)\ln 1.10=\ln 0.67391 \quad \Rightarrow \quad t=-\frac{\ln 0.67391}{\ln 1.10}+\frac{3}{12}=4.39$$

So the discounted payback period is 4.39 years.

Instead of working with accumulated values, we could work with present values. The DPP is the time t when $NPV_{\text{up to } t}=0$. The equation to solve is:

$$NPV=-250+200v^{1/12}-2{,}000v^{3/12}+600v^{3/12}\bar{a}_{\overline{t-3/12|}}=0$$

This gives the same answer as above.

The discounted payback period is often employed when considering a single investment of *C*, say, in return for a series of payments each of *R*, say, payable annually in arrears for *n* years. The discounted payback period t_1 years is clearly the smallest integer *t* such that $A^*(t)\geq 0$, where:

$$A^*(t)=-C(1+j_1)^t+Rs_{\overline{t|}} \qquad \text{at rate } j_1$$

***ie* the smallest integer *t* such that:**

$$Ra_{\overline{t|}}\geq C \qquad \text{at rate } j_1$$

The project is therefore viable if $t_1\leq n$, in which case the accumulated profit after *n* years is clearly:

$$P=A^*(t_1)(1+j_2)^{n-t_1}+Rs_{\overline{n-t_1|}} \quad \text{at rate } j_2$$

Question

A speculator borrows £50,000 at an effective interest rate of 8% per annum to finance a project that is expected to generate £7,500 at the end of each year for the next 15 years.

Calculate the discounted payback period for this investment.

Solution

The accumulated profit at the end of year t will be:

$$AP(t)=-50{,}000(1+i)^t+7{,}500s_{\overline{t|}}$$

We need to find the first time t such that this accumulated profit is greater than or equal to 0:

$$-50{,}000(1+i)^t + 7{,}500 s_{\overline{t|}} \geq 0$$

Since $s_{\overline{n|}} = (1+i)^n a_{\overline{n|}}$, we can simplify this by dividing through by $7{,}500(1+i)^t$:

$$a_{\overline{t|}} \geq \frac{50{,}000}{7{,}500} = 6.6666$$

Looking at the *Tables* (at 8%), we see that $a_{\overline{9|}} = 6.2469$ and $a_{\overline{10|}} = 6.7101$. So the discounted payback period is 10 years.

Instead of working with accumulated values, we could work with present values, in which case we want the first time t when $NPV_{\text{up to } t} \geq 0$. The equation to solve is:

$$-50{,}000 + 7{,}500 a_{\overline{t|}} \geq 0$$

This gives the same answer as above.

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3 Other considerations

At the simplest level, for projects involving similar amounts of money and with similar time horizons, the project that results in the highest accumulated profit will be the most favourable.

Where the project can be funded without the need for external borrowing, this is equivalent to selecting the project with the highest net present value. The internal rate of return will provide a useful secondary criterion.

Where external borrowing is involved, the accumulated profit must be calculated directly by looking at the cashflows and taking into account the precise conditions of the loan. The discounted payback period will provide a useful secondary criterion.

However, it may not be a straightforward decision for the owners of a business to decide between alternative investment projects purely on the basis of net present values, internal rates of return and discounted payback periods. In many cases a comparison of net present values or internal rates of return for alternative projects will not lead to a decisive conclusion, since the values may be very close or they may conflict. So other considerations will have to be brought into the decision.

Other factors for a company to consider when deciding between different projects include:

Cashflows

- Are the cashflow requirements for the project consistent with the business's other needs?
- Over what period will the profits be produced and how will the profits be used?
- Is it worth carrying out the project if the potential profit is very small in money terms?

Borrowing requirements

- Can the business raise the necessary cash at the times required?
- What rate of interest will the business have to pay on borrowed funds?
- Are time limits or other restrictions imposed on borrowing?

Resources

- Are the other resources required for the project available?
- Does the business have the necessary staff, technical expertise and equipment?

Risk

- What are the financial risks involved in going ahead with the project (and in doing nothing)?
- How certain is the business about the appropriate risk discount rate to use?
- Is it possible that the project might make an unacceptably large loss?
- Can suppliers be relied on to fulfil their contracts according to the agreed timetable and budget?

Investment conditions

- What is the economic climate?
- Are interest rates likely to rise or fall?

Indirect benefits

- Will the project bring any additional benefits?
- Will the equipment purchased and the skills developed be of value to the business in the future?

The following question illustrates these points.

Question

A company is considering investing in two projects: Project C and Project E.

The cashflows for Project C, which has a term of 5 years, are:

- Initial outgo: £100,000
- Income (at the end of year 5): £140,000

The cashflows for Project E, which has a term of 3 years, are:

- Initial outgo: £100,000
- Income (at the end of each of the next 3 years): £38,850

(i) For each of the projects, calculate:

(a) the internal rate of return

(b) the range of interest rates at which money can be borrowed in order for the projects to be viable

(c) the accumulated profit at the end of 5 years, assuming that the projects are financed by a loan subject to interest at 6.25% *pa* effective.

(ii) Outline other considerations that might be taken into account when deciding between Project C and Project E.

Solution

(i)(a) The internal rate of return for Project C, i_C, is the interest rate that solves the equation:

$$100,000 = 140,000(1+i_C)^{-5} \quad \Rightarrow \quad i_C = \left(\frac{140,000}{100,000}\right)^{1/5} - 1 = 7.0\% \text{ (to 1 dp)}$$

The internal rate of return for Project E, i_E, is the interest rate that solves the equation:

$$100{,}000 = 38{,}850 a_{\overline{3}|} \quad \Rightarrow \quad a_{\overline{3}|} = 2.574$$

Using trial and error, we find that $i_E = 8.1\%$ (to 1 dp).

(i)(b) If the borrowing rate is less than 7%, both projects will be profitable.

If the borrowing rate is between 7% and 8.1%, Project E will be profitable, but Project C will not.

If the borrowing rate exceeds 8.1%, neither project will be profitable.

(i)(c) The accumulated value of the profits at the end of 5 years, using a rate of interest of 6.25%, are:

Project C: $140{,}000 - 100{,}000 \times 1.0625^5 = £4{,}592$

Project E: $38{,}850 \times 1.0625^2 s_{\overline{3}|} - 100{,}000 \times 1.0625^5 = £4{,}561$

(ii) Other considerations include:

1. Although $i_E > i_C$, the internal rates of return are quite close. So the decision based on this criterion is not clear cut.

2. The higher yield available under Project E applies for only 3 years, whereas Project C will produce profits over a 5-year period. If the company invests in Project C, it will receive a yield of 7.0% for the full 5 years. So, if interest rates turn out to be lower in years 4 and 5, Project C may prove to be a better investment.

3. Project C may not be desirable if it will leave the company short of cash during the next 5 years, until the income is received at the end of the project.

4. If the company is not able to borrow more than £100,000 and is required to make regular repayments on money borrowed, then Project C will not be viable, since it does not produce any income with which to pay the interest.

Chapter 11 Summary

The profitability of an investment project can be assessed by calculating the accumulated profit:

$$\textit{Accumulated profit} = \textit{AV income} - \textit{AV outgo}$$

The net present value (NPV) of an investment project is the present value of the net cashflows, calculated at the risk discount rate.

$$\textit{Net present value} = \textit{PV income} - \textit{PV outgo} \qquad (\textit{@ risk discount rate})$$

The internal rate of return for an investment project is the effective rate of interest that equates the present value of income and outgo, *ie* it makes the net present value of the cashflows equal to zero.

The discounted payback period for an investment project is the earliest time after the start of the project when the accumulated value of the past cashflows (positive and negative), calculated using the borrowing rate, becomes greater or equal to zero.

$$\textit{Acc. profit at discounted payback period} \geq 0 \quad (\textit{@ borrowing rate})$$

The payback period is similar to the discounted payback period, but it ignores any interest paid. It is the earliest time that the total value of the past cashflows (positive and negative) becomes greater than or equal to zero.

There are several other factors that must be taken into account when comparing alternative investment projects. These include the cashflow requirements of the business, the borrowing requirements, other resources required, risks involved, investment conditions and indirect benefits.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 11 Practice Questions

11.1 An investor borrows money at an effective rate of interest of 10% *pa* to invest in a 6-year project. The cashflows for the project are:

- an initial outlay of £25,000
- regular income of £10,000 *pa* during the first 5 years (assumed to be received continuously)
- regular expenditure of £2,000 *pa* during the first 5 years (assumed to be payable continuously)
- a decommissioning expense of £5,000 at the end of the 6th year.

Calculate:

(i) the net present value of the project's cashflows

(ii) the discounted payback period

(iii) the internal rate of return.

11.2 An investor borrows £1,000 by taking out an interest-only loan at an effective rate of interest of 6% *pa*, and invests the money in a project. The loan is to be repaid in full after 2 years (with no early repayment option) and interest on the money borrowed is paid at the end of each month. The project will provide income of £50 at the end of each month for 24 months and the investor can invest spare funds at an interest rate of 5% *pa* effective. Calculate the accumulated profit at the end of 2 years.

11.3 (i) Define:

Exam style

(a) the discounted payback period

(b) the payback period

of an investment project. [2]

(ii) Describe the disadvantages of using these two measures for determining whether to proceed with an investment project. [3]

[Total 5]

11.4 A property development company has just purchased a retail outlet for \$4,000,000. A further \$900,000 will be spent refurbishing the outlet in six months' time.

Exam style

An agreement has been made with a prospective tenant who will occupy the outlet beginning one year after the purchase date. The tenant will pay rent to the owner for five years and will then immediately purchase the outlet from the property development company for \$6,800,000. The initial rent will be \$360,000 per annum and this will be increased by the same percentage compound rate at the beginning of each successive year. The rental income is received quarterly in advance.

Calculate the compound percentage increase in the annual rent required to earn the company an internal rate of return of 12% per annum effective. [9]

11.5 An insurance company borrows £50 million at an effective interest rate of 9% per annum. The insurance company uses the money to invest in a capital project that pays £6 million per annum payable half-yearly in arrears for 20 years. The income from the project is used to repay the loan. Once the loan has been repaid, the insurance company can earn interest at an effective interest rate of 7% per annum.

Exam style

(i) Calculate the discounted payback period for this investment. [4]

(ii) Calculate the accumulated profit the insurance company will have made at the end of the term of the capital project. [5]

[Total 9]

11.6 A company is considering investing in the following project. The company has to make an initial investment of three payments, each of £85,000. The first is due at the start of the project, the second one year later, and the third payment is due two years after the start of the project.

Exam style

After 10 years it is assumed that a major refurbishment of the infrastructure will be required, costing £125,000.

The project is expected to provide no income in the first two years, an income received continuously of £30,000 in the third year, £32,000 in the fourth year, £34,000 in the fifth year and £36,000 in the sixth year. Thereafter the income is expected to increase by 2% per annum (compound) at the start of each year.

The income is expected to cease at the end of the 20th year from the start of the project.

The cashflow within each year is assumed to be received at a constant rate.

(i) Calculate the net present value of the project at a rate of interest of 7% *pa* effective. [6]

(ii) Show that the discounted payback period does not fall within the first 10 years, assuming an effective rate of interest of 7% *pa*. [5]

(iii) Calculate the discounted payback period for the project, assuming an effective rate of interest of 7% *pa*. [5]

[Total 16]

Chapter 11 Solutions

11.1 (i) ***Net present value***

The net present value of the project's cashflows, evaluated using an interest rate of 10% *pa* effective, is:

$$\begin{aligned}NPV &= -25{,}000 + 10{,}000\bar{a}_{\overline{5|}} - 2{,}000\bar{a}_{\overline{5|}} - 5{,}000v^6\\ &= -25{,}000 + 8{,}000\bar{a}_{\overline{5|}} - 5{,}000v^6\\ &= -25{,}000 + 8{,}000 \times 3.97732 - 5{,}000 \times 1.1^{-6}\\ &= £3{,}996.16\end{aligned}$$

(ii) ***Discounted payback period***

To calculate the DPP, we need to find the point in time at which the net present value of the project's cashflows (calculated at the borrowing rate of 10%) equals zero, *ie* we need to find the value of t for which:

$$-25{,}000 + 8{,}000\bar{a}_{\overline{t|}} = 0 \quad @10\%$$

Note that since the NPV of ***all*** *the project's cashflows (as calculated in (i)) exceeds 0, the DPP must fall at some point* ***before*** *the end of the project, so we can exclude the decommissioning expense from the above equation.*

This can be simplified to:

$$\bar{a}_{\overline{t|}}^{@10\%} = \frac{25{,}000}{8{,}000} = 3.125$$

Solving, we find:

$$\frac{1-1.1^{-t}}{\ln 1.1} = 3.125 \quad \Rightarrow \quad 1.1^{-t} = 0.70216 \quad \Rightarrow \quad -t\ln 1.1 = \ln 0.70216 \quad \Rightarrow \quad t = 3.71 \text{ years}$$

So the DPP is 3.71 years.

(iii) ***Internal rate of return***

The internal rate of return is the interest rate at which the net present value equals 0, *ie*:

$$NPV = -25{,}000 + 8{,}000\bar{a}_{\overline{5|}} - 5{,}000v^6 = 0 \quad @IRR$$

Using an interest rate of 10% (in part (i)), gives a positive NPV. To reduce the NPV, we need to increase the interest rate:

At 15%: $NPV = -25{,}000 + 8{,}000 \times 3.59771 - 5{,}000 \times 1.15^{-6} = 1{,}620.06$

At 18%: $NPV = -25,000 + 8,000 \times 3.40086 - 5,000 \times 1.18^{-6} = 354.69$

At 19%: $NPV = -25,000 + 8,000 \times 3.33969 - 5,000 \times 1.19^{-6} = -43.17$

Linearly interpolating between the values at 18% and 19% gives:

$$IRR \approx 18\% + \left(\frac{0 - 354.69}{-43.17 - 354.69} \right)(19\% - 18\%) = 18.9\%$$

11.2 The amount of each monthly interest payment required under the loan is:

$$1,000(1.06^{1/12} - 1) = £4.87$$

So the investor's net monthly income is:

$$50 - 4.87 = £45.13$$

The accumulated profit at the end of 2 years (when the £1,000 borrowed must be repaid) will be:

$$AV = 12 \times 45.13 s_{\overline{2}|}^{(12)} @5\% - 1,000 = 12 \times 45.13 \times 2.0966 - 1,000 = £135$$

11.3 (i) ***Definitions***

(a) The discounted payback period for a project is the smallest time t for which the present (or accumulated) value of the income up to time t exceeds the present (or accumulated) value of the outgo up to time t. [1]

(b) The payback period is the same as the discounted payback period, except that the present value calculation (or accumulation) is carried out using an interest rate of 0%. In other words, it is the earliest time for which the monetary value of the income exceeds the monetary value of the outgo. [1]

[Total 2]

(ii) ***Disadvantages***

Neither the DPP nor the PP give any indication of how profitable a project is, as they ignore cashflows after the accumulated value of zero is reached. [1]

The PP can give misleading results as it does not take into account the time value of money. [1]

There may not be one unique time when the balance in the investor's account changes from negative to positive, so there may not be a unique DPP or PP. [1]

[Total 3]

11.4 *This question is Subject CT1, April 2015, Question 9.*

Working in \$millions, the total present value of the rental payments is:

$$PV = 0.36v\,\ddot{a}^{(4)}_{\overline{1}|}\left[1+(1+b)v+\cdots+(1+b)^4 v^4\right] \qquad [1]$$

where b is the compound increase in the rental rate.

Using the formula for the sum of five terms of a geometric progression with first term 1 and common ratio $(1+b)v$, we have a total present value of the rental payments of:

$$PV = 0.36v\,\ddot{a}^{(4)}_{\overline{1}|}\left[\frac{1-\left[(1+b)v\right]^5}{1-(1+b)v}\right] \qquad [1]$$

The net present value of all the project's cashflows must equal 0 at the internal rate of return of 12%. So:

$$0 = -4 - 0.9v^{\frac{1}{2}} + 0.36v\,\ddot{a}^{(4)}_{\overline{1}|}\left[\frac{1-\left[(1+b)v\right]^5}{1-(1+b)v}\right] + 6.8v^6 \qquad [1]$$

Evaluating the compound interest functions at 12%:

$$v^{\frac{1}{2}} = 0.944911$$

$$v^6 = 0.506631$$

$$d^{(4)} = 4\left[1-1.12^{-\frac{1}{4}}\right] = 0.111738$$

$$\ddot{a}^{(4)}_{\overline{1}|} = \frac{1-v}{d^{(4)}} = \frac{1-1.12^{-1}}{0.111738} = 0.958873$$

So the equation of value becomes:

$$0 = -4 - 0.9\times 0.944911 + 0.36\times\frac{1}{1.12}\times 0.958873\times\frac{1-X^5}{1-X} + 6.8\times 0.506631$$

where $X = \dfrac{1+b}{1.12}$.

Simplifying this, we obtain:

$$0 = -4 - 0.850420 + 0.308209\times\frac{1-X^5}{1-X} + 3.445092$$

Rearranging this equation, we find that:

$$\frac{1-X^5}{1-X}=4.5597$$ [2]

We now solve this by trial and error. Choosing an initial starting value of $X=0.9$, we obtain the following values:

X	$\frac{1-X^5}{1-X}$
0.9	4.0951
0.92	4.2615
0.94	4.4349
0.96	4.6157
0.953	4.5516
0.954	4.5607

[2]

Interpolating, we obtain:

$$X \approxeq 0.953+\frac{4.5597-4.5516}{4.5607-4.5516}\times 0.001=0.95389$$ [1]

Finally this gives us a compound growth rate of:

$$b=1.12\times 0.95389-1=0.06835$$

The required compound growth rate is 6.84%. [1]

[Total 9]

11.5 *This question is Subject CT1, April 2014, Question 8.*

(i) ***Discounted payback period***

The discounted payback period is the point in time when the balance on our account first turns positive. So, we require the smallest value of n for which:

$$NPV=-50+6a_{\overline{n}|}^{(2)}>0 \quad \Rightarrow \quad a_{\overline{n}|}^{(2)}>8.333$$ [1]

Solving for n we find:

$$\frac{1-1.09^{-n}}{2(1.09^{½}-1)}>8.333 \quad \Rightarrow \quad 0.2662>1.09^{-n} \quad \Rightarrow \quad \ln 0.2662>-n\ln 1.09 \quad \Rightarrow \quad n>15.36$$ [2]

Since the payments are half-yearly, the DPP is 15.5 years. [1]

[Total 4]

(ii) ***Accumulated profit***

Working in £millions, the NPV of the cashflows occurring up to and including time 15.5 years is:

$$-50+6a^{(2)}_{\overline{15.5}|}\ @9\% = -50+6\times\frac{1-v^{15.5}}{i^{(2)}} = -50+6\times 8.369627 = 0.217762 \qquad [2]$$

So the accumulated value at time 15.5 years of these cashflows is:

$$0.217762\times 1.09^{15.5} = 0.828119 \qquad [1]$$

So the total accumulated profit at time 20 years will be:

$$0.828119\times 1.07^{4.5} + 6s^{(2)}_{\overline{4.5}|}\ @7\% = 1.122845+6\times 5.171726 = 32.153200 \qquad [2]$$

The accumulated profit at the end of the project is £32.153 million. [Total 5]

11.6 (i) ***Net present value***

The PV of the outgo (in £000s) is:

$$PV\ outgo = 85(1+v+v^2)+125v^{10}\ \ @7\% = 302.225 \qquad [1]$$

The PV of the income (in £000s) is:

$$PV\ income = 30v^2\bar{a}_{\overline{1}|}+32v^3\bar{a}_{\overline{1}|}+34v^4\bar{a}_{\overline{1}|}+36v^5\bar{a}_{\overline{1}|}$$
$$+36\times 1.02v^6\bar{a}_{\overline{1}|}+36\times 1.02^2v^7\bar{a}_{\overline{1}|}+\cdots+36\times 1.02^{14}v^{19}\bar{a}_{\overline{1}|}$$

This can be simplified to:

$$PV\ income = (30v^2+32v^3+34v^4)\bar{a}_{\overline{1}|}+36v^5(1+1.02v+1.02^2v^2+\cdots+1.02^{14}v^{14})\bar{a}_{\overline{1}|} \qquad [2]$$

The first part can be calculated directly and the second part can be summed as a geometric series of 15 terms with first term 1 and common ratio $1.02v$, to give:

$$PV\ income = 78.263\times 0.96692+36\times 1.07^{-5}\times\frac{1-1.02^{15}v^{15}}{1-1.02v}\times 0.96692 = 347.709 \qquad [2]$$

So the NPV is $347,709-302,225 = £45,484$. [1]

[Total 6]

(ii) ***Show the discounted payback period does not fall within the first 10 years***

The PV of the income (in £000s) received up to time 10 years can be calculated similarly as:

$$PV\ income\ (10\,yrs) = 78.263\times 0.96692+36\times 1.07^{-5}\times\frac{1-1.02^5v^5}{1-1.02v}\times 0.96692$$
$$= 188.698 \qquad [3]$$

The PV of the outgo (in £000s) excluding the refurbishment payment at time 10 years is:

$$PV\ outgo\ (10\ yrs) = 85(1+v+v^2) = 238.682 \qquad [1]$$

As the PV of the income is less than the PV of the outgo up to the end of the 10th year, and the initial outgo (*ie* the 3 payments of £85,000) precedes the start of the income, the accumulated value of the project's cashflows is negative throughout the first 10 years. So the DPP does not fall within the first 10 years. [1]

[Total 5]

(iii) ***Discounted payback period***

We first establish in which year the DPP falls, by considering complete years. Equating the PV of the income for the first n complete years of the project ($n>10$) to the PV of the outgo gives the following equation:

$$78.263 \times 0.96692 + 36 \times 1.07^{-5} \times \frac{1-1.02^{n-5}v^{n-5}}{1-1.02v} \times 0.96692 = 302.225$$

$$\Rightarrow \quad 75.674 + 24.818 \times \frac{1-1.02^{n-5}v^{n-5}}{1-1.02v} = 302.225 \qquad [1]$$

Note that this equation will only be exact if n is an integer.

Solving this, we find that:

$$(1.02v)^{n-5} = 0.57344 \quad \Rightarrow \quad (n-5)\ln(1.02v) = \ln 0.57344 \quad \Rightarrow \quad n = 16.6 \qquad [1]$$

So, the DPP ends somewhere between time 16 and time 17. To find the exact value of the DPP, we need to value the payments in the final part year (of length t , say) exactly, which gives the equation:

$$75.674 + 24.818 \times \frac{1-1.02^{11}v^{11}}{1-1.02v} + 36 \times 1.02^{11} v^{16} \bar{a}_{\overline{t|}} = 302.225 \qquad [1]$$

Solving this, we find:

$$\bar{a}_{\overline{t|}} = \frac{1-1.07^{-t}}{\ln 1.07} = 0.60514 \quad \Rightarrow \quad 1.07^{-t} = 0.95906$$

$$\Rightarrow \quad -t\ln 1.07 = \ln 0.95906 \quad \Rightarrow \quad t = 0.618 \qquad [1]$$

So the discounted payback period is 16.618 years. [1]

[Total 5]

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Bonds, equity and property

Syllabus objectives

3.2 Use the concept of equation of value to solve various practical problems.

- 3.2.2 Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.
- 3.2.3 Calculate the running yield and the redemption yield for the financial instrument as described in 3.2.2.
- 3.2.4 Calculate the upper and lower bounds for the present value of the financial instrument as described in 3.2.2, when the redemption date can be a single date within a given range at the option of the borrower.
- 3.2.5 Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

This chapter also deals with real rates of interest as required in syllabus objective 2.2.

0 Introduction

In this chapter, we show how to calculate the price to be paid for, or the yield obtained from, the following investments:

- fixed-interest bonds
- index-linked bonds
- equities (*ie* shares)
- property.

The cashflows relating to the first three of these investments were discussed in Chapter 2.

Investment in property involves outgo in the form of the purchase price of the building, any legal or other fees, and maintenance costs. Income is received in the form of rent. Investors in property also hope to make a gain from selling the property for more than its purchase price.

1 Fixed-interest securities

First, we will apply the theory we have developed to fixed-interest securities, *ie* investments that make regular interest payments at a fixed rate, followed by a capital repayment.

The interest payments on a bond are called coupons and the capital repayment is called the redemption payment. Prices for bonds are often calculated 'per £100 nominal'. The coupon and redemption payments are expressed as a percentage of the nominal amount and not as a percentage of the purchase price.

For example, if an investor owns £100 nominal of a fixed-interest bond, which pays coupons annually in arrears at a rate of 6% per annum and has a redemption rate of 102%, the investor will receive:

- £6 at the end of each year of the bond's term, and
- £102 at maturity.

If the redemption payment is equal to the nominal value, the bond is said to be 'redeemed at par', and an investor owning £100 nominal of the bond would receive £100 at maturity.

As in other compound interest problems, one of two questions may be asked:

(1) What price P per unit nominal, should be paid by an investor to secure a net yield of i per annum?

(2) Given that the investor pays a price P per unit nominal, what net yield per annum will be obtained?

1.1 Calculating the price and yield

The price, P, to be paid to achieve a yield of i per annum is equal to:

$$P = \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net interest payments} \end{pmatrix} + \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net capital payments} \end{pmatrix} \qquad \textbf{(1.1)}$$

In other words, the price paid is the present value of all the payments the investor receives net of any tax.

Question

A tax-exempt investor purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain an effective annual yield of 10%.

Solution

Based on owning £10,000 nominal of the bond, the coupons are $8\% \times 10{,}000 = £800$ per year. The coupons are payable half-yearly in arrears, so £400 is paid at the end of each half-year. Since the bond is newly issued, the first coupon is due in 6 months' time.

As the bond is redeemable at par, the investor receives a redemption payment of £10,000 at the end of the 5-year term.

The investor is tax-exempt and will therefore receive the full coupon and redemption payments. Hence the price the investor should pay for £10,000 nominal of this bond is:

$$P = 800a_{\overline{5}|}^{(2)} + 10{,}000v^5 \qquad @10\%$$

$$= 800 \times 3.8833 + 10{,}000 \times 1.1^{-5}$$

$$= £9{,}315.85$$

The yield available on a stock that can be bought at a given price, P, can be found by solving equation (1.1) for the net yield i.

If the investor is not subject to taxation, the yield i is referred to as a *gross* yield. The yield on a security is sometimes referred to as the *yield to redemption* or the *redemption yield* to distinguish it from the *flat* (or *running*) *yield*, which is defined as D/P, the ratio of the coupon rate to the price per unit nominal of the stock.

The letter D is used here to denote the coupon rate on the bond.

A redemption yield is the yield obtained by an investor who holds the bond until redemption whereas the running yield is the yield generated by the payment of the coupons alone (ignoring the redemption payment).

Question

Calculate the running yield for the investor in the previous question.

Solution

The investor pays £9,315.85 for £10,000 nominal of the bond and receives coupons of £800 each year. So the running yield is:

$$\frac{800}{9{,}315.85} = 8.59\%$$

Since the coupons are paid half-yearly, this running yield is convertible half-yearly.

A redemption yield that is calculated without making any allowance for tax is called a *gross redemption yield* (GRY). If tax is incorporated in the calculation, this gives the *net redemption yield* (NRY).

We can determine the gross redemption yield for a fixed-interest stock by trial and error and then interpolation.

Question

A tax-exempt investor pays £10,500 for £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the gross redemption yield obtained by the investor.

Solution

The gross redemption yield is the interest rate i that satisfies the equation of value:

$$10,500 = 800a_{\overline{5}|}^{(2)} + 10,000v^5$$

This is the bond referred to in earlier questions, and we've already seen that if the yield is 10% *pa* effective, the price is £9,315.85. Here the investor is paying more than £9,315.85, so the yield is lower than 10% *pa* effective. However, we can get a better first guess by considering the coupon rate and the redemption payment.

If the purchase price were £10,000, the redemption yield would be exactly 8% *pa* convertible half-yearly, because an investment of £10,000 would be receiving interest of £800 each year payable half-yearly, followed by a return of the initial investment. Since the investor pays more than £10,000, the redemption yield will be less than 8% *pa* convertible half-yearly. The effective redemption yield will be only slightly higher than the yield convertible half-yearly, so a good first guess is that the effective redemption yield will be lower than 8% *pa*.

At 7%: $RHS = 10,466.45$

At 6%: $RHS = 10,892.28$

We can approximate i by linearly interpolating using these two values:

$$i \approx 6\% + \frac{10,500 - 10,892.28}{10,466.45 - 10,892.28} \times (7\% - 6\%) = 6.9\%$$

So the GRY is approximately 6.9% *pa* effective.

1.2 No tax

Consider an n year fixed-interest security which pays coupons of D per annum, payable $p$thly in arrears and has redemption amount R.

The price of this bond, at an effective rate of interest i per annum, with no allowance for tax (*ie* i represents the gross yield) is:

$$P = Da_{\overline{n}|}^{(p)} + Rv^n \quad \text{at rate } i \text{ per annum} \tag{1.2}$$

We looked at an example of this earlier, as our tax-exempt investor received the full amount of the coupon and redemption payments.

Note: One could also work with a period of half a year. The corresponding equation of value would then be:

$$P = \frac{D}{2} a_{\overline{2n}|} + Rv^{2n} \quad \text{at rate } i' \text{ where } (1+i')^2 = 1+i$$

Question

A tax-exempt investor purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain an effective annual yield of 10%.

Solution

We have already calculated the answer to this question by working in years. The price worked out to be £9,315.85. We will confirm this by working in half-years, using a half-yearly effective interest rate.

First we need to calculate the half-yearly effective rate:

$$1.1^{1/2} - 1 = 4.880885\%$$

So the price to be paid is:

$$P = 400a_{\overline{10}|} + 10{,}000v^{10} \quad @4.880885\%$$

$$= 400 \times 7.7666 + 10{,}000 \times 1.04880885^{-10}$$

$$= £9{,}315.85$$

1.3 Income tax

Suppose an investor is liable to income tax at rate t_1 on the coupons, which is due at the time that the coupons are paid. The price, P', of this bond, at an effective rate of interest i per annum, where i now represents the net yield, is now:

$$P' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n \qquad \text{at rate } i \text{ per annum} \tag{1.3}$$

The coupon payments are income and hence are liable to income tax. Since investors only want to pay for the coupons they actually receive, they calculate the price based on the *net* coupons received after tax has been deducted.

The redemption payment is a return of the capital lent by the investor (*ie* purchaser of the bond) to the borrower (*ie* issuer of the bond) and so is not liable to income tax (even if the redemption proceeds are greater than the price originally paid).

Question

An investor liable to income tax at 25% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 10% *pa* effective.

Solution

Again, this is the same bond referred to in earlier questions. However, this time the investor is liable to tax on income.

Using equation (1.3) we have:

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10{,}000v^5 \qquad @10\%$$

$$= 600 \times 3.8833 + 10{,}000 \times 1.1^{-5}$$

$$= £8{,}539.19$$

We can see that this investor receives only £600 *pa* (=75% of the gross coupons) and so to obtain the same yield as the tax-exempt investor considered before, this investor must buy the stock at a lower price.

As well as calculating the price paid for a stock by an investor subject to income tax, we could be asked to calculate the net yield obtained by such an investor given the price paid.

Question

Suppose the investor in the previous question actually purchases the stock for £9,000.

Calculate the net redemption yield obtained on the investment.

Solution

We saw that the price was £8,539.19 when the net yield was 10% *pa*. If the price is £9,000, the net yield is lower than 10% *pa*.

The net redemption yield is the interest rate i that solves the equation of value:

$$9{,}000 = (1-0.25)800a_{\overline{5}|}^{(2)} + 10{,}000v^5$$

At 8%: $RHS = 9{,}248.45$

At 9%: $RHS = 8{,}884.48$

We can approximate i by linearly interpolating using these two values:

$$i \approx 8\% + \frac{9{,}000 - 9{,}248.45}{8{,}884.48 - 9{,}248.45} \times (9\% - 8\%) = 8.7\%$$

So the net redemption yield is approximately 8.7%.

It is possible in some countries that the tax is paid at some later date, for example at the calendar year end.

This does not cause any particular problems as we follow the usual procedure – identify the cashflow amounts and dates and set out the equation of value.

For example, suppose that income tax on the bond each year **is paid in a single** annual **instalment, due, say, k years after the second half-yearly coupon payment each year. Then the equation of value for a given net yield i and price (or value) P' is, immediately after a coupon payment,**

$$P' = Da_{\overline{n}|}^{(p)} + Rv^n - t_1Dv^k a_{\overline{n}|}$$

Other arrangements may be dealt with similarly from first principles.

Question

An investor purchases £100 nominal of a 10-year bond. The coupon rate is 6% *pa*, and coupons are payable half-yearly in arrears. The bond is redeemable at par. The investor pays 15% tax on income, and tax payments are due four months after each coupon is received.

Calculate the price paid by the investor to achieve a net redemption yield of 9% *pa* effective.

Solution

The price for £100 nominal, *P*, can be calculated from the equation:

$$P = 6a_{\overline{10}|}^{(2)} + 100v^{10} - 6 \times 0.15 \times v^{4/12} a_{\overline{10}|}^{(2)} \quad @9\%$$

$$= 6 \times 6.5589 + 100 \times 1.09^{-10} - 0.9 \times 1.09^{-4/12} \times 6.5589$$

$$= £75.86$$

1.4 Capital gains tax

If the price paid for a bond is less than the redemption (or sale price if sold earlier), then the investor has made a capital gain.

Capital gains tax is a tax levied on the capital gain. In contrast to income tax, this tax is normally payable once only in respect of each disposal, at the date of sale or redemption.

So capital gains tax is paid at redemption or at the time the bond is sold. As with income tax, the investor only wants to pay for the redemption payment actually received, and so will therefore calculate the price based on the *net* redemption payment received after tax has been deducted.

However, the price depends on whether there is a capital gain, and knowing whether there's a capital gain depends on the price. So we use a test to determine whether there is a capital gain *before* we calculate the price.

Capital gains test

Consider an n year fixed-interest security which pays coupons of D per annum, payable p thly in arrears and has redemption amount R. An investor, liable to income tax at rate t_1 (due at the same time the coupons are paid), **purchases the bond at price P'. If $R > P'$ then there is a capital gain and from (1.3), we have:**

$$R > (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n$$

$$\Rightarrow \quad R(1-v^n) > (1-t_1)D\frac{1-v^n}{i^{(p)}}$$

$$\Rightarrow \quad R > (1-t_1)\frac{D}{i^{(p)}}$$

$$\Rightarrow \quad i^{(p)} > (1-t_1)\frac{D}{R} \tag{1.4}$$

An intuitive way of thinking about this is that the overall return on a bond comes from both the coupons and any capital gain. If the return we require is greater than the net coupons we receive, then we must be getting more back than we paid, *ie* by having a capital gain.

If $i^{(p)} < (1-t_1)\frac{D}{R}$, then there is a capital loss, *ie* the investor pays more for the bond than is received back at redemption. If $i^{(p)} = (1-t_1)\frac{D}{R}$, there is neither a capital gain nor a capital loss, *ie* the investor pays the same for the bond as is received back at redemption.

Sometimes g is used in place of $\frac{D}{R}$.

Question

An investor, who is liable to income tax at 25%, purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Determine whether there is a capital gain if the investor requires a net yield of 10% *pa* effective.

Solution

The investor requires a net yield of 10% *pa* and coupons are paid half-yearly, so:

$$i^{(2)} @10\% = 2\left(1.1^{½} - 1\right) = 0.0976177$$

Income tax is 25%, and for £10,000 nominal we receive coupons of £800 each year and a redemption payment of £10,000. So:

$$(1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10,000} = 0.06$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$ we have a capital gain.

We looked at this bond in an earlier question, and found that an investor who is liable to income tax at 25% and requires a net yield of 10% *pa* effective would pay £8,539.19 for £10,000 nominal. Since the investor receives a redemption payment of £10,000, this confirms the result of the test – there is a capital gain.

If the investor were also liable to capital gains tax (as well as income tax), then the price would have to be less than £8,539.19 to receive the same 10% *pa* net return.

If the investor is also subject to tax at rate t_2 $(0 < t_2 < 1)$ on the capital gains, then let the price payable, for a given net yield i, be P''.

If $i^{(p)} > (1-t_1)\frac{D}{R}$ then there is a capital gain. At the redemption date of the loan there is therefore an additional liability of $t_2(R - P'')$.

Capital gains tax is paid *only* on the capital gain and not on the whole of the redemption amount.

In this case:

$$P'' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n \quad \text{at rate } i \text{ per annum} \qquad (1.5)$$

Question

An investor liable to income tax at 25% and capital gains tax at 20% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 10% *pa* effective.

Solution

This follows on from the previous question, in which we showed that there is a capital gain:

$$i^{(2)} \text{ @}10\% = 0.0976177 > (1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10,000} = 0.06$$

Now we can use equation (1.5) to calculate the price paid:

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10,000v^5 - 0.2(10,000-P)v^5 \qquad \text{@}10\%$$

$$= 600a_{\overline{5}|}^{(2)} + 8,000v^5 + 0.2Pv^5$$

The price paid, P , appears on both sides of the equation. Rearranging:

$$(1-0.2v^5)P = 600a_{\overline{5}|}^{(2)} + 8,000v^5$$

$$\Rightarrow P = \frac{600a_{\overline{5}|}^{(2)} + 8,000v^5}{1-0.2v^5} = \frac{600 \times 3.8833 + 8,000 \times 1.1^{-5}}{1-0.2 \times 1.1^{-5}} = £8,332.06$$

As expected, this is less than the price paid by an investor who is liable for income tax at 25% only (which we calculated to be £8,539.19).

The price calculated is critically dependent on the net yield required by the investor – this affects not only the annuity and discount factors in the equation, but also the capital gains test itself.

Question

An investor liable to income tax at 25% and capital gains tax at 20% purchases £10,000 nominal of a newly issued 5-year fixed-interest bond that is redeemable at par and pays coupons of 8% *pa* half-yearly in arrears.

Calculate the price the investor should pay to obtain a net yield of 5% *pa* effective.

Solution

This is the same scenario as the previous question, but the net yield required is now 5% *pa*, rather than 10% *pa*.

Carrying out the capital gains test:

$$i^{(2)} \text{ @}5\% = 0.0493902$$

$$(1-t_1)\frac{D}{R} = (1-0.25)\frac{800}{10{,}000} = 0.06$$

Since $i^{(2)} < (1-t_1)\frac{D}{R}$, there is no capital gain.

Since there is no capital gain, the investor will not be liable for capital gains tax, and so we need to use equation (1.3):

$$P = (1-0.25)800a_{\overline{5}|}^{(2)} + 10{,}000v^5 \qquad \text{@}5\%$$

$$= 600 \times 4.382935 + 10{,}000 \times 1.05^{-5}$$

$$= £10{,}465.02$$

The value of P obtained is greater than £10,000, which confirms that no capital gains tax is payable.

Note that if a stock is sold before the final maturity date, the capital gains tax liability will in general be different, since it will be calculated with reference to the sale proceeds rather than the corresponding redemption amount.

If an investor who paid £9,000 to purchase £10,000 nominal of a bond, sold the bond before maturity for £9,700, the capital gain would be $£9{,}700 - £9{,}000 = £700$. If the investor were liable to pay capital gains tax at a rate of 30%, the capital gains tax liability would be $0.3 \times 700 = £210$.

If $i^{(p)} \le (1-t_1)\frac{D}{R}$ then there is no capital gain and no capital gains tax liability due at redemption. Hence $P'' = P'$ in (1.3), *ie* we only pay income tax. We saw this in the above question.

(We are assuming that it is not permissible to offset the capital loss against any other capital gain.)

Finding the yield when there is capital gains tax

An investor who is liable to capital gains tax may wish to determine the net yield on a particular transaction in which he has purchased a loan at a given price.

One possible approach is to determine the price on two different net yield bases and then estimate the actual yield by interpolation. This approach is not always the quickest method. Since the purchase price is known, so too is the amount of the capital gains tax, and the net receipts for the investment are thus known. In this situation one may more easily write down an equation of value which will provide a simpler basis for interpolation, as illustrated by the next question.

Question

A loan of £1,000 bears interest of 6% per annum payable yearly and will be redeemed at par after ten years. An investor, liable to income tax and capital gains tax at the rates of 40% and 30% respectively, buys the loan for £800. What is his net effective annual yield?

Solution

Note that the net income each year of £36 is 4.5% of the purchase price. Since there is a gain on redemption, the net yield is clearly greater than 4.5%.

The gain on redemption is £200, so that the capital gains tax payable will be £60 and the net redemption proceeds will be £940. The net effective yield *pa* is thus that value of i for which:

$$800 = 36a_{\overline{10|}} + 940v^{10}$$

If the net gain on redemption (*ie* £140) were to be paid in equal instalments over the ten-year duration of the loan rather than as a lump sum, the net receipts each year would be £50 (*ie* £36 + £14). Since £50 is 6.25% of £800, the net yield actually achieved is less than 6.25%.

The net yield is less than 6.25% because, in reality, the capital gain is received right at the end of the term, rather than in equal instalments over the term, and being further in the future, this is less valuable to the investor.

When $i = 0.055$, the right-hand side of the above equation takes the value 821.66, and when $i = 0.06$ the value is 789.85.

By interpolation, we estimate the net yield as:

$$i = 0.055 + \frac{821.66 - 800}{821.66 - 789.85} 0.005 = 0.0584$$

The net yield is thus 5.84% per annum.

Alternatively, we may find the prices to give net yields of 5.5% and 6% per annum. These prices are £826.27 and £787.81, respectively.

For example, using a net yield of 5.5% *pa* effective, the price of the loan is:

$$P = 36a_{\overline{10}|} + 1{,}000v^{10} - 0.3(1{,}000 - P)v^{10}$$

$$\Rightarrow \quad P = \frac{36 \times 7.5376 + 700 \times 1.055^{-10}}{1 - 0.3 \times 1.055^{-10}} = £826.27$$

The yield may then be obtained by interpolation. However, this alternative approach is somewhat longer than the first method.

1.5 Optional redemption dates

Sometimes a security is issued without a fixed redemption date. In such cases the terms of issue may provide that the borrower can redeem the security *at the borrower's option* at any interest date on or after some specified date. Alternatively, the issue terms may allow the borrower to redeem the security at the borrower's option at any interest date on or between two specified dates (or possibly on any one of a series of dates between two specified dates).

In such cases, the loan will be redeemed at the time considered to be most favourable by the borrower (*ie* the issuer). If the interest rate payable on the loan is high relative to market rates, it will be cheaper for the borrower to repay the loan and borrow from elsewhere. Conversely, if the interest rate payable is relatively low, it will be cheaper to allow the loan to continue.

The latest possible redemption date is called the *final redemption date* of the stock, and if there is no such date, then the stock is said to be *undated*. It is also possible for a loan to be redeemable between two specified interest dates, or on or after a specified interest date, at the option of the lender, but this arrangement is less common than when the borrower chooses the redemption date.

The loan will be redeemed at the time when the party with the choice of date will obtain the greatest yield.

An investor who wishes to purchase a loan with redemption dates at the option of the borrower cannot, at the time of purchase, know how the market will move in the future and hence when the borrower will repay the loan. The investor thus cannot know the yield which will be obtained. However, by using (1.4) the investor can determine either:

(1) The maximum price to be paid, if the net yield is to be at least some specified value;

or

(2) The minimum net yield the investor will obtain, if the price is some specified value.

Consider a fixed-interest security which pays coupons of D per annum, payable $p$thly in arrears and has redemption amount R. The security has an outstanding term of n years, which may be chosen by the borrower subject to the restriction that $n_1 \le n \le n_2$. (We assume that n_1 and n_2 are integer multiples of $1/p$.) Suppose that an investor, liable to income tax at rate t_1, wishes to achieve a net annual yield of at least i.

It follows from equations (1.3) and (1.4) that if $i^{(p)} > (1-t_1)\dfrac{D}{R}$, then the purchaser will receive a capital gain when the security is redeemed. From the investor's viewpoint, the sooner a capital gain is received the better. The investor will therefore obtain a greater yield on a security which is redeemed first.

Question

An investor purchases £100 nominal of a zero-coupon bond for £80. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £80, we must find i that solves the equation:

$$80 = 100v^n$$

(a) $n=5 \;\Rightarrow\; (1+i)^{-5} = 0.8 \;\Rightarrow\; i = 0.8^{-1/5} - 1 = 4.56\%\ pa$

(b) $n=10 \;\Rightarrow\; (1+i)^{-10} = 0.8 \;\Rightarrow\; i = 0.8^{-1/10} - 1 = 2.26\%\ pa$

This illustrates that, if there is a capital gain, then the sooner the bond is redeemed, the higher the yield.

So to *ensure* the investor receives a net annual yield of *at least* i, they should assume the worst case result: that the redemption money is paid as late as possible, *ie* $n = n_2$.

Similarly if $i^{(p)} < (1-t_1)\dfrac{D}{R}$ then there will be a capital loss when the security is redeemed. The investor will wish to defer this loss as long as possible, and will therefore obtain a greater yield on a security which is redeemed later.

Question

An investor purchases £100 nominal of a zero-coupon bond for £120. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £120, we must find i that solves the equation:

$$120 = 100v^n$$

(a) $n=5 \;\Rightarrow\; (1+i)^{-5} = 1.2 \;\Rightarrow\; i = 1.2^{-1/5} - 1 = -3.58\%\ pa$

(b) $n=10 \;\Rightarrow\; (1+i)^{-10} = 1.2 \;\Rightarrow\; i = 1.2^{-1/10} - 1 = -1.81\%\ pa$

This illustrates that if there is a capital loss, then the later the bond is redeemed, the higher the yield.

So to *ensure* the investor receives a net annual yield of *at least* i, they should assume the worst case result: that the redemption money is paid as soon as possible, *ie* $n = n_1$.

Finally, if $i^{(p)} = (1-t_1)\dfrac{D}{R}$, then there is neither a capital gain nor a capital loss. So it will make no difference to the investor when the security is redeemed. The net annual yield will be i irrespective of the actual redemption date chosen.

Question

An investor purchases £100 nominal of a zero-coupon bond for £100. Calculate the yield obtained if the bond is redeemed after (a) five years, and (b) ten years.

Solution

If the purchase price is £100, we must find i that solves the equation:

$$100 = 100v^n$$

(a) $n=5 \Rightarrow (1+i)^{-5} = 1 \Rightarrow i = 1^{-1/5} - 1 = 0\%\ pa$

(b) $n=10 \Rightarrow (1+i)^{-10} = 1 \Rightarrow i = 1^{-1/10} - 1 = 0\%\ pa$

This illustrates that if there is neither a capital gain nor a capital loss, then the yield is the same irrespective of when the bond is redeemed. In the question immediately above, the yield is zero regardless of the term to redemption as it is a zero-coupon bond. This would not be the case for a coupon-paying bond.

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income only, wishes to purchase £100 nominal of this stock. Calculate the maximum price the investor should pay in order to obtain a net yield of at least 7% per annum.

Solution

Carrying out the capital gains test:

$$i^{(2)} = 2(1.07^{0.5} - 1) = 0.0688 \qquad (1-t_1)\frac{D}{R} = 0.75 \times \frac{8}{100} = 0.06$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$, there is a capital gain on redemption. The worst case scenario is the latest possible redemption date (*ie* time 15 years). This will give the minimum yield. So the maximum price is:

$$P = 0.75 \times 8a_{\overline{15|}}^{(2)} + 100v^{15} \text{ @7\%} = 6 \times 9.2646 + 100 \times 1.07^{-15} = £91.83$$

Suppose, alternatively, that the price of the loan is given. The minimum net annual yield is obtained by again assuming the worst case result for the investor. So if:

(a) **$P < R$, then the investor receives a capital gain when the security is redeemed. The worst case is that the redemption money is repaid at the *latest* possible date. If this does in fact occur, the net annual yield will be that calculated. If redemption takes place at an earlier date, the net annual yield will be greater than that calculated.**

(b) **$P > R$, then the investor receives a capital loss when the security is redeemed. The worst case is that the redemption money is repaid at the *earliest* possible date. The actual yield obtained will be at least the value calculated on this basis.**

(c) **$P = R$, then the investor receives neither a capital gain nor a capital loss. The net annual yield is i, where $i^{(p)} = (1-t_1)\frac{D}{R}$, irrespective of the actual redemption date chosen.**

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income only, purchases £100 nominal of this stock for £110. Calculate the minimum net yield this investor could expect to obtain.

Solution

Here there will be a loss on redemption (as the redemption payment of £100 is less than the price paid). So the earliest redemption date is the worst case scenario and will give the minimum yield.

This means we need to determine the yield i that satisfies the equation:

$$110 = 0.75 \times 8a_{\overline{10|}}^{(2)} + 100v^{10}$$

In the previous question, we saw that the right-hand side of this equation is equal to £91.83 when $i = 7\%$. So the value of i to give a price of £110 will be lower than 7%.

Using $i=5\%$ gives 108.29 and $i=4.5\%$ gives 112.40. Interpolating gives the minimum net yield:

$$i \approx 4.5\% + \frac{110-112.40}{108.29-112.40} \times (5\%-4.5\%) = 4.8\%$$

Note that a capital gains tax liability does not change any of this. For example, an investment which has a capital gain before allowing for capital gains tax must still have a net capital gain after allowing for the capital gains tax liability, so that the 'worst case' for the investor is still the latest redemption.

Question

A fixed-interest stock with a coupon of 8% per annum payable half-yearly in arrears can be redeemed at the option of the borrower at any time between 10 and 15 years from the date of issue. The stock is redeemable at par.

An investor, who is subject to tax at 25% on income and capital gains, purchases £100 nominal of this stock. Calculate the price the investor should pay in order to obtain a net yield of at least 7% per annum.

Solution

Carrying out the capital gains test:

$$i^{(2)} = 2(1.07^{0.5}-1) = 0.0688 \qquad (1-t_1)\frac{D}{R} = 0.75 \times \frac{8}{100} = 0.06$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$, there is a capital gain on redemption. The worst case scenario is the latest possible redemption date (*ie* time 15 years). This will give the minimum yield. So:

$$P = 0.75 \times 8a^{(2)}_{\overline{15|}} + 100v^{15} - 0.25(100-P)v^{15} \quad @7\%$$

$$= 6a^{(2)}_{\overline{15|}} + 75v^{15} + 0.25Pv^{15}$$

Rearranging gives:

$$P(1-0.25v^{15}) = 6a^{(2)}_{\overline{15|}} + 75v^{15} \quad \Rightarrow \quad P = \frac{6 \times 9.2646 + 75 \times 1.07^{-15}}{1-0.25 \times 1.07^{-15}} = £91.02$$

However, in some cases, such as if the redemption price varies, the simple strategy described above will not be adequate, and several values may need to be calculated to determine which is lowest.

For example, we may have a bond with a redemption rate of:

- 120% if it is redeemed between 5 and 10 years from now
- 110% if it is redeemed between 10 and 15 years from now.

2 Uncertain income securities

Securities with uncertain income include:

1. ***Equities*, which have regular declarations of *dividends*. The dividends vary according to the performance of the company issuing the stocks and may be zero.**

2. ***Property* which carries regular payments of rent, which may be subject to regular review.**

3. ***Index-linked bonds* which carry regular coupon payments and a final redemption payment, all of which are increased in proportion to the increase in a relevant index of inflation.**

For any of these investments, investors may be interested in calculating the yield for a given price, or the price or value of the security for a given yield. In order to calculate the value or the yield it is necessary to make assumptions about the future income.

Given the uncertain nature of the future income, one method of modelling the cashflows is to assume statistical distributions for, say, the inflation or dividend growth rate. In this Subject, however, we will make simpler assumptions – for example that dividends increase at a constant rate. It is important to recognise that modelling random variables deterministically, ignoring the variability of the payments and the uncertainty about the expected growth rate, is not adequate for many purposes and stochastic methods will be required.

In all three cases, using this deterministic approach means that we estimate the future cashflows and then solve the equation of value using the estimated cashflows.

Index-linked bonds differ slightly from the other two in that the income is certain in *real* terms. These are therefore covered separately, in Section 4.

2.1 Equities

This section relates to the material covered in Chapter 7 on *perpetuities*.

Given deterministic assumptions about the growth of dividends, we can estimate the future dividends for any given equity, and then solve the equation of value using estimated cashflows for the yield or the price or value.

So, let the value of an equity just after a dividend payment be P, and let D be the amount of this dividend payment. Assume that dividends grow in such a way that the dividend due at time t is estimated to be D_t. We generally value the equity assuming dividends continue in perpetuity, and without explicit allowance for the possibility that the company will default and the dividend payments will cease. In this case, assuming annual dividends:

$$P = \sum_{t=1}^{\infty} D_t v_i^t$$

where i is the return on the share, given price P.

If we assume a constant dividend growth rate of g, say, then $D_t = (1+g)^t D$ and:

$$P = D\left(\frac{(1+g)}{(1+i)} + \frac{(1+g)^2}{(1+i)^2} + \frac{(1+g)^3}{(1+i)^3} + \cdots\right)$$

This is a compound increasing annuity. So:

$$P = D a_{\overline{\infty}|\,i'} \qquad \text{where} \qquad i' = \frac{1+i}{1+g} - 1$$

$$\Rightarrow \quad P = \frac{D(1+g)}{i-g}$$

since $a_{\overline{\infty}|\,i'} = \frac{1}{i'}$.

At certain times close to the dividend payment date the equity may be offered for sale *excluding* the next dividend. This allows for the fact that there may not be time between the sale date and the dividend payment date for the company to adjust its records to ensure the buyer receives the dividend. An equity which is offered for sale without the next dividend is called ex-dividend or 'xd'. The valuation of ex-dividend stocks requires no new principles.

Question

A French investor, who is taxed at 35% on income, has just purchased 500 shares in a small company ex-dividend. Dividends are paid annually and the next dividend is due in one month's time. The last dividend was €8 per share and dividends are expected to rise by 4% *pa*.

Calculate:

(i) the price paid by the investor if the expected yield is 12% *pa* effective

(ii) the yield the investor would expect to obtain if the price per share is €120.

Solution

(i) ***Price paid***

The last dividend was €8, so the next dividend (due in one month) is $€8 \times 1.04$, and the one after that (due in one year and one month) is $€8 \times 1.04^2$.

The investor does not receive the dividend due in one month because the shares are purchased ex-dividend, so the first dividend received by the investor is after one year and one month.

The price of the shares (after taking off the income tax paid) is therefore:

$$P = 500v^{\frac{1}{12}}\left(8 \times 1.04^2 \times 0.65v + 8 \times 1.04^3 \times 0.65v^2 + \cdots\right)$$

The terms in brackets form an infinite geometric progression with first term $8\times1.04^2\times0.65v$ and common ratio $1.04v$.

So the price of the shares is:

$$P=500v^{\frac{1}{12}}\times\frac{8\times1.04^2\times0.65v}{1-1.04v}=€34,822$$

This price can also be calculated using annuities as follows:

$$P=500v^{\frac{1}{12}}\left(8\times1.04^2\times0.65v+8\times1.04^3\times0.65v^2+\cdots\right)$$

$$=500\times8\times1.04\times0.65v^{\frac{1}{12}}\left(1.04v+1.04^2v^2+\cdots\right)$$

$$=500\times5.408v^{\frac{1}{12}}a_{\overline{\infty}|i'}$$

where $i'=\frac{1.12}{1.04}-1=\frac{0.08}{1.04}$. Therefore:

$$P=500\times5.408\times1.12^{-1/12}\times\frac{1.04}{0.08}=€34,822 \qquad \text{since} \qquad a_{\overline{\infty}|i'}=\frac{1}{i'}=\frac{1.04}{0.08}$$

(ii) ***Yield***

Using the geometric progression approach from above, if the price per share is €120, the yield i can be found from the equation:

$$120=v^{\frac{1}{12}}\times\frac{8\times1.04^2\times0.65v}{1-1.04v}$$

Since a yield of 12% *pa* gives a price of $34,822/500=€69.64$ per share, the yield to give a price of €120 per share will be lower than 12%.

Using $i=8\%$, the right-hand side of the above equation gives €139.71, and using $i=9\%$, the right-hand side gives €111.68. Linearly interpolating, we find the yield to be:

$$i\approx8\%+\frac{120-139.71}{111.68-139.71}\times(9\%-8\%)=8.7\%$$

2.2 Property

The valuation of property by discounting future income follows very similar principles to the valuation of equities. Both require some assumption about the increase in future income; both have income which is related to the rate of inflation (both property rents and company profits will be broadly linked to inflation, over the long term); in both cases we use a deterministic approach.

The major differences between the approach to the property equation of value, compared with the equity equation of value, are:

(1) property rents are generally fixed for a number of years at a time and

(2) some property contracts may be fixed term, so that after a certain period the property income ceases and ownership passes back to the original owner (or another investor) with no further payments.

Let *P* be the price immediately after receipt of the periodic rental payment. Let *m* be the frequency of the rental payments each year. We estimate the future cashflows, such that D_t/m is the rental income at time *t*, $t=\frac{1}{m},\frac{2}{m},\dots$. If the rents cease after some time n then clearly $D_t=0$ for $t>n$.

Then the equation of value is:

$$P=\sum_{k=1}^{\infty}\frac{1}{m}D_{k/m}v^{k/m}$$

It will usually be much easier to work from first principles than try and apply this formula to every question about property values.

Question

The rent for the next five years of an eighty-year property contract is set at £4,000 per month. Thereafter the rent will increase by 20% compound every five years.

Calculate the price that should be paid for the contract by an investor who wishes to achieve a yield of 7% per annum effective.

Solution

The price to pay is given by:

$$P=48{,}000a_{\overline{5}|}^{(12)}+48{,}000\times1.2\times v^5a_{\overline{5}|}^{(12)}+\cdots+48{,}000\times1.2^{15}\times v^{75}a_{\overline{5}|}^{(12)}$$

Note that the monthly rent is £4,000 and so the annual rent is £48,000.

This formula can be simplified quite easily, as follows:

$$P=48{,}000a_{\overline{5}|}^{(12)}\left(1+1.2v^5+1.2^2v^{10}+\cdots+1.2^{15}v^{75}\right)$$

The terms in brackets form a geometric progression of 16 terms, with first term 1 and common ratio $1.2v^5$. So, using $i=7\%$, the price to pay is:

$$P=48{,}000a_{\overline{5}|}^{(12)}\times\frac{1-(1.2v^5)^{16}}{1-1.2v^5}=48{,}000\times4.2301\times\frac{1-(1.2\times1.07^{-5})^{16}}{1-1.2\times1.07^{-5}}=£1{,}290{,}000$$

Alternatively, we could calculate the price by converting the summation in brackets into an annuity:

$$P = 48{,}000 a_{\overline{5|}}^{(12)} \left(1 + 1.2v^5 + 1.2^2 v^{10} + \cdots + 1.2^{15} v^{75}\right) = 48{,}000 a_{\overline{5|}\,7\%}^{(12)} \ddot{a}_{\overline{16|}\,j\%}$$

where $\dfrac{1}{1+j} = \dfrac{1.2}{1.07^5} \quad \Rightarrow \quad j = 16.879\%$.

Therefore:

$$P = 48{,}000 \times 4.2301 \times \frac{1 - 1.16879^{-16}}{0.16879/1.16879} = £1{,}290{,}000$$

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3 Real rates of interest

An investor's basic objective is to maximise the rate of return. So, if an investor has to choose between a number of possible investment opportunities then, other things being equal, the higher the interest rate the better.

However, this is not the whole story. Where price inflation is present, the 'purchasing power' of a specified sum of money tends to be eroded as time passes.

Question

A pensioner has just invested £3,000 in a government savings account that guarantees to provide a rate of return of 7.25% per annum over the next 5 years.

(i) Calculate the accumulated amount in the account at the end of the 5 years.

Toasters currently cost £30 each and are expected to increase in price by 2.5% per annum over the next 5 years.

(ii) Calculate the number of toasters that could be bought now with the initial investment and with the proceeds at the end of the 5 years.

(iii) Comment on your answers to (i) and (ii).

Solution

(i) By the end of the 5 years the account will have accumulated to:

$$3,000 \times 1.0725^5 = 3,000 \times 1.4190 = £4,257$$

(ii) The £3,000 invested now could buy $3,000/30 = 100$ toasters.

At the end of the 5 years toasters will cost $£30 \times 1.025^5 = £33.94$ each. So the proceeds will be able to buy $4,257/33.94 = 125$ toasters.

(iii) In money terms, the fund has grown by 41.9%, but in terms of the number of toasters it will buy it has only grown by 25%, *ie* there has been an erosion in purchasing power caused by price inflation.

When considering investments, it is often more useful to look at the rate of return earned *after* taking into account the erosion caused by inflation. This is done by looking at the *real rate of interest* (or *real rate of return*).

The idea of a real rate of interest, as distinct from a money rate of interest, was introduced in Chapter 5. Ways of calculating real rates of interest will now be examined.

The real rate of interest of a transaction is the rate of interest after allowing for the effect of inflation on a payment series.

3.1 Inflation-adjusted cashflows

The effect of inflation means that a unit of money at, say, time 0 has different purchasing power than a unit of money at any other time. We find the real rate of interest by first adjusting all payment amounts for inflation, so that they are all expressed in units of purchasing power at the same date.

As a simple example, consider a transaction represented by the following payment line:

Time:	**0**	**1**
Payment:	**–100**	**120**

That is, for an investment of 100 at time 0 an investor receives 120 at time 1.

The effective rate of interest on this transaction is clearly 20% per annum. The real rate of interest is found by first expressing both payments in units of the same purchasing power. Suppose that inflation over this one year period is 5% per annum. This means that 120 at time 1 has a value of $120/1.05 = 114.286$ in terms of time 0 money units.

So, in 'real' terms, that is, after adjusting for the rate of inflation, the transaction is represented as:

Time:	**0**	**1**
Payment:	**–100**	**114.286**

Hence, the real rate of interest is 14.286%.

Question

Calculate the real rate of return on this transaction if inflation had been 5% *pa* for the first nine months of the year but only 3% *pa* for the remaining three months.

Solution

In this case, 120 at time 1 has a value of $\dfrac{120}{1.05^{9/12} \times 1.03^{3/12}} = 114.84$ in terms of time 0 money units.

Therefore the real rate of return is 14.84%.

3.2 Calculating real yields using an inflation index

Where the rates of inflation are known (that is, we are looking back in time at a transaction that is complete) we may adjust payments for the rate of inflation by reference to a relevant inflation index.

For example, assume we have an inflation index, $Q(t_k)$ at time t_k, and a payment series as follows:

Time, t:	**0**	**1**	**2**	**3**
Payment:	**–100**	**8**	**8**	**108**
$Q(t)$	**150**	**156**	**166**	**175**

Clearly the rate of interest on this transaction is 8%.

This is because (assuming the time period is 1 year) for an initial investment of 100, we receive interest payments of 8 at the end of each year plus a return of capital at the end of three years. We could check this by showing that 8% solves the equation of value:

$$100 = 8a_{\overline{3}|} + 100v^3$$

Now we can change all these amounts into time 0 money values by dividing the payment at time t by the proportional increase in the inflation index from 0 to t. For example the inflation-adjusted value of the payment of 8 at time 1 is $8 + (Q(1)/Q(0))$. The series of payments in time 0 money values is then as follows:

Time, t:	**0**	**1**	**2**	**3**
Payment:	**–100**	**7.6923**	**7.2289**	**92.5714**

This gives a yield equation for the real yield:

$$-100 + 7.6923v_{i'} + 7.2289v_{i'}^2 + 92.5714v_{i'}^3 = 0$$

where i' is the real rate of interest, which can be solved using numerical methods to give $i' = 2.63\%$.

In general, the real yield equation for a series of cashflows $\{C_{t_1}, C_{t_2}, \dots, C_{t_n}\}$, given associated inflation index values $\{Q(0), Q(t_1), Q(t_2), \dots, Q(t_n)\}$ is, using time 0 money units:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0 \quad \Rightarrow \quad \sum_{k=1}^{n} \frac{C_{t_k}}{Q(t_k)} v_{i'}^{t_k} = 0$$

Until now we have been expressing all payments in terms of time 0 money units. However, this choice was arbitrary, as we could have chosen any date on which to value all the payments.

The second equation here, in which all terms are divided by $Q(0)$, demonstrates that the solution of the yield equation is independent of the date the payment units are adjusted to.

3.3 Calculating real yields given constant inflation assumptions

If we are considering future cashflows, the actual inflation experience will not be known, and some assumption about future inflation will be required. For example, if it is assumed that a constant rate of inflation of j per annum will be experienced, then a cashflow of, say, 100 due at t has value $100(1+j)^{-t}$ in time 0 money values.

So, for a fixed net cashflow series $\{C_{t_k}\}$, $k = 1,2,\ldots,n$, assuming a rate of inflation of j per annum, the real, effective rate of interest, i', is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} v_j^{t_k} v_{i'}^{t_k} = 0$$

We also know that the effective rate of interest with no inflation adjustment, which may be called the 'money yield' to distinguish from the real yield, is i where:

$$\sum_{k=1}^{n} C_{t_k} v_i^{t_k} = 0$$

So the relationship between the real yield i', the rate of inflation j and the money yield i is $v_i = v_j v_{i'}$.

Therefore:

$$\frac{1}{1+i} = \frac{1}{1+j} \times \frac{1}{1+i'} \Rightarrow 1+i' = \frac{1+i}{1+j} \quad \Rightarrow \quad i' = \frac{i-j}{1+j}$$

This is sometimes approximated to $i' \approx i - j$ if only estimates are required, as this is a reasonable approximation if j is small.

Question

Ten years ago, a saver invested £5,000 in an investment fund operated by an insurance company. Over this period the rate of return earned by the fund has averaged 12% per annum. If prices have increased by 80% in total over this period, calculate the average annual real rate of interest earned by the fund over this period.

Solution

The average annual rate of inflation j over the 10-year period can be found from:

$$(1+j)^{10} = 1.8 \quad \Rightarrow \quad j = 1.8^{1/10} - 1 = 0.0605 \quad \textit{ie } 6.05\%$$

The average annual (money) rate of interest is $i = 0.12$.

So the average annual real rate is:

$$i' = \frac{i-j}{1+j} = \frac{0.12-0.0605}{1.0605} = 0.0561 \quad ie\ 5.61\%$$

Conversely, if we know the real yield i' which we have obtained from an equation of value using inflation-adjusted cashflows then we can calculate the money yield as follows.

The real growth factor $(1+i')$ can be viewed as the monetary growth factor $(1+i)$, adjusted for the effects of inflation:

$$1+i' = \frac{1+i}{1+j}$$

Rearranging the above equation gives:

$$i = (1+i')(1+j)-1 \quad \Rightarrow \quad \boldsymbol{i = i' + j(1+i')}$$

Question

For the last 10 years an investor has paid £50 at the start of each month into a savings account that has achieved a real rate of interest of 3% per annum over this period.

If inflation has been at a constant rate of 5% per annum for the 10 years, calculate the balance of the investor's account today.

Solution

The real rate of interest over the period, i', has been 3% *pa* and inflation, j, has been 5% *pa*. So, the actual (money) rate of interest, i, has been:

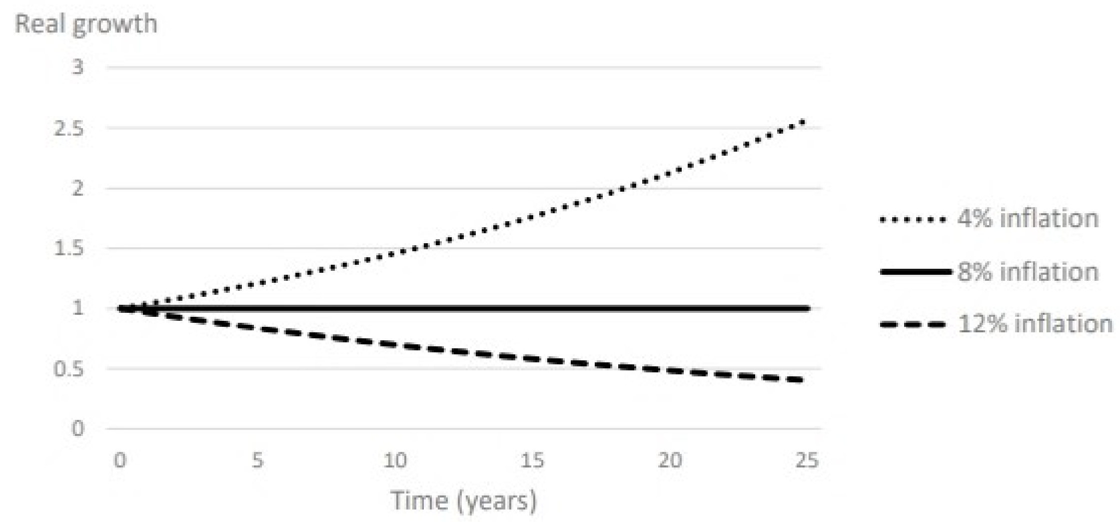
$$i = (1+i')(1+j)-1 = (1.03)(1.05)-1 = 0.0815 \quad ie\ 8.15\%\ pa$$

and the accumulated value of the investor's account today is:

$$(50\times 12)\ddot{s}_{\overline{10}|}^{(12)}\ @8.15\% = 600\times\frac{1.0815^{10}-1}{12(1-1.0815^{-1/12})} = 600\times 15.2265 = £9,136$$

In some cases a combination of known inflation index values and an assumed future inflation rate may be used to find the real rate of interest.

The following graph shows the real growth over time of a single investment when the rate of inflation is (a) higher than, (b) equal to and (c) less than the money rate of interest (taken to be 8% *pa*).



If an investment is achieving a positive real rate of return ($i > j$), then it is outstripping inflation. If it is achieving a negative real rate of return ($i < j$), then it is falling behind inflation (in which case it would be better to spend the money now, rather than 'investing' it).

3.4 Payments related to the rate of inflation

Some contracts specify that the cashflows will be adjusted to allow for future inflation, usually in terms of a given inflation index. The index-linked government security is an example. The actual cashflows will be unknown until the inflation index at the relevant dates are known. The contract cashflows will be specified in terms of some nominal amount to be paid at time *t*, say c_t. If the inflation index at the base date is $Q(0)$ and the relevant value for the time *t* payment is $Q(t)$ then the actual cashflow is:

$$C_t = c_t \frac{Q(t)}{Q(0)}$$

It is easy to show that if the real yield i' is calculated by reference to the same inflation index as is used to inflate the cashflows, then i' is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0 \quad \Rightarrow \quad \sum_{k=1}^{n} c_{t_k} v_{i'}^{t_k} = 0$$

In other words we can solve the yield equation using the nominal amounts.

However, it is *not* always the case that the index used to inflate the cashflows is the same as that used to calculate the real yield. For example the index-linked UK government security has coupons inflated by reference to the inflation index value three months before the payment is made. The real yield, however, is calculated using the inflation index at the actual payment dates.

The lag of 3 months in the UK means that the amount of the next coupon can be determined before it is received. Consider coupons arriving in January and July. The July coupon will use the index from April and the January coupon will use the index from the preceding October.

This makes things a little more complicated and is illustrated in the following question.

Question

A three-year index-linked security is issued at time 0. The security pays nominal coupons of 4% annually in arrears and is redeemed at par. The coupons and capital payment are increased by reference to the inflation index value 3 months before the payment is made. The inflation index value 3 months before issue was 110. The table below shows the index values at other times.

Time	0	$\frac{9}{12}$	1	$1\frac{9}{12}$	2	$2\frac{9}{12}$	3
Index	112	115	116	118	119	122	125

Calculate the real yield if the price of the security is £100.

Solution

First, we calculate the monetary amount of each payment using the inflation index values 3 months before the payment date. We then express these amounts in terms of time 0 money units. The results are shown in the following table.

Time	1	2	3
Nominal payment (*ie* cash actually received)	$4\times\frac{115}{110}=4.1818$	$4\times\frac{118}{110}=4.2909$	$104\times\frac{122}{110}=115.35$
Payment in time 0 units	$4.1818\times\frac{112}{116}=4.0376$	$4.2909\times\frac{112}{119}=4.0385$	$115.35\times\frac{112}{125}=103.35$

The amounts in terms of time 0 money units are calculated using the inflation index values on the actual payment dates, with no 3 month lag.

The real yield is then found by solving for i the equation of value:

$$100=4.0376v+4.0385v^2+103.35v^3$$

We can solve this using trial and error. When $i=3\%$, the right-hand side gives 102.31, and when $i=4\%$, the right-hand side gives 99.49. Linearly interpolating, the real yield is:

$$i\approx 3\%+\frac{100-102.31}{99.49-102.31}\times(4\%-3\%)=3.8\%\ pa$$

We will look further at index-linked bonds in Section 4.

3.5 The effects of inflation

Consider the simplest situation, in which an investor can lend and borrow money at the same rate of interest i_1. In certain economic conditions the investor may assume that some or all elements of the future cashflows should incorporate allowances for inflation (*ie* increases in prices and wages). The extent to which the various items in the cashflow are subject to inflation may differ. For example, wages may increase more rapidly than the prices of certain goods, or vice versa, and some items (such as the income from rent-controlled property) may not rise at all, even in highly inflationary conditions.

The case when *all* items of cashflow are subject to the same rate of escalation j per time unit is of special interest. In this case we find or estimate c_t^j and $\rho^j(t)$, the net cashflow and the net rate of cashflow allowing for escalation at rate j per unit time, by the formulae:

$$c_t^j = (1+j)^t c_t$$

$$\rho^j(t) = (1+j)^t \rho(t)$$

where c_t and $\rho(t)$ are estimates of the net cashflow and the net rate of cashflow respectively at time t without any allowance for inflation. It follows that, with allowance for inflation at rate j per unit time, the net present value of an investment or business project at rate of interest i is:

$$NPV_j(i) = \sum c_t(1+j)^t(1+i)^{-t} + \int_0^\infty \rho(t)(1+j)^t(1+i)^{-t}\,dt$$

$$= \sum c_t(1+i_0)^{-t} + \int_0^\infty \rho(t)(1+i_0)^{-t}\,dt \qquad (3.1)$$

where $1+i_0 = \dfrac{1+i}{1+j}$, or $i_0 = \dfrac{i-j}{1+j}$. **(3.2)**

Here j is the rate of inflation, i is the money yield and i_0 is the real yield. Previously, i' was used for the real yield.

If j is not too large, one sometimes uses the approximation $i_0 \approx i - j$.

Question

A developer buys a property for £90,000, and immediately spends £10,000 to buy materials in order to develop it. The developer expects to make the improvements over the next year and then sell the property. An identical property, with the improvements already made, is worth £110,000 today.

Assuming that the rate of inflation is 5% *pa* for the next year, calculate:

(i) the annual money yield on the investment

(ii) the annual real yield on the investment.

Solution

(i) The developer spends $90{,}000+10{,}000=100{,}000$ at outset. In one year's time, the developer expects the property to be worth $1.05\times 110{,}000$. We can find the money yield on the investment, i, from:

$$100{,}000=\frac{1.05\times 110{,}000}{1+i} \Rightarrow i=15.5\%$$

(ii) In today's money, the developed property is worth 110,000, so we can find the real yield on the investment, i_0, from:

$$100{,}000=\frac{110{,}000}{1+i_0} \Rightarrow i_0=10\%$$

These results are of considerable practical importance, because projects which are apparently unprofitable when rates of interest are high may become highly profitable when even a modest allowance is made for inflation.

It is, however, true that in many ventures the positive cashflow generated in the early years of the venture is insufficient to pay bank interest, so recourse must be had to further borrowing (unless the investor has adequate funds of their own). This does not undermine the profitability of the project, but the investor would require the agreement of his lending institution before further loans could be obtained and this might cause difficulties in practice.

Question

An entrepreneur is considering a business project that will be financed by a flexible loan that can be increased or repaid at any time.

The only outlay required in the project is an initial cost of £80,000. The income from the project will be received annually in arrears. At the end of the first year, the income is expected to be £8,800 and this is expected to increase each year thereafter by 10% *pa* compound.

The entrepreneur may borrow and invest money at 12% *pa* interest. If the project is expected to last for twelve years, calculate:

(i) the largest loan amount outstanding over the term of the project

(ii) the net present value of the project at 12% *pa* effective.

Solution

(i) ***Largest loan amount***

The interest due at the end of the first year is $0.12\times 80{,}000=£9{,}600$. The income is only £8,800 and so the loan increases to:

$$80,000 + 9,600 - 8,800 = £80,800$$

The interest due at the end of the second year is $0.12 \times 80,800 = £9,696$. The income is only $1.1 \times 8,800 = £9,680$ and so the loan increases to:

$$80,800 + 9,696 - 9,680 = £80,816$$

The interest due at the end of the third year is $0.12 \times 80,816 = £9,697.92$, whereas the income is now $1.1^2 \times 8,800 = £10,648$. As the income is more than enough to pay the interest, the loan will decrease.

So the maximum loan amount outstanding is £80,816.

(ii) ***Net present value***

The net present value equals:

$$NPV = -80,000 + 8,800v + 8,800 \times 1.1v^2 + \cdots + 8,800 \times 1.1^{11}v^{12}$$

$$= -80,000 + \frac{8,800v(1 - 1.1^{12}v^{12})}{1 - 1.1v} = £5,555$$

using the formula for the sum of 12 terms of a geometric progression with first term $8,800v$ and common ratio $1.1v$.

4 Index-linked bonds

Index-linked bond cashflows are described in Chapter 2. The coupon and redemption payments are increased according to an index of inflation.

We have also already looked at an example involving index-linked securities in the previous section of this chapter.

Given simple assumptions about the rate of future inflation, it is possible to estimate the future payments. Given these assumptions we may calculate the price or yield by solving the equation of value using the estimated cashflows.

For example, let the nominal annual coupon rate for an *n*-year index-linked bond be *D* per £1 nominal face value with coupons payable half-yearly, and let the nominal redemption price be *R* per £1 nominal face value. We assume that payments are inflated by reference to an index with base value $Q(0)$, such that the coupon due at time t years is:

$$\frac{D}{2}\frac{Q(t)}{Q(0)}$$

Then the equation of value, given an effective (money) yield of *i* per annum, and a present value or price *P* per £1 nominal at issue or immediately following a coupon payment, is:

$$P=\sum_{k=1}^{2n}\frac{D}{2}\frac{Q(k/2)}{Q(0)}v_i^{k/2}+R\frac{Q(n)}{Q(0)}v_i^n$$

We estimate the unknown value of $Q(t)$ using some assumption about future inflation and using the latest known value – which may be $Q(0)$. For example, assume inflation increases at rate j_t per annum in the year $t-1$ to *t*, then we have:

$$Q(\tfrac{1}{2})=Q(0)(1+j_1)^{\frac{1}{2}}$$

$$Q(1)=Q(0)(1+j_1)$$

$$Q(1\tfrac{1}{2})=Q(0)(1+j_1)(1+j_2)^{\frac{1}{2}}$$

$$Q(2)=Q(0)(1+j_1)(1+j_2)$$

etc

Question

An index-linked bond was issued on 1 April 2016. It pays half-yearly coupons and is redeemable on 1 April 2034. The nominal redemption rate is 100%. There is no time lag on the indexation. The coupon paid on 1 April 2018 was £2.10. A non-taxpayer buys £100 nominal of the bond on 1 April 2018, just after the payment of the coupon.

Assuming that past inflation has been 4% *pa* and future inflation is 5.25% *pa*, calculate the price this investor should pay in order to obtain a money rate of return of 10% *pa*.

Solution

To obtain the money rate of return we need to calculate the payments actually received by the investor.

The coupon payments are calculated allowing for inflation from the date of issue of the bond to the date of payment. Since the coupon payment of £2.10 on 1 April 2018 will include an allowance for past inflation, we can calculate the future coupons (*ie* those received by the investor) by increasing £2.10 at the rate of 5.25% *pa*.

The redemption payment is calculated allowing for inflation from the date of issue of the bond to the date of redemption – a period of 18 years. This will include 2 years of past inflation at 4% *pa* and 16 years of future inflation at 5.25% *pa*. Since the nominal redemption rate is 100%, the assumed redemption payment is $100 \times 1.04^2 \times 1.0525^{16}$.

The price the investor should pay is the present value of the payments:

$$\begin{aligned}P &= 2.10 \times 1.0525^{0.5} v^{0.5} + 2.10 \times 1.0525 v + 2.10 \times 1.0525^{1.5} v^{1.5} + \cdots \\ &\qquad + 2.10 \times 1.0525^{16} v^{16} + 100 \times 1.04^2 \times 1.0525^{16} v^{16} \\ &= 2.10 \times 1.0525^{0.5} v^{0.5} \frac{\left(1-(1.0525^{0.5} v^{0.5})^{32}\right)}{1-1.0525^{0.5} v^{0.5}} + 100 \times 1.04^2 \times 1.0525^{16} v^{16} \\ &= 47.6642 + 53.3749 \\ &= £101.04\end{aligned}$$

Here we have calculated the present value of the coupon payments using the formula for the sum of a geometric progression consisting of 32 terms, with first term $2.10 \times 1.0525^{0.5} v^{0.5}$ and common ratio $1.0525^{0.5} v^{0.5}$.

As we have already mentioned, **it is important to bear in mind that the index used may not be the same as the actual inflation index value at time t that one would use, for example, to calculate the real (inflation-adjusted) yield. In the case of UK index-linked bonds, the payments are increased using the index values from three months before the payment date. Real yields would be calculated using the inflation index values *at* the payment date.**

Like equities, index-linked bonds (and fixed-interest bonds) may be offered for sale 'ex-dividend'. No new principles are involved in the valuation of ex-dividend index-linked bonds.

If a bond is offered for sale ex-dividend, then the purchaser will not receive the next coupon.

Question

An inflation index takes the following values:

1/1/16:	121.2	1/1/17:	123.9	1/1/18:	125.2
1/4/16:	122.8	1/4/17:	124.2	1/4/18:	126.0
1/7/16:	123.1	1/7/17:	124.4		
1/10/16:	123.6	1/10/17:	124.9		

A two-year index-linked bond is purchased at issue on 1/4/16 by a tax-exempt investor for a price of £101 per £100 nominal. The nominal redemption rates is 100%. All coupon and redemption payments are linked to the inflation index three months prior to the payment date. The coupons on the bond are of nominal amount 4% *pa* payable half-yearly in arrears and are payable on 1 April and 1 October every year.

Calculate the real yield obtained by the investor.

Solution

The nominal coupon rate is 4% *pa*, so the nominal coupon payment is £4 *pa* payable half-yearly in arrears. This means that, before indexation, the amount of each coupon payment is £2 per £100 nominal. Since the nominal redemption rate is 100%, the redemption payment before indexation is £100 per £100 nominal.

We first use the values of the inflation index to calculate the actual coupon and redemption payments per £100 nominal of the bond:

Coupon on 1/10/16:

$$2\times\frac{index_{1/7/16}}{index_{1/1/16}}=2\times\frac{123.1}{121.2}=2.031353$$

Coupon on 1/4/17:

$$2\times\frac{index_{1/1/17}}{index_{1/1/16}}=2\times\frac{123.9}{121.2}=2.044554$$

Coupon on 1/10/17:

$$2\times\frac{index_{1/7/17}}{index_{1/1/16}}=2\times\frac{124.4}{121.2}=2.052805$$

Coupon on 1/4/18:

$$2\times\frac{index_{1/1/18}}{index_{1/1/16}}=2\times\frac{125.2}{121.2}=2.066007$$

Redemption on 1/4/18:

$$100 \times \frac{index_{1/1/18}}{index_{1/1/16}} = 100 \times \frac{125.2}{121.2} = 103.30033$$

So the total payment made on 1/4/18 is:

$$2.066007 + 103.30033 = 105.366337$$

We can now set up the equation of value for the real yield using the actual coupon and redemption payments, and taking into account inflation. This time there is no lag in the inflation index:

$$\begin{aligned}101 &= 2.031353 \times \frac{122.8}{123.6} \times v^{0.5} + 2.044554 \times \frac{122.8}{124.2} \times v + 2.052805 \times \frac{122.8}{124.9} \times v^{1.5} \\ &\quad + 105.366337 \times \frac{122.8}{126.0} \times v^2 \\ &= 2.018205v^{0.5} + 2.021508v + 2.018291v^{1.5} + 102.690366v^2\end{aligned}$$

The values multiplying the discount factors in this equation represent the real cashflows from the investor's point of view, ie the cashflows expressed in 1/4/16 prices.

As a first guess for the real yield, we can try the nominal coupon rate of 4%.

At 4%, the right-hand side is 100.7688, and at 3%, the right-hand side is 102.6775. Linearly interpolating, we find:

$$i \approx 3\% + \frac{101 - 102.6775}{100.7688 - 102.6775}(4\% - 3\%) = 3.9\%$$

So the real yield is approximately 3.9% *pa*.

---

5 Variable rate securities

Variable rate securities, such as floating rate notes, are investments **on which interest payments may be tied to a reference interest rate.**

In banking, many loans and deposits are variable rate, and banks are the main issuers of floating rate notes (FRNs).

Interest payments on FRNs are normally paid quarterly and are set at the beginning of each period as a stated spread, which remains fixed, above a variable reference rate such as LIBOR (London Inter-bank Offered Rate) or SONIA (Sterling Overnight Index Average).

In the UK, LIBOR has traditionally been used as a 'risk-free' rate against which interest payments may be defined *eg* interest is payable at LIBOR + 0.25%. The spread here is 0.25% *ie* the interest received or payable in excess of the risk-free rate. LIBOR is due to be replaced as the risk-free rate in the UK by SONIA with the transition occurring at the end of 2021 (at the time of writing).

Floating rate securities are covered in more detail in Subject SP5, Investment and Finance.

Chapter 12 Summary

The price of a fixed-interest security can be calculated as the present value of the coupon and redemption payments. When the proceeds are subject to income tax or capital gains tax, the net payments must be used. The formulae used for calculating the price are:

Ignoring tax: $P = Da_{\overline{n}|}^{(p)} + Rv^n$

Allowing for income tax: $P' = D(1-t_1)a_{\overline{n}|}^{(p)} + Rv^n$

Allowing for income and capital gains tax: $P'' = D(1-t_1)a_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n$

Capital gains tax is a tax levied on the difference between the sale or redemption price of a stock (or other asset) and the purchase price, if lower.

The capital gains test can be used to establish whether there is a capital gain or not:

- $i^{(p)} > \frac{D}{R}(1-t_1) \Rightarrow$ capital gain
- $i^{(p)} < \frac{D}{R}(1-t_1) \Rightarrow$ capital loss
- $i^{(p)} = \frac{D}{R}(1-t_1) \Rightarrow$ neither a capital gain nor a capital loss, so the price is equal to the redemption payment.

Based on a given purchase price for a fixed-interest security, the running yield and the redemption yield can be calculated, on either a gross or a net basis.

The running yield is the coupon divided by the price. The redemption yield is the rate of interest that equates the price with the present value of the coupon and redemption payments.

For a security with a redemption date that is selected at the option of the borrower, an investor who wishes to achieve a yield of at least *i* should value the security on the assumption that will give the lowest yield, *ie* the worst case. The worst case is the latest possible redemption date for a gain and the earliest possible redemption date for a loss.

Equities are usually valued by assuming that dividends increase at a constant rate and continue in perpetuity. For an equity that pays annual dividends that are expected to increase at a constant annual rate of g, the price P immediately after a dividend of D has just been paid is given by:

$$P = \sum_{k=1}^{\infty} D(1+g)^k v^k = \frac{D(1+g)}{i-g}$$

Property is valued in a similar way to equities although rents are generally fixed for a number of years at a time and some contracts are for a fixed term. The equation of value for property is:

$$P=\sum_{k=1}^{\infty}\tfrac{1}{m}D_{k/m}\times v^{k/m} \quad \text{where } \tfrac{1}{m}D_{k/m} \text{ is the rental income at time } k/m$$

The real rate of interest (i') on a transaction is the rate of interest after allowing for the effect of inflation (j) on a payment series:

$$i'=\frac{i-j}{1+j}\approx i-j$$

where i is the money rate of interest.

Index-linked bonds can be valued by allowing for the increases in the coupons and the redemption payment. The equation of value for an index-linked bond, with a nominal annual coupon of D payable half-yearly in arrears, and a nominal redemption payment of R , is:

$$P=\sum_{k=1}^{2n}\frac{D}{2}\frac{Q(k/2)}{Q(0)}v_i^{k/2}+R\frac{Q(n)}{Q(0)}v_i^{n} \quad \text{where } Q(t) \text{ is the index value at time } t$$

Chapter 12 Practice Questions

12.1 In a particular country, which uses dollars as its national currency, price inflation has been running at 20% *pa* for the last 20 years. Calculate the average annual real rate of return for each of the following investments:

(i) A set of gold coins purchased for \$14,000 on 1 January 2015 and sold for \$20,000 on 31 December 2017.

(ii) A painting purchased for \$3,000 on 1 March 2017 and sold for \$3,200 on 1 September 2017.

(iii) A diamond purchased for \$13,000 on 1 July 2016 and sold for \$10,000 on 1 July 2018.

(iv) A statuette purchased for \$7,500 on 1 November 2010 and sold for \$19,000 on 31 December 2017.

12.2 On 1 January 2015 an investor purchased £10,000 nominal of a stock that pays coupons half-yearly on 30 June and 31 December each year at the rate of 6% *pa* and is redeemable at par on 31 December 2027. The investor is liable for income tax at the rate of 40%, but is not liable for capital gains tax. Calculate the price paid by the investor in order to achieve a net redemption yield of 5% *pa* effective.

12.3 A fixed-interest security with a coupon rate of 6% *pa* payable half-yearly in arrears is purchased by an investor who is subject to capital gains tax at a rate of 30%. The fixed-interest security is redeemable at par after 15 years. The price paid is such that the investment provides a gross redemption yield of 10% *pa* effective. Calculate the amount of capital gains tax payable by the investor per £100 nominal.

12.4 A stock with a term of 9½ years has a coupon rate of 5% *pa* payable half-yearly in arrears and is redeemable at 105%. An investor who is not subject to tax purchases £100 nominal of the stock for £85. Calculate the yield obtained by the investor.

12.5 An investor purchases a bond 6 months after issue. The bond will be redeemed at 105% eight years after issue and pays coupons of 4% per annum annually in arrears. The investor pays tax of 25% on income and 15% on capital gains.

Exam style

(i) Calculate the purchase price of the bond per £100 nominal to provide the investor with a rate of return of 5% per annum effective. [6]

The real rate of return expected by the investor from the bond is 2% per annum effective.

(ii) Calculate the annual rate of inflation expected by the investor. [2]

[Total 8]

12.6 Exam style

An investor purchased a government bond on 1 January in a particular year. The bond pays coupons of 6% *pa* six monthly in arrears on 30 June and 31 December. The bond is due to be redeemed at 105% 11 years after the purchase date.

The investor pays income tax at the rate of 23% on 1 April on any coupon payments received in the previous year (1 April to 31 March), and also pays capital gains tax on that date at the rate of 40% on any capital gains realised in the previous year.

(i) Calculate the price paid for £100 nominal of the bond, given that the investor achieves a net yield of 5% *pa* effective interest. [7]

(ii) Without doing any further calculations, explain how and why your answer to (i) would alter if tax were collected on 1 October instead of 1 April each year. [2]

[Total 9]

12.7 Exam style

An investor purchases £100 nominal of a fixed-interest stock, which pays coupons of 7% *pa* half-yearly in arrears. The stock is redeemable at par and can be redeemed at the option of the borrower at any time between 5 and 10 years from the date of issue. The investor is subject to tax at the rate of 40% on income and 25% on capital gains.

(i) Calculate the maximum price that the investor should pay in order to obtain a net yield of at least 6% *pa*. [5]

(ii) Given that this was the price paid by the investor, calculate his net annual running yield, convertible half-yearly. [1]

[Total 6]

12.8 An equity pays half-yearly dividends. A dividend of d per share is due in exactly 3 months' time. Subsequent dividends are expected to grow at a compound rate of g per half-year forever.

(i) If i denotes the annual effective rate of return on the equity, show that P, the price per share, is given by:

$$P = \frac{d(1+i)^{1/4}}{(1+i)^{1/2} - (1+g)}$$

(ii) The current price of the share is £3.60, dividend growth is expected to be 2% per half-year and the next dividend payment in 3 months is expected to be 12p.

Calculate the expected annual effective rate of return for an investor who purchases the share.

12.9 An ordinary share pays dividends on each 31 December. A dividend of 35p per share was paid on 31 December 2017. The dividend growth is expected to be 3% in 2018, and a further 5% in 2019. Thereafter, dividends are expected to grow at 6% per annum compound in perpetuity.

Exam style

(i) Calculate the present value of the dividend stream described above at a rate of interest of 8% per annum effective for an investor holding 100 shares on 1 January 2018. [4]

An investor buys 100 shares for £17.20 each on 1 January 2018. He expects to sell the shares for £18 on 1 January 2021.

(ii) Calculate the investor's expected real rate of return.

You should assume that dividends grow as expected and use the following values of the inflation index:

Year:	2018	2019	2020	2021
Inflation index at start of year:	110.0	112.3	113.2	113.8

[5]
[Total 9]

12.10 An index-linked zero-coupon bond was issued on 1 January 2013 for redemption at par on 31 December 2017. The redemption payment was linked to a price inflation index with a 6-month time lag. The value of the price index on different dates is given below:

Date	Index	Date	Index
01.01.12	144	01.01.16	181
01.07.12	148	01.07.16	182
01.01.13	155	01.01.17	188
01.07.13	160	01.07.17	193
01.01.14	162	01.01.18	201
01.07.14	168		
01.01.15	175		
01.07.15	177		

Calculate the annual effective money and real rates of return obtained by an investor who purchased £10,000 nominal of the stock on 1 January 2015 for £10,250 and held it until redemption.

12.11 Exam style

On 25 October 2013 a certain government issued a 5-year index-linked stock. The stock had a nominal coupon rate of 3% per annum payable half-yearly in arrears and a nominal redemption price of 100%. The actual coupon and redemption payments were index-linked by reference to a retail price index as at the month of payment.

An investor, who was not subject to tax, bought £10,000 nominal of the stock on 26 October 2017. The investor held the stock until redemption.

You are given the following values of the retail price index:

	2013	-----	2017	2018
April	-----	-----	-----	171.4
October	149.2	-----	169.4	173.8

(i) Calculate the coupon payment that the investor received on 25 April 2018 and the coupon and redemption payments that the investor received on 25 October 2018. [3]

(ii) Calculate the purchase price that the investor paid on 25 October 2017 if the investor achieved an effective real yield of 3.5% per annum effective on the investment. [4]

[Total 7]

12.12 Exam style

On 15 May 2017 the government of a country issued an index-linked bond of term 10 years. Coupons are payable half-yearly in arrears, and the annual nominal coupon rate is 8%. The nominal redemption price is 102%.

Coupon and redemption payments are indexed by reference to the value of a retail price index with a time lag of 6 months. The retail price index value in November 2016 was 185 and in May 2017 was 190.

The issue price of the bond was such that, if the retail price index were to increase continuously at a rate of 2% *pa* from May 2017, a tax-exempt purchaser of the bond at the issue date would obtain a real yield of 3% *pa* convertible half-yearly.

Show that the issue price of the bond is £146.85 per £100 nominal. [9]

Chapter 12 Solutions

12.1 We can first calculate the average annual money rates of return, which are:

(i) for the set of gold coins: $\left(\frac{20,000}{14,000}\right)^{1/3}-1=12.62\%$

(ii) for the painting: $\left(\frac{3,200}{3,000}\right)^{2}-1=13.78\%$

(iii) for the diamond: $\left(\frac{10,000}{13,000}\right)^{1/2}-1=-12.29\%$

(iv) for the statuette: $\left(\frac{19,000}{7,500}\right)^{1/(7\frac{2}{12})}-1=13.85\%$

The real rates of return, i', can then be calculated using:

$$1+i'=\frac{1+i}{1.2} \quad \Rightarrow \quad i'=\frac{i-0.2}{1.2}$$

where i is the money rate of return. So the answers are:

(i) –6.15%

(ii) –5.19%

(iii) –26.91%

(iv) –5.13%

12.2 The term of the bond is 13 years (from 1 January 2015 to 31 December 2027). Letting P denote the price per £100 nominal, we have:

$$P=6(1-0.4)a_{\overline{13|}}^{(2)}+100v^{13} \quad @5\%$$

$$=3.6\times 9.5096+100\times 1.05^{-13}=£87.2666$$

So the price paid for £10,000 nominal is £8,726.66.

12.3 The *gross* redemption yield tells us the rate of return the investor earns *before* any tax is paid. So the price paid per £100 nominal is:

$$P=6a_{\overline{15|}}^{(2)}+100v^{15} \ @10\%$$

$$=6\times 7.7917+100\times 1.10^{-15}=£70.69$$

Since the stock is redeemed at par, the capital gains tax payable is $0.3\times(100-70.69)=£8.79$.

12.4 The yield is the interest rate that satisfies the equation:

$$85 = 5a^{(2)}_{\overline{9.5}|} + 105v^{9.5}$$

We can find the yield using trial and error.

For a rough guess at the yield, we can use the total payment at a 'typical' time (taken to be close to the end of the term, as the redemption payment is by far the largest cashflow):

$$85 = (9.5 \times 5 + 105)v^8 \quad \Rightarrow \quad i \approx 7.6\%$$

At 8%: $5a^{(2)}_{\overline{9.5}|} + 105v^{9.5} = 5 \times 6.6101 + 105 \times 1.08^{-9.5} = 83.5938$

At 7%: $5a^{(2)}_{\overline{9.5}|} + 105v^{9.5} = 5 \times 6.8902 + 105 \times 1.07^{-9.5} = 89.6645$

Linear interpolation gives the yield:

$$i \approx 7\% + \frac{85 - 89.6645}{83.5938 - 89.6645}(8\% - 7\%) = 7.8\%$$

12.5 (i) ***Purchase price of bond***

The capital gains test tells us that there is a capital gain if:

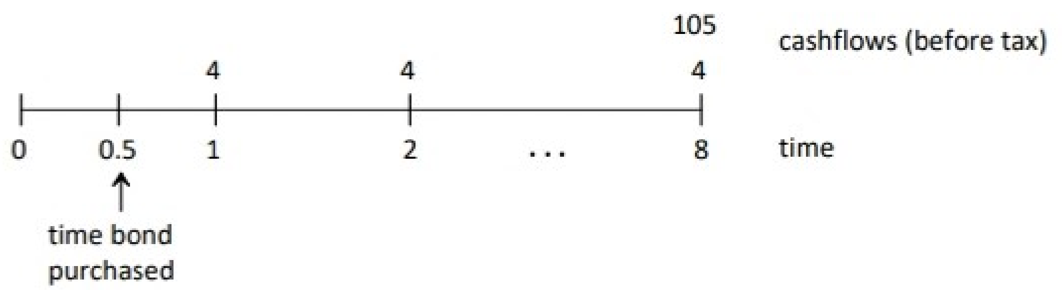
$$i^{(p)} > (1 - t_1)\frac{D}{R}$$

In this case:

$$i = 0.05 > (1 - 0.25)\frac{4}{105} = 0.0286$$

so there is a capital gain. [1]

A timeline for the cashflows is:



The present value of the net coupon payments at time 0 is:

$$0.75 \times 4a_{\overline{8}|}$$

So the present value of the net coupon payments at time $t = 0.5$ (the time of purchase) is:

$$1.05^{0.5} \times 0.75 \times 4a_{\overline{8}|}$$ [1]

Letting P denote the price for £100 nominal of the bond:

$$P = 1.05^{0.5} \times 0.75 \times 4a_{\overline{8}|} + 105v^{7.5} - 0.15(105 - P)v^{7.5}$$ [2]

$$= 1.05^{0.5} \times 3a_{\overline{8}|} + 89.25v^{7.5} + 0.15v^{7.5}P$$

Rearranging gives:

$$P(1 - 0.15v^{7.5}) = 1.05^{0.5} \times 3a_{\overline{8}|} + 89.25v^{7.5}$$

$$\Rightarrow \quad P = \frac{1.05^{0.5} \times 3 \times 6.4632 + 89.25 \times 1.05^{-7.5}}{1 - 0.15 \times 1.05^{-7.5}} = £91.26$$ [2]

[Total 6]

(ii) ***Annual rate of inflation***

To calculate the real rate of return, i', given a money rate of return, i, and constant inflation, j, we use:

$$1 + i' = \frac{1+i}{1+j}$$ [1]

Therefore:

$$1 + j = \frac{1+i}{1+i'} \quad \Rightarrow \quad j = \frac{1.05}{1.02} - 1 = 2.94\% \ pa$$ [1]

[Total 2]

12.6 (i) ***Price paid***

We will first check if there is a capital gain:

$$i^{(2)} @ 5\% = 2\left(1.05^{0.5} - 1\right) = 4.94\%$$

and: $$(1 - t_1)\frac{D}{R} = (1 - 0.23)\frac{6}{105} = 4.4\%$$

Since $i^{(2)} > (1 - t_1)\frac{D}{R}$, there is a capital gain. [1]

The capital gains test is not exact in this case as the income tax is not paid at the time each coupon is received.

To set up the equation of value for the bond, we need to consider the present value of all the relevant cashflows, including the tax payments.

The first payment of income tax is due on 1 April in the second year of ownership. The tax is payable on that date for both coupons in the previous 12 months. The amount of income tax payable on that date is $0.23 \times 6 = 1.38$ per £100 nominal of the bond owned. Income tax of 1.38 will also be paid on 1 April in each of the following 10 years.

The present value of the income tax payments on the date of purchase is $1.38 a_{\overline{11}|} v^{0.25}$. [1]

The capital gains tax is payable on 1 April after the bond is redeemed, *ie* 11.25 years after the purchase date. The present value of the capital gains tax payable is:

$$0.4(105 - P)v^{11.25}$$ [1]

Let the price paid for £100 nominal be *P*. The equation of value is then:

$$P = 6a_{\overline{11}|}^{(2)} - 1.38a_{\overline{11}|}v^{0.25} + 105v^{11} - 0.4(105 - P)v^{11.25}$$ [2]

Using an interest rate of 5% *pa* effective:

$$P = 6 \times 8.40898 - 1.38 \times 8.30641 \times 1.05^{-0.25} + 105 \times 1.05^{-11} - 0.4(105 - P) \times 1.05^{-11.25}$$

$$\Rightarrow \quad P(1 - 0.4 \times 1.05^{-11.25}) = 76.2625$$

$$\Rightarrow \quad P = £99.18$$

So the price for £100 nominal of the bond is £99.18. [2]

[Total 7]

(ii) ***Change in date of payment of tax***

The tax payment is being deferred from April to October, so the present value of the tax will reduce. [1]

As tax is outgo from the investor's point of view, reducing its present value means that the price the investor pays to achieve a given yield will increase. [1]

[Total 2]

12.7 (i) ***Price paid***

We first check whether there is a capital gain on redemption. To do this, we compare:

$$(1 - t_1)\tfrac{D}{R} = 0.6 \times \tfrac{7}{100} = 0.042$$

with:

$$i^{(2)} \ @6\% = 0.059126$$

As $i^{(2)} > (1 - t_1)\tfrac{D}{R}$, there is a capital gain. [1]

We need to look at the worst case scenario to ensure that the investor obtains a minimum yield of 6%. The worst case for an investor receiving a capital gain is to have the latest possible redemption date (*ie* the stock will be redeemed after 10 years). [1]

Let P denote the maximum price that the investor should pay in order to achieve a net yield of at least 6% *pa*. The equation of value is:

$$P = 0.6 \times 7a_{\overline{10}|}^{(2)} + 100v^{10} - 0.25(100 - P)v^{10}$$ [2]

$$\Rightarrow \quad P\left(1 - \frac{0.25}{1.06^{10}}\right) = 4.2 \times 7.46888 + \frac{75}{1.06^{10}}$$

$$\Rightarrow \quad P = £85.13$$ [1]

[Total 5]

(ii) ***Net running yield***

The net running yield is defined to be the net coupon per £100 nominal divided by the price per £100 nominal. So, the investor's net annual running yield is:

$$\frac{7 \times 0.6}{85.13} = 4.93\%$$ [1]

Since the coupons are paid half-yearly, this running yield is convertible half-yearly.

[Total 1]

12.8 (i) ***Formula for share price***

The price of a share is equal to the present value of the dividends it pays:

$$P = dv^{1/4} + d(1+g)v^{3/4} + d(1+g)^2 v^{1\frac{1}{4}} + \cdots$$
$$= dv^{1/4}\left[1 + (1+g)v^{1/2} + (1+g)^2 v + \cdots\right]$$

The expression in square brackets is an infinite geometric progression with first term 1, and common ratio $(1+g)v^{1/2}$:

$$P = dv^{1/4} \times \frac{1}{1-(1+g)v^{1/2}} = \frac{dv^{1/4}}{1-(1+g)v^{1/2}}$$

Multiplying both the numerator and the denominator of the fraction by a factor of $(1+i)^{1/2}$:

$$P = \frac{d(1+i)^{1/4}}{(1+i)^{1/2} - (1+g)}$$

(ii) ***Annual effective rate of return***

Substituting the values into the formula from part (i) gives:

$$3.60=\frac{0.12(1+i)^{1/4}}{(1+i)^{1/2}-1.02}$$

$$\Rightarrow \quad 3.60(1+i)^{1/2}-0.12(1+i)^{1/4}-3.672=0$$

Solving this as a quadratic in $(1+i)^{1/4}$ gives:

$$(1+i)^{1/4}=\frac{0.12\pm\sqrt{0.12^2-4\times 3.6\times -3.672}}{2\times 3.6}=1.02675$$

Hence the rate of return is:

$$i=1.02675^4-1=11.1\%\ pa$$

12.9 *This question is Subject CT1, April 2012, Question 9 (with dates updated).*

(i) ***Present value of dividend stream***

The PV of the future dividends from 100 shares on 1 January 2018 is:

$$\begin{aligned}PV&=35(1.03)v+35(1.03)(1.05)v^2+35(1.03)(1.05)(1.06)v^3+35(1.03)(1.05)(1.06)^2v^4+\cdots\\&=35(1.03)v+35(1.03)(1.05)v^2+35(1.03)(1.05)(1.06)v^3(1+1.06v+1.06^2v^2+\cdots)\end{aligned}$$ [2]

Using the formula for an infinite geometric progression with first term 1 and common ratio $1.06v$ to evaluate the infinite summation gives:

$$\begin{aligned}PV&=35(1.03)v\left[1+1.05v+(1.05)(1.06)v^2\times\frac{1}{1-1.06v}\right]\\&=35\times\frac{1.03}{1.08}\left[1+\frac{1.05}{1.08}+\frac{1.05\times 1.06}{1.08^2}\times\frac{1}{1-\frac{1.06}{1.08}}\right]\\&=£1,785.81\end{aligned}$$ [2]

[Total 4]

(ii) ***Expected real rate of return***

The following table shows the real and money cashflows per 100 shares.

Date	Money cashflows (£)	Inflation index	Real cashflows (£)
1/1/18	−1,720.00	110.0	−1,720.00
31/12/18	+35(1.03) = +36.05	112.3	$36.05\times\frac{110}{112.3}=35.31167$
31/12/19	+35(1.03)(1.05) = +37.8525	113.2	$37.8525\times\frac{110}{113.2}=36.78246$
31/12/20	+35(1.03)(1.05)(1.06) = +40.12365	113.8	$40.12365\times\frac{110}{113.8}=38.78384$
1/1/21	+1,800.00	113.8	$1,800.00\times\frac{110}{113.8}=1,739.89455$

The real rate of return, i', satisfies the following equation of value:

$$1,720=35.31167v+36.78246v^2+1,778.67840v^3$$ [3]

where $v=\frac{1}{1+i'}$.

Using $i'=2.5\%$, RHS = 1,721.14. [½]

Using $i'=3\%$, RHS = 1,696.70. [½]

Interpolating gives:

$$i'=2.5\%+\frac{1,720-1,721.14}{1,696.70-1,721.14}(3\%-2.5\%)=2.52\%$$ [1]

[Total 5]

12.10 The amount of the redemption payment is calculated using the index values from 6 months before the date of issue of the bond, and 6 months before the date of the redemption payment:

$$10{,}000 \times \frac{Index(01.07.17)}{Index(01.07.12)} = 10{,}000 \times \frac{193}{148} = £13{,}040.54$$

So the investor's money rate of return is found from the equation:

$$10{,}250 = 13{,}040.54v^3 \quad \Rightarrow \quad (1+i)^3 = 1.27225 \quad \Rightarrow \quad i = 8.36\%$$

In terms of 01.01.15 prices, the redemption payment is:

$$13{,}040.54 \times \frac{Index(01.01.15)}{Index(01.01.18)} = 13{,}040.54 \times \frac{175}{201} = 11{,}353.70$$

The investor's real rate of return is then found from the equation:

$$10{,}250 = 11{,}353.70v'^3 \quad \Rightarrow \quad (1+i')^3 = 1.10768 \quad \Rightarrow \quad i' = 3.47\%$$

12.11 *This question is Subject CT1, April 2014, Question 5 (with dates updated).*

(i) ***Coupon and redemption payments***

Since the nominal coupon rate is 3% *pa*, the investor actually received 1.5% (before indexation) per half-year. So the coupon payment received on 25 April 2018 was:

$$0.015 \times \frac{171.4}{149.2} \times 10{,}000 = £172.319$$ [1]

The coupon payment received on 25 October 2018 was:

$$0.015 \times \frac{173.8}{149.2} \times 10{,}000 = £174.732$$ [1]

The redemption proceeds also received on 25 October 2018 were:

$$\frac{173.8}{149.2} \times 10{,}000 = £11{,}648.794$$ [1]

[Total 3]

(ii) ***Purchase price***

We first calculate the real values as at 25 October 2017 of the 2018 cashflows.

The April 2018 coupon has real value:

$$172.319 \times \frac{169.4}{171.4} = 170.308$$ [1]

The total of the final coupon and redemption proceeds (in real values) is:

$$(174.732+11{,}648.794)\times\frac{169.4}{173.8}=11{,}524.196$$ [1]

So the real equation of value, using the effective real yield of 3.5% *pa*, is:

$$P=170.308v^{0.5}+11{,}524.196v^{1}$$ [1]

$$=170.308\times1.035^{-0.5}+11{,}524.196\times1.035^{-1}$$

$$=11{,}301.893$$

So the purchase price is £11,301.89. [1]

[Total 4]

12.12 The annual nominal coupon rate is 8% and coupons are payable half-yearly, so each nominal coupon payment is £4 per £100 nominal. The amount of each coupon payment is indexed with reference to the retail price index value with a lag of 6 months. So the actual coupon payment received in November 2017 is:

$$4\times\frac{\text{Index value May 2017}}{\text{Index value November 2016}}=4\times\frac{190}{185}$$

and the actual coupon payment received in May 2018 is:

$$4\times\frac{\text{Index value November 2017}}{\text{Index value November 2016}}=4\times\frac{190}{185}\times1.02^{0.5}$$

assuming that the index value grows at 2% *pa* from its May 2017 value.

The money and real values (as at May 2017) of the purchase price for £100 nominal, P, and the coupon and redemption payments are given in the table below:

Date	Money cashflow	Real cashflow
May 17	$-P$	$-P$
Nov 17	$4\times\frac{190}{185}$	$4\times\frac{190}{185}\times1.02^{-0.5}$
May 18	$4\times\frac{190}{185}\times1.02^{0.5}$	$4\times\frac{190}{185}\times1.02^{0.5}\times1.02^{-1}=4\times\frac{190}{185}\times1.02^{-0.5}$
Nov 18	$4\times\frac{190}{185}\times1.02$	$4\times\frac{190}{185}\times1.02\times1.02^{-1.5}=4\times\frac{190}{185}\times1.02^{-0.5}$
...	...	...
May 27	$(4+102)\times\frac{190}{185}\times1.02^{9.5}$	$(4+102)\times\frac{190}{185}\times1.02^{9.5}\times1.02^{-10}=(4+102)\times\frac{190}{185}\times1.02^{-0.5}$

[3 for money cashflows, 3 for real cashflows]

We need to calculate the price paid to obtain a real yield of 3% *pa* convertible half-yearly. This is equivalent to a real yield of 1.5% per half-year effective. So we will work in half-years:

$$P = 4 \times \frac{190}{185} \times 1.02^{-0.5} \left(v + v^2 + \cdots + v^{20} \right) + 102 \times \frac{190}{185} \times 1.02^{-0.5} v^{20}$$

$$= \frac{190}{185} \times 1.02^{-0.5} \left(4a_{\overline{20}|} + 102v^{20} \right) \quad @1.5\%$$ [2]

Evaluating this gives:

$$P = \frac{190}{185} \times 1.02^{-0.5} \left(4 \times 17.16864 + 102 \times 1.015^{-20} \right) = £146.85$$ [1]

[Total 9]

13

Term structure of interest rates

Syllabus objectives

2.6 Show an understanding of the term structure of interest rates.

- 2.6.1 Describe the main factors influencing the term structure of interest rates.
- 2.6.2 Explain what is meant by, derive the relationships between, and evaluate:
  - discrete spot rates and forward rates
  - continuous spot rates and forward rates.
- 2.6.3 Explain what is meant by the par yield and yield to maturity.

2.7 Show an understanding of duration, convexity and immunisation of cashflows.

- 2.7.1 Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.
- 2.7.2 Evaluate the duration and convexity of a cashflow sequence.
- 2.7.3 Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

0 Introduction

So far, it has generally been assumed that the interest rate *i* or force of interest δ earned on an investment are independent of the term of that investment. In practice the interest rate offered on investments does usually vary according to the term of the investment. It is often important to take this variation into consideration.

In investigating this variation we make use of *unit zero-coupon bond* prices. A unit zero-coupon bond of term *n*, say, is an agreement to pay £1 at the end of *n* years. No coupon payments are paid. It is also called a *pure discount bond.*

We denote the price at issue of a unit zero-coupon bond maturing in *n* years by P_n.

Question

Calculate the yield achieved by investing in a 15-year unit zero-coupon bond if $P_{15} = 0.54$.

Solution

The yield is the value of i that solves the bond's equation of value:

$$0.54 = v^{15} \quad \Rightarrow \quad 1+i = 0.54^{-1/15} \quad \Rightarrow \quad i = 4.19\%$$

Sections 1 and 2 of this chapter look at ways of expressing interest rates that vary by term and Section 3 gives the reasons why interest rates vary by term. Section 4 then looks at some numerical measures that allow us to quantify the effect of a change in interest rates on series of cashflows, and techniques used to minimise the risk of interest rate changes.

1 Discrete-time rates

1.1 Discrete-time spot rates

The yield on a unit zero-coupon bond with term *n* years, y_n, is called the '*n*-year spot rate of interest'.

The *n*-year spot rate is a measure of the average interest rate over the period from now until *n* years' time. Sometimes s_n is used to denote the *n*-year spot rate instead of y_n.

Using the equation of value for the zero-coupon bond we find the yield on the bond, y_n, from:

$$P_n = \frac{1}{(1+y_n)^n} \quad \Rightarrow \quad (1+y_n) = P_n^{-\frac{1}{n}}$$

Question

The prices for £100 nominal of zero-coupon bonds of various terms are as follows:

1 year = £94 5 years = £70 10 years = £47 15 years = £30

Calculate the spot rates for these terms and sketch a graph of these rates as a function of the term.

Solution

The spot rates for the various terms can be found from the equations of value:

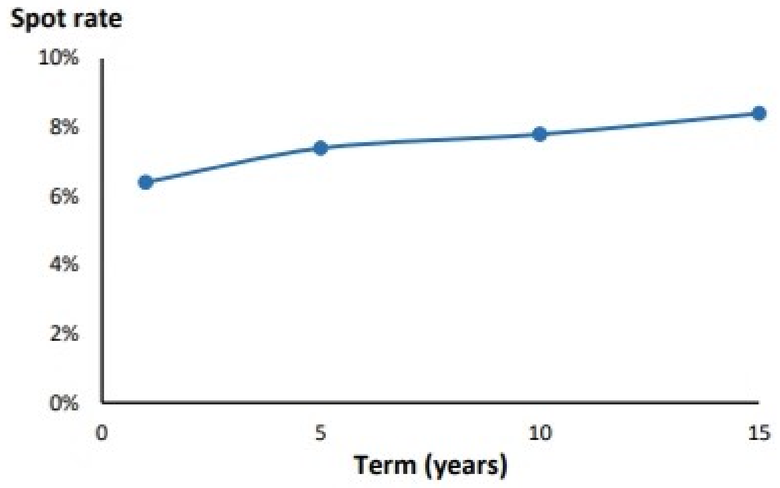
1 year: $100(1+y_1)^{-1} = 94 \quad \Rightarrow \quad y_1 = 6.4\%$

5 years: $100(1+y_5)^{-5} = 70 \quad \Rightarrow \quad y_5 = 7.4\%$

10 years: $100(1+y_{10})^{-10} = 47 \quad \Rightarrow \quad y_{10} = 7.8\%$

15 years: $100(1+y_{15})^{-15} = 30 \quad \Rightarrow \quad y_{15} = 8.4\%$

A graph of these spot rates is shown below:



Since rates of interest differ according to the term of the investment, in general $y_s \neq y_t$ for $s \neq t$. Every fixed-interest investment may be regarded as a combination of (perhaps notional) zero-coupon bonds. For example, a bond paying coupons of *D* every year for *n* years, with a final redemption payment of *R* at time *n* may be regarded as a combined investment of *n* zero-coupon bonds with maturity value *D*, with terms of 1 year, 2 years .., *n* years, plus a zero-coupon bond of nominal value *R* with term *n* years.

Defining $v_{y_t} = (1+y_t)^{-1}$, the price of the bond is:

$$A = D(P_1 + P_2 + \cdots + P_n) + RP_n$$
$$= D(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_n}^n) + Rv_{y_n}^n$$

This is actually a consequence of 'no arbitrage'; the portfolio of zero-coupon bonds has the same payouts as the fixed-interest bond, and the prices must therefore be the same.

Arbitrage is the existence of risk-free profits. This is discussed in great detail in Subject CM2.

If the price of a ten-year fixed-interest security were greater than the price given by the above formula, then investors would spot the anomaly and seek to make risk-free profits by selling holdings of the fixed-interest security (at the high price) and buying an appropriate combination of zero-coupon bonds (which would cost less). The future cashflows of the investors would not change but they would make an immediate risk-free profit. This is an example of arbitrage.

In this scenario, the increased demand for the zero-coupon bonds relative to the fixed-interest security would result in the price of the zero-coupon bonds rising and the price anomaly (and arbitrage opportunity) being removed. In practice, these price changes can happen very quickly, so we assume that no arbitrage opportunities exist.

The key result of the 'no arbitrage' assumption is that if two investments provide identical cashflows in the future, they must have the same price now. This is called the 'Law of One Price'.

The variation by term of interest rates is often referred to as the *term structure of interest rates*. The curve of spot rates $\{y_t\}$ is an example of a *yield curve*.

The graph above is an example of a yield curve. It isn't always spot rates that are plotted in a yield curve. It might instead be redemption yields or forward rates (which we will meet shortly).

Question

The current annual term structure of interest rates is:

(7%, 7.25%, 7.5%, 7.75%, 8%, ...)

ie $y_1 = 7\%$, $y_2 = 7.25\%$, $y_3 = 7.5\%$, $y_4 = 7.75\%$ and $y_5 = 8\%$.

A five-year fixed-interest security has just been issued. It pays coupons of 6% annually in arrears and is redeemable at par. Calculate:

(i) the price per £100 nominal of the security

(ii) the gross redemption yield of the security.

Solution

(i) The price per £100 nominal is given by:

$$P = 6(v_{7\%} + v_{7.25\%}^2 + v_{7.5\%}^3 + v_{7.75\%}^4 + v_{8\%}^5) + 100v_{8\%}^5$$
$$= £92.25$$

(ii) The gross redemption yield is the value of i that solves the equation:

$$92.25 = 6a_{\overline{5}|} + 100v^5$$

The gross redemption yield reflects the overall return from owning the security. It is a weighted average of the interest rates that apply over the term of the security, where the weights are the present values of the cashflows that occur at the different durations.

So the gross redemption yield must lie between 7% and 8% here (the range of spot rates over the five-year term), and will be closer to 8% as that is the spot rate that applies when the largest cashflow occurs, at time 5.

Trying 7.5%, we see that the right-hand side is £93.93, and trying 8%, the right-hand side is £92.01. Using linear interpolation, the gross redemption yield is:

$$i \approx 7.5\% + \frac{92.25 - 93.93}{92.01 - 93.93} \times (8\% - 7.5\%) = 7.94\%$$

1.2 Discrete-time forward rates

So far we have defined spot rates, which tell us about interest rates over a period that starts *now*. In this section we will consider *forward rates*, which tell us about interest rates over *future* periods that may start at a future time.

The discrete-time forward rate, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of r years.

That is, if an investor agrees at time 0 to invest £100 at time t for r years, the accumulated investment at time $t+r$ is:

$$100(1+f_{t,r})^r$$

The forward rate, $f_{t,r}$, is a measure of the average interest rate between times t and $t+r$.

Forward rates, spot rates and zero-coupon bond prices are all connected. The accumulation at time t of an investment of 1 at time 0 is $(1+y_t)^t$. If we agree at time 0 to invest the amount $(1+y_t)^t$ at time t for r years, we will earn an annual rate of $f_{t,r}$. So we know that £1 invested for $t+r$ years will accumulate to $(1+y_t)^t(1+f_{t,r})^r$. But we also know from the $(t+r)$ spot rates that £1 invested for $t+r$ years accumulates to $(1+y_{t+r})^{t+r}$, and we also know from the zero-coupon bond prices that £1 invested for $t+r$ years accumulates to P_{t+r}^{-1}. Hence we know that:

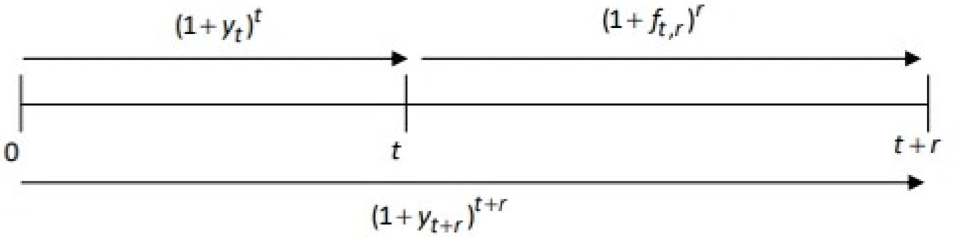
$$(1+y_t)^t(1+f_{t,r})^r = (1+y_{t+r})^{t+r} = P_{t+r}^{-1}$$

from which we find that:

$$(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}$$

so that the full term structure may be determined given the spot rates, the forward rates or the zero-coupon bond prices.

The connection between the spot rates and the forward rates can be represented on a timeline, as follows:



Accumulating payments from time 0 to time $t+r$ using the spot rate y_{t+r} is equivalent to first accumulating to time t using the spot rate y_t, and then accumulating from time t to time $t+r$ using the forward rate $f_{t,r}$.

***One-period forward rates* are of particular interest. The one-period forward rate at time *t* (agreed at time 0) is denoted $f_t = f_{t,1}$. We define $f_0 = y_1$. Comparing an amount of £1 invested for *t* years at the spot rate y_t, and the same investment invested 1 year at a time with proceeds reinvested at the appropriate one-year forward rate, we have:**

$$(1+y_t)^t = (1+f_0)(1+f_1)(1+f_2)\cdots(1+f_{t-1})$$

The one-year forward rate, f_t, is therefore the rate of interest from time *t* to time $t+1$. It can be expressed in terms of spot rates:

$$(1+f_t) = \frac{(1+y_{t+1})^{t+1}}{(1+y_t)^t}$$

Question

The 3, 5 and 7-year spot rates are 6%, 5.7% and 5% *pa* respectively. The 3-year forward rate from time 4 is 5.2% *pa*. Calculate:

(i) f_3

(ii) $f_{5,2}$

(iii) y_4

(iv) $f_{3,4}$

Solution

We are given:

$$y_3 = 6\%, \quad y_5 = 5.7\%, \quad y_7 = 5\%, \quad f_{4,3} = 5.2\%$$

We can present these rates on a timeline as follows: