Subject CM1

Revision Notes

For the 2022 exams

Bonds, equity, and property

Booklet 5

covering

The Actuarial Education Company

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These chapter numbers refer to the 2022 edition of the ActEd course notes.

Syllabus objectives covered in this booklet

The numbering of the syllabus items is the same as that used by the Institute and Faculty of Actuaries.

2.2 Demonstrate a knowledge and understanding of real and money interest rates.

3.2 Use the concept of equation of value to solve various practical problems.

2. Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.

3. Calculate the running yield and the redemption yield for the financial instrument as described in 3.2.2.

4. Calculate the upper and lower bounds for the present value of the financial instrument as described in 3.2.2, when the redemption date can be a single date within a given range at the option of the borrower.

5. Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

OVERVIEW

This booklet covers Syllabus objectives 2.2 and 3.2.2 to 3.2.5, which relate to real rates, and valuing bonds, equity shares and property assets.

Breakdown of topics

In Chapter 12, we describe how to calculate the price and the yield for a fixed-interest security. The situation is complicated slightly if the investor is liable to tax. The types of tax covered here are income tax, which applies to the coupon payments, and capital gains tax, which applies to the capital gain – *ie* the difference between the redemption value and the purchase price, provided this is positive. We also consider how to deal with securities that have optional redemption dates.

Also covered in this booklet are real rates of return and index-linked bonds, where the coupon and redemption payments are linked to an inflation index. This builds on the introductory material on real and money rates covered in Chapter 5.

We also consider the valuation of equities and property by discounting future cashflows.

Exam questions

There are lots of exam questions on calculating the price or the return on a fixed-interest security. By the time of the exam, once you've had some practice, you should find these relatively straightforward. Many students find questions on real yields and/or index-linked bonds much harder.

All of the Core Reading for the topics covered in this booklet is contained in this section.

Chapter 5 – Real and money interest rates

1 **Accumulating an investment of 1 for a period of time t from time 0 produces a new total accumulated value $A(0,t)$, say. Typically the investment of 1 will be a sum of money, say £1 or \$1 or 1 Euro.**

In this case, if we are given the information on the initial investment of 1 in the specified currency, the period of the investment, and the cash amount of money accumulated, then the underlying interest rate is termed a 'money rate of interest'.

More generally, given any series of monetary payments accumulated over a period, a money rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation.

In practice, most such accumulations will take place in economies subject to inflation, where a given sum of money in the future will have less purchasing power than at the present day. It is often useful, therefore, to reconsider what the accumulated value is worth allowing for the eroding effects of inflation.

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2 **Returning to the initial Core Reading example above, suppose the accumulation took place in an economy subject to inflation so that the cash $A(0,t)$ is effectively worth only $A^*(0,t)$ after allowing for inflation, where $A^*(0,t) < A(0,t)$. In this case, the rate of interest at which the original sum of 1 would have to be accumulated to produce the sum A^* is lower than the money rate of interest.**

The sum $A^*(0,t)$ is referred to as the real amount accumulated, and the underlying interest rate, reduced for the effects of inflation, is termed a 'real rate of interest'.

More generally, given any series of monetary payments accumulated over a period, a real rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effects of inflation.

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3 The above descriptions assume that the inflation rate is positive. Where the inflation rate is negative, termed 'deflation', the above theory still applies and $A^*(0,t) > A(0,t)$, giving rise to the conclusion that the real rate of interest in such circumstances would be higher than the money rate of interest.

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4 As might be expected, where there is no inflation $A^*(0,t) = A(0,t)$, and the real and money rates of interest are the same.

We assume here that we have a positive inflation rate.

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5 Which of the two rates of interest, real or money, is the more useful will depend on a two main factors:

- the purpose to which the rate will be put
- whether the underlying data have or have not already been adjusted for inflation.

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6 *The purpose to which the rate will be put*

Generally, where the actuary is performing calculations to determine how much should be invested to provide for future outgo, the first step will be to determine whether the future outgo is real or monetary in nature. The type of interest rate to be assumed would then be, respectively, a real or a monetary rate.

For example, first suppose that an actuary was asked to calculate the sum to be invested by a person aged 40 to provide for a round-the-world cruise when the person reaches 60, and where the person says the cruise costs £25,000.

Unless the person has, for some reason, already made an allowance for inflation in suggesting a figure of £25,000 then that amount is probably today's cost of the cruise. In this case, the actuary would be wise to assume (checking his understanding with the person) an inflation rate and this could be achieved by assuming a real rate of interest.

As an alternative example, suppose that a person has a mortgage of £50,000 to be paid off in twenty years' time. Here, the party that granted the mortgage would contractually be entitled to only £50,000 in twenty years' time. Accordingly, in working out how much should be invested to repay the outgo in this case, a money rate of interest would be assumed.

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7 *Whether the underlying data has or has not already been adjusted for inflation*

In the first example above, we see that the data may already have been adjusted for inflation and in that case it would not be appropriate to allow for inflation again. A money rate would then be assumed.

More generally in actuarial work, the nature of the data provided must be understood before choosing the type and amount of assumptions to be made.

Chapter 12 – Bonds, equity and property

As in other compound interest problems, one of two questions may be asked:

(1) What price P per unit nominal, should be paid by an investor to secure a net yield of i per annum?

(2) Given that the investor pays a price P per unit nominal, what net yield per annum will be obtained?

8 The price, P, to be paid to achieve a yield of i per annum is equal to:

$$P = \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net interest payments} \end{pmatrix} + \begin{pmatrix} \text{Present value, at rate} \\ \text{of interest } i \text{ per annum,} \\ \text{of net capital payments} \end{pmatrix} \qquad (1.1)$$

---

9 The yield available on a stock that can be bought at a given price, P, can be found by solving equation (1.1) for the net yield i.

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10 If the investor is not subject to taxation the yield i is referred to as a gross yield.

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11 The yield on a security is sometimes referred to as the *yield to redemption* or the *redemption yield* to distinguish it from the *flat* (or *running*) *yield*, which is defined as D/P, the ratio of the coupon rate to the price per unit nominal of the stock.

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12 Consider an n year fixed interest security which pays coupons of D per annum, payable $p$thly in arrear and has redemption amount R.

The price of this bond, at an effective rate of interest i per annum, with no allowance for tax (*ie* i represents the gross yield) is:

$$P = Da_{\overline{n}|}^{(p)} + Rv^n \quad \text{at rate } i \text{ per annum} \qquad (1.2)$$

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13 Note: One could also work with a period of half a year. The corresponding equation of value would then be:

$$P = \frac{D}{2} a_{\overline{2n}|} + Rv^{2n} \quad \text{at rate } i' \text{ where } (1+i')^2 = 1+i$$

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14 Suppose an investor is liable to income tax at rate t_1 on the coupons, which is due at the time that the coupons are paid. The price, P', of this bond, at an effective rate of interest *i* per annum, where *i* now represents the net yield, is now:

$$P' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n \quad \text{at rate } i \text{ per annum} \tag{1.3}$$

It is possible in some countries that the tax is paid at some later date, for example at the calendar year end.

This does not cause any particular problems as we follow the usual procedure — identify the cashflow amounts and dates and set out the equation of value.

For example, suppose that income tax on the bond is paid in a single instalment, due, say, *k* years after the second half-yearly coupon payment each year.

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15 Then the equation of value for a given net yield *i* and price (or value) P' is, immediately after a coupon payment,

$$P' = Da_{\overline{n}|}^{(p)} + Rv^n - t_1Dv^k a_{\overline{n}|}$$

Other arrangements may be dealt with similarly from first principles.

Capital gains tax

16 If the price paid for a bond is less than the redemption (or sale price if sold earlier) then the investor has made a capital gain.

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17 Capital gains tax is a tax levied on the capital gain. In contrast to income tax, this tax is normally payable once only in respect of each disposal, at the date of sale or redemption.

Capital gains test

Consider an n year fixed-interest security which pays coupons of D per annum, payable pthly in arrear and has redemption amount R. An investor, liable to income tax at rate t_1, purchases the bond at price P'. If $R > P'$ then there is a capital gain and from (1.3), we have:

$$R > (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n$$

$$\Rightarrow \quad R(1-v^n) > (1-t_1)D\frac{1-v^n}{i^{(p)}}$$

$$\Rightarrow \quad R > (1-t_1)\frac{D}{i^{(p)}}$$

$$\Rightarrow \quad i^{(p)} > (1-t_1)\frac{D}{R} \qquad (1.4)$$

If the investor is also subject to tax at rate t_2 $(0 < t_2 < 1)$ on the capital gains, then let the price payable, for a given net yield i, be P''.

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18 If $i^{(p)} > (1-t_1)\dfrac{D}{R}$ then there is a capital gain.

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19 At the redemption date of the loan there is therefore an additional liability of $t_2(R-P'')$.

In this case:

$$P'' = (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n - t_2(R-P'')v^n \qquad \text{at rate } i \text{ per annum} \qquad (1.5)$$

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20 Note that if a stock is sold before the final maturity date, the capital gains tax liability will in general be different, since it will be calculated with reference to the sale proceeds rather than the corresponding redemption amount.

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21 If $i^{(p)} \leq (1-t_1)\dfrac{D}{R}$ then there is no capital gain and no capital gains tax liability due at redemption. Hence $P'' = P'$ in (1.3). (We are assuming that it is not permissible to offset the capital loss against any other capital gain).

Finding the yield when there is capital gains tax

An investor who is liable to capital gains tax may wish to determine the net yield on a particular transaction in which he has purchased a loan at a given price.

One possible approach is to determine the price on two different net yield bases and then estimate the actual yield by interpolation. This approach is not always the quickest method. Since the purchase price is known, so too is the amount of the capital gains tax, and the net receipts for the investment are thus known. In this situation one may more easily write down an equation of value which will provide a simpler basis for interpolation, as illustrated by the next question.

Question

A loan of £1,000 bears interest of 6% per annum payable yearly and will be redeemed at par after ten years. An investor, liable to income tax and capital gains tax at the rates of 40% and 30% respectively, buys the loan for £800. What is his net effective annual yield?

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22 Solution

Note that the net income each year of £36 is 4.5% of the purchase price. Since there is a gain on redemption, the net yield is clearly greater than 4.5%.

The gain on redemption is £200, so that the capital gains tax payable will be £60 and the net redemption proceeds will be £940. The net effective yield *pa* is thus that value of i for which

$$800 = 36a_{\overline{10|}} + 940v^{10}$$

If the net gain on redemption (*ie* £140) were to be paid in equal instalments over the ten-year duration of the loan rather than as a lump sum, the net receipts each year would be £50 (*ie* £36 + £14). Since £50 is 6.25% of £800, the net yield actually achieved is less than 6.25%. When $i = 0.055$, the right-hand side of the above equation takes the value 821.66, and when $i = 0.06$ the value is 789.85. By interpolation, we estimate the net yield as

$$i = 0.055 + \frac{821.66 - 800}{821.66 - 789.85} 0.005 = 0.0584$$

The net yield is thus 5.84% per annum.

Alternatively, we may find the prices to give net yields of 5.5% and 6% per annum. These prices are £826.27 and £787.81, respectively. The yield may then be obtained by interpolation. However, this alternative approach is somewhat longer than the first method.

Optional redemption dates

Sometimes a security is issued without a fixed redemption date. In such cases the terms of issue may provide that the borrower can redeem the security *at the borrower's option* at any interest date on or after some specified date. Alternatively, the issue terms may allow the borrower to redeem the security at the borrower's option at any interest date on or between two specified dates (or possibly on any one of a series of dates between two specified dates).

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23 The latest possible redemption date is called the *final redemption date* of the stock, and if there is no such date, then the stock is said to be *undated*.

It is also possible for a loan to be redeemable between two specified interest dates, or on or after a specified interest date, at the option of the lender, but this arrangement is less common than when the borrower chooses the redemption date.

An investor who wishes to purchase a loan with redemption dates at the option of the borrower cannot, at the time of purchase, know how the market will move in the future and hence when the borrower will repay the loan. The investor thus cannot know the yield which will be obtained.

However, by using (1.4) the investor can determine either:

(1) The maximum price to be paid, if the net yield is to be at least some specified value;

or

(2) The minimum net yield the investor will obtain, if the price is some specified value.

Consider a fixed interest security which pays coupons of D per annum, payable $p$thly in arrear and has redemption amount R. The security has an outstanding term of n years, which may be chosen by the borrower subject to the restriction that $n_1 \le n \le n_2$. (We assume that n_1 and n_2 are integer multiples of $1/p$.) Suppose that an investor, liable to income tax at rate t_1, wishes to achieve a net annual yield of at least i.

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24 It follows from equations (1.3) and (1.4) that if $i^{(p)} > (1-t_1)\frac{D}{R}$ then the purchaser will receive a capital gain when the security is redeemed. From the investor's viewpoint, the sooner a capital gain is received the better. The investor will therefore obtain a greater yield on a security which is redeemed first.

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25 So to *ensure* the investor receives a net annual yield of *at least i* then they should assume the worst case result: that the redemption money is paid as late as possible, *ie* $n = n_2$.

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26 Similarly if $i^{(p)} < (1-t_1)\frac{D}{R}$ then there will be a capital loss when the security is redeemed. The investor will wish to defer this loss as long as possible, and will therefore obtain a greater yield on a security which is redeemed later.

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27 So to *ensure* the investor receives a net annual yield of *at least* i then they should assume the worst case result: that the redemption money is paid as soon as possible, *ie* $n = n_1$.

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28 Finally, if $i^{(p)} = (1-t_1)\dfrac{D}{R}$ then there is neither a capital gain nor a capital loss. So it will make no difference to the investor when the security is redeemed. The net annual yield will be i irrespective of the actual redemption date chosen.

Suppose, alternatively, that the price of the loan is given. The minimum net annual yield is obtained by again assuming the worst case result for the investor.

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29 So if:

$P < R$, then the investor receives a capital gain when the security is redeemed. The worst case is that the redemption money is repaid at the *latest* possible date. If this does in fact occur, the net annual yield will be that calculated. If redemption takes place at an earlier date, the net annual yield will be greater than that calculated.

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30 $P > R$, then the investor receives a capital loss when the security is redeemed. The worst case is that the redemption money is repaid at the *earliest* possible date. The actual yield obtained will be at least the value calculated on this basis.

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31 $P = R$, then the investor receives neither a capital gain nor a capital loss. The net annual yield is i, where $i^{(p)} = (1-t_1)\dfrac{D}{R}$, irrespective of the actual redemption date chosen.

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32 Note that a capital gains tax liability does not change any of this. For example, an investment which has a capital gain before allowing for capital gains tax must still have a net capital gain after allowing for the capital gains tax liability, so that the 'worst case' for the investor is still the latest redemption. However, in some cases, for example if the redemption price varies, the simple strategy described above will not be adequate, and several values may need to be calculated to determine which is lowest.

Uncertain income securities

33 Securities with uncertain income include:

1. *Equities*, which have regular declarations of *dividends*. The dividends vary according to the performance of the company issuing the stocks and may be zero.

2. *Property* which carries regular payments of rent, which may be subject to regular review.

3. *Index-linked bonds* which carry regular coupon payments and a final redemption payment, all of which are increased in proportion to the increase in a relevant index of inflation.

For all of these investments investors may be interested in calculating the yield for a given price, or the price or value of the security for a given yield. In order to calculate the value or the yield it is necessary to make assumptions about the future income.

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34 Given the uncertain nature of the future income, one method of modelling the cashflows is to assume statistical distributions for, say, the inflation or dividend growth rate. In this course however we will make simpler assumptions – for example that dividends increase at a constant rate. It is important to recognise that modelling random variables deterministically, ignoring the variability of the payments and the uncertainty about the expected growth rate, is not adequate for many purposes and stochastic methods will be required.

In all three cases, using this deterministic approach means that we estimate the future cashflows and then solve the equation of value using the estimated cashflows.

Index linked bonds differ slightly from the other two in that the income is certain in real terms. These are therefore covered separately, later.

Given deterministic assumptions about the growth of dividends, we can estimate the future dividends for any given equity, and then solve the equation of value using estimated cashflows for the yield or the price or value.

So, let the value of an equity just after a dividend payment be P, and let D be the amount of this dividend payment. Assume that dividends grow in such a way that the dividend due at time t is estimated to be D_t.

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35 We generally value the equity assuming dividends continue in perpetuity, and without explicit allowance for the possibility that the company will default and the dividend payments will cease. In this case, assuming annual dividends:

$$P = \sum_{t=1}^{\infty} D_t v_i^t$$

where i is the return on the share, given price P.

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36 If we assume a constant dividend growth rate of g, say, then $D_t = (1+g)^t D$ and:

$$P = Da_{\overline{\infty}|i'} \qquad \text{where } i' = \frac{1+i}{1+g} - 1$$

$$\Rightarrow P = \frac{D(1+g)}{i-g}$$

---

37 At certain times close to the dividend payment date the equity may be offered for sale *excluding* the next dividend. This allows for the fact that there may not be time between the sale date and the dividend payment date for the company to adjust its records to ensure the buyer receives the dividend. An equity which is offered for sale without the next dividend is called ex-dividend or 'xd'. The valuation of ex-dividend stocks requires no new principles.

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38 The valuation of property by discounting future income follows very similar principles to the valuation of equities. Both require some assumption about the increase in future income; both have income which is related to the rate of inflation (both property rents and company profits will be broadly linked to inflation, over the long term); in both cases we use a deterministic approach.

The major differences between the approach to the property equation of value, compared with the equity equation of value, are:

(1) property rents are generally fixed for a number of years at a time and

(2) some property contracts may be fixed term, so that after a certain period the property income ceases and ownership passes back to the original owner (or another investor) with no further payments.

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39 Let P be the price immediately after receipt of the periodic rental payment. Let m be the frequency of the rental payments each year. We estimate the future cashflows, such that D_t/m is the rental income at time t, $t = \frac{1}{m}, \frac{2}{m}, \ldots$. If the rents cease after some time n then clearly $D_t = 0$ for $t > n$.

Then the equation of value is:

$$P = \sum_{k=1}^{\infty} \frac{1}{m} D_{k/m} v^{k/m}$$

Real rates of interest

The idea of a real rate of interest, as distinct from a money rate of interest, was introduced in Chapter 5. Ways of calculating real rates of interest will now be examined.

The real rate of interest of a transaction is the rate of interest after allowing for the effect of inflation on a payment series.

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40 **The effect of inflation means that a unit of money at, say, time 0 has different purchasing power than a unit of money at any other time. We find the real rate of interest by first adjusting all payment amounts for inflation, so that they are all expressed in units of purchasing power at the same date.**

As a simple example, consider a transaction represented by the following payment line:

Time:	**0**	**1**
Payment:	**–100**	**120**

That is, for an investment of 100 at time 0 an investor receives 120 at time 1.

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41 **The effective rate of interest on this transaction is clearly 20% per annum. The real rate of interest is found by first expressing both payments in units of the same purchasing power. Suppose that inflation over this one-year period is 5% per annum. This means that 120 at time 1 has a value of $120/1.05 = 114.286$ in terms of time 0 money units. So, in 'real' terms, that is, after adjusting for the rate of inflation, the transaction is represented as:**

Time:	**0**	**1**
Payment:	**–100**	**114.286**

Hence, the real rate of interest is 14.286%.

42 Where the rates of inflation are known (that is, we are looking back in time at a transaction that is complete) we may adjust payments for the rate of inflation by reference to a relevant inflation index.

For example, assume we have an inflation index, $Q(t_k)$ at time t_k, and a payment series as follows:

Time, t:	0	1	2	3
Payment:	–100	8	8	108
$Q(t)$	150	156	166	175

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43 Clearly the rate of interest on this transaction is 8%.

Now we can change all these amounts into time 0 money values by dividing the payment at time t by the proportional increase in the inflation index from 0 to t. For example the inflation-adjusted value of the payment of 8 at time 1 is $8 \div (Q(1)/Q(0))$. The series of payments in time 0 money values is then as follows:

Time, t:	0	1	2	3
Payment:	–100	7.6923	7.2289	92.5714

This gives a yield equation for the real yield:

$$-100 + 7.6923v_{i'} + 7.2289v_{i'}^2 + 92.5714v_{i'}^3 = 0$$

where i' is the real rate of interest, which can be solved using numerical methods to give $i' = 2.63\%$.

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44 In general, the real yield equation for a series of cashflows $\{C_{t_1}, C_{t_2}, \ldots, C_{t_n}\}$, given associated inflation index values $\{Q(0), Q(t_1), Q(t_2), \ldots, Q(t_n)\}$ is, using time 0 money units:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0$$

$$\Rightarrow \sum_{k=1}^{n} \frac{C_{t_k}}{Q(t_k)} v_{i'}^{t_k} = 0$$

The second equation here, in which all terms are divided by $Q(0)$, demonstrates that the solution of the yield equation is independent of the date the payment units are adjusted to.

If we are considering future cashflows, the actual inflation experience will not be known, and some assumption about future inflation will be required. For example, if it is assumed that a constant rate of inflation of j per annum will be experienced, then a cashflow of, say, 100 due at t has value $100(1+j)^{-t}$ in time 0 money values.

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45 So, for a fixed net cashflow series $\{C_{t_k}\}$, $k = 1, 2, \ldots, n$, assuming a rate of inflation of j per annum, the real, effective rate of interest, i', is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} v_j^{t_k} v_{i'}^{t_k} = 0$$

---

46 We also know that the effective rate of interest with no inflation adjustment which may be called the 'money yield' to distinguish from the real yield, is i where:

$$\sum_{k=1}^{n} C_{t_k} v_i^{t_k} = 0$$

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47 **So the relationship between the real yield i', the rate of inflation j and the money yield i is:**

$$v_i = v_j v_{i'} \Rightarrow i' = \frac{i-j}{1+j}$$

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48 **Conversely, if we know the real yield i' which we have obtained from an equation of value using inflation-adjusted cashflows then we can calculate the money yield as follows:**

$$i = i' + j(1+i')$$

In some cases a combination of known inflation index values and an assumed future inflation rate may be used to find the real rate of interest.

Some contracts specify that the cashflows will be adjusted to allow for future inflation, usually in terms of a given inflation index.

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49 **The index-linked government security is an example.**

The actual cashflows will be unknown until the inflation index at the relevant dates are known.

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50 **The contract cashflows will be specified in terms of some nominal amount to be paid at time t, say c_t. If the inflation index at the base date is $Q(0)$ and the relevant value for the time t payment is $Q(t)$ then the actual cashflow is:**

$$C_t = c_t \frac{Q(t)}{Q(0)}$$

---

51 It is easy to show that if the real yield i' is calculated by reference to the same inflation index as is used to inflate the cashflows, then i' is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} \frac{Q(0)}{Q(t_k)} v_{i'}^{t_k} = 0 \Rightarrow \sum_{k=1}^{n} c_{t_k} v_{i'}^{t_k} = 0$$

In other words we can solve the yield equation using the nominal amounts.

However, it is *not* always the case that the index used to inflate the cashflows is the same as that used to calculate the real yield. For example the index-linked UK government security has coupons inflated by reference to the inflation index value three months before the payment is made. The real yield, however, is calculated using the inflation index at the actual payment dates.

Consider the simplest situation, in which an investor can lend and borrow money at the same rate of interest i_1. In certain economic conditions the investor may assume that some or all elements of the future cashflows should incorporate allowances for inflation (*ie* increases in prices and wages). The extent to which the various items in the cashflow are subject to inflation may differ. For example, wages may increase more rapidly than the prices of certain goods, or vice versa, and some items (such as the income from rent-controlled property) may not rise at all, even in highly inflationary conditions.

The case when *all* items of cashflow are subject to the same rate of escalation j per time unit is of special interest.

52 In this case we find or estimate c_t^j and $\rho^j(t)$, the net cashflow and the net rate of cashflow allowing for escalation at rate j per unit time, by the formulae:

$$c_t^j = (1+j)^t c_t$$

$$\rho^j(t) = (1+j)^t \rho(t)$$

where c_t and $\rho(t)$ are estimates of the net cashflow and the net rate of cashflow respectively at time t without any allowance for inflation.

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53 It follows that, with allowance for inflation at rate j per unit time, the net present value of an investment or business project at rate of interest i is:

$$\begin{aligned} NPV_j(i) &= \sum c_t(1+j)^t(1+i)^{-t} + \int_0^\infty \rho(t)(1+j)^t(1+i)^{-t}\,dt \\ &= \sum c_t(1+i_0)^{-t} + \int_0^\infty \rho(t)(1+i_0)^{-t}\,dt \qquad (3.1) \end{aligned}$$

where:

$$1+i_0 = \frac{1+i}{1+j}$$

or:

$$i_0 = \frac{i-j}{1+j} \qquad (3.2)$$

---

54 If j is not too large, and $1+j$ is close to 1, one sometimes uses the approximation:

$$i_0 \approx i-j$$

These results are of considerable practical importance, because projects which are apparently unprofitable when rates of interest are high may become highly profitable when even a modest allowance is made for inflation. It is, however, true that in many ventures the positive cashflow generated in the early years of the venture is insufficient to pay bank interest, so recourse must be had to further borrowing (unless the investor has adequate funds of their own). This does not undermine the profitability of the project, but the investor would require the agreement of his lending institution before further loans could be obtained and this might cause difficulties in practice.

Index-linked bonds

55 Index-linked bond cashflows are described in Chapter 2. The coupon and redemption payments are increased according to an index of inflation.

Given simple assumptions about the rate of future inflation it is possible to estimate the future payments. Given these assumptions we may calculate the price or yield by solving the equation of value using the estimated cashflows.

For example, let the nominal annual coupon rate for an n-year index-linked bond be D per £1 nominal face value with coupons payable half-yearly, and let the nominal redemption price be R per £1 nominal face value. We assume that payments are inflated by reference to an index with base value $Q(0)$, such that...

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56 ... the coupon due at time t years is:

$$\frac{D}{2}\frac{Q(t)}{Q(0)}$$

---

57 **Then the equation of value, given an effective (money) yield of *i* per annum, and a present value or price *P* per £1 nominal at issue or immediately following a coupon payment, is:**

$$P = \sum_{k=1}^{2n} \frac{D}{2} \frac{Q(k/2)}{Q(0)} v_i^{k/2} + R\frac{Q(n)}{Q(0)} v_i^n$$

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58 **We estimate the unknown value of $Q(t)$ using some assumption about future inflation and using the latest known value – which may be $Q(0)$. For example, assume inflation increases at rate j_t per annum in the year $t-1$ to *t*, then we have:**

$$Q(\tfrac{1}{2}) = Q(0)(1+j_1)^{\frac{1}{2}}$$

$$Q(1) = Q(0)(1+j_1)$$

$$Q(1\tfrac{1}{2}) = Q(0)(1+j_1)(1+j_2)^{\frac{1}{2}}$$

$$Q(2) = Q(0)(1+j_1)(1+j_2)$$

etc

It is important to bear in mind that the index used may not be the same as the actual inflation index value at time t that one would use, for example, to calculate the real (inflation-adjusted) yield. In the case of UK index-linked bonds, the payments are increased using the index values from three months before the payment date. Real yields would be calculated using the inflation index values at the payment date.

Like equities, index-linked bonds (and fixed-interest bonds) may be offered for sale 'ex-dividend'. No new principles are involved in the valuation of ex-dividend index linked bonds.

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59 **Variable rate securities, such as floating rate notes, are investments on which interest payments may be tied to a reference interest rate.**

In banking, many loans and deposits are variable rate, and banks are the main issuers of floating rate notes (FRNs).

Interest payments on FRNs are normally paid quarterly and are set at the beginning of each period as a stated spread, which remains fixed, above a variable reference rate such as LIBOR (London Inter-bank Offered Rate) or SONIA (Sterling Overnight Index Average).

The next section starts on the following page.

PAST EXAM QUESTIONS

This section contains all the relevant parts of the exam questions from April 2011 to April 2021 that are related to the topics covered in this booklet.

Solutions are given later in this booklet. These give enough information for you to check your answer, including working, and also show you what an adequate examination answer should look like. Further information may be available in the Examiners' Report, ASET or Course Notes. (ASET can be ordered from ActEd.)

We first provide you with a **cross-reference grid** that indicates the main subject areas of each exam question. You can use this, if you wish, to select the questions that relate just to those aspects of the topic that you may be particularly interested in reviewing.

Alternatively you can choose to ignore the grid, and instead attempt each question without having any clues as to its content.

Cross-reference grid

		Bonds						
Question	Tick when attempted	Price	Yield	Income/CG tax	Optional redemption	Index-linked	Real yield	Equity price/yield
1							✓	
2		✓		✓	✓			
3				✓			✓	✓
4		✓	✓					
5							✓	✓
6								✓
7		✓	✓	✓	✓			
8		✓						
9		✓	✓					
10			✓	✓			✓	
11		✓		✓	✓			
12		✓				✓	✓	
13		✓		✓		✓	(✓)	
14								✓
15		✓	✓	✓				
16				✓			✓	✓
17		✓		✓	✓			
18						✓	✓	
19							✓	✓
20		✓	✓					
21			✓	✓			✓	
22								✓
23				✓			✓	
24		✓		✓				
25			✓			✓	✓	
26							✓	✓
27		✓	✓	✓	✓			
28		✓		✓	✓			
29							✓	

1 ***Subject CT1, April 2011, Question 3***

An investment trust bought 1,000 shares at £135 each on 1 July 2005. The trust received dividends on its holding on 30 June each year that it held the shares.

The rate of dividend per share was as given in the table below:

30 June in year	Rate of dividend per share (£)	Retail price index
2005	…	121.4
2006	7.9	125.6
2007	8.4	131.8
2008	8.8	138.7
2009	9.4	145.3
2010	10.1	155.2

On 1 July 2010, the investment trust sold its entire holding of the shares at a price of £151 per share.

(i) Using the retail price index values shown in the table, calculate the real rate of return per annum effective achieved by the trust on its investment. [6]

(ii) Explain, without doing any further calculations, how your answer to (i) would alter (if at all) if the retail price index for 30 June 2008 had been greater than 138.7 (with all other index values unchanged). [2]

[Total 8]

2 ***Subject CT1, April 2011, Question 5***

A loan of nominal amount £100,000 was issued on 1 April 2011 bearing interest payable half-yearly in arrear at a rate of 6% per annum. The loan is to be redeemed with a capital payment of £105 per £100 nominal on any coupon date between 20 and 25 years after the date of issue, inclusive, with the date of redemption being at the option of the borrower.

An investor who is liable to income tax at 20% and capital gains tax of 35% wishes to purchase the entire loan on 1 June 2011 at a price which ensures that the investor achieves a net effective yield of at least 5% per annum.

(i) Determine whether the investor would make a capital gain if the investment is held until redemption. [3]

(ii) Explain how your answer to (i) influences the assumptions made in calculating the price the investor should pay. [2]

(iii) Calculate the maximum price the investor should pay. [5]

[Total 10]

3 ***Subject CT1, September 2011, Question 7***

An investment manager is considering investing in the ordinary shares of a particular company.

The current price of the shares is 12 pence per share. It is highly unlikely that the share will pay any dividends in the next five years. However, the investment manager expects the company to pay a dividend of 2 pence per share in exactly six years' time, 2.5 pence per share in exactly seven years' time, with annual dividends increasing thereafter by 1% per annum in perpetuity.

In five years' time, the investment manager expects to sell the shares. The sale price is expected to be equal to the present value of the expected dividends from the share at that time at a rate of interest of 8% per annum effective.

(i) Calculate the effective gross rate of return per annum the investment manager will obtain if he buys the share and then sells it at the expected price in five years' time. [6]

(ii) Calculate the net effective rate of return per annum the investment manager will obtain if he buys the share today and then sells it at the expected price in five years' time if capital gains tax is payable at 25% on any capital gains. [3]

(iii) Calculate the net effective real rate of return per annum the investment manager will obtain if he buys the share and then sells it at the expected price in five years' time if capital gains tax is payable at 25% on any capital gains and inflation is 4% per annum effective. There is no indexation allowance. [3]

[Total 12]

4 ***Subject CT1, April 2012, Question 6***

A fixed-interest bond pays annual coupons of 5% per annum in arrear on 1 March each year and is redeemed at par on 1 March 2025.

On 1 March 2007, immediately after the payment of the coupon then due, the gross redemption yield was 3.158% per annum effective.

(i) Calculate the price of the bond per £100 nominal on 1 March 2007. [3]

On 1 March 2012, immediately after the payment of the coupon then due, the gross redemption yield on the bond was 5% per annum.

(ii) State the new price of the bond per £100 nominal on 1 March 2012. [1]

A tax-free investor purchased the bond on 1 March 2007, immediately after payment of the coupon then due, and sold the bond on 1 March 2012, immediately after payment of the coupon then due.

(iii) Calculate the gross annual rate of return achieved by the investor over this period. [2]

(iv) Explain, without doing any further calculations, how your answer to part (iii) would change if the bond were due to be redeemed on 1 March 2035 (rather than 1 March 2025). You may assume that the gross redemption yield at both the date of purchase and the date of sale remains the same as in parts (i) and (ii) above. [3]

[Total 9]

5 *Subject CT1, April 2012, Question 9*

An ordinary share pays dividends on each 31 December. A dividend of 35p per share was paid on 31 December 2011. The dividend growth is expected to be 3% in 2012, and a further 5% in 2013. Thereafter, dividends are expected to grow at 6% per annum compound in perpetuity.

(i) Calculate the present value of the dividend stream described above at a rate of interest of 8% per annum effective for an investor holding 100 shares on 1 January 2012. [4]

An investor buys 100 shares for £17.20 each on 1 January 2012. He expects to sell the shares for £18 on 1 January 2015.

(ii) Calculate the investor's expected real rate of return.

You should assume that dividends grow as expected and use the following values of the inflation index:

Year:	2012	2013	2014	2015
Inflation index at start of year:	110.0	112.3	113.2	113.8

[5]
[Total 9]

6 *Subject CT1, April 2013, Question 4*

An investor is interested in purchasing shares in a particular company.

The company pays annual dividends, and a dividend payment of 30 pence per share has just been made.

Future dividends are expected to grow at the rate of 5% per annum compound.

(i) Calculate the maximum price per share that the investor should pay to give an effective return of 9% per annum. [4]

(ii) Without doing any further calculations, explain whether the maximum price paid will be higher, lower or the same if:

(a) after consulting the managers of the company, the investor increases his estimate of the rate of growth of future dividends to 6% per annum.

(b) as a result of a government announcement, the general level of future price inflation in the economy is now expected to be 2% per annum higher than previously assumed.

(c) general economic uncertainty means that, whilst the investor still estimates future dividends will grow at 5% per annum, he is now much less sure about the accuracy of this assumption.

You should consider the effect of each change separately. [6]

[Total 10]

7 ***Subject CT1, April 2013, Question 9***

A fixed-interest security pays coupons of 8% per annum half yearly on 1 January and 1 July. The security will be redeemed at par on any 1 January from 1 January 2017 to 1 January 2022 inclusive, at the option of the borrower.

An investor purchased a holding of the security on 1 May 2011, at a price which gave him a net yield of at least 6% per annum effective. The investor pays tax at 30% on interest income and 25% on capital gains.

On 1 April 2013 the investor sold the holding to a fund which pays no tax at a price to give the fund a gross yield of at least 7% per annum effective.

(i) Calculate the price per £100 nominal at which the investor bought the security. [5]

(ii) Calculate the price per £100 nominal at which the investor sold the security. [3]

(iii) Show that the effective net yield that the investor obtained on the investment was between 8% and 9% per annum. [6]

[Total 14]

8 ***Subject CT1, September 2013, Question 3***

A fixed-interest security pays coupons of 4% per annum, half-yearly in arrear and will be redeemed at par in exactly ten years.

(i) Calculate the price per £100 nominal to provide a gross redemption yield of 3% per annum convertible half-yearly. [2]

(ii) Calculate the price, 91 days later, to provide a net redemption yield of 3% per annum convertible half-yearly if income tax is payable at 25%. [2]

[Total 4]

9 ***Subject CT1, September 2013, Question 5***

An investor is considering the purchase of two government bonds, issued by two countries A and B respectively, both denominated in euro.

Both bonds provide a capital repayment of €100 together with a final coupon payment of €6 in exactly one year. The investor believes that he will receive both payments from the bond issued by Country A with certainty. He believes that there are four possible outcomes for the bond from Country B, shown in the table below.

Outcome	*Probability*
No coupon or capital payment	0.1
Capital payment received, but no coupon payment received	0.2
50% of capital payment received, but no coupon payment received	0.3
Both coupon and capital payments received in full	0.4

The price of the bond issued by Country A is €101.

(i) Calculate the price of the bond issued by Country B to give the same expected return as that for the bond issued by Country A. [3]

(ii) Calculate the gross redemption yield from the bond issued by Country B assuming that the price is as calculated in part (i). [1]

(iii) Explain why the investor might require a higher expected return from the bond issued by Country B than from the bond issued by Country A. [2]

[Total 6]

10 Subject CT1, September 2013, Question 8

Mrs Jones invests a sum of money for her retirement which is expected to be in 20 years' time. The money is invested in a zero coupon bond which provides a return of 5% per annum effective. At retirement, the individual requires sufficient money to purchase an annuity certain of £10,000 per annum for 25 years. The annuity will be paid monthly in arrear and the purchase price will be calculated at a rate of interest of 4% per annum convertible half-yearly.

(i) Calculate the sum of money the individual needs to invest at the beginning of the 20-year period. [5]

The index of retail prices has a value of 143 at the beginning of the 20-year period and 340 at the end of the 20-year period.

(ii) Calculate the annual effective real return the individual would obtain from the zero coupon bond. [2]

The government introduces a capital gains tax on zero coupon bonds of 25 per cent of the nominal capital gain.

(iii) Calculate the net annual effective real return to the investor over the 20-year period before the annuity commences. [3]

(iv) Explain why the investor has achieved a negative real rate of return despite capital gains tax only being a tax on the profits from an investment. [2]

[Total 12]

11 Subject CT1, April 2014, Question 4

A company issues a loan stock bearing interest at a rate of 8% per annum payable half-yearly in arrear. The stock is to be redeemed at 103% on any coupon payment date in the range from 20 years after issue to 25 years after issue inclusive, to be chosen by the company.

An investor, who is liable to income tax at 30% and tax on capital gains at 40%, bought the stock at issue at a price which gave her a minimum net yield to redemption of 6% per annum effective.

Calculate the price that the investor paid. [7]

12 *Subject CT1, April 2014, Question 5*

On 25 October 2008 a certain government issued a 5-year index-linked stock. The stock had a nominal coupon rate of 3% per annum payable half-yearly in arrear and a nominal redemption price of 100%. The actual coupon and redemption payments were index-linked by reference to a retail price index as at the month of payment.

An investor, who was not subject to tax, bought £10,000 nominal of the stock on 26 October 2012. The investor held the stock until redemption.

You are given the following values of the retail price index:

	2008	-----	2012	2013
April	-----	-----	-----	171.4
October	149.2	-----	169.4	173.8

(i) Calculate the coupon payment that the investor received on 25 April 2013 and the coupon and redemption payments that the investor received on 25 October 2013. [3]

(ii) Calculate the purchase price that the investor paid on 25 October 2012 if the investor achieved an effective real yield of 3.5% per annum effective on the investment. [4]

[Total 7]

13 *Subject CT1, September 2014, Question 9*

A government issued a number of index-linked bonds on 1 June 2012 which were redeemed on 1 June 2014. Each bond had a nominal coupon of 2% per annum, payable half yearly in arrear and a nominal redemption price of 100%. The actual coupon and redemption payments were indexed according to the increase in the retail price index between three months before the issue date and three months before the relevant payment dates. No adjustment is made to allow for the actual date of calculation of the price index within the month or the precise coupon payment date within the month.

The values of the retail price index in the relevant months were:

Date	Retail Price Index
March 2012	112
June 2012	113
September 2012	116
December 2012	117
March 2013	117
June 2013	118
September 2013	120
December 2013	121
March 2014	121
June 2014	122

An investor purchased £3.5m nominal of the bond at the issue date and held it until it was redeemed. The investor was subject to tax on coupon payments at a rate of 25%.

(i) Calculate the incoming net cashflows the investor received. [5]

(ii) Express the cashflows in terms of 1 June 2012 prices. [4]

(iii) Calculate the purchase price of the bond per £100 nominal if the real net redemption yield achieved by the investor was 1.5% per annum effective. [3]

When the investor purchased the security, he expected the retail price index to rise much more slowly than it did in practice.

(iv) Explain whether the investor's expected net real rate of return at purchase would have been greater than 1.5% per annum effective. [2]

In September 2012, the government indicated that it might change the price index to which payments were linked to one which tends to rise more slowly than the retail price index.

(v) Explain the likely impact of such a change on the market price of index-linked bonds. [2]

[Total 16]

14 ***Subject CT1, April 2015, Question 6***

An ordinary share pays annual dividends. The next dividend is expected to be 6p per share and is due in exactly six months' time. It is expected that subsequent dividends will grow at a rate of 6% per annum compound and that inflation will be 4% per annum. The price of the share is 175p and dividends are expected to continue in perpetuity.

Calculate the expected effective real rate of return per annum for an investor who purchases the share. [6]

Part (i) of Subject CT1 April 2015 Question 8 is about bond pricing. However, later parts of this question concern discounted mean term and volatility, so we have included it in a later booklet.

15 *Subject CT1, September 2015, Question 7*

A special type of loan is to be issued by a company. The loan is made up of 100,000 bonds, each of nominal value €100. Coupons will be paid semi-annually in arrear at a rate of 4% per annum. The bonds are to be issued on 1 October 2015 at a price of €100 per €100 nominal. Income tax will be paid by the bond holders at a rate of 25% on all coupon payments.

Exactly half the bonds will be redeemed after ten years at €100 per €100 nominal. The bonds that are redeemed will be determined by lot (*ie* the bonds will be numbered and half the numbered bonds will be chosen randomly for redemption). Coupon payments on the remaining bonds will be increased to 7% per annum and these bonds will be redeemed 20 years after issue at €130 per €100 nominal.

An individual buys a single bond.

Calculate, as an effective rate of return per annum:

(i) the maximum rate of return the individual can obtain from the bond. [5]

(ii) the minimum rate of return the individual can obtain from the bond. [2]

(iii) the expected rate of return the individual will obtain from the bond [2]

An investor is considering buying the whole loan.

(iv) Show that the rate of return that the investor will obtain is greater than the expected rate of return that the above individual who buys a single bond will receive. [5]

[Total 14]

16 ***Subject CT1, September 2015, Question 8***

(i) State the characteristics of an equity. [4]

An investor was considering investing in the shares of a particular company on 1 August 2014. The investor assumed that the next dividend would be payable in exactly one year and would be equal to 6 pence per share.

Thereafter, dividends will grow at a constant rate of 1% per annum and are assumed to be paid in perpetuity. All dividends will be taxed at a rate of 20%. The investor requires a net rate of return from the shares of 6% per annum effective.

(ii) Derive and simplify as far as possible a general formula which will allow you to determine the value of a share for different values of:

- the next expected dividend.
- the dividend growth rate.
- the required rate of return.
- the tax rate.

(iii) Calculate the value of one share to the investor. [5]

The company announces some news that makes the shares more risky.

(iv) Explain what would happen to the value of the share, using the formula derived in part (ii). [2]

The investor bought 1,000 shares on 1 August 2014 for the price calculated in part (iii). He received the dividend of 6 pence on 1 August 2015 and paid the tax due on the dividend. The investor then sold the share immediately for 120 pence. Capital gains tax was charged on all gains at a rate of 25%. On 1 August 2014, the index of retail prices was 123. On 1 August 2015, the index of retail prices was 126.

(v) Determine the net real return earned by the investor. [3]
[Total 14]

17 *Subject CT1, April 2016, Question 4*

A loan of nominal amount £100,000 is to be issued bearing coupons payable quarterly in arrear at a rate of 7% per annum. Capital is to be redeemed at £108 per £100 nominal on a coupon date between 15 and 20 years inclusive after the date of issue. The date of redemption is at the option of the borrower.

An investor who is liable to income tax at 25% and capital gains tax at 40% wishes to purchase the entire loan at the date of issue.

(i) Determine the price which the investor should pay to ensure a net effective yield of at least 5% per annum. [5]

(ii) Explain the significance of the redemption date being at the option of the borrower in relation to your calculation in part (i). [2]

[Total 7]

18 *Subject CT1, April 2016, Question 9*

In January 2014, the government of a country issued an index-linked bond with a term of two years. Coupons were payable half-yearly in arrear, and the annual nominal coupon rate was 6%. The redemption value, before indexing, was £100 per £100 nominal. Interest and capital payments were indexed by reference to the value of an inflation index with a time lag of six months.

A tax-exempt investor purchased £100,000 nominal at issue and held it to redemption. The issue price was £97 per £100 nominal.

The inflation index was as follows:

Date	*Inflation Index*
July 2013	120.0
January 2014	122.3
July 2014	124.9
January 2015	127.2
July 2015	129.1
January 2016	131.8

(i) Set out a schedule of the investor's cashflows, showing the amount and month of each cashflow. [3]

(ii) Determine the annual effective real yield obtained by the investor to the nearest 0.1% per annum. [5]
[Total 8]

19 *Subject CT1, April 2016, Question 11*

An investor is considering the purchase of 10,000 ordinary shares in Enterprise plc.

Dividends from the shares are payable half-yearly in arrear. The next dividend is due in exactly six months and is expected to be 6.5 pence per share.

The required rate of return is 6% per half-year effective and an estimated rate of future dividend growth is 2% per half-year.

(i) Calculate, showing all working, the maximum price that the investor should pay for the shares. [4]

As a result of a recently announced expansion plan, the investor increases the estimated rate of future dividend growth to 2.5% per half-year.

(ii) (a) Calculate, showing all working, the maximum price the investor should now pay for the shares.

(b) Explain the difference between your answers to part (i) and part (ii)(a). [2]

It is rumoured that new legislation may affect the operation of Enterprise plc.

As a result, the investor decides to increase her required rate of return to 7% per half-year effective. The estimated dividend growth rate remains at 2% per half-year

(iii) (a) Explain why it might be appropriate for the investor to increase her required rate of return.

(b) Calculate the maximum price that the investor should now pay for the shares.

(c) Explain the difference between your answers to part (i) and part (iii)(b). [3]

In the prevailing economic circumstances, investors are expecting lower inflation in the wider economy.

As a result, the investor decides to reduce both the assumed rate of dividend growth and her required rate of return to 1% and 5% per half-year effective respectively.

(iv) (a) Explain why it is appropriate for the investor to reduce both the future dividend growth rate and the required rate of return in this case.

(b) Calculate the maximum price that the investor should now pay for the shares.

(c) Explain the difference between your answers to part (i) and part (iv)(b). [5]

[Total 14]

20 ***Subject CT1, September 2016, Question 5***

A zero-coupon bond was issued on 1 January 1975 with a redemption date of 1 January 2015. An investor bought the bond to provide a yield to maturity of 5% per annum convertible half yearly. On a particular date the borrower defaulted, repaying 80% of the capital to all bondholders. The investor obtained a rate of return until the date of default which was equivalent to a force of interest of 4.8% per annum.

Determine the date on which the borrower defaulted. [5]

21 ***Subject CT1, April 2017, Question 7***

A fixed-interest bond was issued on 1 January 2017 with a term of 20 years and is redeemable at 105%. The security pays a coupon of 4% per annum, payable half-yearly in arrear.

An investor is liable to income tax at the rate of 30% and capital gains tax at the rate of 40%. Income tax and capital gains tax are both collected on 1 June each year in relation to gross payments made during the previous 12 months.

The investor bought £10,000 nominal of the stock at an issue price of £9,800.

(i) Show that the net redemption yield obtained by the investor will be between 3% and 4% per annum effective. [7]

The inflation rate over the term of the bond is assumed to be 2% per annum.

(ii) Calculate the net effective annual real redemption yield that would be obtained by the investor. [3]

(iii) Explain, without doing any further calculations, how your answers to parts (i) and (ii) would alter if the tax were collected on 1 April instead of 1 June each year. [2]
[Total 12]

22 ***Subject CT1, April 2018, Question 2***

(i) Describe what is meant by the term 'ex-dividend'. [1]

An individual purchased 10,000 shares on 1 December 2017. Dividends are payable on 1 January and 1 July each year, and are assumed to be payable in perpetuity. The next dividend, due on 1 January 2018, is \$0.07 per share.

The two dividend payments in any calendar year are expected to be the same, but the dividend payment is expected to increase at the end of each year at a rate of 2% per annum compound.

Assume that the share is ex-dividend on 1 December 2017 and use an effective rate of interest of 7% per annum.

(ii) Calculate the present value of the investment at the date of purchase. [5]

[Total 6]

23 ***Subject CT1, April 2018, Question 6***

On 1 April 2018 a government issues a 10-year bond redeemable at £105 per £100 nominal and paying coupons at the rate of 3% per annum half-yearly in arrear. The price of the bond was £102 per £100 nominal.

An investor subject to income tax of 25% and capital gains tax of 35% purchased £10,000 nominal of the bond at issue.

The investor assumes that inflation will be constant over the term of the bond at a rate of 2% per annum.

(i) Calculate the net effective real redemption yield which the investor expects to earn on the investment. [6]

(ii) Explain how your answer to part (i) would change if inflation were less than 2% per annum throughout the term. [2]

[Total 8]

24 *Subject CT1, September 2018, Question 4*

A company issues a loan stock which pays coupons at a rate of 6% per annum half-yearly in arrears. The stock is to be redeemed at 103% after 25 years.

(i) (a) Calculate the price per £100 nominal at issue which would provide a gross redemption yield of 3% per annum convertible half-yearly.

(b) Calculate the price per £100 nominal three months after issue which would provide a gross redemption yield of 3% per annum convertible half-yearly. [3]

An investor, who is liable to income tax at 30% and capital gains tax at 40%, bought the stock at issue at a price which gave him a net redemption yield of 10% per annum effective.

(ii) Calculate the price the investor paid. [4]

[Total 7]

25 *Subject CT1, September 2018, Question 9 (part)*

An investor bought £1m nominal of an index-linked bond on 31 December 2015 for £100 per £100 nominal. Nominal coupon payments of 1% were received on 30 June and 31 December each year. The bond was sold for £101 per £100 nominal on 31 December 2017 immediately after the coupon due on that date had been received.

The coupon payments from the bond were linked to the retail prices index (RPI) with a three-month lag with cash payments being rounded to the nearest pound. RPI inflation was 2% per annum effective from three months before the bond was issued until three months before it was sold.

Assume that all months are of equal length.

(ii) Calculate the cash payments received by the investor from the index-linked bond. [3]

(iii) Calculate, to the nearest 0.1%, the effective rate of return per annum obtained from the bond over the holding period (before allowing for inflation). [5]

The real rate of return obtained from the bond over the holding period was 1% per annum convertible half-yearly.

(iv) Calculate the rate of inflation in the three months to 31 December 2017, expressing your answer as an annual effective rate. [7]

[Total 15]

26 ***Subject CM1, April 2019, Question 11***

On 1 February 2017, an investor was considering purchasing ordinary shares in Actuaria PLC.

Dividends are payable annually, and a dividend of £0.40 per share had just been paid.

At the date of purchase, dividends were expected to grow each year on a compound basis. The rate of growth was expected to be 5% in the first year, 4% in the second year and 3% per annum thereafter.

The investor was not entitled to the dividend which had just been paid.

(i) Calculate the maximum price per share the investor would have been prepared to pay at this date to give a rate of return of 9% per annum effective, assuming the investor holds the share in perpetuity. [6]

The investor purchased a holding of shares on 1 February 2017 at a price of £7.00 per share and sold the holding at a price of £7.50 per share on 1 February 2019, immediately after receiving the dividend payment then due.

(ii) Calculate the effective annual real rate of return achieved by the investor between 1 February 2017 and 1 February 2019 using the following information:

Date	Inflation index	Dividend per share
1 February 2017	211.0	£0.400
1 February 2018	215.7	£0.428
1 February 2019	221.2	£0.449

[5]

[Total 11]

27 *Subject CM1, September 2019, Question 8*

A loan of £1,000,000 nominal is issued with coupons payable half-yearly in arrears at a rate of 9% per annum. The loan is to be redeemed at £110 per £100 nominal on a single coupon date between 20 and 25 years after the date of issue, inclusive. The date of redemption is at the option of the borrower.

An investor who is liable to income tax at 15% but not liable to capital gains tax wishes to purchase the loan at the date of issue.

(i) Calculate the price the investor should pay to ensure a net effective yield of at least 8% per annum. [5]

The investor purchases the loan for the price calculated in part (i). Exactly ten years later, immediately after the payment of the coupon then due, a second investor, who is liable to income tax at 25% and capital gains tax of 35%, purchases the loan for a price such that the first investor obtained a net effective yield of 8% per annum. The second investor holds the loan to maturity.

(ii) Calculate:

(a) the price paid by the second investor

(b) the minimum net redemption yield earned by the second investor, to the nearest 0.1% per annum. [6]

[Total 11]

28 *Subject CM1, September 2020, Question 9*

A fixed interest security of nominal amount $1,000,000 is to be issued paying coupons quarterly in arrears at a rate of 6% per annum. The security is to be redeemed with a capital payment of $105 per $100 nominal on a coupon date between 20 and 25 years after the date of issue, inclusive. The date of redemption is at the option of the borrower.

An investor, who is liable to income tax at 20% and capital gains tax of 25%, wishes to purchase the entire security at the date of issue, at a price that ensures she receives a net effective yield of at least 4.9% per annum.

(i) Determine whether the investor would make a capital gain if she holds the security until redemption. [3]

(ii) Explain how your answer to part (i) influences the assumptions made in calculating the price the investor should pay. [2]

(iii) Calculate the maximum price that the investor should pay per \$100 nominal. [5]

(iv) Explain, without carrying out any further calculations, how your answer to part (iii) would change if the coupons had been payable half-yearly in arrears. [2]
[Total 12]

29 *Subject CM1, April 2021, Question 3*

A fixed interest security of nominal amount \$100,000 was issued on 1 March 2017 and was redeemed at par on 1 March 2020. Coupons were paid at the rate of 4% *pa* annually in arrears.

The value of the inflation index at various dates during the term of the security was as follows:

Date	Inflation index
1 March 2017	240.5
1 March 2018	256.0
1 March 2019	272.8
1 March 2020	286.6

(i) Demonstrate that the effective annual real rate of return achieved over the term of the security is approximately equal to –1.9% *pa*. [5]

(ii) Comment on the result in part (i). [3]
[Total 8]

The next section starts on the following page.

SOLUTIONS TO PAST EXAM QUESTIONS

The solutions presented here are just outline solutions for you to use to check your answers. See ASET for full solutions.

1 *Subject CT1, April 2011, Question 3*

(i) *The real rate of return*

Date	*Money cashflows*	*RPI*	*Real cashflows*
1/7/05	−135,000	121.4	−135,000
30/6/06	+7,900	125.6	$7{,}900 \times \dfrac{121.4}{125.6} = 7{,}635.828$
30/6/07	+8,400	131.8	$8{,}400 \times \dfrac{121.4}{131.8} = 7{,}737.178$
30/6/08	+8,800	138.7	$8{,}800 \times \dfrac{121.4}{138.7} = 7{,}702.379$
30/6/09	+9,400	145.3	$9{,}400 \times \dfrac{121.4}{145.3} = 7{,}853.820$
30/6/10	+10,100 + 151,000 =161,100	155.2	$161{,}100 \times \dfrac{121.4}{155.2} = 126{,}015.077$

The real rate of return, $\hat{i}$, satisfies the following equation of value:

$$135{,}000 = 7{,}635.828\hat{v} + 7{,}737.178\hat{v}^2 + 7{,}702.379\hat{v}^3 + 7{,}853.820\hat{v}^4 + 126{,}015.077\hat{v}^5$$

where $\hat{v} = \dfrac{1}{1+\hat{i}}$.

Using $\hat{i} = 3\%$, RHS = 137,434.955.

Using $\hat{i} = 3.5\%$, RHS = 134,492.919

Interpolating:

$$\hat{i} = 3\% + \frac{(137{,}434.955 - 135{,}000)}{(137{,}434.955 - 134{,}492.919)} \times 0.5\% = 3.414\%$$

So the real rate of return is approximately 3.4% *pa*.

(ii) *The effect on the real rate of return of a higher RPI in June 2008*

The real value of the June 2008 dividend is calculated as:

$$8,800 \times \frac{121.4}{\text{RPI for June 2008}}$$

So, if the RPI for June 2008 had been higher than 138.7, the real value of the June 2008 payment would have been lower and therefore the real rate of return would have been lower.

2 *Subject CT1, April 2011, Question 5*

(i) *Capital gain*

The investor wishes to make a net (*ie* after-tax) effective return of at least 5%, so $i^{(2)} = 4.939\%$.

The annual coupon per £100 nominal is £6, the investor's income tax rate is 20% and the redeemable value is £105, so:

$$(1-t_1)\frac{D}{R} = 0.8 \times \frac{6}{105} = 4.5714\%$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$, the investor will have to make a capital gain in order to make a net effective return of 5%.

(ii) *Effect of capital gain on assumptions made in calculating the price*

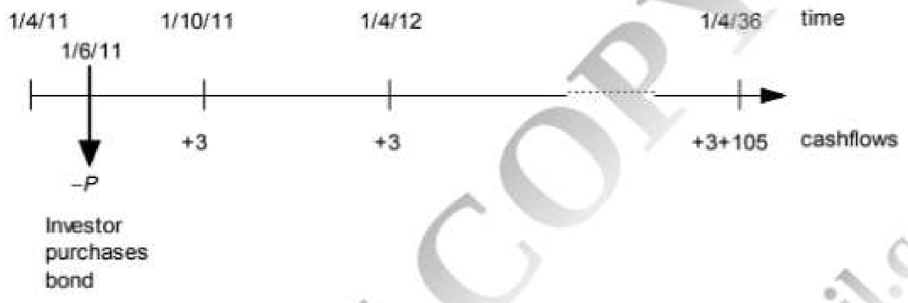
The capital gains test is made for two reasons:

1. to see if the investor needs to pay capital gains tax – in this case, the investor will pay tax at 35% on the capital gain made
2. to work out which of the optional redemption dates we're going to choose when calculating the price – in this case, we assume the latest date possible, ie after 25 years.

Since the investor makes a capital gain, the return would be *greater* the *earlier* the redemption date. Therefore, *assuming the worst case scenario* for the investor, we assume the *latest* redemption date. If we calculate the maximum price that will give the required return to the investor in the worst case scenario, the investor will then earn at least this return whatever the redemption date.

(iii) *The maximum price the investor should pay*

Considering cashflows per £100 nominal of the bond:



We can write the price equation as though the investor were buying it on 1 April 2011 and then multiply it by $1.05^{2/12}$ to accumulate the value forward from 1 April 2011 to 1 June 2011. The maximum price, P, that the investor will be prepared to pay for the bond is:

$$\begin{aligned} P &= \left[(1-t_1)Da_{\overline{n}|}^{(2)} + Rv^n - t_2(R-P)v^n\right](1.05)^{2/12} \text{ @ } 5\% \\ &= \left[0.8\times 6a_{\overline{25}|}^{(2)} + 105v^{25} - 0.35(105-P)v^{25}\right](1.05)^{2/12} \\ &= 4.8a_{\overline{25}|}^{(2)}(1.05)^{2/12} + 105v^{25}(1.05)^{2/12} - 0.35\times 105v^{25}(1.05)^{2/12} + 0.35\times Pv^{25}(1.05)^{2/12} \end{aligned}$$

Rearranging and calculating:

$$P\left(1-0.35v^{25}(1.05)^{2/12}\right) = 4.8a_{\overline{25}|}^{(2)}(1.05)^{2/12} + 0.65\times 105v^{25}(1.05)^{2/12}$$

$$P(1-0.35\times 0.2953\times 1.00816) = (4.8\times 14.0939\times 1.012348\times 1.00816) + (68.25\times 0.2953\times 1.00816)$$

$$P(1-0.1042) = 69.0452 + 20.3187$$

$$\Rightarrow \qquad P = \frac{89.3639}{0.8958} = 99.759$$

The maximum price the investor would pay for the whole £100,000 loan is £99,759.

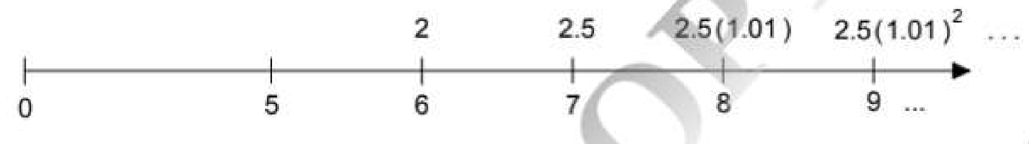
3 *Subject CT1, September 2011, Question 7*

(i) *Gross effective rate of return*

Let P be the sale price of the share in five years' time. If the investment manager buys the share now for 12p, then because no dividends are expected in the next 5 years, the gross rate of return earned on the investment is the interest rate that solves the following equation of value:

$$12(1+i)^5 = P$$

The following timeline shows the expected dividends:



P is the PV at time 5 of the dividends to be received after time 5:

$$P = 2v + 2.5v^2 + 2.5(1.01)v^3 + 2.5(1.01)^2v^4 + \cdots$$
$$= 2v + 2.5v^2\left[1 + (1.01)v + (1.01)^2v^2 + \cdots\right]$$

The infinite summation in the set of square brackets is an infinite geometric progression, with $a = 1$ and $r = (1.01)v$. Hence:

$$P = 2v + 2.5v^2\left(\frac{1}{1-(1.01)v}\right)$$
$$= \frac{2}{1.08} + \frac{2.5}{1.08^2} \times \frac{1}{1-\frac{1.01}{1.08}}$$
$$= 34.9206\text{p}$$

Substituting this value for P into our equation of value, we obtain:

$$12(1+i)^5 = 34.9206 \Rightarrow i = 23.817\%\ pa$$

(ii) *Net effective rate of return*

The capital gain achieved by the investor is $34.9206 - 12 = 22.9206$.

So the investor's CGT liability is 22.9206×0.25.

This tax will need to be paid at time 5, when the share is sold. The net rate of return earned from this investment is the solution of the equation of value:

$$12(1+i)^5 = 34.9206 - 22.9206 \times 0.25 = 29.1905 \quad \Rightarrow \quad i = 19.457\% \ pa$$

(iii) *Net effective real rate of return*

From (ii) we know that the net effective money rate of return (*ie* ignoring any allowance for inflation) is 19.457% *pa*.

Given a constant rate of inflation of 4% *pa*, the net effective real rate of return is:

$$i' = \frac{0.19457 - 0.04}{1.04} = 14.863\% \ pa$$

4 *Subject CT1, April 2012, Question 6*

(i) *The price of the bond on 1 March 2007*

The price, P_1, per £100 nominal of the bond on 1 March 2007 is:

$$\begin{aligned} P_1 &= 5a_{\overline{18|}} + 100v^{18} \ @3.158\% \\ &= £125 \text{ per } £100 \text{ nominal} \end{aligned}$$

(ii) *The price of the bond on 1 March 2012*

The price, P_2, of the bond on 1 March 2012 is:

$$P_2 = 5a_{\overline{13|}} + 100v^{13} \ @5\%$$

If the GRY is 5% and the annual coupon return is 5%, we are earning no capital gain (or loss), so the price we pay must be equal to the redemption.

The price of the bond on 1 March 2012 is £100 per £100 nominal.

(iii) *The gross annual rate of return earned by the investor*

The equation of value is:

$$125 = 5a_{\overline{5}|} + 100v^5 \ @ \ i\%$$

The investor is receiving £5 coupon per year along with £100 at the end of the five-year period. This adds up to £125 with no discounting. Therefore the i that satisfies the equation of value is $i = 0\%$.

(iv) *The effect of a longer term*

In part (i), we calculated the price, P_1, of the bond on 1 March 2007 as:

$$P_1 = 5a_{\overline{18}|} + 100v^{18} \ @ \ 3.158\% = 125$$

With a redemption date of 2035, the new price, P_1' would be found as:

$$P_1' = 5a_{\overline{28}|} + 100v^{28} \ @ \ 3.158\%$$

Since the coupons are being earned for a longer period of time, this will help to offset the capital loss. The investor will be prepared to pay more for the bond, so it will rise in price.

In part (ii), we calculated the price, P_2, of the bond on 1 March 2012 as:

$$P_2 = 5a_{\overline{13}|} + 100v^{13} \ @ \ 5\% = £100$$

With a redemption date of 2035, the new price, P_2' would be found as:

$$P_2' = 5a_{\overline{23}|} + 100v^{23} \ @ \ 5\% = £100$$

The price is unchanged at £100 because the investor is earning an annual return of 5% from the coupons alone.

The longer redemption date has increased the March 2007 price (the purchase price for the investor) but left the March 2012 price unchanged (the selling price for the investor), so the investor would make a larger capital loss. Since the income earned from the coupon payments is unchanged, the investor's return would be lower, *ie* less than 0%.

5 *Subject CT1, April 2012, Question 9*

(i) *The present value of the dividend stream*

The PV of the future dividends on 100 shares on 1 January 2012 is:

$$\begin{aligned} PV &= 35(1.03)v + 35(1.03)(1.05)v^2 + 35(1.03)(1.05)(1.06)v^3 \\ &\quad + 35(1.03)(1.05)(1.06)^2 v^4 + \cdots \\ &= 35(1.03)v + 35(1.03)(1.05)v^2 \\ &\quad + 35(1.03)(1.05)(1.06)v^3(1 + 1.06v + 1.06^2 v^2 + \cdots) \end{aligned}$$

Using the formula for an infinite geometric progression for the final expression:

$$\begin{aligned} PV &= 35(1.03)v\left[1 + 1.05v + (1.05)(1.06)v^2 \times \frac{1}{1-1.06v}\right] \\ &= 35 \times \frac{1.03}{1.08}\left[1 + \frac{1.05}{1.08} + \frac{1.05 \times 1.06}{1.08^2} \times \frac{1}{1-\frac{1.06}{1.08}}\right] \\ &= £1{,}785.81 \end{aligned}$$

(ii) *The expected real rate of return*

The following table shows the real and money cashflows per 100 shares.

Date	Money cashflows (£)	Inflation index	Real cashflows (£)
1/1/12	$-1{,}720.00$	110.0	$-1{,}720.00$
31/12/12	$+35(1.03)$ $= +36.05$	112.3	$36.05\times\dfrac{110}{112.3}=35.31167$
31/12/13	$+35(1.03)(1.05)$ $= +37.8525$	113.2	$37.8525\times\dfrac{110}{113.2}=36.78246$
31/12/14	$+35(1.03)(1.05)(1.06)$ $= +40.12365$	113.8	$40.12365\times\dfrac{110}{113.8}=38.78384$
1/1/15	$+1{,}800.00$	113.8	$1{,}800.00\times\dfrac{110}{113.8}=1{,}739.89455$

The real rate of return, i', satisfies the following equation of value:

$$1{,}720 = 35.31167v + 36.78246v^2 + 1{,}778.67840v^3$$

where $v = \dfrac{1}{1+i'}$.

Using $i' = 2.5\%$, RHS = 1,721.14.
Using $i' = 3\%$, RHS = 1,696.70.

Interpolating gives:

$$i' = 2.5\% + \frac{1{,}720 - 1{,}721.14}{1{,}696.70 - 1{,}721.14}(3\% - 2.5\%) = 2.52\%$$

6 *Subject CT1, April 2013, Question 4*

(i) *Price per share*

	30×1.05	30×1.05^2	30×1.05^3	30×1.05^4	...	Payments
0	1	2	3	4	...	Time

The present value of the future dividends is:

$$PV = 30 \times 1.05v + 30 \times 1.05^2 v^2 + 30 \times 1.05^3 v^3 + \cdots$$
$$= 30 \times 1.05v \left[1 + (1.05v) + (1.05v)^2 + (1.05v)^3 + \cdots \right]$$

Summing using a geometric series:

$$PV = \frac{30 \times 1.05v}{1 - 1.05v} = \frac{30 \times 1.05/1.09}{1 - 1.05/1.09} = 787.5 \text{ pence}$$

(ii)(a) *Higher dividend growth rate*

If the future dividends grow more quickly, they will have a higher present value. So the price the investor is prepared to pay will increase.

(ii)(b) *Higher general future inflation*

The calculation is not affected by the general level of inflation. So theoretically the price the investor is prepared to pay will be the same.

However, if the company can increase its prices in line with the new higher level of inflation, it may be able to generate higher profits, and hence pay higher dividends. In this case the dividend growth rate will increase, and the price the investor should pay may go up.

But this is only an increase in value in nominal terms. If the investor wishes to maintain his *real* rate of return (which is probably more likely), he will now need a nominal return higher than 9%. So the price that he is prepared to pay would stay about the same, depending on the nominal rate of return used in the calculation.

(ii)(c) *Higher general economic uncertainty*

The level of uncertainty is not built into the model. So again the price the investor is prepared to pay will remain the same.

However, in this case the investor may require a higher rate of return to compensate him for the greater uncertainty. If this is true, the price he is prepared to pay should decrease.

7 *Subject CT1, April 2013, Question 9*

(i) *Price at which the investor bought*

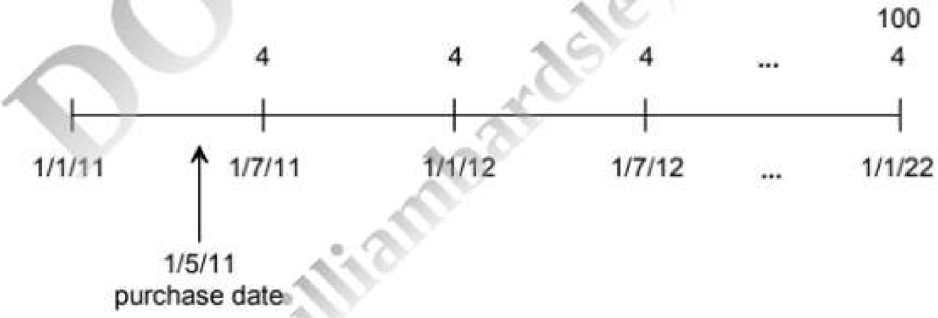
Applying the capital gains tax test, we have:

$$i^{(2)} \text{ at } 6\% = 2\left(1.06^{\frac{1}{2}} - 1\right) = 0.059126$$

and: $(1-t_1)\dfrac{D}{R} = 0.7 \times \dfrac{8}{100} = 0.056$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$ the investor will need to make a capital gain in order to achieve his 6% yield.

So the worst case scenario for the investor is for this capital gain to be deferred for as long as possible. We assume for the purposes of the calculation that the bond is redeemed at the last possible date, on 1 January 2022.



$$P = (1-t_1)(1+i)^{\frac{4}{12}} 8 a_{\overline{11}|}^{(2)} + 100 v^{10\frac{8}{12}} - t_2(100-P)v^{10\frac{8}{12}}$$
$$= 5.6(1+i)^{\frac{4}{12}} a_{\overline{11}|}^{(2)} + 100 v^{10\frac{8}{12}} - 0.25(100-P)v^{10\frac{8}{12}}$$

Rearranging and evaluating at 6% gives:

$$P\left(1-0.25v^{10\frac{8}{12}}\right)=5.6(1.06)^{\frac{4}{12}}\frac{1-1.06^{-11}}{0.059126}+75v^{10\frac{8}{12}}$$

$$0.8657202P=45.69838+40.28395$$

$$P=\frac{85.98233}{0.8657202}=£99.31885 \text{ per } £100 \text{ nominal}$$

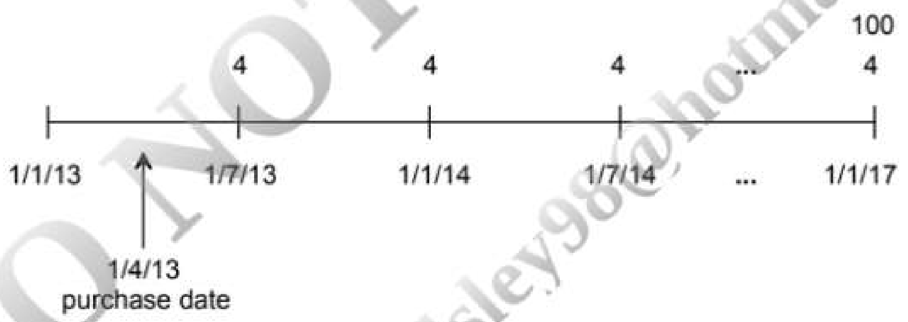
(ii) *Price at which the investor sold*

Carrying out the capital gains tax test again:

$$i^{(2)} \text{ at } 7\% = 2\left(1.07^{½}-1\right)=0.068816$$

and: $(1-t_1)\dfrac{D}{R}=1\times\dfrac{8}{100}=0.08$

Since $i^{(2)}<(1-t_1)\frac{D}{R}$ the fund must make a capital loss to achieve its 7% return. The worst case scenario for the fund will therefore be for the bond to be redeemed as soon as possible. So we must now assume that redemption takes place on 1 January 2017.



The equation of value for the fund is:

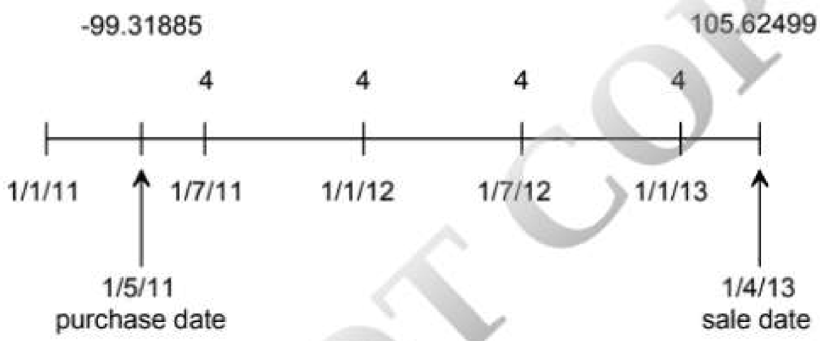
$$P=(1+i)^{\frac{3}{12}}8a_{\overline{4}|}^{(2)}+100v^{3\frac{9}{12}}$$

Evaluating this at 7% gives:

$$\begin{aligned} P &= 8\times 1.07^{\frac{1}{4}}\,\frac{1-1.07^{-4}}{0.068816}+100v^{3\frac{3}{4}} \\ &= 28.03408+77.59091 \\ &= £105.62499 \text{ per } £100 \text{ nominal} \end{aligned}$$

(iii) *Net effective yield*

The cashflows (before tax) for the first investor are as follows:



So the equation of value is:

$$\begin{aligned} 99.31885 &= 5.6(1+i)^{\frac{4}{12}}a^{(2)}_{\overline{2}|}+105.62499v^{1\frac{11}{12}} \\ &\qquad -0.25(105.62499-99.31885)v^{1\frac{11}{12}} \\ &= 5.6(1+i)^{\frac{4}{12}}a^{(2)}_{\overline{2}|}+104.04845v^{1\frac{11}{12}} \end{aligned}$$

Evaluating the right hand side of this equation at both 8% and 9%:

$$i = 8\% \quad \Rightarrow \quad RHS = 5.6\times 1.08^{4/12}\times 1.8182439+89.778722 = 100.226$$

$$i = 9\% \quad \Rightarrow \quad RHS = 5.6\times 1.09^{4/12}\times 1.797839+88.206684 = 98.568$$

We see that the price actually paid by the investor (99.31885) lies in between these two figures. Therefore the yield obtained by the investor must lie between 8% and 9% per annum.

8 *Subject CT1, September 2013, Question 3*

We are given $i^{(2)} = 0.03$, so:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = \left(1 + \frac{0.03}{2}\right)^2 - 1 = 3.0225\%$$

(i) *Price at outset*

$$\begin{aligned} P &= 4a^{(2)}_{\overline{10}|} + 100v^{10} \\ &= 4\frac{1 - 1.030225^{-10}}{0.03} + 100(1.030225)^{-10} \\ &= 34.3373 + 74.2470 \\ &= £108.58 \end{aligned}$$

(ii) *Price after 91 days*

Evaluating the cashflows at the outset and then accumulate for ¼ of a year:

$$\begin{aligned} P &= \left(4(0.75)a^{(2)}_{\overline{10}|} + 100v^{10}\right)(1+i)^{0.25} \\ &= \left(4 \times 0.75\frac{1 - 1.030225^{-10}}{0.03} + 100(1.030225)^{-10}\right)(1.030225)^{0.25} \\ &= (25.7530 + 74.2470)(1.030225)^{0.25} \\ &= £100.75 \end{aligned}$$

9 ***Subject CT1, September 2013, Question 5***

(i) *Price of bond issued by Country B*

For Country A, the rate of return is:

$$101 = 106v \quad \Rightarrow \quad i = 4.9505\%$$

For Country B there are four possible outcomes (as listed in the question):

PV	PV evaluated at $i = 4.9505\%$	Probability
0	0	0.1
$100v$	95.2830	0.2
$50v$	47.6415	0.3
$106v$	101	0.4

$$\begin{aligned} EPV &= \sum PV \times prob \\ &= 95.2830 \times 0.2 + 47.6415 \times 0.3 + 101 \times 0.4 \\ &= €73.75 \end{aligned}$$

So the price paid for the bond issued by Country B is €73.75.

(ii) *GRY*

We can calculate the GRY using the price and the cashflow:

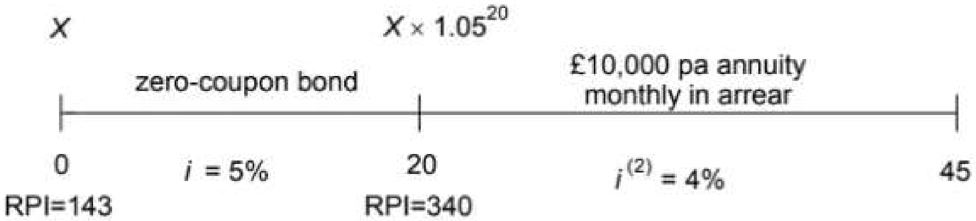
$$73.75 = 106v \quad \Rightarrow \quad i = 43.7307\%$$

(iii) *Higher return from Country B*

There is more uncertainty about the cashflows for the bond from Country B. This means there is more risk and investors will want a higher expected return to compensate for that risk.

10 *Subject CT1, September 2013, Question 8*

Let X be the amount of money Mrs Jones invests. A timeline showing the investment described in the question is as follows:



(i) *Money invested*

The present value of the annuity at time 20 is:

$$10{,}000a_{\overline{25|}}^{(12)}$$

We are given $i^{(2)} = 0.04$, so:

$$i = \left(1+\frac{i^{(2)}}{2}\right)^2 - 1 = \left(1+\frac{0.04}{2}\right)^2 - 1 = 4.04\%$$

$$i^{(12)} = 12\left((1+i)^{\frac{1}{12}} - 1\right) = 12\left((1.0404)^{\frac{1}{12}} - 1\right) = 3.96707\%$$

Equating the accumulated value of the investment to the present value of the annuity, we get:

$$X(1.05)^{20} = 10{,}000a_{\overline{25|}}^{(12)} = 10{,}000\frac{1-(1+i)^{-25}}{i^{(12)}}$$

$$= 10{,}000\frac{1-(1.0404)^{-25}}{0.0396707} = 158{,}422.31$$

Solving this equation:

$$X = 158{,}422.31(1.05)^{-20} = 59{,}707.70$$

In other words Mrs Jones has to invest £59,708 (5SF) in the zero-coupon bond.

(ii) *Annual effective real yield*

The annual rate of inflation over the 20-year period is:

$$j = \left(\frac{340}{143}\right)^{\frac{1}{20}} - 1 = 4.4256\%$$

The real yield is:

$$i' = \frac{i-j}{1+j} = \frac{0.05 - 0.044256}{1.044256} = 0.550\%$$

(iii) *Net effective real yield*

The capital gain is:

$$X(1.05)^{20} - X = 59{,}708\left(1.05^{20} - 1\right) = 98{,}714.60$$

The capital gains tax is 25% of this, so the proceeds from the zero-coupon band are therefore:

$$X(1.05)^{20} - 0.25 \times 98{,}714.60 = 133{,}743.65$$

The return on the bond is given by i, where:

$$X(1+i)^{20} = 133{,}743.65 \quad \Rightarrow \quad i = 4.1147\%$$

The real yield is:

$$\frac{i-j}{1+j} = \frac{0.041147 - 0.044256}{1.044256} = -0.298\%$$

(iv) *Why a negative real rate of return*

The rate of inflation (4.4256%) exceeds the nominal return on the money invested (4.1147%), which leads to a negative real rate of return. Effectively the tax and inflation have eroded the real value of the investor's capital.

11 ***Subject CT1, April 2014, Question 4***

Using the capital gains tax test:

$$i^{(2)} \text{ at } 6\% = 0.059126$$

$$(1-t_1)\frac{D}{R} = 0.7 \times \frac{8}{103} = 0.054369$$

Since $i^{(2)} > (1-t_1)\frac{D}{R}$ we have a capital gain. The worst case would be for this gain to be deferred for as long as possible, *ie* 25 years.

Allowing for both income and capital gains tax, we have:

$$\begin{aligned} P &= 0.7 \times 8a_{\overline{25}|}^{(2)} + 103v^{25} - 0.4(103-P)v^{25} \qquad @6\%\ pa \\ &= 5.6a_{\overline{25}|}^{(2)} + 103v^{25} - 0.4(103-P)v^{25} \\ &= 5.6 \times \frac{1-v^{25}}{i^{(2)}} + 103v^{25} - 41.2v^{25} + 0.4Pv^{25} \end{aligned}$$

Rearranging this:

$$\begin{aligned} P(1-0.4v^{25}) &= 5.6a_{\overline{25}|}^{(2)} + 61.8v^{25} \\ P(1-0.4 \times 1.06^{-25}) &= 5.6 \times 12.972313 + 14.399315 \\ P &= 95.9905 \end{aligned}$$

The price is £95.99 per £100 nominal of stock.

12 *Subject CT1, April 2014, Question 5*

(i) *Coupon and redemption payments*

Since the coupon rate is 3% *pa*, we actually receive 1.5% (indexed) per half year. So the coupon payment received on 25 April 2013 is:

$$0.015 \times \frac{171.4}{149.2} \times 10{,}000 = £172.319$$

The coupon payment received on 25 October 2013 is:

$$0.015 \times \frac{173.8}{149.2} \times 10{,}000 = £174.732$$

The redemption proceeds also received on 25 October 2013 are:

$$\frac{173.8}{149.2} \times 10{,}000 = £11{,}648.794$$

(ii) *Purchase price*

We first calculate the real values as at 25 October 2012 of the 2013 cashflows.

The April 2013 coupon has real value:

$$172.319 \times \frac{169.4}{171.4} = 170.308$$

The total of the final coupon and redemption proceeds (in real values) is:

$$(174.732 + 11{,}648.794) \times \frac{169.4}{173.8} = 11{,}524.196$$

So the real equation of value, using the effective real yield of 3.5% *pa*, is:

$$\begin{aligned} P &= 170.308v^{0.5} + 11{,}524.196v^{1} \\ &= 170.308 \times 1.035^{-0.5} + 11{,}524.196 \times 1.035^{-1} \\ &= 11{,}301.893 \end{aligned}$$

So the purchase price is £11,301.89.

13 *Subject CT1, September 2014, Question 9*

(i) *Net cashflows*

Coupons are 2% each year, so 1% each half year with tax of 25% on them. So for £3.5m nominal the cashflows are:

1/6/12	1/12/12	1/6/13	1/12/13	1/6/14
	$3.5m \times 0.01 \times 0.75$	$3.5m \times 0.01 \times 0.75$	$3.5m \times 0.01 \times 0.75$	$3.5m + 3.5m \times 0.01 \times 0.75$
112	116	117	120	121

We now need to inflate these cashflows using the RPI figures given below each cashflow (which have a three month lag).

The first incoming net cashflow is the coupon paid in December 2012. This is (working in millions of pounds):

$$C_1 = 3.5 \times 0.01 \times 0.75 \times \frac{116}{112} = 0.0271875$$

Similarly, the coupon paid in June 2013 will be:

$$C_2 = 3.5 \times 0.01 \times 0.75 \times \frac{117}{112} = 0.0274219$$

The coupon paid in December 2013 will be:

$$C_3 = 3.5 \times 0.01 \times 0.75 \times \frac{120}{112} = 0.0281250$$

The coupon paid in June 2014 will be:

$$3.5 \times 0.01 \times 0.75 \times \frac{121}{112} = 0.0283594$$

The redemption proceeds (which are not subject to income tax), will be:

$$3.5 \times 1 \times \frac{121}{112} = 3.78125$$

So the total final cashflow is:

$$C_4 = 0.0283594 + 3.78125 = 3.8096094$$

(ii) *In terms of 1 June 2012 prices*

We now convert these to real values (based on 1 June 2012), using the RPI ratios given below each cashflow in this diagram (which has no time lag).

	0.0271875	0.0274219	0.0281250	3.8096094
1/6/12	1/12/12	1/6/13	1/12/13	1/6/14
113	117	118	121	122

The real values in June 2012 terms of the cash flows are:

$$RC_1 = \frac{0.0271875}{117/113} = 0.0262580$$

$$RC_2 = \frac{0.0274219}{118/113} = 0.0262599$$

$$RC_3 = \frac{0.0281250}{121/113} = 0.0262655$$

$$RC_4 = \frac{3.8096094}{122/113} = 3.5285726$$

(iii) *Purchase price*

The equation of value for the price of the bond, working in real terms, is:

$$
\begin{aligned}
P &= RC_1 \times 1.015^{-0.5} + RC_2 \times 1.015^{-1} + RC_3 \times 1.015^{-1.5} \\
&\qquad + RC_4 \times 1.015^{-2} \\
&= 0.0262580 \times 1.015^{-0.5} + 0.0262599 \times 1.015^{-1} + 0.0262655 \times 1.015^{-1.5} \\
&\qquad + 3.5285726 \times 1.015^{-2} \\
&= 3.50267
\end{aligned}
$$

So for £100 nominal (rather than £3.5m nominal) the price is:

$$3.50267 \times \frac{100}{3.5} = £100.08$$

(iv) *Explanation*

If there had been no time lag, the investor's expected net real rate of return would be unchanged. The higher inflation would increase the cashflows received by the investor, and then the real values would equal the original nominal payments.

However, the investor is not protected against inflation during the last three months for which the bond is held, due to the time lag. Since the inflation in this period is higher than expected, the investor's real rate of return will fall a little. However, the effect is unlikely to be great, as the lag is not that great.

(v) *Impact on market prices*

The actual inflation experienced by an investor would have been unaltered. But the money received would be less under the new arrangement.

So an investor's real return from buying such a bond would fall. This would make the bonds less attractive to these investors, and so the price they would be prepared to pay would have fallen. Hence, the market prices would have fallen.

14 *Subject CT1, April 2015, Question 6*

The real equation of value for the cashflows is:

$$PV = \frac{6}{1.04^{½}}v^{½} + \frac{6\times1.06}{1.04^{1½}}v^{1½} + \frac{6\times1.06^2}{1.04^{2½}}v^{2½} + \cdots$$

So we have:

$$175 = \frac{\dfrac{6}{1.04^{½}}v^{½}}{1-\dfrac{1.06v}{1.04}} = \frac{6v^{½}\times1.04^{½}}{1.04-1.06v}$$

At 5%, the value of the expression on the right hand side of our equation is 195.935. At 6%, the corresponding value is 148.578. So the real return is between these two values.

At 5.4%, we obtain a value of 173.724, and at 5.3% we obtain 178.784.

Interpolating between these two figures, we find that the effective real rate of return is approximately:

$$5.3\% + \frac{178.784-175}{178.784-173.724}\times0.1\% = 5.37\%\ pa$$

15 *Subject CT1, September 2015, Question 7*

(i) *Maximum rate of return*

The maximum rate of return will be received after 20 years.

The equation of value is:

$$100 = 0.75\times4a_{\overline{10}|}^{(2)} + 0.75\times7v^{10}a_{\overline{10}|}^{(2)} + 130v^{20}$$

Simplifying this gives:

$$100 = (3 + 5.25v^{10})a_{\overline{10}|}^{(2)} + 130v^{20}$$

Using trial and error:

$$i = 5\% \quad \Rightarrow \quad RHS = 97.642$$
$$i = 4\% \quad \Rightarrow \quad RHS = 112.956$$
$$i = 4.5\% \quad \Rightarrow \quad RHS = 104.953$$

Interpolating between these gives:

$$i = 4.5\% + \frac{100 - 104.953}{97.642 - 104.953}(5\% - 4.5\%) = 4.84\% \text{ pa}$$

(ii) *Minimum rate of return*

The minimum rate of return will be received after 10 years.

The equation of value is:

$$100 = 0.75 \times 4a_{\overline{10}|}^{(2)} + 100v^{10}$$

Simplifying this gives:

$$100 = 3a_{\overline{10}|}^{(2)} + 100v^{10}$$

Since there is no capital gain or loss, we have:

$$i^{(2)} = \frac{D}{R} \quad \Rightarrow \quad i^{(2)} = \frac{3}{100} = 0.03$$

This gives:

$$1 + i = \left(1 + \frac{0.03}{2}\right)^2 \quad \Rightarrow \quad i = 3.0225\%$$

(iii) *Calculate the expected rate of return*

The expected return is:

$$(0.5 \times 4.84) + (0.5 \times 3.02) = 3.93\% \ pa$$

(iv) *Rate of return if the investor purchases the whole loan*

The equation of value is:

$$100 = 0.5\left[3a^{(2)}_{\overline{10}|} + 100v^{10}\right] + 0.5\left[(3 + 5.25v^{10})a^{(2)}_{\overline{10}|} + 130v^{20}\right]$$

Simplifying this gives:

$$100 = (3 + 2.625v^{10})a^{(2)}_{\overline{10}|} + 50v^{10} + 65v^{20}$$

Substituting $i = 3.93\%$ *pa* into the RHS of the above equation gives:

$$RHS = (3 + 2.625 \times 1.0393^{-10})\frac{1 - 1.0393^{-10}}{2(1.0393^{½} - 1)} + 50 \times 1.0393^{-10} + 65 \times 1.0393^{-20}$$

$$= €103.72$$

Since this is greater than €100 the rate of return must be greater than 3.93% *pa*, *ie* it is greater than the average rate of return an individual who purchases a single bond will receive.

16 *Subject CT1, September 2015, Question 8*

(i) *Characteristics of an equity*

Any EIGHT of the following:

- Securities that are held by the owners of an organisation.
- Entitles their holders to a share in the net profits of the company in proportion to the number of shares owned.
- The distribution of profits to shareholders takes the form of regular payments of dividends.
- The cash paid out each year (dividend) depends on the company's profits and how much is retained by the company (which is at the discretion of the directors).

- Since they are related to the company profits that are not known in advance, dividend rates are variable.
- The return on ordinary shares is made up of two components, the dividends received and any increase in the market price of the shares.
- Higher return than for most other classes of security to compensate for the greater risk of default, and for the variability of returns.
- Ordinary shares are the lowest ranking form of finance issued by companies (so only receive payment in a windup after all other creditors have been paid).
- The initial running yield on ordinary shares is low but dividends should increase with inflation and real growth in a company's earnings.
- Marketability of ordinary shares varies according to the size of the company.
- Shareholders get voting rights in proportion to the number of shares held, so shareholders may have the ability to influence the decisions taken by the directors and managers of the company.

(ii) *General formula*

We will assume that dividends are paid annually with the next dividend due in exactly one year.

Using d for the next dividend payment, g for the expected annual growth rate and t for the tax rate on the dividends, we have the following equation of value:

$$P = d(1-t)v + d(1-t)(1+g)v^2 + d(1-t)(1+g)^2v^3 + \cdots$$

Summing this geometric series gives:

$$P = \frac{d(1-t)v}{1-(1+g)v}$$

Let i be the interest rate. Simplifying we get:

$$P = \frac{d(1-t)}{(1+i)-(1+g)} = \frac{d(1-t)}{i-g}$$

(iii) *Value to the investor*

We have $d=6$, $g=0.01$, $t=0.2$ and $i=0.06$. Substituting these into our formula from part (i) gives:

$$P=\frac{6\times 0.8}{0.06-0.01}=96$$

So the value of the share to the investor is 96 pence.

(iv) *Explain*

Investors will demand more return to compensate them for the increased risk.

This will reduce the value of the share (as the denominator will increase).

(v) *Net real return*

The net payment will be $0.8\times 6+120-0.25(120-96)=118.8$. So the real equation of value (in 2014 prices) is:

$$96=118.8v\times\frac{123}{126}$$

Simplifying and rearranging gives:

$$96=115.97v$$
$$1+i=\frac{115.97}{96}\quad\Rightarrow\quad i=0.208$$

So the net real rate of return is 20.8% *pa*.

17 ***Subject CT1, April 2016, Question 4***

(i) *Price to obtain at least 5% pa*

Working in units of £100 nominal, we have $D=7$, $R=108$, $t_1=0.25$, $i=5\%pa$ and $p=4$. So:

$$(1-t_1)\frac{D}{R}=0.75\times\frac{7}{108}=0.048611$$

We compare this with $i^{(4)}$ at 5%:

$$i^{(4)} = 4\left[(1+i)^{1/4} - 1\right] = 4\ [1.05^{0.25} - 1] = 0.049089$$

Since the second amount is a little bigger, the investor will make a small capital gain.

In order to ensure that the investor always makes at least 5% pa, we need to consider the investor's worst case scenario. The worst case is that the gain is deferred for as long as possible, and so we should assume that the bond is redeemed at the latest possible date.

Assuming that the bond is redeemed on the last redemption date (after 20 years), we have the following equation of value for the price of the bond:

$$P = 0.75 \times 7a_{\overline{20}|}^{(4)} + 108v^{20} - 0.4 \times (108 - P)v^{20}$$

Evaluating the annuity at 5% *pa*, we have:

$$a_{\overline{20}|}^{(4)} = \frac{1 - v^{20}}{i^{(4)}} = \frac{1 - 1.05^{-20}}{0.049089} = 12.693502$$

So we have:

$$P = 5.25 \times 12.693502 + 108 \times 1.05^{-20} - 0.4 \times (108 - P) \times 1.05^{-20}$$

Rearranging this equation, we find that:

$$0.849244P = 91.063324$$

This gives us $P = 107.229$, or about £107.23 per £100 nominal.

So the price is £107,229 per £100,000 nominal.

(ii) *Significance of the borrower holding the option*

The calculation in part (i) does not depend on which party has the option.

In either case, we want to calculate the worst case scenario for the investor, in order to defer the investor's capital gain for as long as possible. This will be the case whether the option is with the investor or with the borrower.

18 *Subject CT1, April 2016, Question 9*

(i) *Schedule of cashflows*

The annual nominal coupon rate is 6% *pa*. So each half year the investor will receive a coupon of nominal value £3,000 per £100,000 nominal. However, the actual amounts of these coupons will be indexed using the inflation index, with a time-lag of six months. So the first coupon (payable in July 2014) will be for an amount of:

$$3{,}000 \times \frac{I_{July\ 2014\ -\ 6\ months}}{I_{January\ 2014\ -\ 6\ months}} = 3{,}000 \times \frac{122.3}{120.0} = 3{,}057.50$$

where I is the inflation index. The other cashflows are calculated similarly.

The dates and amounts of the cashflows are summarised as follows:

Date	*Amount*
January 2014	–£97,000
July 2014	$3{,}000 \times \frac{122.3}{120} = £3{,}057.50$
January 2015	$3{,}000 \times \frac{124.9}{120} = £3{,}122.50$
July 2015	$3{,}000 \times \frac{127.2}{120} = £3{,}180$
January 2016	$103{,}000 \times \frac{129.1}{120} = £110{,}810.83$

(ii) *Effective real yield*

Using time units of a year, and working for the moment using £100 nominal of the bond, the real equation of value is:

$$97 = \frac{3.0575}{124.9/122.3} v^{½} + \frac{3.1225}{127.2/122.3} v^{1} + \frac{3.180}{129.1/122.3} v^{1½} + \frac{110.81083}{131.8/122.3} v^{2}$$

Working out these numerical values, we obtain:

$$97 = 2.993853v^{½} + 3.002215v + 3.012502v^{1½} + 102.823709v^{2}$$

Using trial and improvement, and starting with a first guess of $i = 7\%$ *pa*, we find that:

i	RHS of equation
0.07	98.232
0.08	96.499
0.0775	96.928
0.0765	97.100

We see from these values that the real annual rate of return lies between 7.65% *pa* and 7.75% *pa*. So, to the nearest 0.1% *pa*, the real return is 7.7% *pa*.

19 *Subject CT1, April 2016, Question 11*

(i) *Maximum price for the shares*

Working in time units of a half-year, and in amount units of pence, the present value of all the future dividends from one share can be written as:

$$PV = 6.5v + 6.5 \times 1.02v^2 + 6.5 \times 1.02^2 v^3 + \cdots$$

This is an infinite geometric progression, with first term $a = 6.5v$ and common ratio $r = 1.02v$. So the sum to infinity is:

$$PV = \frac{a}{1-r} = \frac{6.5v}{1-1.02v}$$

Evaluating this at $i = 6\%$ *pa*, we have:

$$PV = \frac{6.5/1.06}{1-1.02/1.06} = 162.5 \text{ pence}$$

So the maximum price an investor should pay for 10,000 shares is £16,250.

(ii)(a) *New maximum price*

Using the same approach as before, the new maximum price of a share is:

$$PV = \frac{6.5/1.06}{1-1.025/1.06} = 185.714 \text{ pence}$$

So the new maximum price for 10,000 shares is £18,571.

(ii)(b) *Difference*

If the growth rate is greater, the investor will receive more dividends in the future. So the investor is prepared to pay a higher price for the shares.

(iii)(a) *Why increase the required rate of return?*

The new legislation may adversely affect the operation of the company. If it does so, the investor will require a higher rate of return to compensate for the additional risk of investing in the company.

(iii)(b) *New maximum price*

The maximum price payable is now:

$$PV = \frac{6.5/1.07}{1-1.02/1.07} = 130 \text{ pence}$$

So the price payable for 10,000 shares is now £13,000.

(iii)(c) *Comparison with part (i)*

The price payable is now less than in part (i). If she requires a higher return on her investment, she needs to pay less now for the same quantity of shares. The greater uncertainty of return is compensated for by a lower entry price for the investment.

(iv)(a) *Explanation*

If investors' expectations for future inflation are reduced, then the likely future rate of dividend growth will also be reduced. In times of higher inflation, companies are more able to pass this inflation on the customers by raising their prices. These higher prices should in the long term lead to greater company profits, and hence to a higher rate of growth in the dividends that the company is able to pay.

Higher inflation is also one of the risks facing the investor, since higher inflation reduces the real value of the returns from her investment. So if expectations of inflation are reduced, the inflation risk facing the investor is also reduced. So she can reduce the required rate of return, as the inflation risk she is facing is lower.

(iv)(b) *New maximum price*

The maximum price payable is now:

$$PV = \frac{6.5/1.05}{1-1.01/1.05} = 162.5 \text{ pence}$$

So the price payable for 10,000 shares is now £16,250.

(iv)(c) *Explanation of the difference*

In both cases the maximum price the investor is prepared to pay is the same.

In part (i), we can rewrite the maximum price as:

$$PV = \frac{6.5/1.06}{1-1.02/1.06} = \frac{6.5}{1.06-1.02}$$

In part (iv)(b), we can rewrite the maximum price as:

$$PV = \frac{6.5/1.05}{1-1.01/1.05} = \frac{6.5}{1.05-1.01}$$

In each case we see that the maximum price is determined by the difference between the assumed dividend growth rate and the required rate of return. In both cases here this difference is 4%. So the price the investor should be prepared to pay is the same.

This is intuitively reasonable. An equity share is a real investment. In both cases the investor is expecting a real return of about 4% *pa*. So the fact that the investor is prepared to pay the same price in both cases is not unreasonable.

20 *Subject CT1, September 2016, Question 5*

The zero-coupon bond has a term of 40 years. So, working in terms of £100 nominal, the present value of the redemption proceeds is:

$$PV = 100v^{40}$$

At an interest rate of $i^{(2)} = 0.05$, we have an effective annual rate of:

$$i = \left(1 + \frac{0.05}{2}\right)^2 - 1 = 5.0625\% \ pa$$

So:

$$PV = 100 \times 1.050625^{-40} = 13.87046$$

The price paid for the bond is £13.87 per £100 nominal.

Now assume that the default occurs at time t. Then the equation of value for the actual transaction (using the force of interest, which is what we are given), is:

$$80\ e^{-0.048t} = 13.87046$$

We can solve this equation by dividing through by 80 and taking logs of both sides:

$$t = \frac{1}{-0.048} \ln\left(\frac{13.87046}{80}\right) = 36.50553$$

Since the 37th year of the bond's life is 2011, we don't need to worry about leap years. Converting 0.50553 of a year into days by multiplying by 365, we have a default time of 36 years and 184.519 days.

The 185th day of 2011 is 4th July. So the date on which the borrower defaulted is 4 July 2011.

21 *Subject CT1, April 2017, Question 7*

(i) *Net redemption yield*

Allowing for income tax at a rate of 30% and capital gains tax at a rate of 40%, and for the delay in the collection of the tax payments, the investor's equation of value is:

$$9{,}800 = 400a_{\overline{20}|}^{(2)} + 10{,}500v^{20} - 0.3 \times 400v^{\frac{5}{12}}a_{\overline{20}|} - 0.4(10{,}500 - 9{,}800)v^{20\frac{5}{12}}$$

ie $\quad 9{,}800 = 400a_{\overline{20}|}^{(2)} + 10{,}500v^{20} - 120v^{\frac{5}{12}}a_{\overline{20}|} - 280v^{20\frac{5}{12}}$

When $i = 3\%$:

$$a_{\overline{20}|} = \frac{1 - 1.03^{-20}}{0.03} = 14.87747 \quad \text{and} \quad a_{\overline{20}|}^{(2)} = \frac{1 - 1.03^{-20}}{2(1.03^{1/2} - 1)} = 14.98823$$

So, the right-hand side of the above equation gives:

$$400 \times 14.98823 + 10{,}500(1.03)^{-20} - 120(1.03)^{-\frac{5}{12}} \times 14.87747 - 280(1.03)^{-20\frac{5}{12}}$$

$= 9{,}892.3126$

When $i = 4\%$:

$$a_{\overline{20}|} = \frac{1 - 1.04^{-20}}{0.04} = 13.59033 \quad \text{and} \quad a_{\overline{20}|}^{(2)} = \frac{1 - 1.04^{-20}}{2(1.04^{1/2} - 1)} = 13.72490$$

So, the right-hand side of the above equation gives:

$$400 \times 13.72490 + 10{,}500(1.04)^{-20} - 120(1.04)^{-\frac{5}{12}} \times 13.59033 - 280(1.04)^{-20\frac{5}{12}}$$

$= 8{,}551.9001$

Since the value of the right-hand side at 3% exceeds £9,800, but the value at 4% is less than £9,800, the investor's net redemption yield must lie between 3% *pa* effective and 4% *pa* effective.

(ii) *Net real redemption yield*

Using the values obtained in part (i) and linear interpolation, the net redemption yield obtained by the investor is:

$$i \approx 3\% + \left(\frac{9{,}800 - 9{,}892.3126}{8{,}551.9001 - 9{,}892.3126} \right)(4\% - 3\%) = 3.069\%$$

So, assuming an annual rate of inflation of 2%, the investor's net real redemption yield is:

$$i' = \frac{0.03069 - 0.02}{1.02} = 1.05\% \ pa$$

(iii) *Effect of tax being collected on 1 April rather than 1 June*

If tax were collected on 1 April each year, rather than on 1 June, the tax payments will have been brought forward. This would increase the present value of the tax payments.

Since the tax payments are outgo from the investor's point of view, increasing their present value will cause the net redemption yield and the net real redemption yield to fall.

22 *Subject CT1, April 2018, Question 2*

(i) *Definition of ex-dividend*

The term 'ex-dividend' relates to an equity (or share) and means that the equity is offered for sale without the next dividend (*ie* the purchaser will not receive the next dividend to be paid, but will receive those paid after that).

(ii) *Present value of the investment*

As the share is sold ex-dividend, the first dividend received by the investor will be \$0.07 per share on 1 July 2018. The dividend is expected to increase at the end of each year by 2% *pa*, so the dividends payable in 2019 (on 1 January and 1 July) will both be $\$0.07 \times 1.02$ per share; those payable in 2020 will both be $\$0.07 \times 1.02^2$ per share, and so on.

The present value on 1 December 2017 of the dividends received from 10,000 shares is:

$$PV = 10{,}000\Big[0.07v^{\frac{7}{12}} + 0.07\times1.02\Big(v^{1\frac{1}{12}} + v^{1\frac{7}{12}}\Big)$$
$$+0.07\times1.02^2\Big(v^{2\frac{1}{12}} + v^{2\frac{7}{12}}\Big) + \cdots\Big]$$
$$= 700v^{\frac{7}{12}} + 700\times1.02\Big[\Big(v^{1\frac{1}{12}} + v^{1\frac{7}{12}}\Big) + 1.02v\Big(v^{1\frac{1}{12}} + v^{1\frac{7}{12}}\Big) + \cdots\Big]$$
$$= 700v^{\frac{7}{12}} + 700\times1.02\Big(v^{1\frac{1}{12}} + v^{1\frac{7}{12}}\Big)(1 + 1.02v + \cdots)$$

The infinite summation in the second term can be evaluated using the formula for the sum to infinity of a geometric progression, $\frac{a}{1-r}$, with first term $a = 1$ and common ratio $r = 1.02v$. This gives:

$$PV = 700v^{\frac{7}{12}} + 700\times1.02\Big(v^{1\frac{1}{12}} + v^{1\frac{7}{12}}\Big)\times\frac{1}{1-1.02v}$$

Using an annual effective interest rate of 7%, we have:

$$PV = 700\times1.07^{-\frac{7}{12}} + 700\times1.02\Big(1.07^{-1\frac{1}{12}} + 1.07^{-1\frac{7}{12}}\Big)\times\frac{1}{1-1.02\times1.07^{-1}}$$
$$= \$28{,}600$$

23 Subject CT1, April 2018, Question 6

(i) *Net real redemption yield*

As we are given a constant rate of inflation over the whole term of the investment, we can calculate the net money yield first, and then convert this to the net real yield.

The equation of value in money terms for £100 nominal of the bond, allowing for income tax of 25% on the coupons, and capital gains tax of 35% on the capital gain is:

$$102 = 3(1-0.25)a_{\overline{10|}}^{(2)} + 105v^{10} - 0.35(105-102)v^{10}$$

ie $\quad 102 = 2.25a_{\overline{10|}}^{(2)} + 103.95v^{10}$

The net money yield is the interest rate that solves this equation of value. When $i = 2\%$:

$$2.25a_{\overline{10|}}^{(2)} + 103.95v^{10} = 2.25 \times 9.02728 + 103.95 \times 1.02^{-10} = 105.5866$$

and when $i = 2.5\%$:

$$2.25a_{\overline{10|}}^{(2)} + 103.95v^{10} = 2.25 \times 8.80643 + 103.95 \times 1.025^{-10} = 101.0200$$

Linearly interpolating between these values gives a money yield of:

$$i = 2\% + \left(\frac{102-105.5866}{101.0200-105.5866}\right) \times 0.5\% = 2.39\% \ pa$$

So the net real yield is:

$$\frac{1.0239}{1.02} - 1 = 0.38\% \, pa$$

(ii) *Effect of lower inflation*

If inflation was less than 2% *pa* throughout the term, the money yield would be unchanged (as it is not affected by the rate of inflation), but due to the lower inflation, the real yield would be higher (as less of the return is eroded by inflation).

24 *Subject CT1, September 2018, Question 4*

(i)(a) *Price per £100 nominal at issue*

The gross redemption yield is 3% *pa* convertible half-yearly, *ie* $i^{(2)} = 3\%$.
This is equivalent to an annual effective gross redemption yield of:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = 1.015^2 - 1 = 3.0225\%$$

The price per £100 nominal at issue, P, is equal to the present value at time 0 of the cashflows from the loan stock:

$$P = 6a_{\overline{25}|}^{(2)} + 103v^{25} \quad @3.0225\%$$

Now:

$$a_{\overline{25}|}^{(2)} = \frac{1 - 1.030225^{-25}}{2\left(1.030225^{0.5} - 1\right)} = 17.4998$$

So:

$$P = 6 \times 17.4998 + 103 \times 1.030225^{-25} = £153.92$$

(i)(b) *Price per £100 nominal three months after issue*

The price per £100 nominal three months after issue is equal to the present value at time 3 months of the cashflows from the loan stock, *ie*:

$$P \times (1+i)^{3/12} = 153.92 \times 1.030225^{3/12} = £155.07$$

(ii) *Price paid*

We start by performing the capital gains test:

$$i^{(2)} = 2\left(1.1^{0.5} - 1\right) = 9.7618\%$$

$$\frac{D}{R}(1-t_1) = \frac{6\%}{103\%}(1-0.3) = 4.0777\%$$

Since $i^{(2)} > \frac{D}{R}(1-t_1)$, there is a capital gain.

The price paid per £100 nominal, P, is equal to the present value of the cashflows from the loan stock, allowing for income tax of 30% on the coupons and capital gains tax of 40% on the capital gain. So:

$$P = 6(1-0.3)a^{(2)}_{\overline{25}|} + 103v^{25} - 0.4(103-P)v^{25} \quad @10\%$$

Rearranging gives:

$$P\left(1-0.4v^{25}\right) = 4.2a^{(2)}_{\overline{25}|} + 61.8v^{25} \quad \Rightarrow \quad P = \frac{4.2a^{(2)}_{\overline{25}|} + 61.8v^{25}}{1-0.4v^{25}}$$

Now:

$$a^{(2)}_{\overline{25}|} = \frac{1-1.1^{-25}}{2\left(1.1^{0.5}-1\right)} = 9.2986$$

So:

$$P = \frac{4.2 \times 9.2986 + 61.8 \times 1.1^{-25}}{1-0.4 \times 1.1^{-25}} = £46.47$$

The investor paid £46.47 per £100 nominal of the loan stock.

25 *Subject CT1, September 2018, Question 9 (part)*

(ii) *Cash payments received*

The investor will receive four coupon payments (on 30 June 2016, 31 December 2016, 30 June 2017 and 31 December 2017) and the sale price (on 31 December 2017).

Using a rate of inflation of 2% *pa*, and noting that the coupon payments are rounded to the nearest pound, the coupon received on 30 June 2016 (*ie* 6 months after issue) is:

$$C_{\text{June 16}} = 0.01 \times 1{,}000{,}000 \times 1.02^{0.5} = £10{,}100$$

Similarly:

$$C_{\text{Dec 16}} = 0.01 \times 1{,}000{,}000 \times 1.02 = £10{,}200$$

$$C_{\text{June 17}} = 0.01 \times 1{,}000{,}000 \times 1.02^{1.5} = £10{,}301$$

$$C_{\text{Dec 17}} = 0.01 \times 1{,}000{,}000 \times 1.02^{2} = £10{,}404$$

The proceeds from the sale of the bond are:

$$101 \times 1{,}000{,}000 = £1{,}010{,}000$$

(iii) *Annual effective money rate of return*

The annual effective money (*ie* before allowing for inflation) rate of return earned by the investor is the interest rate that solves the equation of value:

$$1{,}000{,}000 = 10{,}100v^{0.5} + 10{,}200v + 10{,}301v^{1.5} + (10{,}404 + 1{,}010{,}000)v^2$$

ie: $$1{,}000{,}000 = 10{,}100v^{0.5} + 10{,}200v + 10{,}301v^{1.5} + 1{,}020{,}404v^2$$

Using $i = 2.5\%$, the right-hand side equals 1,001,089, and using $i = 3\%$, the right-hand side equals 991,538. Linearly interpolating between these values gives a money rate of return, to the nearest 0.1%, of:

$$i = 2.5\% + \left(\frac{1{,}000{,}000 - 1{,}001{,}089}{991{,}538 - 1{,}001{,}089}\right) \times 0.5\% = 2.6\% \ pa$$

(iv) *Rate of inflation in the three months to 31 December 2017*

The real rate of return from the bond over the period from 31 December 2015 to 31 December 2017 was 1% *pa* convertible half-yearly. This is equivalent to an effective real rate of return of 0.5% per half-year.

Working in half-years, and letting j denote the rate of inflation over the three months to 31 December 2017 expressed as an annual effective rate, the equation of value for the bond in real terms is:

$$1{,}000{,}000 = \frac{10{,}100}{1.02^{0.5}}v + \frac{10{,}200}{1.02}v^2 + \frac{10{,}301}{1.02^{1.5}}v^3$$

$$+\frac{1{,}020{,}404}{1.02^{1.75}(1+j)^{0.25}}v^4 \quad @0.5\%$$

This gives:

$$\frac{1}{(1+j)^{0.25}} = 1.0042626 \quad \Rightarrow \quad j = -1.69\% \ pa$$

26 ***Subject CM1, April 2019, Question 11***

(i) *Price per share*

At 1 February 2017, the present value of the future dividends is:

$$\begin{aligned}PV &= 0.40\times 1.05\times v + 0.40\times 1.05\times 1.04\times v^2\\ &\quad +0.40\times 1.05\times 1.04\times 1.03\times v^3 + 0.40\times 1.05\times 1.04\times 1.03^2\times v^4+\cdots\\ &= 0.40\times 1.05v + 0.40\times 1.05\times 1.04v^2\\ &\quad +0.40\times 1.05\times 1.04\times 1.03v^3\left(1+1.03v+1.03^2v^2+\cdots\right)\end{aligned}$$

So:

$$PV = 0.40 \times 1.05v + 0.40 \times 1.05 \times 1.04v^2$$
$$+0.40 \times 1.05 \times 1.04 \times 1.03v^3 \times \frac{1}{1-1.03v}$$
$$= 0.40 \times \frac{1.05}{1.09} \times \left(1 + \frac{1.04}{1.09} + \frac{1.04 \times 1.03}{1.09^2} \times \frac{1}{1-1.03 \times 1.09^{-1}}\right)$$
$$= 7.0642$$

So the investor would have been prepared to pay a maximum of £7.06 per share.

(ii) *Effective annual real rate of return*

The following table shows the real and money cashflows for the investor:

Date	Money cashflow (£)	Inflation index	Real cashflow (£)
01/02/2017	–7.00	211.0	–7.00
01/02/2018	0.428	215.7	$0.428 \times \frac{211.0}{215.7} = 0.41867$
01/02/2019	$0.449 + 7.50$	221.2	$(0.449 + 7.50) \times \frac{211.0}{221.2}$ $= 7.58245$

The real rate of return is the interest rate that satisfies the following equation of value:

$$7.00 = 0.41867v + 7.58245v^2$$
$$\Rightarrow 7.58245v^2 + 0.41867v - 7.00 = 0$$

This is a quadratic equation, so we can solve for v using the quadratic formula:

$$v = \frac{-0.41867 \pm \sqrt{(-0.41867)^2 + 4 \times 7.58245 \times 7}}{2 \times 7.58245}$$

The positive root gives us $v = 0.933613058$ and converting this to i gives us a real rate of return of 7.11% *pa*.

Alternatively, trial and error can be used to find the interest rate.

27 *Subject CM1, September 2019, Question 8*

(i) *Price the investor should pay to ensure a yield of at least 8% pa*

First we need to apply the capital gains test, *ie* an investor makes a capital gain if:

$$i^{(p)} > (1-t_1)\frac{D}{R}$$

Working in units of £100 nominal, we have $D = 9$, $R = 110$, $t_1 = 0.15$, $i = 8\%$ *pa* and $p = 2$. So:

$$(1-t_1)\frac{D}{R} = 0.85 \times \frac{9}{110} = 0.069545$$

We compare this with $i^{(2)}$ at 8%:

$$i^{(2)} = 2\left[(1+i)^{1/2} - 1\right] = 2\,[1.08^{0.5} - 1] = 0.078461$$

Since the second amount is larger, the investor will make a capital gain.

This means that the worst outcome for the investor is for this gain to be deferred as long as possible, *ie* the bond is redeemed on the latest possible date.

So, assuming that the loan is redeemed on the last redemption date (after 25 years), we have the following equation of value for the price of the loan:

$$P = 0.85 \times 9a^{(2)}_{\overline{25}|} + 110v^{25}$$

Evaluating the annuity at 8% *pa*, we have:

$$a^{(2)}_{\overline{25}|} = \frac{1-v^{25}}{i^{(2)}} = \frac{1-1.08^{-25}}{0.078461} = 10.884165$$

So we have:

$$P = 7.65 \times 10.884165 + 110 \times 1.08^{-25} = 99.3258$$

This gives us a price of around £99.3258 per £100 nominal, or £993,258 for the whole loan of £1,000,000.

(ii)(a) *Price paid by the second investor*

We now know that the investor sells the loan after 10 years. The equation of value for the loan is therefore:

$$99.3258 = 7.65a^{(2)}_{\overline{10}|} + Rv^{10}$$

where R is the price paid by the second investor.

Rearranging this expression:

$$\begin{aligned} R &= \left(99.3258 - 7.65a^{(2)}_{\overline{10}|}\right) \times (1+i)^{10} \\ &= (99.3258 - 7.65 \times 6.841701) \times 1.08^{10} \\ &= 101.4410 \end{aligned}$$

So the second investor paid £101.4410 per £100 nominal, or £1,014,410 for the whole loan.

(ii)(b) *Minimum net redemption yield earned by the second investor*

The equation of value for the loan from the second investor's perspective is:

$$101.4410 = 0.75 \times 9a_{\overline{15}|}^{(2)} + 110v^{15} - 0.35 \times (110 - 101.4410) \times v^{15}$$
$$\Rightarrow 101.4410 - 6.75a_{\overline{15}|}^{(2)} - 107.00435v^{15} = 0$$

Using trial and error and a first guess of 7% to reflect the (approximate) 6.75% net return from coupons plus a small capital gain on redemption:

$$i = 7\% \quad \Rightarrow \quad LHS = 0.121604$$
$$i = 6\% \quad \Rightarrow \quad LHS = -9.734899$$

Interpolating between these gives:

$$i = 6\% + \frac{0 - (-9.734899)}{0.121604 - (-9.734899)} \times (7\% - 6\%) = 6.9877\%$$

So the minimum net redemption yield achieved by the second investor is around 6.99% *pa*.

Alternatively, table mode on a Casio calculator can be used to find the net redemption yield.

28 ***Subject CM1, September 2020, Question 9***

(i) *Capital gains test*

The required return convertible quarterly is:

$$i^{(4)} = 4\left(1.049^{1/4} - 1\right) = 4.8125\%$$

The return provided from the net coupon payments, convertible quarterly, is:

$$\frac{D}{R}(1 - t_1) = \frac{6}{105} \times 0.8 = 4.5714\%$$

Since $i^{(4)} > \frac{D}{R}(1 - t_1)$ there must be a capital gain.

(ii) *Assumption to make in calculating price*

Since there is a capital gain, the security is least valuable to the investor if the repayment is made by the borrower at the latest possible date.

Since that decision is beyond the control of the investor, we must assume that the redemptions occurs after 25 years as this will produce the smallest possible yield from the investment.

(iii) *Maximum price investor should pay*

The maximum price of the bond is:

$$P = 6(1-0.2)a_{\overline{25}|}^{(4)} + 105v^{25} - 0.25(105-P)v^{25}$$

where:

$$a_{\overline{25}|}^{(4)} = \frac{1-v^{25}}{i^{(4)}} = \frac{1-1.049^{-25}}{0.048125} = 14.4953$$

So

$$\begin{aligned} P &= 4.8 \times 14.4953 + 105v^{25} - 0.25(105-P)v^{25} \\ &= 69.5773 + 31.7543 - 7.9386 + 0.075605P \\ &= 93.3930 + 0.075605P \end{aligned}$$

Rearranging to make P the subject and solving:

$$\begin{aligned} &0.924395P = 93.3930 \\ \Rightarrow \quad &P = 101.03 \end{aligned}$$

The maximum price the investor should pay per \$100 nominal is \$101.03.

(iv) *Impact on price if coupons were payable half-yearly*

If the coupons were paid half-yearly rather than quarterly then the investor would have to wait longer, on average, for the coupon payments to be made.

This will make the investment less valuable, and therefore the price would be lower than in part (iii).

29 ***Subject CM1, April 2021, Question 3***

(i) *Effective annual real rate of return*

Interest paid per year is $0.04 \times 100{,}000 = 4{,}000$

Converting the cashflows to real values in 1 March 2017 terms, discounting at the real rate of return i and setting equal to the price of the security:

$$100{,}000 = 4{,}000 \times \frac{240.5}{256.0} \times v + 4{,}000 \times \frac{240.5}{272.8} \times v^2 + 104{,}000 \times \frac{240.5}{286.6} \times v^3$$

$$\Rightarrow \quad 100{,}000 = 3{,}757.81v + 3{,}526.39v^2 + 87{,}271.46v^3$$

Solving this equation using 'Goal Seek' in Excel gives $i = -1.92\%$ which is approximately equal to -1.9%.

(ii) *Comment on the result*

High actual inflation over the term of the security has eroded the return achieved.

The nominal rate of return achieved is 4% per annum. However, as the average inflation rate over the term of the loan (6.02% per annum) has exceeded 4% per annum, the real rate of return achieved by the security holder is negative.

