(i) We have:

$$(1+y_3)^3(1+f_3)(1+f_{4,3})^3=(1+y_7)^7$$

$$\Rightarrow \quad 1+f_3=\frac{1.05^7}{1.06^3\times 1.052^3}=1.0148$$

$$\Rightarrow \quad f_3=1.48\%$$

(ii) We have:

$$(1+y_5)^5(1+f_{5,2})^2=(1+y_7)^7$$

$$\Rightarrow \quad 1+f_{5,2}=\left(\frac{1.05^7}{1.057^5}\right)^{\frac{1}{2}}=\sqrt{1.0665}$$

$$\Rightarrow \quad f_{5,2}=3.27\%$$

(iii) We have:

$$(1+y_4)^4(1+f_{4,3})^3=(1+y_7)^7$$

$$\Rightarrow \quad 1+y_4=\left(\frac{1.05^7}{1.052^3}\right)^{\frac{1}{4}}=1.0485$$

$$\Rightarrow \quad y_4=4.85\%$$

(iv) We have:

$$(1+y_3)^3(1+f_{3,4})^4=(1+y_7)^7$$

$$\Rightarrow \quad 1+f_{3,4}=\left(\frac{1.05^7}{1.06^3}\right)^{\frac{1}{4}}=1.0426$$

$$\Rightarrow \quad f_{3,4}=4.26\%$$

---

2 Continuous-time rates

2.1 Continuous-time spot rates

The continuous-time spot rate is the force of interest that is equivalent to the spot rate expressed as an effective rate of interest.

Let P_t be the price of a unit zero-coupon bond of term *t*. Then the *t*-year *spot force of interest* is Y_t where:

$$P_t = e^{-Y_t t} \quad \Rightarrow \quad Y_t = -\frac{1}{t}\log P_t$$

This is also called the continuously compounded spot rate of interest or the continuous-time spot rate. Y_t and its corresponding discrete annual rate y_t are connected in the same way as δ and *i*; an investment of £1 for *t* years at a discrete spot rate y_t accumulates to $(1+y_t)^t$; at the continuous-time rate it accumulates to $e^{Y_t t}$; these must be equal, so $y_t = e^{Y_t} - 1$.

2.2 Continuous-time forward rates

The continuous-time forward rate $F_{t,r}$ is the force of interest equivalent to the annual forward rate of interest $f_{t,r}$.

A £1 investment of duration *r* years, starting at time *t*, agreed at time $0 \le t$ accumulates using the annual forward rate of interest to $(1+f_{t,r})^r$ at time $t+r$.

Using the equivalent forward force of interest the same investment accumulates to $e^{F_{t,r} r}$.

Hence the annual rate and continuous-time rate are related as:

$$f_{t,r} = e^{F_{t,r}} - 1$$

The relationship between the continuous-time spot and forward rates may be derived by considering the accumulation of £1 at a continuous-time spot rate of Y_t for *t* years, followed by the continuous-time forward rate of $F_{t,r}$ for *r* years. Compare this with an investment of £1 at a continuous-time spot rate of Y_{t+r} for $t+r$ years. The two investments are equivalent, so the accumulated values must be the same. Hence:

$$e^{tY_t} \, e^{rF_{t,r}} = e^{(t+r)Y_{t+r}}$$

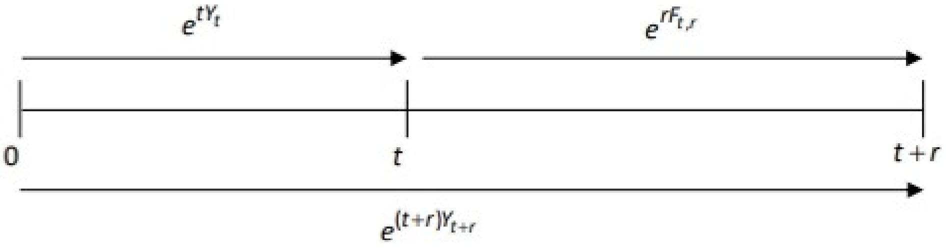
$$\Rightarrow \quad tY_t + rF_{t,r} = (t+r)Y_{t+r}$$

$$\Rightarrow \quad F_{t,r} = \frac{(t+r)Y_{t+r} - tY_t}{r}$$

Also, using $Y_t = -\frac{1}{t}\log P_t$, we have:

$$F_{t,r} = \frac{1}{r}\log\left(\frac{P_t}{P_{t+r}}\right)$$

Once again we can represent the connection between the continuous-time spot and forward rates on a timeline.



Question

The prices for £100 nominal of zero-coupon bonds of various terms are as follows:

1 year = £94   5 years = £70   10 years = £47   15 years = £30

Calculate:

(i) Y_{10}

(ii) $F_{5,10}$

Solution

(i) We can calculate Y_{10} using the price of the 10-year zero-coupon bond:

$$Y_{10} = -\frac{1}{10}\log P_{10} = -\frac{1}{10}\log 0.47 = 7.55\%$$

Instead of using the general formula, we could take a first principles approach and set up the equation of value:

$$47 = 100e^{-10Y_{10}}$$

ie the price paid (of £47) is equal to the present value of the redemption payment of £100 calculated using the 10-year continuous-time spot rate. This gives the same answer as before.

(ii) $F_{5,10}$ is the continuous-time forward rate applying from time 5 to time 15, so we can calculate this using the 5-year and 15-year zero-coupon bond prices:

$$F_{5,10} = \frac{1}{10}\log\left(\frac{P_5}{P_{15}}\right) = \frac{1}{10}\log\left(\frac{0.7}{0.3}\right) = 8.47\%$$

Again, we could take a first principles approach using the equation of value for the 15-year bond:

$$30 = 100e^{-5Y_5}e^{-10F_{5,10}}$$

ie the price paid (of £30) is equal to the present value of the redemption payment of £100, calculated by discounting it for 10 years using the continuous-time forward rate (between time 5 and time 15) and for 5 years using the continuous-time spot rate (between time 0 and time 5). Now, considering the 5-year bond:

$$70 = 100e^{-5Y_5}$$

so the equation of value for the 15-year bond becomes:

$$30 = 70e^{-10F_{5,10}}$$

Solving this gives the same answer as before.

2.3 Instantaneous forward rates

The instantaneous forward rate F_t is defined as:

$$F_t = \lim_{r \to 0} F_{t,r}$$

The instantaneous forward rate may broadly be thought of as the forward force of interest applying in the instant of time $t \to t + \Delta t$.

$$F_t = \lim_{r \to 0} \frac{1}{r}\log\left(\frac{P_t}{P_{t+r}}\right) \qquad (1)$$

$$= \lim_{r \to 0} \frac{\log P_t - \log P_{t+r}}{r}$$

$$= -\lim_{r \to 0} \frac{\log P_{t+r} - \log P_t}{r} \qquad (2)$$

$$= -\frac{d}{dt}\log P_t \qquad (3)$$

$$= -\frac{1}{P_t}\frac{d}{dt}P_t \qquad (4)$$

Step (3) above uses the definition of a derivative in terms of a limit:

$$\frac{d}{dt}f(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

and step (4) uses the chain rule for differentiation.

We also find, by integrating both sides of (3) and using the fact that $P_0 = 1$ (as the price of a unit zero-coupon bond of term zero years must be 1), that:

$$P_t = e^{-\int_0^t F_s\,ds}$$

This formula might look familiar. Earlier in the course we defined $v(t)$ to be the present value of 1 due at time *t* and we expressed $v(t)$ in terms of $\delta(t)$, the force of interest per unit time at time *t*, as follows:

$$v(t) = e^{-\int_0^t \delta(s)\,ds}$$

Comparing these last two equations, we see that P_t is equivalent to $v(t)$ and F_s is equivalent to $\delta(s)$.

Note

We have described in this chapter the *initial term structure*, where everything is fixed at time 0. In practice the term structure varies rapidly over time, and the 5-year spot rate tomorrow may be quite different from the 5-year spot rate today. In more sophisticated treatments we model the change in term structure over time.

In this case all the variables we have used, *ie*:

$$P_t \quad y_t \quad f_{t,r} \quad Y_t \quad F_{t,r}$$

need another argument, *v*, say, to give the 'starting point'. For example, $y_{v,t}$ would be the *t*-year discrete spot rate of interest applying at time *v*; $F_{v,t,r}$ would be the force of interest agreed at time *v*, applying to an amount invested at time $v+t$ for the *r*-year period to time $v+t+r$.

Question

Using the above notation, suppose that $y_{0,5}=6\%$, $y_{5,5}=7.5\%$, $F_{0,5,5}=7\%$ and $F_{5,5,5}=8.25\%$.

Calculate the price for £100 nominal of a ten-year zero-coupon bond issued at:

(i) time 0

(ii) time 5.

Solution

(i) To calculate the price of the bond issued at time 0, we use the initial term structure of interest rates, *ie* $y_{0,5}$ and $F_{0,5,5}$, which are the 5-year spot rate and the 5-year force of interest from time 5 to time 10, as at time 0, respectively.

So the price for £100 nominal of the zero-coupon bond issued at time 0 is:

$$100(1+y_{0,5})^{-5}e^{-5F_{0,5,5}} = 100\times 1.06^{-5}\times e^{-5\times 0.07} = £52.66$$

(ii) In this case, we use the term structure of interest rates at time 5, *ie* $y_{5,5}$ and $F_{5,5,5}$, which are the 5-year spot rate and the 5-year force of interest from time 10 to time 15, as at time 5, respectively.

So the price for £100 nominal of the zero-coupon bond issued at time 5 is:

$$100(1+y_{5,5})^{-5}e^{-5F_{5,5,5}} = 100\times 1.075^{-5}\times e^{-5\times 0.0825} = £46.11$$

3 Theories of the term structure of interest rates

3.1 Why interest rates vary over time

The prevailing interest rates in investment markets usually vary depending on the time span of the investments to which they relate. This variation determines the *term structure* of interest rates.

The variation arises because the interest rates that lenders expect to receive and borrowers are prepared to pay are influenced by the following factors, which are not normally constant over time:

Supply and demand

Interest rates are determined by market forces, *ie* the interaction between borrowers and lenders. If cheap finance is easy to obtain or if there is little demand for finance, this will push interest rates down.

Base rates

In many countries there is a central bank that sets a base rate of interest. This provides a reference point for other interest rates. For example, an interest rate in the UK may be expressed as the Bank of England's base rate plus 4%. Investors will have a view on how this rate is likely to move in the future.

Interest rates in other countries

The interest rates in a particular country will also be influenced by the cost of borrowing in other countries because major investment institutions have the alternative of borrowing from abroad.

Expected future inflation

Lenders will expect the interest rates they obtain to outstrip inflation. So periods of high inflation tend to be associated with high interest rates.

Tax rates

If tax rates are high, interest rates may also be high, because investors will require a certain level of return after tax.

Risk associated with changes in interest rates

In general, rates of interest tend to increase as the term increases because the risk of loss due to a change in interest rates is greater for longer-term investments. This is the idea behind liquidity preference theory, which we meet in the next section.

3.2 The theories

Some examples of typical (spot rate) yield curves are given below.

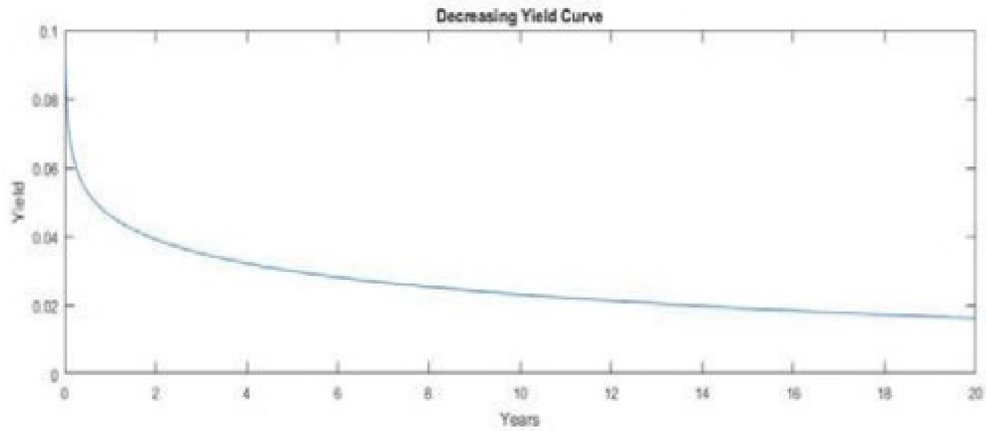


Figure 1: Decreasing yield curve (used with permission from Dominic Cortis)

In Figure 1 the long-term bond yields are lower than the short-term bonds. Since price is a decreasing function of yield, an interpretation is that long-term bonds are more expensive than short-term bonds. There are several possible explanations – for example it is possible that investors believe that they will get a higher overall return from long-term bonds, despite the lower current yields, and the higher demand for long-term bonds has pushed up the price, which is equivalent to pushing down the yield, compared with short-term bonds.

Other explanations for different yield curve shapes are given below.

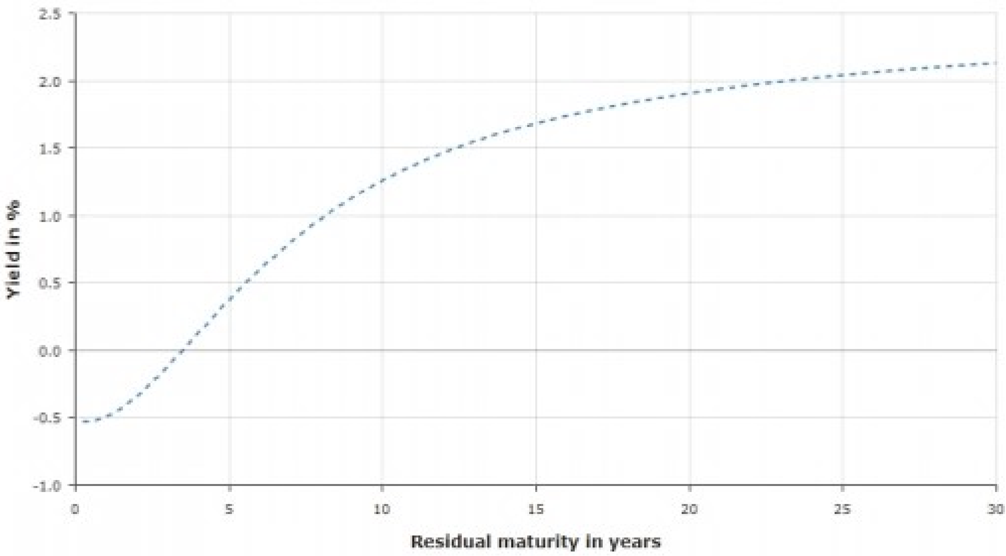


Figure 2: Increasing yield curve: Euro Area Yield Curve for all bonds (obtained from European Central Bank website on 7 February 2018)

In Figure 2 the long-term bonds are higher yielding (or cheaper) than the short-term bonds. This shows the case of an increasing yield curve.

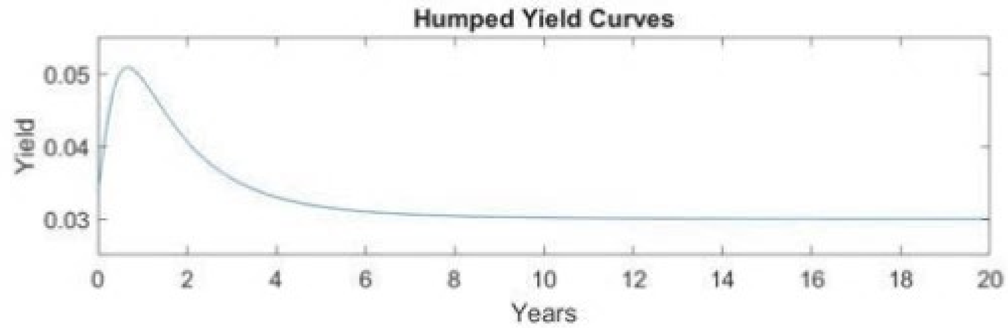


Figure 3: Humped yield curve (used with permission from Dominic Cortis)

In Figure 3 the short-term bonds are generally cheaper than the long bonds, but the very short rates (with terms less than one year) are lower than the one-year rates.

The three most popular explanations for the fact that interest rates vary according to the term of the investment are:

1. **Expectations theory**
2. **Liquidity preference**
3. **Market segmentation**

Expectations theory

The relative attraction of short and longer-term investments will vary according to expectations of future movements in interest rates. An expectation of a fall in interest rates will make short-term investments less attractive and longer-term investments more attractive. In these circumstances yields on short-term investments will rise and yields on long-term investments will fall. An expectation of a rise in interest rates will have the converse effect.

In Figure 1 it appears that the demand for long-term bonds may be greater than for short, implying an expectation that interest rates will fall. By buying long-term bonds investors can continue getting higher rates after a future fall in interest rates, for the duration of the long bond.

In Figure 2 the demand is higher for short-term bonds – perhaps indicating an expectation of a rise in interest rates.

If it is expected that interest rates will rise, investors will wish to avoid locking into the low current rate over the long term, and so will choose shorter-term investments. This increased demand for shorter-term investments increases the price of these bonds and reduces the yield, leading to an increasing yield curve, as in Figure 2.

Liquidity preference

Consider a ten-year and a twenty-year zero-coupon bond. If the spot rate for all terms is 5% *pa* then the prices for £100 nominal are $100 \times 1.05^{-10} = £61.39$ and $100 \times 1.05^{-20} = £37.69$ respectively.

If interest rates rise to 6%, then the price of both bonds will fall:

- the ten-year bond price falls to $100 \times 1.06^{-10} = £55.84$, a 9% drop
- the twenty-year bond price falls to $100 \times 1.06^{-20} = £31.18$, a 17% drop.

Longer-dated bonds are more sensitive to interest rate movements than short-dated bonds. It is assumed that risk averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds. This might explain some of the excess return offered on long-term bonds over short-term bonds in Figure 2.

Later in this chapter we will look at measures that allow us to quantify the effects of changes in interest rates.

Market segmentation

Bonds of different terms are attractive to different investors, who will choose assets that are similar in term to their liabilities. The liabilities of banks, for example, are very short-term (investors may withdraw a large proportion of the funds at very short notice); hence banks invest in very short-term bonds. Many pension funds have liabilities that are very long-term, so pension funds are more interested in the longest-dated bonds. The demand for bonds will therefore differ for different terms.

The supply of bonds will also vary by term, as governments and companies' strategies may not correspond to the investors' requirements.

Remember that governments and companies issue bonds because they need to borrow money, not because they are kind-hearted and want to give investors something to invest in. More bonds will be supplied if more money needs to be borrowed. This will put downward pressure on prices.

The market segmentation hypothesis argues that the term structure emerges from these different forces of supply and demand.

These theories are covered in more detail in Subject CP1, Actuarial Practice.

Question

Outline what would happen to yields of fixed-interest bonds if:

(i) bond prices fall

(ii) demand for fixed-interest bonds falls

(iii) the government issues many more bonds

(iv) institutional investors suddenly decide to invest less in equities and more in fixed-interest bonds.

Solution

(i) If bond prices fall, then bond yields will rise, as investors will now be paying less to receive the same coupon and redemption payments.

(ii) If demand for fixed-interest bonds falls, then the price of the bonds will fall, and bond yields will rise.

(iii) In this case, the supply of fixed-interest bonds will increase, and this puts downward pressure on prices. As bond prices fall, bond yields will rise.

(iv) This is the opposite scenario to (ii). Here, the demand for fixed-interest bonds increases, so the price of the bonds will increase, and bond yields will fall.

3.3 Yields to maturity

The yield to maturity for a coupon-paying bond (also called the redemption yield) has been defined as the effective rate of interest at which the discounted value of the proceeds of a bond equal the price. It is widely used, but has the disadvantage that it depends on the coupon rate of the bond, and therefore does not give a simple model of the relationship between term and yield.

In the UK, yield curves plotting the average (smoothed) yield to maturity of coupon-paying bonds are produced separately for 'low coupon', 'medium coupon' and 'high coupon' bonds.

Question

The current annual term structure of interest rates is:

(6%, 6%, 6%, 6%, 7%)

ie $y_1 = y_2 = y_3 = y_4 = 6\%$ and $y_5 = 7\%$.

Calculate the gross redemption yield of a five-year fixed-interest security that is redeemable at par if the annual coupon, payable in arrears, is:

(i) 2%

(ii) 4%.

Solution

(i) ***2% coupon rate***

The cashflows at times 1, 2, 3 and 4 are discounted using an interest rate of 6% *pa* (as the spot rate is the same at all these durations), and the cashflow at time 5 – made up of the final coupon payment and the redemption payment – is discounted using the 5-year spot rate of 7% *pa*.

Letting P denote the price for £100 nominal of the bond:

$$P = 2a_{\overline{4}|6\%} + 102v_{7\%}^{5} = 2 \times 3.4651 + 102 \times 1.07^{-5} = £79.65$$

The gross redemption yield is the interest rate, i, that satisfies the equation of value:

$$79.65 = 2a_{\overline{5}|} + 100v^{5}$$

The gross redemption yield is a weighted average of the interest rates over the term of the bond, where the weights are the present values of the cashflows that occur at the different durations. The gross redemption yield is likely to be close to 7% here, as that is the spot rate associated with the duration of the largest cashflow (the redemption payment).

At 7%, the right-hand side gives £79.50, and at 6.5%, the right-hand side gives £81.30. Linearly interpolating, we find the gross redemption yield to be:

$$i \approx 6.5\% + \frac{79.65 - 81.30}{79.50 - 81.30} \times (7\% - 6.5\%) = 6.96\%$$

(ii) ***4% coupon rate***

In this case, the price for £100 nominal of the bond is:

$$P = 4a_{\overline{4}|6\%} + 104v_{7\%}^{5} = 4 \times 3.4651 + 104 \times 1.07^{-5} = £88.01$$

The gross redemption yield is the interest rate, i, that satisfies the equation of value:

$$88.01 = 4a_{\overline{5}|} + 100v^{5}$$

At 7%, the right-hand side gives £87.70, and at 6.5% the right-hand side gives £89.61. Linearly interpolating, we find the gross redemption yield to be:

$$i \approx 6.5\% + \frac{88.01 - 89.61}{87.70 - 89.61} \times (7\% - 6.5\%) = 6.92\%$$

The gross redemption yield of the bond with the 4% coupon rate is lower than that for the bond with the 2% coupon rate. This is because the bond with the 4% coupon rate has higher cashflows at the earlier durations, so these gain more weighting in the calculation and, since the spot rates at the earlier durations are lower, the gross redemption yield is lower.

3.4 Par yields

We have already met the *yield to maturity* or the *redemption yield* for a fixed-interest investment. This is just the constant interest rate that satisfies the equation of value. For a zero-coupon bond, this is the same as the spot rate.

The *n-year par yield* represents the coupon per £1 nominal that would be payable on a bond with term *n* years, which would give the bond a current price under the current term structure of £1 per £1 nominal, assuming the bond is redeemed at par.

That is, if yc_n is the *n*-year par yield:

$$1 = (yc_n)(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + \cdots + v_{y_n}^n) + 1v_{y_n}^n$$

The par yields give an alternative measure of the relationship between the yield and term of investments.

The gross redemption yield of a bond depends on its term, price, coupon rate and redemption rate. Different bonds with the same term can have different gross redemption yields, so the gross redemption yield on its own does not provide a simple measure of how interest rates vary by term.

The par yield aims to provide us with that simple measure by considering a *notional bond* with a particular term, rather than specific bonds. This notional bond has a standardised structure in which the price is equal to the redemption payment, so there is no capital gain (or loss) and the only return is provided by the coupons. If the yield curve is flat (*ie* with all the spot/forward rates being equal to one 'market' rate of interest at all terms), then the par yield would be equal to that market interest rate, and would be the same for all terms. The extent to which the par yield *does* vary by term, therefore, reflects the way in which the underlying spot/forward rates of interest vary by term.

The difference between the par yield rate and the spot rate is called the 'coupon bias'.

The spot rate for a given term is the yield on a zero-coupon bond of that term, whilst the par yield for a given term is the yield on a notional coupon-paying bond of that term. The coupon bias is then the difference (or bias) in yields between these two types of bond as a result of the coupons paid.

Question

Calculate the 5-year par yield if the annual term structure of interest rates is:

$$(6\%, 6.25\%, 6.5\%, 6.75\%, 7\%, \ldots)$$

ie $y_1 = 6\%$, $y_2 = 6.25\%$, $y_3 = 6.5\%$, $y_4 = 6.75\%$ and $y_5 = 7\%$.

Solution

The 5-year par yield yc_5 is found from the equation:

$$yc_5(v_{y_1} + v_{y_2}^2 + \cdots + v_{y_5}^5) + v_{y_5}^5 = 1$$

Using the spot rates given, this becomes:

$$yc_5(1.06^{-1} + 1.0625^{-2} + 1.065^{-3} + 1.0675^{-4} + 1.07^{-5}) + 1.07^{-5} = 1$$

ie: $yc_5 \times 4.1401 + 0.71299 = 1 \quad \Rightarrow \quad yc_5 = 6.93\%$

4 Duration, convexity and immunisation

One of the key concerns for the manager of a fixed-interest investment portfolio is how the portfolio would be affected if there were a change in interest rates and, in particular, whether such a movement might compromise the ability of the fund to meet its liabilities.

In this section we consider simple measures of vulnerability to interest rate movements.

We will also look at the technique of immunisation, which is a method of minimising the risks relating to interest rates.

For simplicity we assume a flat yield curve, and that when interest rates change, all change by the same amount, so that the curve stays flat. A flat yield curve implies that $y_t = f_{t,r} = i$ for all *t*, *r* and $Y_t = F_{t,r} = F_t = \delta$ for all *t*, *r*.

4.1 Interest rate risk

Suppose an institution holds assets of value V_A, to meet liabilities of value V_L. Since both V_A and V_L represent the discounted value of future cashflows, both are sensitive to the rate of interest. We assume that the institution is healthy at time 0 so that currently $V_A \geq V_L$.

If $V_A > V_L$, then we say that there is a surplus in the fund equal to $V_A - V_L$. If $V_A < V_L$, then the fund is in deficit.

If rates of interest fall, both V_A and V_L will increase. If rates of interest rise then both will decrease. We are concerned with the risk that following a downward movement in interest rates the value of assets increases by less than the value of liabilities, or that, following an upward movement in interest rates the value of assets decreases by more than the value of the liabilities.

In other words, for a fund currently in surplus, we are concerned that after a movement in interest rates the fund moves into deficit.

In order to examine the impact of interest rate movements on different cashflow sequences, we will use changes in the yield to maturity to represent changes in the underlying term structure. This is approximately (but not exactly) the same as assuming a constant movement of similar magnitude in the one-period forward rates.

Before we can look at a technique used to minimise this interest rate risk we must be familiar with the measures: effective duration, duration and convexity.

4.2 Effective duration

One measure of the sensitivity of a series of cashflows to movements in the interest rates is the *effective duration* (or volatility). Consider a series of cashflows $\{C_{t_k}\}$ for $k = 1,2,\ldots,n$.

Let *A* be the present value of the payments at rate (yield to maturity) i, so that:

$$A = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$$

Then the effective duration is defined to be:

$$\nu(i) = -\frac{1}{A}\frac{d}{di}A = -\frac{A'}{A} \qquad (4.1)$$

$$= \frac{\sum_{k=1}^{n} C_{t_k}\, t_k\, v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

To obtain this last relationship, we need to differentiate A with respect to i. Since:

$$A = \sum_{k=1}^{n} C_{t_k} v^{t_k} = \sum_{k=1}^{n} C_{t_k} (1+i)^{-t_k}$$

it follows that:

$$A' = \sum_{k=1}^{n} C_{t_k} (-t_k)(1+i)^{-t_k-1} = (-1)\sum_{k=1}^{n} C_{t_k} t_k v^{t_k+1}$$

This is a measure of the rate of change of value of A with i, which is independent of the size of the present value. Equation (4.1) assumes that the cashflows do not depend on the rate of interest.

For a small movement ε in interest rates, from i to $i+\varepsilon$, the relative change in value of the present value is approximately $-\varepsilon\nu(i)$ so the new present value is approximately $A(1-\varepsilon\nu(i))$.

Effective duration is denoted by the Greek letter nu, ν.

Question

Consider a three-year fixed-interest bond that pays coupons annually in arrears at a rate of 5% *pa* and is redeemable at par.

Let $P(i)$ denote the price for £100 nominal of this bond based on a yield of i *pa*.

Calculate $\nu(0.05)$, the volatility of the bond at a yield of 5% *pa*, and use this to calculate the approximate price of the bond if the yield falls by 1% to 4% *pa*.

Solution

We have:

$$P(i) = 5v + 5v^2 + 105v^3 = 5(1+i)^{-1} + 5(1+i)^{-2} + 105(1+i)^{-3}$$

and:

$$P'(i) = 5(-1)(1+i)^{-2} + 5(-2)(1+i)^{-3} + 105(-3)(1+i)^{-4} = -5(1+i)^{-2} - 10(1+i)^{-3} - 315(1+i)^{-4}$$

So, the volatility at a yield of 5% *pa* is:

$$\nu(0.05) = -\frac{P'(0.05)}{P(0.05)} = -\left(\frac{-5(1.05)^{-2} - 10(1.05)^{-3} - 315(1.05)^{-4}}{5(1.05)^{-1} + 5(1.05)^{-2} + 105(1.05)^{-3}}\right) = -\left(\frac{-272.325}{100}\right) = 2.723$$

This value tells us that if interest rates change by 1%, the price of this bond will change by approximately 2.723% (in the opposite direction to the interest rate movement).

So, if the yield falls by 1% to 4% *pa*, the bond price will rise by approximately 2.723% to:

$$1.02723 \times P(0.05) = 1.02723 \times 100 = £102.72$$

For comparison, the exact value is:

$$P(0.04) = 5(1.04)^{-1} + 5(1.04)^{-2} + 105(1.04)^{-3} = £102.78$$

4.3 Duration

Another measure of interest rate sensitivity is the duration, also called Macaulay duration or discounted mean term (DMT). This is the mean term of the cashflows $\{C_{t_k}\}$, weighted by present value. That is, at rate i, the duration of the cashflow sequence $\{C_{t_k}\}$ is:

$$\tau = \frac{\sum_{k=1}^{n} t_k \, C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}$$

The discounted mean term for a continuously payable payment stream (or a mixture of discrete and continuous payments) is calculated similarly, but with the summations replaced by integrals for the continuous payments.

Comparing this expression with the equation for the effective duration it is clear that:

$$\tau = (1+i)\nu(i)$$

Note the following points:

- The discounted mean term is dependent on the interest rate used to calculate the present values, as well as the amounts and timings of the cashflows.
- The discounted mean term is calculated as an average, so the DMT of a series of cashflows must take a value that is between the times of the first and last cashflows.

Question

Consider a 10-year fixed-interest bond that pays coupons annually in arrears at a rate of 8% *pa* and is redeemable at par.

Calculate the discounted mean term of this bond using an interest rates of 5% *pa* effective.

Solution

The discounted mean term, calculated at interest rate i, is:

$$DMT(i)=\frac{\sum_{t=1}^{10} t\times 8v^t+10\times 100v^{10}}{\sum_{t=1}^{10} 8v^t+100v^{10}}$$

In terms of annuities, this can be written as:

$$DMT(i)=\frac{8(Ia)_{\overline{10}|}+1{,}000v^{10}}{8a_{\overline{10}|}+100v^{10}}$$

since $a_{\overline{n}|}=\sum_{t=1}^{n} v^t$ and $(Ia)_{\overline{n}|}=\sum_{t=1}^{n} tv^t$.

Evaluating this expression using an interest rate of 5% *pa*, we obtain:

$$DMT(0.05)=\frac{8\times 39.3738+1{,}000\times 1.05^{-10}}{8\times 7.7217+100\times 1.05^{-10}}=7.54 \text{ years}$$

The first cashflow from the bond is at time 1 and the last cashflow is at time 10. As expected, the DMT is between 1 and 10. The DMT is closer to 10 as it is a weighted average of the payment times. The weights are the present values of the cashflows, and the largest cashflow occurs at time 10.

Another way of deriving the Macaulay duration is in terms of the force of interest, δ:

$$\tau=-\frac{1}{A}\frac{d}{d\delta}A=\frac{di}{d\delta}\nu(i)$$

$$i=e^{\delta}-1 \quad\Rightarrow\quad \frac{di}{d\delta}=e^{\delta}$$

$$\Rightarrow\quad \tau=e^{\delta}\nu(i)=(1+i)\nu(i)$$

The equation for τ in terms of the cashflows C_{t_k} may be found by differentiating A with respect to δ, recalling that $v_i^{t_k}=e^{-\delta t_k}$.

The first line above can be obtained by starting with the definition of A written in terms of δ:

$$A = \sum_{k=1}^{n} C_{t_k} v^{t_k} = \sum_{k=1}^{n} C_{t_k} e^{-\delta t_k}$$

Differentiating this expression with respect to δ gives:

$$\frac{dA}{d\delta} = \sum_{k=1}^{n} (-t_k) C_{t_k} e^{-\delta t_k}$$

So, in terms of δ, the DMT can be written as:

$$\tau = \frac{\sum_{k=1}^{n} t_k \, C_{t_k} \, v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} \, v_i^{t_k}} = \frac{\sum_{k=1}^{n} t_k \, C_{t_k} \, e^{-\delta t_k}}{\sum_{k=1}^{n} C_{t_k} \, e^{-\delta t_k}} = -\frac{1}{A}\frac{dA}{d\delta}$$

Then using the chain rule for differentiation, we can write:

$$\tau = -\frac{1}{A}\frac{dA}{d\delta} = -\frac{1}{A}\frac{dA}{di}\frac{di}{d\delta} = \nu(i)\frac{di}{d\delta}$$

The duration of an *n*-year coupon-paying bond, with coupons of *D* payable annually, redeemed at *R*, is:

$$\tau = \frac{D(Ia)_{\overline{n|}} + Rnv^n}{Da_{\overline{n|}} + Rv^n}$$

This is identical to what we found in the previous question.

The duration of an *n*-year zero-coupon bond of nominal amount 100, say, is:

$$\tau = \frac{100nv^n}{100v^n} = n$$

This last result should be intuitively obvious. The average term of a series of cashflows that has only one payment must be the time of that cashflow.

Both the volatility and the discounted mean term provide a measure of the average 'life' of an investment. This is important when considering the effect of changes in interest rates on investment portfolios since an investment with a longer term will in general be more affected by a change in interest rates than an investment with a shorter term.

Question

Consider a fixed-interest security that pays coupons annually in arrears at a rate of 3% *pa* and is redeemable at par.

Determine the effect on the price of £100 nominal of this security if interest rates over all terms increase from 7% *pa* to 8% *pa*, assuming that the term of the security is:

(a) 5 years

(b) 25 years.

Solution

Letting P denote the price of £100 nominal of this security, and assuming a term of n years, we have:

$$P = 3a_{\overline{n}|} + 100v^n$$

(a) ***Term of 5 years***

When $i = 0.07$, $P = 3 \times 4.1002 + 100 \times 1.07^{-5} = £83.60$.

When $i = 0.08$, $P = 3 \times 3.9927 + 100 \times 1.08^{-5} = £80.04$.

This is a fall in price of 4.3%.

(b) ***Term of 25 years***

When $i = 0.07$, $P = 3 \times 11.6536 + 100 \times 1.07^{-25} = £53.39$.

When $i = 0.08$, $P = 3 \times 10.6748 + 100 \times 1.08^{-25} = £46.63$.

This is a fall in price of 12.7%.

So we see that the change in interest rate has a greater effect on the longer 25-year security, than it has on the shorter 5-year security.

Roughly speaking, a change in interest rates has the same effect on the present value of a cashflow series as it has on a zero-coupon bond with the same discounted mean term or volatility.

Note that another definition of duration exists: the *modified duration*. This can be expressed in terms of the Macaulay duration as:

$$\frac{\tau}{1+\frac{i^{(p)}}{p}}$$

where $i^{(p)}$ and p are as defined earlier.

4.4 Convexity

To determine more precisely the effect of a change in the interest rate (which we will need to do to carry out immunisation calculations in the next section), we need another quantity called *convexity*.

The *convexity* of the cashflow series $\{C_{t_k}\}$ is defined as:

$$c(i) = \frac{1}{A}\frac{d^2}{di^2}A = \frac{A''}{A}$$

$$= \left(\frac{1}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}}\right)\left(\sum_{k=1}^{n} C_{t_k}\ t_k\ (t_k+1) v_i^{t_k+2}\right)$$

Question

Consider a share that pays a dividend of D annually in arrears in perpetuity.

Derive an expression for the convexity of the share's cashflows, in terms of the annual effective interest rate i.

Solution

The present value of the dividends is:

$$P(i) = Da_{\overline{\infty}|} = \frac{D}{i}$$

Differentiating:

$$P'(i) = -\frac{D}{i^2} \quad \text{and} \quad P''(i) = \frac{2D}{i^3}$$

So the convexity is:

$$\frac{P''(i)}{P(i)} = \frac{2D}{i^3} \Big/ \frac{D}{i} = \frac{2D}{i^3} \times \frac{i}{D} = \frac{2}{i^2}$$

For series of cashflows with the same discounted mean term, a series consisting of payments paid close together will have a low convexity, whereas a series that is more spread out over time will have a higher convexity.

Question

Consider the following two assets:

- Asset A is an 11-year zero-coupon bond.
- Asset B will provide a lump sum payment of £9,663 in 5 years' time and a lump sum payment of £26,910 in 20 years' time.

For each asset, calculate the volatility and convexity, using an interest rate of 10% *pa* effective.

Solution

Asset A

The present value of £100 nominal of Asset A at interest rate i is:

$$P_A(i) = 100(1+i)^{-11}$$

The volatility and convexity are given by:

$$\nu_A(i) = -\frac{P_A'(i)}{P_A(i)} = -\frac{100(-11)(1+i)^{-12}}{100(1+i)^{-11}} = \frac{11}{1+i}$$

and: $$c_A(i) = \frac{P_A''(i)}{P_A(i)} = \frac{100(-11)(-12)(1+i)^{-13}}{100(1+i)^{-11}} = \frac{11\times 12}{(1+i)^2}$$

When $i = 10\%$, we have:

$$\nu_A(0.1) = \frac{11}{1.1} = 10$$

and: $$c_A(0.1) = \frac{11\times 12}{1.1^2} = 109.1$$

The volatility of 10 corresponds to a discounted mean term of:

$$\tau_A = 1.1\times \nu_A(0.1) = 1.1\times 10 = 11 \text{ years}$$

as expected for an 11-year zero-coupon bond.

Asset B

The present value of Asset B at interest rate i is:

$$P_B(i) = 9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}$$

The volatility and convexity are given by:

$$\nu_B(i) = -\frac{P_B'(i)}{P_B(i)} = -\frac{9{,}663(-5)(1+i)^{-6} + 26{,}910(-20)(1+i)^{-21}}{9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}}$$

and: $$c_B(i) = \frac{P_B''(i)}{P_B(i)} = \frac{9{,}663(-5)(-6)(1+i)^{-7} + 26{,}910(-20)(-21)(1+i)^{-22}}{9{,}663(1+i)^{-5} + 26{,}910(1+i)^{-20}}$$

When $i = 10\%$, we have:

$$\nu_B(0.1) = -\frac{9{,}663(-5)(1.1)^{-6} + 26{,}910(-20)(1.1)^{-21}}{9{,}663(1.1)^{-5} + 26{,}910(1.1)^{-20}} = 10$$

and: $$c_B(0.1) = \frac{9{,}663(-5)(-6)(1.1)^{-7} + 26{,}910(-20)(-21)(1.1)^{-22}}{9{,}663(1.1)^{-5} + 26{,}910(1.1)^{-20}} = 153.7$$

So, Asset A and Asset B have the same volatilities (and hence the same discounted mean term of 11 years), but Asset B has the higher convexity because it involves payments that are more spread out around the discounted mean term.

Combining convexity and duration gives a more accurate approximation to the change in *A* following a small change in interest rates. For small ε:

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{\partial A}{\partial i} \times \frac{1}{A} \times \varepsilon + \tfrac{1}{2} \times \frac{\partial^2 A}{\partial i^2} \times \frac{1}{A} \times \varepsilon^2 + \cdots$$

$$\approx -\varepsilon \nu(i) + \varepsilon^2 \times \tfrac{1}{2} \times c(i)$$

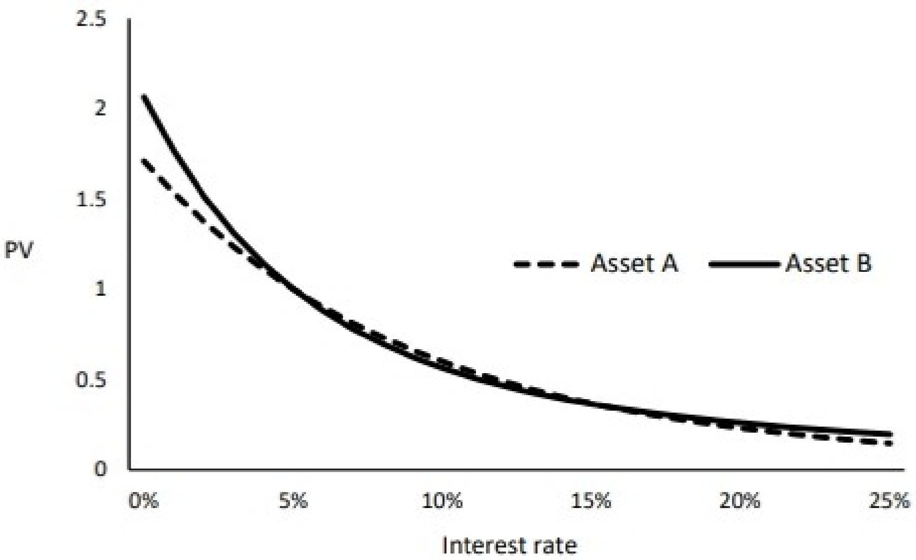
This last result comes from applying Taylor's formula, which appears on page 3 of the *Tables*:

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

Convexity gives a measure of the change in duration of a bond when the interest rate changes. Positive convexity implies that $\tau(i)$ is a decreasing function of i. This means, for example, that A increases more when there is a decrease in interest rates than it falls when there is an increase of the same magnitude in interest rates.

Why is it called 'convexity'?

'Convexity' refers to the shape of the graph of the present value as a function of the interest rate. The following graph shows the present value of the two assets in the previous question (scaled to have a present value of 1 unit at 5% interest). We can see that Asset B (which has the higher convexity) has a more 'curved' graph than Asset A (which has the lower convexity).



4.5 Immunisation

Suppose an organisation has liabilities that will require a known series of cashflows (which we will assume are all negative) and holds assets that will generate a known series of cashflows (which we will assume are all positive) to meet these liabilities.

If it were possible to select a portfolio of assets that generated cashflows that exactly matched the liabilities of the fund (in terms of timing and amount), then the fund would be completely protected against any changes in interest rates. However, this is an idealised scenario and, apart from in very simple cases, perfect matching of this kind cannot be achieved.

It may, however, be possible to choose an asset portfolio that offers the fund a milder form of protection. Suppose the present value of the fund's liabilities and assets, calculated at a valuation rate of interest i, which reflects the interest rate in the market, are $V_L(i)$ and $V_A(i)$, respectively. Then the fund has a *surplus* of $S(i)=V_A(i)-V_L(i)$. We can consider how this surplus would be affected by changes in the interest rate i. In particular, we would be concerned about the downside risk if a change in market interest rates causes the surplus to become negative, *ie* a deficit.

In simple cases, it is possible to select an asset portfolio that will protect this surplus against small changes in the interest rate. This is known as *immunisation*. In the 1950s, the actuary Frank Redington derived the three conditions that are required to achieve immunisation.

Consider a fund with asset cashflows $\{A_{t_k}\}$ and liability cashflows $\{L_{t_k}\}$. Let $V_A(i)$ be the present value of the assets at effective rate of interest i and let $V_L(i)$ be the present value of the liabilities at rate i; let $\nu_A(i)$ and $\nu_L(i)$ be the volatility of the asset and liability cashflows respectively, and let $c_A(i)$ and $c_L(i)$ be the convexity of the asset and liability cashflows respectively.

At rate of interest i_0 the fund is immunised against small movements in the rate of interest of ε if and only if $V_A(i_0)=V_L(i_0)$ and $V_A(i_0+\varepsilon)\geq V_L(i_0+\varepsilon)$.

In words, a fund is said to be *immunised* against small changes in the interest rate if:

- the surplus in the fund at the current interest rate is zero and
- any small change in the interest rate (in either direction) would lead to a positive surplus.

Redington's conditions

Then consider the surplus $S(i) = V_A(i) - V_L(i)$.

From Taylor's theorem:

$$S(i_0 + \varepsilon) = S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2} S''(i_0) + \cdots$$

Consider the terms on the right hand side. We know that $S(i_0) = 0$.

This is because the definition of immunisation requires that the surplus is zero at the current interest rate *ie* $S(i_0) = V_A(i_0) - V_L(i_0) = 0$. In other words, $V_A(i_0) = V_L(i_0)$, *ie* the present value of the assets at the current interest rate is equal to the present value of the liabilities.

The second term, $\varepsilon S'(i_0)$, will be equal to zero for any values of ε (positive or negative) if and only if $S'(i_0) = 0$, that is if $V_A'(i_0) = V_L'(i_0)$.

Since we have already assumed that $V_A(i_0) = V_L(i_0)$, this is equivalent to:

$$-\frac{V_A'(i_0)}{V_A(i_0)} = -\frac{V_L'(i_0)}{V_L(i_0)}$$

In other words, the assets and the liabilities must have the same volatility.

This is equivalent to requiring that $\nu_A(i) = \nu_L(i)$ or (equivalently) that the durations of the two cashflow series are the same.

In the third term, $\frac{\varepsilon^2}{2}$ is always positive, regardless of the sign of ε. Thus, if we ensure that $S''(i_0) > 0$, then the third term will also always be positive.

This is equivalent to requiring that $V_A''(i_0) > V_L''(i_0)$, which is equivalent to requiring that $c_A(i) > c_L(i)$.

For small $|\varepsilon|$ the fourth and subsequent terms in the Taylor expansion will be very small. Hence, given the three conditions above, the fund is protected against small movements in interest rates. This result is known as *Redington's immunisation* after the British actuary who developed the theory.

The conditions for Redington's immunisation may be summarised as follows:

1. $V_A(i_0) = V_L(i_0)$ **– that is, the value of the assets at the starting rate of interest is equal to the value of the liabilities.**

2. **The volatilities of the asset and liability cashflow series are equal, that is,** $\nu_A(i_0) = \nu_L(i_0)$.

3. **The convexity of the asset cashflow series is greater than the convexity of the liability cashflow series – that is,** $c_A(i_0) > c_L(i_0)$.

As mentioned in the above reasoning, the second condition could be replaced by one of the following equivalent conditions:

- the discounted mean terms of the asset and liability cashflow series are equal
- $V_A'(i_0) = V_L'(i_0)$

The third condition could be replaced by the equivalent condition:

- $V_A''(i_0) > V_L''(i_0)$

Question

A fund must make payments of £50,000 at time 6 years and at time 8 years. Interest rates are currently 7% *pa* effective at all durations.

Show that immunisation against small changes in the interest rate can be achieved by purchasing appropriately chosen nominal amounts of a 5-year zero-coupon bond and a 10-year zero-coupon bond.

Solution

Let P denote the nominal amount purchased of the 5-year bond (so that a cashflow of P is received at time 5) and let Q denote the nominal amount purchased of the 10-year (so that a cashflow of Q is received at time 10). Then the present value of the assets is:

$$V_A(0.07) = Pv^5 + Qv^{10} \quad @7\%$$

By Redington's first condition, this must equal the present value of the liabilities, which is:

$$V_L(0.07) = 50{,}000(v^6 + v^8) \ @7\% = £62{,}418$$

This gives us our first equation:

$$Pv^5 + Qv^{10} = 62{,}418 \qquad (1)$$

The negative of the derivative (with respect to i) of the PV of the assets is given by:

$$-V_A'(0.07) = P \times 5v^6 + Q \times 10v^{11} \quad @7\%$$

So the volatility of the assets is:

$$-\frac{V_A'(0.07)}{V_A(0.07)}=\frac{5Pv^6+10Qv^{11}}{V_L(0.07)}=\frac{5Pv^6+10Qv^{11}}{62,418}$$

where we have used equation 1 to set $V_A(0.07)=V_L(0.07)=£62,418$.

By Redington's second condition, this must equal the volatility of the liabilities, which is:

$$-\frac{V_L'(0.07)}{V_L(0.07)}=\frac{50,000(6v^7+8v^9)}{62,418}=\frac{404,398}{62,418}$$

Since the denominators in the volatilities are the same, we have the second equation:

$$5Pv^6+10Qv^{11}=404,398 \qquad (2)$$

We can solve equations (1) and (2) simultaneously. Multiplying equation (1) by $5v$ and then subtracting this from equation (2) gives:

$$5Qv^{11}=404,398-5v\times 62,418 \quad\Rightarrow\quad Q=\frac{404,398-5\times 1.07^{-1}\times 62,418}{5\times 1.07^{-11}}=£47,454$$

Substituting this into equation 1 gives:

$$P=(62,418-Qv^{10})(1+i)^5=(62,418-47,454\times 1.07^{-10})(1.07)^5=£53,710$$

This determines the portfolio of assets we require. We now need to check Redington's third condition. With these values of P and Q, the convexity of the assets is:

$$\frac{V_A''(0.07)}{V_A(0.07)}=\frac{P(-5)(-6)v^7+Q(-10)(-11)v^{12}}{62,418}=\frac{3,321,152}{62,418}=53.21$$

The convexity of the liabilities is:

$$\frac{V_L''(0.07)}{V_L(0.07)}=\frac{50,000((-6)(-7)v^8+(-8)(-9)v^{10})}{62,418}=\frac{50,000\times 61.045}{62,418}=48.90$$

So the convexity of the assets exceeds the convexity of the liabilities. This is what we would expect, since the asset cashflows (at times 5 and 10) are more spread out around the discounted mean term than the liability cashflows (at times 6 and 8).

Since all three of Redington's conditions are satisfied, the fund is immunised against small changes in the interest rate around 7% *pa*.

We can verify numerically that this fund is indeed immunised by calculating the surplus for interest rates a small distance either side of 7% pa.

The surplus, calculated at interest rate i, is:

$$V_A(i) - V_L(i) = (53,710v^5 + 47,454v^{10}) - 50,000(v^6 + v^8)$$

So: $V_A(0.065) - V_L(0.065) = 64,482 - 64,478 = 4 > 0$

and: $V_A(0.075) - V_L(0.075) = 60,437 - 60,433 = 4 > 0$

We see that a 0.5% movement in interest rates in either direction will result in a positive surplus, ie the fund is immunised.

In practice there are difficulties with implementing an immunisation strategy based on these principles. For example, the method requires continuous rebalancing of portfolios to keep the asset and liability volatilities equal.

The asset portfolio required to provide Redington immunisation normally depends on the initial interest rate. Once the interest rates have moved away from the initial rate, it may be necessary to 'rebalance' the portfolio so that it is once again immunised at the new rate. This makes the practical application of the technique quite laborious except in very simple situations.

Other limitations of immunisation include:

- **There may be options or other uncertainties in the assets or in the liabilities, making the assessment of the cashflows approximate rather than known.**
- **Assets may not exist to provide the necessary overall asset volatility to match the liability volatility.**
- The theory relies upon small changes in interest rates. The fund may not be protected against large changes.
- The theory assumes a flat yield curve and requires the same change in interest rates at all terms. In practice, this is rarely the case.
- Immunisation removes the likelihood of making large profits.

Despite these problems, immunisation theory remains an important consideration in the selection of assets.

In practice, actuaries making investment decisions are aware of Redington's theory in a general sense. For example, they are aware of the consequences of investing 'long' (*ie* holding assets with a higher DMT than the liabilities), but they would not normally apply the theory directly. A more open-ended technique called *asset-liability modelling* is often used instead, and this is covered in later subjects.

Chapter 13 Summary

The yield on a unit zero-coupon bond with term *n* years is called the *n*-year spot rate of interest, y_n .

The variation by term of interest rates is often referred to as the term structure of interest rates. The curve of spot rates is an example of a yield curve.

The discrete-time forward rate, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of *r* years.

There is a direct relationship between forward rates of interest and spot rates:

$$(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}$$

where P_t denotes the price at issue of a unit zero-coupon bond with term t years.

The three most popular explanations for the fact that interest rates vary according to the term of the investment are:

1. Expectations theory
2. Liquidity preference
3. Market segmentation

The performance of a fixed-interest investment can be assessed by its yield to maturity or redemption yield.

The *n*-year par yield represents the coupon per £1 nominal that would be payable on a bond with term *n* years, which would give the bond a price under the current term structure of £1 per £1 nominal, assuming the bond is redeemed at par.

The effects of changes in interest rates on the cashflows generated by an asset or required by a liability can be quantified by calculating the discounted mean term (duration), the volatility and the convexity. The discounted mean term and the volatility are related.

$$\textit{Volatility} = \nu(i) = -\frac{A'}{A} = \sum_{k=1}^{n} C_k t_k v^{t_k+1} \Big/ \sum_{k=1}^{n} C_k v^{t_k}$$

$$DMT = \tau(i) = \sum_{k=1}^{n} t_k C_k v^{t_k} \Big/ \sum_{k=1}^{n} C_k v^{t_k} = (1+i) \times \textit{Volatility}$$

$$\textit{Convexity} = c(i) = \frac{A''}{A} = \sum_{k=1}^{n} C_k t_k (t_k+1) v^{t_k+2} \Big/ \sum_{k=1}^{n} C_k v^{t_k}$$

The surplus in a fund can be immunised against small changes in interest rates if Redington's conditions can be met. These require the assets and liabilities to have the same present value and discounted mean term (or volatility), and for the convexity of the assets to exceed that of the liabilities. The conditions are:

1. $V_A(i_0) = V_L(i_0)$ *ie* $PV(Assets) = PV(Liabilities)$

2. $V'_A(i_0) = V'_L(i_0)$ *ie* $Volatility(Assets) = Volatility(Liabilities)$

   or $DMT(Assets) = DMT(Liabilities)$

3. $V''_A(i_0) > V''_L(i_0)$ *ie* $Convexity(Assets) > Convexity(Liabilities)$

Chapter 13 Practice Questions

13.1 The n-year spot rate is estimated using the function $y_n = 0.09 - 0.03e^{-0.1n}$. Calculate the one-year forward rate at time 10.

13.2 Let f_t denote the 1-year forward rate for a transaction beginning at time t. Calculate the 3-year par yield, given that:

$$f_0 = 6\%, \qquad f_1 = 6.5\%, \qquad f_2 = 7\%$$

13.3 In a particular bond market, the two-year par yield at time $t = 0$ is 5.65% and the issue price at time $t = 0$ of a two-year fixed-interest stock, paying coupons of 7% annually in arrears and redeemed at 101, is £103.40 per £100 nominal.

Exam style

Calculate:

(a) the one-year spot rate

(b) the two-year spot rate. [6]

13.4 Three bonds each paying annual coupons in arrears of 6% and redeemable at £103 per £100 nominal reach their redemption dates in exactly one, two and three years' time, respectively. The price of each bond is £97 per £100 nominal.

Exam style

(i) Calculate the gross redemption yield of the 3-year bond. [3]

(ii) Calculate the one-year and two-year spot rates implied by the information given. [3]

[Total 6]

13.5 (i) Explain what is meant by the following theories of the shape of the yield curve:

Exam style

(a) market segmentation theory

(b) liquidity preference theory. [4]

Short-term, one-year annual effective interest rates are currently 6%; they are expected to be 5% in one year's time; 4% in two years' time and 3% in three years' time.

(ii) Calculate the gross redemption yields from one-year, two-year, three-year and four-year zero-coupon bonds using the above expected interest rates. [4]

The price of a coupon-paying bond is calculated by discounting individual payments from the bond at the zero-coupon yields in part (ii).

(iii) Calculate the gross redemption yield of a bond that pays a coupon of 4% per annum annually in arrears and is redeemed at 110% in exactly four years. [5]

(iv) Explain why the gross redemption yield of a bond that pays a coupon of 8% per annum annually in arrears and is redeemed at par would be greater than that calculated in part (iii). [2]

The government introduces regulations that require banks to hold more government bonds with very short terms to redemption.

(v) Explain, with reference to market segmentation theory, the likely effect of this regulation on the pattern of spot rates calculated in part (ii). [2]

[Total 17]

13.6 Consider a fixed-interest security that pays coupons of 10% at the end of each year and is redeemable at par at the end of the third year.

Calculate, using an effective interest rate of 8% *pa*, the:

(i) volatility of the cashflows

(ii) discounted mean term of the cashflows

(iii) convexity of the cashflows.

13.7 Consider the three 20-year annuities described below:

(i) level payments of £1,000 payable annually in arrears

(ii) increasing payments made annually in arrears, where the first payment is £1,000 and the payments increase by 10% *pa* compound each year thereafter

(iii) continuous payments at the rate of £1,000 *pa* over the 20 years.

Calculate the discounted mean term of each annuity using an interest rate of 10% *pa* effective.

13.8 *Exam style*

A company has to pay £ $2,000(10-t)$ at the end of year t, for $t=5,6,7,8,9$. It values these liabilities assuming that there will be a constant effective annual rate of interest of 6% *pa*.

(i) Calculate the present value of the company's liabilities. [3]

The company wants to immunise its exposure to the liabilities by investing in two bonds:

- Bond A pays coupons of 5% *pa* annually in arrears and is redeemable at par in 15 years' time
- Bond B is a zero-coupon bond that is redeemable at par in 5 years' time.

The gross redemption yield on both stocks is the same as the interest rate used to value the liabilities.

(ii) (a) Determine the amount that the company should invest in each of the two bonds to ensure that the present value and discounted mean term of the assets are equal to those of the liabilities.

(b) State the third condition required for immunisation. [9]

[Total 12]

13.9 *Exam style*

An insurance company has liabilities of £10 million due in 10 years' time and £20 million due in 15 years' time. The company's assets consist of two zero-coupon bonds. One pays £7.404 million in 2 years' time and the other pays £31.834 million in 25 years' time. The current interest rate is 7% per annum effective.

(i) Show that Redington's first two conditions for immunisation against small changes in the rate of interest are satisfied for this insurance company. [6]

(ii) Calculate the present value of profit that the insurance company will make if the interest rate increases immediately to 7.5% per annum effective. [2]

(iii) Explain, without any further calculation, why the insurance company made a profit as a result of the change in the interest rate. [2]

[Total 10]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 13 Solutions

13.1 Using the formula given, the 10-year and 11-year spot rates are:

$$y_{10} = 0.09 - 0.03e^{-0.1\times 10} = 0.07896$$

and:

$$y_{11} = 0.09 - 0.03e^{-0.1\times 11} = 0.08001$$

Therefore:

$$1 + f_{10} = \frac{(1+y_{11})^{11}}{(1+y_{10})^{10}} = 1.0906$$

ie the one-year forward rate at time 10 is 9.06%.

13.2 The par yield, c, is the annual coupon that gives a bond price equal to the par value based on the current term structure of interest rates.

Using the 1-year forward rates given, the 3-year par yield is found from the equation:

$$1 = c\left(\frac{1}{1.06} + \frac{1}{1.06\times 1.065} + \frac{1}{1.06\times 1.065\times 1.07}\right) + \frac{1}{1.06\times 1.065\times 1.07}$$

ie $\quad 1 = c(0.94340 + 0.88582 + 0.82787) + 0.82787 \quad \Rightarrow \quad c = 0.06478$

The 3-year par yield is therefore 6.478%.

13.3 Let y_1 be the one-year spot rate and y_2 be the two-year spot rate.

Using the two-year par yield of 5.65%, and assuming that we have £100 nominal, gives:

$$100 = \frac{5.65}{1+y_1} + \frac{100+5.65}{(1+y_2)^2} = \frac{5.65}{1+y_1} + \frac{105.65}{(1+y_2)^2}$$ [1]

Considering the two-year fixed-interest bond gives a second equation:

$$103.40 = \frac{7}{1+y_1} + \frac{101+7}{(1+y_2)^2} = \frac{7}{1+y_1} + \frac{108}{(1+y_2)^2}$$ [1]

Rearranging both equations to get $\dfrac{1}{(1+y_2)^2}$ gives:

$$\frac{1}{(1+y_2)^2} = \frac{1}{105.65}\left(100 - \frac{5.65}{1+y_1}\right) = \frac{1}{108}\left(103.40 - \frac{7}{1+y_1}\right)$$

This can be solved for y_1 :

$$10,800-\frac{610.2}{1+y_1}=10,924.21-\frac{739.55}{1+y_1}$$

$$\Rightarrow \quad \frac{129.35}{1+y_1}=124.21$$

$$\Rightarrow \quad y_1=4.14\%$$ [2]

Substituting this into the equation for the par yield:

$$100=\frac{5.65}{1.0414}+\frac{105.65}{(1+y_2)^2} \quad \Rightarrow \quad (1+y_2)^2=1.11711 \quad \Rightarrow \quad y_2=5.69\%$$ [2]

So:

(a) the one-year spot rate is 4.14%

(b) the two-year spot rate is 5.69%.

[Total 6]

13.4 *This question is Subject CT1, April 2013, Question 3.*

(i) ***GRY of the three-year bond***

The equation of value for the three-year bond is:

$$97=6v+6v^2+109v^3$$ [1]

This equation can be solved by trial and error. To obtain a first guess for the gross redemption yield, we note that the coupons provide an annual return of about 6%, and there is a capital gain at redemption of 6 (the redemption amount minus the price paid), so the overall yield will be greater than 6%.

Using trial and error we get:

$$i=8\% \quad \Rightarrow \quad RHS=97.2273$$

$$i=8.5\% \quad \Rightarrow \quad RHS=95.9637$$ [1]

Interpolating between 8% and 8.5%, we have:

$$GRY \approx 8\%+\frac{97-97.2273}{95.9637-97.2273}\times(8.5\%-8\%)=8.09\%\ pa$$ [1]

[Total 3]

(ii) ***Spot rates***

The equation of value for the one-year bond (with one-year spot rate y_1) is:

$$97 = 109 \times \frac{1}{1+y_1} \Rightarrow y_1 = 12.371\%\ pa \qquad [1]$$

The equation of value for the two-year bond (with two-year spot rate y_2) is:

$$97 = 6 \times \frac{1}{1+y_1} + 109 \times \frac{1}{(1+y_2)^2} \Rightarrow y_2 = 9.049\%\ pa \qquad [2]$$

[Total 3]

13.5 *This question is Subject CT1, September 2014, Question 8.*

(i)(a) ***Market segmentation theory***

Market segmentation theory says that the shape of the yield curve is determined by supply and demand at different terms. So, for example, yields at the short end of the curve will be determined by demand from investors who have a preference for short-dated stocks, *ie* those with short-dated liabilities, as well as by the supply of short-dated stock. Similarly, yields at the long end will be a function of demand from investors interested in buying long bonds to match long-dated liabilities, and of supply of long-dated stock. [2]

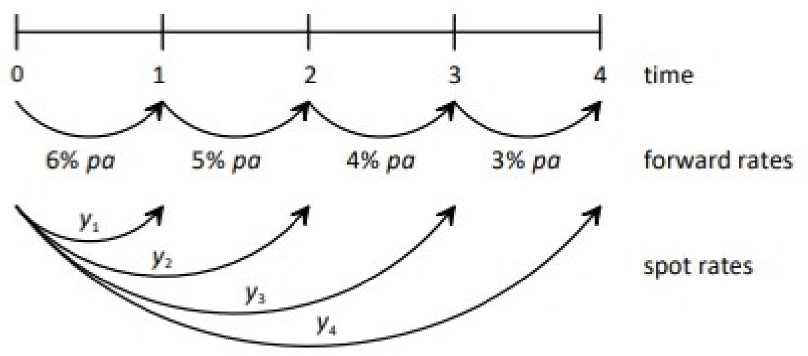
(i)(b) ***Liquidity preference theory***

Liquidity preference theory states that investors will in general prefer more liquid (*ie* shorter) stocks to less liquid ones, as short-dated stocks are less sensitive to changes in interest rates. Hence, investors purchasing long-dated bonds will require higher yields in order to compensate them for the greater volatility of the stock they are purchasing. [2]

[Total 4]

(ii) ***Gross redemption yields from zero-coupon bonds***

The diagram of the rates we are given in the question is:



The gross redemption yields for the zero-coupon bonds are the spot rates. Hence:

$$y_1 = 6\%\ pa \qquad [1]$$

$$(1+y_2)^2 = 1.06 \times 1.05 \;\Rightarrow\; y_2 = 5.499\%\ pa \qquad [1]$$

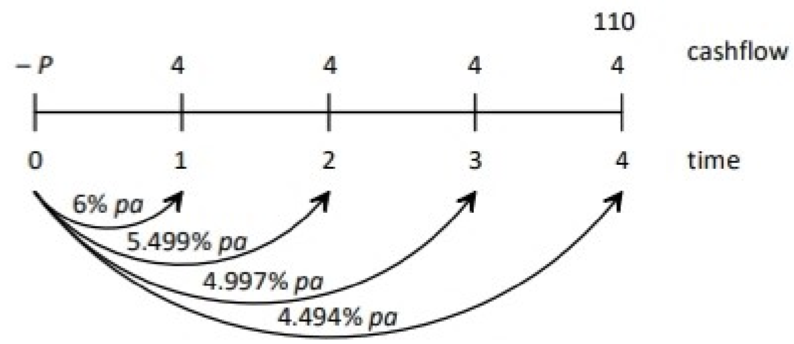
$$(1+y_3)^3 = 1.06 \times 1.05 \times 1.04 \;\Rightarrow\; y_3 = 4.997\%\ pa \qquad [1]$$

$$(1+y_4)^4 = 1.06 \times 1.05 \times 1.04 \times 1.03 \;\Rightarrow\; y_4 = 4.494\%\ pa \qquad [1]$$

[Total 4]

(iii) ***Gross redemption yield from coupon-paying bond***

The cashflow diagram is:



So the price of the bond is given by:

$$P = 4\left[1.06^{-1} + 1.05499^{-2} + 1.04997^{-3}\right] + 114 \times 1.04494^{-4} = 106.441 \qquad [1]$$

The equation for the gross redemption yield is:

$$106.441 = 4a_{\overline{4}|} + 110v^4 \qquad [1]$$

We will use trial and error to solve this for the GRY. Since we discounted the cashflows at spot rates of 6%, 5.499%, 4.997% and 4.494%, the GRY will lie between the smallest and largest of these values. Also, since most of the cashflow arises at the end of four years, it is likely that y_4 will give us a reasonable first approximation.

$$i = 4.5\% \;\Rightarrow\; 4a_{\overline{4}|} + 110v^4 = 106.592 \qquad [1]$$

$$i = 5\% \;\Rightarrow\; 4a_{\overline{4}|} + 110v^4 = 104.681 \qquad [1]$$

Using linear interpolation between these two values, we obtain a gross redemption yield of:

$$4.5\% + \frac{106.441 - 106.592}{104.681 - 106.592} \times (5\% - 4.5\%) = 4.54\%$$ [1]

[Total 5]

(iv) ***Explanation***

For the bond with the 8% coupon redeemed at par, a greater proportion of the proceeds are received earlier, in the coupon payments, rather than in the redemption proceeds. [1]

Since interest rates are higher earlier on, the GRY (which is a weighted average of the rates used to discount the payments) would be higher. [1]

[Total 2]

(v) ***Effect of regulation***

If banks are required to hold more short-dated bonds, demand for these types of bonds will increase. [1]

If the supply of these bonds remains the same, this will push up the price of these bonds, causing their gross redemption yields and hence spot rates to fall. [1]

[Total 2]

13.6 The present value of the cashflows for £100 nominal of this stock is:

$$P(i) = 10v + 10v^2 + 110v^3$$

Differentiating gives:

$$P'(i) = 10(-1)v^2 + 10(-2)v^3 + 110(-3)v^4$$

$$P''(i) = 10(-1)(-2)v^3 + 10(-2)(-3)v^4 + 110(-3)(-4)v^5$$

(i) Using the above expressions, the volatility is:

$$-\frac{P'(0.08)}{P(0.08)} = -\frac{10(-1)(1.08)^{-2} + 10(-2)(1.08)^{-3} + 110(-3)(1.08)^{-4}}{10(1.08)^{-1} + 10(1.08)^{-2} + 110(1.08)^{-3}} = \frac{267.01}{105.15} = 2.54$$

(ii) The discounted mean term is the volatility multiplied by $1+i$:

$$2.54 \times 1.08 = 2.74 \text{ years}$$

(iii) The convexity is:

$$\frac{P''(0.08)}{P(0.08)} = \frac{10(-1)(-2)(1.08)^{-3} + 10(-2)(-3)(1.08)^{-4} + 110(-3)(-4)(1.08)^{-5}}{105.15}$$

$$= \frac{958.35}{105.15} = 9.11$$

13.7 (i) For the level annuity, payable in arrears, we have:

$$DMT=\frac{\sum_{t=1}^{20} t\times 1{,}000v^t}{\sum_{t=1}^{20} 1{,}000v^t}=\frac{\sum_{t=1}^{20} tv^t}{\sum_{t=1}^{20} v^t}=\frac{(Ia)_{\overline{20|}}}{a_{\overline{20|}}}=\frac{63.9205}{8.5136}=7.51 \text{ years}$$

(ii) For the compound increasing annuity, payable in arrears, we have:

$$DMT=\frac{\sum_{t=1}^{20} t\times 1{,}000\times 1.1^{t-1}\times v^t}{\sum_{t=1}^{20} 1{,}000\times 1.1^{t-1}\times v^t}=\frac{\sum_{t=1}^{20} t\times 1.1^{t-1}\times v^t}{\sum_{t=1}^{20} 1.1^{t-1}\times v^t}$$

Here $i=0.1$, so $1.1^{t-1}v^{t-1}=1$, and this expression simplifies to:

$$DMT=\frac{\sum_{t=1}^{20} tv}{\sum_{t=1}^{20} v}=\frac{\sum_{t=1}^{20} t}{\sum_{t=1}^{20} 1}=\frac{\frac{1}{2}\times 20\times 21}{20}=10.5 \text{ years}$$

To evaluate the numerator in the expression above, we have used the formula for the sum of the first n integers:

$$\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$$

(iii) For the continuous annuity, we have:

$$\frac{\int_0^{20} t\times 1{,}000v^t\,dt}{\int_0^{20} 1{,}000v^t\,dt}=\frac{\int_0^{20} tv^t\,dt}{\int_0^{20} v^t\,dt}=\frac{(\bar{Ia})_{\overline{20|}}}{\bar{a}_{\overline{20|}}}=\frac{62.5286}{8.9325}=7.00 \text{ years}$$

13.8 (i) ***Present value of liabilities***

The liabilities are illustrated on the timeline below:

		10,000	8,000	6,000	4,000	2,000	Payment
0	...	5	6	7	8	9	Time (yrs)

The present value of the liabilities is:

$$PV_L=10{,}000v^5+8{,}000v^6+6{,}000v^7+4{,}000v^8+2{,}000v^9$$ [1]

If $i = 0.06$, then:

$$PV_L = 10,000(1.06)^{-5} + 8,000(1.06)^{-6} + 6,000(1.06)^{-7} + 4,000(1.06)^{-8} + 2,000(1.06)^{-9}$$

$$= £20,796$$ [2]

[Total 3]

Alternatively, we could calculate this as:

$$PV_L = 2,000v^4\left[6a_{\overline{5}|} - (Ia)_{\overline{5}|}\right] = 2,000(1.06)^{-4}(25.2742 - 12.1469)$$

(ii)(a) ***Amount to be invested in each bond***

Let A denote the amount invested in Bond A and B denote the amount invested in Bond B.

The present value of the assets is then:

$$PV_A = A + B$$

Setting this equal to the present value of the liabilities, we have:

(1) ... $A + B = 20,796$ [1]

The discounted mean term of the liabilities is:

$$DMT_L = \frac{5\times 10,000v^5 + 6\times 8,000v^6 + 7\times 6,000v^7 + 8\times 4,000v^8 + 9\times 2,000v^9}{PV_L}$$

$$= \frac{129,865}{PV_L} = 6.245 \text{ years}$$ [1]

To calculate the discounted mean term of the assets, we first need to calculate the price of Bond A per £100 nominal. This is:

$$P = 5a_{\overline{15}|} + 100v^{15} = (5\times 9.7122) + \frac{100}{1.06^{15}} = 90.288$$ [1]

So an investment of A in Bond A buys $\frac{A}{90.288}$ lots of £100 nominal.

The discounted mean term of the assets is then:

$$DMT_A = \frac{\frac{A}{90.288}\left(1\times 5v + 2\times 5v^2 + \dots + 15\times 5v^{15} + 15\times 100v^{15}\right) + 5B}{PV_A}$$

$$= \frac{\frac{A}{90.288}\left(5(Ia)_{\overline{15|}} + 1{,}500v^{15}\right) + 5B}{PV_A}$$

$$= \frac{\frac{A}{90.288}\left(5\times 67.2668 + \frac{1{,}500}{1.06^{15}}\right) + 5B}{PV_A}$$

$$= \frac{10.657A + 5B}{PV_A} \qquad [2]$$

Setting the discounted mean term of the assets equal to the discounted mean term of the liabilities we obtain:

(2) ... $10.657A + 5B = 129{,}865$ [1]

Now multiplying (1) by 5, we get:

(3) ... $5A + 5B = 103{,}980$

and subtracting (3) from (2):

$5.657A = 25{,}885 \quad \Rightarrow \quad A = £4{,}576$ [1]

Substituting this back into (1), we find that:

$B = 20{,}796 - 4{,}576 = £16{,}220$ [1]

Alternatively, defining A to be the nominal amount purchased of Bond A, and B to be the nominal amount purchased of Bond B, the present value of the assets is:

$$PV_A = A\left(0.05a_{\overline{15|}} + v^{15}\right) + Bv^5 = 0.90288A + 0.74726B$$

The DMT of the assets is:

$$DMT_A = \frac{A\left(0.05v + 2\times 0.05v^2 + \dots + 15\times 0.05v^{15} + 15v^{15}\right) + 5Bv^5}{PV_A}$$

$$= \frac{A\left(0.05(Ia)_{\overline{15|}} + 15v^{15}\right) + 5Bv^5}{PV_A}$$

$$= \frac{9.6223A + 3.7363B}{PV_A}$$

These give us the simultaneous equations:

$$0.90288A + 0.74726B = 20,796$$

$$9.6223A + 3.7363B = 129,865$$

Solving these gives $A = 5,068$ *and* $B = 21,707$. *Converting these values into the amounts invested in the bonds (ie the present values) asked for in the question, gives the same answers as above.*

(ii)(b) ***Other condition for immunisation***

We also require that the convexity of the assets is greater than the convexity of the liabilities at the current rate of interest, *ie*:

$$\frac{d^2 PV_A}{di^2} \Big/ PV_A > \frac{d^2 PV_L}{di^2} \Big/ PV_L \qquad [1]$$

[Total 9]

13.9 (i) ***Two conditions for immunisation***

The present value of the assets and liabilities at 7% *pa* are:

$$PV_A = 7.404v^2 + 31.834v^{25} = 12.332 \qquad [½]$$

$$PV_L = 10v^{10} + 20v^{15} = 12.332 \qquad [½]$$

Since these are equal, the first condition for immunisation is satisfied. [1]

The discounted mean terms for the assets and liabilities at 7% *pa* are:

$$DMT_A = \frac{7.404v^2 \times 2 + 31.834v^{25} \times 25}{PV_A} = \frac{159.569}{PV_A} \qquad [1½]$$

$$DMT_L = \frac{10v^{10} \times 10 + 20v^{15} \times 15}{PV_L} = \frac{159.569}{PV_L} \qquad [1½]$$

Since we know that the denominators are equal and we can see that the numerators are equal, the DMTs are also equal. [1]

So the first two conditions for immunisation are both satisfied. [Total 6]

Instead of comparing the numerators here, we could have calculated the actual DMTs to be:

$$\frac{159.569}{12.332} = 12.939 \text{ years}$$

Alternatively, we could consider the volatilities:

$$vol_A = -\frac{PV_A'}{PV_A} = -\left[\frac{7.404v^3 \times (-2) + 31.834v^{26} \times (-25)}{PV_A}\right] = -\left[\frac{-149.130}{12.332}\right] = 12.093$$

$$vol_A = -\frac{PV_L'}{PV_L} = -\left[\frac{10v^{11} \times (-10) + 20v^{16} \times (-15)}{PV_L}\right] = -\left[\frac{-149.130}{12.332}\right] = 12.093$$

and conclude that the volatility of the assets is equal to the volatility of the liabilities (or the numerator of the volatility of the assets is equal to the numerator of the volatility of the liabilities).

(ii) ***Present value of profit***

The present value of the profit at the new interest rate of 7.5% *pa* is:

$$PV_{profit} = PV_A - PV_L = 7.404v^2 + 31.834v^{25} - 10v^{10} - 20v^{15} = 0.015773 \qquad [2]$$

This gives us a present value of profit of about £15,800. [Total 2]

(iii) ***Explanation of profit***

In this scenario, we can see that the spread of the times of the asset cashflows around the discounted mean term is greater than the corresponding spread of the liabilities. [1]

So the third condition for immunisation will also be satisfied, and we will be immunised against any losses arising from a small change in the interest rate. So, when the rate changes to 7.5% *pa*, we make a small profit. [1]

[Total 2]

End of Part 2

What next?

1. Briefly **review** the key areas of Part 2 and/or re-read the **summaries** at the end of Chapters 10 to 13.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 2. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X2**.
4. Attempt the questions relating to Chapters 10 to 13 of the **Paper B Online Resources (PBOR)**.
5. Attempt **Assignment Y1**.

14

The life table

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.1 Define the following terms:

- premium
- benefit

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming a constant deterministic interest rate.

4.2.1 Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

4.2.2 Define the following probabilities: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

4.2.3 Express the probabilities defined in 4.2.2 in terms of life table functions defined in 4.2.1.

0 Introduction

So far in this course, we have concentrated on placing a value on cashflows that are certain to happen. For example, $a_{\overline{5}|}$ gives the present value of a payment of 1 at the end of each of the next 5 years, where each payment definitely occurs.

We now start to consider valuing uncertain future cashflows, such as those faced by a life insurance company. These cashflows may be dependent on the survival, or otherwise, of an individual, or on other uncertain future events, such as when an individual becomes sick, or chooses to retire.

Some different types of life insurance contracts, *eg* term assurances and pure endowments, were first introduced in Chapter 2.

Life insurance contracts (also called policies) are made between a life insurance company and one or more persons called the policyholder(s).

The policyholder(s) will agree to pay an amount or a series of amounts to the life insurance company, called premiums.

The premiums may be paid:

- on a regular basis (known as regular premiums), typically paid monthly, quarterly or annually, or
- as one single payment (known as a single premium).

In return the life insurance company agrees to pay an amount or amounts called the benefit(s), to the policyholder(s) on the occurrence of a specified event.

In this subject we first consider contracts with a single policyholder and then later show how to extend the theory to two policyholders.

The benefits payable under simple life insurance contracts are of two main types.

(a) **The benefit may be payable on or following the death of the policyholder.**

For example, under a term assurance contract, the insurance company will make a payment to the policyholder's estate if the policyholder dies during the term of the policy.

(b) **The benefit(s) may be payable provided the life survives for a given term. An example of this type of contract is an annuity, under which amounts are payable at regular intervals as long as the policyholder is still alive.**

Such annuities are called 'life annuities' and we will look at these in Chapter 16. Note that life annuities are different to the annuities we studied in the earlier part of this course, as each payment under a life annuity is not certain to occur.

More generally, the theory of this subject may be applied to 'near-life' contingencies – such as the state of health of a policyholder – or to 'non-life' contingencies – such as the cost of replacing a machine at the time of failure. However, in this Subject, we confine ourselves to cases where the payment is of a known amount only.

1 Present values of payments under life insurance and annuity contracts

1.1 Equations of value

To calculate premiums for life insurance contracts, we can use a similar *equation of value* to the one we met in Chapter 9, and have used extensively since:

Present value of income = Present value of outgo

Now, because each payment is not certain to happen, we should include the probability of its occurrence and this leads us to consider *expected present values*, rather than present values. So, the equation of value becomes:

Expected present value of income = Expected present value of outgo

When we are applying this equation to calculate an insurance premium, we are considering the money into and out of the insurance company. So, the items that we will be considering are:

- premiums coming in
- expected payments to policyholders (*ie* benefits) going out
- expenses that are incurred by the company.

Much actuarial work is concerned with finding a fair price for a life insurance contract. In such calculations we must consider:

(a) the time value of money, and

(b) the uncertainty attached to payments to be made in the future, depending on the death or survival of a given life.

This requires us to bring together the topics covered earlier in this course, in particular compound interest, and the topics covered in Subject CS2, in particular the unknown future lifetime and its associated probabilities.

1.2 Allowance for investment income

Premiums are usually paid in advance (*eg* at the start of each month or year). This is to protect the insurance company against policyholders dying before paying any premium, and therefore receiving a benefit payment 'for free'. In addition, policyholders can lapse their contracts (*ie* stop paying their premiums), so if premiums were payable annually in arrears, policyholders might be tempted to lapse at the end of a year, thus receiving that year's insurance protection without paying the corresponding premium.

The benefits purchased by the premiums will be paid at a later time – sometimes much later, *eg* at the end of the term of the contract, or when the policyholder dies. This means that the insurer will need to allow for the time value of money when calculating the appropriate premium to charge.

In this and subsequent chapters we will usually assume that money can be invested or borrowed at some given rate of interest. We will always assume that the rate of interest is *known*, that is, deterministic, but we will not always assume that the rate of interest is *constant*. When the rate of interest *is* constant, we denote the effective compound rate of interest per annum by i and define $v = (1+i)^{-1}$, and we will use these without further comment.

In practice, life insurance companies may price products using either of the approaches mentioned above, *ie* they may:

- assume a constant interest rate for each future year
- use a deterministic approach where interest rates are assumed to change in a predetermined way.

Alternatively, they may use a stochastic approach, where future interest rates are random and follow a certain statistical distribution, although we will not consider this approach in this Subject.

1.3 Other assumptions

We will also assume knowledge of the basic probabilities introduced in CS2, namely $_tp_x$ and $_tq_x$ and their degenerate quantities, when $t=1$, p_x and q_x, and the force of mortality μ_x. Further, we assume knowledge of the key survival model formulae, the fundamental ones being that:

$$_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds$$

and:

$$_tp_x = \exp\left\{-\int_0^t \mu_{x+s} \, ds\right\}$$

In the notation defined above, $_tp_x$ is the probability that a life aged x survives for at least another t years and $_tq_x$ is the probability that a life aged x dies within the next t years. In addition, p_x is the probability that a life aged x survives for at least another year and q_x is the probability that a life aged x dies within the next year. Note that $_tp_x + {}_tq_x = 1$ and $p_x + q_x = 1$.

The force of mortality at exact age $x+s$, μ_{x+s}, denotes the annual rate of transfer between alive and dead at exact age $x+s$, *ie* it is the annual rate at which people are dying at that exact age. It is probably most helpful to think of this as:

$\mu_{x+s}\, ds \approx$ probability of a life aged $x+s$ dying over the short time interval $(x+s, x+s+ds)$

For example:

$${}_sp_x\,\mu_{x+s}\,ds \approx \text{ probabilty of a life aged } x \text{ living for another } s \text{ years and then dying in the next instant of time}$$

so (remembering that an integral is just the continuous version of a summation):

$${}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,ds = \text{ probability that a life aged } x \text{ dies at any of the possible moments over the next } t \text{ years}$$

Using the two basic building blocks described in Section 1.1, and the assumptions made in this section and the last, we will develop formulae for the means and variances of the present value of contingent benefits.

We will do this in Chapter 15 for assurance contracts and Chapter 16 for annuity contracts.

We will consider ways of assigning probability values to the unknown future lifetime, so as to evaluate the formulae. Two different assumptions are typically used in practice when determining the probability values. The first is to assume that the underlying probability depends on age only. The second is to assume that the underlying probability depends on age plus duration since some specific event. For example, considering mortality, the assumption in the second case can allow for the likely lower level of mortality which might result from having to pass a medical test before the insurer agrees to issue a life insurance contract.

Under the first assumption, all lives aged x are assumed to have the same mortality, regardless of when they took out their policies. Under the second assumption, this is no longer the case. The mortality of the policyholder is now assumed also to depend on the time that has elapsed since the policy was issued.

The first assumption is described as assuming *ultimate mortality*, and the second as assuming *select mortality*. We will return later to discussing select mortality but assume for the moment that ultimate mortality applies.

2 The life table

2.1 Introduction

As mentioned above, to calculate the expected present value of a life insurance contract, we need to consider probabilities of survival and death, *ie* ${}_tp_x$ and ${}_tq_x$.

The calculation of the relevant probabilities can be simplified by use of *life tables*.

The life table is a device for calculating probabilities such as ${}_tp_x$ and ${}_tq_x$ using a one-dimensional array. The key to the definition of a life table is the relationship:

$$ {}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s} $$

This result is called the *principle of consistency* and is introduced in Subject CS2. This result holds because, for example, surviving for $t+s$ years from age x is the same as surviving for t years from age x (to reach age $x+t$) and then surviving a further s years to reach age $x+t+s$.

2.2 Constructing a life table

To construct a life table, we choose a starting age, which will be the lowest age in the table. We denote this lowest age α. This choice of α will often depend on the data which were available. For example, in studies of pensioners' mortality it is unusual to observe anyone younger than (say) 50, so 50 might be a suitable choice for α in a life table that is to represent pensioners' mortality.

We also need to choose the highest age in the table, ω, which is the age beyond which survival is assumed to be impossible. ω is referred to as the *limiting age* of the table.

We next choose an arbitrary positive number and denote it l_α. We call l_α the *radix* of the life table. It is convenient to interpret l_α as being the number of lives starting out at age α in a homogeneous population, but the mathematics does not depend on this interpretation.

The notation here is a lower case l (for '*life table*'), not a capital *i*.

As we will see, the choice of the radix (l_α) is not important. When setting the radix, it is common to choose a large round number such as 100,000. This is purely for presentational purposes.

For $\alpha \le x \le \omega$, define the function l_x by:

$$ l_x = l_\alpha \times {}_{x-\alpha}p_\alpha $$

We assume that the probabilities ${}_{x-\alpha}p_\alpha$ are all known. By definition, $l_\omega = 0$.

Now we see that, for $\alpha \le x \le \omega$ and for $t \ge 0$:

$$ {}_tp_x = \frac{{}_{t+x-\alpha}p_\alpha}{{}_{x-\alpha}p_\alpha} = \frac{l_{x+t}}{l_\alpha} \times \frac{l_\alpha}{l_x} = \frac{l_{x+t}}{l_x} $$

Hence, if we know the function l_x for $\alpha \le x \le \omega$, we can find any probability ${}_tp_x$ or ${}_tq_x$.

The choice of radix is unimportant since the l_α terms cancel. The formula for ${}_tp_x$ involves a ratio of two entries from the life table, so the answer would be the same if the radix (and hence all the other figures in the life table) were multiplied by 2, say.

Question

Write down an expression for ${}_tq_x$ in terms of the function l_x .

Solution

The required expression is:

$${}_tq_x = 1 - {}_tp_x = 1 - \frac{l_{x+t}}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$

The function l_x is called the *life table*. It depends on age only, so it is more easily tabulated than the probabilities ${}_tp_x$ or ${}_tq_x$, although the importance of this has diminished with the widespread use of computers.

The life table is an important tool that enables actuaries to calculate a wide range of useful figures from a single set of tabulated factors. A clear understanding of the underlying principles is still important today since life tables are at the heart of much actuarial valuation software.

2.3 The force of mortality

Earlier in this chapter, we introduced the force of mortality at age x , μ_x , as the annual rate of transfer between alive and dead at exact age x , where:

$$\mu_x h \approx \text{probability of a life aged } x \text{ dying over the short time interval } (x, x+h) \;\; ie \; {}_hq_x$$

More formally, we write this as:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times {}_hq_x$$

In terms of life table functions, this is:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times \frac{l_x - l_{x+h}}{l_x} = -\frac{1}{l_x} \times \lim_{h \to 0+} \frac{l_{x+h} - l_x}{h}$$

The limit in this last expression matches the definition of a derivative. So we have:

$$\mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x = -\frac{d}{dx} \ln l_x$$

where the final equality uses the chain rule for differentiation.

2.4 Interpretation

If we interpret l_α to be the number of lives known to be alive at age α (in which case it has to be an integer) then we can interpret l_x $(x > \alpha)$ as the *expected* number of those lives who survive to age x.

After all, l_x $(x > \alpha)$ is defined as l_α multiplied by ${}_{x-\alpha}p_\alpha$, the probability of a life aged α surviving to age x.

A life table is sometimes given a deterministic interpretation. That is, l_α is interpreted as above, and l_x $(x > \alpha)$ is interpreted as the number of lives who will survive to age x, as if this were a fixed quantity.

Then the symbol ${}_tp_x = l_{x+t}/l_x$ is taken to be the proportion of the l_x lives alive at age x who survive to age $x+t$. This is the so-called 'deterministic model of mortality'. It is not such a fruitful approach as the stochastic model that we have outlined, and we will not use it. In particular, while it is useful in computing quantities like premium rates, it is of no use when we need to analyse mortality data.

This deterministic model of mortality is too inflexible for our needs. After all, lives rarely survive to later ages in the exact proportions dictated by the life table. The stochastic approach allows us to model the inherent variability.

Another point to note is that the probabilities, ${}_tp_x$ *etc*, that are used to construct a life table can only be *estimates* of the true underlying probabilities. So the life table does not *define* the survival probabilities and mortality rates – it is merely one representation of them.

2.5 Using the life table

We now introduce a further life table function d_x. For $\alpha \le x \le \omega - 1$, define:

$$d_x = l_x - l_{x+1}$$

We interpret d_x as the expected number of lives who die between age x and age $x+1$, out of the l_α lives alive at age α.

Note that:

$$q_x = 1 - p_x = \frac{l_x}{l_x} - \frac{l_{x+1}}{l_x} = \frac{d_x}{l_x}$$

We can also see that:

$$d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$$

and that (if x and ω are integers):

$$d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$$

Question

Express in words the two results:

(i) $d_x + d_{x+1} + \cdots + d_{x+n-1} = l_x - l_{x+n}$

(ii) $d_x + d_{x+1} + \cdots + d_{\omega-1} = l_x$

Solution

(i) The difference between the number of lives expected to be alive at age x and the number of lives expected to be alive at age $x+n$ (*ie* $l_x - l_{x+n}$) is equal to the total of the number of lives expected to die between those two ages (*ie* $d_x + d_{x+1} + \cdots + d_{x+n-1}$).

(ii) The expected number of lives alive at age x (*ie* l_x) equals the sum of the expected number of deaths at each age from x onwards, *ie* all those lives alive at age x must die at some point in the future.

It is usual to tabulate values of l_x and d_x at integer ages, and often other functions such as μ_x, p_x or q_x as well. For an example, see the English Life Table No. 15 (Males) in the 'Formulae and Tables for Examinations'.

ELT15 (Males) appears in the *Tables* on pages 68 and 69.

The following is an extract from that table.

Age, x	l_x	d_x	q_x	μ_x
0	100,000	814	0.00814	
1	99,186	62	0.00062	0.00080
2	99,124	38	0.00038	0.00043
3	99,086	30	0.00030	0.00033
4	99,056	24	0.00024	0.00027
5	99,032	22	0.00022	0.00023

(No value is given for μ_0 because of the difficulty of calculating a reasonable estimate from observed data). It is easy to check the relationships:

$$ {}_tp_x = \frac{l_{x+t}}{l_x} \qquad d_x = l_x - l_{x+1} \qquad q_x = \frac{d_x}{l_x} $$

Question

Using the extract from ELT15 (Males) given above, calculate the values of:

(i) p_2

(ii) ${}_2p_3$

(iii) ${}_4q_1$

(iv) l_6

Solution

(i) $p_2 = \dfrac{l_3}{l_2} = \dfrac{99{,}086}{99{,}124} = 0.99962$ or $p_2 = 1 - q_2 = 1 - 0.00038 = 0.99962$

(ii) ${}_2p_3 = \dfrac{l_5}{l_3} = \dfrac{99{,}032}{99{,}086} = 0.99946$

(iii) ${}_4q_1 = 1 - {}_4p_1 = 1 - \dfrac{l_5}{l_1} = 1 - \dfrac{99{,}032}{99{,}186} = 0.00155$

(iv) $l_6 = l_5 - d_5 = 99{,}032 - 22 = 99{,}010$

2.6 Lifetime random variables

Since the remaining lifetime of an individual is of unknown duration, we can model it using a random variable.

Recall from Subject CS2:

T_x **= complete future lifetime of a life aged x**

which has probability density function ${}_tp_x\,\mu_{x+t}$.

T_x represents the exact length of the remaining lifetime a life currently aged x, *ie* the exact length of time the life survives for after age x, before dying.

As we saw earlier in this chapter:

$${}_tq_x = \int_0^t {}_sp_x\,\mu_{x+s}\,ds = \text{probability that a life aged } x \text{ dies at any of the possible moments over the next } t \text{ years}$$

Expressing this in terms of the complete future lifetime random variable:

$${}_tq_x = P(T_x \le t) = \int_0^t {}_sp_x\, \mu_{x+s}\, ds$$

so we see that the probability is obtained by integrating the PDF over the range of possible values, as usual. The probability $P(T_x \le t)$ is the cumulative distribution function (or CDF) of the random variable T_x and is usually denoted $F_x(t)$.

Also recall from Subject CS2 that T_x has expected value:

$$E[T_x] = \int_{t=0}^{\infty} t\, {}_tp_x\, \mu_{x+t}\, dt = \int_{t=0}^{\infty} {}_tp_x\, dt = \mathring{e}_x$$

$\mathring{e}_x$ is called the *complete expected future lifetime* at age x or the *complete expectation of life* at age x, and its value is tabulated in some life tables. For example, based on ELT15 (Females) mortality, $\mathring{e}_{30} = 49.937$.

The first integral formula for $E[T_x]$ is based on the standard approach to calculating the expected value of a continuous random variable, *ie* taking the variable, multiplying by the PDF and then integrating over the range of values the variable can take.

The second integral formula for $E[T_x]$ is much easier to use and can be obtained from the first using integration by parts. Firstly, differentiating the CDF of a random variable gives the PDF, so we have:

$$\frac{d}{dt}P(T_x \le t) = {}_tp_x\, \mu_{x+t}$$

We can also write this derivative as follows:

$$\frac{d}{dt}P(T_x \le t) = \frac{d}{dt}\,{}_tq_x = \frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt}\,{}_tp_x$$

Combining these gives:

$$-\frac{d}{dt}\,{}_tp_x = {}_tp_x\, \mu_{x+t} \quad \Rightarrow \quad \frac{d}{dt}\,{}_tp_x = -{}_tp_x\, \mu_{x+t}$$

Now using the formula on page 3 of the *Tables*, with $u = t$ and $\frac{dv}{dt} = {}_tp_x\, \mu_{x+t}$:

$$\int_{t=0}^{\infty} t\, {}_tp_x\, \mu_{x+t}\, dt = \left[t(-{}_tp_x)\right]_0^{\infty} - \int_{t=0}^{\infty} 1 \times (-{}_tp_x)dt = \int_{t=0}^{\infty} {}_tp_x\, dt$$

since ${}_{\infty}p_x = 0$.

Question

Suppose that in a particular life table:

$$l_x = 100 - x \qquad \text{for } 0 \le x \le 100$$

Calculate the complete expected future lifetime of a life currently aged 50.

Solution

Using the given expression for l_x:

$$_t p_{50} = \frac{l_{50+t}}{l_{50}} = \frac{50-t}{50} \qquad \text{for } 0 \le t \le 50$$

We only consider values of t up to 50, as there is a zero probability of surviving to age 100 in this life table.

So the complete expected future lifetime of a life currently aged 50 is:

$$\mathring{e}_{50} = \int_0^{50} {}_t p_{50}\, dt = \int_0^{50} \frac{50-t}{50} dt = \frac{1}{50}\left[50t - \frac{1}{2}t^2\right]_0^{50} = \frac{1}{50}\left(50^2 - \frac{1}{2}50^2\right) = 25$$

Also recall from Subject CS2:

$$\boldsymbol{K_x} \textbf{ = curtate future lifetime of a life aged } \boldsymbol{x} \textbf{ = int}[\boldsymbol{T_x}]$$

K_x is the integer part of T_x, and so represents the complete number of years a life currently aged x survives for, before dying.

Question

A person is currently exactly 40 years old. Suppose that this person dies aged 76 years and 197 days. State the values of T_{40} and K_{40} for this individual.

Solution

T_{40} is the exact length of time the person survives for from the current age (40 exact) to the age at death. This is therefore 36 years and 197 days.

K_{40} is the integer part of this, and therefore equals 36 years.

2.7 The pattern of human mortality

We now return to the extract from the ELT15 (Males) table:

Age, x	l_x	d_x	q_x	μ_x
0	100,000	814	0.00814	
1	99,186	62	0.00062	0.00080
2	99,124	38	0.00038	0.00043
3	99,086	30	0.00030	0.00033
4	99,056	24	0.00024	0.00027
5	99,032	22	0.00022	0.00023

Notice that $\mu_x > q_x$ at all ages in this part of the table. At some higher ages it is found that $\mu_x < q_x$. In fact, since:

$$q_x = \int_0^1 {}_tp_x \, \mu_{x+t} \, dt$$

we see that if ${}_tp_x\mu_{x+t}$ is increasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x \, \mu_{x+t} \, dt > {}_0p_x\mu_{x+0} = \mu_x$$

while if ${}_tp_x\mu_{x+t}$ is decreasing for $0 \le t \le 1$:

$$q_x = \int_0^1 {}_tp_x \, \mu_{x+t} \, dt < {}_0p_x\mu_{x+0} = \mu_x$$

It is therefore of interest to note the behaviour of the function ${}_tp_x\mu_{x+t}$ for $0 \le t < \omega - x$.

As mentioned above, this function is the probability density function of T_x.

Figure 1 shows ${}_tp_0\mu_t$ (*ie* the density of $T = T_0$) for the English Life Table No. 15 (Males) as an example.

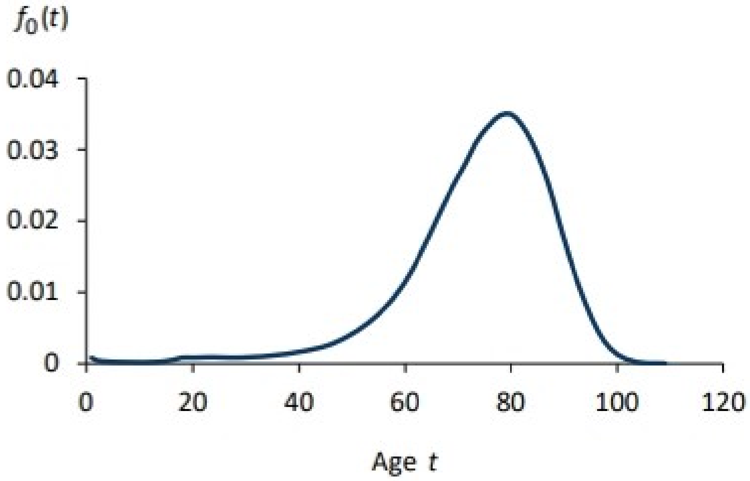


Figure 1

$f_0(t) = {}_tp_0\mu_t$ (ELT15 (Males) Mortality Table)

Typical life tables based on human mortality in modern times tend to display the following features:

(1) Mortality just after birth ('infant mortality') is very high.

(2) Mortality falls during the first few years of life.

(3) There is a distinct 'hump' in the function at ages around 18–25. This is often attributed to a rise in accidental deaths during young adulthood, and is called the 'accident hump'.

(4) From middle age onwards there is a steep increase in mortality, reaching a peak at about age 80.

(5) The probability of death at higher ages falls again (even though μ_x continues to increase) since the probabilities of surviving to these ages are small.

2.8 More notation

We will now introduce some more actuarial notation for probabilities of death, and give formulae for them in terms of the life table l_x.

These definitions relate to the deferred probabilities of death.

Define:

$$ {}_{n|m}q_x = P[\, n < T_x \le n+m \,] $$

In words, ${}_{n|m}q_x$ is the probability that a life age x will survive for n years but die during the subsequent m years.

It can be seen that:

$$_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

or alternatively that:

$$_{n|m}q_x = {_n}p_x \times {_m}q_{x+n}$$

We can see that these two results are equivalent since:

$$_n p_x \times {_m}q_{x+n} = \frac{l_{x+n}}{l_x} \times \frac{l_{x+n} - l_{x+n+m}}{l_{x+n}} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

This probability can also be expressed in the following ways:

- $_{n|m}q_x = {_n}p_x \times {_m}q_{x+n} = {_n}p_x \times (1 - {_m}p_{x+n}) = {_n}p_x - {_{n+m}}p_x$, *ie* the probability that the life survives for another n years, but not for another $n+m$ years
- $_{n|m}q_x = {_n}p_x - {_{n+m}}p_x = (1 - {_n}q_x) - (1 - {_{n+m}}q_x) = {_{n+m}}q_x - {_n}q_x$, *ie* the probability that the life dies within the next $n+m$ years, but does not die within the next n years.

Question

Using ELT15 (Males) mortality, calculate the probability of a 37-year old dying between age 65 and age 75.

Solution

The required probability is:

$$_{28|10}q_{37} = \frac{l_{65} - l_{75}}{l_{37}}$$

Looking up the values in the *Tables* gives:

$$_{28|10}q_{37} = \frac{79{,}293 - 53{,}266}{96{,}933} = 0.26851$$

An important special case for actuarial calculations is $m = 1$, since we often use probabilities of death over one year of age. By convention, we drop the '*m*' and write:

$$_{n|1}q_x = {_{n|}}q_x$$

In words, $_{n|}q_x$ is the probability that a life aged x will survive for n years but die during the subsequent year, *ie* die between ages $x+n$ and $x+n+1$.

So:

$$_{n|}q_x = {}_np_x \times q_{x+n} = \frac{l_{x+n} - l_{x+n+1}}{l_x} = \frac{d_{x+n}}{l_x}$$

Recall that K_x, the curtate future lifetime random variable, is the integer part of T_x, and represents the complete number of years a life currently aged x survives for, before dying.

We now see that the probability function of K_x can be written:

$$P[K_x = k] = {}_{k|}q_x$$

$P[K_x = k]$ is the probability that a life aged x dies between time k years and time $k+1$ years in the future, *ie* between age $x+k$ and age $x+k+1$.

Question

Using ELT15 (Males) mortality, calculate:

(i) $P(K_{30} = 40)$

(ii) $P(T_{30} > 40)$

Solution

(i) $P(K_{30} = 40)$ is the probability that the curtate future lifetime of a 30-year-old is equal to 40 years, *ie* it is the probability that a life aged exactly 30 survives for 40 years and then dies in the following year (between age 70 and age 71):

$$P(K_{30} = 40) = {}_{40|}q_{30} = \frac{d_{70}}{l_{30}} = \frac{2,674}{97,645} = 0.02738$$

(ii) $P(T_{30} > 40)$ is the probability that the complete future lifetime of a 30-year-old exceeds 40 years, *ie* it is the probability that a life aged exactly 30 survives for at least another 40 more years:

$$P(T_{30} > 40) = {}_{40}p_{30} = \frac{l_{70}}{l_{30}} = \frac{68,055}{97,645} = 0.69696$$

Also:

$$E[K_x] = \sum_{k=0}^{\infty} k\; {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x = e_x \approx \mathring{e}_x - ½$$

e_x is called the *curtate expected future lifetime* at age x or the *curtate expectation of life* at age x, and its value is tabulated in some life tables. For example, based on AM92 mortality, $e_{20} = 58.447$.

The first summation formula for $E[K_x]$ is based on the standard approach to calculating the expected value of a discrete random variable, *ie* taking each value the variable can take, multiplying by the probability that the random variable takes that value and then summing over all possible values.

The second summation formula for $E[K_x]$ is much easier to use and can be obtained from the first as follows:

$$\begin{aligned}\sum_{k=0}^{\infty} k\,{}_{k|}q_x &= 1\times{}_{1|}q_x + 2\times{}_{2|}q_x + 3\times{}_{3|}q_x + \cdots \\ &= {}_{1|}q_x + {}_{2|}q_x + {}_{3|}q_x + \cdots \\ &\qquad + {}_{2|}q_x + {}_{3|}q_x + \cdots \\ &\qquad\qquad + {}_{3|}q_x + \cdots\end{aligned}$$

The first row in the summation above, *ie*:

$${}_{1|}q_x + {}_{2|}q_x + {}_{3|}q_x + \cdots$$

is the probability that the life dies between age $x+1$ and age $x+2$, plus the probability that the life dies between age $x+2$ and age $x+3$, plus the probability that the life dies between age $x+3$ and age $x+4$, and so on. This is the probability that the life dies at some time after age $x+1$, which is equal to the probability that the life survives until at least age $x+1$, p_x.

Similarly, ${}_{2|}q_x + {}_{3|}q_x + \cdots$ represents the probability that the life dies at some time after age $x+2$, which is equal to the probability that the life survives until at least age $x+2$, ${}_2p_x$.

So the summation becomes:

$$E[K_x] = \sum_{k=0}^{\infty} k\,{}_{k|}q_x = p_x + {}_2p_x + {}_3p_x + \cdots = \sum_{k=1}^{\infty} {}_kp_x$$

Question

Suppose that in a particular life table:

$$l_x = 100 - x \qquad \text{for } 0 \le x \le 100$$

Calculate the curtate expected future lifetime of a newborn life.

Solution

The curtate expected future lifetime of a newborn life is:

$$e_0 = p_0 + {}_2p_0 + \cdots + {}_{99}p_0 = \frac{l_1}{l_0} + \frac{l_2}{l_0} + \cdots + \frac{l_{99}}{l_0}$$

We do not include terms from ${}_{100}p_0$ onwards in the summation, as there is a zero probability of surviving to age 100 in this life table.

Using the given expression for l_x, we have:

$$e_0 = \frac{99}{100} + \frac{98}{100} + \cdots + \frac{1}{100} = \frac{99 + 98 + \cdots + 1}{100}$$

The numerator can be evaluated using the sum of the first n positive integers, $\frac{1}{2}n(n+1)$, so:

$$e_0 = \frac{\frac{1}{2} \times 99 \times 100}{100} = \frac{4,950}{100} = 49.5$$

The approximate relationship between the complete and curtate expectations of life:

$$e_x \approx \mathring{e}_x - ½$$

is based on the assumption that deaths occur halfway between birthdays. Under this assumption, the total length of time a life survives for will be half a year longer than the number of complete years it survives for, *ie*:

$$K_x = T_x - \tfrac{1}{2}$$

Taking the expectation of both sides of this leads to the relationship:

$$e_x = \mathring{e}_x - \tfrac{1}{2}$$

This relationship is exact if the assumption holds, but in practice it is just an approximation.

3 Life table functions at non-integer ages

3.1 Introduction

Life table functions such as l_x , q_x or μ_x are usually tabulated at integer ages only, but sometimes we need to compute probabilities involving non-integer ages or durations, such as ${}_{2.5}p_{37.5}$. We can do so using approximate methods. We will show two methods.

In both cases, we suppose that we split up the required probability so that we need only approximate over single years of age.

Whilst the underlying force of mortality can vary greatly at different ages, it should not change significantly within a single year of age, so by taking this approach our approximations should be reasonably accurate.

For example, we would write:

$${}_3p_{55.5} \text{ as } {}_{0.5}p_{55.5} \times {}_2p_{56} \times {}_{0.5}p_{58}$$

The middle factor can be found from the life table. To approximate the other two factors we need only consider single years of age.

3.2 Method 1 – uniform distribution of deaths (UDD)

The first method is based on the assumption that, for integer x and $0 \le t \le 1$, the function ${}_tp_x\mu_{x+t}$ is a constant.

Since this is the density (PDF) **of the time to death from age x , it is seen that this assumption is equivalent to a uniform distribution of the time to death, conditional on death falling between these two ages. Hence it is called the *Uniform Distribution of Deaths* (or UDD) assumption.**

In other words, for an individual aged exactly x , the probability of dying on one particular day over the next year is the same as that of dying on any other day over the next year.

The UDD assumption implicitly assumes that μ_x increases over the year of age. This follows because the quantity ${}_tp_x$ is a decreasing function of t , so if the function ${}_tp_x\,\mu_{x+t}$ is constant, μ_{x+t} must be an increasing function of t .

Since ${}_sq_x = \int_0^s {}_tp_x\mu_{x+t}\,dt$, by putting $s = 1$ we must have:

$${}_tp_x\mu_{x+t} = q_x \quad (0 \le t \le 1)$$

Remember that we are assuming that ${}_tp_x\mu_{x+t}$ is constant over the year.

Therefore:

$${}_sq_x = \int_0^s q_x\,dt = s\,q_x$$

This is sometimes taken as the definition of the UDD assumption.

Since q_x can be found from the life table, we can use this to approximate any ${}_sq_x$ or ${}_sp_x$ $(0 \le s \le 1)$.

Question

Calculate the value of ${}_{0.5}p_{58}$ using ELT15 (Females) mortality, assuming a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$$ {}_{0.5}p_{58} = 1 - {}_{0.5}q_{58} = 1 - 0.5 \times q_{58} = 1 - 0.5 \times 0.00660 = 0.99670 $$

Note that we must have an integer age x in the above formula, so (in our example) we can now estimate ${}_{0.5}p_{58}$ but not ${}_{0.5}p_{55.5}$.

However, using ${}_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$, it can be shown that for integer age x and $0 \le s < t \le 1$:

$$ {}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x} $$

Rearranging ${}_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$ gives:

$$ {}_{t-s}p_{x+s} = \frac{{}_tp_x}{{}_sp_x} $$

and using the earlier result that under the UDD assumption ${}_sq_x = sq_x$, we can derive the above result as follows:

$$ {}_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{{}_tp_x}{{}_sp_x} = 1 - \frac{1 - {}_tq_x}{1 - {}_sq_x} = 1 - \frac{1 - tq_x}{1 - sq_x} $$

$$ = \frac{(1 - sq_x) - (1 - tq_x)}{1 - sq_x} = \frac{(t-s)q_x}{1 - sq_x} $$

This result, with $s = 0.5$, $t = 1$, can now be used to estimate ${}_{0.5}p_{55.5}$, for example.

Question

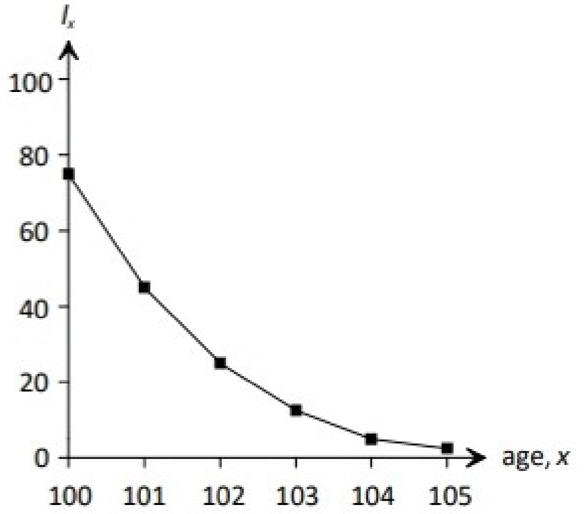
Calculate the value of ${}_{0.5}p_{55.5}$ using ELT15 (Females) mortality, assuming a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$$_{0.5}p_{55.5} = 1 - {}_{0.5}q_{55.5} = 1 - \frac{0.5q_{55}}{1-0.5q_{55}} = 1 - \frac{0.5 \times 0.00475}{1-0.5\times 0.00475} = 0.99762$$

Under the UDD assumption l_x is made up of straight-line segments between integer ages, as shown in the following graph of l_x between ages 100 and 105:



The following question justifies why this is the case.

Question

Show that, under the UDD assumption over each year of age:

(i) $l_{x+t} = l_x - t\,d_x$

(ii) $l_{x+t} = (1-t)l_x + t\,l_{x+1}$

for $x = 0, 1, 2, \dots, \omega - 1$ and for $0 \le t \le 1$.

Solution

(i) We have:

$$l_{x+t} = l_x \times {}_tp_x = l_x(1 - {}_tq_x) = l_x(1 - t\,q_x) = l_x - t(l_x\,q_x) = l_x - t\,d_x$$

(ii) Using part (i):

$$l_{x+t} = l_x - t\,d_x = l_x - t(l_x - l_{x+1}) = (1-t)l_x + t\,l_{x+1}$$

The result in part (ii) means that the values of l_{x+t} for $0<t<1$ can be found using linear interpolation, and provides an alternative approach to using the formulae:

$$_sq_x = s\,q_x \quad \text{and} \quad {}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-s\,q_x}$$

Question

Calculate the value of ${}_{0.5}p_{55.5}$ using ELT15 (Females) mortality and linear interpolation, based on the assumption of a uniform distribution of deaths between integer ages.

Solution

Using the UDD assumption, we have:

$${}_{0.5}p_{55.5} = \frac{l_{56}}{l_{55.5}} = \frac{l_{56}}{0.5l_{55}+0.5l_{56}} = \frac{94{,}082}{0.5\times 94{,}532+0.5\times 94{,}082} = \frac{94{,}082}{94{,}307} = 0.99761$$

This answer is slightly different from the answer obtained previously for this probability, due to the fact that the values shown in the *Tables* are rounded.

3.3 Method 2 – constant force of mortality (CFM)

The second method of approximation is based on the assumption of a *constant force of mortality*. That is, for integer x and $0 \le t < 1$, we suppose that:

$$\mu_{x+t} = \mu = \text{constant}$$

Then for $0 \le s < t < 1$ the formula:

$${}_{t-s}p_{x+s} = \exp\left\{-\int_s^t \mu_{x+r}\,dr\right\} = e^{-(t-s)\mu}$$

can be used to find the required probabilities. We do this by first noting that, under this assumption, $p_x = e^{-\mu}$, so we can simply write:

$${}_{t-s}p_{x+s} = (p_x)^{t-s}$$

where p_x can be found from the life table. Hence we can easily calculate any required probability.

Usually, the value of q_x will be tabulated and $p_x = 1-q_x$, meaning we don't need to work out the value of the constant force of mortality to calculate the probabilities.

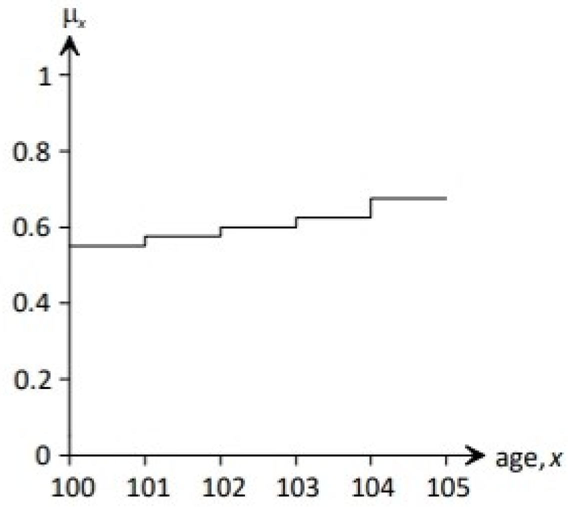
If we did wish to calculate the constant force of mortality over the year of age, then:

$$p_x = e^{-\mu} \quad \Rightarrow \quad \mu = -\ln p_x = -\ln(1-q_x)$$

Also, note that when $s=0$, the formula ${}_{t-s}p_{x+s} = (p_x)^{t-s}$ simplifies a little to:

$$ {}_t p_x = (p_x)^t $$

Under this assumption μ_x has a stepped shape, as shown in the following graph of μ_x between ages 100 and 105:



Question

Calculate ${}_3p_{62.5}$ based on PFA92C20 mortality using:

(i) the CFM assumption

(ii) the UDD assumption.

Solution

First of all, we can split up the probability at integer ages as follows:

$$ {}_3p_{62.5} = {}_{0.5}p_{62.5} \times {}_2p_{63} \times {}_{0.5}p_{65} $$

Looking up values from PFA92C20 in the *Tables*:

$$ {}_2p_{63} = \frac{l_{65}}{l_{63}} = \frac{9,703.708}{9,775.888} = 0.992617 $$

(i) Under the constant force of mortality assumption:

$$ {}_{0.5}p_{62.5} = (p_{62})^{0.5} = (1-q_{62})^{0.5} = (1-0.002885)^{0.5} = 0.998556 $$

Also:

$$ {}_{0.5}p_{65} = \left(p_{65}\right)^{0.5} = \left(1-q_{65}\right)^{0.5} = \left(1-0.004681\right)^{0.5} = 0.997657 $$

So, overall:

$$ {}_{3}p_{62.5} = 0.998556 \times 0.992617 \times 0.997657 = 0.988861 $$

(ii) Under the UDD assumption:

$$ {}_{0.5}p_{62.5} = 1 - {}_{0.5}q_{62.5} = 1 - \frac{0.5q_{62}}{1-0.5q_{62}} = 1 - \frac{0.5 \times 0.002885}{1 - 0.5 \times 0.002885} = 0.998555 $$

and:

$$ {}_{0.5}p_{65} = 1 - {}_{0.5}q_{65} = 1 - 0.5q_{65} = 1 - 0.5 \times 0.004681 = 0.997660 $$

So overall:

$$ {}_{3}p_{62.5} = 0.998555 \times 0.992617 \times 0.997660 = 0.988863 $$

Alternatively, assuming UDD, we could take the following approach:

$$ \begin{aligned} {}_{3}p_{62.5} &= \frac{l_{65.5}}{l_{62.5}} = \frac{0.5l_{65} + 0.5l_{66}}{0.5l_{62} + 0.5l_{63}} \\ &= \frac{0.5(9,703.708 + 9,658.285)}{0.5(9,804.173 + 9,775.888)} = \frac{9,680.9965}{9,790.0305} = 0.988863 \end{aligned} $$

4 Evaluating probabilities without use of the life table

The main alternative to using a life table is to postulate a formula and/or parameter values for the probability $_tp_x$ and then evaluate the expressions directly.

This means that we hypothesise a formula for $_tp_x$, estimate the values of the parameters in the formula, and then calculate the probabilities for different values of x and t.

Equivalently a formula or values for $_tq_x$ or μ_{x+t} could be postulated.

The difficulty of adopting this approach is that the postulation would need to be valid across the whole age range for which the formulae might be applied. As can be seen from the discussion in Section 2.7 above, the shape of human mortality may make a simple postulation difficult. Simple formulae may, however, be more appropriate and expedient for non-life contingencies.

The following example shows how mortality and survival probabilities can be evaluated without using a life table.

Here, we assume that the formula for μ_{x+t} is a very simple one – simply that it takes a constant value at all ages. (We are also assuming that there is no upper limit to age.)

Question

In a certain population, the force of mortality equals 0.025 at all ages.

Calculate:

(i) the probability that a new-born baby will survive to age 5

(ii) the probability that a life aged exactly 10 will die before age 12

(iii) the probability that a life aged exactly 5 will die between ages 10 and 12.

Solution

(i) $${}_5p_0 = \exp\left(-\int_0^5 0.025\,dt\right) = e^{-0.125} = 0.88250$$

(ii) $${}_2q_{10} = 1 - {}_2p_{10} = 1 - \exp\left(-\int_0^2 0.025\,dt\right) = 1 - e^{-0.05} = 0.04877$$

(iii) $${}_{5|2}q_5 = {}_5p_5 \times {}_2q_{10} = e^{-0.125} \times 0.04877 = 0.88250 \times 0.04877 = 0.04304$$

In general, where the force of mortality is constant over the whole of an n-year period:

$${}_np_x = e^{-n\mu} \quad \text{and} \quad {}_nq_x = 1 - e^{-n\mu}$$

5 Select mortality

5.1 Introduction

So far, we have made an assumption of ultimate mortality, that is, that mortality varies by age only. In real life, many other factors other than just age might affect observed mortality rates. In practice, therefore, the evaluation of assurance and annuity benefits is often modified to allow for factors other than just age, which affect the survival probabilities.

Examples of factors (other than age) that might affect observed mortality rates are sex, smoker status and occupation.

Many factors can be allowed for by segregating the population, the assumption being that an age pattern of mortality can be discerned in the sub-population. For example, a population may well be segregated by sex and then sex-specific mortality rates (and mortality tables) can be used directly by using the techniques so far described.

Where, however, the pattern of mortality is assumed to depend not just on age, consequently slightly more complicated survival probabilities are employed. The most important, in the case of human mortality, is where the mortality rates depend upon duration as well as age, called *select rates*.

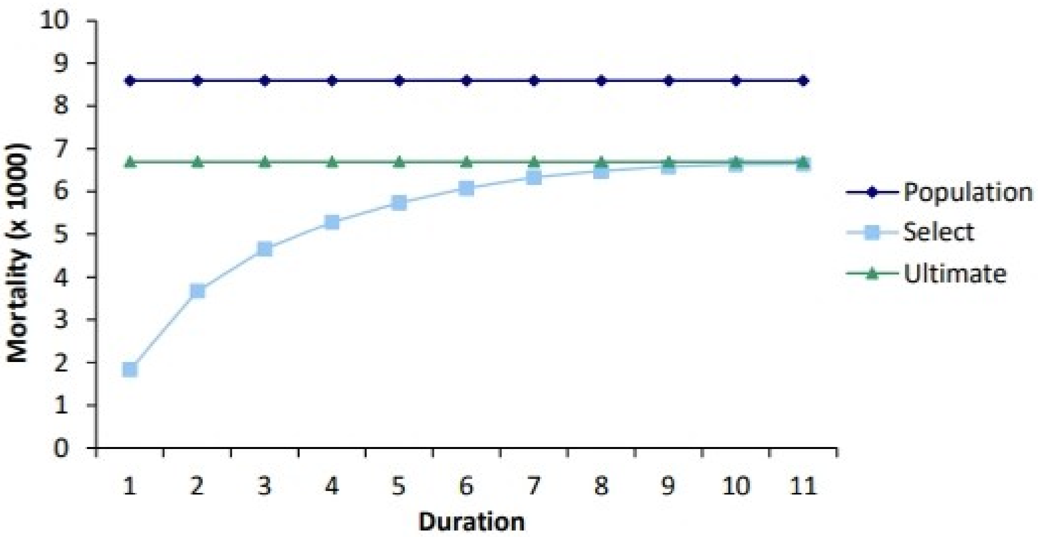
To understand why the concept of duration in the population is important, we will consider life assurance policyholders (*ie* those who hold policies that make a payment on death).

In order to take out their policies, policyholders will have been subject to some *medical underwriting*. This means that they provide evidence about their recent state of health, answering simple questions on the policy proposal form, and perhaps also attending a medical examination if the sum insured is very high. The aim of this underwriting is to allow the company to screen out very bad risks, and to charge appropriately higher premiums for worse than average risks.

So policyholders with duration of one year, say, will have recently satisfied the company about their state of health. We would therefore expect their mortality to be better than that of policyholders of the same age with duration of, say, 3 years who passed the medical underwriting hurdle several years ago. Moreover, we would expect both these sets of policyholders to display better mortality than policyholders of the same age with duration 10 or 20 years, for instance. The mortality of the recently joined policyholders is called *select* mortality, and we expect it to be better than that of longer duration policyholders, whose mortality we call *ultimate* mortality.

However, it is important to realise that even ultimate mortality may be good when compared with the mortality of the whole population of a country. This is because those taking out life insurance contracts tend to be the more affluent in society, who can afford to purchase insurance products, and financially better-off individuals tend to have lower mortality (*eg* they may be able to access better health care).

If we were to plot mortality for any given age x by varying duration, and compare it also against population mortality, we might expect something like:



When creating a life table based on select mortality, we adopt a pragmatic approach and look for the duration beyond which there is no significant change in mortality with further increases in duration. This is called the *select period*, and we examine the choice of the select period in Section 5.3.

We take this approach because, intuitively, we would not expect a difference in mortality between someone who answered some health questions on a policy proposal form 10 years ago and someone else (of the same age) who answered such questions 20 years ago. In addition, in practice, it would be very difficult to measure mortality rates if we want to split the population up by every year of duration as well as by age and sex because in each group we might see only 50 to 100 policyholders, even for a very large company.

Above, we have discussed select mortality in the context of life assurance policyholders. In fact, the effect of selection is also found with annuity business: the mortality of policyholders who have recently bought an annuity policy is lighter than that of policyholders who took out their policy some time ago.

5.2 Mortality rates that depend on both age and duration

Select rates are usually studied by modelling the force of mortality μ as a function of the age at joining the population and the duration since joining the population. The usual notation is:

$[x]+r$	**age at date of transition**
$[x]$	**age at date of joining population**
r	**duration from date of joining the population until date of transition**

$\boldsymbol{\mu_{[x]+r}}$	**the transition rate (force of mortality) at exact duration r having joined the population at age $[x]$**
$\boldsymbol{l_{[x]+r}}$	**expected number of lives alive at duration r having joined the population at age $[x]$, based on some assumed radix**

The important principle to grasp with this notation is that the term in square brackets [ ] denotes the age at joining the population, so the age 'now' (*ie* the moment of transition, or when exposed to possible transition) will be $x+r$. In some circumstances, we may be given the age 'now' of y and the duration r; in this case the transition rate 'now' would be expressed as $\mu_{[y-r]+r}$.

In effect, a model showing how μ varies with r is constructed for each value of $[x]$. Instead of the single life age-specific life table described earlier we have a series of life tables, one for each value of $[x]$.

5.3 Displaying select rates

Once select rates have been estimated, it is conventional to display estimated rates for each age at entry into the population, $[x]$, by age attained at the date of transition *ie* $[x]+r$. This can be done in an array:

$$\begin{array}{ccccc}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} & \mu_{[x-3]+3} & \cdots \\
\mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} & \mu_{[x-2]+3} & \cdots \\
\mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} & \mu_{[x-1]+3} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Question

Assuming that the force of mortality increases with both age and policy duration, comment on the expected relationship between the following pairs of values:

(i) $\mu_{[x+1]}$ and $\mu_{[x]+1}$

(ii) $\mu_{[x+2]+2}$ and $\mu_{[x+4]}$

(iii) $\mu_{[x+4]}$ and $\mu_{[x+1]+2}$

Solution

(i) We would expect to see $\mu_{[x+1]} < \mu_{[x]+1}$, because the rates both relate to the same age, $x+1$, but $\mu_{[x]+1}$ relates to a later duration (1 instead of 0).

(ii) Here, we would expect to see $\mu_{[x+2]+2} > \mu_{[x+4]}$, again because they relate to the same age, but the first term is for duration 2 rather than 0.

We might also expect the difference to be higher than in part (i), because:

- it involves twice the duration difference (2 instead of 1)
- the current age is three years older ($x+4$ instead of $x+1$), which should increase the size of the effect of duration differences on mortality.

(iii) How $\mu_{[x+4]}$ compares with $\mu_{[x+1]+2}$ will depend on how duration and age affect mortality, because the current age has decreased (reducing mortality) while duration has increased (increasing mortality). It is therefore impossible to give a general answer in this case.

Each diagonal (↘) of the array represents a model of how rates vary with duration since joining the population for a particular age at the date of joining, *ie* each is a set of life table mortality rates.

The diagonals referred to here are those going from top left to bottom right.

The rates displayed on the rows of the array are rates for lives that have a common age attained at the time of transition, but different ages at the date of joining the population.

For example, the second row in the table above contains $\mu_{[x+1]}$ and $\mu_{[x]+1}$. These both apply to lives aged $x+1$, but with differing durations. Similarly, the next row relates to lives age $x+2$ but with differing durations, and so on.

If the rates did not depend on the duration since the date of joining the population, then, apart from sampling error, the rates on each row would be equal.

Usually it is the case that rates are assumed to depend on duration until duration s, and after s they are assumed to be independent of duration. This phenomenon is termed *temporary initial selection* and s is called the length of the select period. In any investigation s is determined empirically by considering the statistical significance of the differences in transition rates along each row and the substantive impact of the different possible values of s.

We examine rows to ensure that we consider different durations but equal ages.

Typical select periods seen in life company investigations range from one to five years. For instance, the AM92 tables (based on UK assured lives data gathered over the period 1991-1994) in the *Tables* are based on a two-year select period.

The UK 1980 assured lives table AM80 uses a five-year select period, although durations 2 to 4 are grouped together. This is because, when the underlying data values were analysed, no significant differences in mortality rates at these durations were apparent. So a typical row shows the rates:

Duration 0	Duration 1	Duration 2-4	Duration 5+
$\mu_{[34]}$	$\mu_{[33]+1}$	$\mu_{[32]+2} = \mu_{[31]+3} = \mu_{[30]+4}$	μ_{34}

Once a value of s has been determined, the estimates of the rates for duration $\geq s$ are pooled to obtain a common estimated value that is used in all the life tables in which it is needed. The array can be written:

$$\begin{array}{llllll}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mu_{[x]} & \mu_{[x-1]+1} & \mu_{[x-2]+2} & \cdots & \mu_{[x-s+1]+s-1} & \mu_x \\
\mu_{[x+1]} & \mu_{[x]+1} & \mu_{[x-1]+2} & \cdots & \mu_{[x-s+2]+s-1} & \mu_{x+1} \\
\mu_{[x+2]} & \mu_{[x+1]+1} & \mu_{[x]+2} & \cdots & \mu_{[x-s+3]+s-1} & \mu_{x+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

The right hand column of the above array represents a set of rates that are common to all the constituent life tables in the model of rates by age and duration. It is called an ultimate table.

In the *Tables*, the AM92 Select mortality rates are tabulated in the way shown above.

The initial select probabilities can be displayed in a similar way.

'Initial probabilities' refers to q_x values, as q_x is sometimes called 'the initial rate of mortality'.

5.4 Constructing select and ultimate life tables

The first step in constructing life tables is to refine the crude estimated rates (μ or q) into a smooth set of rates that statistically represent the true underlying mortality rates. This refinement process, called 'graduation', is dealt with in CS2, and here we assume that we have a set of suitable graduated mortality rates available.

Knowledge of the graduation process is not required in this subject.

Using the graduated values of the initial probabilities displayed in the array:

$$\begin{array}{llllll}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q_{[x]} & q_{[x-1]+1} & q_{[x-2]+2} & \cdots & q_{[x-s+1]+s-1} & q_x \\
q_{[x+1]} & q_{[x]+1} & q_{[x-1]+2} & \cdots & q_{[x-s+2]+s-1} & q_{x+1} \\
q_{[x+2]} & q_{[x+1]+1} & q_{[x]+2} & \cdots & q_{[x-s+3]+s-1} & q_{x+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

a table representing the select and ultimate experience can be constructed.

This is achieved by firstly constructing the ultimate life table based on the final column of the array and the formula described in Section 2.2 above:

- **choose the starting age of the table, α**
- **choose an arbitrary radix for the table, l_α**
- **recursively calculate the values of l_x using $l_{x+1} = l_x(1-q_x)$.**

Beginning with the appropriate ultimate value in the final column, the select life table functions for each row of the array are then determined. This is achieved by 'working backwards' up each diagonal using:

$$l_{[x]+t} = \frac{l_{[x]+t+1}}{(1-q_{[x]+t})}$$

for $t = s-1, s-2, ..., 1, 0$, and noting in the first iteration that $l_{[x]+s-1+1} = l_{x+s}$.

Question

Given the following select and ultimate q_x values, calculate the values of $l_{[x]}$ and $l_{[x]+1}$ for $x=45$ and $x=46$, assuming that $l_{47}=1,000$.

Age	Duration 0	Duration 1	Duration 2+ (*ie* Ultimate)
45	0.000838		
46	0.000924	0.001158	
47	0.001018	0.001284	0.001415
48		0.001423	0.001564
49			0.001729

Solution

Starting from l_{47}, we work back diagonally upwards to the left to $l_{[45]}$:

$$l_{[45]+1} = \frac{l_{47}}{\left(1-q_{[45]+1}\right)} = \frac{1,000}{(1-0.001158)} = 1,001.16$$

$$l_{[45]} = \frac{l_{[45]+1}}{\left(1-q_{[45]}\right)} = \frac{1,001.16}{(1-0.000838)} = 1,002.00$$

Next, working down the column from l_{47} to l_{48}:

$$l_{48} = l_{47}\left(1-q_{47}\right) = 1,000\left(1-0.001415\right) = 998.59$$

This then allows us to calculate $l_{[46]}$ and $l_{[46]+1}$:

$$l_{[46]+1} = \frac{l_{48}}{\left(1-q_{[46]+1}\right)} = \frac{998.59}{(1-0.001284)} = 999.87$$

$$l_{[46]} = \frac{l_{[46]+1}}{\left(1-q_{[46]}\right)} = \frac{999.87}{(1-0.000924)} = 1,000.79$$

5.5 Using tabulated select life table functions

Some probabilities are particularly useful for life contingencies calculations. We have already defined:

$$_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

$$_{n|}q_x = \frac{l_{x+n} - l_{x+n+1}}{l_x}$$

representing the m-year and 1-year probabilities of transition when the event of transition is deferred for n years.

Similar probabilities can be defined for each select mortality table:

$$_{n|m}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+m}}{l_{[x]+r}}$$

$$_{n|}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+1}}{l_{[x]+r}}$$

with the special case of $n = 0$ and $m = n$ being of particular interest:

$$_nq_{[x]+r} = \frac{l_{[x]+r} - l_{[x]+r+n}}{l_{[x]+r}}$$

and the complement of this n year transition probability, the n year survival probability, is:

$$_np_{[x]+r} = \frac{l_{[x]+r+n}}{l_{[x]+r}}$$

The above probabilities may also be expressed in terms of the expected number of deaths in the select mortality table by defining:

$$d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}$$

Question

Calculate the following probabilities using AM92 mortality:

(i) $_2p_{[42]}$

(ii) $_2p_{42}$

(iii) $_3q_{[40]+1}$

(iv) $_{2|}q_{[41]+1}$

Solution

(i) We have:

$$ {}_2p_{[42]} = \frac{l_{44}}{l_{[42]}} = \frac{9{,}814.3359}{9{,}834.7030} = 0.997929 $$

As the select period of the AM92 life table is 2 years, $l_{[x]+k} = l_{x+k}$ for $k \geq 2$, so $l_{[42]+2} = l_{44}$.

We can alternatively calculate this as:

$$ {}_2p_{[42]} = p_{[42]} \times p_{[42]+1} = \left(1 - q_{[42]}\right)\left(1 - q_{[42]+1}\right) $$

Looking up the relevant values in the *Tables*, we have:

$$ {}_2p_{[42]} = (1 - 0.000922)(1 - 0.001150) = 0.997929 $$

as before.

(ii) This is a probability based solely on ultimate mortality:

$$ {}_2p_{42} = \frac{l_{44}}{l_{42}} = \frac{9{,}814.3359}{9{,}837.0661} = 0.997689 $$

This could alternatively be calculated as:

$$ {}_2p_{42} = p_{42} \times p_{43} = \left(1 - q_{42}\right)\left(1 - q_{43}\right) $$

to obtain the same answer.

The value obtained here is lower than the value of ${}_2p_{[42]}$ *from part (i), because the select life,* $[42]$*, will have undergone some medical underwriting at that age, and so should be in generally better health, and less likely to die within the next two years, than a member of the ultimate population.*

(iii) We have:

$$ {}_3q_{[40]+1} = \frac{l_{[40]+1} - l_{44}}{l_{[40]+1}} = \frac{9{,}846.5384 - 9{,}814.3359}{9{,}846.5384} = 0.003270 $$

(iv) We have:

$$ {}_{2|}q_{[41]+1} = \frac{l_{44} - l_{45}}{l_{[41]+1}} = \frac{d_{44}}{l_{[41]+1}} = \frac{13.0236}{9{,}836.5245} = 0.001324 $$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 14 Summary

Modelling mortality

We can model mortality by assuming that the complete future lifetime of a life aged x is a continuous random variable, T_x. Assuming some limiting age for the population, ω, T_x can take values between 0 and $\omega - x$.

We also model the curtate future lifetime of a life aged x using K_x, which is the integer part of T_x. The expected values of these lifetime random variables are given by:

$$E[T_x] = \mathring{e}_x = \int_{t=0}^{\infty} {}_tp_x \, dt \qquad E[K_x] = e_x = \sum_{k=1}^{\infty} {}_kp_x \approx \mathring{e}_x - ½$$

From this starting point, we can calculate probabilities of survival $({}_tp_x)$ and death $({}_tq_x)$ for an individual aged x over a period of t years.

Definitions of probabilities of death and survival

$${}_tq_x = F_x(t) = P[T_x \le t]$$

$${}_tp_x = 1 - {}_tq_x = 1 - F_x(t) = P[T_x > t]$$

$${}_{t+s}p_x = {}_tp_x \times {}_sp_{x+t} = {}_sp_x \times {}_tp_{x+s}$$

Force of mortality

The force of mortality μ_x is the instantaneous rate of mortality at age x. It is defined as:

$$\mu_x = \lim_{h \to 0+} \frac{1}{h} \times {}_hq_x \quad \text{or, equivalently} \quad \mu_x = -\frac{1}{l_x} \times \frac{d}{dx} l_x = -\frac{d}{dx} \ln l_x$$

In addition:

$${}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} \, ds \qquad \text{and} \qquad {}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} \, ds \right\}$$

Using a life table

We can tabulate probabilities and other quantities for each year of age in a life table. Simple formulae allow us to use the entries in the life table to calculate other useful quantities:

$$l_x = l_\alpha \times {}_{x-\alpha}p_\alpha \qquad {}_tp_x = \frac{l_{x+t}}{l_x} \qquad d_x = l_x - l_{x+1} \qquad q_x = \frac{d_x}{l_x}$$

$${}_{n|m}q_x = P[n < T_x \le n+m] = {}_np_x \times {}_mq_{x+n} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

Dealing with non-integer ages

By making an assumption about mortality within a year of age, we can calculate the probability of survival for individuals at non-integer ages and for periods other than a whole number of years.

Uniform distribution of deaths

Assumption: ${}_tp_x\mu_{x+t}$ is a constant for integer x and $0 \le t \le 1$

$${}_sq_x = sq_x \quad \text{and} \quad {}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x} \quad (0 \le s \le t \le 1)$$

$$l_{x+t} = (1-t)l_x + tl_{x+1} \quad (0 \le t \le 1)$$

Constant force of mortality

Assumption: μ_{x+t} is a constant for integer x and $0 \le t \le 1$

$${}_tp_x = e^{-t\mu} = (p_x)^t \quad \text{and} \quad {}_{t-s}p_{x+s} = e^{-(t-s)\mu} = (p_x)^{t-s} \quad (0 \le s \le t \le 1)$$

Select mortality

Lives who are 'selected' from a larger group of people, in some non-random way with respect to their mortality, will experience different mortality at any given age from the group as a whole. They are then referred to as having select mortality.

An example occurs when applicants for life assurance are underwritten, so that only the 'better' risks are accepted for cover at the insurer's standard premium rates.

Lives subject to select mortality are denoted by $[x]+r$, where x is the age at selection and r is the number of years since selection, meaning that $x+r$ is the current age.

A select mortality table (such as AM92) has select functions (*eg* $q_{[x]+r}$, $l_{[x]+r}$) for $r = 0, 1, \ldots, s-1$, where s is the select period of the table. The select period is the number of years since selection during which mortality rates are assumed to be dependent upon the duration since selection as well as on current age.

For $r \ge s$, mortality is assumed to be a function of age only (called ultimate mortality), *eg*:

$$q_x = q_{[x-r]+r} \quad \text{for } r \ge s$$

The version of the AM92 table quoted in the *Tables* has a 2-year select period. This means that, for example, the expected progression of survivors through the table is:

$$l_{[x]} \to l_{[x]+1} \to l_{x+2} \to l_{x+3} \to l_{x+4} \to \cdots$$

Chapter 14 Practice Questions

14.1 Calculate the following probabilities assuming AM92 mortality applies:

(i) ${}_{10}p_{90}$

(ii) ${}_{5}p_{[79]+1}$

(iii) ${}_{2}q_{[70]}$

(iv) ${}_{5|}q_{[60]}$

(v) ${}_{10|15}q_{50}$

14.2 The mortality of a certain population is governed by the life table function $l_x = 100 - x$, $0 \le x \le 100$. Calculate the values of the following expressions:

(i) ${}_{10}p_{30}$

(ii) μ_{30}

(iii) $P(T_{30} < 20)$

(iv) $P(K_{30} = 20)$

14.3 A population is subject to a constant force of mortality of 0.015 *pa*. Calculate:

(i) the probability that a life aged exactly 20 dies before age 21.25.

(ii) the probability that a life aged exactly 22.5 dies between the ages of 25 and 27

(iii) the complete expectation of life for a life aged exactly 28.

14.4 Examine the column of d_x shown in the English Life Table No.15 (Males) in the Formulae and Tables for Examinations (pages 68-69).

Exam style

Describe the key characteristics of this mortality table using the data to illustrate your points. [6]

14.5 The AM92 table is based on the mortality of assured male lives in the UK.

(i) As there is no corresponding mortality table for female lives, in order to calculate survival probabilities for female policyholders, an actuary decides to use the AM92 table, but with an 'age rating' of 4 years applied, *ie* a female aged x is considered to experience mortality equivalent to a male aged $x-4$.

(a) Explain the rationale underlying this approach.

(b) Calculate the probability that a female policyholder aged 62 survives for at least the next 10 years.

(ii) A male policyholder aged 65 is known to be in poor health, and it has been determined that his mortality is 200% of AM92 Ultimate, *ie* he is subject to q_x values equal to twice those of the AM92 Ultimate table.

Calculate the probability that this policyholder will die before age 67.

14.6 A select life table is to be constructed with a select period of two years added to the ELT15 (Males) table, which is to be treated as the ultimate table. Select rates are to be derived by applying the same ratios select : ultimate seen in the AM92 table, *ie*:

$$q_{[x]} = \frac{q'_{[x]}}{q'_x} q_x \text{ and } q_{[x]+1} = \frac{q'_{[x]+1}}{q'_{x+1}} q_{x+1}$$

where the dash notation q'_x refers to AM92 mortality.

Calculate the value of $l_{[60]}$.

14.7 The table below is part of a mortality table used by a life insurance company to calculate probabilities for a special type of life insurance policy.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}
51	1,537	1,517	1,502	1,492	1,483
52	1,532	1,512	1,497	1,487	1,477
53	1,525	1,505	1,490	1,480	1,470
54	1,517	1,499	1,484	1,474	1,462
55	1,512	1,492	1,477	1,467	1,453

(i) Calculate the probability that a policyholder who was accepted for insurance exactly 2 years ago and is now aged exactly 55 will die between age 56 and age 57.

(ii) Calculate the corresponding probability for an individual of the same age who has been a policyholder for many years.

(iii) Comment on your answers to (i) and (ii).

14.8 Given that $p_{80} = 0.988$, calculate ${}_{0.5}p_{80}$ assuming:

(i) a uniform distribution of deaths between integer ages

(ii) a constant force of mortality between integer ages.

14.9 Calculate the value of ${}_{1.75}p_{45.5}$ using AM92 Ultimate mortality and assuming that:

Exam style

(i) deaths are uniformly distributed between integer ages. [3]

(ii) the force of mortality is constant between integer ages. [3]

[Total 6]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 14 Solutions

14.1 (i) $${}_{10}p_{90} = \frac{l_{100}}{l_{90}} = \frac{95.8476}{1,658.5545} = 0.057790$$

(ii) $${}_{5}p_{[79]+1} = \frac{l_{85}}{l_{[79]+1}} = \frac{3,385.2479}{5,176.2224} = 0.654000$$

(iii) $${}_{2}q_{[70]} = 1 - {}_{2}p_{[70]} = 1 - \frac{l_{72}}{l_{[70]}} = 1 - \frac{7,637.6208}{7,960.9776} = 0.040618$$

(iv) $${}_{5|}q_{[60]} = \frac{d_{65}}{l_{[60]}} = \frac{125.6412}{9,263.1422} = 0.013564$$

(v) $${}_{10|15}q_{50} = \frac{l_{60} - l_{75}}{l_{50}} = \frac{9,287.2164 - 6,879.1673}{9,712.0728} = 0.247944$$

14.2 (i) $${}_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{60}{70} = \frac{6}{7}$$

(ii) We have:

$$\mu_x = -\frac{d}{dx}\ln l_x = -\frac{d}{dx}\ln(100 - x) = \frac{1}{100 - x}$$

So:

$$\mu_{30} = \frac{1}{70}$$

(iii) $$P(T_{30} < 20) = {}_{20}q_{30} = 1 - \frac{l_{50}}{l_{30}} = 1 - \frac{50}{70} = \frac{2}{7}$$

(iv) $$P(K_{30} = 20) = {}_{20}p_{30} \times q_{50} = \frac{l_{50} - l_{51}}{l_{30}} = \frac{50 - 49}{70} = \frac{1}{70}$$

14.3 (i) $${}_{1.25}q_{20} = 1 - {}_{1.25}p_{20} = 1 - e^{-1.25\mu} = 1 - e^{-0.01875} = 0.018575$$

(ii) $${}_{2.5|2}q_{22.5} = {}_{2.5}p_{22.5}\,{}_{2}q_{25} = e^{-2.5\mu}\left(1 - e^{-2\mu}\right) = e^{-0.0375}\left(1 - e^{-0.03}\right) = 0.028467$$

(iii) $$\mathring{e}_{28} = \int_0^\infty {}_{t}p_{28}\,dt = \int_0^\infty e^{-t\mu}\,dt = \left[-\frac{1}{\mu}e^{-t\mu}\right]_0^\infty = \frac{1}{\mu} = \frac{1}{0.015} = 66.67 \text{ years}$$

14.4 *This question is Subject CT5, September 2012, Question 8.*

- d_x is the number of lives expected to die between exact ages x and $x+1$ in the mortality table, and is equal to $l_x q_x$ at each age. [½]
- In ELT15 (Males), $d_0 = 814$. This relatively high figure reflects the high mortality occurring in the first few weeks of life. [1]
- For ages from one to fourteen, d_x takes low values, reflecting the low mortality in childhood, because of the protected environment in which most children live. [1]
- In the late teens, d_x increases. This reflects the higher mortality rate at these ages, primarily as a consequence of car accidents and other accidental deaths. [1]
- d_x remains fairly constant in the twenties. From $x = 30$ onwards, d_x starts to rise, reflecting the increasing underlying mortality with age. [½]
- d_x peaks at age 79, which is the most likely age at which a newborn life will die. [½]
- At ages beyond 79, values of d_x start to decrease. The underlying mortality rates are still rising steeply at this point, but since fewer lives survive to these higher ages the number of deaths falls off, since the population size is smaller. [1]
- d_x falls to zero by age 110, which is the age beyond which survival is assumed to be impossible for this group of lives. [½]

[Total 6]

14.5 (i)(a) The rationale is that females experience lower mortality than males. If the pattern of mortality is similar for the two sexes, but the average age at death of females exceeds that of males by 4 years, then we can approximate the mortality of a female by that of a male 4 years younger.

(i)(b) The probability that a female policyholder aged 62 survives for at least the next 10 years is the same as the probability that a male policyholder aged 58 survives for at least the next 10 years:

$${}_{10}p_{58} = \frac{l_{68}}{l_{58}} = \frac{8,404.4916}{9,413.8004} = 0.892784$$

(ii) The adjusted annual probabilities of death are:

$$q'_{65} = 2q_{65} = 2 \times 0.014243 = 0.028486$$

and: $$q'_{66} = 2q_{66} = 2 \times 0.015940 = 0.031880$$

The probability of surviving for 2 years can then be calculated as:

$${}_2p'_{65} = p'_{65} \times p'_{66} = (1 - 0.028486)(1 - 0.031880) = 0.940542$$

So the probability of dying within 2 years is:

$${}_2q'_{65} = 1 - 0.94054 = 0.059458$$

14.6 We can derive $l_{[60]}$ from:

$$l_{[60]}\left(1 - q_{[60]}\right) = l_{[60]+1}$$

and:

$$l_{[60]+1}\left(1 - q_{[60]+1}\right) = l_{62}$$

Now:

$$q_{[60]+1} = \frac{q'_{[60]+1}}{q'_{61}} \times q_{61} = \frac{0.008680}{0.009009} \times 0.01560 = 0.01503$$

and:

$$q_{[60]} = \frac{q'_{[60]}}{q'_{60}} \times q_{60} = \frac{0.005774}{0.008022} \times 0.01392 = 0.01002$$

So we have:

$$l_{[60]+1} = \frac{l_{62}}{1 - q_{[60]+1}} = \frac{84,173}{1 - 0.01503} = 85,457.5$$

and then:

$$l_{[60]} = \frac{l_{[60]+1}}{1 - q_{[60]}} = \frac{85,457.5}{1 - 0.01002} = 86,322.3$$

14.7 (i) *The table in the question is not laid out in the same way as AM92 in the Tables.*

The policyholder is currently aged $[53]+2$. So the probability of dying between ages 56 and 57 is:

$$\frac{l_{[53]+3} - l_{57}}{l_{[53]+2}} = \frac{1,480 - 1,470}{1,490} = 0.00671$$

(ii) The corresponding probability for an ultimate policyholder is:

$$\frac{l_{56} - l_{57}}{l_{55}} = \frac{1,477 - 1,470}{1,483} = 0.00472$$

(iii) For the usual types of policies (life assurance and annuities), policyholders experience lighter mortality during the select period.

Here, however, the mortality rate is higher in (i) than in (ii), so these policyholders experience heavier mortality during the select period. This could occur, for example, if the policy was sold to individuals who had recently had a particular form of medical treatment that increased mortality rates during the first few years.

14.8 (i) Assuming a uniform distribution of deaths:

$$_{0.5}p_{80} = 1 - {}_{0.5}q_{80} = 1 - 0.5q_{80} = 1 - 0.5(1 - p_{80}) = 1 - 0.5 \times 0.012 = 0.99400$$

(ii) Assuming a constant force of morality:

$$_{0.5}p_{80} = (p_{80})^{0.5} = 0.988^{0.5} = 0.99398$$

14.9 (i) ***Uniform distribution of deaths***

To calculate the value of $_{1.75}p_{45.5}$, we first split the age range up into single years of age:

$$_{1.75}p_{45.5} = {}_{0.5}p_{45.5} \times p_{46} \times {}_{0.25}p_{47}$$ [½]

Now:

$$p_{46} = 1 - q_{46} = 1 - 0.001622 = 0.998378$$ [½]

$$_{0.25}p_{47} = 1 - {}_{0.25}q_{47} = 1 - 0.25q_{47} = 1 - 0.25 \times 0.001802 = 0.999550$$ [½]

and:

$$_{0.5}p_{45.5} = 1 - {}_{0.5}q_{45.5} = 1 - \frac{0.5q_{45}}{1 - 0.5q_{45}} = 1 - \frac{0.5 \times 0.001465}{1 - 0.5 \times 0.001465} = 0.999267$$ [½]

So we have:

$$_{1.75}p_{45.5} = 0.999267 \times 0.998378 \times 0.999550 = 0.997197$$ [1]

[Total 3]

Alternatively, we could calculate this as follows:

$$_{1.75}p_{45.5} = \frac{l_{47.25}}{l_{45.5}} = \frac{0.75l_{47} + 0.25l_{48}}{0.5l_{45} + 0.5l_{46}}$$

$$= \frac{0.75 \times 9{,}771.0789 + 0.25 \times 9{,}753.4714}{0.5 \times 9{,}801.3123 + 0.5 \times 9{,}786.9534}$$

$$= \frac{9{,}766.677025}{9{,}794.13285} = 0.997197$$

(ii) ***Constant force of mortality***

To calculate the value of ${}_{1.75}p_{45.5}$, we first split the age range up into single years of age:

$${}_{1.75}p_{45.5} = {}_{0.5}p_{45.5} \times p_{46} \times {}_{0.25}p_{47}$$ [½]

Now:

$$p_{46} = 1 - q_{46} = 1 - 0.001622 = 0.998378$$ [½]

$${}_{0.25}p_{47} = \left(p_{47}\right)^{0.25} = \left(1 - q_{47}\right)^{0.25} = \left(1 - 0.001802\right)^{0.25} = 0.999549$$ [½]

and:

$${}_{0.5}p_{45.5} = \left(p_{45}\right)^{0.5} = \left(1 - q_{45}\right)^{0.5} = \left(1 - 0.001465\right)^{0.5} = 0.999267$$ [½]

So we have:

$${}_{1.75}p_{45.5} = 0.999267 \times 0.998378 \times 0.999549 = 0.997197$$ [1]

[Total 3]

15

Life assurance contracts

Syllabus objectives

4.1 Define various assurance contracts.

4.1.1 Define the following terms:

- whole-life assurance
- term assurance
- pure endowment
- endowment assurance

including assurance contracts where the benefits are deferred.

4.2 Develop formulae for the means and variances of the payments under various assurance contracts, assuming a constant deterministic interest rate.

4.2.4 Define the assurance factors and their select and continuous equivalents.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming (constant) contingent benefits are payable at the middle or end of the year of contingent event or continuously. Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In the previous chapter, we looked at how life tables can be used to calculate probabilities of survival and death for an individual.

We shall now define the present values of all the main life insurance and annuity contracts, and show how the means and variances of these present values can be calculated.

We will look at assurance contracts in this chapter, and annuity contracts in the next.

The simplest life insurance contract is the *whole life assurance*. The benefit under such a contract is an amount, called the *sum assured*, which will be paid on the policyholder's death.

A *term assurance* contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

A *pure endowment* contract provides a sum assured at the end of a fixed term, provided the policyholder is alive.

An *endowment assurance* is a combination of:

(i) a term assurance, and

(ii) a pure endowment.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

1 Whole life assurance contracts

We begin by looking at the simplest assurance contract, the whole life assurance, which pays the sum assured on the policyholder's death. For the moment we ignore the premiums which the policyholder might pay.

We will use this simple contract to introduce important concepts, in particular the expected present value (EPV) of a payment contingent on an uncertain future event. We will then apply these concepts to other types of life insurance contracts.

We briefly introduced the idea of an expected present value in the previous chapter, where we set out the basic equation of value for an insurance contract.

The present value at time 0 of a certain payment of 1 to be made at time t is v^t.

In this case, we know precisely when the payment will be made, *ie* at time t.

Suppose, however, that the time of payment is not certain but is a random variable, say H. Then the present value of the payment is v^H, which is also a random variable. A whole life assurance benefit is a payment of this type.

For a whole life assurance, the time to payment is a random variable, as we do not know in advance when the policyholder will die.

1.1 Present value random variable

For the moment we will introduce two conventions, which can be relaxed later on.

Convention 1: **we suppose that we are considering a benefit payable to a life who is currently aged x, where x is an integer.**

Convention 2: **we suppose that the sum assured is payable, not *on* death, but at the end of the year of death (based on policy year).**

These will not always hold in practice, of course, but they simplify the application of life table functions.

Under these conventions we see that the whole life sum assured will be paid at time $K_x + 1$.

We will often use the notational convention '(x)' to indicate the phrase 'a life who is currently aged x'.

Recall that K_x is the number of complete years (x) survives for, before dying. For example, if K_x takes the value 5, then (x) dies between time 5 years and time 6 years. So we see that, for our whole life assurance, the duration $K_x + 1$ takes us to the *end* of the year in which (x) dies.

If the sum assured is S, then the present value of the benefit is Sv^{K_x+1}, a random variable.

We now wish to consider the expected value and the variance of Sv^{K_x+1}.

1.2 The expected present value

Since K_x is a discrete random variable that must take a non-negative integer value:

$$E(K_x)=\sum_{k=0}^{\infty} k\,P(K_x=k)$$

and:

$$E\left[g(K_x)\right]=\sum_{k=0}^{\infty} g(k)P(K_x=k)$$

for any function g (assuming that the sum exists, *ie* is finite).

In particular, for $g(k)=v^{k+1}$, **we have:**

$$\mathbf{E\left[v^{K_x+1}\right]=\sum_{k=0}^{\infty} v^{k+1}P(K_x=k)=\sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x}$$

The deferred probability of death ${}_{k|}q_x$ was introduced in the previous chapter. It is the probability that a life aged x survives for k years, but dies before time $k+1$, and is the probability function of the random variable K_x.

To construct the above formula, we start with the present value of the benefit assuming that K_x takes the value k, which is v^{k+1}. We then multiply this by $P(K_x=k)$, which is the probability that death occurs between time k and time $k+1$, and then sum over all possible values of k. As death could occur in *any* future year, this involves summing to infinity.

This is the standard way to calculate the expected value of a random variable, and so gives us an expression for the expectation of the present value of the benefit (or just 'expected present value' for the benefits, for short).

Question

Calculate $E\left(v^{K_x+1}\right)$ at an interest rate of 0%.

Solution

If $i=0\%$, then $v=1$, so:

$$E\left(v^{K_x+1}\right)=E\left(1^{K_x+1}\right)=\sum_{k=0}^{\infty} 1^{k+1}P(K_x=k)=\sum_{k=0}^{\infty} P(K_x=k)=P(K_x\geq 0)=1$$

ie it is the probability that the life will die eventually.

Actuarial notation for the expected present value

$E[v^{K_x+1}]$ is the *expected present value* (EPV) of a sum assured of 1, payable at the end of the year of death. Such functions play a central role in life insurance mathematics and are included in the standard actuarial notation. We define:

$$A_x = E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x$$

[Note that, for brevity, we have written the sum as $\sum_{k=0}^{\infty}$ instead of $\sum_{k=0}^{\omega-x-1}$. This should cause no confusion, since ${}_kp_x = 0$ for $k \geq \omega - x$.]

As we saw when considering life tables, we use the symbol ω to indicate the maximum possible age for a population. So the probability of surviving from any age x to age ω (and beyond) is 0.

In standard actuarial notation, a capital A function denotes the expected present value of a single payment of 1 unit. The subscript x (in the A_x symbol) indicates that the payment is contingent (*ie* depends on) the status of 'x' in some way.

The status 'x' is called a *life status*, and relates to a person who is currently x years old exactly. The status remains in existence (*ie* remains 'active') for as long as the person stays alive into the future. The status ceases to exist (or 'fails') at the point in the future when this person, currently aged x, dies. So, in summary, the life status 'x':

- remains active for as long as (x) remains alive in the future
- fails at the moment at which (x) dies in the future.

So, returning to A_x, we can now say that this symbol represents the EPV of a single payment of 1 unit paid *at the end of the year* in which the status x fails or, in other words, the payment is made at the end of the year in which a person, currently aged x, dies.

We will see later in this chapter how we vary the notation when the payment is made immediately on death, rather than at the end of the year.

Also $E\left[Sv^{K_x+1}\right] = SE\left[v^{K_x+1}\right]$, so if the sum assured is S, then the EPV of the benefit is SA_x.

Values of A_x at various rates of interest are tabulated in (for example) the AM92 tables, which can be found in 'Formulae and Tables for Examinations'.

This is the yellow *Tables* book.

Question

Find A_{40} (AM92 at 6%).

Solution

0.12313

(This appears on page 102 of the *Tables*.)

For comparison purposes, using the same mortality and interest rate (AM92, 6%), we find:

$$A_{30} = 0.07328$$

and: $$A_{70} = 0.48265$$

We see that the values increase with age. This is because the 70-year-old is more likely to die in the near future, so the benefit has a higher expected present value (since we are discounting for a shorter period).

Alternatively, the values of:

$$\sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x \ (= A_x)$$

can be obtained by direct computation from a life table, most conveniently using a computer. This has the advantage that any rate of interest can be used, including the possibility of having an interest rate that varies over time, or with duration.

Question

A whole life assurance pays a benefit of $50,000 at the end of the policy year of death of a life now aged exactly 90. Mortality is assumed to follow the life table given below:

Age, x	l_x	d_x
90	100	25
91	75	35
92	40	40
93	0	0

Calculate the expected present value of this benefit using an effective rate of interest of 5% *pa*.

Solution

The expected present value is:

$$\begin{aligned}50{,}000A_{90} &= 50{,}000\sum_{k=0}^{2} v^{k+1}\ {}_{k|}q_{90} \\ &= 50{,}000\sum_{k=0}^{2} v^{k+1}\frac{d_{90+k}}{l_{90}} \\ &= 50{,}000\left(1.05^{-1}\times\frac{25}{100}+1.05^{-2}\times\frac{35}{100}+1.05^{-3}\times\frac{40}{100}\right) \\ &= \$45{,}054.53\end{aligned}$$

The summation here contains only three terms, because under this mortality table the life will definitely have died before age 93.

1.3 Variance of the present value random variable

Recall that:

$$\text{var}[X]=E\left[X^2\right]-[E(X)]^2$$

for any random variable X, and:

$$\text{var}\left[g(X)\right]=E\left[\left(g(X)\right)^2\right]-\left[E\left(g(X)\right)\right]^2$$

for any random variable X and any function g.

Turning now to the variance of v^{K_x+1}, we have:

$$\text{var}\left[v^{K_x+1}\right]=E\left[\left(v^{K_x+1}\right)^2\right]-\left[E\left(v^{K_x+1}\right)\right]^2$$

This can also be written as:

$$\mathbf{var}[v^{K_x+1}]=\sum_{k=0}^{\infty}(v^{k+1})^2\ {}_{k|}q_x-\left(A_x\right)^2$$

But since $\left(v^{k+1}\right)^2=\left(v^2\right)^{k+1}$, the first term is just ${}^2A_x$ where the '2' prefix denotes an EPV calculated at a rate of interest $(1+i)^2-1$.

So:

$$\operatorname{var}\left[v^{K_x+1}\right] = {}^2A_x - (A_x)^2$$

The 'trick' used here is to notice that $(v^2)^{k+1}$ is the same as v^{k+1}, except that we have replaced v^2 by v. So we are effectively using a new interest rate (i^*, say) for which $v^* = v^2$, *ie*:

$$\frac{1}{1+i^*} = \left(\frac{1}{1+i}\right)^2 \quad \Rightarrow \quad i^* = (1+i)^2 - 1$$

Alternatively, since $v = e^{-\delta}$, we could say that we are using a new force of interest (δ^*, say) for which $v^* = v^2$, *ie*:

$$e^{-\delta^*} = (e^{-\delta})^2 = e^{-2\delta} \quad \Rightarrow \quad \delta^* = 2\delta$$

So, we could say that the '2' prefix on the symbol ${}^2A_x$ indicates that it is evaluated at twice the original force of interest.

So provided we can calculate EPVs at any rates of interest, it is easy to find the variance of a whole life benefit.

Note that:

$$\operatorname{var}\left[Sv^{K_x+1}\right] = S^2 \operatorname{var}\left[v^{K_x+1}\right]$$

Values of ${}^2A_x$ at various rates of interest are tabulated in AM92 'Formulae and Tables for Examinations', but of course any rate of interest can be assumed when a computer is used.

Question

Claire, aged exactly 30, buys a whole life assurance with a sum assured of £50,000 payable at the end of the year of her death.

Calculate the standard deviation of the present value of this benefit using AM92 Ultimate mortality and 6% *pa* interest.

Solution

The standard deviation of the present value is:

$$\sqrt{50{,}000^2\left[{}^2A_{30} - (A_{30})^2\right]} = 50{,}000\sqrt{0.01210 - 0.07328^2} = £4{,}102$$

2 Term assurance contracts

A term assurance contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

If the life survives to the end of the term of the contract there will be no payment made by the life insurer to the policyholder.

2.1 Present value random variable

Consider such a contract, which is to pay a sum assured at the end of the year of death of a life aged x, provided this occurs during the next n years. We assume that n is an integer.

Let F denote the present value of this payment. F is a random variable.

We are just using the letter F here for notational convenience. It is not standard notation.

If the policyholder dies within the n-year term, then $F = v^{K_x+1}$. If the policyholder is still alive at the end of the n-year term, then $F = 0$. We can express this more mathematically as follows:

$$F = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

2.2 Expected present value

The expected present value of the term assurance is:

$$E[F] = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + 0 \times P(K_x \geq n)$$

Writing this using actuarial notation:

$$\boldsymbol{E[F] = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x + 0 \times {}_n p_x}$$

$$\boldsymbol{= \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x}$$

Actuarial notation for the expected present value

Before we look at this, we need to introduce a second kind of status, the duration status $\overline{n|}$. This is easily distinguished from the life status because the number (*n*) is enclosed by the corner symbol ($\overline{|}$), which we used extensively in earlier chapters.

The status $\overline{n|}$ remains active for as long as the duration of time from the valuation date does not exceed *n* years. The status fails at the moment the elapsed time reaches *n* years in duration.

Question

Describe in words the difference in meaning between $A_{\overline{10|}}$ and A_{10}.

Solution

$A_{\overline{10|}}$ is the present value of 1 unit paid in exactly 10 years' time (*ie* the payment is made when the 10-year duration status fails). Since this is a certain payment, we don't use the phrase 'expected present value' for this. In addition, since $A_{\overline{10|}} = v^{10}$, we don't usually have much call to use the $A_{\overline{n|}}$ notation, as it is just as easy to write v^n every time we need it.

A_{10}, on the other hand, is the expected present value of a payment of 1 unit, paid at the end of the year of death of a person currently aged exactly 10 years old.

In actuarial notation, we define:

$$A^{1}_{x:\overline{n|}} = E[F] = \sum_{k=0}^{n-1} v^{k+1}\,{}_{k|}q_x$$

to be the EPV of a term assurance benefit of 1, payable at the end of the year of death of a life x, provided this occurs during the next n years.

The summation formula here is of the same form as that for the whole life assurance, except that under a term assurance the benefit is only paid if the life dies within *n* years, so the summation is only for *n* years.

Now let's explain the more complex notation used here ($A^{1}_{x:\overline{n|}}$).

This still represents the EPV of a single payment, because it is a 'big A' symbol. However, the payment is now contingent on what happens to *two* statuses in some way (multiple statuses are shown in the subscript separated by a colon).

The exact condition for payment is identified by the number that is placed above the statuses (*ie* the 1) and where it is placed:

- the number is positioned above the life status x: this indicates that the payment is made *only* when the life status x fails (*ie* dies)
- the number over the x is 1: this tells us that the life status x has to fail *first* out of the two statuses involved, in order for the payment to be made.

As the $\overline{n|}$ status will fail after n years, then (x) has to die within n years for the payment to be made. In other words, $A^{1}_{x:\overline{n|}}$ is the EPV of 1 unit paid only on the death of (x), provided that occurs within n years.

As before, the symbol also indicates that the payment is made at the end of the year of death. This aspect of the notation will become clearer later in the chapter.

2.3 Variance of the present value random variable

Along the same lines as for the whole life assurance:

$$\text{var}[F] = {}^2A^{\,1}_{x:\overline{n}|} - \left(A^{\,1}_{x:\overline{n}|}\right)^2$$

where the '2' prefix means that the EPV is calculated at rate of interest $(1+i)^2 - 1$, as before.

3 Pure endowment contracts

A pure endowment contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

3.1 Present value random variable

Consider a pure endowment contract to pay a sum assured of 1 after n years, provided a life aged x is still alive. We assume that n is an integer.

Let G denote the present value of the payment.

Again, we are using the letter G for notational convenience. This is not standard notation.

The payment is equal to zero if the policyholder dies within n years, but 1 if the policyholder survives to the end of the term. So, if the policyholder dies within the n-year term, then $G=0$. On the other hand, if the policyholder is still alive at the end of the n-year term, then $G=v^n$.

This can be written mathematically as:

$$G=\begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

3.2 Expected present value

The expected present value of the pure endowment is:

$$E[G]=0\times P(K_x<n)+v^n P(K_x \geq n)$$

In actuarial notation (and reversing the order of the two terms), this is:

$$\mathbf{E[G]=v^n\,{}_np_x+(0\times{}_nq_x)}$$

Question

At a certain company, the probability of each employee leaving during any given year is 5%, independent of the other employees. Those who remain with the company for 25 years are given $1,000,000.

Calculate the expected present value of this payment to a new starter, assuming an effective interest rate of 7% *pa* and ignoring the possibility of death.

Solution

The probability of remaining with the company in any given year is 0.95, so the probability of remaining with the company for 25 years is 0.95^{25}. Therefore, the expected present value of the benefit is equal to:

$$1,000,000 \times \frac{1}{1.07^{25}} \times 0.95^{25} = \$51,109$$

Actuarial notation for the expected present value

In actuarial notation, we define:

$$A_{\substack{1\\x:\overline{n}|}} = E[G] = v^n \, {}_np_x$$

to be the EPV of a pure endowment benefit of 1, payable after n years to a life aged x.

The notation here is $A_{\substack{1\\x:\overline{n}|}}$ because:

- the benefit is paid *only* when the term of n years ends, *ie* at the moment at which the $\overline{n}|$ status fails, so the number (whatever it may be) needs to be placed above the $\overline{n}|$
- the benefit will only be paid (at time n) if the person is still alive at that time: this requires the status $\overline{n}|$ to be the *first* of the two statuses to fail, and hence we need the number to be a '1'.

3.3 Variance of the present value random variable

Following the same lines as before:

$$\text{var}[G] = {}^2A_{\substack{1\\x:\overline{n}|}} - \left(A_{\substack{1\\x:\overline{n}|}}\right)^2$$

where, as usual, the '2' prefix denotes an EPV calculated at a rate of interest of $(1+i)^2 - 1$.

Question

Calculate the standard deviation of the present value of the payment described in the previous question.

Solution

In the previous question, we calculated the expected present value of the payment to be:

$$E(PV) = 1,000,000 \times \frac{1}{1.07^{25}} \times 0.95^{25} = \$51,109$$

To calculate the standard deviation, we will also need to calculate $E(PV^2)$. This is:

$$E(PV^2)=\left(1{,}000{,}000\times\frac{1}{1.07^{25}}\right)^2\times 0.95^{25}=1{,}000{,}000^2\times\frac{1}{1.07^{50}}\times 0.95^{25}=\$9{,}416{,}754{,}493$$

The standard deviation is:

$$\begin{aligned}sd(PV)&=\sqrt{E(PV^2)-\left[E(PV)\right]^2}\\&=\sqrt{9{,}416{,}754{,}493-51{,}109^2}\\&=\$82{,}490\end{aligned}$$

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4 Endowment assurance contracts

An endowment assurance is a combination of:

- a term assurance and
- a pure endowment.

That is, a sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

4.1 Present value random variable

Consider an endowment assurance contract to pay a sum assured of 1 to a life now aged x at the end of the year of death, if death occurs during the next n years, or after n years if the life is then alive. We assume that n is an integer.

Let H be the present value of this payment.

Again, we are using the letter H for notational convenience. This is not standard notation.

The benefit is paid on death or survival, so its value must be the sum of the values of a benefit paid on death and a benefit paid on survival.

In terms of the present values already defined, $H = F + G$.

Hence:

$$H = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$$

This can also be written as:

$$H = v^{\min\{K_x+1,\, n\}}$$

4.2 Expected present value

The expected present value of the endowment assurance is:

$$E[H] = \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) + v^n P(K_x \geq n)$$

This can also be written as:

$$\begin{aligned} E[H] &= E[F] + E[G] \\ &= \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x + v^n {}_n p_x \\ &= \sum_{k=0}^{n-2} v^{k+1} {}_{k|}q_x + v^n {}_{n-1} p_x \end{aligned}$$

The last expression holds because payment at time n is certain if the life survives to age $x+n-1$.

Actuarial notation for the expected present value

In actuarial notation we define:

$$\begin{aligned} A_{x:\overline{n}|} &= E[H] \\ &= E[F] + E[G] \\ &= A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} \end{aligned}$$

The interpretation of the $A_{x:\overline{n}|}$ symbol is that where we have no number above either status, it implies that the payment is made on the *first* of the two statuses to fail, regardless of order. So $A_{x:\overline{n}|}$ is the EPV of 1 unit, paid after *n* years, or at the end of the year of death of (*x*), whichever occurs first.

Values of $A_{x:\overline{n}|}$ are tabulated where $x+n=60$ or $x+n=65$, for AM92 mortality at both 4% and 6% interest.

Question

Allan, aged exactly 40, has just bought a 20-year endowment assurance policy. The sum assured is £100,000 and this is payable on survival to age 60 or at the end of the year of earlier death.

Calculate the expected present value of the benefit paid to Allan, assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The expected present value is:

$$100,000 A_{40:\overline{20}|} = 100,000 \times 0.46433 = £46,433$$

The value of $A_{40:\overline{20}|}$ (AM92 Ultimate mortality, 4% interest) is taken from page 100 of the Tables.

4.3 Variance of the present value random variable

Note that F and G are not independent random variables (one must be zero and the other non-zero).

This is because the life will either survive to the end of the *n*-year period or die during it.

Therefore:

$$\text{var}[H] \neq \text{var}[F] + \text{var}[G]$$

We must find $\text{var}[H]$ from first principles.

As before, we find that:

$$\text{var}[H] = {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2$$

where the '2' prefix denotes an EPV calculated at rate of interest $(1+i)^2 - 1$.

Question

Derive the formula above for $\text{var}[H]$ from first principles.

Solution

We can use the expression $H = v^{\min\{K_x+1,\, n\}}$, and the same approach as we've used elsewhere in this chapter:

$$\begin{aligned}
\text{var}[H] &= \text{var}\left[v^{\min\{K_x+1,\, n\}}\right] \\
&= E\left[\left(v^{\min\{K_x+1,\, n\}}\right)^2\right] - \left(E\left[v^{\min\{K_x+1,\, n\}}\right]\right)^2 \\
&= E\left[\left(v^2\right)^{\min\{K_x+1,\, n\}}\right] - \left(A_{x:\overline{n}|}\right)^2 \\
&= {}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2
\end{aligned}$$

where ${}^2A_{x:\overline{n}|}$ is calculated at rate of interest $(1+i)^2 - 1$.

It is necessary to express our random variable using a single term if we wish to derive a variance like this, ie as $H = v^{\min\{K_x+1,\, n\}}$ rather than $H = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$.

5 Deferred assurance benefits

Although not as common as deferred annuities, deferred assurance benefits can be defined in a similar way.

Deferred annuities will be introduced in Chapter 16.

5.1 Present value random variable

For example, a whole life assurance with sum assured 1, payable to a life aged x but deferred n years is a contract to pay a death benefit of 1 provided death occurs after age $x+n$.

If we let J denote the present value of this benefit, then:

$$J=\begin{cases}0 & \text{if } K_x<n\\ v^{K_x+1} & \text{if } K_x\geq n\end{cases}$$

5.2 Expected present value

If the benefit is payable at the end of the year of death (if at all), the EPV of this assurance is denoted $_{n|}A_x$.

As usual, the subscript of $n|$ to the left of the symbol indicates that the event is deferred for n years. The subscript x to the right of the symbol means that the payment will be made on the failure of the life status (but now we have to 'wait' at least n years before anything can be paid).

It can be shown that:

$$_{n|}A_x = A_x - A^1_{x:\overline{n|}} = v^n\,{}_np_x\,A_{x+n}$$

Note the appearance of $v^n\,{}_np_x$. The factor $v^n\,{}_np_x$ is important and useful in developing EPVs. It plays the role of the pure interest discount factor v^n, where now the payment or present value being discounted depends on the survival of a life aged x.

Question

Prove that $_{n|}A_x = A_x - A^1_{x:\overline{n|}} = v^n\,{}_np_x\,A_{x+n}$.

Solution

The first result is proved as follows:

$$_{n|}A_x = E(J) \qquad \text{where:} \qquad J=\begin{cases}0 & \text{if } K_x<n\\ v^{K_x+1} & \text{if } K_x\geq n\end{cases}$$

Now $A_x = E(X)$, say, where:

$$X = v^{K_x+1}$$

and $A^1_{x:\overline{n}|} = E(Y)$, say, where:

$$Y = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

So:

$$X - Y = \begin{cases} v^{K_x+1} - v^{K_x+1} & = 0 & \text{if } K_x < n \\ v^{K_x+1} - 0 & = v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

$$= J$$

Therefore:

$$E(J) = E(X - Y) = E(X) - E(Y) = A_x - A^1_{x:\overline{n}|}$$

Furthermore:

$$A_x - A^1_{x:\overline{n}|} = \sum_{k=0}^{\infty} v^{k+1} P(K_x = k) - \sum_{k=0}^{n-1} v^{k+1} P(K_x = k)$$

$$= \sum_{k=n}^{\infty} v^{k+1} P(K_x = k)$$

$$= \sum_{k=n}^{\infty} v^{k+1}\, {}_k p_x\, q_{x+k}$$

$$= v^n\, {}_n p_x \sum_{k=n}^{\infty} v^{k+1-n}\, {}_{k-n} p_{x+n}\, q_{x+k}$$

since ${}_k p_x = {}_n p_x\, {}_{k-n} p_{x+n}$ (for $k \geq n$).

If we let $j = k - n$ in the summation, we can write:

$$A_x - A^1_{x:\overline{n}|} = v^n \, {}_n p_x \sum_{j=0}^{\infty} v^{j+1} \, {}_j p_{x+n} \, q_{x+n+j}$$

$$= v^n \, {}_n p_x \sum_{j=0}^{\infty} v^{j+1} P(K_{x+n} = j)$$

$$= v^n \, {}_n p_x \, A_{x+n}$$

as required.

We can evaluate the expected present value of deferred assurances using values from the *Tables*.

Question

A life assurance policy pays a benefit of £20,000 at the end of the policy year of death of a life now aged exactly 55, provided that death occurs after exact age 60.

Calculate the expected present value of this benefit assuming that the effective annual rate of interest is 4% and mortality follows the AM92 Ultimate table.

Solution

The expected present value is:

$$20{,}000 \, {}_{5|}A_{55} = 20{,}000 v^5 \, {}_5 p_{55} \, A_{60}$$

$$= 20{,}000(1+i)^{-5} \times \frac{l_{60}}{l_{55}} \times A_{60}$$

$$= 20{,}000 \times 1.04^{-5} \times \frac{9{,}287.2164}{9{,}557.8179} \times 0.45640$$

$$= £7{,}290$$

5.3 Variance of the present value random variable

One useful feature of deferred assurances is that it is easier to find their variances directly than is the case for (deferred) **annuities. For example, let X be the present value of a whole life assurance and Y the present value of a temporary assurance with term n years, both for a sum assured of 1 payable at the end of the year of death of a life aged x. Then $E[X] = A_x$, $E[Y] = A^1_{x:\overline{n}|}$ and:**

$$E[X - Y] = {}_{n|}A_x = A_x - A^1_{x:\overline{n}|}$$

We took this approach in the previous question.

Moreover:

$$\mathrm{var}[X-Y]=\mathrm{var}[X]+\mathrm{var}[Y]-2\,\mathrm{cov}[X,Y]$$

and it can be shown by considering the distributions of X and Y, that

$$\mathrm{cov}[X,Y]= {}^2A^{1}_{x:\overline{n}|}-A_x\,A^{1}_{x:\overline{n}|}$$

where the '2' superscript has its usual meaning.

Question

Show that $\mathrm{cov}(X,Y)={}^2A^{1}_{x:\overline{n}|}-A_xA^{1}_{x:\overline{n}|}$.

Solution

From Subject CS1, the covariance of two random variables X and Y is defined by the equation:

$$\mathrm{cov}(X,Y)=E(XY)-E(X)E(Y)$$

For a whole life assurance with a sum assured of 1 payable at the end of the year of death of a life currently aged x, the present value random variable is:

$$X=v^{K_x+1}$$

and: $E(X)=A_x$

Furthermore, for an n-year term assurance issued to the same life, the present value random variable is:

$$Y=\begin{cases} v^{K_x+1} & \text{if } K_x<n \\ 0 & \text{otherwise} \end{cases}$$

and: $E(Y)=A^{1}_{x:\overline{n}|}$

Multiplying X and Y gives:

$$XY=\begin{cases} v^{2(K_x+1)} & \text{if } K_x<n \\ 0 & \text{otherwise} \end{cases}$$

So we see that XY is just the same as Y, except for the fact that v has been replaced by v^2. Hence $E(XY)={}^2A^{1}_{x:\overline{n}|}$, where, as usual, the superscript of 2 to the left of the assurance symbol indicates that v has been replaced by v^2, or in other words that we are valuing the benefit using an interest rate of $(1+i)^2-1$ (or, equivalently, twice the force of interest).

So we have:

$$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)={}^2A^{1}_{x:\overline{n}|}-A_x A^{1}_{x:\overline{n}|}$$

This covariance result can now be used in the formula:

$$\operatorname{var}[X-Y]=\operatorname{var}[X]+\operatorname{var}[Y]-2\operatorname{cov}[X,Y]$$

Hence:

$$\begin{aligned}\mathbf{var}[X-Y] &= {}^2A_x-(A_x)^2+{}^2A^{1}_{x:\overline{n}|}-\left(A^{1}_{x:\overline{n}|}\right)^2-2\left({}^2A^{1}_{x:\overline{n}|}-A_x A^{1}_{x:\overline{n}|}\right)\\ &= {}^2A_x-\left(A_x-A^{1}_{x:\overline{n}|}\right)^2-{}^2A^{1}_{x:\overline{n}|}\\ &= {}^2A_x-\left({}_{n|}A_x\right)^2-{}^2A^{1}_{x:\overline{n}|}\\ &= {}_{n|}{}^2A_x-\left({}_{n|}A_x\right)^2\end{aligned}$$

This result for the variance of a deferred whole life assurance can alternatively be obtained by considering the present value random variable:

$$J=\begin{cases}0 & \text{if } K_x<n\\ v^{K_x+1} & \text{if } K_x\ge n\end{cases}$$

Now:

$$E[J]=\sum_{k=n}^{\infty} v^{k+1}\,{}_{k|}q_x={}_{n|}A_x$$

so:

$$\begin{aligned}\operatorname{var}[J] &= E\left[J^2\right]-\left(E[J]\right)^2\\ &= \sum_{k=n}^{\infty}\left(v^{k+1}\right)^2\,{}_{k|}q_x-\left({}_{n|}A_x\right)^2\\ &= \sum_{k=n}^{\infty}\left(v^2\right)^{k+1}\,{}_{k|}q_x-\left({}_{n|}A_x\right)^2\\ &= {}_{n|}{}^2A_x-\left({}_{n|}A_x\right)^2\end{aligned}$$

as before.

5.4 Deferred term assurance

So far in this section, we have looked at a deferred whole life assurance.

We will now consider a deferred term assurance, where a benefit of 1 is payable at the end of the year of death of a life currently aged x, provided that death occurs between age $x+m$ and age $x+m+n$.

Present value random variable

If we let M denote the present value of this benefit, then:

$$M=\begin{cases}0 & \text{if } K_x<m \text{ or } K_x\geq m+n \\ v^{K_x+1} & \text{if } m\leq K_x<m+n\end{cases}$$

Expected present value

The expected present value of the benefit is denoted ${}_{m|}A^{1}_{x:\overline{n}|}$. Similar to the deferred whole life assurance above, it can be shown that:

$${}_{m|}A^{1}_{x:\overline{n}|}=A^{1}_{x:\overline{m+n}|}-A^{1}_{x:\overline{m}|}=v^m\,{}_mp_x\,A^{1}_{x+m:\overline{n}|}$$

Variance of the present value random variable

This is given by the formula:

$$\text{var}[M]={}_{m|}^{2}A^{1}_{x:\overline{n}|}-\left({}_{m|}A^{1}_{x:\overline{n}|}\right)^2$$

6 Benefits payable immediately on death

So far we have assumed that assurance death benefits have been paid at the end of the year of death. In practice, assurance death benefits are paid a short time after death, as soon as the validity of the claim can be verified.

Assuming a delay until the end of the year of death is therefore not a prudent approximation, but assuming that there is no delay and that the sum assured is paid immediately on death is a prudent approximation.

This is due to the fact that if we delay payment until the end of the year, the 'expected present value of the benefits' part of the equation of value will be smaller. Thus the premium charged for those benefits will be lower. Likewise, assuming that claims are paid immediately on death would give a higher (*ie* more prudent) premium.

The present value of death benefits payable immediately on the death of the policyholder can be expressed in terms of the policyholder's complete future lifetime, T_x. This random variable was introduced in Chapter 14, along with its PDF ${}_tp_x\,\mu_{x+t}$.

Related to such assurance benefits are annuities under which payment is made in a continuous stream instead of at discrete intervals. Of course, this does not happen in practice, but such an assumption is reasonable if payments are very frequent, say weekly or daily. Later in the course we will consider annuities with a payment frequency between continuous and annual.

We will look at continuously payable annuities in Chapter 16, and those with a payment frequency between continuous and annual in Chapter 17.

6.1 Whole life assurance

Consider a whole life assurance with sum assured 1, payable immediately on the death of a life aged *x*.

Present value random variable

The payments will be made exactly T_x years from now (as T_x is the time from now until the exact moment of death for a life currently aged *x*). So:

The present value of this benefit is v^{T_x}.

Expected present value

As shown in Subject CS1, for any continuous random variable *Y*, with probability density function $f_Y(y)$, the expectation is:

$$E(Y)=\int_{-\infty}^{\infty} y\, f_Y(y)\,dy$$

Also, for any function g:

$$E\left(g(Y)\right)=\int_{-\infty}^{\infty} g(y)\, f_Y(y)\, dy$$

Since the density function of T_x is ${}_tp_x\,\mu_{x+t}$, the EPV of the benefit, denoted by $\bar{A}_x$ is:

$$\bar{A}_x = E\left[v^{T_x}\right] = \int_0^{\infty} v^t\, {}_tp_x \mu_{x+t}\, dt$$

and its variance can be shown to be:

$$\text{var}\left[v^{T_x}\right] = {}^2\bar{A}_x - (\bar{A}_x)^2$$

Intuitively, the integral expression for the expectation can be built up as follows. The expression ${}_tp_x\mu_{x+t}\,dt$ is the probability that the life, currently aged x, survives to time t and then dies during the short interval $(t, t+dt)$. The factor v^t gives the present value of the payment made if the life dies in this interval, and the integral sums this over all future time periods. As usual, this sum of all the possible values multiplied by their probabilities gives us the expectation.

The variance formula given above is derived from the general formula $\text{var}[X]=E[X^2]-E[X]^2$, as follows:

$$\begin{aligned}
\text{var}\left[v^{T_x}\right] &= E\left[\left(v^{T_x}\right)^2\right]-\left(E\left[v^{T_x}\right]\right)^2 \\
&= E\left[\left(v^2\right)^{T_x}\right]-\left(\bar{A}_x\right)^2 \\
&= \int_0^{\infty}\left(v^2\right)^t\, {}_tp_x\,\mu_{x+t}\,dt-\left(\bar{A}_x\right)^2 \\
&= {}^2\bar{A}_x-\left(\bar{A}_x\right)^2
\end{aligned}$$

where ${}^2\bar{A}_x$ is calculated at rate of interest $(1+i)^2-1$

The standard actuarial notation for the EPVs of assurances with a death benefit payable immediately on death, or of annuities payable continuously, is a bar added above the symbol for the EPV of an assurance with a death benefit payable at the end of the year of death, or an immediate annuity with annual payments, respectively.

6.2 Term assurance

Term assurance contracts with a death benefit payable immediately on death can be defined in a similar way, with the obvious notation for their EPVs and deferred assurance benefits likewise.

Consider a term assurance with a sum assured of 1 payable immediately upon the death of a life now aged x, provided that this life dies within n years.

Present value random variable

Let $\bar{F}$ denote the present value of this benefit. Then:

$$\bar{F}=\begin{cases} v^{T_x} & \text{if } T_x<n \\ 0 & \text{if } T_x\geq n \end{cases}$$

Expected present value

The expected present value is:

$$E\left(\bar{F}\right)=\int_0^n v^t f_{T_x}(t)\,dt=\int_0^n v^t\,{}_tp_x\,\mu_{x+t}\,dt$$

where $f_{T_x}(t)={}_tp_x\,\mu_{x+t}$ is the probability density function of T_x, as mentioned earlier.

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\bar{A}^{\,1}_{x:\overline{n|}}$.

Variance of the present value random variable

Its variance is ${}^2\bar{A}^{\,1}_{x:\overline{n|}}-(\bar{A}^{\,1}_{x:\overline{n|}})^2$.

6.3 Endowment assurance

Consider an endowment assurance with a sum assured of 1 payable after n years or immediately upon the earlier death of a life now aged x.

Present value random variable

Let $\bar{H}$ denote the present value of this benefit. Then:

$$\bar{H}=\bar{F}+G=\begin{cases} v^{T_x} & \text{if } T_x<n \\ v^n & \text{if } T_x\geq n \end{cases}$$

Recall that the random variable G is equal to the present value of the pure endowment. As this can never be paid earlier than the maturity date, there is no need for a random variable $\bar{G}$.

Expected present value

The expected present value is:

$$E(\bar{H}) = \int_0^n v^t f_{T_x}(t)\,dt + \int_n^\infty v^n f_{T_x}(t)\,dt$$

$$= \int_0^n v^t\, {}_tp_x\, \mu_{x+t}\,dt + v^n \int_n^\infty f_{T_x}(t)\,dt$$

$$= \int_0^n v^t\, {}_tp_x\, \mu_{x+t}\,dt + v^n P(T_x > n)$$

$$= \int_0^n v^t\, {}_tp_x\, \mu_{x+t}\,dt + v^n\, {}_np_x$$

Actuarial notation for the expected present value

The EPV of the benefit is denoted $\bar{A}_{x:\overline{n|}}$.

Variance of the present value random variable

Its variance is ${}^2\bar{A}_{x:\overline{n|}} - (\bar{A}_{x:\overline{n|}})^2$.

6.4 Other relationships

We leave the reader to supply the definitions and proofs of the following:

$$\bar{A}_x = \bar{A}^{\,1}_{x:\overline{n|}} + {}_{n|}\bar{A}_x$$

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{\,1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;\;\,1}$$

$${}_{n|}\bar{A}_x = v^n\, {}_np_x\, \bar{A}_{x+n}$$

Note in the second of these that it is only death benefits that are affected by the changed time of payment. Survival benefits such as a pure endowment are not affected.

Question

Explain each of the formulae shown above by general reasoning.

Solution

In the first equation, a whole life assurance is equal to a term assurance for *n* years (which pays out on death provided this occurs within *n* years) plus a whole life assurance that is deferred for *n* years (which pays out on death provided this occurs after *n* years). All benefits are paid immediately on death.

In the second equation, an endowment assurance, with a benefit paid immediately on death within *n* years or on survival to the end of the *n*-year term, is equal to an *n*-year term assurance with a benefit paid immediately on death plus a pure endowment benefit paid if the policyholder survives for the *n*-year period. Note that the pure endowment symbol does not have a bar over it – a bar represents a payment made immediately on death, and a pure endowment incorporates no death benefit whatsoever.

In the third equation, we have a deferred whole life assurance, under which the benefit is paid immediately on death, but only if death happens after *n* years. This is equal to a whole life assurance from age $x+n$, allowing for discounting for interest and the fact that the life aged x must survive for *n* years in order for the benefit to be paid.

6.5 Claims acceleration approximation

It is convenient to be able to estimate $\bar{A}_x$, $\bar{A}_{x:\overline{n}|}$ and so on in terms of commonly tabulated functions. One simple approximation is *claims acceleration*. Of deaths occurring between ages $x+k$ and $x+k+1$, say, ($k = 0, 1, 2, \ldots$) roughly speaking the average age at death will be $x+k+\tfrac{1}{2}$. Under this assumption claims are paid on average 6 months before the end of the year of death.

So, for example:

$$\begin{aligned}\bar{A}_x &\approx v^{\frac{1}{2}} q_x + v^{1\frac{1}{2}}\, p_x\, q_{x+1} + v^{2\frac{1}{2}}\, {}_2p_x\, q_{x+2} + \cdots \\ &= (1+i)^{\frac{1}{2}} \left(v q_x + v^2\, p_x\, q_{x+1} + v^3\, {}_2p_x\, q_{x+2} + \cdots \right) \\ &= (1+i)^{\frac{1}{2}} A_x\end{aligned}$$

and a similar result holds for term assurances.

Therefore we obtain the approximate EPVs:

$$\bar{A}_x \cong (1+i)^{\frac{1}{2}} A_x$$

$$\bar{A}^{1}_{x:\overline{n}|} \cong (1+i)^{\frac{1}{2}} A^{1}_{x:\overline{n}|}$$

$$\bar{A}_{x:\overline{n}|} \cong (1+i)^{\frac{1}{2}} A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;1}$$

Note again that, in the case of the endowment assurance, only the death benefit is affected by the claims acceleration.

6.6 Further approximation

A second approximation is obtained by considering a whole life or term assurance to be a sum of deferred term assurances, each for a term of one year. Then, taking the whole life case as an example:

$$\bar{A}_x = {}_{0|}\bar{A}^{1}_{x:\overline{1}|} + {}_{1|}\bar{A}^{1}_{x:\overline{1}|} + {}_{2|}\bar{A}^{1}_{x:\overline{1}|} + \dots$$

$$= \bar{A}^{1}_{x:\overline{1}|} + vp_x\bar{A}^{\;1}_{x+1:\overline{1}|} + v^2\,{}_2p_x\bar{A}^{\;1}_{x+2:\overline{1}|} + \dots$$

Now:

$$\bar{A}^{\;1}_{x+k:\overline{1}|} = \int_0^1 v^t\,{}_tp_{x+k}\,\mu_{x+k+t}\,dt$$

If we now make the assumption that deaths are uniformly distributed between integer ages, then as we saw in Chapter 14, the PDF of the complete future lifetime random variable is constant between integer ages. So for an integer age $x+k$, we have:

$$f_{T_{x+k}}(t) = {}_tp_{x+k}\,\mu_{x+k+t} = \text{constant} \qquad (0 \le t \le 1)$$

Now, using the formula for ${}_tq_x$ introduced in Section 1.3 of Chapter 14:

$$q_{x+k} = \int_0^1 {}_tp_{x+k}\,\mu_{x+k+t}\,dt = {}_tp_{x+k}\,\mu_{x+k+t} \qquad (0 \le t \le 1)$$

If we assume that deaths are uniformly distributed between integer ages, such that:

$${}_tp_{x+k}\,\mu_{x+k+t} = q_{x+k} \qquad (0 \le t < 1)$$

then:

$$\bar{A}^{\;1}_{x+k:\overline{1}|} \cong q_{x+k}\int_0^1 v^t\,dt = q_{x+k}\,\frac{iv}{\delta}$$

Here, the integration has been carried out as follows:

$$\int_0^1 v^t\,dt = \int_0^1 e^{-\delta t}\,dt = \left[-\frac{e^{-\delta t}}{\delta}\right]_0^1 = \frac{1-e^{-\delta}}{\delta} = \frac{1-v}{\delta} = \frac{d}{\delta} = \frac{iv}{\delta}$$

Hence:

$$\bar{A}_x \cong \frac{i}{\delta}\left(vq_x + v^2p_x\,q_{x+1} + v^3\,{}_2p_x\,q_{x+2} + \dots\right)$$

$$= \frac{i}{\delta}A_x$$

Similarly:

$$\bar{A}^{1}_{x:\overline{n|}} \cong \frac{i}{\delta} A^{1}_{x:\overline{n|}}$$

Question

Evaluate A_{40} and $\bar{A}_{40}$ based on AM92 Ultimate mortality at 4% *pa* interest.

Solution

We can look up the value of A_{40} in the *Tables*:

$$A_{40} = 0.23056$$

Then we can approximate $\bar{A}_{40}$ as:

$$\bar{A}_{40} \approx (1+i)^{\frac{1}{2}} A_{40} = 1.04^{\frac{1}{2}} \times 0.23056 = 0.23513$$

or: $$\bar{A}_{40} \approx \frac{i}{\delta} A_{40} = \frac{0.04}{\ln 1.04} \times 0.23056 = 0.23514$$

Despite the difference in the fifth decimal place, both of these approximations would be acceptable in the exam.

7 Evaluating means and variances using select mortality

Corresponding to the assurances defined earlier in this chapter are select equivalents defined as before, but assumed to be issued to a select life denoted $[x]$ rather than x.

So, for example, $A_{[x]} = \sum_{k=0}^{k=\infty} v^{k+1} {}_{k|}q_{[x]}$ can be used to calculate the EPV of benefits of a whole life assurance issued to a select life aged $[x]$ at entry.

The variance formulae established earlier also apply replacing x with $[x]$.

Question

A whole life assurance contract, under which the sum assured of £40,000 is payable immediately on death, is issued to a life aged exactly 35.

Using AM92 Select mortality and an interest rate of 6% *pa* effective, calculate:

(i) the expected present value of the benefits

(ii) the variance of the present value of the benefits.

Solution

(i) The EPV is given by:

$$40{,}000\bar{A}_{[35]} \approx 40{,}000 \times 1.06^{0.5} A_{[35]} = 40{,}000 \times 1.06^{0.5} \times 0.09475 = £3{,}902$$

(ii) The variance is given by:

$$40{,}000^2\left({}^2\bar{A}_{[35]} - \left(\bar{A}_{[35]}\right)^2\right)$$

where ${}^2\bar{A}_{[35]}$ is evaluated using an interest rate of $1.06^2 - 1 = 12.36\%$. So the variance is:

$$40{,}000^2\left(1.1236^{0.5} \times {}^2A_{[35]} - \left(1.06^{0.5} \times A_{[35]}\right)^2\right)$$

$$= 40{,}000^2\left(1.1236^{0.5} \times 0.01765 - \left(1.06^{0.5} \times 0.09475\right)^2\right)$$

$$= (£3{,}835)^2$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 15 Summary

Types of contracts

Assurance contracts are contracts where the insurer makes a payment on death. These might be term assurances, whole life assurances or endowment assurances. The benefits can be payable at the end of the year of death or immediately on death.

Pure endowment contracts make a payment on survival to a given age.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

For each of the following benefits we assume that the sum assured is 1 and the policyholder is aged x at the outset.

Whole life assurance with benefit payable at the end of the year of death

Present value: v^{K_x+1}

Expected present value: $E\left[v^{K_x+1}\right]=A_x$

Variance of present value: $\text{var}\left[v^{K_x+1}\right]={}^2A_x-(A_x)^2$

Whole life assurance with benefit payable immediately on death

Present value: v^{T_x}

Expected present value: $E\left[v^{T_x}\right]=\bar{A}_x$

Variance of present value: $\text{var}\left[v^{T_x}\right]={}^2\bar{A}_x-(\bar{A}_x)^2$

Term assurance (*n*-year term) with benefit payable at the end of the year of death

Present value: $F=\begin{cases} v^{K_x+1} & \text{if } K_x<n \\ 0 & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[F]=A^{1}_{x:\overline{n}|}$

Variance of present value: $\text{var}[F]={}^2A^{1}_{x:\overline{n}|}-\left(A^{1}_{x:\overline{n}|}\right)^2$

Term assurance (*n*-year term) with benefit payable immediately on death

Present value: $\bar{F}=\begin{cases} v^{T_x} & \text{if } T_x<n \\ 0 & \text{if } T_x \geq n \end{cases}$

Expected present value: $E[\bar{F}]=\bar{A}^{1}_{x:\overline{n}|}$

Variance of present value: $\text{var}[\bar{F}]={}^2\bar{A}^{1}_{x:\overline{n}|}-\left(\bar{A}^{1}_{x:\overline{n}|}\right)^2$

Pure endowment (*n*-year term)

Present value: $G=\begin{cases} 0 & \text{if } K_x<n \\ v^n & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[G]=A_{x:\overline{n}|}^{\;\;1}$

Variance of present value: $\text{var}[G]={}^2A_{x:\overline{n}|}^{\;\;1}-\left(A_{x:\overline{n}|}^{\;\;1}\right)^2$

Endowment assurance (*n*-year term) with benefit payable on survival to the maturity date or at the end of the year of earlier death

Present value: $H=F+G=\begin{cases} v^{K_x+1} & \text{if } K_x<n \\ v^n & \text{if } K_x \geq n \end{cases}$

Expected present value: $E[H]=A_{x:\overline{n}|}$

Variance of present value: $\text{var}[H]={}^2A_{x:\overline{n}|}-\left(A_{x:\overline{n}|}\right)^2$

Endowment assurance (n-year term) with benefit payable on survival to the maturity date or immediately on earlier death

Present value: $\bar{H} = \bar{F} + G = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$

Expected present value: $E\left[\bar{H}\right] = \bar{A}_{x:\overline{n|}}$

Variance of present value: $\text{var}\left[\bar{H}\right] = {}^2\bar{A}_{x:\overline{n|}} - \left(\bar{A}_{x:\overline{n|}}\right)^2$

Deferred whole life assurance with benefit paid at end of year of death

Present value: $J = v^{K_x+1} - F = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$

Expected present value: $E\left[J\right] = {}_{n|}A_x = A_x - A^{1}_{x:\overline{n|}}$

Variance of present value: $\text{var}\left[J\right] = {}^2_{n|}A_x - \left({}_{n|}A_x\right)^2$

The letters F, G, H and J have simply been used for notational convenience. They do not represent standard notation.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 15 Practice Questions

15.1 If T_x and K_x are random variables measuring the complete and curtate future lifetimes, respectively, of a life aged x, write down an expression for each of the following symbols as the expectation of a random variable:

(i) A_x

(ii) $\bar{A}^{1}_{x:\overline{n}|}$

(iii) $A_{x:\overline{n}|}^{\;\;\;1}$

15.2 The benefit payable under a special assurance policy has present value random variable:

$$W = v^{\max\{K_x+1,\, n\}}$$

where K_x is the curtate future lifetime of a person currently aged exactly x.

(i) Describe the benefit paid under this policy.

(ii) Express $E[W]$ in terms of standard actuarial notation.

(iii) Express $\text{var}[W]$ in terms of standard actuarial notation.

15.3 Calculate the expectation and standard deviation of the present value of the benefits from each of the following contracts issued to a life aged exactly 40, assuming that the annual effective interest rate is 4% and AM92 Ultimate mortality applies:

(i) a 20-year pure endowment, with a benefit of £10,000

(ii) a deferred whole life assurance with a deferred period of 20 years, under which the death benefit of £20,000 is paid at the end of the year of death, as long as this occurs after the deferred period has elapsed.

15.4 Using an interest rate of 6% *pa* effective and AM92 Ultimate mortality, calculate:

(i) $A_{50:\overline{15}|}^{\;\;\;\;1}$

(ii) $\bar{A}_{50:\overline{15}|}$

15.5 If $l_{40} = 1{,}000$ and $l_{40+t} = l_{40} - 5t$ for $t = 1,2,\ldots,10$, calculate the value of $A_{40:\overline{10}|}$ at 6% *pa* interest.

15.6 A life insurance company issues a 3-year term assurance contract to a life aged exactly 42. The sum assured of 10,000 is payable at the end of the policy year of death.

Calculate the expected present value of these benefits assuming AM92 Select mortality and an interest rate of 5% *pa* effective.

15.7 A whole life assurance policy pays £10,000 immediately on death of a policyholder currently aged 50 exact, but only if death occurs after the age of 60.

(i) Write down an expression for the present value random variable of the benefit payable under this policy.

(ii) Determine an expression, in the form of an integral, for the expected present value of the benefit payment, and express your answer using standard actuarial notation.

(iii) Determine an expression for the variance of the present value of the benefit payment, expressing your answer using standard actuarial notation.

15.8 An endowment assurance contract with a term of 10 years pays a sum assured of £100,000 immediately on death and a sum of £50,000 on survival for 10 years.

Exam style

Calculate the expected present value and variance of this contract.

Basis:

Mortality:	$\mu_x = 0.03$ throughout
Rate of interest:	5% per annum

[8]

Chapter 15 Solutions

15.1 (i) $A_x = E\left(v^{K_x+1}\right)$

(ii) $\bar{A}^{\,1}_{x:\overline{n}|} = E\left[g(T_x)\right]$ where $g(T_x) = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \ge n \end{cases}$

(iii) $A_{x:\overline{n}|}^{\;\;1} = E\left[h(K_x)\right]$ where $h(K_x) = \begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \ge n \end{cases}$

Alternatively, we could replace K_x with T_x here.

15.2 (i) ***Description of benefit***

This assurance policy pays 1 in n years' time if the life dies before time n years, or it pays 1 at the end of the year of death of the life if this occurs after time n years.

(ii) $E[W]$

We can write:

$$W = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \ge n \end{cases}$$

So:

$$\begin{aligned} E[W] &= v^n P(K_x < n) + \sum_{k=n}^{\infty} v^{k+1} P(K_x = k) \\ &= v^n\,{}_nq_x + \sum_{k=n}^{\infty} v^{k+1}\,{}_{k|}q_x \\ &= v^n\,{}_nq_x + {}_{n|}A_x \\ &= v^n\,{}_nq_x + v^n\,{}_np_x\,A_{x+n} \end{aligned}$$

(iii) $\text{var}[W]$

Now:

$$\text{var}[W] = E\left[W^2\right] - \left(E[W]\right)^2$$

We have:

$$E\left[W^2\right]=\left(v^n\right)^2 P(K_x<n)+\sum_{k=n}^{\infty}\left(v^{k+1}\right)^2 P(K_x=k)$$

$$=v^{2n}\,{}_nq_x+\sum_{k=n}^{\infty}\left(v^2\right)^{k+1}\,{}_{k|}q_x$$

$$=v^{2n}\,{}_nq_x+{}_{n|}^{2}A_x$$

$$=v^{2n}\,{}_nq_x+v^{2n}\,{}_np_x\,{}^2A_{x+n}$$

where the pre-superscript of 2 indicates a function evaluated at the interest rate $(1+i)^2-1$.

So:

$$\text{var}\left[W\right]=v^{2n}\,{}_nq_x+v^{2n}\,{}_np_x\,{}^2A_{x+n}-\left(v^n\,{}_nq_x+v^n\,{}_np_x\,A_{x+n}\right)^2$$

15.3 (i) ***Pure endowment***

The expected value of the benefits is:

$$10{,}000A_{\substack{1\\40:\overline{20}|}}=10{,}000v^{20}\,{}_{20}p_{40}$$

Now:

$${}_{20}p_{40}=\frac{l_{60}}{l_{40}}=\frac{9{,}287.2164}{9{,}856.2863}=0.94226$$

so the expected present value is:

$$10{,}000v^{20}\,{}_{20}p_{40}=10{,}000\times 1.04^{-20}\times 0.94226=£4{,}300.37$$

The standard deviation of the benefits is:

$$\sqrt{10{,}000^2\left({}^2A_{\substack{1\\40:\overline{20}|}}-\left(A_{\substack{1\\40:\overline{20}|}}\right)^2\right)}$$

where the function with the pre-superscript of 2 is evaluated at the interest rate $1.04^2-1=8.16\%$.

So, the standard deviation is:

$$10{,}000\sqrt{1.0816^{-20}\,{}_{20}p_{40}-\left(1.04^{-20}\,{}_{20}p_{40}\right)^2}$$

$$=10{,}000\sqrt{1.0816^{-20}\times 0.94226-\left(1.04^{-20}\times 0.94226\right)^2}$$

$$=£1{,}064.50$$

(ii) ***Deferred whole life assurance***

The expected value of the benefits is:

$$20{,}000\;{}_{20|}A_{40}=20{,}000v^{20}\,{}_{20}p_{40}\,A_{60}=20{,}000\times 1.04^{-20}\times 0.94226\times 0.45640=£3{,}925.37$$

The standard deviation of the benefits is:

$$\sqrt{20{,}000^2\left({}_{20|}^{\;2}A_{40}-\left({}_{20|}A_{40}\right)^2\right)}$$

where the function with the pre-superscript of 2 is evaluated at the interest rate $1.04^2-1=8.16\%$.

So, the standard deviation is:

$$20{,}000\sqrt{1.0816^{-20}\,{}_{20}p_{40}\,{}^2A_{60}-\left(1.04^{-20}\,{}_{20}p_{40}\,A_{60}\right)^2}$$

$$=20{,}000\sqrt{1.0816^{-20}\times 0.94226\times 0.23723-\left(1.04^{-20}\times 0.94226\times 0.45640\right)^2}$$

$$=£1{,}793.11$$

15.4 (i) $A_{50:\overline{15|}}^{\;\;\;\;\;1}=v^{15}\,{}_{15}p_{50}=v^{15}\times\dfrac{l_{65}}{l_{50}}=1.06^{-15}\times\dfrac{8{,}821.2612}{9{,}712.0728}=0.37899$

(ii) We have:

$$\bar{A}_{50:\overline{15|}}=\bar{A}^{1}_{50:\overline{15|}}+A_{50:\overline{15|}}^{\;\;\;\;\;1}$$

where:

$$\bar{A}^{1}_{50:\overline{15|}}\approx 1.06^{0.5}A^{1}_{50:\overline{15|}}=1.06^{0.5}\left(A_{50:\overline{15|}}-A_{50:\overline{15|}}^{\;\;\;\;\;1}\right)$$

So, using the *Tables* and the result of (i):

$$\bar{A}_{50:\overline{15|}}\approx 1.06^{0.5}(0.43181-0.37899)+0.37899=0.43337$$

15.5 We have:

$$A_{40:\overline{10}|} = \sum_{k=0}^{9} v^{k+1}\ {}_{k|}q_{40} + v^{10}\ {}_{10}p_{40} = \sum_{k=0}^{9} v^{k+1} \frac{d_{40+k}}{l_{40}} + v^{10} \frac{l_{50}}{l_{40}}$$

Using the definition of l_x given, we see that:

$$d_{40+k} = l_{40+k} - l_{40+k+1} = l_{40} - 5k - \left(l_{40} - 5(k+1)\right) = 5 \quad \text{for} \quad k = 0,1,2,\ldots,9$$

and: $l_{50} = l_{40} - 5\times 10 = 1{,}000 - 50 = 950$

So the value of the endowment assurance is:

$$A_{40:\overline{10}|} = \frac{5}{1{,}000}\sum_{k=0}^{9} v^{k+1} + v^{10} \times \frac{950}{1{,}000}$$

$$= \frac{5a_{\overline{10}|} + 950v^{10}}{1{,}000} \text{@6\%}$$

$$= \frac{5\times 7.3601 + 950\times 1.06^{-10}}{1{,}000} = 0.5673$$

15.6 The expected present value of the benefits is:

$$10{,}000A^{\ 1}_{[42]:\overline{3}|} = 10{,}000\sum_{k=0}^{2} v^{k+1}\ {}_{k|}q_{[42]}$$

$$= 10{,}000\sum_{k=0}^{2} v^{k+1} \frac{d_{[42]+k}}{l_{[42]}}$$

$$= 10{,}000\left(v\frac{d_{[42]}}{l_{[42]}} + v^2\frac{d_{[42]+1}}{l_{[42]}} + v^3\frac{d_{44}}{l_{[42]}}\right)$$

$$= \frac{10{,}000}{9{,}834.7030}\left(1.05^{-1}\times 9.0676 + 1.05^{-2}\times 11.2995 + 1.05^{-3}\times 13.0236\right)$$

$$= 30.64$$

15.7 (i) ***Present value random variable***

We can write this as:

$$X = \begin{cases} 0 & \text{if } T_{50} \le 10 \\ 10{,}000v^{T_{50}} & \text{if } T_{50} > 10 \end{cases}$$

(ii) ***Expected present value***

Now:

$$E\left[X\right]=\int_0^{10} 0\times f_{T_{50}}(t)\,dt+\int_{10}^{\infty} 10{,}000v^t f_{T_{50}}(t)\,dt$$

$$=0+10{,}000\int_{10}^{\infty} v^t\,{}_tp_{50}\,\mu_{50+t}\,dt$$

$$=10{,}000\times {}_{10|}\bar{A}_{50}$$

(iii) ***Variance***

We need:

$$\text{var}\left[X\right]=E\left[X^2\right]-\left(E\left[X\right]\right)^2$$

$$=\int_{10}^{\infty} 10{,}000^2\,(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt-\left(10{,}000\;{}_{10|}\bar{A}_{50}\right)^2$$

Now:

$$\int_{10}^{\infty} 10{,}000^2\,(v^t)^2\,{}_tp_{50}\,\mu_{50+t}\,dt=10{,}000^2\int_{10}^{\infty}(v^2)^t\,{}_tp_{50}\,\mu_{50+t}\,dt$$

$$=10{,}000^2\;{}_{10|}^{\;2}\bar{A}_{50}$$

where ${}^2\bar{A}$ is calculated at rate of interest $(1+i)^2-1$.

So:

$$\text{var}\left[X\right]=10{,}000^2\left[{}_{10|}^{\;2}\bar{A}_{50}-\left({}_{10|}\bar{A}_{50}\right)^2\right]$$

15.8 *This question is Subject CT5, April 2012, Question 12.*

Expected present value

Consider the present value random variable for this endowment assurance contract. This is:

$$PV=\begin{cases}100{,}000v^{T_x} & T_x<10\\ 50{,}000v^{10} & T_x\geq 10\end{cases}$$ [1]

The probability density function for T_x is ${}_tp_x\,\mu_{x+t}$. Here the force of mortality is constant, so μ_{x+t} is just μ, and ${}_tp_x$ is equal to $e^{-\mu t}$.

The expected present value of the benefits is:

$$E(PV)=100,000\int_0^{10} {}_tp_x\,\mu_{x+t}\,v^t\,dt+50,000v^{10}\times {}_{10}p_x$$

$$=100,000\int_0^{10} {}_tp_x\,\mu_{x+t}\,e^{-\delta t}\,dt+50,000e^{-10\delta}\times {}_{10}p_x$$

$$=100,000\int_0^{10} \mu e^{-(\mu+\delta)t}\,dt+50,000e^{-10(\mu+\delta)}$$

$$=100,000\left[\frac{\mu e^{-(\mu+\delta)t}}{-(\mu+\delta)}\right]_0^{10}+50,000e^{-10(\mu+\delta)}$$

$$=100,000\left[\frac{\mu\left(1-e^{-10(\mu+\delta)}\right)}{\mu+\delta}\right]+50,000e^{-10(\mu+\delta)} \qquad [2]$$

Substituting in $\mu=0.03$ and $\delta=\ln 1.05$, we find that:

$$E(PV)=100,000\left[\frac{0.03\left(1-e^{-10(0.03+\ln 1.05)}\right)}{0.03+\ln 1.05}\right]+50,000e^{-10(0.03+\ln 1.05)}$$

$$=20,759.008+22,739.906=43,498.914$$

So the expected present value of the contract is about £43,499. [1]

Variance of the present value

For the variance, we first need to calculate the second moment:

$$E\left(PV^2\right)=100,000^2\int_0^{10} {}_tp_x\,\mu_{x+t}\,e^{-2\delta t}\,dt+50,000^2e^{-20\delta}\times {}_{10}p_x$$

$$=100,000^2\int_0^{10} \mu e^{-(\mu+2\delta)t}\,dt+50,000^2e^{-10(\mu+2\delta)}$$

$$=100,000^2\left[\frac{\mu e^{-(\mu+2\delta)t}}{-(\mu+2\delta)}\right]_0^{10}+50,000^2e^{-10(\mu+2\delta)}$$

$$=100,000^2\left[\frac{\mu\left(1-e^{-10(\mu+2\delta)}\right)}{\mu+2\delta}\right]+50,000^2e^{-10(\mu+2\delta)} \qquad [2]$$

Again, substituting in $\mu = 0.03$ and $\delta = \ln 1.05$, we find that:

$$E\left(PV^2\right) = 100{,}000^2 \left[\frac{0.03\left(1 - e^{-10(0.03 + 2\ln 1.05)}\right)}{0.03 + 2\ln 1.05} \right] + 50{,}000^2 e^{-10(0.03 + 2\ln 1.05)}$$

$$= 1{,}694{,}916{,}638 + 698{,}016{,}490 = 2{,}392{,}933{,}128$$ [1]

So the variance of the present value of the contract is:

$$Var\left(PV\right) = 2{,}392{,}933{,}128 - 43{,}498.914^2 = 500{,}777{,}609$$

or about $\left(£22{,}378\right)^2$. [1]

[Total 8]

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Life annuity contracts

Syllabus objectives

4.1 Define various annuity contracts.

4.1.1 Define the following terms:

- whole-life level annuity
- temporary level annuity
- guaranteed level annuity

including annuity contracts where the benefits are deferred.

4.2 Develop formulae for the means and variances of the payments under various annuity contracts, assuming a constant deterministic interest rate.

4.2.4 Define the annuity factors and their continuous equivalents.

4.2.5 Understand and use the relations between annuities payable in advance and in arrears, and between temporary, deferred and whole life annuities.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- annuities are paid in advance, in arrears or continuously, and the amount is constant
- premiums are payable in advance, in arrears or continuously; and for the full policy term or for a limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

1 Life annuity contracts

In the previous chapter, we described the first broad type of contract sold by a life insurance company: assurances. In this chapter we describe the other main type: annuities.

A *life annuity contract* provides payments of amounts, which might be level or variable, at stated times, provided a life is still then alive.

Here we consider four varieties of life annuity contract:

(1) Annuities under which payments are made for the whole of life, with level payments, called a *whole life level annuity* or, more usually, an immediate annuity.

(2) Annuities under which level payments are made only during a limited term, called a *temporary level annuity* or, more usually, just a temporary annuity.

(3) Annuities under which the start of payment is deferred for a given term, called a *deferred annuity*.

(4) Annuities under which payments are made for the whole of life, or for a given term if longer, called a *guaranteed annuity*.

The income from a life annuity may be:

- level, *eg* $£X$ per annum
- increasing, *eg* starting at $£Y$, but increasing at 5% per annum
- paid for the whole of a person's life, *ie* until the policyholder dies
- paid for a limited term, *eg* for at most 5 years
- paid for a minimum term, *eg* for at least 5 years
- deferred, *eg* $£Z$ per annum paid from the policyholder's 60th birthday

or a combination of the above.

Further, we consider the possibilities that payments are made in advance or in arrears. For the moment we consider only contracts under which level payments are made at yearly intervals.

Increasing life annuities will be dealt with in Chapter 18.

2 Whole life annuities payable annually in arrears

An *immediate* annuity is one under which the first payment is made within the first year. For the purposes of this Section we will assume that payments are made in arrears.

The word *immediate* is used to distinguish these from *deferred* annuities, where the payments start some years in the future (*eg* on reaching age 65).

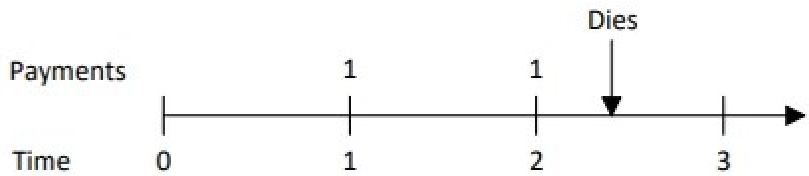
2.1 Present value random variable

Consider an annuity contract to pay 1 at the end of each future year, provided a life now aged x is then alive.

For example, this could be a pension of £1 per annum paid annually in arrears to a life until death.

If the life dies between ages $x+k$ and $x+k+1$ ($k=0,...,\omega-x-1$) which is to say, $K_x=k$, the present value at time 0 of the annuity payments which are made is $a_{\overline{k|}}$. (We define $a_{\overline{0|}}=0$.) Therefore the present value at time 0 of the annuity payments is $a_{\overline{K_x|}}$.

For example, suppose the person dies in the third year after taking out the annuity, *ie* death occurs between time 2 and time 3.



In this case, the life has survived for two complete years before dying in the third year, so the curtate future lifetime, K_x, takes the value 2. Since payments are made at the end of each year but only as long as the person is living (as shown in the diagram), exactly 2 payments are made here, and so the present value is $a_{\overline{2|}}$.

That is, when $K_x=2$, $PV=a_{\overline{2|}}$. Or, in general:

$$PV=a_{\overline{K_x|}}$$

Since we know the distribution of K_x, we can compute moments of $a_{\overline{K_x|}}$.

2.2 Expected present value

The expected value of $a_{\overline{K_x}|}$ is:

$$E\left[a_{\overline{K_x}|}\right]=\sum_{k=0}^{\infty} a_{\overline{k}|} P(K_x=k)$$

where we are again using the general formula for expectation:

$$E\left[g(x)\right]=\sum_{x} g(x) P(X=x)$$

Actuarial notation for the expected present value

We met the symbol $a_{\overline{n}|}$ earlier in the course. It can be interpreted as:

$a_{\overline{n}|}=$ *PV* of an immediate annuity of 1 unit *pa* paid in arrears *for as long as the status* $\overline{n}|$ *remains active*

In other words, this implies an annuity paid annually in arrears until the failure of the $\overline{n}|$ status, that is, until *n* years have elapsed.

We now introduce a_x, which has a similar meaning except that it relates to the life status *x*. So:

$a_x=$ *EPV* of an immediate annuity of 1 unit *pa* paid in arrears for as long as (*x*) remains alive in the future

So the payments stop at the moment that (*x*) dies. We describe a_x as an expected present value ('EPV'), not a present value ('PV'), as the payments are dependent on the life's survival and so are not certain to occur. The word 'expected' really means that we're making an allowance for the probability of payment.

The expectation of $a_{\overline{K_x}|}$ defines the expected present value (EPV) a_x, and:

$$a_x = E[a_{\overline{K_x}|}] = \sum_{k=0}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x$$

We can write this in a form that is easier to calculate.

We begin by writing:

$$a_x = \sum_{k=0}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x = \sum_{k=1}^{\infty}\left(\sum_{j=0}^{k-1} v^{j+1}\right) {}_{k|}q_x$$

This result holds since $a_{\overline{0}|}=0$ and $a_{\overline{k}|}=v+v^2+\cdots+v^k=\sum_{j=0}^{k-1} v^{j+1}$.

If we write out the sum more fully:

$$\begin{aligned} a_x &= v \times {}_{1|}q_x \\ &\quad + \left(v + v^2\right) \times {}_{2|}q_x \\ &\quad + \left(v + v^2 + v^3\right) \times {}_{3|}q_x \\ &\quad + \ldots \end{aligned}$$

Now, reversing the order of summation:

$$\begin{aligned} a_x &= v\left[{}_{1|}q_x + {}_{2|}q_x + \ldots\right] + v^2\left[{}_{2|}q_x + {}_{3|}q_x + \ldots\right] + \ldots \\ &= v\sum_{k=1}^{\infty} {}_{k|}q_x + v^2 \sum_{k=2}^{\infty} {}_{k|}q_x + \ldots \end{aligned}$$

So:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty}\left(\sum_{k=j+1}^{\infty} {}_{k|}q_x\right) v^{j+1}}$$

Now, for example, $\sum_{k=1}^{\infty} {}_{k|}q_x$ is the probability that the life dies at some point after time 1 year, which is equal to the probability that the life is still alive at time 1 year, ${}_1p_x$, and in general, $\sum_{k=j+1}^{\infty} {}_{k|}q_x$ is equal to ${}_{j+1}p_x$. Hence:

$$\boldsymbol{a_x = \sum_{j=0}^{\infty} {}_{j+1}p_x\, v^{j+1} = \sum_{j=1}^{\infty} {}_{j}p_x\, v^{j}}$$

We can interpret the final formula for a_x above as follows.

Each payment is conditional on whether the policyholder is alive or not at the time the payment is due. The present value of the annuity payment made at time j is v^j. The expected present value of this payment is v^j multiplied by the probability that the policyholder is alive at this time, ${}_jp_x$. Summing over all future years gives the expected present value of all the future annuity payments.

Although the result $a_x = \sum_{j=1}^{\infty} {}_jp_x v^j$ is very useful, it is important to realise that it is *not* the *definition* of a_x. The definition is $a_x = \sum_{k=0}^{\infty} a_{\overline{k|}}\, {}_{k|}q_x$.

Question

A whole life annuity of 1 *pa* is payable annually in arrears to a life aged 90. The effective annual rate of interest is 5%. Mortality is assumed to follow the life table given below:

Age, x	l_x	d_x
90	100	25
91	75	35
92	40	40
93	0	0

Calculate the expected present value of this annuity.

Solution

Using the formula $a_x = \sum_{j=1}^{\infty} v^j \, {}_jp_x$, the expected present value is:

$$a_{90} = v\,p_{90} + v^2\,{}_2p_{90} = v\frac{l_{91}}{l_{90}} + v^2\frac{l_{92}}{l_{90}}$$

There are only 2 non-zero terms in the summation since ${}_jp_{90} = 0$ *for* $j \geq 3$.

Using the values from the table above, we have:

$$a_{90} = 1.05^{-1} \times \frac{75}{100} + 1.05^{-2} \times \frac{40}{100} = 1.07710$$

2.3 Variance of the present value random variable

The relationships derived in the previous chapter (for assurances) **provide the easiest approach to finding the variances of the present values of annuity benefits.**

Recall that:

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

Using this result and the properties of variance, we can write:

$$\text{var}\left[a_{\overline{K_x}|}\right] = \text{var}\left[\frac{1-v^{K_x}}{i}\right] = \frac{1}{i^2}\text{var}\left[v^{K_x}\right]$$

In Chapter 15 we showed that:

$$\text{var}\left[v^{K_x+1}\right] = {}^2A_x - \left(A_x\right)^2$$

where the 2 to the top left-hand corner of the first A indicates that the function is valued using the interest rate $(1+i)^2 - 1$.

So:

$$\text{var}\left[a_{\overline{K_x}|}\right] = \frac{1}{i^2}\text{var}\left[\frac{v^{K_x+1}}{v}\right]$$

$$= \frac{1}{i^2 v^2}\text{var}\left[v^{K_x+1}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - \left(A_x\right)^2\right]$$

Question

A whole life annuity of 100 *pa* is payable annually in arrears to a life aged 65.

Calculate the standard deviation of the benefits from this annuity, assuming AM92 Ultimate mortality and an annual effective rate of interest of 4%.

Solution

The variance of the benefits from this annuity is:

$$\text{var}\left[100a_{\overline{K_{65}}|}\right] = \frac{100^2}{d^2}\left[{}^2A_{65} - \left(A_{65}\right)^2\right]$$

Using values from the *Tables*, and recalling that $d = iv$, this gives the variance as:

$$\frac{100^2}{(0.04/1.04)^2}\left[0.30855 - (0.52786)^2\right] = (449.69)^2$$

So the standard deviation is 449.69.

3 Whole life annuities payable annually in advance

An *annuity-due* is one under which payments are made in advance.

3.1 Present value random variable

Consider an annuity contract to pay 1 at the start of each future year, provided a life now aged x is then alive. By similar reasoning to that above, we see that the present value of these payments is $\ddot{a}_{\overline{K_x+1}|}$.

The annuity-due provides the following payments:

- 1 at age x (*ie* at the start of the first year), which is certain to be paid, and
- a further payment of 1 at the start of each subsequent policy year as long as the policyholder is still alive.

The second of these alone results in the same number of payments as under the whole life annuity paid in arrears, *ie* K_x payments. For the annuity in advance, therefore, there will be one additional payment at the start of the contract, so there are K_x+1 payments altogether.

Actuarial notation for the expected present value

In actuarial notation we denote the EPV, $E\left[\ddot{a}_{\overline{K_x+1}|}\right]$, by $\ddot{a}_x$.

$\ddot{a}_x$ has identical meaning to a_x – in that payments continue for as long as (x) survives – except that the payments are made in advance.

We can again write down $\ddot{a}_x$ in a form that is simple to calculate:

$$\begin{aligned}
\ddot{a}_x &= E\left[\ddot{a}_{\overline{K_x+1}|}\right] \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j \\
&= \sum_{j=0}^{\infty} {}_j p_x v^j
\end{aligned}$$

Question

Develop a simple relationship between $\ddot{a}_x$ and a_x, and explain the result by general reasoning.

Solution

Using the simplified formulae we have obtained for $\ddot{a}_x$ and a_x, we see:

$$\ddot{a}_x - a_x = \sum_{j=0}^{\infty} {}_j p_x \, v^j - \sum_{j=1}^{\infty} {}_j p_x \, v^j = {}_0 p_x \, v^0 = 1 \quad \Rightarrow \quad \ddot{a}_x = a_x + 1$$

Intuitively, the annuity-due is the same as the annuity payable in arrears, except for the additional payment of 1 unit made at outset. This payment has present value 1, and will definitely be paid, so its expected present value must also be 1.

As for assurances, a selection of the above functions are tabulated in the Formulae and Tables for Examinations.

Question

Find:

(i) $\ddot{a}_{30}$ **(AM92 at 4%)**

(ii) $\ddot{a}_{75}$ **(PMA92C20 at 4%)**

Solution

(i) **21.834** (This value appears on page 96 of the *Tables*.)

(ii) **9.456** (This value appears on page 114 of the *Tables*.)

3.2 Variance of the present value random variable

The variance is:

$$\begin{aligned}\text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right] &= \text{var}\left[\frac{1-v^{K_x+1}}{d}\right] \\ &= \frac{1}{d^2}\text{var}\left[v^{K_x+1}\right] \\ &= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]\end{aligned}$$

As usual, the '2' superscript denotes an assurance function calculated at a rate of interest of $(1+i)^2 - 1$.

It is simple to prove the formula for the variance of an immediate whole life annuity payable annually in arrears starting from this result.

As shown earlier, **for an immediate annuity payable annually in arrears, we obtain:**

$$\text{var}\left[a_{\overline{K_x}|}\right] = \text{var}\left[\ddot{a}_{\overline{K_x+1}|} - 1\right]$$

$$= \text{var}\left[\ddot{a}_{\overline{K_x+1}|}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

4 Temporary annuities payable annually in arrears

A *temporary immediate annuity* differs from a whole life immediate annuity in that the payments are limited to a specified term.

4.1 Present value random variable

Consider a temporary immediate annuity contract to pay 1 at the end of each of the next n years, provided a life now aged x is then alive.

This contract differs from a whole life annuity since the payments stop after n years, even if the life is still alive.

If we let X denote the present value of the temporary annuity payable annually in arrears, then:

$$X = \begin{cases} a_{\overline{K_x}|} & \text{if } K_x < n \\ a_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

From this we can see that the number of payments made is K_x or n, whichever is the smaller. So, alternatively we can write:

The present value of this benefit is $a_{\overline{\min\{K_x,n\}}|}$.

4.2 Expected present value

The expected present value is:

$$E\left[a_{\overline{\min\{K_x,n\}}|}\right] = \sum_{k=0}^{\infty} a_{\overline{\min\{k,n\}}|} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} \sum_{k=n}^{\infty} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x \geq n)$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|}\, {}_{k|}q_x + a_{\overline{n}|}\, {}_n p_x$$

Looking at the formula above, we see that the benefit paid is:

- 0 if death occurs in the first year
- a payment of 1 (with present value $a_{\overline{1}|} = v$) if death occurs in the second year
- two payments of 1 (with present value $a_{\overline{2}|} = v + v^2$) if death occurs in the third year
- and so on, up until a payment of 1 every year for *n* years (present value $a_{\overline{n}|}$) if the life survives for *n* years, with no further payments.

Actuarial notation for the expected present value

In actuarial notation, $E\left[a_{\overline{\min\{K_x,n\}}|}\right]$ is denoted $a_{x:\overline{n}|}$.

Compare this symbol $a_{x:\overline{n}|}$ with the symbol $A_{x:\overline{n}|}$ we described in the previous chapter. Where $A_{x:\overline{n}|}$ relates to a *single* payment made *when* the first out of x and $\overline{n}|$ fails, $a_{x:\overline{n}|}$ relates to a *series* of payments that continues *until* the first out of x and $\overline{n}|$ fails.

As for a whole life annuity, we would like a formula to simplify the calculation of the expected present value. Writing the EPV as a summation, we have:

$$\begin{aligned} a_{x:\overline{n}|} &= E\left[a_{\overline{\min\{K_x,n\}}|}\right] \\ &= \sum_{k=1}^{n-1} a_{\overline{k}|}\,{}_{k|}q_x + a_{\overline{n}|}\,{}_n p_x \\ &= \sum_{k=1}^{n-1}\left(\sum_{j=0}^{k-1} v^{j+1}\right){}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x \end{aligned}$$

Reversing the order of summation then gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2}\left(\sum_{k=j+1}^{n-1} {}_{k|}q_x\right)v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1}\right){}_n p_x$$

Note that ${}_n p_x = {}_{n|}q_x + {}_{n+1|}q_x + \cdots$, so:

$$\sum_{k=j+1}^{n-1} {}_{k|}q_x = {}_{j+1|}q_x + {}_{j+2|}q_x + \cdots + {}_{n-1|}q_x = {}_{j+1}p_x - {}_n p_x$$

Substituting this into the previous equation gives:

$$a_{x:\overline{n}|} = \sum_{j=0}^{n-2} \left({}_{j+1}p_x - {}_np_x \right) v^{j+1} + \left(\sum_{j=0}^{n-1} v^{j+1} \right) {}_np_x$$

$$= \sum_{j=0}^{n-2} v^{j+1} \, {}_{j+1}p_x + {}_np_x \left(\sum_{j=0}^{n-1} v^{j+1} - \sum_{j=0}^{n-2} v^{j+1} \right)$$

$$= \sum_{j=0}^{n-2} v^{j+1} \, {}_{j+1}p_x + v^n \, {}_np_x$$

$$= \sum_{j=0}^{n-1} v^{j+1} \, {}_{j+1}p_x$$

So, to calculate $a_{x:\overline{n}|}$, we use the following:

$$\boldsymbol{a_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j+1}p_x v^{j+1} = \sum_{j=1}^{n} {}_jp_x \, v^j}$$

This expression seems logical given that $a_x = \sum_{j=1}^{\infty} {}_jp_x v^j$. $a_{x:\overline{n}|}$ is the same summation but allows only for the first n payments. It is this final result for $a_{x:\overline{n}|}$ that is most useful when calculating expected present values in practice.

Question

Calculate the value of $a_{40:\overline{3}|}$ when the effective annual rate of interest is 6% and:

$$l_x = 100 - x \qquad \text{at all ages } x \le 100$$

Solution

Using the given formula, we have:

$$a_{40:\overline{3}|} = \sum_{j=1}^{3} v^j \, {}_jp_{40} = v\frac{l_{41}}{l_{40}} + v^2\frac{l_{42}}{l_{40}} + v^3\frac{l_{43}}{l_{40}}$$

$$= 1.06^{-1} \times \frac{59}{60} + 1.06^{-2} \times \frac{58}{60} + 1.06^{-3} \times \frac{57}{60}$$

$$= 2.58564$$

4.3 Variance of the present value random variable

For a temporary immediate annuity payable annually in arrears we have:

$$\text{var}\left[a_{\overline{\min\{K_x,n\}}|}\right]=\frac{1}{d^2}\left[{}^2A_{x:\overline{n+1}|}-(A_{x:\overline{n+1}|})^2\right]$$

The easiest way to prove this formula is to use the result for a temporary annuity-due. So we will defer the proof until the end of the next section.

5 Temporary annuities payable annually in advance

A *temporary immediate annuity-due* has payments that are made in advance and are limited to a specified term.

5.1 Present value random variable

Consider a temporary immediate annuity-due contract to pay 1 at the start of each of the next n years, provided a life now aged x is then alive.

The present value of the benefit is $\ddot{a}_{\overline{\min[K_x+1,n]}|}$.

Another way to write this is as follows. If we let Y denote the present value of the temporary annuity-due, then:

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

and so we can see that the number of payments is the smaller of $K_x + 1$ and n.

5.2 Expected present value

The expected present value is:

$$E\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{\min\{k+1,n\}}|} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} P(K_x = k) + \ddot{a}_{\overline{n}|} \sum_{k=n}^{\infty} P(K_x = k)$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\, {}_{k|}q_x + \ddot{a}_{\overline{n}|}\, {}_n p_x$$

Actuarial notation for the expected present value

In actuarial notation, $E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$ is denoted $\ddot{a}_{x:\overline{n}|}$.

Then:

$$\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right]$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\ {}_{k|}q_x + \ddot{a}_{\overline{n}|}\ {}_n p_x$$

$$= \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x + \left(\sum_{j=0}^{n-1} v^j\right){}_n p_x$$

$$= \sum_{j=0}^{n-1}\left(\sum_{k=j}^{n-1} {}_{k|}q_x + {}_n p_x\right)v^j$$

$$= \sum_{j=0}^{n-1} {}_j p_x v^j$$

Again, this seems logical. It is similar to a whole life annuity-due, but with payments continuing only up to time $n-1$ (making n payments in total).

Question

Explain why $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} \neq 1$.

Solution

The identity $\ddot{a}_x - a_x = 1$ works for whole life annuities, but when considering temporary annuities $\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} \neq 1$.

In terms of summations, we have:

$$a_{x:\overline{n}|} = v\,p_x + v^2\,{}_2p_x + \cdots + v^n\,{}_np_x$$

and: $$\ddot{a}_{x:\overline{n}|} = 1 + v\,p_x + \cdots + v^{n-1}\,{}_{n-1}p_x$$

Comparing these, we see that the first and the last terms are different. In fact:

$$\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} = 1 - v^n\,{}_np_x$$

Following on from this, we have:

$$a_{x:\overline{n}|} + 1 = \ddot{a}_{x:\overline{n}|} + v^n\,{}_np_x = 1 + v\,p_x + v^2\,{}_2p_x + \ldots + v^{n-1}\,{}_{n-1}p_x + v^n\,{}_np_x = \ddot{a}_{x:\overline{n+1}|}$$

In other words, $\ddot{a}_{x:\overline{n+1}|} - a_{x:\overline{n}|} = 1$.

For functions dependent upon age as well as term, tabulations are restricted in order to save space. For example, in AM92, $\ddot{a}_{x:\overline{n}|}$ is tabulated for $x+n=60$ and for $x+n=65$.

Question

A 35-year-old purchases an endowment assurance with a term of 30 years. The premiums for the policy are payable annually in advance while the policy is in force, and each premium is £2,500.

Calculate the expected present value of the premiums paid, using AM92 Ultimate mortality, and an interest rate of 4% *pa* effective.

Solution

The endowment assurance will remain in force while the policyholder is alive, for at most 30 years. So the expected present value of the premiums payable can be calculated using a temporary annuity-due, with a term of 30 years:

$$2{,}500\ddot{a}_{35:\overline{30}|} = 2{,}500 \times 17.629 = £44{,}072.50$$

The value of $\ddot{a}_{35:\overline{30}|}$ (AM92 Ultimate mortality, 4% interest) appears on page 101 of the Tables.

5.3 Variance of the present value random variable

For a temporary immediate annuity-due:

$$\text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n]}|}\right] = \frac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right]$$

This is proved in the same way as the variance formula for the whole life annuity-due:

$$\text{var}\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right] = \text{var}\left[\frac{1-v^{\min\{K_x+1,n\}}}{d}\right]$$

$$= \frac{1}{d^2}\text{var}\left[v^{\min\{K_x+1,n\}}\right]$$

$$= \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

The last line above uses the result for the variance of an endowment assurance from the previous chapter.

We can now use this to prove the corresponding result for a temporary immediate annuity payable in arrears.

For a temporary immediate annuity (payable annually in arrears) we have:

$$\text{var}\left[a_{\overline{\min[K_x,n]}|}\right] = \text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|} - 1\right]$$

$$= \text{var}\left[\ddot{a}_{\overline{\min[K_x+1,n+1]}|}\right]$$

$$= \frac{1}{d^2}\left[\,{}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right]$$

This proof first of all uses the relationship $a_{\overline{m}|} = \ddot{a}_{\overline{m+1}|} - 1$, with m replaced by $\min\{K_x, n\}$. The second line above follows because subtracting 1 from a random variable does not alter its variance. We then use the formula for the variance of a temporary annuity-due, derived earlier in this section, with n replaced by $n+1$, to give the required result.

6 Deferred annuities

Deferred annuities are annuities under which payment does not begin immediately but is deferred for one or more years.

6.1 Present value random variable

Consider, for example, an annuity of 1 per annum payable annually in arrears to a life now aged x, deferred for n years. Payment will be at ages $x+n+1$, $x+n+2$, .. , provided that the life survives to these ages, instead of at ages $x+1$, $x+2$, .. .

Let X represent the (random) present value of the annuity.

Here are three different ways of representing X:

$$X = \begin{cases} 0 & \text{if } K_x \le n \\ v^n a_{\overline{K_x - n}|} & \text{if } K_x > n \end{cases}$$

$$= \begin{cases} 0 & \text{if } K_x \le n \\ a_{\overline{K_x}|} - a_{\overline{n}|} & \text{if } K_x > n \end{cases}$$

$$= v^n a_{\overline{\max(K_x - n, 0)}|}$$

6.2 Expected present value

Then, by considering the distribution of X, and noting that $v^n a_{\overline{k-n}|} = {}_{n|}a_{\overline{k-n}|}$, **we have that:**

$$E(X) = \sum_{k=0}^{n} 0 \times P(K_x = k) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} \times P(K_x = k)$$

Now adding in the terms $\sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)$ and $a_{\overline{n}|} P(K_x > n)$ and subtracting them again, we can write:

$$E(X) = \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k) + a_{\overline{n}|} P(K_x > n) - \sum_{k=0}^{n} a_{\overline{k}|} P(K_x = k)$$

$$- a_{\overline{n}|} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n}|} P(K_x = k)$$

We can combine the first, second and last terms in the line above as follows:

$$\sum_{k=0}^{n} a_{\overline{k|}} P(K_x = k) + a_{\overline{n|}} P(K_x > n) + \sum_{k=n+1}^{\infty} {}_{n|}a_{\overline{k-n|}} P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k|}} P(K_x = k) + \sum_{k=n+1}^{\infty} \left(a_{\overline{n|}} + v^n a_{\overline{k-n|}} \right) P(K_x = k)$$

$$= \sum_{k=0}^{n} a_{\overline{k|}} P(K_x = k) + \sum_{k=n+1}^{\infty} a_{\overline{k|}} P(K_x = k)$$

$$= \sum_{k=0}^{\infty} a_{\overline{k|}} P(K_x = k)$$

Hence:

$$\boldsymbol{E(X) = \sum_{k=0}^{\infty} a_{\overline{k|}} P(K_x = k) - \sum_{k=0}^{n} a_{\overline{k|}} P(K_x = k) - a_{\overline{n|}} P(K_x > n)}$$

$$\boldsymbol{= a_x - a_{x:\overline{n|}}}$$

Alternatively, we could obtain the same result by considering the random variables. If we let:

$$Y = a_{\overline{K_x|}} = \begin{cases} a_{\overline{K_x|}} & \text{if } K_x \le n \\ a_{\overline{K_x|}} & \text{if } K_x > n \end{cases} \qquad \text{so that } \; E(Y) = a_x$$

and: $$Z = \begin{cases} a_{\overline{K_x|}} & \text{if } K_x \le n \\ a_{\overline{n|}} & \text{if } K_x > n \end{cases} \qquad \text{so that } \; E(Z) = a_{x:\overline{n|}}$$

it is then easy to see that:

$$Y - Z = \begin{cases} a_{\overline{K_x|}} - a_{\overline{K_x|}} = 0 & \text{if } K_x \le n \\ a_{\overline{K_x|}} - a_{\overline{n|}} & \text{if } K_x > n \end{cases}$$

$$= X$$

Therefore:

$$E(X) = E(Y - Z) = E(Y) - E(Z) = a_x - a_{x:\overline{n|}}$$

This formula $E(X) = a_x - a_{x:\overline{n|}}$ is intuitively correct: the value of the deferred annuity is equal to a series of payments paid for the whole of life, less the value of the payments that will not be made for the first n years.

Actuarial notation for the expected present value

In actuarial notation, the EPV of this deferred annuity is denoted ${}_{n|}a_x$, so:

$$ {}_{n|}a_x = a_x - a_{x:\overline{n}|} $$

The notation ${}_{n|}a_x$ represents the expected present value of an annuity of 1 unit *pa* payable in arrears until the failure (death) of life status *x*, with a waiting period of *n* years before payments can begin. The subscript to the right of the symbol always denotes the *current* age, *not* the age of the policyholder when payments begin.

Similarly, expressions can be derived for ${}_{m|}a_{x:\overline{n}|}$, the expected present value of an n-year temporary annuity deferred for m years (assuming survival to that point).

Alternative approach to evaluating the expected present value

An alternative way to evaluate ${}_{n|}a_x$ follows from:

$$ \begin{aligned} {}_{n|}a_x &= a_x - a_{x:\overline{n}|} \\ &= \sum_{k=1}^{\infty} v^k \, {}_kp_x - \sum_{k=1}^{n} v^k \, {}_kp_x \\ &= \sum_{k=n+1}^{\infty} v^k \, {}_kp_x \end{aligned} $$

Letting $j = k - n$:

$$ {}_{n|}a_x = \sum_{j=1}^{\infty} v^{j+n} \, {}_{j+n}p_x = v^n \, {}_np_x \sum_{j=1}^{\infty} v^j \, {}_jp_{x+n} $$

where ${}_{j+n}p_x = {}_np_x \, {}_jp_{x+n}$, using the principle of consistency. Renaming the variable as 'k' gives:

$$ {}_{n|}a_x = v^n \, {}_np_x \sum_{k=1}^{\infty} v^k \, {}_kp_{x+n} = v^n \, {}_np_x \, a_{x+n} $$

Again, this final result is intuitively obvious. The expected present value of the deferred annuity benefit is equal to the expected present value of a life annuity for a *survivor* to age $x+n$, discounted back for n years to allow for interest, multiplied by the probability that the policyholder survives to age $x+n$.

Note the appearance once more of the 'discount factor' $v^n \, {}_np_x$.

7 Deferred annuities-due

Deferred annuities-due can be defined similarly, with the corresponding formulae such as:

$$ {}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n|}} = v^n \, {}_n p_x \ddot{a}_{x+n} $$

Note that $a_x = {}_{1|}\ddot{a}_x$.

Question

A 50-year-old woman purchases a deferred annuity to provide herself with an income of £15,000 *pa*, paid annually in advance from age 70 until death.

Calculate the expected present value of the benefits from this deferred annuity, using PFA92C20 mortality, and an interest rate of 4% *pa* effective.

Solution

The expected present value of the benefits is:

$$ \begin{aligned} 15{,}000\,{}_{20|}\ddot{a}_{50} &= 15{,}000 v^{20}\,{}_{20}p_{50}\,\ddot{a}_{70} \\ &= 15{,}000 v^{20} \frac{l_{70}}{l_{50}} \ddot{a}_{70} \\ &= 15{,}000 \times 1.04^{-20} \times \frac{9{,}392.621}{9{,}952.697} \times 12.934 \\ &= £83{,}561 \end{aligned} $$

In Chapter 15 we showed that the variance of a deferred whole life assurance is:

$$ {}_{n|}^{2}A_x - ({}_{n|}A_x)^2 $$

This can be used to find the variance of the corresponding deferred annuity-due.

However, it is easier to proceed using a first principles approach.

Question

Write down a single term expression to represent the present value of a deferred annuity-due of 1 *pa* payable to a life now aged x, with a deferment period of n years.

Hence derive a formula for the variance of the present value of this contract.

Solution

The present value random variable can be written as:

$$v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}$$

We can check this as follows:

- If $K_x < n$, the present value is $v^n \ddot{a}_{\overline{0}|} = 0$.
- If $K_x = n$, the present value is $v^n \ddot{a}_{\overline{1}|} = v^n$, as it should be.
- If $K_x = n+1$, the present value is $v^n \ddot{a}_{\overline{2}|} = v^n(1+v) = v^n + v^{n+1}$, again as it should be.

The formula for larger values of K_x can be checked in a similar way.

It is always necessary to express our random variable using a single term if we wish to derive a variance.

The variance is:

$$\begin{aligned}
\text{var}\left(v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) &= v^{2n}\,\text{var}\left(\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) \\
&= v^{2n}\,\text{var}\left(\frac{1-v^{\max\{K_x+1-n,0\}}}{d}\right) \\
&= \frac{v^{2n}}{d^2}\text{var}\left(v^{\max\{K_x+1-n,0\}}\right) \\
&= \frac{v^{2n}}{d^2}\text{var}\left(\frac{v^{\max\{K_x+1,n\}}}{v^n}\right) \\
&= \frac{1}{d^2}\text{var}\left(v^{\max\{K_x+1,n\}}\right) \\
&= \frac{1}{d^2}\left[E\left(v^{2\max\{K_x+1,n\}}\right) - \left(E\left(v^{\max\{K_x+1,n\}}\right)\right)^2\right]
\end{aligned}$$

Now:

$$v^{\max\{K_x+1,n\}} = \begin{cases} v^n & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

In other words, 1 is paid at time n if the life dies before time n, or 1 is paid at the end of the year of death if the life dies after time n.

The payment at time n if the life dies before time n is not a form of contract we have met, but a payment at the end of the year of death if the life dies after time n is a deferred whole life assurance. So:

$$E\left(v^{\max\{K_x+1,n\}}\right) = v^n \, {}_nq_x + {}_{n|}A_x$$

and:

$$E\left(v^{2\max\{K_x+1,n\}}\right) = v^{2n} \, {}_nq_x + {}_{n|}^{2}A_x$$

Hence, the variance of the present value of the deferred annuity is:

$$\text{var}\left(v^n \ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}\right) = \frac{1}{d^2}\left[v^{2n} \, {}_nq_x + {}_{n|}^{2}A_x - \left(v^n \, {}_nq_x + {}_{n|}A_x\right)^2\right]$$

---

8 Guaranteed annuities payable annually in advance

A *guaranteed annuity* differs from a whole life annuity in that the payments have a minimum specified term.

A *guaranteed annuity-due* has payments that are made in advance and have a minimum specified term.

Question

Suggest a reason why guaranteed annuities are commonplace.

Solution

When people buy annuities they are often investing large amounts of their life savings. Should they die soon after purchasing the annuity, and the annuity is not guaranteed, they would effectively lose nearly all of their life savings. Relatives of the deceased are likely to find this distressing, in addition to the emotional distress they would be experiencing at the time. The insurer could also experience bad publicity and suffer reputational damage as a result.

Issuing annuities with guaranteed payment periods (typically of five or ten years) reduces the financial loss on early death and so goes a long way in mitigating these problems.

8.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the start of each future year for the next n years, and at the start of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $\ddot{a}_{\overline{\max[K_x+1,n]}|}$.

If (x) dies within n years, then exactly n payments will be made. If (x) lives for longer than n years, then K_x+1 payments will be made. So the present value can also be written as:

$$\begin{cases} \ddot{a}_{\overline{n}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x \geq n \end{cases}$$

8.2 Expected present value

In actuarial notation, $E[\ddot{a}_{\overline{\max[K_x+1,n]}|}]$ is denoted $\ddot{a}_{\overline{\overline{x:n}|}}$.

The combined status $\overline{u:v}$ (*ie* with a bar) means a status that is active while *either or both* of the individual statuses u and v are active. It is known as the *last survivor* status, and we will meet it again when we study multiple lives in a later chapter. When applied to an annuity, $\ddot{a}_{\overline{u:v}}$ implies that payments continue until the last surviving status fails, so in the case of $\ddot{a}_{\overline{x:\overline{n}|}}$ this means payments continue until the *later* of the death of (x), or the expiry of n years.

To calculate $\ddot{a}_{\overline{x:\overline{n}|}}$, we use the following:

$$\ddot{a}_{\overline{x:\overline{n}|}} = E[\ddot{a}_{\overline{\max[K_x+1,n]}|}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n}|}\; {}_{k|}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\; {}_{k|}q_x$$

This is because the present value is $\ddot{a}_{\overline{n}|}$ if (x) dies in any of the first n years (*ie* for $0 \le k < n$), and $\ddot{a}_{\overline{K_x+1}|}$ if (x) dies in any year thereafter (*ie* for $K_x \ge n$).

Using $\ddot{a}_{\overline{t}|} = v^0 + v^1 + \cdots + v^{t-1}$, we obtain:

$$\begin{aligned}
\ddot{a}_{\overline{x:\overline{n}|}} &= \sum_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} v^j\right){}_{k|}q_x + \sum_{k=n}^{\infty}\left(\sum_{j=0}^{k} v^j\right){}_{k|}q_x \\
&= \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{0|}q_x + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{1|}q_x + \\
&\qquad \cdots + \left(v^0 + v^1 + \cdots + v^{n-1}\right) \times {}_{n-1|}q_x \\
&\quad + \left(v^0 + v^1 + \cdots + v^{n-1} + v^n\right) \times {}_{n|}q_x \\
&\quad + \left(v^0 + v^1 + \cdots + v^{n-1} + v^n + v^{n+1}\right) \times {}_{n+1|}q_x \\
&\quad + \cdots \\
&= v^0 \sum_{k=0}^{\infty} {}_{k|}q_x + v^1 \sum_{k=0}^{\infty} {}_{k|}q_x + \cdots + v^{n-1} \sum_{k=0}^{\infty} {}_{k|}q_x \\
&\quad + v^n \sum_{k=n}^{\infty} {}_{k|}q_x + v^{n+1} \sum_{k=n+1}^{\infty} {}_{k|}q_x + \cdots \\
&= \sum_{j=0}^{n-1}\left(\sum_{k=0}^{\infty} {}_{k|}q_x\right)v^j + \sum_{j=n}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j
\end{aligned}$$

Now $\sum_{k=0}^{\infty} {}_{k|}q_x = 1$ and $\sum_{k=j}^{\infty} {}_{k|}q_x = {}_jp_x$, and so:

$$\ddot{a}_{x:\overline{n|}} = \sum_{j=0}^{n-1} 1.v^j + \sum_{j=n}^{\infty} {}_jp_x\, v^j$$

$$= \ddot{a}_{\overline{n|}} + \sum_{j=n}^{\infty} {}_jp_x\, v^j$$

$$= \ddot{a}_{\overline{n|}} + {}_{n|}\ddot{a}_x$$

Question

Calculate $\ddot{a}_{60:\overline{10|}}$.

Basis:

Mortality: AM92 Ultimate

Interest: 6% *pa* effective

Solution

We can evaluate this as follows:

$$\ddot{a}_{60:\overline{10|}} = \ddot{a}_{\overline{10|}} + {}_{10|}\ddot{a}_{60}$$

$$= \ddot{a}_{\overline{10|}} + v^{10}\,{}_{10}p_{60}\,\ddot{a}_{70}$$

$$= \left(\frac{1-v^{10}}{d}\right) + v^{10}\frac{l_{70}}{l_{60}}\ddot{a}_{70}$$

$$= \left(\frac{1-1.06^{-10}}{0.06/1.06}\right) + 1.06^{-10} \times \frac{8{,}054.0544}{9{,}287.2164} \times 9.140$$

$$= 7.80169 + 4.42605$$

$$= 12.2277$$

8.3 Variance of the present value random variable

The variance of this benefit is:

$$\begin{aligned}\text{var}[\ddot{a}_{\overline{\max[K_x+1,n]}|}] &= \text{var}\left[\frac{1-v^{\max[K_x+1,n]}}{d}\right] \\ &= \frac{1}{d^2}\text{var}\left[v^{\max[K_x+1,n]}\right] \\ &= \frac{1}{d^2}\left(\text{E}\left[\left(v^{\max[K_x+1,n]}\right)^2\right]-\left(\text{E}\left[v^{\max[K_x+1,n]}\right]\right)^2\right) \\ &= \frac{1}{d^2}\left(v^{2n}\,{}_nq_x + {}_{n|}^{2}A_x - \left(v^n\,{}_nq_x + {}_{n|}A_x\right)^2\right)\end{aligned}$$

The last step above uses the same approach that we used when deriving the formula for the variance of a deferred annuity-due in the question at the end of Section 7.

In fact, we see that the variance of the deferred annuity-due and the variance of the guaranteed annuity-due are the same. This is because all the uncertainty in these contracts stems from the payments made if the policyholder is alive after time n, and these are identical for the two annuities. In both cases, the payments before time n have no uncertainty associated with them (the payments are 0 for the deferred annuity, and are 1 every year for the guaranteed annuity), and so contribute nothing to the variance.

9 Guaranteed annuities payable annually in arrears

9.1 Present value random variable

Consider a guaranteed annuity contract to pay 1 at the end of each future year for the next n years, and at the end of each subsequent future year provided a life now aged x is then alive.

The present value of this benefit is $a_{\overline{\max[K_x,n]}|}$.

This can alternatively be written:

$$\begin{cases} a_{\overline{n}|} & \text{if } K_x \le n \\ a_{\overline{K_x}|} & \text{if } K_x > n \end{cases} = \begin{cases} a_{\overline{n}|} & \text{if } K_x < n \\ a_{\overline{K_x}|} & \text{if } K_x \ge n \end{cases}$$

9.2 Expected present value

In actuarial notation, $E[a_{\overline{\max[K_x,n]}|}]$ is denoted $a_{\overline{x:\overline{n}|}}$.

To obtain a formula for this, we follow similar logic to Section 8.2.

To calculate $a_{\overline{x:\overline{n}|}}$, we use the following:

$$\begin{aligned}
a_{\overline{x:\overline{n}|}} &= E[a_{\overline{\max[K_x,n]}|}] \\
&= \sum_{k=0}^{n-1} a_{\overline{n}|}\ {}_{k|}q_x + \sum_{k=n}^{\infty} a_{\overline{k}|}\ {}_{k|}q_x \\
&= \sum_{k=0}^{n-1}\left(\sum_{j=1}^{n} v^j\right){}_{k|}q_x + \sum_{k=n}^{\infty}\left(\sum_{j=1}^{k} v^j\right){}_{k|}q_x \\
&= \sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} {}_{k|}q_x\right)v^j + \sum_{j=n+1}^{\infty}\left(\sum_{k=j}^{\infty} {}_{k|}q_x\right)v^j \\
&= a_{\overline{n}|} + \sum_{j=n+1}^{\infty} {}_jp_x\, v^j \\
&= a_{\overline{n}|} + {}_{n|}a_x
\end{aligned}$$

9.3 Variance of the present value random variable

The variance of this benefit is:

$$\begin{aligned}\text{var}[a_{\overline{\max[K_x,n]}|}] &= \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|} - 1] \\ &= \text{var}[\ddot{a}_{\overline{\max[K_x+1,n+1]}|}] \\ &= \frac{1}{d^2}\left(v^{2(n+1)}\,{}_{n+1}q_x + {}_{n+1|}^{\;\;\,2}A_x - \left(v^{n+1}\,{}_{n+1}q_x + {}_{n+1|}A_x\right)^2\right)\end{aligned}$$

10 Continuous annuities

So far we have concentrated on annuities payable annually.

We will now consider annuities that are payable continuously. In practice, these may be used to approximate annuities under which payments are very frequent, *eg* weekly or daily.

10.1 Immediate annuity

Consider an immediate annuity of 1 per annum payable continuously during the lifetime of a life now aged x .

Present value random variable

The present value of this annuity is $\bar{a}_{\overline{T_x}|}$.

Expected present value

Recall that T_x is a continuous random variable with PDF $f_{T_x}(t) = {}_tp_x\,\mu_{x+t}$.

The EPV, denoted $\bar{a}_x$, is:

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\,\mu_{x+t}\,dt$$

This is saying that the expected present value is the present value of an annuity paid to time *t*, multiplied by the probability of surviving to *t* and dying in the next instant after time t . The integral is 'summing' this over all future instants at which death could occur.

We can derive a more useful formula than that above.

Note that $\bar{a}_{\overline{t}|} = \int_0^t e^{-\delta s}ds$, so that $\frac{d}{dt}\bar{a}_{\overline{t}|} = e^{-\delta t} = v^t$, and then integrate by parts.

The formula for integrating by parts (given on page 3 of the *Tables*) is:

$$\int_0^\infty u\frac{dw}{dt}dt = \left[uw\right]_0^\infty - \int_0^\infty w\frac{du}{dt}dt$$

In this case, we set:

$$u = \bar{a}_{\overline{t}|} \quad \text{and} \quad \frac{dw}{dt} = {}_tp_x\,\mu_{x+t}$$

so that $\frac{du}{dt} = v^t$ and $w = -{}_tp_x$. The result for w follows because $f_{T_x}(t)$ is the PDF of T_x, and is therefore equal to the derivative of the CDF of T_x, giving:

$$f_{T_x}(t) = {}_tp_x\,\mu_{x+t} = \frac{d}{dt}F_{T_x}(t) = \frac{d}{dt}P(T_x \le t) = \frac{d}{dt}{}_tq_x = \frac{d}{dt}(1 - {}_tp_x) = \frac{d}{dt}(-{}_tp_x)$$

Putting this together gives:

$$\boldsymbol{\bar{a}_x = -\left[\bar{a}_{\overline{t}|}\,{}_tp_x\right]_0^\infty + \int_0^\infty v^t\,{}_tp_x dt = \int_0^\infty v^t\,{}_tp_x dt}$$

Again this final result follows by general reasoning. We take the present value of the payment made at time t, multiply by the probability that the life is alive at that time to receive the payment, and then integrate (*ie* sum) over all future times t.

Another way to prove this result, which is analogous to the proof of the discrete annuity result, is to write $\bar{a}_{\overline{t}|} = \int_0^t v^s ds$ and reverse the order of integration. Then:

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|} f_{T_x}(t)dt = \int_0^\infty \left(\int_0^t v^s\, ds\right) f_{T_x}(t)dt$$

$$= \int_0^\infty \left(\int_s^\infty f_{T_x}(t)dt\right) v^s\, ds = \int_0^\infty v^s P(T_x > s) ds$$

$$= \int_0^\infty v^s\,{}_sp_x\, ds$$

Variance of the present value random variable

The variance of $\bar{a}_{\overline{T_x}|}$ is:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\left[{}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right]$$

Question

Prove this result.

Solution

Using the formula for continuous annuities-certain:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \text{var}\left(\frac{1-v^{T_x}}{\delta}\right)$$

and using the properties of variance:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \frac{1}{\delta^2}\text{var}\left(v^{T_x}\right)$$

In Chapter 15 we saw that:

$$\text{var}\left(v^{T_x}\right) = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$$

Hence:

$$\text{var}\left(\bar{a}_{\overline{T_x}|}\right)=\frac{1}{\delta^2}\left[{}^2\bar{A}_x-\left(\bar{A}_x\right)^2\right]$$

10.2 Other annuities

Temporary, deferred and guaranteed continuous annuities can be defined, and their EPVs calculated, in a similar way. Using the obvious notation, for example:

$$\bar{a}_{x:\overline{n}|}=E\left[\bar{a}_{\overline{\min[T_x,n]}|}\right]=\int_0^n \bar{a}_{\overline{t}|}\,{}_tp_x\mu_{x+t}dt+\bar{a}_{\overline{n}|}\,{}_np_x=\int_0^n v^t\,{}_tp_x dt$$

$$\bar{a}_x=\bar{a}_{x:\overline{n}|}+{}_{n|}\bar{a}_x$$

$$\bar{a}_{\overline{\overline{x:n}|}}=\bar{a}_{\overline{n}|}+{}_{n|}\bar{a}_x$$

$${}_{n|}\bar{a}_x=v^n\,{}_np_x\,\bar{a}_{x+n}$$

10.3 Approximations

To evaluate these annuities, use the approximation:

$$\bar{a}_x\approx\ddot{a}_x-½$$

or: $$\bar{a}_x\approx a_x+½$$

The rationale here is that $\ddot{a}_x$ and a_x represent two extremes, in which the payments are made at the beginning and at the end of each year, respectively. With $\bar{a}_x$, the payments are spread uniformly over the year, so we might expect its value to lie roughly midway between $\ddot{a}_x$ and a_x. Since $\ddot{a}_x$ differs from a_x by 1, $\bar{a}_x$ differs from them both by ½.

Question

A level annuity of £1,000 *pa* is to be paid continuously to a 40-year-old male for the rest of his life.

On the basis of 4% *pa* interest and AM92 Ultimate mortality, calculate the expected present value of this annuity.

Solution

The expected present value of the annuity is:

$$1{,}000\bar{a}_{40}\approx 1{,}000\left(\ddot{a}_{40}-½\right)=1{,}000\left(20.005-½\right)=£19{,}505$$

For temporary annuities:

$$\boldsymbol{\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \tfrac{1}{2}(1 - v^n\,{}_np_x)}$$

We can prove this using the result $\bar{a}_{x:\overline{n}|} = \bar{a}_x - {}_{n|}\bar{a}_x = \bar{a}_x - v^n\,{}_np_x\,\bar{a}_{x+n}$ from Section 10.2, and the approximations for continuously payable whole life annuities we have just met. This gives:

$$\begin{aligned}\bar{a}_{x:\overline{n}|} &= \bar{a}_x - v^n\,{}_np_x\,\bar{a}_{x+n} \\ &\approx \ddot{a}_x - \tfrac{1}{2} - v^n\,{}_np_x\,(\ddot{a}_{x+n} - \tfrac{1}{2}) \\ &= \ddot{a}_x - v^n\,{}_np_x\,\ddot{a}_{x+n} - \tfrac{1}{2}(1 - v^n\,{}_np_x)\end{aligned}$$

But $\ddot{a}_x - v^n\,{}_np_x\,\ddot{a}_{x+n} = \ddot{a}_x - {}_{n|}\ddot{a}_x = \ddot{a}_{x:\overline{n}|}$. So we have:

$$\bar{a}_{x:\overline{n}|} \approx \ddot{a}_{x:\overline{n}|} - \tfrac{1}{2}(1 - v^n\,{}_np_x)$$

11 Evaluating means and variances using select mortality

Corresponding to the annuities defined earlier in this chapter are select equivalents defined as before, but assumed to be issued to a select life denoted $[x]$ rather than x.

So, for example, $\ddot{a}_{[x]} = \sum_{k=0}^{k=\infty} {}_k p_{[x]} v^k$ can be used to calculate the EPV of benefits of a whole life annuity-due, with level annual payments, issued to a select life aged $[x]$ at entry.

The variance formulae established earlier also apply replacing x with $[x]$.

Question

A continuously payable temporary annuity is sold to a life aged exactly 40. The annuity makes payments at a rate of 5,000 *pa* until age 60 or the policyholder's earlier death.

Calculate the expected present value of the annuity payments, using AM92 Select mortality and an interest rate of 4% *pa* effective.

Solution

The expected present value of the annuity payments is:

$$5{,}000\bar{a}_{[40]:\overline{20}|} \approx 5{,}000\left(\ddot{a}_{[40]:\overline{20}|} - \frac{1}{2}\left(1 - v^{20}\,{}_{20}p_{[40]}\right)\right)$$

$$= 5{,}000\left(\ddot{a}_{[40]:\overline{20}|} - \frac{1}{2}\left(1 - v^{20}\frac{l_{60}}{l_{[40]}}\right)\right)$$

$$= 5{,}000\left(13.930 - \frac{1}{2}\left(1 - 1.04^{-20} \times \frac{9{,}287.2164}{9{,}854.3036}\right)\right)$$

$$= 68{,}225$$

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 16 Summary

Annuities

Annuity contracts pay a regular income to the policyholder. The income might be deferred to a future date and could be paid in advance, in arrears or continuously.

For each type of contract we can write down expressions for:

- the present value of the benefits, which is a random variable
- the expected present value of the benefits
- the variance of the present value of the benefits.

Whole life immediate annuity in arrears

Present value: $a_{\overline{K_x}|}$

Expected present value: $E\left(a_{\overline{K_x}|}\right) = a_x$

Variance of present value: $\text{var}\left(a_{\overline{K_x}|}\right) = \dfrac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$

Whole life immediate annuity-due

Present value: $\ddot{a}_{\overline{K_x+1}|}$

Expected present value: $E\left(\ddot{a}_{\overline{K_x+1}|}\right) = \ddot{a}_x$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{K_x+1}|}\right) = \dfrac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$

Continuously payable whole life annuity

Present value: $\bar{a}_{\overline{T_x}|}$

Expected present value: $E\left(\bar{a}_{\overline{T_x}|}\right) = \bar{a}_x$

Variance of present value: $\text{var}\left(\bar{a}_{\overline{T_x}|}\right) = \dfrac{1}{\delta^2}\left[{}^2\bar{A}_x - (\bar{A}_x)^2\right]$

Temporary immediate annuity in arrears

Present value: $a_{\overline{\min\{K_x,n\}}|}$

Expected present value: $E\left(a_{\overline{\min\{K_x,n\}}|}\right) = a_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(a_{\overline{\min\{K_x,n\}}|}\right) = \dfrac{1}{d^2}\left[{}^2A_{x:\overline{n+1}|} - \left(A_{x:\overline{n+1}|}\right)^2\right]$

Temporary immediate annuity-due

Present value: $\ddot{a}_{\overline{\min\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \ddot{a}_{x:\overline{n}|}$

Variance of present value: $\text{var}\left(\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right) = \dfrac{1}{d^2}\left[{}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right]$

Deferred annuity-due

Present value: $Y = v^n\,\ddot{a}_{\overline{\max\{K_x+1-n,0\}}|}$

Expected present value: $E(Y) = {}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n\,{}_np_x\,\ddot{a}_{x+n}$

Variance of present value: $\dfrac{1}{d^2}\left[v^{2n}\,{}_nq_x + {}_{n|}^{2}A_x - \left(v^n\,{}_nq_x + {}_{n|}A_x\right)^2\right]$

Guaranteed annuity-due

Present value: $\ddot{a}_{\overline{\max\{K_x+1,n\}}|}$

Expected present value: $E\left(\ddot{a}_{\overline{\max\{K_x+1,n\}}|}\right) = \ddot{a}_{\overline{x:\overline{n}|}}$

Variance of present value: As for the deferred annuity-due

Chapter 16 Practice Questions

16.1 If T_x and K_x are random variables measuring the complete and curtate future lifetimes, respectively, of a life aged x, write down an expression for each of the following symbols as the expectation of a random variable:

(i) $\bar{a}_x$

(ii) $\ddot{a}_{x:\overline{n}|}$

(iii) $\bar{a}_{\overline{x:\overline{n}|}}$

16.2 Calculate the expectation of the present value of the benefits from each of the following contracts issued to a life aged exactly 45, assuming that the annual effective interest rate is 4%, and AM92 Select mortality applies:

(i) a deferred whole life annuity-due, with a deferred period of 15 years, under which payments of £5,000 are made annually in advance while the policyholder is alive after the deferred period has elapsed

(ii) a guaranteed annuity, with a guarantee period of 15 years, under which payments of £5,000 are made annually in arrears for a minimum of 15 years and for life thereafter.

16.3 Exam style (i) Let Z be a random variable representing the present value of the benefits payable under an immediate life annuity that pays 1 per year in advance, issued to a life aged x.

Show that $\text{var}(Z) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right)$, where ${}^2A_x$ is an assurance calculated at a rate of interest which you should specify. [4]

(ii) A life office issues such a policy to a life aged exactly 65. The benefit is £275 per annum. Calculate the standard deviation of the annuity.

Basis:	Mortality:	AM92 Ultimate
	Interest:	6% per annum throughout

[3]

[Total 7]

16.4 A special 25-year life insurance policy is issued to a life aged x and provides the following benefits:

- a lump sum of £75,000 (payable at the end of the policy year) if death occurs during the first 10 years
- a dependants' pension (payable in the form of an annuity certain) of £5,000 *pa* payable on each remaining policy anniversary during the term (including the 25th anniversary) if death occurs after 10 years but before the end of the term of the policy
- a pension of £7,500 *pa* commencing on the day after the term of the policy expires and with payments on each subsequent policy anniversary while the policyholder is still alive.

Write down an expression for the present value random variable of the benefits under this policy.

16.5 A life currently aged x is subject to a constant force of mortality of 0.02 *pa*. The constant force of interest is 0.03 *pa*. Calculate:

(i) a_x

(ii) $\ddot{a}_x$

16.6 An annuity is payable continuously throughout the lifetime of a person now aged exactly 60, but for at most 10 years. The rate of payment at all times t during the first 5 years is £10,000 *pa*, and thereafter it is £12,000 *pa*.

Exam style

The force of mortality of this life is 0.03 *pa* between the ages of 60 and 65, and 0.04 *pa* between the ages of 65 and 70.

Calculate the expected present value of this annuity assuming a force of interest of 0.05 *pa*. [5]

Chapter 16 Solutions

16.1 (i) $\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right]$

(ii) $\ddot{a}_{x:\overline{n}|} = E\left[f(K_x)\right]$ where $f(K_x) = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$

Alternatively, we can write $\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\min\{K_x+1,n\}}|}\right]$.

(iii) $\bar{a}_{\overline{x:\overline{n}|}} = E\left[g(T_x)\right]$ where $g(T_x) = \begin{cases} \bar{a}_{\overline{n}|} & \text{if } T_x < n \\ \bar{a}_{\overline{T_x}|} & \text{if } T_x \geq n \end{cases}$

Alternatively, we can write $\bar{a}_{\overline{x:\overline{n}|}} = E\left[\bar{a}_{\overline{\max\{T_x,n\}}|}\right]$.

16.2 (i) ***Deferred whole life annuity-due***

The expected present value of the benefits is:

$$5{,}000\ {}_{15|}\ddot{a}_{[45]} = 5{,}000v^{15}\ {}_{15}p_{[45]}\ \ddot{a}_{60}$$

Now:

$${}_{15}p_{[45]} = \frac{l_{60}}{l_{[45]}} = \frac{9{,}287.2164}{9{,}798.0837} = 0.94786$$

So:

$$5{,}000v^{15}\ {}_{15}p_{[45]}\ \ddot{a}_{60} = 5{,}000 \times 1.04^{-15} \times 0.94786 \times 14.134 = £37{,}195$$

(ii) ***Guaranteed annuity in arrears***

The expected present value of the benefits is:

$$\begin{aligned}
5{,}000a_{\overline{[45]:\overline{15}|}} &= 5{,}000\left(a_{\overline{15}|} + v^{15}\ {}_{15}p_{[45]}\ a_{60}\right) \\
&= 5{,}000\left(\left(\frac{1-1.04^{-15}}{0.04}\right) + 1.04^{-15}\ {}_{15}p_{[45]}\left(\ddot{a}_{60} - 1\right)\right) \\
&= 5{,}000\left(11.11839 + 1.04^{-15} \times 0.94786\left(14.134 - 1\right)\right) \\
&= £90{,}155
\end{aligned}$$

16.3 (i) ***Proof***

The random variable Z is defined as:

$$Z = \ddot{a}_{\overline{K_x+1}|}$$ [½]

where K_x is the curtate future lifetime random variable of a life currently aged x.

So:

$$\text{var}(Z) = \text{var}\left(\ddot{a}_{\overline{K_x+1}|}\right) = \text{var}\left(\frac{1-v^{K_x+1}}{d}\right) = \frac{1}{d^2}\text{var}\left(v^{K_x+1}\right)$$ [1]

Expressing the variance in terms of expectations, we have:

$$\text{var}(Z) = \frac{1}{d^2}\left[E\left(v^{2(K_x+1)}\right) - \left(E\left(v^{K_x+1}\right)\right)^2\right]$$ [1]

By definition:

$$A_x = E\left(v^{K_x+1}\right)$$ [½]

and:

$$^2A_x = E\left(v^{2(K_x+1)}\right)$$ [½]

So:

$$\text{var}(Z) = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right]$$

where ^2A_x is evaluated at the rate of interest $i' = (1+i)^2 - 1$. [½]

[Total 4]

(ii) ***Standard deviation of the annuity***

The variance of the present value of this annuity is:

$$\frac{275^2}{d^2}\left[{}^2A_{65} - (A_{65})^2\right] = \frac{275^2}{(0.06/1.06)^2}\left[0.19985 - 0.40177^2\right] = (952.42)^2$$ [2]

So the standard deviation of the present value is £952.42. [1]

[Total 3]

16.4 The present value random variable is:

$$\begin{cases} 75{,}000v^{K_x+1} & \text{if } 0 \le K_x \le 9 \\ 5{,}000v^{K_x+1}\ddot{a}_{\overline{25-K_x}|} & \text{if } 10 \le K_x \le 24 \\ 7{,}500v^{25}\ddot{a}_{\overline{K_x-24}|} & \text{if } K_x \ge 25 \end{cases}$$

A possible alternative solution is:

$$\begin{cases} 75{,}000v^{K_x+1} & \text{if } 0 \le K_x \le 9 \\ 5{,}000\left(a_{\overline{25}|} - a_{\overline{K_x}|}\right) & \text{if } 10 \le K_x \le 24 \\ 7{,}500\left(a_{\overline{K_x}|} - a_{\overline{24}|}\right) & \text{if } K_x \ge 25 \end{cases}$$

16.5 (i) We have:

$$a_x = \sum_{j=1}^{\infty} {}_jp_x\, v^j$$

where ${}_jp_x = e^{-0.02j}$ and $v^j = e^{-\delta j} = e^{-0.03j}$. So:

$$a_x = \sum_{j=1}^{\infty} e^{-0.03j} \times e^{-0.02j} = \sum_{j=1}^{\infty} e^{-0.05j}$$

This is the sum to infinity of a geometric progression with first term $a = e^{-0.05}$ and common ratio $r = e^{-0.05}$. So, using the formula:

$$S_\infty = \frac{a}{1-r}$$

gives:

$$a_x = \frac{e^{-0.05}}{1-e^{-0.05}} = 19.5042$$

(ii) $\ddot{a}_x = a_x + 1 = 19.5042 + 1 = 20.5042$

16.6 The expected present value of this annuity is:

$$10{,}000\,\bar{a}_{60:\overline{5}|} + 12{,}000\,{}_{5|}\bar{a}_{60:\overline{5}|} = 10{,}000\,\bar{a}_{60:\overline{5}|} + 12{,}000\,v^5\,{}_5p_{60}\,\bar{a}_{65:\overline{5}|}$$ [1]

Since the force of mortality is constant between age 60 and age 65:

$$v^5\,{}_5p_{60} = e^{-5\delta}e^{-5\mu} = e^{-5(0.05+0.03)} = e^{-0.4} = 0.67032$$ [1]

Also, using the constant force of mortality of 0.03 between the ages of 60 and 65:

$$\bar{a}_{60:\overline{5}|} = \int_0^5 v^t\,{}_tp_{60}\,dt = \int_0^5 e^{-(0.05+0.03)t}\,dt = \left[\frac{e^{-0.08t}}{-0.08}\right]_0^5 = \frac{1}{0.08}\left(1-e^{-0.4}\right) = 4.12100$$ [1]

and similarly using the constant force of mortality of 0.04 between the ages of 65 and 70:

$$\bar{a}_{65:\overline{5}|} = \int_0^5 v^t\,{}_tp_{65}\,dt = \int_0^5 e^{-(0.05+0.04)t}\,dt = \left[\frac{e^{-0.09t}}{-0.09}\right]_0^5 = \frac{1}{0.09}\left(1-e^{-0.45}\right) = 4.02635$$ [1]

So the expected present value of the annuity is:

$$(10{,}000\times 4.12100) + (12{,}000\times 0.67032\times 4.02635) = £73{,}597$$ [1]

[Total 5]

17

Evaluation of assurances and annuities

Syllabus objectives

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming a constant deterministic interest rate.

- 4.2.4 Define the assurance and annuity factors and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.
- 4.2.5 Understand and use the relations between annuities payable in advance and in arrears, and between temporary, deferred and whole life annuities.
- 4.2.6 Understand and use the relations between assurance and annuity factors using equation of value, and their select and continuous equivalents.

Syllabus objectives continued

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- (constant) contingent benefits are payable at the middle or end of the year of contingent event or continuously
- annuities are paid in advance, in arrears or continuously, and the amount is constant
- premiums are payable in advance, in arrears or continuously; and for the full policy term or for a limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In Chapters 15 and 16, we introduced the basic functions of life insurance mathematics – expected present values of assurance and annuity contracts. The next step is to explore useful relationships between these EPVs. We can then apply the same ideas to other types of life insurance contracts.

The formulae we have derived for EPVs can be interpreted in a simple way, which is often useful in practice. Consider, for example:

$$A_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_{k|}q_x \qquad \text{or} \qquad \ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_k p_x$$

Each term of these sums can be interpreted as:

an amount payable at time k

× a discount factor for k years

× the probability that a payment will be made at time k.

The first term in each case is just 1, but it should be easy to see that this interpretation can be applied to *any* benefit, level or not, payable on death or survival. This makes it easy to write down formulae for EPVs.

For example, consider an annuity-due, under which an amount k will be payable at the start of the kth year provided a life aged x is then alive (an increasing annuity-due). With this interpretation of EPVs, we can write down the EPV of this benefit (which is denoted $(I\ddot{a})_x$):

$$(I\ddot{a})_x = \sum_{k=0}^{\infty} (k+1)v^k\,{}_k p_x$$

Increasing benefits will be covered in Chapter 18.

1 Evaluating assurance benefits

It is important to become familiar with the *Tables* and know which annuity and assurance functions are included in them.

As we have seen, the AM92 table, for example, contains the values of A_x at 4% and 6% *pa* interest. It also contains the values of $A_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x+n=60$ and $x+n=65$. However, we need to be able to calculate the values of assurance functions, *eg* $A_{30:\overline{25}|}$ and $A^{1}_{40:\overline{25}|}$, that are not listed in the *Tables*.

There are, in fact, several ways to proceed. One possibility is to use the relationships that exist between the assurance functions in order to write the required function in terms of functions that are listed in the *Tables*.

We saw in Chapter 15 that:

$${}_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_np_x \, A_{x+n}$$

Rearranging this gives:

$$A^{1}_{x:\overline{n}|} = A_x - v^n \, {}_np_x \, A_{x+n}$$

So we can calculate the value of a term assurance by writing it in terms of whole life assurances.

Question

Calculate the values of:

(i) $A^{1}_{40:\overline{25}|}$

(ii) $A_{30:\overline{25}|}$

(iii) $\bar{A}_{30:\overline{25}|}$

using AM92 mortality and 4% *pa* interest.

Solution

(i) The value of the term assurance is:

$$A^{1}_{40:\overline{25}|} = A_{40} - v^{25} \, {}_{25}p_{40} \, A_{65}$$

$$= 0.23056 - \frac{1}{1.04^{25}} \times \frac{8,821.2612}{9,856.2863} \times 0.52786$$

$$= 0.05334$$

Alternatively, we could note that because 40 and 25 sum to 65, the value of the endowment assurance $A_{40:\overline{25|}}$ is tabulated. Recalling that an endowment assurance is the sum of a term assurance and a pure endowment, we can say:

$$A^{1}_{40:\overline{25|}} = A_{40:\overline{25|}} - A_{40:\overline{25|}}^{\ \ \ \ 1}$$

$$= A_{40:\overline{25|}} - v^{25}\ {}_{25}p_{40}$$

$$= 0.38907 - \frac{1}{1.04^{25}} \times \frac{8,821.2612}{9,856.2863}$$

$$= 0.05334$$

(ii) The value of the endowment assurance is:

$$A_{30:\overline{25|}} = A^{1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\ \ \ \ 1}$$

$$= A_{30} - v^{25}\ {}_{25}p_{30}\, A_{55} + v^{25}\ {}_{25}p_{30}$$

$$= 0.16023 - \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094} \times 0.38950 + \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094}$$

$$= 0.38076$$

(iii) The value of the endowment assurance under which the death benefit is payable immediately on death is:

$$\bar{A}_{30:\overline{25|}} = \bar{A}^{1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\ \ \ \ 1}$$

$$\approx 1.04^{1/2} A^{1}_{30:\overline{25|}} + v^{25}\ {}_{25}p_{30}$$

$$= 1.04^{1/2}\left(0.16023 - \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094} \times 0.38950\right) + \frac{1}{1.04^{25}} \times \frac{9,557.8179}{9,925.2094}$$

$$= 0.38115$$

Remember that it is only the death benefit part that gets multiplied by the acceleration factor, $(1+i)^{1/2}$.

We can use the same techniques, along with the formulae developed in earlier chapters, to calculate the variance of the present value of the benefits payable. Recall that this involves functions with a pre-superscript of 2 to indicate that they are evaluated at the interest rate $(1+i)^2 - 1$.

Question

A life office has just sold a 25-year term assurance policy to a life aged 40. The sum assured is £50,000 and is payable at the end of the year of death.

Calculate the variance of the present value of this benefit, assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

The variance of the present value is:

$$50{,}000^2\left[{}^2A^{\,1}_{40:\overline{25|}}-\left(A^{\,1}_{40:\overline{25|}}\right)^2\right]$$

From part (i) of the preceding question, we know that $A^{\,1}_{40:\overline{25|}}=0.05334$.

The function ${}^2A^{\,1}_{40:\overline{25|}}$ is evaluated using an interest rate of $1.04^2-1=8.16\%$. This gives:

$$\begin{aligned}{}^2A^{\,1}_{40:\overline{25|}} &= {}^2A_{40}-\frac{1}{1.0816^{25}}\times {}_{25}p_{40}\times {}^2A_{65}\\ &=0.06792-\frac{1}{1.0816^{25}}\times\frac{8{,}821.2612}{9{,}856.2863}\times 0.30855\\ &=0.02906\end{aligned}$$

So the variance of the present value is:

$$50{,}000^2\left[0.02906-0.05334^2\right]=(£8{,}096)^2$$

2 Evaluating annuity benefits

The following relationships are easy to prove:

$$\ddot{a}_x = 1 + a_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v\,p_x\,\ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = v p_x\,\ddot{a}_{x+1:\overline{n}|}$$

The question below illustrates the first of the formulae above, which we introduced in Chapter 16.

Question

Find a_{65} (PFA92C20 at 4%).

Solution

$$a_{65} = \ddot{a}_{65} - 1 = 13.871$$

We also obtained the result $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$ in Chapter 16.

We derive the relationship $a_{x:\overline{n}|} = v p_x \ddot{a}_{x+1:\overline{n}|}$ below. The relationship $a_x = v p_x \ddot{a}_{x+1}$ can be derived in the same way.

Question

Prove that $a_{x:\overline{n}|} = v p_x \ddot{a}_{x+1:\overline{n}|}$.

Solution

Starting with the definition of $a_{x:\overline{n}|}$, we have:

$$a_{x:\overline{n}|} = v\,p_x + v^2\,{}_2p_x + \ldots + v^n\,{}_np_x$$

Now, using the principle of consistency ${}_kp_x = {}_np_x\;{}_{k-n}p_{x+n}$ (for $k \geq n$) with $n=1$, we are able to take out the factor $v\,p_x$, giving:

$$\begin{aligned} a_{x:\overline{n}|} &= v\,p_x\left(1 + v\,p_{x+1} + v^2\,{}_2p_{x+1} + \ldots + v^{n-1}\,{}_{n-1}p_{x+1}\right) \\ &= v\,p_x\,\ddot{a}_{x+1:\overline{n}|} \end{aligned}$$

As we have seen, the AM92 table contains the values of $\ddot{a}_x$ at 4% and 6% *pa* interest. It also contains the values of $\ddot{a}_{x:\overline{n}|}$ (at 4% and 6% *pa* interest) for ages x and terms n such that $x+n=60$ and $x+n=65$.

Other values can be calculated using formulae such as:

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n}$$

This formula enables us to calculate the value of a temporary annuity-due in terms of whole life functions. It is a rearrangement of a relationship we met in Chapter 16 for a deferred annuity-due:

$${}_{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = v^n \, {}_np_x \, \ddot{a}_{x+n}$$

Similar formulae hold for temporary annuities payable annually in arrears and temporary annuities payable continuously:

$$a_{x:\overline{n}|} = a_x - v^n \, {}_np_x \, a_{x+n}$$

$$\bar{a}_{x:\overline{n}|} = \bar{a}_x - v^n \, {}_np_x \, \bar{a}_{x+n}$$

Question

Calculate the values of $\ddot{a}_{60:\overline{10}|}$ and $\bar{a}_{60:\overline{10}|}$ using AM92 mortality and 6% *pa* interest.

Solution

The value of the temporary annuity-due is:

$$\ddot{a}_{60:\overline{10}|} = \ddot{a}_{60} - v^{10} \, {}_{10}p_{60} \, \ddot{a}_{70}$$

$$= 11.891 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164} \times 9.140$$

$$= 7.465$$

The value of the temporary continuously-payable annuity can be written as:

$$\bar{a}_{60:\overline{10}|} = \bar{a}_{60} - v^{10} \, {}_{10}p_{60} \, \bar{a}_{70}$$

$$= \left(\ddot{a}_{60} - ½\right) - v^{10} \, {}_{10}p_{60} \left(\ddot{a}_{70} - ½\right)$$

$$= \ddot{a}_{60:\overline{10}|} - ½\left(1 - v^{10} \, {}_{10}p_{60}\right)$$

So, using the value of the temporary annuity-due, we have:

$$\bar{a}_{60:\overline{10}|} = 7.465 - ½\left(1 - \frac{1}{1.06^{10}} \times \frac{8,054.0544}{9,287.2164}\right) = 7.207$$

For the temporary continuously payable annuity in the question above, we used the relationship:

$$\bar{a}_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \tfrac{1}{2}\left(1 - v^n \,{}_np_x\right)$$

In Chapter 16, we also developed the corresponding formula for a temporary annuity in arrears:

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - \left(1 - v^n \,{}_np_x\right)$$

These two results can be particularly useful when $x+n=60$ or $x+n=65$, as the value of $\ddot{a}_{x:\overline{n}|}$ is tabulated (at 4% and 6% *pa* interest) for AM92 mortality.

Question

Calculate the value of $a_{[40]:\overline{25}|}$ using AM92 mortality and 4% *pa* interest.

Solution

We can write:

$$\begin{aligned} a_{[40]:\overline{25}|} &= \ddot{a}_{[40]:\overline{25}|} - \left(1 - v^{25}\,{}_{25}p_{[40]}\right) \\ &= \ddot{a}_{[40]:\overline{25}|} - 1 + v^{25}\frac{l_{65}}{l_{[40]}} \\ &= 15.887 - 1 + 1.04^{-25} \times \frac{8,821.2612}{9,854.3036} \\ &= 15.223 \end{aligned}$$

Another way to write the factor $v^n \,{}_np_x$ is as $\dfrac{D_{x+n}}{D_x}$. D_x is an example of a commutation function, and is defined as follows:

$$D_x = v^x\, l_x$$

so that:

$$\frac{D_{x+n}}{D_x} = \frac{v^{x+n}\, l_{x+n}}{v^x\, l_x} = v^n \frac{l_{x+n}}{l_x} = v^n \,{}_np_x$$

In the past, commutation functions were widely used in the calculation of annuities and assurances. Values of the functions D_x, N_x, S_x, C_x, M_x and R_x are listed for AM92 mortality at 4% *pa* interest, but the only one of these that we will use in this course is D_x. We use it purely because it is quicker to calculate $\frac{D_{x+n}}{D_x}$ than $v^n\,{}_np_x$. However, if we are not using AM92 mortality and an assumed rate of interest of 4% *pa*, D_x is not available and we have to use $v^n\,{}_np_x$.

Question

Calculate a_{30} and ${}_{10|}a_{30}$, and also a_{70} and ${}_{10|}a_{70}$, based on AM92 mortality and 4% *pa* interest. Comment on your results.

Solution

For age 30 we have:

$$a_{30} = \ddot{a}_{30} - 1 = 21.834 - 1 = 20.834$$

$${}_{10|}a_{30} = v^{10}\,{}_{10}p_{30}\,a_{40} = \frac{D_{40}}{D_{30}}(\ddot{a}_{40} - 1) = \frac{2,052.96}{3,060.13}(20.005 - 1) = 12.750$$

Similarly for age 70 we have:

$$a_{70} = \ddot{a}_{70} - 1 = 10.375 - 1 = 9.375$$

$${}_{10|}a_{70} = v^{10}\,{}_{10}p_{70}\,a_{80} = \frac{D_{80}}{D_{70}}(\ddot{a}_{80} - 1) = \frac{228.48}{517.23}(6.818 - 1) = 2.570$$

a_{30} is much bigger than a_{70} since the payments are expected to continue for a much longer period.

${}_{10|}a_{30}$ is lower than a_{30}, because no payments are made during the first 10 years, and since these payments are very likely to made (and are discounted least), the difference is quite close to 10.

${}_{10|}a_{70}$ is lower than a_{70} for the same reason. The difference is less than is seen in the age 30 functions, however, since the payments during the first 10 years are less likely to be made (as the life is more likely to die between ages 70 and 80 than between ages 30 and 40).

It's important to be aware that sometimes the assumptions made about mortality or interest will be different in different time periods. In the following question, the mortality assumption is different pre- and post-retirement.

Question

A male pension policyholder is currently aged 50 and he will retire at age 65, from which age a pension of £5,000 *pa* will be paid annually in advance. Before retirement he is assumed to experience mortality in line with AM92 Ultimate and after retirement in line with PMA92C20.

Calculate the expected present value of the benefits assuming interest of 4% *pa*.

Solution

The expected present value of the benefits is:

$$EPV = 5{,}000 \times \frac{D'_{65}}{D'_{50}} \times \ddot{a}''_{65}$$

where $\frac{D'_{65}}{D'_{50}}$ is calculated using AM92 Ultimate mortality, and $\ddot{a}''_{65}$ uses PMA92C20 mortality.

Therefore:

$$EPV = 5{,}000 \times \frac{689.23}{1{,}366.61} \times 13.666 = £34{,}461$$

3 Premium conversion formulae

There are both discrete and continuous versions of the premium conversion formulae.

3.1 Discrete version

There is a simple and very useful relationship between the EPVs of certain assurance contracts and the EPVs of annuities-due:

$$\ddot{a}_x = E\left[\ddot{a}_{\overline{K_x+1}|}\right] = E\left[\frac{1-v^{K_x+1}}{d}\right] = \frac{1-E\left[v^{K_x+1}\right]}{d} = \frac{1-A_x}{d}$$

Hence $A_x = 1 - d\ddot{a}_x$ **.**

Question

Verify that $A_{65} = 1 - d\ddot{a}_{65}$ using AM92 mortality and 4% *pa* interest.

Solution

From the AM92 table with 4% *pa* interest:

$$\ddot{a}_{65} = 12.276$$

So using the premium conversion formula:

$$A_{65} = 1 - d\ddot{a}_{65} = 1 - \frac{0.04}{1.04} \times 12.276 = 0.52785$$

The slight difference between this and the *Tables* value of $A_{65} = 0.52786$ is due to rounding.

Along similar lines, we find that:

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

and as we shall see, similar relationships hold for all of the whole life and endowment assurance contracts that we consider.

These relationships also apply replacing x with $[x]$, *ie*:

$$A_{[x]} = 1 - d\ddot{a}_{[x]}$$

and:

$$A_{[x]:\overline{n}|} = 1 - d\ddot{a}_{[x]:\overline{n}|}$$

3.2 Continuous version

Similar relationships hold between level annuities payable continuously and assurance contracts with death benefits payable immediately on death. In these, we use δ instead of d.

For whole life benefits:

$$\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = E\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_x)$$

Hence $\bar{A}_x = 1-\delta\,\bar{a}_x$.

For temporary benefits:

$$\bar{a}_{x:\overline{n}|} = E\left[\bar{a}_{\overline{\min[T_x,n]}|}\right] = E\left[\frac{1-v^{\min[T_x,n]}}{\delta}\right] = \frac{1}{\delta}(1-\bar{A}_{x:\overline{n}|})$$

Hence $\bar{A}_{x:\overline{n}|} = 1-\delta\,\bar{a}_{x:\overline{n}|}$.

These formulae also apply replacing x with $[x]$.

The premium conversion formulae are given on page 37 of the *Tables*.

3.3 Variance of benefits

We can use a similar approach to express the variances of annuities payable continuously. For example:

$$\mathrm{var}\left[\bar{a}_{\overline{T_x}|}\right] = \mathrm{var}\left[\frac{1-v^{T_x}}{\delta}\right] = \frac{1}{\delta^2}\mathrm{var}\left[v^{T_x}\right] = \frac{1}{\delta^2}\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$$

We have already seen this result in Chapter 16.

4 Expected present values of annuities payable *m* times each year

Very often, premiums are not paid annually but are instead paid with some other frequency, *eg* every quarter, or every month. Likewise, we may want to value annuity benefits payable more frequently than annually.

We now consider the question of how annuities, with payments made more than once each year but less frequently than continuously, may be evaluated.

We define the expected present value of an immediate annuity of 1 per annum, payable m times each year in arrears **to a life aged x, as $a_x^{(m)}$. This comprises payments, each of $\frac{1}{m}$, at ages:**

$$x+\frac{1}{m},\ x+\frac{2}{m},\ x+\frac{3}{m} \text{ and so on.}$$

The expected present value may be written:

$$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} v^{t/m}\frac{l_{x+t/m}}{l_x}$$

If a mathematical formula for l_x is known, this expression may be evaluated directly.

So, for example, if $l_x = 1{,}000 - 10x$ for $90 \le x \le 100$, we have:

$$a_{90}^{(4)} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4}\frac{l_{90+t/4}}{l_{90}} = \frac{1}{4}\sum_{t=1}^{\infty} v^{t/4}\left(\frac{100-2.5t}{100}\right)$$

More often, an approximation will be needed to evaluate the expression.

We now consider an annuity-due.

An expression for $\ddot{a}_x^{(m)}$ can be valued as a series of deferred annuities with annual payments of $\frac{1}{m}$ and deferred period of $\frac{t}{m}$; $t = 0, 1, 2, \ldots, m-1$:

$$\sum_{t=0}^{m-1}\frac{1}{m}\ {}_{\frac{t}{m}|}\ddot{a}_x$$

Using the approximation that a sum of 1 payable a proportion k $(0 < k < 1)$ through the year is equivalent to an amount of $(1-k)$ paid at the start of the year and an amount of k paid at the end of the year, we can write:

$${}_{\frac{t}{m}|}\ddot{a}_x \approx \ddot{a}_x - \frac{t}{m}$$

To explain this result, let's think about an example. Suppose that $t=3$ and $m=4$. Then the annuity-due is deferred for ¾ of a year. So there are payments of £1 at times ¾, 1¾, 2¾, *etc*. In symbols we have:

$${}_{\frac{3}{4}|}\ddot{a}_x = v^{\frac{3}{4}}\,{}_{\frac{3}{4}}p_x + v^{1\frac{3}{4}}\,{}_{1\frac{3}{4}}p_x + v^{2\frac{3}{4}}\,{}_{2\frac{3}{4}}p_x + \cdots$$

Now the EPV of £1 at time ¾ is very similar to the EPV of £¼ paid at time 0, plus the EPV of ¾ paid at time 1, *ie*:

$$v^{\frac{3}{4}}\,{}_{\frac{3}{4}}p_x \approx \tfrac{1}{4} + \tfrac{3}{4} v\, p_x$$

Similarly, the EPV of £1 at time 1¾ is very similar to the EPV of £¼ paid at time 1, plus the EPV of ¾ paid at time 2, *ie*:

$$v^{1\frac{3}{4}}\,{}_{1\frac{3}{4}}p_x \approx \tfrac{1}{4} v\, p_x + \tfrac{3}{4} v^2\,{}_2p_x$$

Continuing similarly and adding up all these payments we get:

$${}_{\frac{3}{4}|}\ddot{a}_x \approx \left(\tfrac{1}{4} + \tfrac{3}{4} v\, p_x\right) + \left(\tfrac{1}{4} v\, p_x + \tfrac{3}{4} v^2\,{}_2p_x\right) + \cdots$$

$$= \tfrac{1}{4} + v\, p_x + v^2\,{}_2p_x + \cdots$$

$$= \ddot{a}_x - \tfrac{3}{4}$$

In general, we have:

$${}_{\frac{t}{m}|}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$$

We now return to our expression for $\ddot{a}_x^{(m)}$ and substitute in this result for the deferred annuity.

So the expected present value of the *m*thly annuity is approximately:

$$\sum_{t=0}^{t=m-1} \frac{1}{m}\left(\ddot{a}_x - \frac{t}{m}\right) = m.\frac{1}{m}\ddot{a}_x - \frac{1}{m}.\frac{1}{2}\frac{(m-1)m}{m}$$

This uses the result that the sum of the first n integers is $\tfrac{1}{2}n(n+1)$, so:

$$\frac{1}{m}\sum_{t=0}^{m-1}\frac{t}{m} = \frac{1}{m}\times\frac{1+2+\cdots+(m-1)}{m} = \frac{1}{m}\times\frac{\tfrac{1}{2}(m-1)m}{m}$$

That is:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

This formula is given on page 36 of the *Tables*.

The corresponding expression for $a_x^{(m)}$ then follows from the relationship:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

that is:

$$a_x^{(m)} \approx a_x + \frac{m-1}{2m}$$

(Note that, letting $m \to \infty$, we obtain the expression $\bar{a}_x \approx \ddot{a}_x - ½$, as referred to in Chapter 16.)

The relationship:

$$\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$$

holds because each payment under the annuities $a_x^{(m)}$ and $\ddot{a}_x^{(m)}$ is for amount $\frac{1}{m}$, and so the only difference between them is the payment of $\frac{1}{m}$ made at time 0 under the annuity-due that is not made under the annuity in arrears.

These approximations may be used to develop equivalent expressions for temporary and deferred annuities.

For example:

$$\begin{aligned}\ddot{a}_{x:\overline{n}|}^{(m)} &= \ddot{a}_x^{(m)} - v^n\,{}_np_x\,\ddot{a}_{x+n}^{(m)} \\ &= \left(\ddot{a}_x - \frac{m-1}{2m}\right) - v^n\,{}_np_x\left(\ddot{a}_{x+n} - \frac{m-1}{2m}\right) \\ &= \ddot{a}_x - v^n\,{}_np_x\,\ddot{a}_{x+n} - \frac{m-1}{2m}\left(1 - v^n\,{}_np_x\right) \\ &= \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right)\end{aligned}$$

This formula is also given on page 36 of the *Tables.*

Question

Using AM92 mortality and 4% *pa* interest, calculate:

(i) $\ddot{a}_{60}^{(2)}$

(ii) $a_{60}^{(12)}$

(iii) $\ddot{a}_{50:\overline{15}|}^{(4)}$

Solution

(i) $\ddot{a}_{60}^{(2)} \approx \ddot{a}_{60} - \frac{1}{4} = 14.134 - 0.25 = 13.884$

(ii) $a_{60}^{(12)} \approx a_{60} + \frac{11}{24} = \ddot{a}_{60} - 1 + \frac{11}{24} = 14.134 - 1 + \frac{11}{24} = 13.592$

(iii) $\ddot{a}_{50:\overline{15}|}^{(4)} \approx \ddot{a}_{50:\overline{15}|} - \frac{3}{8}\left(1 - \frac{D_{65}}{D_{50}}\right) = 11.253 - \frac{3}{8}\left(1 - \frac{689.23}{1,366.61}\right) = 11.067$

5 Expected present values under a constant force of mortality

So far in this chapter we have evaluated assurance and annuity functions assuming that the underlying mortality is given by a mortality table (often AM92 Ultimate mortality).

We can also calculate these functions in the (admittedly somewhat artificial) situation where a life is subject to a constant force of mortality.

In this particular situation, it will usually be easier to calculate what we might call 'continuous time' mortality functions – for example, assurances payable immediately on death, or annuities payable continuously. We do this by first writing the expression we need as an integral, and then evaluating it.

We can also calculate the corresponding 'discrete time' functions – although these will often involve a summation of a series, usually a geometric progression.

Question

A life is subject to a constant force of mortality of 0.008 *pa* at all ages above 50. The constant force of interest of 4% *pa*. Calculate the exact values of:

(i) $\bar{A}_{50}$

(ii) A_{50}

Solution

(i) Since the PDF of the complete future lifetime random variable T_x is ${}_tp_x\ \mu_{x+t}$, we can calculate $\bar{A}_{50}$ as follows:

$$\bar{A}_{50} = E\left(v^{T_{50}}\right) = \int_0^\infty v^t\ {}_tp_{50}\ \mu_{50+t}\ dt$$

We know that $\mu_{50+t} = 0.008$ for all t, so that ${}_tp_{50} = e^{-0.008t}$. Writing $v^t = e^{-\delta t}$, we have:

$$\bar{A}_{50} = \int_0^\infty e^{-0.04t} \times e^{-0.008t} \times 0.008\ dt = \int_0^\infty 0.008e^{-0.048t}\ dt = \left[\frac{0.008e^{-0.048t}}{-0.048}\right]_0^\infty$$

$$= \frac{0.008}{0.048} = 0.16667$$

(ii) We now need to calculate:

$$A_{50} = E\left(v^{K_{50}+1}\right) = v\,q_{50} + v^2 p_{50}\,q_{51} + v^3\ {}_2p_{50}\,q_{52} + \cdots$$

Since $p_{50+t} = e^{-0.008}$ for all t , $q_{50+t} = 1 - e^{-0.008}$. Also using $v^t = e^{-0.04t}$, we find that:

$$A_{50} = e^{-0.04}\left(1 - e^{-0.008}\right) + e^{-0.08}e^{-0.008}\left(1 - e^{-0.008}\right) + e^{-0.12}e^{-0.016}\left(1 - e^{-0.008}\right) + \cdots$$

This is a geometric progression with first term $a = e^{-0.04}\left(1 - e^{-0.008}\right)$ and common ratio $r = e^{-0.048}$. So the sum to infinity is:

$$A_{50} = \frac{a}{1-r} = \frac{e^{-0.04}\left(1 - e^{-0.008}\right)}{1 - e^{-0.048}} = 0.16335$$

The answers obtained here are consistent with the use of the acceleration factor $(1+i)^{0.5}$ *.*
Since $\delta = 0.04$ *,* $1 + i = e^{0.04}$ *and:*

$$(1+i)^{0.5} A_{50} = e^{0.02} \times 0.16335 = 0.16665 \approx \bar{A}_{50}$$

We can use a similar approach to calculate the expected present value of annuity functions, where the life is subject to a constant force of mortality.

Question

A life is subject to a constant force of mortality of 0.02 *pa* at all ages above 40. The constant force of interest of 5% *pa*. Calculate the exact values of:

(i) $\bar{a}_{40:\overline{10|}}$

(ii) $\bar{a}_{\overline{40:\overline{10|}}}$

Solution

(i) Here we have:

$$\bar{a}_{40:\overline{10|}} = \int_0^{10} e^{-\delta t}\, {}_tp_{40}\, dt = \int_0^{10} e^{-0.05t} e^{-0.02t}\, dt = \left[\frac{e^{-0.07t}}{-0.07}\right]_0^{10} = \frac{1 - e^{-0.7}}{0.07} = 7.192$$

(ii) Using the standard formula for a guaranteed annuity, we have:

$$\bar{a}_{\overline{40:\overline{10|}}} = \bar{a}_{\overline{10|}} + {}_{10}p_{40}\, e^{-10\delta}\, \bar{a}_{50}$$