The whole life annuity at age 50 has expected present value:

$$\bar{a}_{50} = \int_0^\infty e^{-0.05t} e^{-0.02t}\, dt = \left[\frac{e^{-0.07t}}{-0.07} \right]_0^\infty = \frac{1}{0.07} = 14.28571$$

The present value of the continuously payable annuity-certain is:

$$\bar{a}_{\overline{10|}} = \frac{1 - e^{-0.05\times 10}}{0.05} = 7.86939$$

So the expected present value of the guaranteed annuity is:

$$\bar{a}_{\overline{\overline{40:\overline{10}|}}} = 7.86939 + e^{-0.02\times 10} e^{-0.05\times 10} \times 14.28571 = 14.963$$

Chapter 17 Summary

Relationships between assurances

$$A^{1}_{x:\overline{n}|} = A_x - v^n \, {}_np_x \, A_{x+n}$$

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n}|} = v^n \, {}_np_x \, A_{x+n}$$

Relationships between annuities

$$a_x = \ddot{a}_x - 1 \qquad \bar{a}_x \approx \ddot{a}_x - ½$$

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n \, {}_np_x \, \ddot{a}_{x+n} \qquad a_{x:\overline{n}|} = a_x - v^n \, {}_np_x \, a_{x+n}$$

$$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \, {}_np_x \qquad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$$

$$a_x = v \, p_x \, \ddot{a}_{x+1} \qquad a_{x:\overline{n}|} = v \, p_x \, \ddot{a}_{x+1:\overline{n}|}$$

Premium conversion formulae

$$A_x = 1 - d\ddot{a}_x \qquad A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$$

$$\bar{A}_x = 1 - \delta\bar{a}_x \qquad \bar{A}_{x:\overline{n}|} = 1 - \delta\bar{a}_{x:\overline{n}|}$$

Annuities payable *m* times a year

The expected present value of a life annuity of 1 *pa*, payable in arrears m times a year to a life aged x is:

$$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} v^{t/m} \, {}_{t/m}p_x \approx a_x + \frac{m-1}{2m}$$

The corresponding annuity-due has expected present value:

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{t=0}^{\infty} v^{t/m} \, {}_{t/m}p_x = \frac{1}{m} + a_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 17 Practice Questions

17.1 Calculate the value of each of the following functions, using the given basis:

(i) $\ddot{a}_{65:\overline{20}|}$ AM92 mortality and 4% *pa* interest

(ii) $A_{68:\overline{2}|}$ AM92 mortality and 6% *pa* interest

(iii) $a_{\overline{62:\overline{5}|}}$ PFA92C20 mortality and 4% *pa* interest

(iv) $\ddot{a}_{[55]+1:\overline{4}|}$ AM92 mortality and 4% *pa* interest

(v) $A_{[60]+1}$ AM92 mortality and 4% *pa* interest

(vi) $A^{\,1}_{[40]:\overline{10}|}$ AM92 mortality and 6% *pa* interest

17.2 Calculate the values of:

(a) $\ddot{a}^{(12)}_{50:\overline{15}|}$

(b) $\ddot{a}^{(12)}_{\overline{50:\overline{15}|}}$

using AM92 mortality and 4% *pa* interest.

17.3 Calculate the expected present value of a payment of £2,000 made 6 months after the death of a life now aged exactly 60, assuming AM92 Select mortality and 6% *pa* interest.

17.4 (i) Calculate $\bar{A}_{30:\overline{25}|}$ and $\bar{a}_{30:\overline{25}|}$ independently, assuming AM92 mortality and 6% *pa* interest.

(ii) Hence verify that the usual premium conversion relationship holds approximately between these two functions.

17.5 Let K be a random variable representing the curtate future lifetime of a life aged 40, and let $g(K)$ be the function defined as:

$$g(K)=\begin{cases} 0 & \text{if } 0 \le K < 10 \\ v^{10}\,\ddot{a}_{\overline{K-9}|} & \text{if } 10 \le K < 35 \\ v^{10}\,\ddot{a}_{\overline{25}|} & \text{if } 35 \le K \end{cases}$$

Calculate $E[g(K)]$, assuming mortality follows AM92 Ultimate and interest is 4% *pa*.

17.6 A 10-year temporary annuity is payable continuously to a life now aged x. The rate of payment is 100 *pa* for the first 5 years and 150 *pa* for the next 5 years.

Exam style

(i) Write down an expression in terms of T_x for the present value of this annuity. [3]

(ii) Calculate the expected present value of the annuity assuming a constant force of interest of 0.04 *pa* and a constant force of mortality of 0.01 *pa*. [5]

[Total 8]

17.7 An impaired life aged 40 experiences 5 times the force of mortality of a life of the same age subject to standard mortality. A two-year term assurance policy is sold to this impaired life, and another two-year term assurance is sold to a standard life aged 40. Both policies have a sum assured of £10,000 payable at the end of the year of death.

Calculate the expected present value of the benefits payable to each life assuming that standard mortality is AM92 Ultimate and interest is 4% *pa*.

17.8 Assuming that the force of mortality between consecutive integer ages is constant in the AM92 Ultimate table, calculate the exact value of $\bar{A}_{50:\overline{2|}}$ using a rate of interest of 4% *pa*. [6]

Exam style

17.9 In a mortality table with a one-year select period $q_{[x]} = a.q_x$ for all $x \geq 0$ and for a certain constant $0 < a < 1$.

Exam style

(i) Let:

$K_{[x]+t}$ = curtate future lifetime of a person aged $x+t$ who became select t years ago

K_x = curtate future lifetime of a person aged x whose mortality is reflected by the ultimate part of the mortality table.

Explain whether the expected value of $K_{[x]+1}$ is less than, greater than, or equal to the expected value of K_{x+1}. [1]

(ii) Calculate $\bar{A}_{[45]:\overline{20|}}$ based on the following assumptions:

- $a = 0.9$
- Interest: 4% *pa*
- Ultimate mortality: ELT15 (Males)

Financial functions using ELT15 (Males) mortality are given on page 136 of the Tables. [6]

[Total 7]

Chapter 17 Solutions

17.1 (i) We can calculate the value of this temporary annuity-due as:

$$\ddot{a}_{\overline{65:20|}} = \ddot{a}_{65} - \frac{D_{85}}{D_{65}}\ddot{a}_{85} = 12.276 - \frac{120.71}{689.23} \times 5.333 = 11.342$$

(ii) The value of this endowment assurance is not tabulated, but since the term is 2 years only, we can calculate its value using a summation from first principles.

If death occurs during the first year of the contract, the payment will be made at the end of the first year. Otherwise, it will be made at the end of the second year. So the value is:

$$A_{\overline{68:2|}} = vq_{68} + v^2 p_{68}$$

$$= 1.06^{-1} \times 0.019913 + 1.06^{-2} \times (1 - 0.019913) = 0.89106$$

(iii) This is the EPV of a guaranteed annuity, payable annually in arrears for a minimum of 5 years and for the remaining lifetime of a person currently aged 62 exact. We can write:

$$a_{\overline{\overline{62:5|}}} = a_{\overline{5|}} + {}_{5|}a_{62}$$

$$= \frac{1-v^5}{i} + v^5 \frac{l_{67}}{l_{62}}(\ddot{a}_{67} - 1)$$

$$= \frac{1-1.04^{-5}}{0.04} + 1.04^{-5} \times \frac{9,605.483}{9,804.173} \times (14.111 - 1)$$

$$= 15.010$$

(iv) The value of $\ddot{a}_{[55]+1:\overline{4|}}$ is not tabulated, so to calculate its value we express it in terms of $\ddot{a}_{\overline{57:3|}}$, which is tabulated, as follows:

$$\ddot{a}_{[55]+1:\overline{4|}} = 1 + v\, p_{[55]+1}\, \ddot{a}_{\overline{57:3|}} = 1 + \frac{1}{1.04}(1 - 0.004903) \times 2.870 = 3.746$$

(v) As in part (iv), the value of $A_{[60]+1}$ is not tabulated, so to calculate its value we express it in terms of A_{62}, which is tabulated, as follows:

$$A_{[60]+1} = v\, q_{[60]+1} + v\, p_{[60]+1}\, A_{62}$$

$$= 1.04^{-1}\left(0.008680 + (1 - 0.008680) \times 0.48458\right)$$

$$= 0.47024$$

(vi) We can calculate the value of this term assurance as:

$$A^{\;1}_{[40]:\overline{10|}} = A_{[40]} - v^{10}\frac{l_{50}}{l_{[40]}}A_{50}$$

$$= 0.12296 - \frac{1}{1.06^{10}} \times \frac{9{,}712.0728}{9{,}854.3036} \times 0.20508$$

$$= 0.01010$$

17.2 (a) We can calculate the value of this temporary annuity-due as:

$$\ddot{a}^{(12)}_{50:\overline{15|}} = \ddot{a}_{50:\overline{15|}} - \frac{11}{24}\left(1 - \frac{D_{65}}{D_{50}}\right) = 11.253 - \frac{11}{24}\left(1 - \frac{689.23}{1{,}366.61}\right) = 11.026$$

(b) We can calculate the value of this guaranteed annuity-due as:

$$\ddot{a}^{(12)}_{\overline{\overline{50:15|}}} = \ddot{a}^{(12)}_{\overline{15|}} + \frac{D_{65}}{D_{50}}\ddot{a}^{(12)}_{65}$$

$$= \frac{1-v^{15}}{d^{(12)}} + \frac{D_{65}}{D_{50}}\left(\ddot{a}_{65} - \frac{11}{24}\right)$$

$$= \frac{1-1.04^{-15}}{0.039157} + \frac{689.23}{1{,}366.61} \times \left(12.276 - \frac{11}{24}\right)$$

$$= 17.318$$

17.3 The EPV of a payment of £2,000 made immediately on the death of a select life now aged 60 is:

$$2{,}000\,\bar{A}_{[60]}$$

So the EPV of a payment of £2,000 made 6 months after the death of this life is:

$$2{,}000\,v^{0.5}\,\bar{A}_{[60]} \approx 2{,}000\,v^{0.5}(1+i)^{0.5}A_{[60]} = 2{,}000A_{[60]} = 2{,}000 \times 0.32533 = £650.66$$

17.4 (i) ***Calculations***

The endowment assurance is calculated as:

$$\bar{A}_{30:\overline{25|}} = \bar{A}^{\;1}_{30:\overline{25|}} + A_{30:\overline{25|}}^{\;\;\;\;\;1} \approx 1.06^{0.5}A^{\;1}_{30:\overline{25|}} + v^{25}\,{}_{25}p_{30}$$

Remember that only the death benefit is accelerated.

Now:

$$v^{25}\,{}_{25}p_{30} = v^{25}\frac{l_{55}}{l_{30}} = 1.06^{-25} \times \frac{9,557.8179}{9,925.2094} = 0.22437$$

and:

$$A^{\;1}_{30:\overline{25|}} = A_{30} - v^{25}\,{}_{25}p_{30}\,A_{55} = 0.07328 - 0.22437 \times 0.26092 = 0.01474$$

So:

$$\bar{A}_{30:\overline{25|}} \approx 1.06^{0.5} \times 0.01474 + 0.22437 = 0.23955$$

The temporary annuity is calculated as:

$$\bar{a}_{30:\overline{25|}} = \bar{a}_{30} - v^{25}\,{}_{25}p_{30}\,\bar{a}_{55} \approx (16.372 - 0.5) - 0.22437 \times (13.057 - 0.5) = 13.055$$

(ii) ***Verification of premium conversion formula***

We would expect the following premium conversion relationship to hold:

$$\bar{A}_{30:\overline{25|}} = 1 - \delta\bar{a}_{30:\overline{25|}}$$

In part (i), we calculated $\bar{A}_{30:\overline{25|}}$ to be 0.23955. Calculating this using the right-hand side of the premium conversion relationship gives:

$$1 - \ln 1.06 \times 13.055 = 0.23933$$

So, the answers are approximately equal.

The slight discrepancy is due to the different assumptions underlying the two calculations in part (i).

17.5 The function $g(K)$ corresponds to a benefit that is deferred for 10 years, and then makes payments annually in advance during the remaining lifetime of a life now aged 40, up to a maximum of 25 payments, *ie* it is a deferred temporary annuity-due.

In terms of actuarial symbols, the expected present value is:

$$E[g(K)] = \frac{D_{50}}{D_{40}}\ddot{a}_{50:\overline{25|}} = \frac{D_{50}}{D_{40}}\left(\ddot{a}_{50} - \frac{D_{75}}{D_{50}}\ddot{a}_{75}\right)$$

So:

$$E[g(K)] = \frac{1,366.61}{2,052.96}\left(17.444 - \frac{363.11}{1,366.61} \times 8.524\right) = 10.104$$

17.6 (i) The present value of the annuity is $g(T_x)$, where:

$$g(T_x)=\begin{cases}100\,\bar{a}_{\overline{T_x}|} & \text{if } T_x<5\\ 150\,\bar{a}_{\overline{T_x}|}-50\,\bar{a}_{\overline{5}|} & \text{if } 5\le T_x<10\\ 150\,\bar{a}_{\overline{10}|}-50\,\bar{a}_{\overline{5}|} & \text{if } T_x\ge 10\end{cases}$$

[Total 3]

This could also be written as:

$$g(T_x)=\begin{cases}100\,\bar{a}_{\overline{T_x}|} & \text{if } T_x<5\\ 100\,\bar{a}_{\overline{5}|}+150v^5\,\bar{a}_{\overline{T_x-5}|} & \text{if } 5\le T_x<10\\ 100\,\bar{a}_{\overline{5}|}+150v^5\,\bar{a}_{\overline{5}|} & \text{if } T_x\ge 10\end{cases}$$

or as:

$$g(T_x)=\int_0^{\min\{T_x,10\}}\rho(s)v^s\,ds \text{ where } \rho(s)=\begin{cases}100 & \text{for } 0\le s<5\\ 150 & \text{for } 5\le s<10\end{cases}$$

(ii) In terms of actuarial notation, the expected present value of the annuity is:

$$100\bar{a}_{x:\overline{10}|}+50\,{}_{5|}\bar{a}_{x:\overline{5}|}=100\bar{a}_{x:\overline{10}|}+50v^5\,{}_5p_x\,\bar{a}_{x+5:\overline{5}|} \quad [1]$$

We could alternatively express the EPV of the annuity as:

$$100\bar{a}_{x:\overline{5}|}+150\,{}_{5|}\bar{a}_{x:\overline{5}|}=100\bar{a}_{x:\overline{5}|}+150v^5\,{}_5p_x\,\bar{a}_{x+5:\overline{5}|}$$

Since we have a constant force of interest of 0.04 *pa*:

$$v^t=e^{-\delta t}=e^{-0.04t} \quad [½]$$

and since we have a constant force of mortality of 0.01 *pa*:

$${}_tp_x=e^{-\mu t}=e^{-0.01t} \quad [½]$$

So:

$$\bar{a}_{x:\overline{10}|}=\int_0^{10}v^t\,{}_tp_x\,dt=\int_0^{10}e^{-0.05t}\,dt=\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{10}=\frac{1}{0.05}\left(1-e^{-0.5}\right) \quad [1]$$

and:

$$\bar{a}_{x+5:\overline{5}|}=\int_0^{5}v^t\,{}_tp_{x+5}\,dt=\int_0^{5}e^{-0.05t}\,dt=\left[\frac{e^{-0.05t}}{-0.05}\right]_0^{5}=\frac{1}{0.05}\left(1-e^{-0.25}\right) \quad [1]$$

Therefore, the EPV of the annuity is:

$$100\bar{a}_{x:\overline{10}|} + 50v^5 \, {}_5p_x \, \bar{a}_{x+5:\overline{5}|}$$

$$= 100 \times \frac{1}{0.05}\left(1 - e^{-0.5}\right) + 50e^{-0.04\times 5}e^{-0.01\times 5} \times \frac{1}{0.05}\left(1 - e^{-0.25}\right)$$

$$= 959.21$$

[1]

[Total 5]

17.7 ***Standard life***

The expected present value of the benefit is:

$$10{,}000A^{1}_{40:\overline{2}|} = 10{,}000\left(A_{40} - \frac{D_{42}}{D_{40}}A_{42}\right) = 10{,}000\left(0.23056 - \frac{1{,}894.37}{2{,}052.96}\times 0.24787\right) = £18.38$$

Impaired life

From first principles, we can write the expected present value of the term assurance as:

$$10{,}000\left(vq^*_{40} + v^2 p^*_{40} q^*_{41}\right)$$

where * denotes impaired mortality. Now using the formula for ${}_tp_x$ from page 32 of the *Tables*:

$$p^*_{40} = \exp\left(-\int_0^1 \mu^*_{40+t}\,dt\right) = \exp\left(-5\int_0^1 \mu_{40+t}\,dt\right) = \left(\exp\left(-\int_0^1 \mu_{40+t}\,dt\right)\right)^5 = (p_{40})^5$$

using $e^{AB} = \left(e^A\right)^B$. Similarly:

$$q^*_{41} = 1 - p^*_{41} = 1 - (p_{41})^5$$

Now:

$$p_{40} = 1 - q_{40} = 1 - 0.000937 = 0.999063$$

and: $p_{41} = 1 - q_{41} = 1 - 0.001014 = 0.998986$

So the expected present value of the benefit payable to the impaired life is:

$$10{,}000\left(v\left[1-(p_{40})^5\right] + v^2 (p_{40})^5\left[1-(p_{41})^5\right]\right)$$

$$= 10{,}000\left(\frac{1}{1.04}\times\left(1 - 0.999063^5\right) + \frac{1}{1.04^2}\times\left(0.999063^5\right)\left(1 - 0.998986^5\right)\right)$$

$$= £91.53$$

17.8 We can write:

$$\bar{A}_{50:\overline{2}|} = \bar{A}^{\,1}_{50:\overline{2}|} + A_{50:\overline{2}|}^{\;\;\;\;\;1}$$

$$= \int_0^2 v^t \, {}_tp_{50}\mu_{50+t}\,dt + v^2 \, {}_2p_{50}$$

$$= \int_0^1 v^t \, {}_tp_{50}\mu_{\overline{50}}\,dt + vp_{50}\int_0^1 v^t \, {}_tp_{51}\mu_{\overline{51}}\,dt + v^2 \, {}_2p_{50} \quad [2]$$

where we have written $\mu_{\bar{x}}$ to indicate the assumed constant force of mortality operating between integer ages x and $x+1$.

Now, since $p_x = e^{-\mu_{\bar{x}}}$:

$$\mu_{\overline{50}} = -\ln p_{50} = -\ln(1-q_{50}) = -\ln 0.997492 = 0.00251115 \quad [½]$$

$$\mu_{\overline{51}} = -\ln p_{51} = -\ln(1-q_{51}) = -\ln 0.997191 = 0.00281295 \quad [½]$$

So, using $v^t = e^{-\delta t}$ where $\delta = \ln 1.04$, and ${}_tp_x = e^{-\mu_{\bar{x}}t}$:

$$\int_0^1 v^t \, {}_tp_{50}\mu_{\overline{50}}\,dt = \mu_{\overline{50}}\int_0^1 e^{-(\delta+\mu_{\overline{50}})t}\,dt = \frac{\mu_{\overline{50}}}{\delta+\mu_{\overline{50}}}\left(1-e^{-(\delta+\mu_{\overline{50}})}\right) = 0.00245947 \quad [1]$$

$$\int_0^1 v^t \, {}_tp_{51}\mu_{\overline{51}}\,dt = \frac{\mu_{\overline{51}}}{\delta+\mu_{\overline{51}}}\left(1-e^{-(\delta+\mu_{\overline{51}})}\right) = 0.00275465 \quad [1]$$

So:

$$\bar{A}_{50:\overline{2}|} = 0.00245947 + \frac{1}{1.04}\times 0.997492 \times 0.00275465 + \frac{1}{1.04^2}\times 0.997492 \times 0.997191$$

$$= 0.924748 \quad [1]$$

[Total 6]

17.9 (i) ***Longevity comparison***

The mortality of a life aged $[x]+1$ is equal to that of a life aged $x+1$ because the table has a one-year select period. Thus:

$$E\left[K_{[x]+1}\right] = E\left[K_{x+1}\right] \quad \text{[Total 1]}$$

(ii) ***Calculation***

Since the select period is one year, we can write:

$$\bar{A}_{[45]:\overline{20}|} = \bar{A}^{\,1}_{[45]:\overline{1}|} + {}_{1|}\bar{A}_{[45]:\overline{19}|} = \bar{A}^{\,1}_{[45]:\overline{1}|} + v\,p_{[45]}\,\bar{A}_{46:\overline{19}|} \quad [2]$$

Now, assuming deaths occur on average halfway between age 45 and age 46:

$$\bar{A}^{\;1}_{[45]:\overline{1}|} = v^{0.5} q_{[45]} = v^{0.5} \times 0.9 \times q_{45} = 1.04^{-0.5} \times 0.9 \times 0.00266 = 0.0023475 \qquad [1]$$

Also:

$$v\,p_{[45]} = v\left(1 - q_{[45]}\right) = 1.04^{-1} \times \left(1 - 0.9 \times 0.00266\right) = 0.959236 \qquad [1]$$

and, using a premium conversion relationship with standard ELT15 (Males) mortality (as we are now outside the one-year select period):

$$\bar{A}_{46:\overline{19}|} = 1 - \delta \bar{a}_{46:\overline{19}|} = 1 - \ln(1.04) \times 12.740 = 0.500328 \quad (\textit{Tables}\text{, page 136}) \qquad [1]$$

This gives:

$$\bar{A}_{[45]:\overline{20}|} = 0.0023475 + 0.959236 \times 0.500328 = 0.48228 \qquad [1]$$

[Total 6]

18

Variable benefits and conventional with-profits policies

Syllabus objectives

4.1 Define various assurance and annuity contracts.

4.1.2 Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses. Describe the benefits payable under the above assurance-type contracts.

Syllabus objectives continued

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rates.

4.2.7 Obtain expressions in the form of sums/integrals for the mean of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- contingent benefits (increasing or decreasing) are payable at the middle or end of the year of contingent event or continuously
- annuities are paid in advance, in arrears or continuously, and the amount increases or decreases by a constant monetary amount or by a fixed or time-dependent variable rate.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

0 Introduction

In this chapter we look at policies for which the benefit amount varies over time, including conventional with-profits contracts. In particular, we examine how to calculate the expected present value of the benefits when they increase by a constant amount or constant percentage each year.

1 Variable payments

The simple contracts discussed so far have had constant payments of the form:

- **a benefit of 1 payable on death; or**
- **a benefit of 1 payable, at some frequency, on survival to the date of payment.**

We consider now the possibility that the payment amount varies.

In the case of a benefit payable on death, let the payment be Y_x if death occurs in the year of age $(x, x+1)$.

Assume first that the benefit is payable at the end of the year of death. The EPV of this death benefit, to a life currently aged x, will be given by, using life table notation:

$$Y_x v \frac{d_x}{l_x} + Y_{x+1} v^2 \frac{d_{x+1}}{l_x} + \cdots + Y_{x+t} v^{t+1} \frac{d_{x+t}}{l_x} + \cdots$$

This can also be written using summations as:

$$\sum_{j=0}^{\infty} Y_{x+j} v^{j+1} \frac{d_{x+j}}{l_x} = \sum_{j=0}^{\infty} Y_{x+j} v^{j+1} \ {}_j p_x \, q_{x+j}$$

Equivalent integral expressions apply if the benefit is payable immediately on death.

In this case the expected present value of the benefit would be:

$$\int_0^1 Y_x v^t \ {}_t p_x \, \mu_{x+t} \, dt + \int_1^2 Y_{x+1} v^t \ {}_t p_x \, \mu_{x+t} \, dt + \int_2^3 Y_{x+2} v^t \ {}_t p_x \, \mu_{x+t} \, dt + \cdots$$

$$= \sum_{j=0}^{\infty} Y_{x+j} \int_j^{j+1} v^t \ {}_t p_x \, \mu_{x+t} \, dt$$

The above expression may be evaluated directly, and this would be the usual approach when computer power is available. This direct evaluation would be the only approach when no set pattern to the Y_x is imposed.

In this chapter we will describe ways of evaluating the EPV when Y_x is:

- **constant – this evaluation has already been dealt with in an earlier chapter, and the function of interest is A_x**
- **changing by a constant compound rate – we will use A values at an adjusted interest rate**
- **changing by a constant monetary amount – we will use tabulated factors which increase by 1 per annum.**

Corresponding to the assurance evaluation above, we will discuss a similar approach to evaluating annuity benefits where the variation follows one of the patterns just described.

In general, the EPV of an annuity of amount F_{x+t} payable on survival to age $x+t$ to a life currently aged x, assuming, for example, immediate annual payments in arrears, would be evaluated directly from:

$$F_{x+1} v \frac{l_{x+1}}{l_x} + F_{x+2} v^2 \frac{l_{x+2}}{l_x} + \ldots + F_{x+t} v^t \frac{l_{x+t}}{l_x} + \ldots$$

Having evaluated the appropriate assurance and annuity factors, the equivalence principle may then be used to calculate the required premiums and reserves.

Premiums and reserves are covered in a later chapter.

The notation in this chapter for age x can be taken to mean ultimate mortality is being assumed. The algebra and definitions are identical if we assume select mortality, in which case x will be replaced with $[x]$.

2 Payments varying at a constant compound rate

Consider first a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,\ldots$.

The (random) present value of these benefits is:

$$(1+b)^{K_x} v^{K_x+1}$$

This is because K_x is the (integer) policy duration at the start of the year in which death occurs.

Then the EPV of these benefits is:

$$\sum_{k=0}^{\infty} (1+b)^k v^{k+1} {}_{k|}q_x = \frac{1}{1+b} A_x^j$$

where the assurance function is determined on the normal mortality basis but using an interest rate, j, where:

$$j = \frac{(1+i)}{(1+b)} - 1$$

A similar approach may be derived for other types of assurance.

Where b is negative this approach may be used to allow for compound-decreasing benefits.

Question

Calculate the expected present value of a whole life assurance taken out by a life aged 50, where:

- the basic sum assured is £100,000
- the sum assured increases by 1.9231% at the start of each year excluding the first
- the benefits are payable at the end of the year of death.

Assume AM92 Ultimate mortality and 6% *pa* interest.

Solution

The expected present value of the benefit is:

$$100{,}000\left(vq_{50}+1.019231v^2\,{}_{1|}q_{50}+1.019231^2v^3\,{}_{2|}q_{50}+\cdots\right)$$

$$=\frac{100{,}000}{1.019231}\left(\frac{1.019231}{1.06}q_{50}+\frac{1.019231^2}{1.06^2}\,{}_{1|}q_{50}+\frac{1.019231^3}{1.06^3}\,{}_{2|}q_{50}+\cdots\right)$$

$$=\frac{100{,}000}{1.019231}A_{50}\ @4\%$$

$$=\frac{100{,}000}{1.019231}\times 0.32907$$

$$=£32{,}286$$

You will often see increases of 1.9231% and $i=6\%$ in questions because it means we end up evaluating the benefit at 4% interest. However, do take care with questions like these, as you will often have to pull a factor out of the EPV to obtain a standard assurance function. (In the example above, we took out 1.019231^{-1}.)

To consider the evaluation of compound-varying survival benefits, consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k=1,2,\ldots$.

The (random) present value of the annuity is:

$$\sum_{k=1}^{K_x}(1+c)^k v^k$$

This is because the life survives to every (integer) policy duration up to and including duration K_x.

Then the EPV of the annuity is:

$$\sum_{k=1}^{\infty}(1+c)^k v^k\,{}_kp_x = a_x^j$$

where the annuity function a_x^j is determined on the normal mortality basis but using an interest rate j, where $j=\dfrac{1+i}{1+c}-1$.

Where c is negative this approach may be used to allow for compound-decreasing annuities.

3 Payments varying by a constant monetary amount

3.1 Whole life assurance

Consider, for example, a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $(x+k, x+k+1)$, $k = 0,1,\ldots$.

The random present value is:

$$(K_x+1)v^{K_x+1}$$

The EPV of this assurance benefit is then:

$$\sum_{k=0}^{\infty}(k+1)v^{k+1}\,{}_{k|}q_x$$

which is given the actuarial symbol $(IA)_x$.

Values of this function are tabulated in the AM92 examination *Tables*.

Where tabulations are available, this provides the quickest way of evaluating such functions. The values are also easy to calculate using a computer.

It is not logical to define a whole life assurance with constant decreases.

If we did, the benefits would eventually become negative if the person lived long enough.

3.2 Term assurance

An increasing temporary assurance, with term n years can now be evaluated using the formula:

$$(IA)^1_{x:\overline{n}|} = (IA)_x - v^n \frac{l_{x+n}}{l_x}\left[(IA)_{x+n} + n\,A_{x+n}\right]$$

$(IA)^1_{x:\overline{n}|}$ is the EPV of payments of $1,2,\ldots,n$ occurring on death in years $1,2,\ldots,n$. In this formula, $(IA)_x$ calculates the EPV of death benefits of $1,2,\ldots,n,n+1,n+2,\ldots$ paid in years $1,2,\ldots,n,n+1,n+2,\ldots$. So we need to deduct the value of death benefits of $n+1,n+2,\ldots$ paid in years $n+1,n+2,\ldots$.

By deducting $v^n \frac{l_{x+n}}{l_x}(IA)_{x+n}$ we deduct the EPV of death benefits of $1,2,\ldots$ paid in years $n+1,n+2,\ldots$. The EPV of the remaining death benefits of $n,n,\ldots$ paid in years $n+1,n+2,\ldots$ are deducted using $v^n \frac{l_{x+n}}{l_x} n A_{x+n}$.

Question

A man aged exactly 40 buys a special 25-year endowment assurance policy that pays £30,000 on maturity. The man pays a premium of £670 at the start of each year throughout the 25 years, or until death if that happens first. Should that happen, all premiums paid so far are returned without interest at the end of the year of death.

Calculate the expected present value of the benefits payable under this policy assuming AM92 Select mortality and 4% *pa* interest.

Solution

The expected present value of the maturity benefit is:

$$\text{EPV} = 30{,}000 \times \frac{D_{65}}{D_{[40]}} = 30{,}000 \times \frac{689.23}{2{,}052.54} = 10{,}073.81$$

The expected present value of the death benefit is:

$$\text{EPV} = 670(IA)^{\ 1}_{[40]:\overline{25|}} = 670 \times \left[(IA)_{[40]} - \frac{D_{65}}{D_{[40]}} \left[(IA)_{65} + 25A_{65} \right] \right]$$

$$= 670 \times \left[7.95835 - \frac{689.23}{2{,}052.54} (7.89442 + 25 \times 0.52786) \right]$$

$$= 587.02$$

So the total EPV is:

$$10{,}073.81 + 587.02 = 10{,}660.83$$

3.3 Endowment assurance

An increasing endowment assurance, with term n years, can be defined.

An example is $(IA)_{x:\overline{n|}}$, which is the EPV of a payment of k paid at the end of the year of death of (x) if the life dies in policy year k $(k = 1, 2, ..., n)$, or a payment of n if (x) is still alive at the end of the term. The EPV of this can be evaluated using:

$$(IA)_{x:\overline{n|}} = (IA)^{1}_{x:\overline{n|}} + n\,A_{x:\overline{n|}}^{\ \ 1}$$

3.4 Decreasing term assurance

A decreasing temporary assurance with a term of b_t years can also be defined. For example, suppose the benefit is n in the first year, and decreases by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$(n+1)A^{1}_{x:\overline{n|}} - (IA)^{1}_{x:\overline{n|}}$$

3.5 Increasing assurances payable immediately on death

Increasing assurances payable immediately on death can also be defined. For example, $(I\bar{A})_x$ is the expected present value of a payment of $k+1$ paid immediately on death occurring in the year of age $(x+k, x+k+1), k=0,1,\ldots$. It can be calculated using the usual approximations, for example:

$$(I\bar{A})_x \approx (1+i)^{1/2}\,(IA)_x$$

The formulae in Sections 3.2, 3.3 and 3.4 can be adjusted in a similar way to allow for immediate payment of the benefit on death.

Question

Calculate the value of $(I\bar{A})_{50:\overline{10|}}$ assuming AM92 Ultimate mortality and 4% *pa* interest.

Solution

We want:

$$(I\bar{A})_{50:\overline{10|}} = (I\bar{A})^{\ 1}_{50:\overline{10|}} + 10A_{50:\overset{\ \ 1}{\overline{10|}}}$$

This is the EPV of:

- k paid immediately on death if the life dies in policy year k $(k \le 10)$
- 10 on surviving to the end of 10 years.

The acceleration of the payment only applies to the death benefit, which is why we need to split the payment into the death benefit and survival benefit components.

So:

$$\begin{aligned}(I\bar{A})_{50:\overline{10|}} &\approx (1+i)^{1/2}(IA)^{\ 1}_{50:\overline{10|}} + 10A_{50:\overset{\ \ 1}{\overline{10|}}}\\ &= (1+i)^{1/2}\left\{(IA)_{50} - \frac{D_{60}}{D_{50}}\left[(IA)_{60} + 10\,A_{60}\right]\right\} + 10\frac{D_{60}}{D_{50}}\\ &= 1.04^{1/2} \times \left\{8.55929 - \frac{882.85}{1{,}366.61} \times \left[8.36234 + 10 \times 0.45640\right]\right\} + 10 \times \frac{882.85}{1{,}366.61}\\ &= 6.673\end{aligned}$$

3.6 Whole life annuity payable annually in arrears

In the case of an annuity that increases by a constant amount each year consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x+k$ being k, $k=1,2,\ldots$.

The (random) present value is:

$$\sum_{k=1}^{K_x} kv^k$$

The EPV of this annuity benefit is:

$$\sum_{k=1}^{\infty} k\,v^k\,{}_kp_x$$

which is given the actuarial symbol $(Ia)_x$.

3.7 Whole life annuity payable annually in advance

Similarly we can define the actuarial symbol:

$$(I\ddot{a})_x$$

to represent the EPV of an annuity-due with the first payment being 1 and subsequent payments increasing by 1 per annum.

Values of this function are tabulated in AM92 in the 'Formulae and Tables for Examinations', although again they can be easily calculated using a computer.

It is not logical to define an immediate annuity with constant decreases.

Question

Calculate the value of $(Ia)_{50}$ assuming AM92 mortality and 4% *pa* interest.

Solution

We want:

$$(Ia)_{50} = vp_{50} + 2v^2\,{}_2p_{50} + 3v^3\,{}_3p_{50} + \ldots$$

But we have:

$$(I\ddot{a})_{50} = 1 + 2vp_{50} + 3v^2\,{}_2p_{50} + \ldots$$

So:

$$\begin{aligned}(I\ddot{a})_{50} - (Ia)_{50} &= 1 + vp_{50} + v^2\,{}_2p_{50} + \ldots \\ &= \ddot{a}_{50}\end{aligned}$$

Therefore:

$$(Ia)_{50} = (I\ddot{a})_{50} - \ddot{a}_{50} = 231.007 - 17.444 = 213.563$$

3.8 Temporary annuities

Increasing temporary annuities can now be evaluated. For example, an increasing temporary annuity-due has an EPV given by:

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right]$$

A decreasing temporary annuity with a term of n years can also be defined. For example, suppose we have an annuity-due with a payment of n in the first year, and decreasing by 1 per subsequent year. Then the EPV can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

3.9 Annuities payable continuously

Increasing annuities payable continuously can also be defined. For example, $(I\bar{a})_x$ is the EPV of an immediate annuity payable continuously, with the (level) benefit payable over the year of age $(x+k,\ x+k+1)$ being $1+k,\ k=0,1,\ldots$. The approximate calculation of this is:

$$(I\bar{a})_x \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

The formulae in Section 3.8 can be adjusted in a similar way to allow for continuous annuity payments.

Question

Obtain an approximate expression for $(I\bar{a})_{x:\overline{n}|}$ in terms of discrete ($\ddot{a}$) annuity functions.

Solution

First we express the annuity as the difference between whole life annuities:

$$(I\bar{a})_{x:\overline{n}|} = (I\bar{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\bar{a})_{x+n} + n\bar{a}_{x+n}\right]$$

Next we put in the relevant approximations for the continuous annuities:

$$(I\bar{a})_{x:\overline{n}|} \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} - \tfrac{1}{2}\ddot{a}_{x+n} + n\left(\ddot{a}_{x+n} - \tfrac{1}{2}\right)\right]$$

Bringing together related terms:

$$(I\bar{a})_{x:\overline{n}|} \approx (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right] - \tfrac{1}{2}\left[\ddot{a}_x - v^n \frac{l_{x+n}}{l_x}\ddot{a}_{x+n}\right] + \tfrac{1}{2} n v^n \frac{l_{x+n}}{l_x}$$

$$= (I\ddot{a})_{x:\overline{n}|} - \tfrac{1}{2}\ddot{a}_{x:\overline{n}|} + \tfrac{1}{2} n v^n \frac{l_{x+n}}{l_x}$$

Question

Calculate the value of $(I\bar{a})_x$ assuming that (x) is subject to a constant force of mortality of 0.02 *pa* and that $\delta = 0.04$ *pa*.

Solution

$(I\bar{a})_x$ is the expected present value of a whole life annuity payable continuously to a life now aged exactly x, where the rate of payment is 1 *pa* in the first year, 2 *pa* in the second year, 3 *pa* in the third year, and so on. So:

$$(I\bar{a})_x = \int_0^1 v^t \,{}_tp_x \, dt + \int_1^2 2v^t \,{}_tp_x \, dt + \int_2^3 3v^t \,{}_tp_x \, dt + \cdots$$

Assuming a constant force of mortality of 0.02 *pa*, we have ${}_tp_x = e^{-0.02t}$.

Also since $v^t = e^{-\delta t} = e^{-0.04t}$, it follows that:

$$\begin{aligned}
(I\bar{a})_x &= \int_0^1 e^{-0.06t} \, dt + \int_1^2 2e^{-0.06t} \, dt + \int_2^3 3e^{-0.06t} \, dt + \cdots \\
&= \left[-\frac{1}{0.06}e^{-0.06t}\right]_0^1 + \left[-\frac{2}{0.06}e^{-0.06t}\right]_1^2 + \left[-\frac{3}{0.06}e^{-0.06t}\right]_2^3 + \cdots \\
&= \frac{1}{0.06}\left[1 - e^{-0.06}\right] + \frac{2}{0.06}\left[e^{-0.06} - e^{-0.06\times 2}\right] \\
&\qquad + \frac{3}{0.06}\left[e^{-0.06\times 2} - e^{-0.06\times 3}\right] + \cdots \\
&= \frac{1}{0.06}\left[1 - e^{-0.06} + 2e^{-0.06} - 2e^{-0.06\times 2} + 3e^{-0.06\times 2} - 3e^{-0.06\times 3} + \cdots\right] \\
&= \frac{1}{0.06}\left[1 + e^{-0.06} + e^{-0.06\times 2} + e^{-0.06\times 3} + \cdots\right]
\end{aligned}$$

The expression in brackets in the line above is the sum to infinity of the terms in a geometric progression with first term 1 and common ratio $e^{-0.06}$. So we have:

$$(I\bar{a})_x = \frac{1}{0.06}\left[\frac{1}{1 - e^{-0.06}}\right] = 286.19$$

Assurances and annuities that increase (or decrease) continuously at constant monetary rates are not covered by the CM1 syllabus.

4 Conventional with-profits contracts

In this section we see how conventional with-profits business operates. These ideas will be extended in the life insurance course, SP2.

A conventional whole life or endowment policy can be issued on a without-profit or a with-profits basis. On a without-profit basis, both the premiums and benefits under the policy are usually fixed and guaranteed at the date of issue.

When the Core Reading says that the benefits are fixed, it does not mean they are constant. The benefit amounts may be different from one year to the next, but the amount of the benefit in each year is known from the outset. All the contracts with increasing or decreasing benefits described in Sections 2 and 3 , for example, are examples of conventional without-profits contracts.

On a with-profits basis the premiums and/or the benefits can be varied to give an additional benefit to the policyholder in respect of any emerging surplus of assets over liabilities following a valuation. For example, surplus might be used to reduce the premium payable for the same benefit or to increase the sum assured without any additional premium becoming payable.

An alternative way of distributing surplus is to make a cash payment to policyholders.

Where surplus is distributed so as to increase benefits, additions to the sum assured are called bonuses.

At first glance it might seem undesirable from the perspective of both the policyholder and the life insurance company to have contracts whose final benefit is very uncertain, so this begs the question as to why with-profits business has arisen. To understand this, consider a life insurance company deciding on the premium basis for an endowment assurance contract. (The 'premium basis' is just the set of assumptions – such as interest and mortality – that is used to calculate the premium charged.) The future interest rate assumed will be a critical parameter. If the company calculates premiums assuming an interest rate of, say, 2% over the next twenty years, but actual interest rates fall below that level, then the company will almost certainly make a loss on those policies. On the other hand, if interest rates over the term are significantly in excess of 2%, then the policyholders will feel hard done by in comparison with the benefits that they might have received from other mediums such as bank savings.

So it makes sense for both the company and the policyholder to assume a low rate of interest in determining the premiums that are required to meet the initial sum assured, and then to distribute bonuses to policyholders when investment returns exceed those assumed in the premium basis.

So far we have discussed only interest. However, the same principles also apply to mortality and expenses; under with-profits business, the company can make slightly pessimistic assumptions about mortality and future expenses when setting premiums, and then pay bonuses to policyholders when mortality and expense experience prove better than that assumed in the premium basis.

The major types of contract for which a with-profits treatment is suitable are:

- endowment assurance
- whole life assurance
- deferred annuity
- immediate annuity.

With-profits contracts can be regular or single premium.

One type of with-profits contract is an endowment assurance with bonuses added to the sum assured. So, suppose someone takes out a policy with a sum assured of £10,000. At the end of the first year, the life insurance company declares a bonus of 4% of the sum assured. The sum assured is now increased to £10,400. If there were to be no more bonus declarations, the sum assured would remain at £10,400 until the claim was eventually paid out, either on maturity or on earlier death. On the other hand, there may be other bonuses added in future years, and these would increase further the amount that would be eventually paid.

When the company receives a premium from a policyholder with a with-profits contract, that premium (or part of it) will be invested. Over the course of the year the company expects experience to be slightly better than assumed, because its assumptions – for instance, the investment return it assumed when calculating the premiums – were prudent.

Thus we would normally expect at least one of the following events to have occurred:

- investment returns on assets are greater than assumed
- number of death claims is lower than assumed
- the amount of expenses is lower than assumed.

So by the end of the year, the assets in respect of the contract will usually have grown by more than originally assumed. This is what we mean by the creation of surplus.

Once the company has this surplus, it will want to distribute it to the policyholder as a bonus. If the company wants to distribute all of the surplus there and then to the policyholder, it will calculate an increase to the sum assured such that its EPV is exactly equal to the amount of surplus that has been created.

Question

A life company has calculated that the amount of surplus attributable to a with-profits single premium whole life contract issued nine years ago to a 44-year-old male with guaranteed sum assured of $90,000 is $2,800. Write down an expression from which the bonus amount of sum assured could be calculated from this surplus.

Solution

The bonus amount of sum assured S satisfies the equation:

$$SA_{53} = 2{,}800$$

As we shall see below, the bonus will normally be presented as a percentage increase to the sum assured.

In practice, companies will determine the surplus generated and bonus to be distributed using very broad groupings (*eg* all endowment assurances of term 10 years issued 4 years ago) rather than on a policy-by-policy basis.

Question

Bonuses are usually funded by distributing part of the surplus. Explain why a life insurance company might distribute just part of the surplus rather than all of it.

Solution

If the company is proprietary (*ie* owned by shareholders), then it will want to make a profit on its business. A common procedure both in the UK and in many overseas markets is to give 90% of surplus to policyholders, 10% to shareholders.

In addition, the company may deliberately want to under-distribute surplus now in order:

- to defer eventual surplus distribution (we consider the reasons for this below), and
- to allow smoothing of payouts (the company deliberately pays out less than the surplus generated when investment markets are booming but deliberately pays out more when investment markets are declining). The smoother the bonus payments, the 'safer' the policyholders feel. Ultimately, this can drive up the life insurance company's share price.

4.1 Types of bonus

Various methods of allocating bonuses have been developed, each intended to provide a way of matching the surplus emerging over the duration of the policy. In the past the choice of methods was restricted by the difficulties of completing a valuation quickly and cheaply and by the difficulties of allocating complex bonuses to individual policies. Modern record keeping systems have largely removed these difficulties, and current systems are chosen to match the bonus distribution philosophy conveyed to the policyholders when the policy was issued (in the UK this is called policyholders' reasonable expectations).

By bonus distribution philosophy, we mean matters such as:

- what form bonuses take (of the various forms described below)
- what portion of surplus the company distributes to policyholders
- what degree of smoothing the company operates (*eg* if investment returns in one year are very good, we might not distribute all of the resultant surplus immediately, but instead hold some back to compensate poorer investment returns in future years) and
- to some extent, the broad investment strategy of the company (*eg* the company may aim to pay bonuses that broadly reflect equity market performance).

Clearly what the company says to potential policyholders at the policy sale stage about these aspects will create justifiable expectations, which the company should then try to meet over the lifetime of the policy.

Bonuses are usually allocated annually, which is likely to tie in with the minimum required frequency of valuation for each insurer.

These bonuses are often referred to as *reversionary* bonuses.

When the company declares a reversionary bonus of, say, 3% of sum assured, there is no immediate cash payment to the policyholder. The company is merely promising that the amount of money paid on claim or maturity will be 3% greater than was previously the case. So, cashflows are unchanged until the moment when the policies in question terminate, *ie* when the claims are paid out.

Once added to the sum assured, bonuses become guaranteed benefits, which then need to be reserved for.

Insurance companies are required to hold reserves for each policy sold to ensure that they can meet claim payments when necessary. We will study reserves in more detail later in the course, but for the time being, it is sufficient to know that the higher the benefit promised to the policyholder, the higher the reserve held by the insurance company needs to be.

The implication of this is that the sooner, and greater, the rate at which bonuses are added, the more conservative the insurer is likely to be in choosing its investments.

The last sentence of the above Core Reading is important for insurance companies in practice, as conservative investments generally lead to lower returns in the long run. This will in turn translate into lower payouts for policyholders.

Bonuses can be distributed more slowly, or at a lower rate, which may allow the insurer to choose investments that are more volatile in the short term, but are expected to be more profitable in the long term. A highly effective example of this is choosing to distribute part of the available surplus as a terminal bonus, rather than as an annual bonus. Terminal bonuses are allocated when a policy matures or becomes a claim as a result of the death of the life assured.

This is an example of the insurance company *deferring* the distribution of its surplus to its policyholders, and as the Core Reading states, having a terminal bonus is generally the most effective way of doing this.

Question

Explain why, everything else being equal, a higher amount of profit deferral should lead to a higher expected maturity payment on a with-profits endowment policy.

Solution

A high level of profit deferral (*eg* as a result of paying lower annual bonuses and a higher terminal bonus) will mean that the insurance company has more freedom to invest in volatile investments that have a higher expected return. Should these higher investment returns occur, then this should ultimately translate into higher total benefit payments (which includes the guaranteed sum assured, the annually added reversionary bonuses, and the terminal bonus).

Terminal bonuses are usually allocated as a percentage of the basic sum assured and the bonuses allocated prior to termination. The percentage rate will vary with the term of the policy at the date of payment. Because the policy is being terminated, the terminal bonus rate is usually chosen so as to distribute all the surplus available to the policy, based on asset share.

The asset share of a policy at maturity is essentially the amount of money that the company has accumulated from the policy, net of that policy's share of the company's expenses and of the cost of death benefits that had been covered over the policy term. If the sum assured plus all the declared annual bonuses at the time of maturity is less than this asset share, then the excess can be distributed as terminal bonus.

Typically, bonuses are added by a mixture of annual and terminal components. The annual bonuses will be at variable rates determined from time to time by the insurer based on actual arising surpluses. These bonuses are typically added according to one of the following methods:

- **Simple – the rate of bonus each year is a percentage of the initial (basic) sum assured under the policy. The sum assured will increase linearly over the term of the policy.**

The increase will be linear provided the same (simple) bonus rate is declared each year.

- **Compound – the rate of bonus each year is a percentage of the basic sum assured and the bonuses added in the past. The sum assured increases exponentially over the term of the policy.**

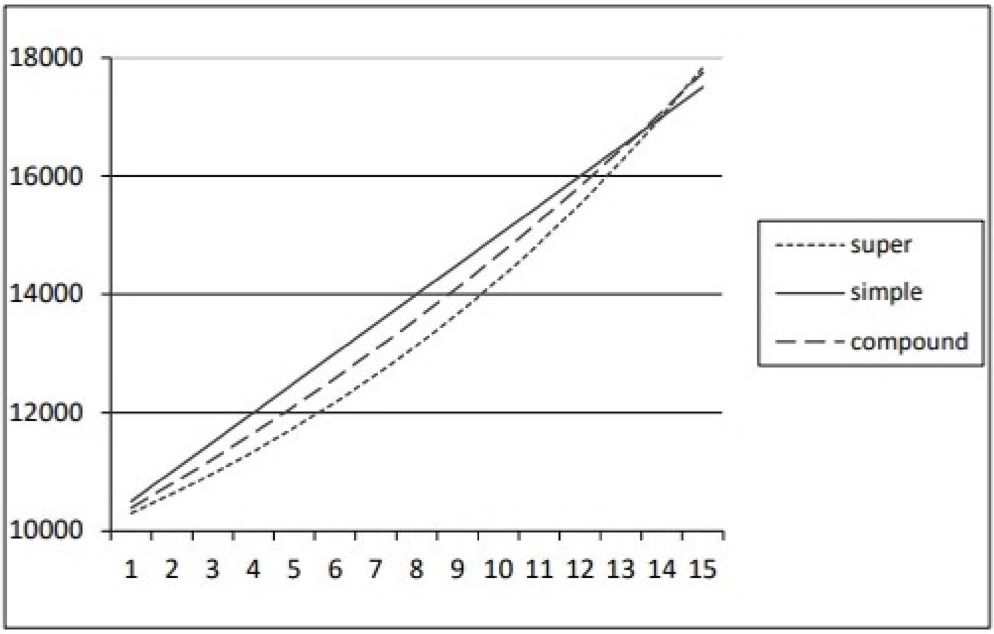
The exponential increase would occur if the same (compound) bonus rate is declared each year.

- **Super compound – two compound bonus rates are declared each year. The first rate (usually the lower) is applied to the basic sum assured. The second rate is applied to the bonuses added to the policy in the past. The sum assured increases exponentially over the term of the policy. The sum assured including bonuses increases more slowly than under a compound allocation in the earlier years, but faster in the later years.**

This pattern of increase would occur if the lower bonus rate was applied to the sum assured.

The graph below shows the build up of the sum assured over the lifetime of a 15-year policy with the following alternative methods, assuming an initial sum assured of £10,000:

- simple bonus at 5% *pa*
- compound bonus at 3.9% *pa*
- super compound bonus at 3% *pa* on basic sum assured, 7.5% *pa* on bonuses.



We see from the graph that:

- the increase in sum assured under a simple bonus arrangement is linear
- under a compound bonus arrangement, the sum assured increases slowly at first and more quickly later in the policy
- under a super compound arrangement, this feature becomes more pronounced, with lower increases earlier on being compensated for by greater increases later on in the policy.

The rates have been chosen so that the final value of the sum assured is approximately the same at the end of 15 years under each of the three different arrangements.

Question

For the three alternative bonus allocation methods specified above, and given an initial guaranteed sum assured of £10,000 in each case, calculate the sum assured (including bonuses) as at the end of Years 1, 2 and 3.

Solution

Simple:	Year 1:	$10,000 \times 1.05 = 10,500$
	Year 2:	$10,000 \times 1.10 = 11,000$
	Year 3:	$10,000 \times 1.15 = 11,500$
Compound:	Year 1:	$10,000 \times 1.039 = 10,390$
	Year 2:	$10,390 \times 1.039 = 10,795$
	Year 3:	$10,795 \times 1.039 = 11,216$
Super-compound:	Year 1:	$10,000 \times 1.03 = 10,300$
	Year 2:	$10,000 \times 1.03 + (10,300 - 10,000) \times 1.075 = 10,623$
	Year 3:	$10,000 \times 1.03 + (10,623 - 10,000) \times 1.075 = 10,969$

Annual bonuses are an allocation in arrears to reflect the growth in the available surplus since the last valuation. Where annual bonuses are given, it is usual at each valuation to declare a rate of interim bonus which will be applied to policies becoming claims before the next valuation. This provides an allocation of bonus for the period from the last valuation to the date of the claim. This rate is applied in the same way as that used for annual bonuses.

An anticipated bonus rate is usually loaded in premium rates by choosing (conservative) rates of bonus allocation and valuing these as benefits in determining the premium to be charged for the policy. The additional premium (above that that would be charged for a without-profit contract with the same (basic) sum assured) is termed the bonus loading.

We saw earlier how with-profits business arose because it allowed companies to price products using conservative interest rate assumptions, which did not hurt the policyholder because as experience proved better than these assumptions, the policyholders would see their benefit increased accordingly. It is now usual to go a step further than this. Companies want to be reasonably sure of being able to pay significant bonuses, which are of similar size to the bonuses which competitors will be paying.

So rather than choose a prudent interest rate (which in reality we expect to exceed by, say 1 to 2% *pa* over the policy's lifetime, giving an equivalent amount of bonus) companies will load explicitly for some specific level of future bonus.

For example, suppose a life insurance company expects an investment return of 4.5% *pa* on the assets backing its with-profits business over the foreseeable future. The company wants to price its contracts to ensure that it can support reversionary bonuses of 2% *pa* (added at the start of each year, using the simple distribution system). For an endowment assurance issued to a life aged x, with a term of n years and a sum assured S payable at the end of the year of death, the premiums will need to meet a benefit with expected present value:

$$S A_{x:\overline{n|}} + 0.02\, S (IA)_{x:\overline{n|}} \quad \text{when } i = 4.5\%$$

Question

Give an expression for the expected present value of the benefits provided by a with-profits endowment assurance issued to a life aged x with a sum assured of S payable in n years' time or at the end of the year of earlier death. Assume that compound reversionary bonuses of 2% *pa* are declared at the start of each year and interest is 4.5% *pa* effective.

Solution

The EPV of the benefits is:

$$EPV = 1.02 S v q_x + 1.02^2 S v^2 \, {}_{1|}q_x + \cdots + 1.02^n S v^n \, {}_{n-1|}q_x + 1.02^n S v^n \, {}_n p_x$$

$$= S\left[\left(\frac{1.02}{1.045}\right) q_x + \left(\frac{1.02}{1.045}\right)^2 {}_{1|}q_x + \cdots + \left(\frac{1.02}{1.045}\right)^n {}_{n-1|}q_x + \left(\frac{1.02}{1.045}\right)^n {}_n p_x\right]$$

$$= S\left[v' q_x + (v')^2 \, {}_{1|}q_x + \cdots + (v')^n \, {}_{n-1|}q_x + (v')^n \, {}_n p_x\right]$$

where $v' = \dfrac{1.02}{1.045} = 1.0245^{-1}$

So:

$$EPV = S\, A^{@i'}_{x:\overline{n}|}$$

where $i' = 2.45\%$ *pa*.

In each of these examples, any excess of investment return actually achieved above the 4.5% *pa* assumed would either be paid out in the form of higher annual bonuses (*ie* greater than 2% *pa*), or as a terminal bonus when the contract terminates.

If the actual return achieved is less than the assumed 4.5% *pa*, then the company could declare annual bonuses averaging less than 2% *pa*, and so avoid making losses on the business.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes

Chapter 18 Summary

Variable benefits

In this chapter we have developed ways of calculating the expected present value of benefits that vary from one year to another according to some predetermined pattern.

Payments varying at a constant compound rate

Assurances

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k=0,1,\dots$, is:

$$\sum_{k=0}^{\infty}(1+b)^k v^{k+1}\,{}_{k|}q_x = \frac{1}{1+b}A_x^j$$

where A_x^j is a whole life assurance function determined using the normal mortality basis, but using an interest rate j such that $j=\dfrac{1+i}{1+b}-1$.

A similar approach can be used for other types of assurance.

Annuities

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being $(1+c)^k$, $k=1,2,\dots$, is:

$$\sum_{k=1}^{\infty}(1+c)^k v^k\,{}_kp_x = a_x^j$$

where the annuity function a_x^j is evaluated using the normal mortality basis, but using an interest rate j, such that $j=\dfrac{1+i}{1+c}-1$.

Payments changing by a constant monetary amount

Whole life assurance

The expected present value of a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k+1$ if death occurs in the year of age $\left(x+k, x+k+1\right)$, $k=0,1,\ldots$, is:

$$\sum_{k=0}^{\infty}(k+1)v^{k+1}\,{}_{k|}q_x=(IA)_x$$

Term assurance

An increasing term assurance can be evaluated using the formula:

$$(IA)^{1}_{x:\overline{n}|}=(IA)_x-v^n\frac{l_{x+n}}{l_x}\left[(IA)_{x+n}+nA_{x+n}\right]$$

The expected present value of a decreasing term assurance with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)A^{1}_{x:\overline{n}|}-(IA)^{1}_{x:\overline{n}|}$$

Endowment assurance

An increasing endowment assurance can be evaluated using the formula:

$$(IA)_{x:\overline{n}|}=(IA)^{1}_{x:\overline{n}|}+nA_{x:\overline{n}|}^{\;\;\,1}=(IA)^{1}_{x:\overline{n}|}+n\frac{D_{x+n}}{D_x}$$

Assurances with death benefits payable immediately

The expected present value for an increasing whole life assurance, which pays immediately on death, is:

$$\left(I\bar{A}\right)_x\approx(1+i)^{1/2}\left(IA\right)_x$$

Similar formulae can be used for the other assurance types.

Whole life annuity

The expected present value of a whole life annuity payable annually in arrears to a life aged x, with the benefit payable on survival to age $x+k$ being k, $k=1,2,\ldots$, is:

$$\sum_{k=1}^{\infty} k v^k \, {}_k p_x = (Ia)_x$$

The symbol $(I\ddot{a})_x$ denotes the expected present value of the corresponding annuity-due, where the first payment is 1 and payments increase at the rate of 1 *pa*, and:

$$(I\ddot{a})_x = (Ia)_x + \ddot{a}_x$$

Temporary annuity

An increasing temporary annuity-due can be evaluated using the formula:

$$(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\ddot{a})_{x+n} + n\ddot{a}_{x+n}\right]$$

The expected present value of a decreasing temporary annuity-due with a term of n years, where the benefit is n in the first year and decreases by 1 *pa*, can be evaluated using the formula:

$$(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$$

Annuities payable continuously

The expected present value of a whole life annuity payable continuously to a life aged x, which pays at the rate of k in year k, $k=1,2,\ldots$, is:

$$(I\bar{a})_x \approx (I\ddot{a})_x - \tfrac{1}{2}\ddot{a}_x$$

For continuously payable temporary annuities, a good approach is to start from:

$$(I\bar{a})_{x:\overline{n}|} = (I\bar{a})_x - v^n \frac{l_{x+n}}{l_x}\left[(I\bar{a})_{x+n} + n\bar{a}_{x+n}\right]$$

and then replace each of the annuity factors in the formula by their discrete approximations.

Conventional with-profits contracts

Conventional with-profits contracts are priced on a prudent basis. If experience proves better than the prudent assumptions, surplus arises. This surplus is distributed to policyholders as an increase to the benefit.

This increase can be applied as an annual reversionary bonus during the lifetime of the policy, using one of the following methods:

- simple
- compound
- super-compound

and also at claim or maturity as a terminal bonus.

The more a company defers the distribution of surplus as bonus, the more potential the company has to invest in more volatile assets with higher expected investment returns, and thus obtain greater eventual payouts to policyholders. Terminal bonus can be viewed as an extreme example of such deferment.

Reversionary bonuses are often allowed for explicitly when determining the premium basis for a contract.

Chapter 18 Practice Questions

18.1 For each of the following expected present values, describe the payments involved and calculate the value assuming AM92 mortality and 6% *pa* interest.

(i) $(I\ddot{a})_{[45]:\overline{20}|}$

(ii) $(I\bar{A})_{49:\overline{6}|}$

(iii) $(Ia)_{[50]:\overline{10}|}$

18.2 Graham, aged 40, purchases a conventional with-profits whole life assurance with sum assured £2,000 plus attaching bonuses, payable at the end of the year of death. Assuming allowance for simple bonuses of 3% *pa*, which are added at the start of each policy year, calculate the expected present value of Graham's policy benefits.

Basis: AM92 Ultimate mortality, 4% *pa* interest.

18.3 A special term assurance policy is to be issued to a life currently aged 52 exact. The policy term is 8 years, and the sum assured is paid at the end of the year of death. The benefit is 120,000 in the first year, increasing by 10,000 at the end of each year, so that if death occurs in the final policy year 190,000 will be paid.

Calculate the EPV of this policy benefit using the following basis:

Mortality: AM92 Ultimate
Interest: 4% *pa*

18.4 The payments under a special deferred annuity are payable continuously from age 60 and increase continuously at the rate of 5% *pa* compound. The payment stream starts at the rate of £200 *pa*. Assuming AM92 Select mortality before age 60 and PFA92C20 mortality after age 60, calculate the expected present value of the annuity for a female life now aged 40, if interest is 5% *pa* effective.

18.5 A life insurance company has issued a decreasing term assurance with a 20-year term to a person aged exactly 40. The sum assured is 100,000 in the first year, decreasing by 1,000 each year, so that the death benefit in the final policy year is 81,000. The benefit is paid at the end of the policy year of death.

Calculate the EPV of this benefit assuming AM92 Select mortality and 4% *pa* interest.

18.6 A 17-year with-profits endowment assurance is issued to a life aged exactly 48, having a basic guaranteed sum assured of 25,000. The sum assured plus all declared reversionary bonuses to date are paid on survival to the end of the term or immediately on earlier death. Calculate the expected present value of this policy benefit assuming:

- future bonuses are declared at the rate of 2.5% *pa* compound, being added in full at the start of each policy year
- AM92 Select mortality
- 6.6% *pa* interest.

18.7 Exam style

A woman aged 67 exact takes out an annuity that makes monthly payments in arrears. The first monthly payment is £1,500, and payments increase by 0.23726% each month.

Calculate the expected present value of the annuity using the following basis:

Mortality: PFA92C20
Interest: 7% per annum [4]

18.8 Exam style

A whole life assurance policy pays 20,000 on death in Year 1, 20,100 on death in Year 2, and so on increasing by 100 each year. The payment is made immediately on death of a life currently aged 35 exact.

(i) Write down an expression for the present value random variable of this payment, in terms of the curtate future lifetime K_x, and/or the complete future lifetime T_x. [1]

(ii) Calculate the expected present value of these benefits, assuming:

(a) AM92 Select mortality and 6% *pa* interest

(b) a constant force of mortality of 0.015 *pa* and force of interest 0.03 *pa*. [11]
[Total 12]

18.9 Exam style

A whole life annuity with continuous payments is due to commence in 15 years' time. It will be payable to a life that is currently aged exactly 50, provided that person is still alive when the annuity is due to start. Payments commence at the rate of 20,000 *pa*, and increase continuously thereafter at a rate of 2% *pa* compound.

Calculate the expected present value of these payments on the following basis:

Mortality: PMA92C20 prior to age 65
A constant force of 0.038 *pa* at ages over 65
Interest: 3% *pa* effective [6]

18.10 Exam style

A life insurance company is considering selling with-profit endowment policies with a term of twenty years and initial sum assured of £100,000. Death benefits are payable at the end of the policy year of death. Bonuses will be added at the end of each policy year.

The company is considering three different bonus structures:

(1) simple reversionary bonuses of 4.5% per annum

(2) compound reversionary bonuses of 3.84615% per annum

(3) super compound bonuses where the original sum assured receives a bonus of 3% each year and all previous bonuses receive an additional bonus of 6% each year.

(i) Calculate the amount payable at maturity under the three structures. [4]

(ii) Calculate the expected present value of benefits under structure (2) for an individual aged 45 exact at the start, using the following basis:

Interest	8% per annum
Mortality	AM92 Select
Expenses	ignore

[4]

(iii) Calculate the expected present value of benefits, using the same policy and basis as in (ii) but reflecting the following changes:

(a) Bonuses are added at the start of each policy year (the death benefit is payable at the end of the policy year of death).

(b) The death benefit is payable immediately on death (bonuses are added at the end of each policy year).

(c) The death benefit is payable immediately on death, and bonuses are added continuously. [3]

[Total 11]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 18 Solutions

18.1 (i) $(I\ddot{a})_{[45]:\overline{20}|}$

This is the EPV of a 20-year increasing temporary annuity-due. Payments are 1 at the start of Year 1, 2 at the start of Year 2, and so on, so that 20 is paid at the start of Year 20. Payments continue for 20 years or until the earlier death of a select life that is currently aged exactly 45.

We have:

$$(I\ddot{a})_{[45]:\overline{20}|} = (I\ddot{a})_{[45]} - v^{20}\frac{l_{65}}{l_{[45]}}\left[(I\ddot{a})_{65} + 20\ddot{a}_{65}\right]$$

$$= 185.197 - 1.06^{-20} \times \frac{8,821.2612}{9,798.0837} \times \left[89.374 + 20 \times 10.569\right]$$

$$= 100.77$$

(ii) $(I\bar{A})_{49:\overline{6}|}$

This is the EPV of a six-year increasing endowment assurance. On death in the first year the payment would be 1, in the second year the payment would be 2, and so on so that on death in Year 6 the payment would be 6. A payment of 6 would alternatively be paid if the person (currently aged 49) survived to the end of the term. The death benefits are payable immediately on death.

We have:

$$(I\bar{A})_{49:\overline{6}|} \approx (1+i)^{½}\left((IA)_{49} - v^{6}\frac{l_{55}}{l_{49}}\left[(IA)_{55} + 6A_{55}\right]\right) + 6v^{6}\frac{l_{55}}{l_{49}}$$

$$= 1.06^{½} \times \left(4.75618 - 1.06^{-6} \times \frac{9,557.8179}{9,733.8865} \times \left[5.22868 + 6 \times 0.26092\right]\right)$$

$$+ 6 \times 1.06^{-6} \times \frac{9,557.8179}{9,733.8865}$$

$$= 4.20800$$

(iii) $(Ia)_{[50]:\overline{10}|}$

This is the EPV of a 10-year increasing temporary annuity. Payments are 1 at the end of Year 1, 2 at the end of Year 2, and so on, so that 10 is paid at the end of Year 10. Payments continue for 10 years or until the earlier death of a select life that is currently aged exactly 50.

We can use the same formula for the annuity in arrears as we did for the annuity-due, provided we remove the 'double-dots' above all the annuity symbols:

$$(Ia)_{[50]:\overline{10}|} = (Ia)_{[50]} - v^{10}\frac{l_{60}}{l_{[50]}}\left[(Ia)_{60} + 10a_{60}\right]$$

Now:

$$(Ia)_x = (I\ddot{a})_x - \ddot{a}_x$$

and

$$a_x = \ddot{a}_x - 1$$

So:

$$(Ia)_{[50]:\overline{10}|} = (I\ddot{a})_{[50]} - \ddot{a}_{[50]} - v^{10}\frac{l_{60}}{l_{[50]}}\left[(I\ddot{a})_{60} - \ddot{a}_{60} + 10\left(\ddot{a}_{60} - 1\right)\right]$$

$$= 162.597 - 14.051 - 1.06^{-10} \times \frac{9,287.2164}{9,706.0977} \times \left[113.516 - 11.891 + 10 \times 10.891\right]$$

$$= 36.058$$

18.2 The expected present value is:

$$2,000A_{40} + 0.03 \times 2,000(IA)_{40} = 2,000 \times 0.23056 + 60 \times 7.95699 = £938.54$$

18.3 The EPV is:

$$110,000A^{\;1}_{52:\overline{8}|} + 10,000(IA)^{\;1}_{52:\overline{8}|}$$

where:

$$A^{\;1}_{52:\overline{8}|} = A_{52:\overline{8}|} - \frac{D_{60}}{D_{52}} = 0.73424 - \frac{882.85}{1,256.80} = 0.031781$$

and:

$$(IA)^{\;1}_{52:\overline{8}|} = (IA)_{52} - \frac{D_{60}}{D_{52}}\left[(IA)_{60} + 8A_{60}\right]$$

$$= 8.59412 - \frac{882.85}{1,256.80} \times \left[8.36234 + 8 \times 0.45640\right]$$

$$= 0.155105$$

So:

$$EPV = 110,000 \times 0.031781 + 10,000 \times 0.155105 = 5,047$$

18.4 The expected present value of the deferred annuity is:

$$v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty 200 \times 1.05^t v^t \,{}_tp_{60}\,dt = 200v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty 1.05^t \times 1.05^{-t} \times {}_tp_{60}\,dt = 200v^{20}\frac{l_{60}}{l_{[40]}}\int_0^\infty {}_tp_{60}\,dt$$

Now, recall from Chapter 14:

$$\int_0^\infty {}_tp_{60}\,dt = \mathring{e}_{60}$$

So the EPV is now:

$$200v^{20}\frac{l_{60}}{l_{[40]}}\mathring{e}_{60} = 1.05^{-20} \times \frac{9,287.2164}{9,854.3036} \times 27.41 = £1,947$$

18.5 The expected present value is:

$$EPV = 101,000A^{\;1}_{[40]:\overline{20|}} - 1,000(IA)^{\;1}_{[40]:\overline{20|}}$$

where:

$$A^{\;1}_{[40]:\overline{20|}} = A_{[40]:\overline{20|}} - \frac{D_{60}}{D_{[40]}} = 0.46423 - \frac{882.85}{2,052.54} = 0.034104$$

and:

$$\begin{aligned}(IA)^{\;1}_{[40]:\overline{20|}} &= (IA)_{[40]} - \frac{D_{60}}{D_{[40]}}\left[(IA)_{60} + 20A_{60}\right] \\ &= 7.95835 - \frac{882.85}{2,052.54} \times \left[8.36234 + 20 \times 0.45640\right] \\ &= 0.435307\end{aligned}$$

So:

$$EPV = 101,000 \times 0.034104 - 1,000 \times 0.435307 = 3,009$$

18.6 We need the expected present values of the survival benefit and of the death benefit.

Survival benefit

The expected present value of the survival benefit is:

$$EPV = 25,000 \times 1.025^{17} v^{17} {}_{17}p_{[48]} = 25,000 \times 1.04^{-17} {}_{17}p_{[48]}$$

as $1.025 \times v = \frac{1.025}{1.066} = \frac{1}{1.04}$.

So:

$$EPV = 25,000 \times \frac{D_{65@4\%}}{D_{[48]@4\%}} = 25,000 \times \frac{689.23}{1,483.73} = 11,613.13$$

Death benefit

The expected present value of the death benefit is:

$$EPV = 25,000 \times \left(1.025v^{½}\,{}_{0|}q_{[48]} + 1.025^2 v^{1½}\,{}_{1|}q_{[48]} + \cdots + 1.025^{17} v^{16½}\,{}_{16|}q_{[48]}\right)$$

It would be helpful if the first term in the bracket began with $1.025v$. We can achieve this by multiplying and dividing by $v^{½}$, so:

$$EPV = \frac{25,000}{v^{½}} \times \left(1.025v \; {}_{0|}q_{[48]} + (1.025v)^2 \; {}_{1|}q_{[48]} + \cdots + (1.025v)^{17} \; {}_{16|}q_{[48]}\right)$$

$$= 25,000 \times 1.066^{½} \times \left(1.04^{-1} \; {}_{0|}q_{[48]} + 1.04^{-2} \; {}_{1|}q_{[48]} + \cdots + 1.04^{-17} \; {}_{16|}q_{[48]}\right)$$

$$= 25,000 \times 1.066^{½} \times A^{\;1}_{[48]:\overline{17|}} @4\%$$

Now:

$$A^{\;1}_{[48]:\overline{17|}} = A_{[48]:\overline{17|}} - \frac{D_{65}}{D_{[48]}} = 0.52596 - \frac{689.23}{1,483.73} = 0.061435$$

So:

$$EPV = 25,000 \times 1.066^{½} \times 0.061435 = 1,585.74$$

Total EPV

Adding the expected present values of the death benefit and survival benefit together we obtain:

$$11,613.13 + 1,585.74 = 13,198.87$$

18.7 First define:

$$f = 0.0023726$$

The EPV of the annuity payments is:

$$EPV = 1,500\left[v^{1/12}\left({}_{1/12}p_{67}\right) + (1+f)v^{2/12}\left({}_{2/12}p_{67}\right) + (1+f)^2 v^{3/12} \; {}_{3/12}p_{67} + \cdots\right]$$

$$= \frac{1,500}{1+f}\left[(1+f)v^{1/12}\left({}_{1/12}p_{67}\right) + (1+f)^2 v^{2/12}\left({}_{2/12}p_{67}\right) + (1+f)^3 v^{3/12} \; {}_{3/12}p_{67} + \cdots\right] \qquad [1]$$

Now:

$$(1+f)v^{1/12} = \frac{1.0023726}{1.07^{1/12}} \Rightarrow (1+f)^{12} v = \frac{1.0023726^{12}}{1.07} = \frac{1.0288457}{1.07} = \frac{1}{1.04} \qquad [½]$$

So:

$$EPV = \frac{1,500}{1.0023726}\left[1.04^{-1/12}\left({}_{1/12}p_{67}\right) + 1.04^{-2/12}\left({}_{2/12}p_{67}\right) + 1.04^{-3/12} \; {}_{3/12}p_{67} + \cdots\right]$$

$$= \frac{12 \times 1,500}{1.0023726} \times a_{67}^{(12)} @4\% \qquad [1]$$

From the *Tables*:

$$a_{67}^{(12)} = \ddot{a}_{67}^{(12)} - \frac{1}{12} \approx \ddot{a}_{67} - \frac{11}{24} - \frac{1}{12} = \ddot{a}_{67} - \frac{13}{24} = 14.111 - \frac{13}{24} = 13.5693$$ [1]

So, the EPV of the benefit is:

$$\frac{£18,000}{1.0023726} \times 13.5693 = £243,670$$ [½]

[Total 4]

18.8 (i) ***Present value random variable***

This is:

$$(20,000 + 100K_{35})v^{T_{35}}$$ [1]

(ii)(a) ***Expected present value using AM92 Select and 6% pa interest***

The expected present value of the benefits is:

$$19,900\bar{A}_{[35]} + 100(I\bar{A})_{[35]}$$ [1]

where:

$$\bar{A}_{[35]} \approx 1.06^{½} \times A_{[35]} = 1.06^{½} \times 0.09475 = 0.097551$$ [1]

$$(I\bar{A})_{[35]} \approx 1.06^{½} \times (IA)_{[35]} = 1.06^{½} \times 3.33735 = 3.43601$$ [½]

So the EPV is:

$$19,900 \times 0.097551 + 100 \times 3.43601 = 2,284.87$$ [½]

(ii)(b) ***Expected present value using force of mortality 0.015 pa and force of interest 0.03 pa***

The expected present value of the benefits is:

$$19,900\bar{A}_{35} + 100(I\bar{A})_{35}$$

With constant forces of interest and mortality we can use integrals to calculate the values of the functions.

We have:

$$\bar{A}_{35} = \int_{t=0}^{\infty} v^t \, {}_tp_{35} \mu_{35+t} dt$$

$$= \int_{t=0}^{\infty} e^{-0.03t} e^{-0.015t} \times 0.015 dt$$

$$= 0.015 \int_{t=0}^{\infty} e^{-0.045t} dt$$

$$= 0.015 \left[\frac{e^{-0.045t}}{-0.045} \right]_0^{\infty}$$

$$= \frac{0.015}{0.045} = \frac{1}{3} \qquad [2]$$

Now consider $(I\bar{A})_{35}$. This is:

$$(I\bar{A})_{35} = \int_{t=0}^{1} v^t \, {}_tp_{35} \, \mu_{35+t} \, dt + 2\int_{t=1}^{2} v^t \, {}_tp_{35} \, \mu_{35+t} \, dt + 3\int_{t=2}^{3} v^t \, {}_tp_{35} \, \mu_{35+t} \, dt + \cdots$$

$$= 0.015 \left(\int_{t=0}^{1} e^{-0.045t} \, dt + 2\int_{t=1}^{2} e^{-0.045t} \, dt + 3\int_{t=2}^{3} e^{-0.045t} \, dt + \cdots \right)$$

$$= 0.015 \left(\left[\frac{e^{-0.045t}}{-0.045} \right]_0^1 + 2\left[\frac{e^{-0.045t}}{-0.045} \right]_1^2 + 3\left[\frac{e^{-0.045t}}{-0.045} \right]_2^3 + \cdots \right)$$

$$= \frac{0.015}{0.045} \left(1 - e^{-0.045} + 2\left(e^{-0.045} - e^{-0.045\times 2} \right) + 3\left(e^{-0.045\times 2} - e^{-0.045\times 3} \right) + \cdots \right)$$

$$= \frac{0.015}{0.045} \left(1 + e^{-0.045} + e^{-0.045\times 2} + e^{-0.045\times 3} + \cdots \right) \qquad [4]$$

The amount in the bracket is the sum of a geometric progression with first term 1 and common ratio $e^{-0.045}$. So we have:

$$(I\bar{A})_{35} = \frac{0.015}{0.045} \times \frac{1}{1 - e^{-0.045}} = 7.57532 \qquad [1]$$

and the expected present value is:

$$19{,}900 \times \frac{1}{3} + 100 \times 7.57532 = 7{,}390.87 \qquad [1]$$

[Total 11]

18.9 The expected present value is:

$$v^{15} \times \frac{l_{65}}{l_{50}} \times 20{,}000 \times \int_{t=0}^{\infty} 1.02^t v^t \, {}_tp_{65} \, dt$$ [2]

Now:

$$1.02^t v^t = \left(\frac{1.02}{1.03}\right)^t = 0.9902913^t = e^{(\ln 0.9902913)t} = e^{-0.0097562t}$$ [1]

So:

$$1.02^t v^t \, {}_tp_{65} = e^{-0.0097562t} e^{-0.038t} = e^{-0.0477562t}$$ [1]

Then:

$$EPV = 1.03^{-15} \times \frac{9{,}647.797}{9{,}941.923} \times 20{,}000 \times \int_{t=0}^{\infty} e^{-0.0477562t} \, dt$$

where:

$$\int_{t=0}^{\infty} e^{-0.0477562t} dt = \left[\frac{e^{-0.0477562t}}{-0.0477562}\right]_0^{\infty} = -\frac{0 - e^0}{0.0477562} = \frac{1}{0.0477562}$$ [1]

So:

$$EPV = 1.03^{-15} \times \frac{9{,}647.797}{9{,}941.923} \times 20{,}000 \times \left(\frac{1}{0.0477562}\right)$$

$$= 260{,}855$$ [1]

[Total 6]

18.10 *This question is Subject CT5, September 2008, Question 12.*

(i)(1) ***Simple bonuses maturity amount***

After 20 years, the maturity value is:

$$100{,}000(1 + 0.045 \times 20) = £190{,}000$$ [1]

(i)(2) ***Compound bonuses maturity amount***

After 20 years, the maturity value is:

$$100{,}000(1.0384615)^{20} = £212{,}719.84$$ [1]

(i)(3) ***Super compound bonuses maturity amount***

Each year, a bonus of 3,000 is added to the benefit. Each of these bonuses then increases by 6% *pa* compound from the date they are declared up to the maturity date.

So the first bonus of 3,000 is added at time 1. This increases by 6% *pa* compound over the next 19 years and so produces an amount at maturity of $3{,}000 \times 1.06^{19}$.

The second bonus of 3,000 is added at time 2. This also increases by 6% *pa* compound over the next 18 years and so produces $3{,}000 \times 1.06^{18}$ by the maturity date.

Each subsequent bonus of 3,000 is treated in a similar way, with the last of these added at time 20, *ie* at the maturity date itself.

The total maturity value is therefore:

$$100{,}000 + 3{,}000\left(1.06^{19} + 1.06^{18} + \dots + 1\right) \qquad [1]$$

The function in brackets is $s_{\overline{20|}}$ calculated at 6% *pa*, and so the maturity value is:

$$100{,}000 + 3{,}000 s_{\overline{20|}@6\%} = 100{,}000 + 3{,}000 \times 36.7856$$

$$= £210{,}356.8 \qquad [1]$$

Alternatively, we could calculate the function in brackets using the formula for the sum of the terms of a geometric progression, ie as:

$$1 + 1.06 + 1.06^2 + \dots + 1.06^{19} = \frac{1 - 1.06^{20}}{1 - 1.06} = \frac{1.06^{20} - 1}{0.06} = 36.7856$$

[Total 4]

(ii) ***EPV of benefits under structure (2)***

The expected present value of the maturity benefit is:

$$100{,}000 \times 1.0384615^{20} v^{20} \, {}_{20}p_{[45]} = 100{,}000 \times 1.04^{-20} \, {}_{20}p_{[45]}$$

as $1.0384615 \times v = \dfrac{1.0384615}{1.08} = \dfrac{1}{1.04}$.

So:

$$EPV = 100{,}000 \times \frac{D_{65@4\%}}{D_{[45]@4\%}} = 100{,}000 \times \frac{689.23}{1{,}677.42} = 41{,}088.70 \qquad [1]$$

Writing 1.0384615 as $1 + b$, the expected present value of the death benefit is:

$$100{,}000 \times \left(v \, {}_{0|}q_{[45]} + (1+b)v^2 \, {}_{1|}q_{[45]} + (1+b)^2 v^3 \, {}_{2|}q_{[45]} + \dots + (1+b)^{19} v^{20} \, {}_{19|}q_{[45]} \right)$$

It would be helpful if the first term in the bracket began with $(1+b)v$. We can achieve this by multiplying and dividing by $(1+b)$, so:

$$EPV = \frac{100,000}{1+b} \times \left((1+b)v \; {}_{0|}q_{[45]} + (1+b)^2 v^2 \; {}_{1|}q_{[45]} + \cdots + (1+b)^{20} v^{20} \; {}_{19|}q_{[45]} \right)$$

$$= \frac{100,000}{1.0384615} \times \left(1.04^{-1} \; {}_{0|}q_{[45]} + 1.04^{-2} \; {}_{1|}q_{[45]} + \cdots + 1.04^{-20} \; {}_{19|}q_{[45]} \right)$$

$$= \frac{100,000}{1.0384615} \times A^{\;1}_{[45]:\overline{20|}} @4\% \qquad [2]$$

Now:

$$A^{\;1}_{[45]:\overline{20|}} = A_{[45]:\overline{20|}} - \frac{D_{65}}{D_{[45]}} = 0.46982 - \frac{689.23}{1,677.42} = 0.058933$$

So:

$$EPV = \frac{100,000}{1.0384615} \times 0.058933 = 5,675.03$$

Adding the EPV of the maturity benefit and death benefit together we find the total expected present value to be:

$$41,088.70 + 5,675.03 = £46,764 \qquad [1]$$

[Total 4]

(iii)(a) ***Bonuses are added at the start of the year***

If we were to write out the EPV of the death benefit as a summation, we would find that the powers of $1+b$ are all increased by 1 compared to part (ii), so the EPV of the death benefit will just be the previous value multiplied by 1.0384615. So the total EPV is now:

$$41,088.70 + 1.0384615 \times 5,675.03 = £46,982 \qquad [1]$$

(iii)(b) ***Death benefit payable immediately on death***

The death benefit can be assumed to be paid on average half a year earlier than in part (ii), so the EPV of the death benefit will just be the previous value multiplied by $1.08^{1/2}$. So the total EPV is now:

$$41,088.70 + 1.08^{1/2} \times 5,675.03 = £46,986 \qquad [1]$$

(iii)(c) ***Bonuses are added continuously***

The sum assured payable on death is increasing continuously at a compound rate of 3.84615% *pa* up to the moment the claim is paid, from which point it is discounted over exactly the same period at a compound interest rate of 8% *pa*. So the EPV of the death benefits is equivalent to the EPV of a level term assurance discounted at a net effective interest rate of 4% *pa*.

Hence the EPV of the death benefit is:

$$100{,}000\bar{A}^{\;1}_{[45]:\overline{20|}@4\%} = 100{,}000 \times 1.04^{½} \times A^{\;1}_{[45]:\overline{20|}@4\%}$$

Now $A^{\;1}_{[45]:\overline{20|}@4\%} = 0.058933$ from part (ii), so the total EPV becomes:

$$41{,}088.70 + 100{,}000 \times 1.04^{½} \times 0.058933 = £47{,}099$$ [1]

[Total 3]

End of Part 3

What next?

1. Briefly **review** the key areas of Part 3 and/or re-read the **summaries** at the end of Chapters 14 to 18.
2. Ensure you have attempted some of the **Practice Questions** at the end of each chapter in Part 3. If you don't have time to do them all, you could save the remainder for use as part of your revision.
3. Attempt **Assignment X3**.
4. Attempt the questions relating to Chapters 14 to 18 of the **Paper B Online Resources (PBOR)**.

19

Gross premiums

Syllabus objectives

6.1 Define the gross random future loss under an insurance contract, and state the principle of equivalence.

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts.

6.2.1 Define and calculate gross premiums for the insurance contract benefits as defined in objective 4.1 under various scenarios using the equivalence principle or otherwise:

- contracts may accept only single premium
- regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously
- death benefits (which increase or decrease by a constant compound rate or by a constant monetary amount) may be payable at the end of the year of death, or immediately on death
- survival benefits (other than annuities) may be payable at defined intervals other than at maturity.

0 Introduction

In this chapter we describe some of the methods that can be used to calculate the premiums for conventional life assurance and annuity contracts.

We show how an equation of value can be used to do this. We first met this method in Chapter 9 in relation to certain (as supposed to random) cashflows. This approach will involve calculating expected present values, as described in Chapters 15-18.

We also discuss an alternative method, where we calculate premiums that satisfy specified probabilities.

This chapter includes the concept of the future loss random variable, on which both of these approaches depend.

While studying this chapter, bear in mind that all of the approaches covered here suffer from a common shortcoming in that they do not allow for the cost of the capital that is necessary for insurers to be able to sell such policies in real life. The methods required to take this into account, which involve projecting cashflows and profit testing, are described later, in Chapters 26 and 27.

1 The gross premium

The income to a life insurer comes from the payments made by policyholders, called the *premiums*. The outgo arises from benefits paid to policyholders and the insurer's expenses.

By expenses we mean the costs incurred in administering a life insurance contract and, more broadly, in managing the insurance company's business as a whole. These costs involve such things as:

- staff salaries
- commission payments to sales intermediaries
- costs of infra-structure (*eg* office buildings, computer systems)
- medical underwriting costs
- costs of managing investments
- other running costs (*eg* policy administration, heating, lighting, internet use).

Some of these costs will be *direct* expenses, which are incurred as a result of activities specifically associated with a particular policy (like commission); other expenses will be *overheads*, which are incurred whether or not a particular policy is actually in force (such as the actuary's salary).

Question

Identify the following costs as overheads or direct:

(i) board of directors' remuneration

(ii) £10,000 bonus payable to sales manager on completion of target new business levels

(iii) head office canteen workers' salaries

(iv) new business administration department's salary costs.

Solution

(i) The board of directors' remuneration is an overhead (unless linked in any way to new business volumes).

(ii) The production bonus is a direct expense (it varies with production).

(iii) Head office canteen workers' salaries are an overhead.

(iv) This is a good example of a 'grey' area where there is no clear right or wrong answer. In the very short term, writing one more policy will not change the costs. But a rush of business over a few weeks may require overtime payments, and a rush of business over a few months may require more staff to be taken on.

> So some companies will treat this as an overhead, other companies will treat it as a direct expense. Others could do both by treating the basic salaries of normal staff as overheads, and overtime and/or salaries of temporary staff as a direct expense.

The future levels of expenses are also significantly affected by future inflation, so we should allow for this when calculating premiums.

The *gross premium* is the premium required to meet all the costs under an insurance contract, and is the premium that the policyholder pays. When we talk of 'the premium' for a contract, we mean the gross premium. It is also sometimes referred to as the *office premium*.

In practice, the office premium also contains impact for loaded profits, cost of capital, adjustments for competition and other loadings. This is currently beyond the scope of this subject.

Some of these aspects are, however, taken into account when premiums are calculated using cashflow methods, as we describe in Chapters 26 and 27.

2 Gross future loss random variable

Consider the net random future loss (or just 'net loss') from a policy which is in force – where the net loss, L, is defined to be:

$$L = \text{present value of the future outgo} - \text{present value of the future income}$$

Now L is a random variable, since both terms are random variables which depend on the policyholder's future lifetime. (If premiums are not being paid, the second term is zero but the first term is a random variable, so L is still a random variable.) When the outgo includes benefits and expenses, and the income is the gross premiums, then L is referred to as the *gross future loss random variable*.

We will use FLRV for short.

We illustrate the form of the definition of the gross future loss random variable using the example of a whole life assurance. This approach can be extended to any of the standard contract types for a given allocation of expenses and, in the case of with-profits contracts, method of bonus allocation.

Example: Whole life assurance

Suppose we can allocate expenses as:

- I **initial expenses in excess of those occurring regularly each year**
- e **level annual expenses**
- f **additional expenses incurred when the contract terminates**

We are assuming here that level expenses of e are incurred every year *including* the first, *ie* that I represents the amount by which the total initial expense amount *exceeds* the subsequent regular expense amount.

We need to be careful when dealing with examination questions that talk about 'regular expenses' or 'renewal expenses'. We need to be very clear as to whether the expense amount applies at the start of every year, or only from the start of Year 2 onwards. If the renewal expenses start in Year 2, it may be easier to think of them as starting in Year 1 and then change I to $I-e$. The question should always make it clear what is required. If it doesn't, then we need to state the assumption we are making very clearly.

The gross future loss random variable when a policy is issued to a life aged x is:

$$Sv^{T_x} + I + e\bar{a}_{\overline{T_x}|} + fv^{T_x} - G\bar{a}_{\overline{T_x}|}$$

where a gross premium of G secures a sum assured of S, the sum assured is paid immediately on death and the premium is payable continuously.

It also assumes that the renewal expenses are paid continuously at a constant rate of e *pa* until the death of the policyholder (*ie* until the contract terminates).

Example: Endowment assurance

The FLRV for an endowment assurance is similar to the above, except that the term of the contract is limited (say to n years), and the survival benefit needs to be included.

Question

Write down the gross future loss random variable for an n-year endowment assurance. Assume that death benefits are payable immediately on death, level premiums are payable continuously, and the expenses are as defined in the above example. Use the symbols defined in the above example as appropriate.

Solution

In a case like this it is usually easiest to write down the FLRV in two parts: the first being the outcome if the person dies during the policy term (*ie* where $T_x < n$), and the second where the person is still alive at the maturity date (*ie* where $T_x \geq n$). So the FLRV is:

$$\begin{cases} Sv^{T_x} + I + e\,\bar{a}_{\overline{T_x}|} + f\,v^{T_x} - G\bar{a}_{\overline{T_x}|} & T_x < n \\ Sv^{n} + I + e\,\bar{a}_{\overline{n}|} + f\,v^{n} - G\bar{a}_{\overline{n}|} & T_x \geq n \end{cases}$$

where I, e, f, S and G are defined as before.

Sometimes we can combine these into a single expression. Here we can do this using the min{…} function, so the FLRV can alternatively be written:

$$Sv^{\min\{T_x,n\}} + I + e\,\bar{a}_{\overline{\min\{T_x,n\}}|} + f\,v^{\min\{T_x,n\}} - G\bar{a}_{\overline{\min\{T_x,n\}}|}$$

If death benefits were payable at the end of the year of death, then T_x would be replaced by $K_x + 1$ in the multipliers of S and f. If premiums and regular expenses were payable annually in advance, then T_x would be replaced by $K_x + 1$ in the multipliers of e and G, and the annuity functions would be $\ddot{a}$ rather than $\bar{a}$.

Gross future loss random variable for an in-force policy

We can also consider the FLRV for a policy that is currently in force, *ie* as at some policy duration $t > 0$. For example, for the policy in the previous question the FLRV as at the end of the fourth policy year $(n > 4)$ is:

$$\begin{cases} Sv^{T_{x+4}} + e\,\bar{a}_{\overline{T_{x+4}}|} + f\,v^{T_{x+4}} - G\bar{a}_{\overline{T_{x+4}}|} & T_{x+4} < n-4 \\ Sv^{n-4} + e\,\bar{a}_{\overline{n-4}|} + f\,v^{n-4} - G\bar{a}_{\overline{n-4}|} & T_{x+4} \geq n-4 \end{cases}$$

$$= Sv^{\min\{T_{x+4},n-4\}} + e\,\bar{a}_{\overline{\min\{T_{x+4},n-4\}}|} + f\,v^{\min\{T_{x+4},n-4\}} - G\bar{a}_{\overline{\min\{T_{x+4},n-4\}}|}$$

This reflects the fact that the policyholder is now aged $x+4$, and there are only $n-4$ years of the policy left to run.

Question

Explain why the value of I does not appear in the above FLRV.

Solution

At time 4, the initial expense cashflow is in the past, *ie* it has already been incurred (at time 0), and so it is not part of the *future* loss.

Example: with-profits policy

We can also apply these ideas to with-profits policies (or to any other type of policy with a non-level benefit). If we do so, we need to allow correctly for the number of bonuses that have been added by the current date. (Note that we sometimes describe the adding of bonuses as the *vesting* of bonuses so, if a bonus 'vests', it means that the policy benefit level has been increased by the amount of that bonus.) We must also consider carefully the timing of the bonus additions (start of the year, end of the year, *etc*), as always.

For example, suppose a company sells with-profits endowment assurances with an initial sum assured of £20,000 to lives aged 40 exact. Each policy has a term of 20 years, and the death benefit is payable at the end of the year of death. Annual premiums of £2,500 are payable in advance during the term of the policy, or until earlier death of the policyholder. The company assumes that:

(a) bonuses of £2,000 will be added to the sum assured each year, vesting at the start of each year except the first;

(b) expenses of £200 will be incurred initially, together with renewal expenses of 2% of every premium except the first.

We will write down the gross future loss random variable for the policy at the start of the contract, and also just before the tenth premium is paid.

FLRV at the start of the policy

If bonuses were added at the start of every year *including* the first year, and the policyholder dies during the term, *ie* $K_{40} < 20$, then the sum assured will be increased by $K_{40}+1$ bonuses. For example, if the policyholder were to die in policy year 3 (at which point $K_{40}=2$), a bonus would be added at the start of each of the three years, *ie* three bonuses in total. However, in this case bonuses are added at the start of each year *except* the first, and so the number of bonuses added is one fewer than this, *ie* equal to K_{40}. If the policyholder survives to the end of the term, a total of 19 bonuses will be paid altogether (*ie* one for each policy year except the first).

So the present value of the future benefits can be written:

$$\begin{cases} \left[20{,}000+2{,}000K_{40}\right]v^{K_{40}+1} & K_{40}<20 \\ \left[20{,}000+2{,}000\times 19\right]v^{20} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},19\}\right]v^{\min\{K_{40}+1,\,20\}}$$

Premiums of £2,500 are payable annually in advance. So the present value of the future premiums is:

$$\begin{cases} 2{,}500\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ 2{,}500\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=2{,}500\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

The present value of the future expenses is:

$$\begin{cases} 200+0.02\times 2{,}500a_{\overline{K_{40}}|} & K_{40}<20 \\ 200+0.02\times 2{,}500a_{\overline{19}|} & K_{40}\geq 20 \end{cases}$$

$$=200+50a_{\overline{\min\{K_{40},\,19\}}|}$$

$$=200+50\left(\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}-1\right)$$

$$=150+50\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

Putting all these together, we have:

$$\begin{cases} \left[20{,}000+2{,}000K_{40}\right]v^{K_{40}+1}+150-2{,}450\,\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ \left[20{,}000+2{,}000\times 19\right]v^{20}+150-2{,}450\,\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},19\}\right]v^{\min\{K_{40}+1,\,20\}}+150-2{,}450\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

FLRV just before the tenth premium is paid

Now suppose the policy is still in force at time 9, *ie* just before the 10th premium is paid. The policyholder is 49 at this point and there are 11 years remaining on the policy.

Eight bonuses will have vested so far (the ninth bonus is about to vest). The sum assured before the ninth bonus vests is £36,000, and, since the next bonus is about to vest immediately, the number of future bonuses is $K_{49}+1$ (if the policyholder dies during the term), or 11 (if the policyholder survives to the end of the term).

So the present value of the future benefits is now:

$$\begin{cases} \left[36{,}000+2{,}000\{K_{49}+1\}\right]v^{K_{49}+1} & K_{49}<11 \\ \left[36{,}000+2{,}000\times 11\right]v^{11} & K_{49}\geq 11 \end{cases}$$

$$=\left[36{,}000+2{,}000\min\{K_{49}+1,11\}\right]v^{\min\{K_{49}+1,11\}}$$

The present value of the future premiums is:

$$\begin{cases} 2{,}500\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ 2{,}500\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=2{,}500\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

The present value of the future expenses is:

$$\begin{cases} 50\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ 50\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=50\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

(We no longer need to include the initial expenses.)

So the gross future loss random variable is:

$$\begin{cases} \left[36{,}000+2{,}000\{K_{49}+1\}\right]v^{K_{49}+1}-2{,}450\,\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ \left[36{,}000+2{,}000\times 11\right]v^{11}-2{,}450\,\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=\left[36{,}000+2{,}000\min\{K_{49}+1,\,11\}\right]v^{\min\{K_{49}+1,\,11\}}-2{,}450\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

Question

Now assume in the above example that the bonuses vest at the end of each year (*including* the first year). Write down revised expressions for the gross future loss random variables at times 0 and 9.

Solution

At outset, the number of bonuses on death is still K_{40}, but there are now 20 bonuses altogether (the 20th bonus is payable on survival only). So the gross future loss random variable at time zero will now be:

$$\begin{cases} [20{,}000+2{,}000K_{40}]v^{K_{40}+1}+150-2{,}450\,\ddot{a}_{\overline{K_{40}+1}|} & K_{40}<20 \\ [20{,}000+2{,}000\times 20]v^{20}+150-2{,}450\,\ddot{a}_{\overline{20}|} & K_{40}\geq 20 \end{cases}$$

$$=\left[20{,}000+2{,}000\min\{K_{40},20\}\right]v^{\min\{K_{40}+1,\,20\}}+150-2{,}450\,\ddot{a}_{\overline{\min\{K_{40}+1,\,20\}}|}$$

By time 9, exactly 9 policy years have been completed and so 9 bonuses have been added to the sum assured, making the total benefit level at this point equal to $20{,}000+9\times 2{,}000=38{,}000$. So the future loss random variable at time 9 is:

$$\begin{cases} [38{,}000+2{,}000K_{49}]v^{K_{49}+1}-2{,}450\,\ddot{a}_{\overline{K_{49}+1}|} & K_{49}<11 \\ [38{,}000+2{,}000\times 11]v^{11}-2{,}450\,\ddot{a}_{\overline{11}|} & K_{49}\geq 11 \end{cases}$$

$$=\left[38{,}000+2{,}000\min\{K_{49},11\}\right]v^{\min\{K_{49}+1,\,11\}}-2{,}450\,\ddot{a}_{\overline{\min\{K_{49}+1,\,11\}}|}$$

Example: term assurance policy

Let's assume the same details as for the (without-profit) endowment assurance example we described earlier, except that the sum assured is now only payable on death during the n-year term (there is no payment made on survival), and f is the claim expense.

The FLRV is:

$$\begin{cases} S v^{T_x}+I+e\,\bar{a}_{\overline{T_x}|}+f v^{T_x}-G\bar{a}_{\overline{T_x}|} & T_x<n \\ I+e\,\bar{a}_{\overline{n}|}-G\bar{a}_{\overline{n}|} & T_x\geq n \end{cases}$$

Note that it is not possible to express a term assurance FLRV as a single-line expression.

Question

Explain why the term involving f does not appear in the FLRV when $T_x \geq n$.

Solution

f is the claim expense, and if $T_x \geq n$ the policyholder has survived the term and so no claim is payable. Hence there is no claim expense to pay.

2.1 Calculating premiums that satisfy probabilities, using the gross future loss random variable

Premiums (and reserves) can be calculated which satisfy probabilities involving the gross future loss random variable.

(The more important method of calculating premiums, however, uses the principle of equivalence – this is covered in Section 3 below.)

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming 3% per annum interest, AM92 Ultimate mortality and expenses of 4% of every premium, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

Solution

We will make a loss on this policy if the total future outgo it generates exceeds the total future income received from it. Future investment returns will affect the overall loss, and one way of taking account of these is to measure the future loss in present value terms, *ie* with all cashflows discounted at the effective interest rate. So, allowing for investment returns, we will make a loss under the policy if the present value of the future loss, *ie* L_0, is greater than zero.

If the annual premium is G, the future loss (random variable) of the policy at outset is:

$$L_0 = 10{,}000\, v^{K_{50}+1} - 0.96\, G\, \ddot{a}_{\overline{K_{50}+1}|}$$

We need to find the smallest value of G such that:

$$P(L_0 > 0) \le 0.05$$

***ie* such that**

$$P(L_0 \le 0) \ge 0.95$$

Define G_n to be the annual premium that ensures $L_0 = 0$ for $K_{50} = n$.

The value of n has to be a valid value for K_{50}, which means that $n = 0, 1, 2, \ldots$. So, G_n is the premium we need to charge to cover the benefits *exactly*, if the policyholder were to die between (integer) times n and $n+1$.

This means:

$$G_n = \frac{10{,}000\, v^{n+1}}{0.96\, \ddot{a}_{\overline{n+1}|}}$$

Now:

$$P\left(L_0 \le 0 \middle| G = G_n\right) = P\left(K_{50} \ge n\right)$$

which reads as 'the probability of not making a loss when the premium is G_n, equals the probability of the policyholder surviving for at least n years'.

To understand why this probability is true, suppose we *do* charge a premium of G_n (meaning that the loss will equal zero if $K_{50} = n$). If death occurs later than this (*ie* if $K_{50} > n$), the present value of the benefit will reduce (because it will be more heavily discounted), and the present value of the premium income will increase (because more premiums are received). So, if $K_{50} \ge n$, and G_n is the premium, we cannot make a loss on the contract (*ie* $L_0 \le 0$).

However, this probability needs to be at least 0.95…

and we need to find the *smallest* premium for which this is true. If we look at the formula for G_n, we can see that as n increases, the premium G_n decreases.

We therefore find the largest value of n that satisfies this condition, and the corresponding value of G_n is then the minimum premium required.

The condition is that $P\left(K_{50} \ge n\right) \ge 0.95$.

So:

$$P\left(K_{50} \ge n\right) \ge 0.95$$

$$\Rightarrow \ {}_n p_{50} \ge 0.95 \ \Rightarrow \ \frac{l_{50+n}}{l_{50}} \ge 0.95$$

$$\Rightarrow \ l_{50+n} \ge 0.95 \times l_{50} = 0.95 \times 9{,}712.0728 = 9{,}226.4692$$

From the *Tables*, $l_{60} = 9{,}287.2164$ and $l_{61} = 9{,}212.7143$, and so the largest value of n that satisfies the required probability is $n = 10$. Hence, the smallest premium that satisfies the required probability is:

$$G_{10} = \frac{10{,}000 v^{11}}{0.96 \ddot{a}_{\overline{11|}}} = \frac{10{,}000 \times 0.72242}{0.96 \times 9.5302} = 790$$

A premium calculated in this way is sometimes called a *percentile premium*.

A reserve at policy duration t can be calculated in a similar way, to satisfy a probability specified in terms of the gross future loss random variable at time t. Reserves are described in the next chapter.

Question

Hubert, aged 60, is applying to buy a whole life immediate annuity from an insurance company, with his life savings of £200,000. Calculate the largest amount of level annuity, payable annually in arrear, that the insurer could pay if it requires a probability of loss from the contract of no more than 10%.

Assume PMA92C20 mortality, interest of 5% *pa*, and expenses of 1% of each annuity payment.

Solution

The present value of the loss from the policy is:

$$L = 1.01 X a_{\overline{K_{60}|}} - 200{,}000$$

where X is the annual annuity benefit. If we set $X_{(k)}$ such that:

$$L = 1.01 X_{(k)}\, a_{\overline{k|}} - 200{,}000 = 0$$

then:

$$X_{(k)} = \frac{200{,}000}{1.01 a_{\overline{k|}}}$$

and the insurer will only make a loss if $K_{60} > k$ (for annuities, the loss will increase the longer the person lives).

So, an annuity of $X_{(k)}$ implies $P(L>0) = P(K_{60} > k)$. To find the largest value of $X_{(k)}$ that satisfies:

$$P(L>0) \le 0.1$$

we need the smallest value of k that satisfies:

$$P(K_{60} > k) \le 0.1 \qquad ie \qquad P(K_{60} \ge k+1) \le 0.1 \qquad \text{or} \qquad {}_{k+1}p_{60} \le 0.1$$

that is, the smallest value of k for which:

$$l_{60+k+1} \le 0.1 l_{60} = 0.1 \times 9{,}826.131 = 982.6131$$

From the *Tables* we find:

$$l_{95} = 1{,}020.409 \qquad (k = 34)$$

$$l_{96} = 798.003 \qquad (k = 35)$$

So the required (smallest) value of k is 35, and hence the largest amount of level annuity is:

$$X_{(35)} = \frac{200{,}000}{1.01 a_{\overline{35}|}^{5\%}} = \frac{200{,}000}{1.01 \times 16.3742} = £12{,}093$$

---

3 Principle of equivalence

3.1 Definition

The equation of value where payments are certain has already been introduced in Chapter 9. In most actuarial contexts some or all of the cashflows in a contract (usually long-term) are uncertain, depending on the death or survival (or possibly the state of health) of a life.

We therefore extend the concept of the equation of value to deal with this uncertainty, by equating expected present values of uncertain cashflows. The equation of expected present values for a contract, usually referred to as the equation of value, is:

The expected present value of the income

= The expected present value of the outgo

This can also be written as:

$$E[\text{present value of future outgo}] - E[\text{present value of future income}] = 0$$

$$\Rightarrow \quad E[\text{present value of future loss}] = 0$$

ie: $E[L_0] = 0$

Alternatively, this is referred to as the *principle of equivalence*.

3.2 Determining gross premiums using the equivalence principle

Given a suitable set of assumptions, which we call the basis, we may use the equation of value to calculate the premium or premiums which a policyholder must pay in return for a given benefit.

This is sometimes referred to as calculating a premium using the principle of equivalence.

We may also calculate the amount of benefit payable for a given premium.

A basis is a set of assumptions regarding expected future experience, *eg*:

- mortality experience
- investment returns
- future expenses
- bonus rates.

The set of assumptions used to calculate a premium is called the *pricing basis*.

The gross premium for a contract, given suitable mortality, interest and expense assumptions would be found from the equation of expected present value.

The equivalence principle states that:

$$E[\text{gross future loss}] = 0$$

which implies that:

$$\text{EPV premiums} = \text{EPV benefits} + \text{EPV expenses}$$

so that in an expected present value context the premiums are equal (equivalent) in value to the expenses and the benefits. This relationship is usually called an equation of (gross expected present) value.

3.3 The basis

The basis for applying the equation of value for a life insurance contract will specify the mortality, interest rates and expenses to be assumed.

Usually the assumptions will not be our best estimates of the individual basis elements, but will be more cautious than the best estimates.

For example, if we expect to earn a rate of interest of 8% *pa* on the invested premiums, a more cautious basis might be to calculate the premiums assuming we earn only 6% *pa*. This lower rate of interest will result in a higher premium than that calculated using the expected rate of 8% *pa*.

Some reasons for the element of caution in the basis are:

1. **To allow a contingency margin, to ensure a high probability that the premiums plus interest income meet the cost of benefits, allowing for random variation. In other words, to ensure a high probability of making a profit.**
2. **To allow for uncertainty in the estimates themselves.**

The size of the appropriate margin to incorporate will depend upon a number of factors, such as:

- The level of risk that the insurer is undertaking. The higher the risk the bigger the margin.
- The competitiveness of the market in which the product needs to be sold. The more competitive the market, the lower the margin.

In the case of the mortality rate, the more cautious basis will depend on the nature of the contract. For example, for a contract which pays out the benefit on death, a heavier mortality rate than that expected will give rise to a higher premium and is therefore the more cautious basis. For an annuity contract, a lighter mortality rate is a more cautious basis.

In actuarial jargon, we refer to this as 'being prudent'.

3.4 Premium payment structures

The premium payment structure will commonly be one of the following:

- **A *single premium* contract, under which benefits are paid for by a single lump sum premium paid at the time the contract is effected. This payment is certain, so that the left-hand side of the equation of value is the certain payment, not the expected value of a payment.**

- **An *annual premium* contract, under which benefits are paid for by a regular annual payment, usually of a level amount, the first premium being due at the time the contract is effected. Premiums continue to be paid each year in advance until the end of some agreed maximum premium term, often the same as the contract term, or until the life dies if this is sooner. Therefore, there would not usually be a premium payment at the end of the contract term.**
- **A *true* m*thly premium* contract, under which benefits are paid for by m level payments made every $1/m$ years. As in the annual premium case, premiums continue to be paid in advance until the end of some agreed maximum premium term or until the life dies if this is sooner. Again, there would not usually be a premium paid at the end of the contract term. Often the premium is paid monthly (that is, $m=12$). For some types of contract, weekly premiums are possible.**

The use of the word 'true' here refers to a technical distinction (between so-called 'true' and 'instalment' premiums), which we needn't worry about here.

***Premiums are always paid in advance*, so the first payment is always due at the time the policy is effected.**

Given a basis specifying mortality, interest and expenses to be assumed, and given details of the benefits to be purchased, we can use the equation of value to calculate the premium payable.

3.5 Annual premium contracts

We will illustrate the equation of value methodology by using the example of a whole life assurance. The same method can be used for other standard contracts.

In particular, the same method can also be used for single premium contracts with the simplification that the expected present value of future premiums will just be equal to the single premium that the company receives on day 1.

Suppose we have premiums of G payable annually in advance, a benefit of S payable at the end of the year of death, initial expenses of I, regular expenses of e payable annually in advance in each year including the first year, and claim expenses of f.

The equation of value is:

$$S\,A_x + I + e\,\ddot{a}_x + f\,A_x = G\,\ddot{a}_x$$

and, with a basis (to determine the annuity and assurance values) and an expense allocation, the value of G can be determined.

If the sum assured is paid immediately on death and the premium is paid continuously we can use functions defined in Chapters 15 and 16 to write:

$$S\,\bar{A}_x + I + e\,\bar{a}_x + f\,\bar{A}_x = G\,\bar{a}_x$$

which can be solved for G as before.

Let's consider Gordon, who is aged 45, and who wishes to buy a whole life assurance policy with a sum assured of £10,000 payable immediately on death. We are going to calculate the gross premium that Gordon needs to pay annually in advance for ten years or until his earlier death, using the equivalence principle according to the following basis:

- Initial expenses of £160 plus 75% of the annual premium
- Renewal expenses of £50 (incurred throughout life from year 2 onwards) plus 4% of the annual premium (incurred at the time of payment of each premium from year 2 onwards)
- Claim expenses of 2.5% of sum assured incurred when the benefit is payable
- AM92 Ultimate mortality
- 4% *pa* interest

The equation of value is:

EPV premiums = EPV benefits + EPV expenses

Now:

$$\text{EPV premiums} = P\ddot{a}_{45:\overline{10|}} = P\left(\ddot{a}_{45} - \frac{D_{55}}{D_{45}}\ddot{a}_{55}\right)$$

$$= \left(18.823 - \frac{1,105.41}{1,677.97} \times 15.873\right)P = 8.366P$$

$$\text{EPV benefits} = 10,000\bar{A}_{45} \approx 10,000 \times (1+i)^{\frac{1}{2}} A_{45}$$

$$= 10,000 \times 1.04^{\frac{1}{2}} \times 0.27605 = 2,815.17$$

and:

$$\text{EPV expenses} = 160 + 0.75P + 50\left(\ddot{a}_{45} - 1\right) + 0.04P\left(\ddot{a}_{45:\overline{10|}} - 1\right) + 0.025 \times 10,000\bar{A}_{45}$$

$$= 160 + 0.75P + 50 \times 17.823 + 0.04 \times 7.366P + 0.025 \times 2,815.17$$

$$= 1.045P + 1,121.53$$

So:

$$8.366P = 2,815.17 + 1.045P + 1,121.53$$

$$\Rightarrow P = £537.69$$

(These figures use full accuracy and are sensitive to rounding.)

We can also calculate gross premiums using select mortality, as in the following question.

Question

A 25-year endowment assurance policy provides a payment of £75,000 on maturity or at the end of the year of earlier death. Calculate the annual premium payable for a policyholder who effects this insurance at exact age 45.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Select mortality and 4% *pa* interest.

Solution

If the annual premium is P, then:

$$\begin{aligned}\text{EPV premiums} &= P\ddot{a}_{[45]:\overline{25}|} \\ &= P\left(\ddot{a}_{[45]} - \frac{D_{70}}{D_{[45]}}\ddot{a}_{70}\right) \\ &= P\left(18.829 - \frac{517.23}{1,677.42}\times 10.375\right) \\ &= 15.630P\end{aligned}$$

$$\begin{aligned}\text{EPV benefits} &= 75,000A_{[45]:\overline{25}|} \\ &= 75,000\left(A^{\;1}_{[45]:\overline{25}|} + A_{[45]:\overline{25}|}^{\;\;\;\;\;\;\;1}\right) \\ &= 75,000\left(A_{[45]} - \frac{D_{70}}{D_{[45]}}A_{70} + \frac{D_{70}}{D_{[45]}}\right) \\ &= 75,000\left(0.27583 - \frac{517.23}{1,677.42}\times 0.60097 + \frac{517.23}{1,677.42}\right) \\ &= 75,000\times 0.39887 = £29,915\end{aligned}$$

$$\begin{aligned}\text{EPV expenses} &= 0.75P + 0.05P(\ddot{a}_{[45]:\overline{25}|} - 1) + 250 \\ &= 0.75P + 0.05\times 14.630P + 250 \\ &= 1.4815P + 250\end{aligned}$$

So the equation of value is:

$$15.630P = 29,915 + 1.4815P + 250 \Rightarrow P = 30,165 / 14.1484 = £2,132$$

In the above, we could have calculated $a_{[45]:\overline{24|}}$ to value the renewal expenses. Instead, we used $\ddot{a}_{[45]:\overline{25|}}-1$, which saved work since $\ddot{a}_{[45]:\overline{25|}}$ had already been calculated. Another way of saving work would be to combine the renewal expenses and initial expenses together as:

$$\text{EPV expenses} = 0.7P + 0.05P\ddot{a}_{[45]:\overline{25|}} + 250$$

3.6 Conventional with-profits contracts

Gross premiums for with-profits contracts will include not only loadings for expenses, but also for future bonuses.

Extending the preceding example (the whole life assurance described at the start of Section 3.5 above), recall that the equation of value for this contract was:

$$S\,A_x + I + e\ddot{a}_x + f\,A_x = G\,\ddot{a}_x$$

Suppose the contract is now with-profits, and a compound bonus of b per annum is assumed. A compound bonus is added to the basic sum assured and existing bonuses. Let us assume that premiums are annual and that the first bonus is awarded at the end of the first policy year (*ie* no bonus is available to deaths in the first year). In this case, the equation of value is:

$$Svq_x + S(1+b)v^2\,{}_{1|}q_x + S(1+b)^2v^3\,{}_{2|}q_x + \cdots + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

or:

$$S\frac{1}{1+b}\left\{(1+b)vq_x + (1+b)^2v^2\,{}_{1|}q_x + (1+b)^3v^3\,{}_{2|}q_x + \cdots\right\} + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$$

Here we are defining b as a proportion (of the sum assured plus previously added bonuses).

This can then be simplified to:

$$S\frac{1}{1+b}A_x^j + I + e\ddot{a}_x + f\,A_x = G\ddot{a}_x$$

where the assurance function is determined using an interest rate j where $j = \dfrac{(1+i)}{(1+b)} - 1$.

We first saw this approach in Chapter 18. What we have done is to take out a factor of $\dfrac{1}{1+b}$ so that the powers of v match the powers of $1+b$ in all the terms in the big bracket. So we can now think of $(1+b)v$ as the 'v' for some different interest rate, and the expression reduces to an A_x term, but calculated at a new rate of interest, in this case at $j = \dfrac{1+i}{1+b} - 1$.

In the case of a simple bonus, the bonus is added to the sum assured only. Simple bonus cases may therefore be valued using the increasing assurance function defined in Chapter 18.

Question

A man aged 45 buys a 15-year with-profits endowment assurance with a basic sum assured of £25,000. Determine the single premium to be paid for this assurance, assuming that simple reversionary bonuses of 6% *pa* vest at the end of each policy year and that death benefits are payable at the end of the year of death. Assume AM92 Ultimate mortality and 4% *pa* interest. Initial expenses are £200 and renewal expenses are £30 at the start of each policy year, excluding the first.

Solution

The single premium is equal to the expected present value of the benefits and the expenses.

On death during the first year, the benefit is just the basic sum assured of £25,000. The benefit on death in subsequent years will increase each year by $0.06 \times 25,000 = £1,500$ because of the bonuses. So:

$$\text{EPV benefits} = 23,500 A^{\,1}_{45:\overline{15}|} + 1,500 (IA)^{\,1}_{45:\overline{15}|} + 47,500 A_{45:\overline{15}|}^{\;\;\;\,\,1}$$

Since the subscripts sum to 60, the endowment assurance factor is given in the *Tables*. We can write the expected present value using the endowment assurance factor as follows:

$$23,500 A_{45:\overline{15}|} + 1,500 (IA)^{\,1}_{45:\overline{15}|} + 24,000 A_{45:\overline{15}|}^{\;\;\;\,\,1}$$

Now:

$$A_{45:\overline{15}|} = 0.56206$$

$$A_{45:\overline{15}|}^{\;\;\;\,\,1} = \frac{D_{60}}{D_{45}} = \frac{882.85}{1,677.97} = 0.52614$$

$$(IA)^{\,1}_{45:\overline{15}|} = (IA)_{45} - \frac{D_{60}}{D_{45}} \left[(IA)_{60} + 15 A_{60} \right]$$

$$= 8.33628 - 0.52614 \left[8.36234 + 15 \times 0.45640 \right]$$

$$= 0.33454$$

So:

$$\text{EPV benefits} = 23,500 \times 0.56206 + 1,500 \times 0.33454 + 24,000 \times 0.52614$$

$$= 26,337.62$$

Also:

$$\text{EPV expenses} = 200 + 30 \left(\ddot{a}_{45:\overline{15}|} - 1 \right) = 200 + 30 \times 10.386 = 511.58$$

So the single premium for the policy is:

$$26,337.62+511.58=£26,849$$

3.7 Premiums payable *m* times per year

If premiums are payable m times per year, then the expected present values of premiums and level annual expenses must be determined using expressions for m thly annuities as derived in Chapter 17.

Example: Whole life assurance

The equation of value is:

$$S\bar{A}_x + I + e\ddot{a}_x^{(m)} + f\bar{A}_x = G\ddot{a}_x^{(m)}$$

with a given basis and expenses. Using the approximations from Chapter 17 the equation can be solved for G.

Expenses and premiums can be assumed to be payable at different frequencies.

Question

Sam, aged 40, buys a 20-year term assurance with a sum assured of £150,000 payable immediately on death. Calculate the quarterly premium payable by Sam for this policy. Assume that initial expenses are 60% of the total annual premium plus £110, and renewal expenses are £30 *pa* from Year 2 onwards.

Basis: AM92 Select, 4% *pa* interest

Solution

Let P denote Sam's quarterly premium. Then:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20}|} \approx 4P\left[\ddot{a}_{[40]:\overline{20}|} - \frac{3}{8}\left(1-\frac{D_{60}}{D_{[40]}}\right)\right]$$

using the formula on page 36 of the *Tables*. From the *Tables*:

$$\ddot{a}_{[40]:\overline{20}|} = 13.930$$

and:

$$\frac{D_{60}}{D_{[40]}} = \frac{882.85}{2,052.54} = 0.43013$$

So:

$$\text{EPV premiums} = 4P\ddot{a}^{(4)}_{[40]:\overline{20}|} \approx 4P\left[13.930 - \frac{3}{8}(1-0.43013)\right] = 54.865P$$

Also:

$$\text{EPV benefits} = 150,000\bar{A}^{\ 1}_{[40]:\overline{20}|}$$

$$\approx 150,000 \times 1.04^{½}\left(A_{[40]:\overline{20}|} - \frac{D_{60}}{D_{[40]}}\right)$$

$$= 150,000 \times 1.04^{½}(0.46423 - 0.43013)$$

$$= 5,216.97$$

and:

$$\text{EPV expenses} = 0.6 \times 4P + 110 + 30\left(\ddot{a}_{[40]:\overline{20}|} - 1\right)$$

$$= 2.4P + 110 + 30 \times 12.930$$

$$= 2.4P + 497.90$$

Alternatively we can think of the expense terms as 60% of the total annual premium plus £80 initially, and then £30 per year every year including the first. This will give us:

$$0.6 \times 4P + 80 + 30\ddot{a}_{[40]:\overline{20}|}$$

for the expenses. Both give the same numerical value. Use whichever method seems easier.

So the equation of value is:

$$54.865P = 5,216.97 + 2.4P + 497.90$$

$$\Rightarrow P = £108.93$$

The same approach can be used to determine equations of value for the gross premiums of annuity contracts with annuity benefits payable *m* thly.

Question

An annuity of £8,500 *pa* is to be paid monthly in advance for the remaining lifetime of Mrs S, who is currently aged exactly 60. Calculate the single premium that should be paid for this annuity, allowing for initial expenses of 1.5% of the premium and administration expenses payable at the start of each year, including the first. Administration expenses are £120 at the start of Year 1 and increase at the rate of 1.9231% *pa*. Assume AM92 Select mortality and 6% *pa* interest.

Solution

The single premium P is equal to the expected present value of benefits and expenses. We have:

$$\text{EPV benefits} = 8{,}500\ddot{a}_{[60]}^{(12)} \text{ @6\%} \approx 8{,}500\left(\ddot{a}_{[60]} - \frac{11}{24}\right)$$

$$= 8{,}500\left(11.919 - \frac{11}{24}\right) = 97{,}415.67$$

and:

$$\text{EPV expenses} = 0.015P + 120\left(1 + 1.019231 v\, p_{[60]} + 1.019231^2 v^2\, {}_2p_{[60]} + \cdots\right)$$

$$= 0.015P + 120\ddot{a}_{[60]} \text{ @4\%}$$

$$= 0.015P + 120 \times 14.167$$

$$= 0.015P + 1{,}700.04$$

So:

$$P = 97{,}415.67 + 0.015P + 1{,}700.04 \Rightarrow P = £100{,}625$$

4 Calculating gross premiums using simple criteria other than the equivalence principle

So far we have seen two ways of calculating a gross premium:

- to satisfy a probability
- using the equivalence principle.

Using the equivalence principle implies that:

E[present value of future loss] = 0

so that on average (provided the assumptions used are true) the contract will 'break even'. It is usual to load premiums for profit so that:

E[present value of future loss] < 0

If a criterion based on this expected value is chosen to reflect the 'loading for profit' required, then a gross premium including a profit loading can be determined.

Example: Whole life assurance

If the criterion specifies an expected present value of future loss of $-\pi$, then the equation of value becomes:

$$S\bar{A}_x + I + e\ddot{a}_x^{(m)} + f\bar{A}_x + \pi = G\ddot{a}_x^{(m)}$$

Here we are assuming that premiums and renewal expenses are payable $m$thly in advance, the death benefit is payable immediately on death, and the expense claim is incurred at the time that the death benefit is paid. We are adding a loading π for profit, which means that, in addition to covering benefits and expenses, our premiums must also cover this profit requirement. So the profit element will have the same sign as benefits and expenses, even though we might intuitively think that profit is 'nice' and should therefore have the opposite sign.

Thinking about the impact on premium, this will give us the right result: adding an explicit profit requirement will give larger premiums if we treat the profit requirement in the same way as benefits and expenses, and we would expect to need greater premiums if the shareholders are demanding greater profits.

Question

Calculate the annual premium, payable monthly in advance, for a deferred annuity of £12,400 *pa* to be paid quarterly in advance from age 60 to a male now aged 40. Initial expenses are 80% of the annual premium, renewal expenses are 4% of the annual premium incurred at the start of each year from Year 2 onwards, annuity payment expenses are £15 per payment, and the EPV of profit is 2% of the annual premium. Assume AM92 Ultimate mortality and 4% *pa* interest.

Solution

Making an allowance for profit, the equation of value is:

EPV premiums = EPV benefits + EPV expenses + EPV profit

For an annual premium P:

$$
\begin{aligned}
\text{EPV premiums} &= P\ddot{a}^{(12)}_{40:\overline{20}|} \\
&\approx P\left(\ddot{a}_{40:\overline{20}|} - \frac{11}{24}\left(1-\frac{D_{60}}{D_{40}}\right)\right) \\
&= P\left(13.927-\frac{11}{24}\left(1-\frac{882.85}{2{,}052.96}\right)\right) \\
&= 13.666P
\end{aligned}
$$

$$
\begin{aligned}
\text{EPV benefits} &= 12{,}400\ {}_{20|}\ddot{a}^{(4)}_{40} \\
&= 12{,}400\frac{D_{60}}{D_{40}}\ddot{a}^{(4)}_{60} \\
&\approx 12{,}400\times\frac{882.85}{2{,}052.96}\left(14.134-\frac{3}{8}\right) \\
&= 12{,}400\times 5.91687 \\
&= 73{,}369.40
\end{aligned}
$$

$$
\begin{aligned}
\text{EPV expenses} &= 0.8P+0.04P\left(\ddot{a}_{40:\overline{20}|}-1\right)+4\times 15\ {}_{20|}\ddot{a}^{(4)}_{40} \\
&= 0.8P+0.04P\times 12.927+60\times 5.91687 \\
&= 1.31708P+355.01
\end{aligned}
$$

$$\text{EPV profit} = 0.02P$$

So we have:

$$13.666P = 73{,}369.40+1.31708P+355.01+0.02P$$

giving:

$$P=\frac{73{,}724.42}{12.3287}=£5{,}980$$

Chapter 19 Summary

Future loss random variable

This is defined as:

$$L_t = \{\text{present value of future outgo}\} - \{\text{present value of future income}\}$$

as at exact policy duration t .

Principle of equivalence

The principle states that $E[L_0]=0$. It leads to the equation of value:

Expected present value of income = Expected present value of outgo

This equation is solved to calculate the premium.

The premium depends on the set of assumptions regarding the future experience that has been used. Such a set of assumptions is called the pricing (or premium) basis.

Gross premiums

Gross premiums (also called office premiums) are calculated allowing for expenses, and are the actual premiums payable for a contract. Using the equivalence principle, we find the gross premium by solving:

EPV of premiums = EPV of benefits + EPV of expenses

This may be modified if we have an explicit profit criterion to:

EPV of premiums = EPV of benefits + EPV of expenses + EPV profit

Premiums that satisfy probabilities

A premium for a new policy can be found that satisfies a probability defined in terms of the future loss random variable, L_0. For example, we can define the premium as being the smallest premium for which $P(L_0 > 0) \leq \alpha$, where α is some acceptably small probability.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 19 Practice Questions

19.1 Write down an expression for the gross future loss random variable at issue for a deferred life annuity with a deferred period of n years. Premiums of amount G are paid annually in advance for a maximum of n years, the annuity benefit of B is paid for life annually in advance starting in n years' time, and no benefit is paid if the life does not survive to time n. Assume that there are regular expenses payable annually in advance during the premium payment term of e, additional initial expenses of I at the start of Year 1, and regular benefit payment expenses of e' payable annually in advance while the benefit is being paid.

19.2 Show how you would modify the premium equation below to allow for the expenses indicated:

$$P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|}$$

(i) initial expenses of 2% of the sum assured

(ii) renewal expenses of 2% of each premium, including the first

(iii) claim expenses of 2% of the sum assured

(iv) initial expenses of 50% of first premium plus renewal expenses of 3% of each premium excluding the first.

19.3 A life office sells 5-year term assurance policies to lives aged 60. Each policy has a sum assured of £10,000 payable at the end of the year of death. Premiums of £200 are payable annually in advance throughout the 5-year term or until earlier death.

Exam style

Let L denote the present value of the insurer's loss on one of these policies, at policy outset, ignoring expenses.

(i) Write down an expression for L. [2]

(ii) Assuming AM92 Ultimate mortality and 5½% *pa* interest, calculate the expected value and standard deviation of L. [9]

[Total 11]

19.4 Calculate the annual premium payable in advance by a life now aged exactly 32, in respect of a deferred annuity payable from age 60 for 5 years certain and for life thereafter. The amount of the annuity is 400 *pa*, payable annually in arrear, and the insurer incurs an additional administration cost of 2 when each annuity payment is made. The premium is paid throughout the deferred period or until the earlier death of the policyholder.

Basis: AM92 Ultimate mortality, 6% *pa* interest

19.5 An insurer issues a combined term assurance and annuity contract to a life aged 35. Level premiums are payable monthly in advance for a maximum of 30 years.

Exam style

On death before age 65 a benefit is paid immediately. The benefit is £200,000 on death in the first year of the contract, £195,000 on death in the second year, £190,000 on death in the third year, *etc*, with the benefit decreasing by £5,000 each year until age 65. No benefit is payable on death after age 65.

On attaining age 65 the life receives a whole life annuity of £10,000 *pa* payable monthly in arrears.

Calculate the monthly premium using the following basis:

Mortality:	up to age 65:	AM92 Select
	over age 65:	PFA92C20
Interest:	4% *pa*	
Expenses:	Initial:	£350
	Regular:	Assumed incurred annually at the start of each year during the deferred period, equal to 45 in the first year, inflating at the rate of 4% *pa*
	Claim:	0.5% of each annuity payment

[12]

19.6 A life insurance company sells term assurance policies with a term of 2 years, with level premiums paid annually in advance, to male lives aged 60 exact at policy commencement. Each policy has a sum assured of £50,000, which is payable at the end of the year of death. The company prices the product assuming AM92 Ultimate mortality.

The premium is calculated from an equation of value, in which the expected present value of the premiums is set to equal the expected present value of the benefit payments plus 10% of the standard deviation of the present value of the benefit payments. Calculate the premium assuming 4% *pa* interest. Ignore expenses.

19.7 An annual premium conventional with-profits 20-year endowment assurance policy, issued to a life aged exactly 40 has a basic sum assured of £10,000 payable at the end of the year of death. Premiums are calculated assuming AM92 Select mortality, 4% *pa* interest, initial expenses of £150 and claim-related expenses of 3% of the basic sum assured (payable at the end of the year of death or on maturity).

Exam style

(i) Calculate the premium if the policy is assumed to provide simple bonuses of 2% of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by £200 at the end of each policy year for future claims). [6]

(ii) Calculate the premium if the policy is assumed to provide compound bonuses of 4% *pa* of the sum assured vesting at the end of each policy year (*ie* the basic benefit amount will be increased by a factor of 1.04 at the end of each policy year for future claims). [5]

[Total 11]

19.8 A life insurance company issues whole life assurance policies to lives aged 50 exact for a sum assured of £75,000 payable at the end of the year of death. Premiums are payable annually in advance.

Exam style

(i) Calculate the annual gross premium for each policy using the basis below. [4]

(ii) Calculate the minimum annual gross premium that the company should charge in order that the probability of making a loss on any one policy would be 10% or less. [6]

Basis:

Mortality:	AM92 Select
Interest:	6% per annum
Initial commission:	100% of the annual gross premium
Initial expenses:	£325
Renewal commission:	2.5% of each annual gross premium excluding the first
Renewal expenses:	£75 per annum at the start of the second and subsequent policy years

[Total 10]

19.9 A life aged 60 exact purchases a special deferred term assurance policy for an overall term of 20 years.

Exam style

Under this policy a sum assured of £100,000 is paid on death but only on death from age 65 exact up to the end of the term. On death between age 60 and 65 the benefit is equal to the total premiums paid without interest.

All payments on death are made at the end of the year of death. An annual premium paid in advance is payable for the full 20 year term.

Calculate the annual premium payable. [7]

Basis:	Mortality	AM92 Ultimate
	Interest	4% per annum
	Expenses	Ignore

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 19 Solutions

19.1 As premiums, benefits and expenses are payable annually, we can use K_x, the curtate future lifetime, rather than T_x.

It is easiest to split the gross future loss random variable (FLRV) by whether the person dies before or after time n. So:

$$FLRV = I + e\ddot{a}_{\overline{K_x+1}|} - G\,\ddot{a}_{\overline{K_x+1}|} \qquad \text{if } K_x < n$$

(*ie* if the life does not survive to the end of the deferred period), and:

$$I + e\ddot{a}_{\overline{n}|} - G\,\ddot{a}_{\overline{n}|} + B\ddot{a}_{\overline{K_x+1-n}|}v^n + e'\ddot{a}_{\overline{K_x+1-n}|}v^n \qquad \text{if } K_x \ge n$$

(*ie* if the life survives to receive payments under the annuity).

Alternatively we can write:

$$\ddot{a}_{\overline{K_x+1-n}|}v^n = \ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|}$$

and collecting together terms we obtain:

$$FLRV = \begin{cases} I - (G-e)\,\ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ I - (G-e)\,\ddot{a}_{\overline{n}|} + (B+e')\left(\ddot{a}_{\overline{K_x+1}|} - \ddot{a}_{\overline{n}|}\right) & \text{if } K_x \ge n \end{cases}$$

19.2 (i) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02S$

(ii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02P\ddot{a}_{x:\overline{n}|}$ which simplifies to $0.98P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|}$

(iii) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.02SA_{x:\overline{n}|}$ which simplifies to $P\ddot{a}_{x:\overline{n}|} = 1.02SA_{x:\overline{n}|}$

(iv) $P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.5P + 0.03P(\ddot{a}_{x:\overline{n}|} - 1)$

This simplifies to:

$$0.97P\ddot{a}_{x:\overline{n}|} = SA_{x:\overline{n}|} + 0.47P$$

19.3 (i) ***Future loss random variable***

The future loss random variable is:

$$L=\begin{cases}10{,}000v^{K_{60}+1}-200\ddot{a}_{\overline{K_{60}+1|}} & \text{if } K_{60}=0,1,2,3,4\\ -200\ddot{a}_{\overline{5|}} & \text{if } K_{60}\geq 5\end{cases}$$

where K_{60} denotes the curtate future lifetime of a life currently aged 60. [2]

(ii) ***Expectation and variance of future loss***

Since we don't have any tabulated functions for 5½% interest, we will need to do the calculations for the expected value numerically (by considering the possible cases for the time of death), rather than attempting an algebraic method using A functions etc.

The table below shows the possible values of L and their associated probabilities:

Curtate future lifetime, K_{60}	Loss, L	Probability
0	9,278.67	$q_{60}=0.0080220$
1	8,594.95	$p_{60}\,q_{61}=0.0089367$
2	7,946.87	${}_2p_{60}\,q_{62}=0.0099405$
3	7,332.58	${}_3p_{60}\,q_{63}=0.011039$
4	6,750.31	${}_4p_{60}\,q_{64}=0.012234$
≥ 5	−901.03	${}_5p_{60}=0.94983$

[5]

The probabilities can alternatively be calculated as $\dfrac{d_{60}}{l_{60}},\dfrac{d_{61}}{l_{60}},\dfrac{d_{62}}{l_{60}},\dfrac{d_{63}}{l_{60}},\dfrac{d_{64}}{l_{60}}$ *and* $\dfrac{l_{65}}{l_{60}}$.

The expected present value of the future loss random variable is:

$$E(L)=(9{,}278.67\times 0.0080220)+(8{,}594.95\times 0.0089367)+\cdots$$

$$+(-901.03\times 0.94983)$$

$$=-462.06$$

[1]

The variance of L is given by:

$$E\left(L^2\right)-\left[E(L)\right]^2$$

Now:

$$E\left(L^2\right)=\left(9{,}278.67^2\times 0.0080220\right)+\cdots+\left(901.03^2\times 0.94983\right)$$

$$=3{,}900{,}700 \text{ (5 } sf\text{)}$$

So:

$$\text{var}(L)=3{,}900{,}700-(-462.06)^2=(\pounds 1{,}920)^2$$

ie the standard deviation of the present value of the loss is £1,920. [3]

[Total 9]

19.4 The premium equation is:

$$P\ddot{a}_{32:\overline{28}|}=(400+2)v^{28}\frac{l_{60}}{l_{32}}\left(a_{\overline{5}|}+v^5\frac{l_{65}}{l_{60}}a_{65}\right)$$

The factors are:

$\ddot{a}_{32:\overline{28}|}=14.053$ (tabulated on page 106 of the *Tables*)

$$v^{28}\frac{l_{60}}{l_{32}}=\frac{1}{1.06^{28}}\times\frac{9{,}287.2164}{9{,}913.3821}=0.18327$$

$$a_{\overline{5}|}=4.2124$$

$$v^5\frac{l_{65}}{l_{60}}=\frac{1}{1.06^5}\times\frac{8{,}821.2612}{9{,}287.2164}=0.70977$$

$$a_{65}=\ddot{a}_{65}-1=9.569$$

So the premium equation becomes:

$$14.053P=402\times 0.18327\times 11.004=810.74$$

$$\Rightarrow P=810.74/14.053=57.69$$

19.5 If the monthly premium is P, the premium equation is:

$$12P\ddot{a}^{(12)}_{[35]:\overline{30}|}=205{,}000\bar{A}^{1}_{[35]:\overline{30}|}-5{,}000(I\bar{A})^{\,1}_{[35]:\overline{30}|}+1.005\times 10{,}000\frac{D_{65}}{D_{[35]}}a^{(12)}_{65}$$

$$+350+45\ddot{a}_{[35]:\overline{30}|@0\%}$$ [3]

The regular expenses inflate at the same compound rate at which they are discounted, so we can value them using an annuity factor calculated at a zero interest rate.

The factors (calculated using the appropriate mortality and interest rates) are:

$$\ddot{a}^{(12)}_{[35]:\overline{30}|} \approx \ddot{a}_{[35]:\overline{30}|} - \frac{11}{24}\left(1-\frac{D_{65}}{D_{[35]}}\right) = 17.631 - \frac{11}{24}\times\left(1-\frac{689.23}{2,507.02}\right) = 17.299 \quad [1]$$

using the formula on page 36 of the *Tables.*

$$\bar{A}^{\,1}_{[35]:\overline{30}|} \approx 1.04^{½}\left(A_{[35]:\overline{30}|} - \frac{D_{65}}{D_{[35]}}\right) = 1.04^{½}\left(0.32187 - \frac{689.23}{2,507.02}\right) = 0.04788 \quad [1]$$

$$\begin{aligned}(I\bar{A})^{\,1}_{[35]:\overline{30}|} &\approx 1.04^{½}\left((IA)_{[35]} - \frac{D_{65}}{D_{[35]}}\left((IA)_{65} + 30A_{65}\right)\right)\\ &= 1.04^{½}\left(7.47005 - \frac{689.23}{2,507.02}\left(7.89442 + 30\times 0.52786\right)\right)\\ &= 0.96487 \quad [2]\end{aligned}$$

$$a^{(12)}_{65} = \ddot{a}^{(12)}_{65} - \frac{1}{12} \approx \ddot{a}_{65} - \frac{11}{24} - \frac{1}{12} = 14.871 - \frac{13}{24} = 14.329 \quad [1]$$

$$\ddot{a}_{[35]:\overline{30}|@0\%} = \ddot{a}_{[35]@0\%} - (v@0\%)^{30}\frac{l_{65}}{l_{[35]}}\ddot{a}_{65@0\%} \quad [1]$$

But:

$$v@0\% = 1$$

$$\ddot{a}_{x@0\%} = 1 + e_x$$

So, using AM92 Select mortality:

$$\begin{aligned}\ddot{a}_{[35]:\overline{30}|@0\%} &= 1 + e_{[35]} - \frac{l_{65}}{l_{[35]}}(1+e_{65})\\ &= 1 + 43.909 - \frac{8,821.2612}{9,892.9151}\times(1+16.645)\\ &= 29.175 \quad [2]\end{aligned}$$

So the premium equation becomes:

$$\begin{aligned}12P\times 17.299 = {}&205,000\times 0.04788 - 5,000\times 0.96487 + 10,050\times\frac{689.23}{2,507.02}\times 14.329\\ &+350 + 45\times 29.175\end{aligned}$$

So: $$P = \frac{46,245.05}{12\times 17.299} = £222.78 \text{ per month} \quad [1]$$

[Total 12]

19.6 The expected present value of the benefit outgo is:

$$50{,}000A^{1}_{60:\overline{2}|} = 50{,}000\left(A_{60} - \frac{D_{62}}{D_{60}}A_{62}\right)$$

$$= 50{,}000\left(0.45640 - \frac{802.40}{882.85} \times 0.48458\right)$$

$$= 50{,}000 \times 0.0159775$$

$$= £798.88 \quad [1]$$

Alternatively we could calculate this from first principles as:

$$50{,}000A^{1}_{60:\overline{2}|} = 50{,}000 \times \left(vq_{60} + v^2 p_{60} q_{61}\right)$$

$$= 50{,}000 \times \left(1.04^{-1} \times 0.008022 + 1.04^{-2} \times (1 - 0.008022) \times 0.009009\right)$$

$$= 798.80$$

The answer is sensitive to the rounding used in the Tables.

The variance per unit sum assured of the present value is equal to ${}^{2}A^{1}_{60:\overline{2}|} - (A^{1}_{60:\overline{2}|})^2$.

From first principles, the variance is:

$$v' q_{60} + v'^2 p_{60} q_{61} - (0.0159775)^2 \quad [1]$$

where v' is calculated at $1.04^2 - 1 = 8.16\%$. So the variance is:

$$\left(1.0816^{-1} \times 0.008022 + 1.0816^{-2} \times 0.991978 \times 0.009009\right) - (0.0159775)^2$$

$$= 0.014801 \quad [2]$$

Therefore, the variance for a sum assured of £50,000 is:

$$50{,}000^2 \times 0.014801 = 6{,}082.9^2$$

and so the standard deviation is $6{,}082.9$. [1]

The expected present value of the premiums is:

$$P\ddot{a}_{60:\overline{2}|} = P\left(1 + v\,p_{60}\right) = P\left(1 + \frac{0.991978}{1.04}\right) = 1.953825P \quad [1]$$

Hence:

$$P = \frac{798.88 + 608.29}{\ddot{a}_{60:\overline{2}|}} = \frac{1{,}407.17}{1.953825} = £720.21 \quad [1]$$

[Total 7]

19.7 (i) ***Simple bonuses***

The benefit amount on death will be £10,000 for the first year, £10,200 for the second year, £10,400 for the third year, *etc*, and the benefit amount on maturity will be £14,000.

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20}|} = 9{,}800A_{[40]:\overline{20}|} + 200(IA)^{\,1}_{[40]:\overline{20}|} + 4{,}200\frac{D_{60}}{D_{[40]}} + 150 + 300A_{[40]:\overline{20}|} \qquad [2]$$

The increasing assurance function can be calculated as:

$$\begin{aligned}(IA)^{\,1}_{[40]:\overline{20}|} &= (IA)_{[40]} - \frac{D_{60}}{D_{[40]}}\left[(IA)_{60} + 20A_{60}\right] \\ &= 7.95835 - \frac{882.85}{2{,}052.54}\left[8.36234 + 20\times 0.45640\right] \\ &= 0.43531 \qquad [2]\end{aligned}$$

So the premium equation becomes:

$$\begin{aligned}13.930P &= 9{,}800\times 0.46423 + 200\times 0.43531 + 4{,}200\times\frac{882.85}{2{,}052.54} \\ &\quad + 150 + 300\times 0.46423 \\ &= 6{,}732.31 \qquad [1]\end{aligned}$$

Hence:

$$P = \frac{6{,}732.31}{13.930} = £483.30 \qquad [1]$$

[Total 6]

(ii) ***Compound bonuses***

The benefit amount on death will be £10,000 for the first year, $£10{,}000\times 1.04$ for the second year, $£10{,}000\times 1.04^2$ for the third year, *etc*, and the benefit amount on maturity will be $£10{,}000\times 1.04^{20}$.

The expected present value of the death benefit is:

$$\begin{aligned}&10{,}000\times\left(v\,{}_{0|}q_{[40]} + 1.04v^2\,{}_{1|}q_{[40]} + 1.04^2v^3\,{}_{2|}q_{[40]} + \cdots + 1.04^{19}v^{20}\,{}_{19|}q_{[40]}\right) \\ &= \frac{10{,}000}{1.04}\times\left(1.04v\,{}_{0|}q_{[40]} + 1.04^2v^2\,{}_{1|}q_{[40]} + 1.04^3v^3\,{}_{2|}q_{[40]} + \cdots + 1.04^{20}v^{20}\,{}_{19|}q_{[40]}\right)\end{aligned}$$

But:

$$1.04v = \frac{1.04}{1.04} = 1$$

So the EPV of the death benefit becomes:

$$\frac{10,000}{1.04}\times\left({}_{0|}q_{[40]}+{}_{1|}q_{[40]}+{}_{2|}q_{[40]}+\dots+{}_{19|}q_{[40]}\right)=\frac{10,000}{1.04}\times{}_{20}q_{[40]}$$

The EPV of the maturity benefit is:

$$10,000\times1.04^{20}v^{20}\,{}_{20}p_{[40]}=10,000\,{}_{20}p_{[40]}$$

So the premium equation can be expressed as:

$$P\ddot{a}_{[40]:\overline{20|}}=\frac{10,000}{1.04}\times{}_{20}q_{[40]}+10,000\,{}_{20}p_{[40]}+150+300A^{@4\%}_{[40]:\overline{20|}}$$ [2]

The probabilities are:

$${}_{20}p_{[40]}=\frac{l_{60}}{l_{[40]}}=\frac{9,287.2164}{9,854.3036}=0.94245$$ [½]

and:

$${}_{20}q_{[40]}=1-{}_{20}p_{[40]}=1-0.94245=0.05755$$ [½]

So the premium equation becomes:

$$13.930P=\frac{10,000}{1.04}\times0.05755+10,000\times0.94245+150+300\times0.46423$$
$$=10,267.14$$ [1]

and hence:

$$P=\frac{10,267.14}{13.930}=£737.05$$ [1]

[Total 5]

19.8 *This question is Subject CT5, April 2013, Question 12.*

(i) ***Gross premium***

Letting G denote the gross premium, the expected present value of the gross premiums is:

$$G\ddot{a}_{[50]}=14.051G$$ [1]

The expected present value of the benefits is:

$$75,000A_{[50]}=75,000\times0.20463=15,347.25$$ [1]

The expected present value of the commission and expenses is:

$$G+325+0.025G(\ddot{a}_{[50]}-1)+75(\ddot{a}_{[50]}-1)$$

Using $\ddot{a}_{[50]} = 14.051$, this simplifies to:

$$1.326275G + 1,303.825 \qquad [1]$$

The premium equation is therefore:

$$14.051G = 15,347.25 + 1.326275G + 1,303.825$$

$$\Rightarrow \quad G = \frac{16,651.075}{12.724725} = £1,308.56$$

The gross premium is £1,308.56. [1]

[Total 4]

(ii) ***Minimum gross premium so that the probability of loss is 10% or less***

The insurer's gross future loss random variable for the policy is:

$$L = 75,000v^{K_{[50]}+1} + G + 325 + (0.025G + 75)a_{\overline{K_{[50]}}|} - G\ddot{a}_{\overline{K_{[50]}+1}|}$$

$$= 75,000v^{K_{[50]}+1} + 325 + 75a_{\overline{K_{[50]}}|} - 0.975Ga_{\overline{K_{[50]}}|} \qquad [2]$$

where G is the gross annual premium. Let G_n denote the premium such that $L = 0$ when $K_{[50]} = n$, *ie* to make the policy breakeven if the person dies in year $[n, n+1]$. Then:

$$0 = 75,000v^{n+1} + 325 + 75a_{\overline{n}|} - 0.975G_n a_{\overline{n}|}$$

$$\Rightarrow G_n = \frac{75,000v^{n+1} + 325 + 75a_{\overline{n}|}}{0.975a_{\overline{n}|}} \qquad [1]$$

Now we make a loss on the policy if the policyholder dies before time n. So:

$$P\left(L > 0 \middle| G = G_n\right) = P\left(K_{[50]} < n\right)$$

and this probability needs to be at most 10%. G_n reduces as n increases, so to find the minimum premium we need to find the largest value of n such that:

$$P\left(K_{[50]} < n\right) \le 0.1$$

This is equivalent to finding n such that:

$$P\left(K_{[50]} \ge n\right) = {}_n p_{[50]} = \frac{l_{[50]+n}}{l_{[50]}} \ge 0.9$$

Using the AM92 table, $l_{[50]} = 9,706.0977$, so we have:

$$l_{[50]+n} \ge 0.9 \times 9,706.0977 = 8,735.488$$

From the *Tables*: $l_{65} = 8,821.2612$ and $l_{66} = 8,695.6199$.

So, the largest value of n to satisfy this inequality is 15. [2]

The minimum premium required is then:

$$G_{15} = \frac{75,000v^{16} + 325 + 75a_{\overline{15}|}}{0.975a_{\overline{15}|}}$$

Using page 58 of the *Tables*:

$$a_{\overline{15}|} = 9.7122$$

Substituting this into our equation gives:

$$G_{15} = \frac{75,000v^{16} + 325 + 75 \times 9.7122}{0.975 \times 9.7122} = \frac{30,576.89}{9.4694} = £3,229.02$$

So, the minimum premium to ensure that the probability of making a loss is at most 10% is £3,229.02. [1]

[Total 6]

19.9 *This question is Subject CT5, September 2014, Question 9.*

The premium equation is:

$$P\ddot{a}_{60:\overline{20}|} = 100,000 \times \frac{D_{65}}{D_{60}} \times A^{\;\;1}_{65:\overline{15}|} + P(IA)^{\;\;1}_{60:\overline{5}|} \qquad [2]$$

Evaluating the various terms:

$$\ddot{a}_{60:\overline{20}|} = \ddot{a}_{60} - \frac{D_{80}}{D_{60}}\ddot{a}_{80} = 14.134 - \frac{228.48}{882.85} \times 6.818 = 12.370 \qquad [1]$$

$$A^{\;\;1}_{65:\overline{15}|} = A_{65} - \frac{D_{80}}{D_{65}}A_{80} = 0.52786 - \frac{228.48}{689.23} \times 0.73775 = 0.28330 \qquad [1]$$

and:

$$(IA)^{\;\;1}_{60:\overline{5}|} = (IA)_{60} - \frac{D_{65}}{D_{60}}\left[(IA)_{65} + 5A_{65}\right]$$

$$= 8.36234 - \frac{689.23}{882.85}\left[7.89442 + 5 \times 0.52786\right] = 0.13880 \qquad [2]$$

Putting this all together, we find that:

$$12.370P = 100,000 \times \frac{689.23}{882.85} \times 0.28330 + 0.13880P$$

Solving this, we find that $P = 1,808.28$, or about £1,808. [1]

[Total 7]

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Gross premium reserves

Syllabus objectives

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rates.

4.2.8 Define and evaluate the expected accumulations in terms of expected values for the contracts described in 4.1.1 and contract structures described in 4.2.7.

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts.

6.2.2 State why an insurance company will set up reserves.

6.2.3 Define and calculate gross prospective and retrospective reserves.

6.2.4 State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.

6.2.5 Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all fixed benefit and increasing / decreasing benefit contracts.

6.2.6 Obtain recursive relationships between successive periodic gross premium reserves, and use this relationship to calculate the profit earned from a contract during the period.

Syllabus objectives continued

6.2.7 Outline the concepts of net premiums and net premium valuation and how they relate to gross premiums and gross premium valuation respectively.

0 Introduction

In this chapter we introduce the concept of a reserve, and how it can be calculated using either prospective or retrospective approaches. The prospective formula is based on the gross future loss random variable, which we introduced in Chapter 19, while the retrospective formula requires the use of accumulations, which are described in this chapter.

A reserve represents the amount of money that an insurer sets aside in respect of a policy that is currently in force, in order to meet future payments on that policy. It is important in real life, because an insurer that holds insufficient reserves will ultimately run out of money before all of its policyholders have made their claims, meaning that the company becomes insolvent and policyholders lose money, which can potentially be very large amounts.

Reserve calculations are required for all the kinds of contracts covered in the course so far. The same approaches will be used to calculate reserves for contracts on multiple lives, which we describe in Chapters 21 and 22. The reserve is an essential part of calculating mortality profit, an important topic which is covered in Chapter 23. In the profit testing chapters (Chapters 26 and 27), we will also see an important alternative methodology for calculating reserves that uses cashflow projections.

We also show in this chapter how thinking about the change in the reserve from one year to the next leads to a recursive relationship between successive years' reserves, and which also leads to an important definition of the profit earned by a contract during any given year. The whole concept of profit testing is based on this result.

We generally talk throughout about *gross premium* reserves. These, like gross premiums themselves (Chapter 19), include assumptions about future expenses in their calculation. At the end of this chapter we briefly describe the idea of *net premium* reserves, which make no explicit allowance for expenses. These are included in the course mainly for historical interest, as they are no longer much used in practice; however, a basic knowledge of what they are, and how they should be calculated for conventional without-profit contracts, is required for this subject.

1 Why hold reserves?

In many life insurance contracts, the expected cost of paying benefits increases over the term of the contract. Let us consider an endowment assurance as an example. The probability that the benefit will be paid in the first few years is relatively low because the life is young and expected to be in good health. Later, the expected cost increases as the life ages and the probability of a claim by death increases. In the final year the payment is certain, either on death during the year or on maturity at the end of that year, therefore the expected cost of the benefit will be close to the actual sum assured.

If we ignore discounting, the expected cost in the final year is *equal* to the sum assured.

Although on average the cost of the benefit increases over the term, the premiums which pay for these benefits are usually level. This means that premiums received in the early years of a contract are more than enough to pay the expected benefits that fall due in those early years, but in the later years the premiums are too small to pay for the expected benefits.

It is, therefore, prudent for the premiums which are not required in the early years of a contract to be set aside, or reserved, to fund the shortfall in the later years. While funds are reserved, they are invested so that interest also contributes to the cost of benefits.

If the life insurance company was to spend all the premiums received, perhaps by distributing the surplus to shareholders, in later years it may not have sufficient funds to pay for the excess of the cost of benefits over the premiums received. The company sets up reserves to ensure (as far as possible) that this does not happen, and to remain solvent.

Consider the case of a term assurance sold to a 30-year old British male, with a 10-year term and a sum assured of £500,000. The premium is £48 per month, paid over the 10-year term or until the earlier death of the policyholder. The premium is level (£48 per month) but the expected cost of the benefits is increasing, since the policyholder is getting older and UK mortality rates increase after age 30.

For example, at age 30 the expected cost is $500{,}000q_{30}$, but at age 39 it is $500{,}000q_{39}$, which is bigger than $500{,}000q_{30}$.

When designing the products sold by the insurer, the actuary will always try to ensure that in the later years the benefit payments are greater than the premiums payable.

To understand why this is the case, suppose now that the product is redesigned so that the sum assured is falling over the term of the contract, say by £50,000 per year. The premium is recalculated to be £23 per month.

In that case, the average cost of benefits may well be falling over time (not rising as in the previous contract) since the sum assured is falling. For example, at age 30 the expected cost is $500{,}000q_{30}$, but at age 39 it is $50{,}000q_{39}$.

This can lead to an issue for the insurer if the policyholder decides to stop paying premiums before the full term of the policy has expired. (This event is referred to as a surrender, or lapse, of the policy. Generally we use the term 'surrender' to refer to the case where the policyholder is given some cash payment (the 'surrender value') at the point where premiums cease, and we use the term 'lapse' where no cash payment is made. For term assurances it is normal practice for such policies to lapse without any surrender value being payable.)

Returning to our example, if the premiums are greater than the expected value of the benefits, say in the final few years of the contract, then this may encourage the policyholder to stop paying premiums, and hence cause the policy to lapse. The result is that the policyholder will have held the contract for the time period for which the benefit is greater than the premium but not for the subsequent period when premiums exceed benefits.

Question

Consider the situation described in the last paragraph above. Explain the effect this would have on the insurer.

Solution

We can explain this by thinking of the future period for which premiums are not going to be paid. During the final few years of this contract the expected present value of the premiums payable exceed the expected present value of the benefits that would be paid out – in other words, this period of the policy term, considered on its own, is expected to produce a profit for the insurer. So if we remove this profitable period, then overall the profitability of the contract must fall.

So, whenever any policy is lapsed in this situation, the insurer will expect to lose money.

2 Prospective reserves

The prospective reserve for a life insurance contract which is in force (that is, has been written but has not yet expired through claim or reaching the end of the term) is defined to be, for a given basis:

The expected present value of the future outgo

less

the expected present value of the future income

This is the prospective reserve because it looks forward to the future cashflows of the contract. The prospective reserve is important because if the company holds funds equal to the reserve, and the future experience follows the reserve basis, then, averaging over many policies, the combination of reserve and future income will be sufficient to pay the future liabilities.

This last sentence is very important and forms the foundation of life insurance reserving and hence the rest of this chapter. That is to say that the money an insurer needs to set aside to meet its future payments is the expected present value of future outgo (benefits plus expenses) less the expected present value of future income (premiums).

The reserve, therefore, gives the office a measure of the minimum funds it needs to hold at any point *during* the term of a contract. The process of calculating a reserve is called the *valuation* of the policy.

Reserves are also calculated for other reasons, such as the calculation of surrender values. The insurer may set the surrender value (*ie* the amount paid to the policyholder) by reference to the reserve.

Question

An annual premium endowment policy is surrendered one year before the maturity date. Explain why the reserve is a sensible surrender value for the insurer to pay.

Solution

If the policy remains in force, the insurer has to make a benefit payment in the next year (either on maturity or on the death of the policyholder). However, if the policy is surrendered, no benefit payment will be made and also the final renewal expense will not be incurred. It is therefore reasonable to set the surrender benefit equal to the sum assured plus final year's renewal expenses, less the final premium and the cost of administrative expenses associated with processing the surrender.

2.1 Calculating gross premium prospective reserves

The expression for the gross future loss at policy duration t can be used to determine the gross premium prospective reserve at policy duration t. This is done by determining the expected value of the gross future loss random variable.

In general we use the notation ${}_tV$ to represent the reserve held at policy duration t.

It should be noted that the calculation of gross premium prospective reserves will use mortality, interest, expense, and (if applicable) future bonus assumptions specifically chosen for this purpose. This set of assumptions, called the gross premium valuation basis, may differ from the underlying basis used to calculate the actual gross premiums which the company charges to policyholders.

The premium used in the calculation of the gross premium reserve is always the actual gross premium being charged.

Example: Whole life assurance

Consider a whole life assurance contract that has:

- a sum assured of S payable immediately on death
- annual premiums of G payable for the duration of the contract
- regular expenses of e *pa* incurred at the start of each year while the policy is in force
- claim expenses of f
- age at entry equal to x.

The gross future loss random variable after exactly t years ($t = 1, 2, \ldots$) is:

$$L_t = Sv^{T_{x+t}} + e\ddot{a}_{\overline{K_{x+t}+1}|} + fv^{T_{x+t}} - G\ddot{a}_{\overline{K_{x+t}+1}|}$$

The prospective reserve after exactly t years is therefore:

$$\begin{aligned} {}_tV^{pro} &= E[L_t] \\ &= S \times E\left[v^{T_{x+t}}\right] + e \times E\left[\ddot{a}_{\overline{K_{x+t}+1}|}\right] + f \times E\left[v^{T_{x+t}}\right] - G \times E\left[\ddot{a}_{\overline{K_{x+t}+1}|}\right] \\ &= S\bar{A}_{x+t} + e\ddot{a}_{x+t} + f\bar{A}_{x+t} - G\ddot{a}_{x+t} \end{aligned}$$

where 'pro' stands for 'prospective'.

So we can think of the gross premium prospective reserve as:

EPV of future benefits + EPV of future expenses – EPV of future premiums

Now consider the more general case for this contract, where premiums and renewal expenses are payable $m$thly in advance.

The expected value of the gross future loss random variable at policy duration t is:

$$S\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} + f\bar{A}_{x+t} - G\ddot{a}^{(m)}_{x+t}$$

This can be evaluated using an assumed valuation basis to determine the values of the annuity and assurance functions, including an assumption for future expenses. G is the gross (*ie* actual) premium payable under the contract.

In order to calculate a gross premium reserve for regular premium products, we therefore first need to know the amount of the gross premium. (In the exam this may mean the first thing we might have to do is to *calculate* the gross premium.) This will not be necessary for single premium policies.

Question

Calculate the gross premium prospective reserve that should be held at the end of the tenth policy year of a 25-year regular premium endowment assurance policy with a sum assured of £75,000 payable on maturity or at the end of the year of earlier death. The policy was taken out by a person aged exactly 45 at entry.

The gross premium, which is being paid annually in advance for the duration of the policy, is £2,132 *pa*.

Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Assume AM92 Ultimate mortality and 4% *pa* interest.

Solution

The prospective reserve at the end of Year 10 is:

$${}_{10}V^{pro} = \text{EPV future benefits and expenses} - \text{EPV future premiums}$$

The components are:

$$\text{EPV future premiums} = P\ddot{a}_{55:\overline{15}|}$$

where:

$$\ddot{a}_{55:\overline{15}|} = \ddot{a}_{55} - \frac{D_{70}}{D_{55}}\ddot{a}_{70} = 15.873 - \frac{517.23}{1{,}105.41} \times 10.375 = 11.0185$$

So:

$$\text{EPV future premiums} = 2{,}132 \times 11.0185 = 23{,}491$$

Now:

$$\text{EPV future benefits} = 75,000A_{55:\overline{15}|}$$

Using premium conversion:

$$A_{55:\overline{15}|} = 1 - d\ddot{a}_{55:\overline{15}|} = 1 - \left(\frac{0.04}{1.04}\right) \times 11.0185 = 0.57621$$

So:

$$\text{EPV future benefits} = 75,000 \times 0.57621 = 43,216$$

Also:

$$\text{EPV future expenses} = 0.05P\ddot{a}_{55:\overline{15}|} = 0.05 \times 23,491 = 1,175$$

So we have:

$$_{10}V^{pro} = 43,216 + 1,175 - 23,491 = £20,899$$

We now consider a prospective reserve for an annuity contract.

Question

Andy, aged 40, purchases a single premium whole life annuity of £8,400 *pa* payable monthly in advance from age 60. Initial expenses are 2% of the premium and renewal expenses are £60 *pa* from Year 2 onwards, including during payment of the annuity, assumed incurred annually in advance throughout.

Calculate the reserve for Andy's policy at the end of the tenth policy year. Assume interest of 4% *pa*, mortality AM92 Ultimate in deferment and PMA92C20 from age 60.

Solution

It is a single premium contract so we do not need to calculate the premium in order to calculate the prospective reserve. The gross premium prospective reserve is:

EPV of future benefits + EPV of future expenses

The expected present value of the future benefits is:

$$8,400\ {}_{10|}\ddot{a}_{50}^{(12)} = 8,400\frac{D_{60}}{D_{50}}\ddot{a}_{60}^{(12)}$$

where D_{50} and D_{60} are taken from the AM92 Table, and $\ddot{a}_{60}^{(12)}$ is taken from the PMA92C20 Table.

Therefore the EPV of the future benefits is:

$$8,400\times\frac{882.85}{1,366.61}\times\left(15.632-\tfrac{11}{24}\right)=82,340$$

The expected present value of the future expenses is:

$$60\ddot{a}_{50}=60\left(\ddot{a}_{50:\overline{10|}}+\frac{D_{60}}{D_{50}}\ddot{a}_{60}\right)$$

with $\ddot{a}_{50:\overline{10|}}$ taken from the AM92 Table.

So the expected value of the future expenses is:

$$60\left(8.314+\frac{882.85}{1,366.61}\times15.632\right)=1,105$$

We need to be careful about valuing the pre-age 60 and the post-age 60 elements separately due to the different mortality assumptions before and after age 60.

Thus the reserve is:

$$82,340+1,105=£83,445$$

2.2 Calculating prospective reserves that satisfy probabilities

In the previous chapter, we explained how gross premiums can be calculated that satisfy probabilities. A similar approach can be used to calculate prospective reserves.

Suppose an insurance company is carrying out a valuation of its in-force business. The following details relate to one particular policy that is in force on the valuation date:

Policy type:	Whole life assurance
Benefit:	£75,000 payable at the end of the year of death
Entry age:	50 exact
Current duration in force:	8 years exact
Annual premium:	£1,500 (paid at the start of each year)

Let us now calculate the smallest reserve that could be held at the valuation date, which will ensure that the insurance company can cover the liability under this contract with a probability of at least 97.5%, assuming interest of 3% *pa* and that mortality follows the AM92 Ultimate table. We shall ignore expenses for simplicity.

The future loss that needs to be covered by the reserve at the present time is:

$$L=75,000v^{K_{58}+1}-1,500\ddot{a}_{\overline{K_{58}+1|}}$$

If V is the reserve, then we need the smallest value of V such that:

$$P(L \le V) \ge 0.975$$

Let V_n be the reserve needed to cover the loss *exactly* when $K_{58} = n$ $(n = 0, 1, 2, \ldots)$, that is, assuming the policyholder dies in year $n+1$, counting from now. So:

$$V_n = 75{,}000v^{n+1} - 1{,}500\ddot{a}_{\overline{n+1|}} \qquad (*)$$

Suppose we were now to *hold* a reserve of amount V_n. The probability that this reserve is large enough to cover the loss is:

$$P(L \le V | V = V_n) = P(K_{58} \ge n)$$

because the larger the value of K_{58} (*ie* the later that death occurs) the smaller will be the value of L.

However, we also need:

$$P(K_{58} \ge n) \ge 0.975$$

Now, as:

$$P(K_{58} \ge n) = {}_np_{58} = \frac{l_{58+n}}{l_{58}}$$

we require:

$$\frac{l_{58+n}}{l_{58}} \ge 0.975 \;\Rightarrow\; l_{58+n} \ge 0.975 \times l_{58} = 0.975 \times 9{,}413.8004 = 9{,}178.4554 \; (**)$$

We now need to choose the particular value of n that produces the smallest reserve that satisfies this condition.

Question

Without performing any calculations, by considering expression (*) above explain whether a larger or a smaller value of n would cause the reserve amount V_n to reduce.

Solution

A larger value of n will reduce the present value of the benefits, by discounting the payment over a longer period; and it will increase the number (and hence present value) of the premiums deducted. Both of these effects will cause the value of V_n to reduce.

So, to find the smallest reserve to cover the loss, we need the largest value of n that satisfies (**). From the *Tables*, we find:

$$l_{61} > 9{,}178.4554 > l_{62}$$

and so the largest value of n that satisfies (**) is 3. Therefore the smallest reserve we can hold to cover the loss with a probability of at least 97.5% is:

$$V_3 = 75{,}000v^4 - 1{,}500\ddot{a}_{\overline{4}|} = 75{,}000 \times 1.03^{-4} - 1{,}500 \times \left[\frac{1-1.03^{-4}}{0.03/1.03}\right] = £60{,}894$$

Question

Suppose that the insurer in the above example has two of these policies in force, on independent lives.

The insurer now wishes to calculate the smallest total reserve that would ensure it could meet its liabilities under both policies with a probability of at least 97.5%.

Without performing any more calculations, explain whether the total reserve would be:

I less than double

II exactly double, or

III more than double

the reserve that is required for one policy when considered on its own.

Solution

Option I is correct.

The two losses are independent and identically distributed. If L_1 and L_2 are the present values of the losses on the two policies, then:

$$E[L_1 + L_2] = 2E[L] \quad \text{and} \quad \text{var}[L_1 + L_2] = 2\,\text{var}[L]$$

This means that the standard deviation of the sum of the losses is:

$$SD[L_1 + L_2] = \sqrt{2\,\text{var}[L]} = 1.414\,SD[L]$$

As the standard deviation has increased by (considerably) less than double, then $P(L_1 + L_2 > 2 \times 60{,}894)$ will be much smaller than 2.5%. That is, we can hold a total reserve that is considerably smaller than $2 \times 60{,}894$ and still be able to meet the required probability of loss.

In the above question, we see the beginnings of the effect of the 'law of large numbers', by which all insurance companies are able to pool their risks by insuring a large number of independent lives or policies (according to the Central Limit Theorem of statistics). The greater the number of independent risks the insurer has on its books, the smaller the reserve required *per policy* that will leave the insurer with an acceptable risk of loss. With enough policies in force, the reserve required per policy will be close in size to the expectation of the present value of the loss – that is, a prospective reserve calculation.

2.3 Gross premium prospective reserves for conventional with-profits policies

In the case of conventional with-profits policies, we must define carefully how bonuses are to be allowed for, both in the future benefits to be valued and the gross premium.

If asked to calculate a gross premium reserve for a with-profits policy in the CM1 exam, just follow the instructions given in the question.

Normally, the future benefits to be valued will include at least the level of bonuses added to the point of calculation of the reserve. However, normally also the full value of the future gross premium payments will be deducted, and this will effectively discount all the loadings for bonuses contained within the gross premiums that are still to be received. In the context of a reserve calculated at some point in a policy's life, the historical premiums paid to date plus the discounted value of the future premiums will effectively capitalise all the premium loadings for bonuses. This would include both those already added as at the date of the reserve calculation and any which could be added thereafter. It can then be seen that calculating benefits allowing just for bonuses to date, but then deducting the full value of the gross premiums, may produce a rather weak, *ie* low, reserve.

For this reason, some level of future bonus is normally also valued as a prospective future benefit. This may or may not include an allowance for terminal bonus, depending mainly on the purpose of the calculation.

Question

A conventional with-profits whole life assurance policy with an initial sum assured of £100,000 was taken out 5 years ago by a man who was then aged exactly 45, for an annual premium of £2,500. The insurance company has declared compound reversionary bonuses of 3% *pa* each year during this period. All benefits are paid at the end of the year of death.

Calculate the prospective gross premium reserve at the present time using the following assumptions:

Mortality:	AM92 Ultimate
Interest:	6% *pa*
Future bonuses:	1.9231% *pa* compound, vesting at the end of each year
Future expenses:	£40 *pa* paid at the start of each year, plus £300 payable on a claim

Solution

The prospective gross premium reserve is given by:

EPV of future benefits + EPV of future expenses – EPV of future premiums

The current sum assured, allowing for five years of past bonuses of 3% *pa* compound, is:

$$100{,}000 \times 1.03^5 = 115{,}927.41$$

Defining $b = 0.019231$ the EPV of the future benefit payments is:

$$115{,}927.41 \times \left(v\ {}_{0|}q_{50} + (1+b)v^2\ {}_{1|}q_{50} + (1+b)^2 v^3\ {}_{2|}q_{50} + \cdots \right)$$

$$= \frac{115{,}927.41}{1+b} \times \left((1+b)v\ {}_{0|}q_{50} + (1+b)^2 v^2\ {}_{1|}q_{50} + (1+b)^3 v^3\ {}_{2|}q_{50} + \cdots \right)$$

$$= 115{,}927.41 \times \frac{A_{50}^{4\%}}{1.019231}$$

because $(1+b)v = \frac{1.019231}{1.06} = \frac{1}{1.04} = v$ calculated at 4% interest.

So the expected present value of the benefits is:

$$115{,}927.41 \times \frac{0.32907}{1.019231} = 37{,}428.45$$

The EPV of the future expenses is:

$$40\ddot{a}_{50} + 300A_{50} \quad @6\%$$

$$= 40 \times 14.044 + 300 \times 0.20508 = 623.28$$

The EPV of the future premium payments is:

$$2{,}500\ddot{a}_{50} = 2{,}500 \times 14.044 = 35{,}110$$

So the gross premium prospective reserve is:

$${}_5V^{pro} = 37{,}428.45 + 623.28 - 35{,}110 = 2{,}941.73$$

2.4 Reserve conventions

We often calculate reserves at integer durations, *ie* whole years. In this case, we calculate the reserve just before any payment of premium due on that date, and just after any payment of benefit payable in arrears due on that date.

The benefit being referred to here is any payment that's made at the end of the year if the policyholder survives to the end of the year, such as an annuity payable in arrears, for example. We don't have to worry about the timing of *death* benefits because reserves are only required for policies that are still in force.

The general rule is, for valuation on the t th policy anniversary, payments in respect of the year $t-1$ to t payable in arrears (*ie* on the t th anniversary) are assumed to have been paid, payments in respect of the year t to $t+1$ payable in advance (and so are also due on the t th anniversary) are assumed not yet to have been paid.

3 Retrospective reserves

The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

The accumulated value allowing for interest and survivorship of the premiums received to date

less

the accumulated value allowing for interest and survivorship of the benefits and expenses paid to date

By 'accumulated value' we mean the *average* amount of money that would be accumulated per policy by a group of identical (but independent) policies over a period of time. In essence this reserve is calculated by 'looking backwards' to the payments that were expected to have occurred under the policy up to now.

The retrospective reserve on a given basis tells us how much the premiums less expenses and claims have accumulated to, averaging over a large number of policies.

To calculate this, we need to understand retrospective accumulations.

3.1 Retrospective accumulations

In mathematics of finance there are two common viewpoints from which a stream of cashflows may be considered.

(1) Prospectively, leading to the calculation of present values

(2) Retrospectively, leading to the calculation of accumulations.

In this section we discuss the latter approach allowing for the presence of mortality.

The 'retrospective accumulation' can be thought of as the 'pot of money' accumulated in respect of a policy, *ie* premiums paid, plus interest, less expenses, less the cost of life cover. The amount is often referred to as the 'asset share' of a policy.

The basic idea is that we consider a *group* of lives, who are regarded as identical and stochastically independent as far as mortality is concerned. At age x, each life transacts an identical life insurance contract. Under these contracts, payments will be made (the direction of the payments is immaterial), depending on the experience of the members of the group. We imagine these payments being accumulated in a fund at rate of interest i. After n years, we divide this fund equally among the surviving members of the group. (If the fund is negative we imagine charging the survivors in equal shares.) The *retrospective accumulation* is defined as the amount that each survivor would receive, as the group size tends towards infinity.

Question

A fund of £1,000,000 has 10,000 members aged 40. The fund accumulates at an interest rate of 4% per annum effective and will be divided between all members who survive to age 60.

Based on AM92 Ultimate mortality, calculate the expected payout for each survivor.

Solution

The accumulated fund will be:

$$1{,}000{,}000 \times 1.04^{20} = 2{,}191{,}123$$

The expected number of survivors will be:

$$10{,}000 \times \frac{l_{60}}{l_{40}} = 10{,}000 \times \frac{9{,}287.2164}{9{,}856.2863} = 9{,}422.63$$

So the expected payout per survivor is:

$$\frac{2{,}191{,}123}{9{,}422.63} = £232.54$$

In the solution to this question we divided the projected fund (£2,191,123) by the *expected* number of survivors (9,422.63). However, the number of survivors, N say, is actually a random variable. So logically, we should be looking at the expected value of $\frac{2{,}191{,}123}{N}$, not $\frac{2{,}191{,}123}{E(N)}$.

From Subject CS1, we know that, in general, $E\left(\frac{2{,}191{,}123}{N}\right) \neq \frac{2{,}191{,}123}{E(N)}$. So we need to justify the approach we have just used, which we can do as follows.

Suppose that there are L_n survivors at age $x+n$ out of L 'starters' at age x, and that the accumulated fund at age $x+n$ is $F_n(L)$. The retrospective accumulation of the benefit under consideration is defined to be:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n}$$

Clearly L_n and $F_n(L)$ are random variables. The process of taking the limit eliminates the awkward possibility that $L_n = 0$, but also since:

$$\lim_{L \to \infty} \frac{F_n(L)}{L} = E[F_n(1)]$$

and:

$$\lim_{L \to \infty} \frac{L_n}{L} = {}_n p_x$$

by the law of large numbers:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n} = \frac{E[F_n(1)]}{{}_n p_x}$$

This proof justifies the method we used in the last question.

Example 1: pure endowment

Consider the case for a pure endowment, that pays 1 on survival to time n. We need only consider a single life and calculate $E[F_n(1)]$. We can see that $F_n(1)$ has the following distribution:

$$F_n(1) = 0 \qquad \text{if } K_x < n$$

$$F_n(1) = 1 \qquad \text{if } K_x \geq n$$

So

$$E[F_n(1)] = 0 \times {}_nq_x + 1 \times {}_np_x = {}_np_x$$

Hence the accumulation of the pure endowment benefit is:

$$\frac{E[F_n(1)]}{{}_np_x} = \frac{{}_np_x}{{}_np_x} = 1$$

Question

Write down the retrospective accumulation after 10 years of the benefit payable under a pure endowment contract, which has a benefit amount of 50,000 and a term of 10 years.

Solution

50,000

We can explain this result intuitively as follows:

- the payment (of 50,000) is made at the accumulation date, so no interest will be added to accumulate it
- the payment is only paid to those who survive to the accumulation date, and so we are directly given the amount 'per survivor' without having to make any further adjustments for survival.

Example 2: term assurance

For an example less trivial than the pure endowment, consider a term assurance that pays 1 at the end of the year of death occurring within n years. Now:

$$F_n(1) = (1+i)^{n-(K_x+1)} \qquad \text{if } K_x < n$$

$$F_n(1) = 0 \qquad \text{if } K_x \geq n$$

That is, if (x) dies during the n years ($K_x < n$), the payment (of 1) is made at time $K_x + 1$ (which is the end of the year of death). We then accumulate this for the remaining period, *ie* over $n-(K_x+1)$ years.

If (x) survives the n years ($K_x \geq n$), no payment is made as there is no survival benefit for a term assurance contract.

So:

$$E[F_n(1)] = \sum_{k=0}^{n-1} (1+i)^{n-(k+1)} \, {}_{k|}q_x = (1+i)^n A^{1}_{x:\overline{n}|}$$

Hence the accumulation of the term assurance benefit is:

$$\frac{(1+i)^n A^{1}_{x:\overline{n}|}}{{}_n p_x}$$

This can also be written as $A^{1}_{x:\overline{n}|} \dfrac{D_x}{D_{x+n}}$.

We can calculate accumulated values at any point in time, not just at the end of the term.

Question

Write down an expression for the retrospective accumulation at the end of t years (where $t < n$) of the payments made under the term assurance contract by that time.

Solution

The retrospective accumulation after t years is $A^{1}_{x:\overline{t}|} \dfrac{D_x}{D_{x+t}}$, *ie* we just change the n's to t's.

Now we will do an example calculation of a retrospective accumulation.

Question

John, aged exactly 35, buys a term assurance policy that pays a benefit of £100,000 at the end of the year of his death if he dies before age 65. Calculate the retrospective accumulation of the benefits to time 10.

Basis: AM92 Ultimate, 6% *pa* interest

Solution

The retrospective accumulation to the end of 10 years is:

$$100{,}000A^{\ \ 1}_{35:\overline{10|}}\times\frac{(1+i)^{10}}{{}_{10}p_{35}}=100{,}000\left(A_{35}-v^{10}\,{}_{10}p_{35}A_{45}\right)\times\frac{(1+i)^{10}}{{}_{10}p_{35}}$$

$$=100{,}000\left(A_{35}\times(1+i)^{10}\times\frac{l_{35}}{l_{45}}-A_{45}\right)$$

$$=100{,}000\left(0.09488\times1.06^{10}\times\frac{9{,}894.4299}{9{,}801.3123}-0.15943\right)$$

$$=£1{,}210$$

We now look at one more example of a retrospective accumulation, this time for an annuity.

Example 3: temporary annuity-due

Now consider a temporary annuity-due paying 1 per annum with a term of n years. $F_n(1)$ has the following distribution:

$$F_n(1)=(1+i)^{n-(K_x+1)}\ddot{s}_{\overline{K_x+1|}} \qquad \text{if } K_x<n$$

$$F_n(1)=\ddot{s}_{\overline{n|}} \qquad \text{if } K_x\ge n$$

Recall that $\ddot{s}_{\overline{n|}}$ is the accumulated value of certain payments of 1 *pa*, payable annually in advance for n years, accumulated to the end of the n years.

Hence:

$$E[F_n(1)]=\sum_{k=0}^{n-1}(1+i)^{n-(k+1)}\ddot{s}_{\overline{k+1|}}\,{}_{k|}q_x+\ddot{s}_{\overline{n|}}\,{}_np_x$$

$$=(1+i)^n\left(\sum_{k=0}^{n-1}\ddot{a}_{\overline{k+1|}}\,{}_{k|}q_x+\ddot{a}_{\overline{n|}}\,{}_np_x\right)$$

$$=(1+i)^n\ddot{a}_{x:\overline{n|}}$$

Therefore the accumulation of the temporary annuity-due is:

$$\frac{(1+i)^n\ddot{a}_{x:\overline{n|}}}{{}_np_x}$$

Notation

To each annuity EPV there corresponds an accumulation denoted by an 's' symbol instead of an 'a' symbol. Thus in Example 3 above we define:

$$\ddot{s}_{x:\overline{n}|} = \frac{(1+i)^n \ddot{a}_{x:\overline{n}|}}{{}_n p_x}$$

and we define symbols for the accumulation of other annuities similarly. There is no actuarial notation for the accumulation of assurance benefits.

Question

Consider an annuity of 1 *pa* that has expected present value, at the start of the contract, of $\ddot{a}_{x:\overline{n}|}$.

Write down an expression for the retrospective accumulation to integer time t $(t < n)$, of the payments made by that date.

Solution

The retrospective accumulation after t years is $\dfrac{(1+i)^t}{{}_t p_x}\ddot{a}_{x:\overline{t}|}$, *ie* we just change the n's to t's.

It is easy to see that we can write down the retrospective accumulation of any benefit after n years in the same way – we multiply its EPV by $\dfrac{(1+i)^n}{{}_n p_x}$. If $i = 4\%$ and we are assuming AM92 mortality, the accumulation can be calculated more efficiently from the *Tables* by multiplying the EPV by the commutation function expression $\dfrac{D_x}{D_{x+n}}$.

3.2 Gross premium retrospective reserve

Gross premium retrospective reserves at policy duration t take account of expected expenses incurred between policy duration 0 and policy duration t, as well as the expected premiums and benefits paid. All the retrospective accumulations are determined at policy duration t.

With a retrospective reserve we are determining the 'pot of money' represented by the retrospective accumulation of premiums and interest, less the average cost of cover provided and of expenses incurred to date.

So a generic definition of a gross premium retrospective reserve is:

Retrospective accumulation of past premiums – *Retrospective accumulation of past benefits and expenses*

Example: Whole life assurance

The gross premium retrospective reserve at policy duration t is:

$$\frac{l_x}{l_{x+t}}(1+i)^t\left\{G\ddot{a}^{(m)}_{x:\overline{t}|}-S\,\bar{A}^{1}_{x:\overline{t}|}-I-e\ddot{a}^{(m)}_{x:\overline{t}|}-f\,\bar{A}^{1}_{x:\overline{t}|}\right\}$$

Again, we are assuming that the premiums and renewal expenses are payable $m$thly in advance, and that claims and claim expenses are payable immediately on death.

The expression in brackets gives us 'the expected present value at time 0 of the premiums less benefits and expenses payable in the first t policy years'. The factors outside the brackets then translate that into a value at time t.

To evaluate this expression we need assumptions to determine the assurance and annuity functions and information or assumptions about expenses incurred.

Similar expressions can be determined using the same approach for other standard contracts.

Question (*)

Calculate the gross premium retrospective reserve at the end of the second policy year for a 5-year single premium endowment assurance with sum assured £30,000 payable on maturity or at the end of the year of earlier death, issued to a 48-year old. Assume AM92 Select mortality, interest of 4% *pa* effective, initial expenses of £360 and renewal expenses of £45 at the start of each year excluding the first.

Solution

We first need to calculate the premium paid at inception. This is given by:

$$P=30{,}000A_{[48]:\overline{5}|}+360+45\left(\ddot{a}_{[48]:\overline{5}|}-1\right)$$

Now:

$$A_{[48]:\overline{5}|}=A_{[48]}-v^5\,{}_5p_{[48]}\,A_{53}+v^5\,{}_5p_{[48]}$$

where:

$$v^5\,{}_5p_{[48]}=(1+i)^{-5}\times\frac{l_{53}}{l_{[48]}}=1.04^{-5}\times\frac{9{,}630.0522}{9{,}748.8603}=0.81191$$

So:

$$\begin{aligned}A_{[48]:\overline{5}|}&=A_{[48]}-v^5\,{}_5p_{[48]}\,A_{53}+v^5\,{}_5p_{[48]}\\&=0.30664-0.81191\times0.36448+0.81191\\&=0.82263\end{aligned}$$

Also:

$$\ddot{a}_{[48]:\overline{5}|} = \ddot{a}_{[48]} - v^5 \, {}_5p_{[48]} \, \ddot{a}_{53} = 18.027 - 0.81191 \times 16.524 = 4.611$$

So:

$$P = 30{,}000 \times 0.82263 + 360 + 45(4.611 - 1) = £25{,}201.25$$

The retrospective reserve at the end of the second policy year is:

$$\left(25{,}201.25 - 30{,}000 A^{\;1}_{[48]:\overline{2}|} - 360 - 45 a_{[48]:\overline{1}|}\right)(1+i)^2 \frac{l_{[48]}}{l_{50}}$$

where:

$$A^{\;1}_{[48]:\overline{2}|} = A_{[48]} - v^2 \frac{l_{50}}{l_{[48]}} A_{50} = 0.30664 - 1.04^{-2} \times \frac{9{,}712.0728}{9{,}748.8603} \times 0.32907 = 0.0035444$$

and:

$$a_{[48]:\overline{1}|} = v \frac{l_{[48]+1}}{l_{[48]}} = 1.04^{-1} \times \frac{9{,}733.1938}{9{,}748.8603} = 0.95999$$

The required retrospective reserve is then:

$$= (25{,}201.25 - 30{,}000 \times 0.0035444 - 360 - 45 \times 0.95999) \times 1.04^2 \times \frac{9{,}748.8603}{9{,}712.0728}$$

$$= £26{,}808$$

We could have alternatively calculated the $v^n \left(l_{x+n} / l_x\right)$ *factors using* D_{x+n} / D_x *throughout, which could have saved a bit of time. The numerical answer will differ slightly due to rounding in the Tables.*

The assurance function we use to value the past claim payments in this reserve is a term assurance function. This is because the only benefit that could have been paid out in the first two years is a death benefit.

We can also see how this formula develops recursively from policy inception. The retrospective reserve at the end of Year 1 is the premium less initial expenses, plus interest and less the expected cost of cover, divided by the probability of surviving the year:

$$\frac{(25{,}201.25 - 360) \times 1.04 - 30{,}000 q_{[48]}}{p_{[48]}} = \frac{25{,}834.90 - 30{,}000 \times 0.001607}{1 - 0.001607} = £25{,}828.20$$

The retrospective reserve at the end of Year 2 is then the end-of-Year-1 reserve less renewal expenses, plus interest and less the expected cost of cover, divided by the probability of surviving that year:

$$\frac{(25{,}828.20-45)\times 1.04-30{,}000\times q_{[48]+1}}{p_{[48]+1}}=\frac{26{,}814.53-30{,}000\times 0.002170}{1-0.002170}=£26{,}807.60$$

which rounds to the same answer of £26,808 that we obtained before.

Later in the chapter we shall return to this idea of calculating the reserve at the end of a given policy year from the reserve at the end of the previous year.

Question

Calculate the gross premium retrospective reserve at the end of the third policy year for the policy described in the previous question. Justify the answer by reconciling it with the reserve at the end of the second policy year.

Solution

The retrospective reserve at the end of the third policy year is:

$${}_3V^{retro}=\left(25{,}201.25-30{,}000A^{1}_{[48]:\overline{3}|}-360-45a_{[48]:\overline{2}|}\right)(1+i)^3\frac{l_{[48]}}{l_{51}}$$

where 'retro' stands for 'retrospective'.

The required factors are:

$$(1+i)^3\frac{l_{[48]}}{l_{51}}=1.04^3\times\frac{9{,}748.8603}{9{,}687.7149}=1.131964$$

$$A^{1}_{[48]:\overline{3}|}=A_{[48]}-v^3\,{}_3p_{[48]}\,A_{51}=0.30664-\frac{1}{1.131964}\times 0.34058=0.00576$$

$$\begin{aligned}a_{[48]:\overline{2}|}&=\ddot{a}_{[48]:\overline{3}|}-1\\&=\ddot{a}_{[48]}-v^3\,{}_3p_{[48]}\,\ddot{a}_{51}-1\\&=18.027-\frac{1}{1.131964}\times 17.145-1\\&=1.881\end{aligned}$$

So:

$$\begin{aligned}{}_3V^{retro}&=(25{,}201.25-30{,}000\times 0.00576-360-45\times 1.881)\times 1.131964\\&=£27{,}828\end{aligned}$$

If instead we start from the end-of-Year-2 reserve, we obtain:

$$\frac{(26,807.60-45)\times 1.04-30,000q_{50}}{p_{50}}=\frac{27,833.10-30,000\times 0.002508}{1-0.002508}=£27,827.66$$

which again rounds to the same answer as before.

---

4 Equality of prospective and retrospective reserves

4.1 Conditions for equality

If:

1. **the retrospective and prospective reserves are calculated on the same basis, and**

2. **this basis is the same as the basis used to calculate the premiums used in the reserve calculation,**

then the retrospective reserve will be equal to the prospective reserve.

Question

Calculate the prospective reserve at the end of the second policy year for the policy considered in the question marked (*) in Section 3.2 above.

Solution

The prospective reserve at the end of the second policy year is:

$${}_2V^{pro} = 30,000A_{50:\overline{3}|} + 45\ddot{a}_{50:\overline{3}|}$$

Now:

$$A_{50:\overline{3}|} = A_{50} - v^3\,{}_3p_{50}\,A_{53} + v^3\,{}_3p_{50}$$

where:

$$v^3\,{}_3p_{50} = (1+i)^{-3} \times \frac{l_{53}}{l_{50}} = 1.04^{-3} \times \frac{9,630.0522}{9,712.0728} = 0.88149$$

So:

$$A_{50:\overline{3}|} = 0.32907 - 0.88149 \times 0.36448 + 0.88149 = 0.88927$$

and:

$$\ddot{a}_{50:\overline{3}|} = \ddot{a}_{50} - v^3\,{}_3p_{50}\,\ddot{a}_{53} = 17.444 - 0.88149 \times 16.524 = 2.878$$

So:

$${}_2V^{pro} = 30,000 \times 0.88927 + 45 \times 2.878 = £26,808$$

This is equal to the retrospective reserve calculated in the earlier question, because both reserves have been calculated using the assumptions of the gross premium basis.

In practice these conditions rarely hold, since the assumptions that are appropriate for the retrospective calculation are not generally appropriate for the prospective calculation. The conditions experienced over the duration of the contract up to the valuation date may no longer be suitable assumptions for the remainder of the policy term. Furthermore, the assumptions considered appropriate at the time the premium was calculated may not be appropriate for either the retrospective or prospective reserve some years later.

This last point is important and is worth reiterating. The retrospective reserve is based on actual experience, in terms of interest, expenses, mortality and so on. This will *not* usually be the same as that assumed when the premiums were set. For example:

- The number of people who actually die will differ from the number expected.
- Investment returns will not exactly match what was assumed.
- We may have had margins in our assumptions (as discussed in the Core Reading in Chapter 19). For example, we may have been expecting 8% interest but assumed only 6%. If we actually did earn 8% then the prospective and retrospective reserves would not be the same.
- The actuary's expectation of future interest rates may have changed, so that now the expected return on money held is 5%.

Question

Explain how the situation in the last bullet point above would affect each of the retrospective and prospective reserves.

Solution

The retrospective reserve would be unchanged (because it is based on the past experience). The prospective reserve would increase. (If we expect to earn the lower rate of 5% interest on future premiums, we will need to set aside more money now in order to meet the cost of the benefits.)

4.2 Demonstrating the equality of prospective and retrospective reserves

We will demonstrate that under the conditions given above, the retrospective and prospective gross premium reserves for a whole life assurance are equal. This method can be extended to all standard contracts, including those with increasing or decreasing benefits.

Example: Whole life assurance

The three key expressions are those for prospective reserves, retrospective reserves and the original equation of value used to determine the gross premium.

These are:

(i) $S\,\bar{A}_{x+t} + e\,\ddot{a}^{(m)}_{x+t} + f\,\bar{A}_{x+t} - G\,\ddot{a}^{(m)}_{x+t}$

(ii) $\dfrac{l_x}{l_{x+t}}(1+i)^t\left\{G\,\ddot{a}^{(m)}_{x:\overline{t}|} - S\,\bar{A}^{1}_{x:\overline{t}|} - I - e\,\ddot{a}^{(m)}_{x:\overline{t}|} - f\,\bar{A}^{1}_{x:\overline{t}|}\right\}$

(iii) $G\,\ddot{a}^{(m)}_{x} - S\,\bar{A}_{x} - I - e\,\ddot{a}^{(m)}_{x} - f\,\bar{A}_{x} = 0$

Then, if we add:

$$\frac{l_x}{l_{x+t}}(1+i)^t\left\{G\,\ddot{a}^{(m)}_{x} - S\,\bar{A}_{x} - I - e\,\ddot{a}^{(m)}_{x} - f\,\bar{A}_{x}\right\}$$

(*ie* zero) to the expression for the prospective reserve, and rearrange the terms, we will obtain the expression for the retrospective reserve.

Alternatively, we could split the premium equation up at time t and rearrange as follows:

$$G\ddot{a}^{(m)}_{x} = (S+f)\bar{A}_x + I + e\ddot{a}^{(m)}_{x}$$

$$\Rightarrow G\left(\ddot{a}^{(m)}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}^{(m)}_{x+t}\right) = (S+f)\left(\bar{A}^{1}_{x:\overline{t}|} + v^t\,{}_tp_x\bar{A}_{x+t}\right) + I + e\left(\ddot{a}^{(m)}_{x:\overline{t}|} + v^t\,{}_tp_x\,\ddot{a}^{(m)}_{x+t}\right)$$

$$\Rightarrow G\ddot{a}^{(m)}_{x:\overline{t}|} - (S+f)\bar{A}^{1}_{x:\overline{t}|} - I - e\ddot{a}^{(m)}_{x:\overline{t}|} = v^t\,{}_tp_x\left((S+f)\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} - G\ddot{a}^{(m)}_{x+t}\right)$$

Dividing both sides by $v^t\,{}_tp_x$ then gives:

$$\left(G\ddot{a}^{(m)}_{x:\overline{t}|} - (S+f)\bar{A}^{1}_{x:\overline{t}|} - I - e\ddot{a}^{(m)}_{x:\overline{t}|}\right)\frac{(1+i)^t}{{}_tp_x} = (S+f)\bar{A}_{x+t} + e\ddot{a}^{(m)}_{x+t} - G\ddot{a}^{(m)}_{x+t}$$

The expression on the left-hand side of this equation is the gross premium retrospective reserve at time t and the expression on the right is the gross premium prospective reserve at time t.

A similar result can be shown when all expenses are assumed to be zero ($I = e = f = 0$).

Question

Consider an immediate annuity paying an amount of B at the end of each year to a life currently aged x exact, in return for a single premium of P. Assume initial expenses of I, and renewal expenses of R (which are incurred at the start of every year except the first year).

Prove that the gross premium retrospective reserve at the end of the t th policy year is equal to the gross premium prospective reserve at the end of the t th policy year, assuming that both of these reserves are calculated using the same basis and this basis is also used to calculate the gross premium.

Solution

The retrospective reserve after exactly t (integer) policy years is:

$${}_tV^{retro} = \left[P - Ba_{x:\overline{t}|} - I - R\left(\ddot{a}_{x:\overline{t}|} - 1\right)\right]\frac{(1+i)^t}{{}_tp_x}$$

and the prospective reserve at the same duration is:

$${}_tV^{pro} = Ba_{x+t} + R\ddot{a}_{x+t}$$

To prove these are equal, we start with the premium equation, which is:

$$P = Ba_x + I + R\left(\ddot{a}_x - 1\right)$$

Now split the premium equation at time t, taking care to include t past annuity payments but only $t-1$ past renewal expense payments. We obtain:

$$P = B\left(a_{x:\overline{t}|} + v^t\,{}_tp_x\,a_{x+t}\right) + I + R\left(\ddot{a}_{x:\overline{t}|} - 1 + v^t\,{}_tp_x\,\ddot{a}_{x+t}\right)$$

Rearranging:

$$P - Ba_{x:\overline{t}|} - I - R\left(\ddot{a}_{x:\overline{t}|} - 1\right) = v^t\,{}_tp_x\left(Ba_{x+t} + R\ddot{a}_{x+t}\right)$$

Finally dividing through by $v^t\,{}_tp_x$:

$$\left[P - Ba_{x:\overline{t}|} - I - R\left(\ddot{a}_{x:\overline{t}|} - 1\right)\right]\frac{(1+i)^t}{{}_tp_x} = Ba_{x+t} + R\ddot{a}_{x+t}$$

ie ${}_tV^{retro} = {}_tV^{pro}$.

The fact that prospective and retrospective reserves are equal for equality of bases has two immediate uses:

- some policies with complicated and varying future benefit levels may require complex calculations to arrive at the reserve prospectively, but a retrospective calculation may be much easier, and
- you may find that retrospective reserves are a more tangible concept than prospective reserves, in which case thinking about aspects of life insurance involving reserves may be easier if you think in retrospective terms.

On the other hand, under some circumstances the prospective calculation will be easier. For example for policies where there are no further premiums to be paid, the prospective calculation is often simpler, because the term 'expected present value of future premiums' disappears.

For equality of prospective and retrospective reserves we require equality of bases. In practice, we shall often find that the bases are not equal. The retrospective basis may be just the past experience – for instance, the mortality experienced by our policyholders was 86% of AM92 – while the prospective basis may be our estimate of future experience, and it might be deliberately prudent (especially in the context of calculating reserves to demonstrate the solvency of the company).

However, we might want to ignore recent experience and 'arbitrarily' set the past basis to be equal to the future basis, in order to use the retrospective method to calculate reserves rather than the prospective method.

However, this will only give a valid answer if the reserving basis is identical to the pricing basis of the gross premium involved. While in most of our examples so far the two bases have been assumed to be identical, in real life they are usually different. On the other hand, *in the exam* it is quite common to have the same basis. So, unless a question specifically states the approach to be adopted, you would then be free to choose retrospective or prospective calculations. In such cases, however, *the prospective method is usually the easier method to do*.

Very often, companies will price products using 'best estimates' of future experience for interest, mortality and expenses, or best estimate with a small margin for prudence. However, the supervisory authority may insist that the reserving basis is chosen to be much more prudent, in order to be more certain that life companies will be capable of honouring their financial commitments in the event of deteriorating future conditions. For instance, if our best estimate of future interest rates was 6½% we might calculate premiums using 5%, but then might need to calculate statutory reserves using 3% interest.

In this case we would *have* to calculate the reserve prospectively, as the reserving basis is different from the pricing basis.

The effect of these bases being different is studied in Chapter 27.

5 Recursive relationship between reserves for annual premium contracts

If the expected cashflows (*ie* premiums, benefits and expenses) during the policy year $(t, t+1)$ are evaluated and allowance is made for the time value of money, we can develop a recursive relationship linking gross premium policy values in successive years.

(Policy value is just another name for reserve.)

We illustrate this using a whole life assurance secured by level annual premiums, but the method extends to all standard contracts.

Example: Whole life assurance

Gross premium policy value at duration t	${}_tV'$
Premium less expenses paid at t	$G-e$
Expected claims plus expenses paid at $t+1$	$q_{x+t}(S+f)$
Gross premium policy value at duration $t+1$	${}_{t+1}V'$

Then the equation of value at time $t+1$ for these cashflows is:

$$({}_tV'+G-e)(1+i)-q_{x+t}(S+f)=(1-q_{x+t})\,{}_{t+1}V' \qquad (*)$$

Question

(i) Summarise the above relationship in words.

(ii) Write down the relationship for a single premium whole life assurance policy.

Solution

(i) The reserves at the end of the year for those policies still in force, are equal to the reserves at the beginning of the year plus premiums and investment income, less expenses and the expected cost of cover.

(ii) $({}_tV'-e)(1+i)-q_{x+t}(S+f)=(1-q_{x+t})\,{}_{t+1}V'$

Equation (*) above will only be satisfied if all quantities are calculated on mutually consistent bases: *ie* using the same interest, mortality and expense assumptions for the reserves, premium and experience over the year.

Dividing through by $p_{x+t} = 1 - q_{x+t}$, we obtain a formula for calculating the reserve at the end of the year from the reserve at the beginning of the year:

$$\frac{\left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S+f\right)}{p_{x+t}} = {}_{t+1}V'$$

In this case, the equation then gives a recursive relationship between policy values in successive years.

Question

Consider a 25-year regular premium endowment assurance policy with a sum assured of £75,000 payable on maturity (or at the end of the year of earlier death), taken out by a 45-year old. Expenses are 75% of the first premium and 5% of each subsequent premium, plus an initial expense of £250.

Given a gross annual premium of £2,132 and a gross premium reserve at the end of the 10th policy year of £20,898, calculate the gross premium reserve at the end of the 11th policy year.

Assume AM92 Select mortality and 4% *pa* interest.

Solution

The recursive relationship is:

$${}_{11}V' = \frac{\left({}_{10}V' + G - e\right)(1+i) - Sq_{[45]+10}}{1 - q_{[45]+10}} = \frac{\left({}_{10}V' + G - e\right)(1+i) - Sq_{55}}{1 - q_{55}}$$

So:

$${}_{11}V' = \frac{\left(20{,}898 + 2{,}132 \times 0.95\right) \times 1.04 - 75{,}000 \times 0.004469}{1 - 0.004469} = £23{,}611$$

The recursive relationship can also be used in reverse, *ie* to calculate the reserve at the start of a policy year given the reserve at the end of that year.

Question

For the policy described in the previous question, calculate ${}_9V'$ using the recursive relationship and the value ${}_{10}V' = £20{,}898$.

Solution

We have:

$$\left({}_9V' + G - e\right)(1+i) - Sq_{54} = p_{54}\ {}_{10}V'$$

This gives:

$$\left({}_{9}V' + 2{,}132 \times 0.95\right) \times 1.04 - 75{,}000 \times 0.003976 = (1 - 0.003976) \times 20{,}898$$

So:

$${}_{9}V' = £18{,}276$$

We can also rearrange equation (*) to give:

$$\left({}_{t}V' + G - e\right)\left(1+i\right) - q_{x+t}\left(S + f - {}_{t+1}V'\right) = {}_{t+1}V'$$

So the end-year reserve is the start-year reserve, plus premium and less expenses, accumulated with one year's interest, less the expected cost of claims *in excess of the end-year reserve*. This excess claim cost is known as the *sum at risk*, or the *death strain at risk*, and is important in the calculation of mortality profit, which we cover in Chapter 23.

Where any of these bases differ, the equation (*) can be reformulated to represent the profit over the year, *ie*:

$$\boldsymbol{PRO_t = \left({}_{t}V' + G - e\right)(1+i) - q_{x+t}(S+f) - (1 - q_{x+t})\,{}_{t+1}V'}$$

This profit relates to the year *starting* at policy duration t, that is for policy *year* $t+1$.

We can consider the elements of this formula in different ways, depending on what kind of 'profit' we wish to calculate.

(1) Actual profit over the year

In this case, the elements will represent the actual experience *that has happened* over the year, rather than being the same as the premium basis assumptions. The reserves may also be calculated using a different basis from the gross premium basis, which is something that happens very often in practice.

Question

For the same 25-year endowment assurance described in the previous two questions, calculate the profit emerging at the end of the 11th policy year if:

- the start and end of year reserves were as previously given/calculated
- the insurer earned 3.8% on its investments during the year
- renewal expenses of £78 were incurred at the start of the year
- claim expenses of £150 per death claim were incurred
- 73% of expected mortality occurred during the year.

Solution

The actual profit emerging at the end of the 11th policy year is:

$$PRO_{10} = \left(20{,}898 + 2{,}132 - 78\right) \times 1.038 - (75{,}000 + 150) \times 0.73 \times q_{55}$$

$$- \left(1 - 0.73 \times q_{55}\right) \times 23{,}611$$

From the *Tables*, $q_{55} = 0.004469$, which gives:

$$PRO_{10} = £45.04$$

(2) Expected profit over the year

Alternatively, we may want to calculate the profit we would *expect* to earn during the year if the actual experience turns out in a particular way, *ie* according to our 'expected future experience' assumptions. The approach is identical to the calculation in (1), but it is based on *assumptions* of what might happen during the year, rather than on what has *actually* happened.

6 Net premium reserves for conventional without-profit contracts

6.1 Difference from gross premium reserve

The differences between the net premium reserve and the gross premium reserve for any conventional without-profit contract are:

1. **all expenses are ignored; and**
2. **the premium used in the reserve calculation is the *net premium*, as defined below.**

The net premium reserve is the prospective reserve, where we make no allowance for future expenses, and where the premium used in the calculation is a notional net premium. This net premium is calculated using the equivalence principle and using the same assumptions as the reserve basis, and again making no allowance for future expenses.

So, with a net premium reserve, we have just one basis, which we first use to calculate a premium using the equivalence principle in the normal way, but with no expenses in the equation of value (this gives us the so-called 'net premium'). Then we calculate the reserve prospectively as the expected present value of the future benefits less the expected present value of these future net premiums. No explicit allowance is made anywhere either for expenses or for the actual amount of premium received from the policyholder. The result is the 'net premium reserve'.

Though this may appear artificial, the net premium valuation has been an important feature in life insurance for many years.

One reason why this is the case is that the reserve is simple to calculate. The net premium method was used before computers, spreadsheets or calculators were available.

The notional net premium calculated and valued as the *future income* element of the reserve is generally considerably smaller than the actual premium being paid. It is considered that the excess of the actual premium over the notional premium will be sufficient to cover the expenses that are not specifically valued.

Example: whole life assurance

The net premium reserve at policy duration t is:

$$S\bar{A}_{x+t} - P\ddot{a}_{x+t}^{(m)}$$

where

$$P = \frac{S\bar{A}_x}{\ddot{a}_x^{(m)}}$$

and the notional net premium P is calculated on the same assumptions as the reserve basis.

In this example the sum assured is payable immediately on death and net premiums of P/m are payable m times a year.

Here P is found by solving the equation of value for this policy at duration 0, ignoring expenses. That is, we find P from:

$$P\ddot{a}_x^{(m)} = S\bar{A}_x$$

Net premium reserves for with-profits contracts are not covered in the CM1 syllabus.

6.2 A special result for the net premium reserve for some endowment and whole life assurance contracts

A special result for the net premium reserve for some endowment and whole life assurance contracts follows from the fact that the net premium is calculated on the same basis as the reserve basis.

Example: Whole life assurance

${}_tV_x$ is the net premium reserve at duration t for a whole life assurance policy, with sum assured of 1 payable at the end of the year of death, and with level annual premiums payable during the duration of the policy. The net premium P_x for this contract is:

$$P_x = \frac{A_x}{\ddot{a}_x}$$

and so:

$$\begin{aligned} {}_tV_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} \\ &= A_{x+t} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t} \\ &= (1-d\,\ddot{a}_{x+t}) - (1-d\,\ddot{a}_x)\frac{\ddot{a}_{x+t}}{\ddot{a}_x} \end{aligned}$$

This uses the premium conversion formula for A_x and A_{x+t} as given on page 37 of the *Tables*.

Since the net premium and the net premium reserve are both calculated using the same basis, the formula for the net premium reserve simplifies to:

$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

The net premium reserve for the above contract with a sum assured of S would then be:

$$S\,{}_tV_x = S\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right) \quad (**)$$

${}_tV_x$ is international actuarial notation for the net premium reserve for this particular type of whole life assurance, as defined at the start of this example. Net premium reserves for other contract variations have different notation.

Similar formulae to (**) can be obtained for endowment assurances with level annual premiums and death benefits paid at the end of the year of death, also for whole life and endowment assurances with continuously payable premiums and death benefits payable immediately on death. These results are shown in the *Formulae and Tables for Examinations*.

(They are to be found on page 37.)

It is important to understand the particular symbols listed below, and to use the special formulae to calculate their values, where appropriate, in the CM1 exam. In all cases the symbol represents the amount of reserve held at time t :

${}_tV_x$ whole life assurance, pays at end of year of death, premium payable annually in advance

${}_tV_{x:\overline{n}|}$ endowment assurance with a term of n years $(t<n)$, pays at end of year of death or at end of term, premium payable annually in advance

${}_t\bar{V}_x$ whole life assurance, pays immediately on death, premium payable continuously

${}_t\bar{V}_{x:\overline{n}|}$ endowment assurance with a term of n years $(t<n)$, pays immediately on death or at end of term, premium payable continuously.

In each case listed above, the sum assured is assumed to be 1.

In all other cases, including m -thly premium cases, these special formulae do not hold and the net premium reserves would need to be calculated from the standard formulae such as shown in Section 6.1 above.

Question

Calculate the net premium reserve at the end of the 10th policy year for a 15-year endowment assurance issued to a life aged exactly 50 at entry, with a sum assured of 10,000 paid at the end of the term or at the end of the year of death if earlier:

(i) where premiums are paid annually

(ii) where premiums are paid monthly.

Basis: AM92 Ultimate mortality and 4% *pa* interest.

Solution

(i) ***Premiums paid annually***

We can use the special result for this contract, as it is one of the examples cited in the above Core Reading (and included on page 37 of the *Tables*). The required net premium reserve is:

$$S\,{}_tV_{x:\overline{n}|}=S\left(1-\frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}\right)$$

where $S=10{,}000$, $x=50$, $n=15$ and $t=10$.

So, we need:

$$10{,}000\,{}_{10}V_{50:\overline{15|}}=10{,}000\times\left(1-\frac{\ddot{a}_{60:\overline{5|}}}{\ddot{a}_{50:\overline{15|}}}\right)=10{,}000\times\left(1-\frac{4.550}{11.253}\right)=5{,}957$$

(ii) ***Premiums paid monthly***

This is not one of the specified contract types, so we cannot use the special result for this.

First we need to calculate the net annual premium from the equation of value. We have:

$$P\ddot{a}^{(12)}_{50:\overline{15|}}=10{,}000A_{50:\overline{15|}}$$

where:

$$A_{50:\overline{15|}}=0.56719$$

$$\ddot{a}^{(12)}_{50:\overline{15|}}\approx\ddot{a}_{50:\overline{15|}}-\frac{11}{24}\left(1-\frac{D_{65}}{D_{50}}\right)$$ (using the formula on page 36 of the *Tables*)

Putting in the numbers:

$$\ddot{a}^{(12)}_{50:\overline{15|}}\approx 11.253-\frac{11}{24}\times\left(1-\frac{689.23}{1{,}366.61}\right)=11.026$$

The annual amount of net premium is then:

$$P=10{,}000\frac{A_{50:\overline{15|}}}{\ddot{a}^{(12)}_{50:\overline{15|}}}=10{,}000\times\frac{0.56719}{11.026}=514.42$$

The net premium reserve at time 10 is then:

$$10{,}000\times A_{60:\overline{5|}}-514.42\times\ddot{a}^{(12)}_{60:\overline{5|}}$$

where:

$$A_{60:\overline{5|}}=0.82499$$

$$\ddot{a}^{(12)}_{60:\overline{5|}}\approx\ddot{a}_{60:\overline{5|}}-\frac{11}{24}\left(1-\frac{D_{65}}{D_{60}}\right)=4.550-\frac{11}{24}\times\left(1-\frac{689.23}{882.85}\right)=4.449$$

Hence the required reserve is:

$$10{,}000\times 0.82499-514.42\times 4.449=5{,}961$$

Chapter 20 Summary

Reserves

A reserve is money set aside by the insurer, to pay policyholders' benefits and, where appropriate, future expenses.

Gross premium prospective reserves

Gross premium prospective reserves can be calculated as:

EPV of future benefits + EPV of future expenses – EPV of future premiums

Retrospective accumulations

The retrospective accumulation of a t-year payment stream can be calculated as:

$$(AV)_t = \left\{EPV \text{ of the payments as at the start of the } t \text{ years}\right\} \times \frac{(1+i)^t}{{}_tp_x}$$

for a life that is initially aged x. Retrospective accumulations are defined as the amount per survivor to time t.

Gross premium retrospective reserves

Gross premium retrospective reserves can be calculated as:

Retrospective accumulation of (past premiums – past benefits and expenses)

Calculating reserves that satisfy probabilities

A reserve at time t can be found that satisfies a probability defined in terms of the future loss random variable, L_t. For example, we can define the reserve as being the smallest reserve ${}_tV$ for which $P\left(L_t \le {}_tV\right) \ge \alpha$, where α is some acceptably large probability.

Equality of reserves

Given equality of bases used to calculate the premium and the reserves, the prospective and retrospective reserves of any policy at any given time t will be equal.

Recursive formula for reserves

Reserves at successive values of time t are related by the equation:

$$\left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S+f\right) = p_{x+t}\ {}_{t+1}V'$$

Profit

The profit for the year between policy durations t and $t+1$ (*ie* for policy year $t+1$) can be calculated using:

$$PRO_t = \left({}_tV' + G - e\right)(1+i) - q_{x+t}\left(S+f\right) - p_{x+t}\ {}_{t+1}V'$$

Net premium reserves for without-profit contracts

These are identical to gross premium reserves except that expenses are not included in any of the calculations.

The net premiums used in the formulae are calculated from the premium equation of value, with the same interest and mortality assumptions as used for the reserves, and no expenses.

Calculations can sometimes be speeded up using special formulae, which apply only to certain endowment and whole life assurances. The net premium reserve at time t for an annual premium n-year endowment assurance policy, with discrete-time payments and a sum assured of 1, is given by:

$${}_tV_{x:\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x:\overline{n}|} \times \ddot{a}_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$$

Similar formulae apply (a) for whole life assurances (by removing the policy term status) and (b) for the equivalent contracts with continuous-time payments (by putting bars over all the actuarial symbols). These are shown on page 37 of the *Tables*.

Chapter 20 Practice Questions

20.1 A 10-year term assurance with a sum assured of £500,000 payable at the end of the year of death, is issued to a male aged 30 for a level annual premium of £330.05 payable in advance. Calculate the prospective and retrospective reserves at the end of the fifth policy year, *ie* just before the sixth premium has been paid, assuming AM92 Ultimate mortality and 4% *pa* interest. Ignore expenses.

20.2 An annual premium conventional with-profits endowment assurance policy is issued to a life aged 35. The initial sum assured is £50,000, the gross premium is £1,500 and the term of the policy is 25 years. The death benefit of the sum assured and attaching bonuses is payable at the end of the year of death. The office declares compound reversionary bonuses, vesting at the end of each policy year. Given that bonuses of 3% *pa* have been declared for each year of the contract so far, calculate the prospective gross premium reserve at the end of the fifth policy year.

Basis: Future bonuses of 1.92308% *pa* compound
AM92 Ultimate mortality
6% *pa* interest
Renewal expenses of 5% of each premium
Claim expenses of £350

20.3 A 10-year regular-premium term assurance policy is issued to a life aged 40. The sum assured is £20,000 and is payable at the end of the year of death. Expenses of £72 are assumed to be incurred at the start of each year in which the policy is in force, except at the start of the first year when the expense is £425. The gross premium is £1,700 *pa*.

Write down an expression for the gross premium retrospective reserve immediately before the 6th premium is due.

20.4 A temporary annuity of £3,000 *pa* payable annually in arrears for a term of 10 years was purchased one year ago by a life then aged exactly 60 by the payment of a single premium. Show algebraically that the current gross premium prospective reserve is equal to the current gross premium retrospective reserve, assuming that the pricing and reserving bases are the same. The company assumes that each policy incurs initial expenses of £200 and annual expenses of 1.5% of each annuity payment.

20.5 A special ten-year increasing endowment assurance policy payable by level annual premiums provides a sum assured of £10,000 during the first year, which increases by £1,000 in each subsequent year. The sum payable on maturity at age 60 is £25,000. Write down an expression for the net premium reserve immediately before the 5th premium is paid.

20.6 An annual premium whole life assurance policy provides a sum assured of £30,000 payable immediately on death. Write down an expression for the gross premium retrospective reserve after 20 years in respect of a life aged 30 at entry, who is paying a gross annual premium of £250. Expenses are £100 payable initially, with renewal expenses of 5% of each premium except the first.

20.7 The premiums under a whole life assurance with sum assured S issued to a life aged x are payable annually in advance throughout life. The annual premium P is calculated assuming that the following expenses will be incurred:

Initial expenses:	I
Renewal expenses:	$100k$ % of each premium after the first
Claim related expenses:	$100c$ % of the sum assured

Write down equations linking the gross premium reserves at the end of successive policy years.

20.8 Calculate the retrospective accumulation of £200 paid by a person aged exactly 25, assuming ELT15 (Females) mortality and 7.5% *pa* interest, at the end of:

(i) 5 years

(ii) 20 years.

20.9 Exam style

A whole life assurance policy pays a benefit of £50,000 at the end of the year of death. The policyholder is currently aged 30 and is paying an annual premium of £700 at the start of each year. A premium has just been paid.

Use the following basis to calculate the reserve the company needs to hold at the present time so that the probability of covering the liability in full is at least 99%.

Mortality:	AM92 Select
Interest:	3% *pa*
Expenses:	5% of each future premium

[5]

20.10 Exam style

A life office sells decreasing term assurance policies with an initial sum assured of £150,000 to lives aged 50 exact. The term of the policies is 10 years, and the sum assured decreases by £10,000 at the start of each year from Year 2 onwards. The benefit is payable immediately on death. Premiums are payable annually in advance throughout the term of the policy. The office calculates premiums using AM92 Ultimate mortality and 4% *pa* interest, initial expenses of £300, renewal expenses of £43 at the start of each year except the first, and claim expenses of £400.

(i) Using P for the annual premium, write down the future loss random variable for the policy at the start of the term, and also just before the payment of the fifth premium, assuming that the policy is still in force at that time. [4]

(ii) Show that the premium for the policy is £491.31. [4]

(iii) Calculate the gross premium prospective reserve for the policy just before the payment of the fifth premium. Assume that the reserving basis is the same as that used to calculate the premium. [4]

(iv) Comment on your answer to part (iii). [2]

[Total 14]

20.11 A life insurance company issues a 30-year with profits endowment assurance policy to a life aged 35 exact. The sum assured of £100,000 plus declared reversionary bonuses are payable on survival to the end of the term or immediately on death if earlier. The quarterly premium payable in advance throughout the term of the policy or until earlier death is £615.61.

Exam style

At the end of the 25th policy year, the actual past bonus additions to the policy have been £145,000.

Calculate the gross prospective policy reserve at the end of that policy year immediately before the premium then due.

Policy reserving basis:

Mortality:	AM92 Ultimate
Interest:	4% per annum
Bonus loading:	4% of the sum assured and attaching bonuses, compounded and vesting at the end of each policy year
Renewal commission:	2.5% of each quarterly premium
Renewal expenses:	£90 at the start of each policy year
Claim expense:	£1,000 on death; £500 on maturity

[6]

20.12 A life insurance company issues a with profit whole life assurance policy to a life aged 55 exact. The sum assured is £75,000 together with any attaching bonuses and is payable immediately on death. Level premiums are payable monthly in advance ceasing on the policyholder's death or on reaching age 85 if earlier.

Exam style

Simple annual bonuses are added at the end of each policy year (*ie* the death benefit does not include any bonus relating to the policy year of death).

The company calculates the premium on the following basis:

Mortality	AM92 Select
Interest	4% per annum
Expenses	
Initial	£275
Renewal	£65 at the start of the second and subsequent policy years and payable until death
Claim	£200 on death
Commission	
Initial	75% of the total premium payable in the first policy year
Renewal	2.5% of the second and subsequent monthly premiums
Bonuses	Simple bonus of 2.0% of basic sum assured per annum

(i) Calculate the monthly premium for this policy. [6]

(ii) Calculate the gross prospective policy value at the end of the 30th policy year given that the total actual past bonus additions to the policy have followed the assumptions stated in the premium basis above (including the bonus just vested).

Policy value basis:

Mortality	AM92 Ultimate
Interest	4% per annum
Expenses	
Renewal	£80 at the start of each policy year and payable until death
Claim	£250 on death
Commission	
Renewal	2.5% of the monthly premiums
Bonuses	Simple bonus of 2.5% of basic sum assured per annum

[4]
[Total 10]

20.13 Exam style

On 1 January 2008, a life insurance company issued a number of without-profit endowment policies maturing at age 60 to lives then aged 40 exact. The sum assured is payable at the end of year of death or on survival to the end of the term and level premiums are payable annually in advance throughout the term of the contract.

Premiums and reserves on each policy are both calculated on the following basis:

Mortality:	AM92 Select
Interest:	4% per annum
Initial commission:	60% of the first premium
Renewal commission:	6% of each annual premium excluding the first

(i) Calculate the annual office premium per £1,000 sum assured for each policy. [2]

(ii) Calculate the gross premium prospective reserve per £1,000 sum assured for each policy in force at 31 December 2012. [2]

(iii) Calculate the profit or loss to the company in 2013 in respect of these policies given the following information:

- The total sums assured in force on 1 January 2013 were £15,500,000.
- The company incurred expenses relating to these policies of £76,500 on 1 January 2013 (including renewal commission).
- The total sums assured paid on 31 December 2013 in respect of deaths during 2013 were £295,000.
- The total sums assured surrendered during 2013 were £625,000. The surrender value on each policy (which was paid on 31 December 2013) was calculated as 85% of the gross premium prospective reserve applicable at the date of payment of the surrender value.
- The company earned interest of 3.5% per annum on its assets during 2013. [10]

[Total 14]

20.14 Under a policy issued by a life insurance company, the death benefit payable at the end of year of death is a return of premiums paid without interest. A level premium of £3,000 is payable annually in advance throughout the term of the policy.

Exam style

For a policy in force at the start of the 12th policy year, you are given the following information:

Reserve at the start of the policy year	£25,130
Reserve at the end of the policy year per survivor	£28,950
Probability of death during the policy year	0.03
Expenses incurred at the start of the policy year	£90
Rate of interest earned	4% per annum

Reserves given above are immediately before payment of the premium due.

Calculate the profit/loss expected to emerge at the end of the 12th policy year per policy in force at the start of that year. [3]

20.15 Calculate $s_{55:\overline{10}|}$.

Exam style

Basis:

Mortality: PFA92C20

Interest: 4% per annum [1]

Chapter 20 Solutions

20.1 ***Prospective calculation***

The prospective reserve is equal to the expected present value (EPV) at time 5 of the future benefit outgo, minus the EPV at time 5 of the future premium income.

The expected present value of the future benefits is:

$$\begin{aligned}500{,}000A^{1}_{35:\overline{5}|} &= 500{,}000\left(A_{35}-\frac{D_{40}}{D_{35}}A_{40}\right)\\ &= 500{,}000\left(0.19219-\frac{2{,}052.96}{2{,}507.40}\times 0.23056\right)\\ &= 500{,}000\times 0.00341659\\ &= 1{,}708.29\end{aligned}$$

The EPV of the future premiums is:

$$\begin{aligned}330.05\ddot{a}_{35:\overline{5}|} &= 330.05\left(\ddot{a}_{35}-\frac{D_{40}}{D_{35}}\ddot{a}_{40}\right)\\ &= 330.05\left(21.003-\frac{2{,}052.96}{2{,}507.40}\times 20.005\right)\\ &= 330.05\times 4.6237\\ &= 1{,}526.05\end{aligned}$$

Hence the prospective reserve is:

$$1{,}708.29-1{,}526.05=£182 \text{ to the nearest £1}$$

Retrospective calculation

The retrospective reserve is equal to the EPV of the first 5 years' premiums minus the EPV of the first 5 years' benefit payments, all accumulated to time 5.

The EPV of the first 5 years' premiums is:

$$\begin{aligned}330.05\ddot{a}_{30:\overline{5}|} &= 330.05\left(\ddot{a}_{30}-\frac{D_{35}}{D_{30}}\ddot{a}_{35}\right)\\ &= 330.05\left(21.834-\frac{2{,}507.40}{3{,}060.13}\times 21.003\right)\\ &= 330.05\times 4.6246\\ &= 1{,}526.36\end{aligned}$$

The EPV of the first 5 years' benefits is:

$$500{,}000\,A^{\;1}_{30:\overline{5}|} = 500{,}000\left(A_{30} - \frac{D_{35}}{D_{30}}A_{35}\right)$$

$$= 500{,}000\left(0.16023 - \frac{2{,}507.40}{3{,}060.13}\times 0.19219\right)$$

$$= 500{,}000 \times 0.00275394$$

$$= 1{,}376.97$$

Hence the retrospective reserve is:

$$\left(1{,}526.36 - 1{,}376.97\right)\times\frac{D_{30}}{D_{35}} = \left(1{,}526.36 - 1{,}376.97\right)\times\frac{3{,}060.13}{2{,}507.40} = £182$$

which is the same as the prospective reserve.

20.2 The reserve at the end of the fifth policy year is:

$${}_5V^{pro} = EPV \text{ future benefits} + EPV \text{ future expenses} - EPV \text{ future premiums}$$

Now, writing $b = 0.0192308$:

EPV future benefits $=$

$$50{,}000\times 1.03^5 \times\left[\frac{1}{1.06}\times {}_{0|}q_{40} + \frac{1+b}{1.06^2}\times {}_{1|}q_{40} + \cdots + \frac{(1+b)^{19}}{1.06^{20}}\times {}_{19|}q_{40}\right]$$

$$+50{,}000\times 1.03^5\times\frac{(1+b)^{20}}{1.06^{20}}\times {}_{20}p_{40}$$

$$= \frac{50{,}000\times 1.03^5}{1+b}\times\left[\frac{1+b}{1.06}\times {}_{0|}q_{40} + \left(\frac{1+b}{1.06}\right)^2\times {}_{1|}q_{40} + \cdots + \left(\frac{1+b}{1.06}\right)^{20}\times {}_{19|}q_{40}\right]$$

$$+50{,}000\times 1.03^5\times 1.04^{-20}\times {}_{20}p_{40}$$

$$= 50{,}000\times 1.03^5\times\left[\frac{A^{\;1\;\;@4\%}_{40:\overline{20}|}}{1+b} + \frac{D^{@4\%}_{60}}{D^{@4\%}_{40}}\right]$$

where:

$$\frac{D^{@4\%}_{60}}{D^{@4\%}_{40}} = \frac{882.85}{2{,}052.96} = 0.43004$$

$$A^{\;1\;\;@4\%}_{40:\overline{20}|} = A^{\;@4\%}_{40:\overline{20}|} - \frac{D^{@4\%}_{60}}{D^{@4\%}_{40}} = 0.46433 - 0.43004 = 0.03429$$

So, the expected present value of the future benefits is:

$$50{,}000\times 1.03^5\times\left[\frac{0.03429}{1.0192308}+0.43004\right]=26{,}876.78$$

Hence:

$$_5V^{pro}=26{,}876.78+350A_{40:\overline{20}|}^{@6\%}-(1-0.05)\times 1{,}500\times\ddot{a}_{40:\overline{20}|}^{@6\%}$$

$$=26{,}876.78+350\times 0.32088-0.95\times 1{,}500\times 11.998$$

$$=£9{,}892$$

20.3 The gross premium retrospective reserve can be calculated as the expected present value at the outset of (premiums – benefits – expenses), accumulated with interest and allowing for survival to age 45. This gives:

$$\frac{(1+i)^5}{_5p_{40}}\times\left[1{,}700\ddot{a}_{40:\overline{5}|}-20{,}000A^{\ \ 1}_{40:\overline{5}|}-425-72\times\left(\ddot{a}_{40:\overline{5}|}-1\right)\right]$$

20.4 The prospective reserve is the expected present value at time 1 of the future benefits and expenses:

$$_1V^{pro}=1.015\times 3{,}000a_{61:\overline{9}|}$$

The retrospective reserve is the retrospective accumulation of the premium less benefits and expenses:

$$_1V^{retro}=\frac{D_{60}}{D_{61}}\left(P-1.015\times 3{,}000a_{60:\overline{1}|}-200\right)$$

where P is the single premium.

The premium equation is:

$$P=1.015\times 3{,}000a_{60:\overline{10}|}+200$$

Splitting this at time 1:

$$P=1.015\times 3{,}000\left[a_{60:\overline{1}|}+\frac{D_{61}}{D_{60}}a_{61:\overline{9}|}\right]+200$$

Rearranging:

$$P-1.015\times 3{,}000a_{60:\overline{1}|}-200=1.015\times 3{,}000\frac{D_{61}}{D_{60}}a_{61:\overline{9}|}$$

Accumulating to time 1:

$$\frac{D_{60}}{D_{61}}\left(P-1.015\times 3{,}000a_{\overline{60:1}|}-200\right)=1.015\times 3{,}000a_{\overline{61:9}|}$$

That is:

$${}_1V^{retro}={}_1V^{pro}$$

20.5 The policy starts when the policyholder is age 50. The 5th premium is paid on the policyholder's 54th birthday, when the remaining term will be 6 years.

If death occurs at age 54 last birthday, the benefit amount will be £14,000, which will increase by £1,000 each year. The maturity value is £25,000. So the net premium reserve is:

$${}_4V^{pro}=13{,}000A^{\,1}_{54:\overline{6}|}+1{,}000(IA)^{\,1}_{54:\overline{6}|}+25{,}000\frac{D_{60}}{D_{54}}-P\ddot{a}_{54:\overline{6}|}$$

where the net premium is given by:

$$P\ddot{a}_{50:\overline{10}|}=9{,}000A^{\,1}_{50:\overline{10}|}+1{,}000(IA)^{\,1}_{50:\overline{10}|}+25{,}000\frac{D_{60}}{D_{50}}$$

In order to specify the calculation of the reserve precisely, it is necessary to state how the premium is calculated. This is because, for a net premium reserve, the net premium is always calculated on the same basis as the reserve.

20.6 The gross premium retrospective reserve at the end of year 20 is:

$${}_{20}V^{retro}=\frac{D_{30}}{D_{50}}\left[250\ddot{a}_{30:\overline{20}|}-30{,}000\bar{A}^{\,1}_{30:\overline{20}|}-100-0.05\times 250\times\left(\ddot{a}_{30:\overline{20}|}-1\right)\right]$$

20.7 The reserve at time 0 is zero, and so in the first year the equation of equilibrium is:

$$(P-I)(1+i)=(1+c)Sq_x+{}_1V\,p_x$$

For subsequent years, *ie* $t=1,2,\ldots$:

$$({}_tV+P-kP)(1+i)=(1+c)Sq_{x+t}+{}_{t+1}V\,p_{x+t}$$

20.8 (i) The retrospective accumulation at the end of 5 years is:

$$200(1.075)^5\frac{l_{25}}{l_{30}}=200(1.075)^5\times\frac{98{,}797}{98{,}617}=£287.65$$

(ii) The retrospective accumulation at the end of 20 years is:

$$200(1.075)^{20}\frac{l_{25}}{l_{45}}=200(1.075)^{20}\times\frac{98{,}797}{97{,}315}=£862.51$$

20.9 The reserve V should be such that the probability of making a positive future loss is less than 1%, *ie* such that:

$$P\left(50{,}000v^{K_{[30]}+1} - 0.95\times 700a_{\overline{K_{[30]}}|} - V > 0\right) < 0.01 \quad [1]$$

noting that the next premium is due in one year's time, hence we use the annuity function for payments in arrears.

We need to calculate the reserve as:

$$V(r) = 50{,}000v^{r+1} - 0.95\times 700a_{\overline{r}|} \quad [½]$$

for a value of r such that:

$$P\left(K_{[30]} < r\right) < 0.01 \quad \text{and} \quad P\left(K_{[30]} < r+1\right) \geq 0.01 \quad [1½]$$

Rearranging the above we require:

$$P\left(K_{[30]} \geq r\right) > 0.99 \quad \text{and} \quad P\left(K_{[30]} \geq r+1\right) \leq 0.99$$

or $\quad {}_rp_{[30]} > 0.99 \quad$ and $\quad {}_{r+1}p_{[30]} \leq 0.99$

or $\quad l_{[30]+r} > 0.99l_{[30]} \quad$ and $\quad l_{[30]+r+1} \leq 0.99l_{[30]}$

From the *Tables* we find $l_{[30]} = 9{,}923.7497$ which makes $0.99l_{[30]} = 9{,}824.5122$. From the *Tables* we also find that $l_{[30]+13} = l_{43} = 9{,}826.2060$ and $l_{[30]+14} = l_{44} = 9{,}814.3359$, which means that we take $r = 13$. [1½]

So the required reserve is:

$$V(13) = 50{,}000v^{14} - 0.95\times 700\left(\frac{1-v^{13}}{0.03}\right) = £25{,}984 \quad [½]$$

[Total 5]

20.10 (i) ***Future loss random variables***

The future loss random variable at time zero is the present value of the future benefits and expenses paid out, less the present value of the future premiums received.

If the life dies, we need to use K_{50} to determine the value of the sum assured (since this decreases by discrete amounts), but T_{50} for the discount factor (since the benefit is paid immediately on death). We add 400 to the total paid on death for the claim expenses.

The future loss random variable at the outset is:

$$\begin{cases} (150{,}400-10{,}000K_{50})v^{T_{50}}+300+43a_{\overline{K_{50}}|}-P\ddot{a}_{\overline{K_{50}+1}|} & K_{50}<10 \\ 300+43a_{\overline{9}|}-P\ddot{a}_{\overline{10}|} & K_{50}\geq 10 \end{cases}$$ [2]

Just before the payment of the fifth premium, the life is aged 54, and the balance of the term is 6 years. So the future loss random variable is now:

$$\begin{cases} (110{,}400-10{,}000K_{54})v^{T_{54}}+43\ddot{a}_{\overline{K_{54}+1}|}-P\ddot{a}_{\overline{K_{54}+1}|} & K_{54}<6 \\ 43\ddot{a}_{\overline{6}|}-P\ddot{a}_{\overline{6}|} & K_{54}\geq 6 \end{cases}$$ [2]

[Total 4]

(ii) ***Premium***

The premium equation using the equivalence principle is:

$$P\ddot{a}_{50:\overline{10}|}=160{,}400\bar{A}^{\;1}_{50:\overline{10}|}-10{,}000(I\bar{A})^{\;1}_{50:\overline{10}|}+300+43\times\left(\ddot{a}_{50:\overline{10}|}-1\right)$$ [1]

The level annuity factor is given on page 100 of the *Tables*, and is equal to 8.314.

For the other factors we shall need:

$$\frac{D_{60}}{D_{50}}=\frac{882.85}{1{,}366.61}=0.64601$$

The level term assurance factor is:

$$\bar{A}^{\;1}_{50:\overline{10}|}\approx(1+i)^{½}A^{\;1}_{50:\overline{10}|}=\sqrt{1.04}\left[A_{50:\overline{10}|}-\frac{D_{60}}{D_{50}}\right]$$

$$=\sqrt{1.04}\left[0.68024-0.64601\right]=0.03490$$ [1]

The increasing term assurance factor is:

$$(I\bar{A})^{\;1}_{50:\overline{10}|}\approx(1+i)^{½}(IA)^{\;1}_{50:\overline{10}|}=\sqrt{1.04}\left[(IA)_{50}-\frac{D_{60}}{D_{50}}\left((IA)_{60}+10A_{60}\right)\right]$$

$$=\sqrt{1.04}\left[8.55929-0.64601\times(8.36234+10\times 0.45640)\right]=0.21282$$ [1½]

So we have:

$$8.314P=160{,}400\times 0.03490-10{,}000\times 0.21282+300+43\times 7.314$$

This gives us a premium of £491.31. [½]

[Total 4]

(iii) ***Gross premium prospective reserve***

The gross premium prospective reserve for the policy at time 4 is the expected present value of the future benefits and expenses less the expected present value of the future premiums:

$$_4V = 120{,}400\,\bar{A}^{\;1}_{54:\overline{6}|} - 10{,}000(I\bar{A})^{\;1}_{54:\overline{6}|} - (491.31-43)\ddot{a}_{54:\overline{6}|} \qquad [1]$$

We shall need:

$$\frac{D_{60}}{D_{54}} = \frac{882.85}{1{,}154.22} = 0.76489$$

We calculate the term assurance factors as before:

$$\bar{A}^{\;1}_{54:\overline{6}|} \approx (1+i)^{½} A^{\;1}_{54:\overline{6}|} = \sqrt{1.04}\left[A_{54:\overline{6}|} - \frac{D_{60}}{D_{54}}\right]$$

$$= \sqrt{1.04}\,[0.79264 - 0.76489] = 0.02830 \qquad [1]$$

and:

$$(I\bar{A})^{\;1}_{54:\overline{6}|} \approx (1+i)^{½} (IA)^{\;1}_{54:\overline{6}|} = \sqrt{1.04}\left[(IA)_{54} - \frac{D_{60}}{D_{54}}\left((IA)_{60} + 6A_{60}\right)\right]$$

$$= \sqrt{1.04}\,[8.59381 - 0.76489\times(8.36234 + 6\times 0.45640)] = 0.10502 \qquad [1½]$$

The annuity function is again given on page 100 of the *Tables*, and is equal to 5.391.

So the reserve is:

$$_4V = 120{,}400\times 0.02830 - 10{,}000\times 0.10502 - 448.31\times 5.391 = -59.60 \qquad [½]$$

[Total 4]

(iv) ***Comment***

The gross premium prospective reserve at the end of the fourth policy year is negative. This is typical of a decreasing term assurance where the cost of claims and expenses are relatively high in the early years of the policy and low in the later years. [½]

As the premiums are level, the future premiums have greater expected present value than the future claims and expenses, producing the negative reserve value. [1]

The insurer would incur an overall loss from a group of policies of this type, should some of them lapse when their reserves are negative. [½]

The insurer can avoid this problem if the premium paying term is reduced, *eg* to 6 years for a 10-year policy, so for reserves at later policy durations the EPV of the future premiums is reduced, causing the reserve value to rise. [½]

[Maximum 2]

20.11 *This question is Subject CT5, April 2010, Question 14, part (ii).*

We want the expected present value of the future benefits, expenses and commission, less the expected present value of the future premiums.

The basic sum assured plus past declared bonuses is £245,000. Allowing for a compound future bonus rate of 4% (and remembering that the bonuses do not vest until the end of the year), the EPV of the death benefits is approximately:

$$245{,}000\left[q_{60}v^{½}+1.04\,{}_{1|}q_{60}v^{1½}+1.04^2\,{}_{2|}q_{60}v^{2½}+1.04^3\,{}_{3|}q_{60}v^{3½}+1.04^4\,{}_{4|}q_{60}v^{4½}\right] \qquad [½]$$

Taking out a factor of $v^{½}$, this is:

$$245{,}000v^{½}\left[q_{60}+1.04\,{}_{1|}q_{60}v^1+1.04^2\,{}_{2|}q_{60}v^2+1.04^3\,{}_{3|}q_{60}v^3+1.04^4\,{}_{4|}q_{60}v^4\right] \qquad [½]$$

But since $v=\dfrac{1}{1.04}$, this simplifies to:

$$245{,}000v^{½}\left[q_{60}+{}_{1|}q_{60}+{}_{2|}q_{60}+{}_{3|}q_{60}+{}_{4|}q_{60}\right]$$

$$=245{,}000v^{½}\times{}_5q_{60}=245{,}000\times\frac{1}{1.04^{½}}\times\left(1-\frac{8{,}821.2612}{9{,}287.2164}\right)$$

$$=12{,}053.357 \qquad [1]$$

Similarly, the EPV of the survival benefit is:

$$245{,}000\times{}_5p_{60}\times v^5\times1.04^5=245{,}000\times{}_5p_{60}=245{,}000\times\frac{8{,}821.2612}{9{,}287.2164}=232{,}707.940 \qquad [1]$$

The EPV of the claim expense on death is:

$$1{,}000\bar{A}^{\,1\,@4\%}_{60:\overline{5}|}=1{,}000\times1.04^{½}\times\left[A_{60:\overline{5}|}-\frac{D_{65}}{D_{60}}\right]$$

$$=1{,}000\times1.04^{½}\times\left[0.82499-\frac{689.23}{882.85}\right]=45.180 \qquad [1]$$

The EPV of the claim expense on maturity is:

$$500A_{60:\overline{5}|}^{\;\;\,1}=500\times\frac{D_{65}}{D_{60}}=390.344 \qquad [½]$$

So the gross prospective reserve at time 25 can be written as follows:

$${}_{25}V^{pro}=12{,}053.357+232{,}707.94+90\ddot{a}^{@4\%}_{60:\overline{5}|}+45.180+390.344$$

$$+0.025\times4\times615.61\ddot{a}^{(4)@4\%}_{60:\overline{5}|}-4\times615.61\ddot{a}^{(4)@4\%}_{60:\overline{5}|} \qquad [½]$$

Calculating the values of the annuity factors:

$$\ddot{a}_{60:\overline{5}|} = 4.550$$

$$\ddot{a}^{(4)}_{60:\overline{5}|} \approx 4.550 - \frac{3}{8}\left(1 - \frac{D_{65}}{D_{60}}\right) = 4.468$$ [½]

So:

$$_{25}V^{pro} = 12{,}053.357 + 232{,}707.940 + 90 \times 4.550 + 45.180 + 390.344$$
$$+ 0.025 \times 2{,}462.44 \times 4.468 - 2{,}462.44 \times 4.468$$
$$= 234{,}880$$

or about £234,900. [½]

[Total 6]

20.12 *This question is Subject CT5, October 2012, Question 13.*

(i) ***Monthly premium***

Let G be the total amount of premium paid in a year. The expected present value of the premiums is:

$$G\ddot{a}^{(12)}_{[55]:\overline{30}|}$$ [½]

The expected present value of the benefits is:

$$73{,}500\bar{A}_{[55]} + 1{,}500(I\bar{A})_{[55]}$$ [1]

The expected present value of the expenses and commission is:

$$275 + 65\left[\ddot{a}_{[55]} - 1\right] + 200\bar{A}_{[55]} + 0.75G + 0.025G\left[\ddot{a}^{(12)}_{[55]:\overline{30}|} - \frac{1}{12}\right]$$ [1½]

We shall need:

$$\frac{D_{85}}{D_{[55]}} = \frac{120.71}{1{,}104.05} = 0.10933$$

Now:

$$\ddot{a}_{[55]:\overline{30}|} = \ddot{a}_{[55]} - \frac{D_{85}}{D_{[55]}}\ddot{a}_{85} = 15.891 - 0.10933 \times 5.333 = 15.308$$ [½]

So:

$$\ddot{a}^{(12)}_{[55]:\overline{30}|} \approx \ddot{a}_{[55]:\overline{30}|} - \frac{11}{24}\left(1 - \frac{D_{85}}{D_{[55]}}\right) = 15.308 - \frac{11}{24}(1 - 0.10933) = 14.900$$ [½]

We also need:

$$\bar{A}_{[55]} \approx 1.04^{½} \times A_{[55]} = 1.04^{½} \times 0.38879 = 0.39649$$

$$(I\bar{A})_{[55]} \approx 1.04^{½} \times (IA)_{[55]} = 1.04^{½} \times 8.58908 = 8.75918$$ [½]

So, using the equivalence principle, we have:

$$G\ddot{a}^{(12)}_{[55]:\overline{30|}} = 73{,}500\bar{A}_{[55]} + 1{,}500(I\bar{A})_{[55]} + 275 + 65\left[\ddot{a}_{[55]} - 1\right]$$

$$+200\bar{A}_{[55]} + 0.75G + 0.025G\left[\ddot{a}^{(12)}_{[55]:\overline{30|}} - \frac{1}{12}\right]$$ [½]

Substituting in the values:

$$14.900G = 73{,}500 \times 0.39649 + 1{,}500 \times 8.75918 + 275 + 65 \times 14.891$$

$$+200 \times 0.39649 + 0.75G + 0.025G \times 14.816$$ [½]

So $G = \dfrac{43{,}602.96}{13.7793} = 3{,}164.38$ and the monthly premium is $\dfrac{3{,}164.38}{12} = £263.70$. [½]

[Total 6]

(ii) ***Gross prospective policy value***

At the end of the 30th policy year, the 30th past bonus has just been added, bringing the current policy benefit to £120,000. [½]

The future bonus rate is assumed to be 2.5% *pa* of the basic sum assured, which means that £1,875 is assumed to be added to the policy benefit at the end of each future year. So, the gross premium prospective reserve at time 30 is:

$${}_{30}V = 118{,}125\bar{A}_{85} + 1{,}875(I\bar{A})_{85} + 80\ddot{a}_{85} + 250\bar{A}_{85}$$ [2]

Evaluating:

$$\bar{A}_{85} \approx 1.04^{½} \times A_{85} = 1.04^{½} \times 0.79490 = 0.81064$$

$$(I\bar{A})_{85} \approx 1.04^{½} \times (IA)_{85} = 1.04^{½} \times 4.40856 = 4.49587$$

$$\ddot{a}_{85} = 5.333$$ [1]

So:

$${}_{30}V = 118{,}125 \times 0.81064 + 1{,}875 \times 4.49587 + 80 \times 5.333 + 250 \times 0.81064 = 104{,}816$$ [½]

or about £104,800. [Total 4]

20.13 *This question is Subject CT5, April 2014, Question 11.*

(i) ***Annual office premium***

Let P be the annual office premium for a sum assured of £1,000.

The equation of value, for a sum assured of £1,000, is:

$$P\ddot{a}_{[40]:\overline{20}|} = 1{,}000A_{[40]:\overline{20}|} + 0.6P + 0.06P\left(\ddot{a}_{[40]:\overline{20}|} - 1\right) \qquad [1]$$

Looking up the values in the AM92 tables, using an interest rate of 4%:

$$13.930P = 1{,}000 \times 0.46423 + 0.6P + 0.06P(13.930 - 1) \qquad [½]$$

Rearranging:

$$P(13.930 - 0.6 - 0.06 \times 12.930) = 464.23$$

$$\Rightarrow \quad P = \frac{464.23}{12.5542} = 36.98 \qquad [½]$$

So, the annual office premium for a sum assured of £1,000 is £36.98.

[Total 2]

(ii) ***Gross premium prospective reserve at time 5***

The policies were issued on 1 January 2008, so 31 December 2012 is 5 years later. This means we need the gross premium prospective reserve at time 5, when the life is aged 45 and the remaining term is 15 years.

The gross premium prospective reserve at time 5 is:

$$1{,}000A_{45:\overline{15}|} + 0.06 \times 36.98\ddot{a}_{45:\overline{15}|} - 36.98\ddot{a}_{45:\overline{15}|}$$

$$= 1{,}000A_{45:\overline{15}|} - 0.94 \times 36.98\ddot{a}_{45:\overline{15}|} \qquad [1½]$$

Looking up values in the AM92 tables using an interest rate of 4% gives:

$$1{,}000 \times 0.56206 - 0.94 \times 36.98 \times 11.386 = 166.27 \qquad [½]$$

So, the gross premium prospective reserve at time 5 for a sum assured of £1,000 is £166.27.

[Total 2]

(iii) ***Overall profit or loss in 2013***

The total reserve held at the start of 2013 for policies with total sum assured £15,500,000 is:

$$166.27 \times \frac{15{,}500{,}000}{1{,}000} = £2{,}577{,}185 \qquad [1]$$

The total premiums received at the start of 2013 are:

$$36.98 \times \frac{15,500,000}{1,000} = £573,190$$ [½]

Interest is earned over 2013 at a rate of 3.5%, so by the end of the year, the accumulated value of the opening reserve plus the premiums less the expenses will be:

$$(2,577,185 + 573,190 - 76,500) \times 1.035 = £3,181,460.63$$ [1½]

This represents the total funds available to the company at the end of 2013.

The total amount paid out at the end of 2013 in respect of surrenders is:

$$0.85 \times {}_6V \times \frac{625,000}{1,000}$$ [1]

where ${}_6V$ denotes the gross premium prospective reserve at time 6 (*ie* 31 December 2013) for a sum assured of £1,000. We can calculate ${}_6V$ as:

$$1,000 A_{46:\overline{14|}} - 0.94 \times 36.98 \ddot{a}_{46:\overline{14|}}$$ [1]

Using AM92 mortality and an interest rate of 4%:

$${}_6V = 1,000 \times 0.58393 - 0.94 \times 36.98 \times 10.818 = 207.88$$ [½]

So, the total amount paid out at the end of 2013 in respect of surrenders is:

$$0.85 \times 207.88 \times \frac{625,000}{1,000} = £110,436.25$$ [½]

The total sum assured in force at the end of 2013 (after death claims and surrenders) is:

$$15,500,000 - 295,000 - 625,000 = £14,580,000$$ [1]

So, the total reserve required at the end of 2013 for the policies remaining in force is:

$$207.88 \times \frac{14,580,000}{1,000} = £3,030,890.40$$ [1]

This means that the total funds needed at the end of 2013 to pay death and surrender claims, and to set up the reserves needed for surviving policies is:

$$295,000 + 110,436.25 + 3,030,890.40 = £3,436,326.65$$ [1]

The overall profit in 2013 will be the funds available at the end of the year minus the funds needed at the end of the year:

$$3,181,460.63 - 3,436,326.65 = -£254,866.02$$ [1]

ie there is a loss of £254,866.

[Total 10]

20.14 *This question is Subject CT5, April 2012, Question 2.*

The left-hand side of the equation of equilibrium takes the form:

Reserve + premium – expenses + interest [½]

Here, this gives us:

$$(25{,}130+3{,}000-90)\times 1.04=29{,}161.60$$ [1]

The required cashflow is:

Expected reserve at the end of the year + expected death benefit [½]

On death in the 12th policy year, an amount equal to 12 times the annual premium (*ie* 36,000) would be paid at the end of the year. So the required cashflow is:

$$=0.97\times 28{,}950+0.03\times 36{,}000=29{,}161.50$$ [½]

The expected cashflow is greater than the required cashflow by £0.10, which is our expected profit emerging at the end of the 12th policy year (and is approximately zero). [½]

[Total 3]

20.15 *This question is Subject CT5, September 2013, Question 11, part (c).*

This is evaluated as follows:

$$s_{55:\overline{10|}}=(1+i)^{10}\frac{l_{55}}{l_{65}}a_{55:\overline{10|}}=(1+i)^{10}\frac{l_{55}}{l_{65}}\left[a_{55}-v^{10}\frac{l_{65}}{l_{55}}a_{65}\right]$$

$$=(1+i)^{10}\frac{l_{55}}{l_{65}}a_{55}\;-\;a_{65}$$

$$=1.04^{10}\times\frac{9{,}917.623}{9{,}703.708}(18.210-1)-(14.871-1)$$

$$=26.03659-13.871=12.166$$ [1]

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Joint life and last survivor functions

Syllabus objectives

5.1 Define and use assurance and annuity functions involving two lives.

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

0 Introduction

In this chapter, we extend the concepts developed earlier to deal with situations involving two lives; for instance, 'what is the value of an annuity payable until the last of two lives dies?'

This chapter covers the following topics:

- random variables describing basic joint life and last survivor functions
- determining simple joint life and last survivor probabilities
- present values of simple joint life and last survivor policies
- calculating premiums and reserves for policies based on two lives.

1 Random variables to describe joint life functions

1.1 Single life functions

So far we have described annuity and assurance functions that depend upon the death or survival of a single life aged x. Central to the development of these functions is the random variable measuring the future lifetime of a life now aged x, T_x, or its curtate counterpart K_x.

Recall that K_x (the curtate future lifetime of x) is the integer part of T_x (the complete future lifetime of x).

1.2 Joint life functions

We now consider annuity and assurance functions, which depend upon the death or survival of two lives. The random variables of interest are T_x and T_y, the future lifetimes of two lives, one aged x and the other aged y. Throughout the analysis of these problems, we assume that T_x and T_y are independent random variables.

So we assume that the mortality of life x is independent of the mortality of life y. In practice, this is seldom the case since many joint life policies involve husband/wife, or similar, partnerships. The mortality of these partners is not independent because of the significant possibility of both lives dying as a result of the same accident or illness. However, the assumption of independence makes the theory far more manageable.

The random variable T_{xy} measures the joint lifetime of (x) and (y), *ie* the time while both (x) and (y) remain alive, which is the time until the first death of (x) and (y). We can write:

$$T_{xy} = \min\{T_x, T_y\}$$

The cumulative distribution function of this random variable can be written:

$$F_{T_{xy}}(t) = P\left[T_{xy} \le t\right]$$

which we write as:

$$\begin{aligned} F_{T_{xy}}(t) &= 1 - {}_tp_{xy} \\ &= P\left[\min\left\{T_x, T_y\right\} \le t\right] \\ &= 1 - P\left[T_x > t \text{ and } T_y > t\right] \\ &= 1 - P\left[T_x > t\right]P\left[T_y > t\right] \end{aligned}$$

since the random variables T_x and T_y are independent.

So:

$$\boldsymbol{F_{T_{xy}}(t) = 1 - {}_t p_x \, {}_t p_y}$$

using the life table notation.

It is helpful at this point to recall the idea of a *status*, which we met earlier when considering single life assurances and annuities.

The random variable T_{xy} is characterised by the *joint life* status xy, which is the status of both lives x and y being alive. If either one of x or y dies, the joint life status is said to *fail*. The random variable T_{xy} represents the future time until the failure of the status, which in this case is the joint life status xy.

With this in mind, the probability notation ${}_t p_{xy}$ introduced above represents the probability that the joint life status of x and y survives for t years, *ie* both lives survive for at least t years.

In the same way, when we consider the single life temporary annuity $\ddot{a}_{x:\overline{n}|}$, we are looking at an annuity payable while the joint status $x:\overline{n}|$ is still active, that is, the life x is still alive and the n-year period has not yet expired. In this case the joint status is made up of an active life and a time period, but the underlying logic is still the same.

The density function of T_{xy} can be obtained by differentiating the cumulative distribution function:

$$\boldsymbol{f_{T_{xy}}(t) = \frac{d}{dt}\left[1 - {}_t p_x \, {}_t p_y\right]}$$

$$= -\frac{d}{dt} {}_t p_x \, {}_t p_y$$

$$= -{}_t p_x \frac{d}{dt} {}_t p_y - {}_t p_y \frac{d}{dt} {}_t p_x \qquad \text{(using the product rule)}$$

$$\boldsymbol{= -{}_t p_x \left(-{}_t p_y \mu_{y+t}\right) - {}_t p_y \left(-{}_t p_x \mu_{x+t}\right)}$$

The last line above follows from the fact that the PDF of the random variable T_x is ${}_t p_x \, \mu_{x+t}$, so this must be equal to the derivative of the cumulative distribution function:

$$f_{T_x}(t) = {}_t p_x \, \mu_{x+t} = \frac{d}{dt} F_{T_x}(t) = \frac{d}{dt} P(T_x \le t) = \frac{d}{dt} {}_t q_x = \frac{d}{dt}\left(1 - {}_t p_x\right) = -\frac{d}{dt} {}_t p_x$$

Tidying up the expression above, the PDF of T_{xy} is:

$$\boldsymbol{f_{T_{xy}}(t) = {}_t p_x \, {}_t p_y \left(\mu_{x+t} + \mu_{y+t}\right)}$$

By considering the infinitesimally small time interval $(t, t+\delta t)$, we can interpret:

$${}_tp_x\,{}_tp_y\left(\mu_{x+t}+\mu_{y+t}\right)\delta t$$

as the approximate probability that the joint life status xy fails over the interval of time δt. This is the product of:

- the probability of both x and y surviving to time t, ${}_tp_x\,{}_tp_y\left(={}_tp_{xy}\right)$, and
- the probability of x or y dying in the interval from time t to time $t+\delta t$, $\left(\mu_{x+t}+\mu_{y+t}\right)\delta t$.

1.3 Joint lifetime random variables and joint life table functions

When functions of a single life *eg* T_x are considered, it is helpful to introduce the life table functions l_x, d_x and q_x as an aid to the calculation of the numerical values of expressions that are the solution of actuarial problems.

In an exactly similar way it is helpful to develop the joint life functions l_{xy}, d_{xy} and q_{xy} to help in the numerical evaluation of expressions that are the solution to problems involving more than one life. We define these functions in terms of the single life functions. Recall that:

$${}_tp_{xy}={}_tp_x\,{}_tp_y$$

Using the independence assumption:

$${}_tp_{xy}=\frac{l_{x+t}}{l_x}\cdot\frac{l_{y+t}}{l_y}$$

So we write:

$$l_{xy}=l_xl_y$$

and:

$${}_tp_{xy}=\frac{l_{x+t:y+t}}{l_{xy}}$$

only separating the subscripts with colons when the exact meaning of the function would be unclear if the colons were omitted.

Then:

$$d_{xy}=l_{xy}-l_{x+1:y+1}$$

$$q_{xy}=\frac{d_{xy}}{l_{xy}}$$

q_{xy} is the probability that the joint life status fails by the end of the year, *ie* it is the probability that *at least one* of x and y dies within the year.

Similarly, ${}_tq_{xy}$ represents the probability that the joint life status fails within the next t years, *ie* by the end of t years there is at least one death.

The force of failure of the joint life status can be derived in the usual way:

$$\mu_{x+t:y+t} = -\frac{1}{l_{x+t:y+t}}\frac{d}{dt}l_{x+t:y+t}$$

This can then be related to the forces of mortality in the life tables for the single lives *x* and *y*:

$$\begin{aligned}\mu_{x+t:y+t} &= -\frac{d}{dt}\log_e l_{x+t:y+t} \\ &= -\frac{d}{dt}\log_e l_{x+t}\, l_{y+t} \\ &= -\frac{d}{dt}\left\{\log_e l_{x+t} + \log_e l_{y+t}\right\} \\ &= \mu_{x+t} + \mu_{y+t}\end{aligned}$$

Notice that:

- **this relationship is additive, which is in contrast to the previous relationships, which were multiplicative, and**
- **that there is no 'simple' relationship for d_{xy} .**

So to get the joint *force of mortality* we *add* the forces for the individual lives. For the joint *probability of survival* we *multiply* the individual probabilities of survival.

So we can write:

$$\begin{aligned}f_{T_{xy}}(t) &= {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) \\ &= {}_tp_{xy}\,\mu_{x+t:y+t}\end{aligned}$$

We can define a discrete random variable, which measures the curtate joint future lifetime of x and y:

$$K_{xy} = \text{integer part of } T_{xy}$$

and develop the probability function of K_{xy}:

$$\begin{aligned}
P\left[K_{xy} = k\right] &= P\left[k \le T_{xy} < k+1\right] \\
&= F_{T_{xy}}(k+1) - F_{T_{xy}}(k) \\
&= \left(1 - {}_{k+1}p_{xy}\right) - \left(1 - {}_{k}p_{xy}\right) \\
&= {}_{k}p_{xy} - {}_{k+1}p_{xy} \\
&= {}_{k}p_{xy} - {}_{k}p_{xy}\, p_{x+k:y+k} \\
&= {}_{k}p_{xy}\, q_{x+k:y+k} \\
&= {}_{k|}q_{xy}
\end{aligned}$$

This is a deferred probability that the joint life status fails. Specifically, it is the probability that the joint life status fails within a one-year period, starting in k years' time, *ie* it is the probability that both lives survive for k years, and then at least one of the lives dies in the following year.

The joint life table functions l_{xy}, d_{xy}, q_{xy} and μ_{xy} are not tabulated in the Formulae and Tables for Examinations. However, these functions can be evaluated using the tabulated single life functions l_x, q_x and μ_x.

There is limited information in the *Tables* for joint life functions. The only values provided are those of joint life annuities-due for males and females of different ages based on PMA92C20 mortality for the male life, PFA92C20 for the female life, and 4% *pa* interest. These appear on page 115 of the *Tables*, and we will use these later on in this chapter.

Question

Assuming that both lives are independently subject to AM92 mortality, calculate the following:

(i) ${}_{3}p_{45:41}$

(ii) $q_{66:65}$

(iii) $\mu_{38:30}$

Solution

(i) $${}_3p_{45:41} = \frac{l_{48:44}}{l_{45:41}} = \frac{l_{48}}{l_{45}} \times \frac{l_{44}}{l_{41}} = \frac{9{,}753.4714}{9{,}801.3123} \times \frac{9{,}814.3359}{9{,}847.0510} = 0.991813$$

(ii) $$q_{66:65} = 1 - p_{66:65} = 1 - \frac{l_{67:66}}{l_{66:65}} = 1 - \frac{l_{67}}{l_{66}} \times \frac{l_{66}}{l_{65}} = 1 - \frac{l_{67}}{l_{65}} = 1 - \frac{8{,}557.0118}{8{,}821.2612} = 0.029956$$

(iii) $$\mu_{38:30} = \mu_{38} + \mu_{30} = 0.000788 + 0.000585 = 0.001373$$

1.4 Last survivor lifetime random variables

Two common types of policy are:

- an annuity payable to a couple while at least one of them is alive, and
- an assurance payable on the second death of a couple.

These are both examples of last survivor policies, where the payment is contingent on what happens to the second life to die, rather than the first.

The random variable $T_{\overline{xy}}$ measures the time until the last death of (x) and (y), *ie* the time while at least one of (x) and (y) remains alive. We can write:

$$T_{\overline{xy}} = \max\{T_x, T_y\}$$

So the last survivor status fails on the second death. The last survivor status is indicated by $\overline{xy}$.

The cumulative distribution function of this random variable can be written:

$$F_{T_{\overline{xy}}}(t) = P\left[T_{\overline{xy}} \le t\right] = {}_tq_{\overline{xy}}$$

This is the probability that the last survivor status fails within t years, *ie* the probability that both lives are dead by the end of t years. Now:

$$\begin{aligned} F_{T_{\overline{xy}}}(t) &= P\left[\max\{T_x, T_y\} \le t\right] \\ &= P\left[T_x \le t \text{ and } T_y \le t\right] \\ &= P\left[T_x \le t\right]P\left[T_y \le t\right] \\ &= {}_tq_x \times {}_tq_y \end{aligned}$$

since the random variables T_x and T_y are independent.

So:

$$\begin{aligned} F_{T_{\overline{xy}}}(t) &= (1 - {}_tp_x)(1 - {}_tp_y) \\ &= 1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y \\ &= (1 - {}_tp_x) + (1 - {}_tp_y) - (1 - {}_tp_x\,{}_tp_y) \\ &= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) \end{aligned}$$

using the life table notation.

This last result is a particular example of a general relationship that will be key to enabling us to calculate multiple life functions easily.

Important result

$$\begin{pmatrix}\text{function of last} \\ \text{survivor status } \overline{xy}\end{pmatrix} = \begin{pmatrix}\text{function of single} \\ \text{life status } x\end{pmatrix} + \begin{pmatrix}\text{function of single} \\ \text{life status } y\end{pmatrix} - \begin{pmatrix}\text{function of joint} \\ \text{life status } xy\end{pmatrix}$$

The function involved must be the same for all statuses. We will subsequently refer to this relationship as just:

last survivor (L) = single (S) + single (S) − joint (J)

In the above Core Reading, the function is the cumulative distribution function of the lifetime of the status at time t, *ie* the probability of the status failing by time t. So we have:

$${}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}$$

The density function of $T_{\overline{xy}}$ can be obtained by differentiating the cumulative distribution function:

$$\begin{aligned} f_{T_{\overline{xy}}}(t) &= \frac{d}{dt}\left[1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y\right] \\ &= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) \end{aligned}$$

Using the results from Section 1.3 above, we can write:

$$\begin{aligned} f_{T_{\overline{xy}}}(t) &= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t} \\ &= f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) \end{aligned}$$

In other words, we have the $L = S + S - J$ relationship again, but this time for the probability density function at time t.

We could have alternatively found the above result from:

$$f_{T_{\overline{xy}}}(t) = \frac{d}{dt} F_{T_{\overline{xy}}}(t) = \frac{d}{dt}\left[F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) \right] = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$

Tip

For joint life statuses, it is often easier to work with p -type (survival) functions, since ${}_tp_{xy} = {}_tp_x \times {}_tp_y$.

For last survivor statuses, it is often easier to work with q -type (mortality) functions, since ${}_tq_{\overline{xy}} = {}_tq_x \times {}_tq_y$.

2 Simple probabilities involving two lives

We now apply the random variable theory developed above to see how joint life and last survivor probabilities of death and survival can be evaluated.

2.1 Evaluating probabilities of death or survival of either or both of two lives

We can define a discrete random variable, which measures the curtate last survivor lifetime of x and y :

$$K_{\overline{xy}} = \text{integer part of } T_{\overline{xy}}$$

and develop the probability function of $K_{\overline{xy}}$:

$$\begin{aligned} P\left[K_{\overline{xy}} = k\right] &= P\left[k \le T_{\overline{xy}} < k+1\right] \\ &= {}_{k|}q_{\overline{xy}} \\ &= F_{T_{\overline{xy}}}(k+1) - F_{T_{\overline{xy}}}(k) \\ &= F_{T_x}(k+1) + F_{T_y}(k+1) - F_{T_{xy}}(k+1) - \left\{F_{T_x}(k) + F_{T_y}(k) - F_{T_{xy}}(k)\right\} \\ &= P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right] \\ &= {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy} \end{aligned}$$

Note that this is again the $L = S + S - J$ relationship:

$${}_{k|}q_{\overline{xy}} = {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}$$

As we saw in the last section, **the cumulative distribution function of $T_{\overline{xy}}$ is given by:**

$${}_tq_{\overline{xy}} = 1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y$$

and so the survival function of $T_{\overline{xy}}$ is:

$${}_tp_{\overline{xy}} = P\left[T_{\overline{xy}} > t\right] = 1 - P\left[T_{\overline{xy}} \le t\right] = {}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y$$

(*ie* $L = S + S - J$).

This can be factorised into:

$$_tp_x\,{}_tp_y + \left(1-{}_tp_x\right){}_tp_y + \left(1-{}_tp_y\right){}_tp_x$$

where each of the three terms corresponds to one of the mutually exclusive and exhaustive events which result in the last survivor of (*x*) and (*y*) living for at least *t* years, *ie*:

- **both (x) and (y) alive after t years,**
- **(x) dead, but (y) alive after t years, and**
- **(x) alive, but (y) dead after t years.**

These probabilities can be evaluated directly. The probability of the complementary event that both lives die within t years is:

$$_tq_{\overline{xy}} = (1-{}_tp_x)\,(1-{}_tp_y) = {}_tq_x\,{}_tq_y$$

as we saw earlier.

In summary, there are three alternatives for calculating $_tp_{\overline{xy}}$:

- $1-{}_tq_{\overline{xy}}$ (1)
- $_tp_x + {}_tp_y - {}_tp_{xy}$ (2)
- $_tp_x\,{}_tp_y + \left(1-{}_tp_x\right){}_tp_y + \left(1-{}_tp_y\right){}_tp_x$ (3)

All three of these can be useful, but it's generally best to try using (1) first and (3) last, as more calculations are involved as we move down the list.

We also have $p_{\overline{xy}} = p_x + p_y - p_{xy}$ (for example), where $p_{\overline{xy}}$ is the probability that the last survivor status remains active for at least one year.

Question

Calculate $P\left(5 < K_{\overline{50:60}} < 10\right)$ assuming that the two lives are both independently subject to AM92 mortality.

Solution

Here we want the curtate joint future lifetime to be 6, 7, 8 or 9 years. This means that the first death of the two lives must occur between time 6 years and time 10 years. So both lives must survive for 6 years, and then at least one life must die in the next 4 years.

We can evaluate this as follows:

$$P(5 < K_{50:60} < 10) = {}_{6|4}q_{50:60} = \frac{l_{56}}{l_{50}} \times \frac{l_{66}}{l_{60}} \times \left[1 - \left(\frac{l_{60}}{l_{56}} \times \frac{l_{70}}{l_{66}}\right)\right]$$

$$= \frac{9{,}515.1040}{9{,}712.0728} \times \frac{8{,}695.6199}{9{,}287.2164} \times \left[1 - \left(\frac{9{,}287.2164}{9{,}515.1040} \times \frac{8{,}054.0544}{8{,}695.6199}\right)\right]$$

$$= 0.088028$$

2.2 Evaluating last survivor functions

We have already derived:

$$F_{T_{\overline{xy}}}(t) = F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t) = 1 - {}_tp_x - {}_tp_y + {}_tp_{xy}$$

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t) = {}_tp_x\,\mu_{x+t} + {}_tp_y\,\mu_{y+t} - {}_tp_{xy}\,\mu_{x+t:y+t}$$

and so it seems that all last survivor functions can be expressed in terms of single life and joint life functions. This is true and provides a method of evaluating such functions without the need to develop any additional functions to help in computation.

This is the result of the relationship between the joint lifetime and last survivor lifetime random variables:

$$T_{xy} + T_{\overline{xy}} = \min\{T_x, T_y\} + \max\{T_x, T_y\} = T_x + T_y$$

Similarly:

$$K_{xy} + K_{\overline{xy}} = \min\{K_x, K_y\} + \max\{K_x, K_y\} = K_x + K_y$$

which gives the result:

$$P\left[K_{\overline{xy}} = k\right] = P\left[K_x = k\right] + P\left[K_y = k\right] - P\left[K_{xy} = k\right]$$

This is the rationale underlying the important relationship that we stated in Section 1.4 above.

So curtate last survivor functions can be evaluated from the corresponding joint life and single life functions.

Question

Calculate:

(i) $p_{\overline{62:65}}$

(ii) ${}_3q_{\overline{50:50}}$

assuming that the two lives are both independently subject to AM92 Ultimate mortality.

Solution

(i) We can calculate this as:

$$p_{\overline{62:65}} = 1 - q_{\overline{62:65}} = 1 - q_{62}\, q_{65} = 1 - 0.010112 \times 0.014243 = 0.999856$$

or, alternatively:

$$\begin{aligned} p_{\overline{62:65}} &= p_{62} + p_{65} - p_{62:65} \\ &= \frac{l_{63}}{l_{62}} + \frac{l_{66}}{l_{65}} - \frac{l_{63:66}}{l_{62:65}} \\ &= \frac{9{,}037.3973}{9{,}129.7170} + \frac{8{,}695.6199}{8{,}821.2612} - \frac{9{,}037.3973}{9{,}129.7170} \times \frac{8{,}695.6199}{8{,}821.2612} \\ &= 0.999856 \end{aligned}$$

(ii) We can calculate this as:

$${}_3q_{\overline{50:50}} = \left({}_3q_{50}\right)^2 = \left(1 - \frac{l_{53}}{l_{50}}\right)^2 = \left(1 - \frac{9{,}630.0522}{9{,}712.0728}\right)^2 = 0.000071$$

3 Present values involving two lives

There is no new theory in this section, as we re-use the same assurance and annuity functions that we met when considering single lives, but using multiple life statuses instead of the single life status.

3.1 Present values of joint life and last survivor assurances

Consider an assurance under which the benefit (of 1) is paid immediately on the ending (failure) of a status u. This status u could be any joint lifetime or last survivor status, *eg* $xy, \overline{xy}$. Let T_u be a continuous random variable representing the future lifetime of the status u and let $f_{T_u}(t)$ be the probability density function of T_u.

The present value of the assurance can be represented by the random variable:

$$\bar{Z}_u = v_i^{T_u}$$

where i is the valuation rate of interest. The expected value of $\bar{Z}_u$ is denoted by $\bar{A}_u$ where:

$$E\left[\bar{Z}_u\right] = \bar{A}_u = \int_{t=0}^{t=\infty} v^t f_{T_u}(t)dt$$

and the variance can be written as:

$$\begin{aligned}\text{var}\left(\bar{Z}_u\right) &= E\left[\bar{Z}_u^2\right] - \left(E\left[\bar{Z}_u\right]\right)^2 \\ &= \int_{t=0}^{t=\infty} v^{2t} f_{T_u}(t)dt - \bar{A}_u^2 \\ &= {}^2\bar{A}_u - \left(\bar{A}_u\right)^2\end{aligned}$$

where ${}^2\bar{A}_u$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

The final expression above follows because:

$$\int_{t=0}^{\infty} v^{2t} f_{T_u}(t)dt = \int_{t=0}^{\infty} \left(v^2\right)^t f_{T_u}(t)dt = \int_{t=0}^{\infty} (v^*)^t f_{T_u}(t)dt$$

which is equal to ${}^2\bar{A}_u$ where the pre-superscript of 2 denotes a rate of interest i^* such that:

$$i^* = \frac{1}{v^*} - 1 = \frac{1}{v^2} - 1 = (1+i)^2 - 1 = 2i + i^2$$

Equivalently, the pre-superscript of 2 can be thought of as denoting a force of interest δ^* such that:

$$\delta^* = -\ln v^* = -\ln v^2 = -2\ln v = 2\delta$$

For example, using the results from Sections 1.2 and 1.3, if $u = xy$ we can write the mean (*ie* the expected value) and variance of the present value of an assurance payable immediately on the ending of the joint lifetime of (x) and (y) as:

Mean: $$\bar{A}_{xy} = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \mu_{x+t:y+t} dt = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy} \left(\mu_{x+t} + \mu_{y+t}\right) dt$$

Variance: $${}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2$$

Again, using the results of Section 1.4, if $u = \overline{xy}$ we can write the mean and variance of the present value of an assurance payable immediately on the death of the last survivor of (x) or (y) as:

Mean: $$\bar{A}_{\overline{xy}} = \int_{t=0}^{t=\infty} v^t \left({}_tp_x \mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_{xy} \mu_{x+t:y+t}\right) dt = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Variance: $${}^2\bar{A}_{\overline{xy}} - \left(\bar{A}_{\overline{xy}}\right)^2 = \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$$

So last survivor functions can be evaluated in terms of single and joint life functions.

The integral expression given above for the mean can also be written as:

$$\begin{aligned}
\bar{A}_{\overline{xy}} &= \int_{t=0}^{t=\infty} v^t \left({}_tp_x \mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_{xy} \mu_{x+t:y+t}\right) dt \\
&= \int_{t=0}^{t=\infty} v^t \left({}_tp_x \mu_{x+t} + {}_tp_y \mu_{y+t} - {}_tp_x \, {}_tp_y \left(\mu_{x+t} + \mu_{y+t}\right)\right) dt \\
&= \int_{t=0}^{t=\infty} v^t \left({}_tp_x \mu_{x+t} \left(1 - {}_tp_y\right) + {}_tp_y \mu_{y+t} \left(1 - {}_tp_x\right)\right) dt \\
&= \int_{t=0}^{t=\infty} v^t \left({}_tp_x \mu_{x+t} \, {}_tq_y + {}_tp_y \mu_{y+t} \, {}_tq_x\right) dt
\end{aligned}$$

The integrand in this expression reflects the payment of the benefit at time t, requiring a discount factor of v^t, and the two events that would cause the benefit to be paid at this time, namely:

- x dies at time t (with probability density ${}_tp_x \mu_{x+t}$), after y has already died (with probability ${}_tq_y$)
- y dies at time t (with probability density ${}_tp_y \mu_{y+t}$), after x has already died (with probability ${}_tq_x$).

Integrating over all possible values of t gives the EPV of the benefit.

If the assurance benefit is paid at the end of the year in which the status ends, then we can use a discrete random variable K_u with a present value function:

$$Z_u = v_i^{K_u+1}$$

where i is the valuation rate of interest.

In the above expression we need the extra '1' to take us to the end of the year in which the status fails, which is when the payment is made, because K_u will take us only to the start of that year. For example, if the status fails in the first year, the present value of the benefit is v and not $v^K = v^0 = 1$.

A similar analysis for K_{xy} and $K_{\overline{xy}}$ gives the means and variances of the present values of the joint life and last survivor assurances with sums assured payable at the end of the year of death.

Joint life

Mean: $$A_{xy} = \sum_{t=0}^{t=\infty} v^{t+1} {}_{t|}q_{xy}$$

Variance: $${}^2A_{xy} - \left(A_{xy}\right)^2$$

Question

Prove the above results for the mean and variance.

Solution

Using the result for the probability function of K_{xy} developed in Section 1.3, the mean is:

$$E\left[v^{K_{xy}+1}\right] = \sum_{t=0}^{\infty} v^{t+1} P(K_{xy}=t) = \sum_{t=0}^{\infty} v^{t+1} {}_tp_{xy}\, q_{x+t:y+t} = \sum_{t=0}^{\infty} v^{t+1} {}_{t|}q_{xy}$$

The variance is:

$$\begin{aligned} \text{var}\left[v^{K_{xy}+1}\right] &= E\left[\left(v^{K_{xy}+1}\right)^2\right] - \left(E\left[v^{K_{xy}+1}\right]\right)^2 \\ &= E\left[\left(v^2\right)^{K_{xy}+1}\right] - \left(A_{xy}\right)^2 \\ &= {}^2A_{xy} - \left(A_{xy}\right)^2 \end{aligned}$$

Last survivor

Mean: $A_{\overline{xy}} = A_x + A_y - A_{xy}$

Variance: ${}^2A_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \left({}^2A_x + {}^2A_y - {}^2A_{xy}\right) - \left(A_x + A_y - A_{xy}\right)^2$

3.2 Present values of joint life and last survivor annuities

Consider an annuity under which a benefit of 1 *pa* is paid continuously so long as a status u continues. The present value of these annuity payments can be represented by the random variable:

$$\bar{a}_{\overline{T_u}|}$$

The expected present value of this benefit is denoted by $\bar{a}_u$ where:

$$E\left[\bar{a}_{\overline{T_u}|}\right] = \bar{a}_u = \int_{t=0}^{t=\infty} \bar{a}_{\overline{t}|}\, f_{T_u}(t)\,dt$$

This is most simply expressed by using assurance functions:

$$E\left[\bar{a}_{\overline{T_u}|}\right] = E\left[\frac{1-v^{T_u}}{\delta}\right] = \frac{1-E\left[v^{T_u}\right]}{\delta} = \frac{1-\bar{A}_u}{\delta}$$

The variance can be expressed in a similar way:

$$\begin{aligned}
\text{var}\left(\bar{a}_{\overline{T_u}|}\right) &= \text{var}\left(\frac{1-v^{T_u}}{\delta}\right) \\
&= \text{var}\left(\frac{1}{\delta} - \frac{1}{\delta}v^{T_u}\right) \\
&= \frac{1}{\delta^2}\text{var}\left(v^{T_u}\right) \\
&= \frac{1}{\delta^2}\left\{{}^2\bar{A}_u - \left(\bar{A}_u\right)^2\right\}
\end{aligned}$$

The results from Section 3.1 can be used to determine the means and variances for $u = xy$ (the joint life annuity) and $u = \overline{xy}$ (the last survivor annuity).

The means and variances of the present values of annuities payable in advance and in arrears can be evaluated using the (discrete) random variables:

- **in advance:** $\ddot{a}_{\overline{K_u+1}|}$
- **in arrears:** $a_{\overline{K_u}|}$

giving the results:

	In advance	**In arrears**
Mean	$\ddot{a}_u = \dfrac{1-A_u}{d}$	$a_u = \ddot{a}_u - 1 = \dfrac{(1-d)-A_u}{d}$
Variance	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$

which can be applied to joint lifetime and last survivor statuses.

The relationship $\ddot{a}_u = \dfrac{1-A_u}{d}$ is a rearrangement of the premium conversion formula we met earlier in the course.

Question

Prove the above results for:

(i) an annuity payable annually in advance

(ii) an annuity payable annually in arrears.

Solution

(i) ***Annuity in advance***

The mean of the present value of an annuity payable in advance is:

$$E\left[\ddot{a}_{\overline{K_u+1}|}\right] = E\left[\frac{1-v^{K_u+1}}{d}\right] = \frac{1-E\left[v^{K_u+1}\right]}{d} = \frac{1-A_u}{d}$$

The variance of the present value of an annuity payable in advance is:

$$\text{var}\left[\ddot{a}_{\overline{K_u+1}|}\right] = \text{var}\left[\frac{1-v^{K_u+1}}{d}\right] = \frac{1}{d^2}\text{var}\left[v^{K_u+1}\right] = \frac{1}{d^2}\left\{{}^2A_u - \left(A_u\right)^2\right\}$$

(ii) ***Annuity in arrears***

The expected present value of an annuity payable in arrears is:

$$E\left[a_{\overline{K_u}|}\right]=E\left[\frac{1-v^{K_u}}{i}\right]=\frac{1-E\left[v^{K_u}\right]}{i}=\frac{1-\frac{1}{v}E\left[v^{K_u+1}\right]}{i}=\frac{v-A_u}{iv}=\frac{(1-d)-A_u}{d}$$

The variance of the present value of an annuity payable in arrears is:

$$\text{var}\left[a_{\overline{K_u}|}\right]=\text{var}\left[\frac{1-v^{K_u}}{i}\right]=\frac{1}{i^2}\text{var}\left[v^{K_u}\right]=\frac{1}{i^2v^2}\text{var}\left\{v^{K_u+1}\right\}=\frac{1}{d^2}\left\{{}^2A_u-\left(A_u\right)^2\right\}$$

4 Calculations, premiums, reserves

We are now in a position to calculate numerical values for some joint life functions. Once we have done this, we can apply the equivalence principle to calculate premiums for policies involving more than one life. In addition, we will adapt the techniques we have covered in earlier chapters for calculating reserves. This will enable us to calculate reserves for these types of policy.

In many of our examples, we will use the PA92C20 mortality tables, where the life represented by the first subscript is subject to PMA92C20 mortality, and the life represented by the second subscript is subject to PFA92C20 mortality. If the interest rate is 4% *pa*, we will be able to use the functions tabulated on pages 114 and 115 of the *Tables*.

Question

Calculate $\ddot{a}_{\overline{55:51}}$, assuming that the 55-year-old's mortality follows PMA92C20, the 51-year-old's mortality follows PFA92C20, and the annual effective interest rate is 4%.

Solution

Using a superscript of m to denote an annuity based on male mortality and a superscript of f to denote an annuity based on female mortality, we have:

$$\ddot{a}_{\overline{55:51}} = \ddot{a}_{55}^{m} + \ddot{a}_{51}^{f} - \ddot{a}_{55:51}$$

The single life annuity factors appear on page 114 of the *Tables*, and the value of the joint life annuity factor appears on page 115 of the *Tables* (with $x = 55$ and $y = 51$, so the age difference $d = y - x = -4$). We have:

$$\ddot{a}_{\overline{55:51}} = 17.364 + 19.291 - 16.506 = 20.149$$

We can also calculate actuarial functions using a constant force of mortality (possibly different) for each of the two lives.

Question

Calculate $\bar{A}_{60:50}$.

Basis: a constant force of mortality for (60) of 0.005 *pa*,

a constant force of mortality for (50) of 0.002 *pa*,

a force of interest of 5% *pa*.

Solution

This is a joint life assurance, where the benefit is paid immediately on the first death of (60) and (50) whenever that occurs in the future.

We can calculate this using the integral expression:

$$\bar{A}_{60:50} = \int_0^\infty v^t \, {}_tp_{60:50} \, \mu_{60+t:50+t} \, dt = \int_0^\infty e^{-\delta t} \, {}_tp_{60} \, {}_tp_{50} \left(\mu_{60+t} + \mu_{50+t} \right) dt$$

Now, $\delta = 0.05$, $\mu_{60+t} = 0.005$ for all t and $\mu_{50+t} = 0.002$ for all t, so:

$$\bar{A}_{60:50} = \int_0^\infty e^{-0.05t} \times e^{-0.005t} \times e^{-0.002t} \times (0.005 + 0.002) \, dt = \int_0^\infty 0.007 e^{-0.057t} \, dt$$

$$= \left[\frac{0.007}{-0.057} e^{-0.057t} \right]_0^\infty = \frac{0.007}{0.057} = 0.122807$$

An alternative, and possibly quicker, approach here is to find the value of the corresponding continuous annuity, and then use a premium conversion formula.

By analogy with the single life expression:

$$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x \, dt$$

for a continuous joint life annuity we have:

$$\bar{a}_{xy} = \int_0^\infty v^t \, {}_tp_{xy} \, dt = \int_0^\infty e^{-\delta t} \, {}_tp_x \, {}_tp_y \, dt$$

So, here:

$$\bar{a}_{60:50} = \int_0^\infty e^{-0.05t} \times e^{-0.005t} \times e^{-0.002t} \, dt = \int_0^\infty e^{-0.057t} \, dt = \left[\frac{e^{-0.057t}}{-0.057} \right]_0^\infty = \frac{1}{0.057}$$

Using a premium conversion relationship, we have:

$$\bar{A}_{60:50} = 1 - \delta \bar{a}_{60:50} = 1 - \frac{0.05}{0.057} = 0.122807$$

as before.

To calculate the value of $\bar{A}_{60:50}$ in the case that the two lives are subject to PMA92C20 and PFA92C20 mortality, we would have to use the premium conversion approach as there are no joint life assurance functions for this mortality basis given in the *Tables*.

So, we would look up $\ddot{a}_{60:50}$ in the *Tables*, calculate the approximate value of $\bar{a}_{60:50}$ as $\ddot{a}_{60:50} - 0.5$, and then use the premium conversion formula $\bar{A}_{60:50} = 1 - \delta \bar{a}_{60:50}$.

Alternatively, we could use premium conversion to calculate $A_{60:50} = 1 - d\ddot{a}_{60:50}$, and then use $\bar{A}_{60:50} \approx (1+i)^{0.5} A_{60:50}$.

4.1 Evaluating premiums

We are now able to calculate premiums for policies involving two lives. As for single life policies, the usual method is to use the equivalence principle.

Question

A life office sells joint whole life assurances to male lives aged 60 and female lives aged 55 exact. The benefits, payable at the end of the year of death in each case, are £100,000 on the first death and £50,000 on the second death. Level premiums are paid annually in advance while the policy is in force.

Calculate the annual premium payable.

Basis: PMA92C20/PFA92C20 mortality, 4% *pa* interest. Ignore expenses.

Solution

The policy will remain in force until the second death. So the annuity we use to value the premiums is a last survivor annuity.

Using the equivalence principle:

$$P\ddot{a}_{\overline{60:55}} = 100{,}000 A_{60:55} + 50{,}000 A_{\overline{60:55}}$$

The relevant annuity from page 115 of the *Tables* is $\ddot{a}_{60:55} = 14.756$. So, using the relevant single life annuities (tabulated on page 114):

$$\ddot{a}_{\overline{60:55}} = \ddot{a}^m_{60} + \ddot{a}^f_{55} - \ddot{a}_{60:55} = 15.632 + 18.210 - 14.756 = 19.086$$

We can calculate the assurance factors using a premium conversion formula:

$$A_{60:55} = 1 - d\ddot{a}_{60:55} = 1 - \frac{0.04}{1.04} \times 14.756 = 0.43246$$

$$A_{\overline{60:55}} = 1 - d\ddot{a}_{\overline{60:55}} = 1 - \frac{0.04}{1.04} \times 19.086 = 0.26592$$

So the premium equation becomes:

$$19.086P = 100{,}000 \times 0.43246 + 50{,}000 \times 0.26592$$

This gives us an annual premium of £2,962.50.

4.2 Calculating reserves

We can calculate reserves for policies based on two lives, again adapting the methods we have met earlier for reserve calculations for single life policies. The prospective calculation involves calculating the EPV of the future benefits and expenses (if appropriate), and subtracting the EPV of any future premiums.

Question

For the policy in the previous question, calculate the prospective reserve just before the payment of the 10th annual premium, assuming both lives are still alive at that point, and using the same basis as was used to calculate the premium.

Solution

Just before the 10th premium is paid (*ie* at time 9) we have:

$${}_9V = 100{,}000A_{69:64} + 50{,}000A_{\overline{69:64}} - 2{,}962.50\ddot{a}_{\overline{69:64}}$$

The last survivor annuity is calculated in the usual way:

$$\ddot{a}_{\overline{69:64}} = \ddot{a}^m_{69} + \ddot{a}^f_{64} - \ddot{a}_{69:64} = 11.988 + 15.242 - 10.933 = 16.297$$

The assurance factors are calculated using premium conversion:

$$A_{69:64} = 1 - d\ddot{a}_{69:64} = 1 - \frac{0.04}{1.04} \times 10.933 = 0.57950$$

$$A_{\overline{69:64}} = 1 - d\ddot{a}_{\overline{69:64}} = 1 - \frac{0.04}{1.04} \times 16.297 = 0.37319$$

So the reserve is:

$$ {}_9V = 100{,}000 \times 0.57950 + 50{,}000 \times 0.37319 - 2{,}962.50 \times 16.297 = £28{,}329.75 $$

In this question, we were told to assume that both lives are still alive at the time of the reserve calculation.

It is important when calculating a reserve for a last survivor assurance to remember that it would be necessary to establish whether both lives remain alive or one has already died. This is because, of course, the contract still remains in force whether one or two lives remain alive. The premium being paid will still be the original premium calculated on a last survivor basis.

Question

For a policy where the male and female are both aged 63, list the three possible states that a last survivor assurance may be in after a few years.

Identify which of these three states requires the largest reserve.

Solution

The three possible states are:

- both lives are still alive
- only the male life is still alive
- only the female life is still alive.

We would expect the sum assured to be paid out earliest on the policy where only the male is still alive (as male life expectancy is shorter than female life expectancy for lives of the same age). This policy is therefore expected to have the largest reserve.

Thus for example if both *x* and *y* are alive, a net premium reserve for a whole life last survivor contract would be:

$$ {}_tV_{\overline{x:y}} = A_{\overline{x+t:y+t}} - P_{\overline{x:y}}\, \ddot{a}_{\overline{x+t:y+t}} $$

whereas if say (*y*) had previously died the reserve would be:

$$ {}_tV_{\overline{x:y}} = A_{x+t} - P_{\overline{x:y}}\, \ddot{a}_{x+t} $$

Thus on first death a significant increase in reserve will take place.

Here $P_{\overline{x:y}}$ is the net premium (*ie* ignoring expenses) payable annually in advance for the last survivor assurance contract, and ${}_tV_{\overline{x:y}}$ is the net premium reserve for this contract at time t.

Above, we have used a prospective approach to calculating reserves. We can also calculate reserves retrospectively, determining the expected accumulated value of the premiums, less the expected accumulated value of the benefits and expenses (if appropriate). However, other than for simple joint life assurances and annuities, the retrospective calculation is very complex, so the prospective method is generally used.

4.3 Future loss random variable

As we have seen in previous chapters, the prospective reserve is the expected value of an underlying random variable, known as the future loss random variable. We can apply the same techniques as before to write down future loss random variables for policies based on two lives.

Question

A life office sells joint whole life assurances to male lives aged 60 and female lives aged 55 exact. The benefits, payable at the end of the year of death in each case, are £100,000 on the first death and £50,000 on the second death. Level premiums of £2,962.50 are paid annually in advance while the policy is in force.

Write down the future loss random variable at the end of the 9th policy year for this policy, in the case where both lives are still alive, ignoring expenses.

Solution

We use a 'min' function for the benefit payable on first death, and a 'max' function for the benefit payable on second death. Since the premiums are payable until the second death, a 'max' function will also be needed for these.

So the future loss random variable at the end of the 9th policy year is:

$$100{,}000v^{\min\{K_{69}+1,K_{64}+1\}}+50{,}000v^{\max\{K_{69}+1,K_{64}+1\}}-2{,}962.50\ddot{a}_{\overline{\max\{K_{69}+1,K_{64}+1\}}|}$$

We could alternatively write this using the joint curtate future lifetime random variable and the corresponding last survivor future lifetime random variable, *ie*:

$$100{,}000v^{K_{69:64}+1}+50{,}000v^{K_{\overline{69:64}}+1}-2{,}962.50\ddot{a}_{\overline{K_{\overline{69:64}}+1}|}$$

Chapter 21 Summary

We have developed the following random variables:

Random variable	***Modelling***
$T_{xy} = \min\{T_x, T_y\}$	Time to failure of the joint life status xy, *ie* the time until the first death of x and y
$T_{\overline{xy}} = \max\{T_x, T_y\}$	Time to failure of the last survivor status $\overline{xy}$, *ie* the time until the second death of x and y
$K_{xy} = \min\{K_x, K_y\}$	Curtate time to failure of the joint life status xy, *ie* the curtate time until the first death of x and y
$K_{\overline{xy}} = \max\{K_x, K_y\}$	Curtate time to failure of the last survivor status $\overline{xy}$, *ie* the curtate time until the second death of x and y

We have also developed the following notation, with associated formulae:

Symbol	***Description***	***Formula***
l_{xy}	Life table survival function for two independent lives x and y	$l_x\, l_y$
μ_{xy}	Force of failure of the joint life status xy	$\mu_x + \mu_y$
${}_tp_{xy}$	Probability that the joint life status xy is still active in t years' time, *ie* the probability that both x and y survive for at least t years	${}_tp_x\, {}_tp_y$
${}_tp_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ is still active in t years' time, *ie* the probability that at least one of x and y survive for at least t years	$1 - {}_tq_x\, {}_tq_y$ $= {}_tp_x + {}_tp_y - {}_tp_x\, {}_tp_y$
${}_tq_{xy}$	Probability that the joint life status xy fails within t years, *ie* the probability that at least one of x and y dies in next t years	$1 - {}_tp_x\, {}_tp_y$ $= {}_tq_x + {}_tq_y - {}_tq_x\, {}_tq_y$
${}_tq_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ fails within t years, *ie* the probability that both x and y die in next t years	${}_tq_x\, {}_tq_y$

The most common benefits, and their values, are:

Function	*Value (algebraic)*	*Value (stochastic)*
$\bar{a}_{xy}$	$\int_0^\infty v^t \, {}_tp_{xy} \, dt$	$E\left[\bar{a}_{\overline{\min\{T_x,T_y\}}\rceil}\right]$
$\bar{a}_{\overline{xy}}$	$\int_0^\infty v^t \, {}_tp_{\overline{xy}} \, dt$	$E\left[\bar{a}_{\overline{\max\{T_x,T_y\}}\rceil}\right]$
$\bar{A}_{xy}$	$\int_0^\infty v^t \, {}_tp_{xy} (\mu_{x+t} + \mu_{y+t}) \, dt$	$E\left[v^{\min\{T_x,T_y\}}\right]$
$\bar{A}_{\overline{xy}}$	$\int_0^\infty v^t ({}_tp_y \, \mu_{y+t} \, {}_tq_x + {}_tp_x \, \mu_{x+t} \, {}_tq_y) \, dt$	$E\left[v^{\max\{T_x,T_y\}}\right]$

Equivalent results hold for discrete functions, *eg*:

$$a_{xy} = \sum_{t=1}^{\infty} v^t \, {}_tp_{xy} = E\left[a_{\overline{\min\{K_x,K_y\}}\rceil}\right]$$

We can adapt the premium conversion relationships for single life functions to help us to calculate the corresponding joint life and last survivor values. For example:

$$A_{xy} = 1 - d\ddot{a}_{xy} \qquad \bar{A}_{xy} = 1 - \delta \bar{a}_{xy}$$

and: $$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}} \qquad \bar{A}_{\overline{xy}} = 1 - \delta \bar{a}_{\overline{xy}}$$

Chapter 21 Practice Questions

21.1 (i) Explain what it means for the last survivor status $\overline{50:60}$ to remain active for at least 10 years.

(ii) Calculate the probability that the event described in part (i) occurs, assuming the two lives are independent with respect to mortality and:

(a) the mortality of each life follows the ELT15 (Males) table

(b) each life is subject to a constant force of mortality of 0.025 *pa*.

21.2 (i) Explain what it means for the joint life status $50:60$ to fail within the next 10 years.

(ii) Calculate the probability that the event described in part (i) occurs, assuming the two lives are independent with respect to mortality and:

(a) the mortality of each life follows the ELT15 (Females) table

(b) each life is subject to a constant force of mortality of 0.025 *pa*.

21.3 Given that ${}_nq_x = 0.3$ and ${}_nq_y = 0.5$, calculate ${}_nq_{xy}$ and ${}_nq_{\overline{xy}}$.

21.4 Consider each of the symbols listed below:

(a) $p_{\overline{xy}}$

(b) $\bar{A}_{xy}$

(c) $\bar{A}_{\overline{xy}}$

Explain carefully the meaning of each of these symbols and calculate the value of each, assuming that:

- (x) is subject to a constant force of mortality of 0.01 *pa*
- (y) is subject to a constant force of mortality of 0.02 *pa*
- the force of interest is 0.04 *pa*.

21.5 A life insurance company issues 1,000 last survivor annuities to pairs of lives aged 60. Each pair comprises one male and one female, and the annuity pays £5,000 *pa* continuously until the second of the two lives dies. The single premium charged is £90,000.

Calculate the expected present value of the profit to the life office and the standard deviation of this profit in respect of this group of policies.

Basis: Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective

Using this mortality assumption, $\bar{A}_{\overline{60:60}} = 0.20021$ at an interest rate of 8.16% *pa*.

21.6 *Exam style*

William, aged 75, and Laura, aged 80, are the guardians of a child. They take out a life assurance policy that provides a payment of £25,000 immediately when the second of them dies. Level annual premiums are payable in advance whilst the policy is in force.

(i) Calculate the annual gross premium, using the basis given below. [4]

(ii) Calculate the gross premium prospective reserve just before the sixth premium is paid, using the basis given below, assuming that both William and Laura are still alive at that time. [3]

[Total 7]

Basis: Mortality: PMA92C20 for William, PFA92C20 for Laura

Interest: 4% *pa* effective

Expenses: Initial: £250

Renewal: 5% of each premium, excluding the first

21.7 *Exam style*

A life insurance company issues an annuity to a male, aged 68, and a female, aged 65. The annuity of £10,000 *pa* is payable annually in arrears and continues until both lives have died.

The insurance company values this benefit using PMA92C20 mortality for the male life, PFA92C20 mortality for the female life and 4% *pa* interest.

(i) Calculate the expected present value of this annuity. [2]

(ii) Derive an expression for the variance of the present value of this annuity in terms of appropriate single life and joint life assurance functions. [4]

Let X be the present value of the insurer's profit from this policy.

(iii) If the insurance company charges a premium of £150,000 for this policy, calculate $P(X>0)$. [4]

[Total 10]

21.8 *Exam style*

A man aged 60 exact and a woman aged 65 exact wish to purchase an annuity that provides:

- £25,000 *pa* payable while they are both alive,
- £20,000 *pa* payable for the remainder of the woman's life, if the man dies first,
- £15,000 *pa* payable for the remainder of the man's life, if the woman dies first.

All the annuity payments are made annually in arrears.

(i) Write down an expression for the present value random variable of this benefit. [2]

(ii) Calculate the expected present value of this annuity benefit, using the following basis:

Mortality: PMA92C20 for the male life, PFA92C20 for the female life

Interest: 4% *pa* effective [2]

[Total 4]

21.9 The random variable T_{xy} represents the time to failure of the joint life status (xy). (x) is subject to a constant force of mortality of 0.02 and (y) is subject to a constant force of mortality of 0.03. (x) and (y) are independent with respect to mortality.

Exam style

Calculate the value of $E\left[T_{xy}\right]$. [5]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 21 Solutions

21.1 (i) For the last survivor status $\overline{50:60}$ to remain active for at least 10 years, at least one of the two lives, currently aged 50 and 60, must survive for at least 10 years. So, the time until the second death must be greater than or equal to 10 years.

(ii)(a) The probability that the last survivor status $\overline{50:60}$ remains active for at least 10 years can be written as:

$$_{10}p_{\overline{50:60}} = 1 - {}_{10}q_{\overline{50:60}} = 1 - {}_{10}q_{50}\ {}_{10}q_{60}$$

Using ELT15 (Males) mortality:

$$_{10}q_{50} = 1 - \frac{l_{60}}{l_{50}} = 1 - \frac{86,714}{93,925} = 0.076774$$

and: $$_{10}q_{60} = 1 - \frac{l_{70}}{l_{60}} = 1 - \frac{68,055}{86,714} = 0.215179$$

So:

$$_{10}p_{\overline{50:60}} = 1 - 0.076774 \times 0.215179 = 0.983480$$

(ii)(b) Assuming a constant force of mortality of 0.025 *pa*, we have:

$$_{10}q_{50} = {}_{10}q_{60} = 1 - e^{-0.025\times 10} = 1 - e^{-0.25}$$

So:

$$_{10}p_{\overline{50:60}} = 1 - \left(1 - e^{-0.25}\right)^2 = 0.951071$$

21.2 (i) For the joint life status $50:60$ to fail within the next 10 years, at least one of the two lives, currently aged 50 and 60, must die within 10 years. So, the time until the first death must be less than 10 years.

This does not exclude the possibility that the second death might also occur within 10 years.

(ii)(a) The probability that the joint life status $50:60$ fails within the next 10 years can be written as:

$$_{10}q_{50:60} = 1 - {}_{10}p_{50:60} = 1 - {}_{10}p_{50}\ {}_{10}p_{60}$$

Using ELT15 (Females) mortality:

$$_{10}p_{50} = \frac{l_{60}}{l_{50}} = \frac{91,732}{96,247} = 0.953089$$

and: $$_{10}p_{60} = \frac{l_{70}}{l_{60}} = \frac{79,970}{91,732} = 0.871779$$

So:

$$_{10}q_{50:60} = 1 - 0.953089 \times 0.871779 = 0.169117$$

(ii)(b) Assuming a constant force of mortality of 0.025 *pa*, we have:

$$_{10}p_{50} = {}_{10}p_{60} = e^{-0.025\times 10} = e^{-0.25}$$

So:

$$_{10}q_{50:60} = 1 - \left(e^{-0.25}\right)^2 = 0.393469$$

21.3 We have:

$$_nq_{xy} = 1 - {}_np_{xy} = 1 - {}_np_x\,{}_np_y = 1 - (1 - {}_nq_x)\left(1 - {}_nq_y\right) = 1 - 0.7 \times 0.5 = 0.65$$

and: $$_nq_{\overline{xy}} = {}_nq_x\,{}_nq_y = 0.15$$

21.4 (a) $p_{\overline{xy}}$ is the probability that the last survivor status based on lives (x) and (y) is still active in one year's time, *ie* it is the probability that at least one of (x) and (y) is alive in one year's time.

We can calculate this as follows:

$$p_{\overline{xy}} = 1 - q_{\overline{xy}} = 1 - q_x\,q_y = 1 - \left(1 - e^{-0.01}\right)\left(1 - e^{-0.02}\right) = 0.999803$$

(b) $\bar{A}_{xy}$ is the expected present value of a benefit of 1 payable immediately on the failure of the joint life status xy. So the benefit is paid immediately upon the first death of the two lives.

We can calculate this using the integral expression:

$$\bar{A}_{xy} = \int_0^\infty v^t\,{}_tp_{xy}\,\mu_{x+t:y+t}\,dt = \int_0^\infty v^t\,{}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right)dt$$

So this gives:

$$\bar{A}_{xy} = \int_0^\infty e^{-0.04t} e^{-0.01t} e^{-0.02t} (0.01 + 0.02)\, dt$$

$$= 0.03 \int_0^\infty e^{-0.07t}\, dt$$

$$= 0.03 \left[\frac{-e^{-0.07t}}{0.07} \right]_0^\infty = \frac{3}{7}$$

(c) $\bar{A}_{\overline{xy}}$ is the expected present value of a benefit of 1 payable immediately on the failure of the last survivor status $\overline{xy}$. So the benefit is paid immediately upon the second death.

We can write:

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

where:

$$\bar{A}_x = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t}\, dt = 0.01 \int_0^\infty e^{-0.04t} e^{-0.01t}\, dt = 0.01 \left[\frac{-e^{-0.05t}}{0.05} \right]_0^\infty = \frac{1}{5}$$

$$\bar{A}_y = \int_0^\infty v^t \, {}_tp_y \, \mu_{y+t}\, dt = 0.02 \int_0^\infty e^{-0.04t} e^{-0.02t}\, dt = 0.02 \left[\frac{-e^{-0.06t}}{0.06} \right]_0^\infty = \frac{1}{3}$$

So:

$$\bar{A}_{\overline{xy}} = \frac{1}{5} + \frac{1}{3} - \frac{3}{7} = \frac{11}{105} = 0.104762$$

21.5 The present value of the profit, X, on a single policy is:

$$X = 90{,}000 - 5{,}000\, \bar{a}_{\overline{T_{\overline{xy}}}|}$$

The expected value of X is:

$$E\left[90{,}000 - 5{,}000\, \bar{a}_{\overline{T_{\overline{xy}}}|} \right] = 90{,}000 - 5{,}000\, E\left[\bar{a}_{\overline{T_{\overline{xy}}}|} \right] = 90{,}000 - 5{,}000\, \bar{a}_{\overline{xy}}$$

Here x and y are both equal to 60, and we can calculate the last survivor annuity from:

$$\begin{aligned}\bar{a}_{\overline{60:60}} &= \bar{a}^m_{60} + \bar{a}^f_{60} - \bar{a}_{60:60}\\ &\approx \left(\ddot{a}^m_{60} - 0.5\right) + \left(\ddot{a}^f_{60} - 0.5\right) - \left(\ddot{a}_{60:60} - 0.5\right)\\ &= \left(15.632 - 0.5\right) + \left(16.652 - 0.5\right) - \left(14.090 - 0.5\right)\\ &= 17.694\end{aligned}$$

So, the expected present value of the profit on a single policy is:

$$90{,}000 - 5{,}000 \times 17.694 = £1{,}530$$

The expected present value of the profit on the block of 1,000 policies is £1,530,000.

The variance of X is:

$$\text{var}\left[90{,}000 - 5{,}000\,\bar{a}_{\overline{T_{\overline{xy}}}|}\right] = 5{,}000^2\,\text{var}\left[\frac{1 - v^{T_{\overline{xy}}}}{\delta}\right] = \left(\frac{5{,}000}{\delta}\right)^2 \text{var}\left[v^{T_{\overline{xy}}}\right]$$

where:

$$\text{var}\left[v^{T_{\overline{xy}}}\right] = {}^2\bar{A}_{\overline{xy}} - (\bar{A}_{\overline{xy}})^2$$

Using the premium conversion relationship $\bar{A}_{\overline{xy}} = 1 - \delta\,\bar{a}_{\overline{xy}}$, we obtain:

$$\bar{A}_{\overline{60:60}} = 1 - \ln 1.04 \times 17.694 = 0.30603$$

and we are given ${}^2\bar{A}_{\overline{60:60}} = 0.20021$ in the question.

So the variance of the present value of the profit for a single policy is:

$$\frac{5{,}000^2}{\delta^2}(0.20021 - 0.30603^2) = 1{,}731{,}764{,}026$$

The variance of the present value of the profit on the block of 1,000 policies is then:

$$1{,}000 \times 1{,}731{,}764{,}026$$

assuming independent policies. So the standard deviation of the total profit is:

$$\sqrt{1{,}000 \times 1{,}731{,}764{,}026} = £1{,}315{,}965$$

21.6 (i) ***Gross annual premium***

The premium equation is:

$$P\ddot{a}_{\overline{75:80}} = 25{,}000\bar{A}_{\overline{75:80}} + 250 + 0.05P\left(\ddot{a}_{\overline{75:80}} - 1\right) \qquad [1]$$

We can calculate $\ddot{a}_{\overline{75:80}}$ using:

$$\ddot{a}_{\overline{75:80}} = \ddot{a}_{75}^{m} + \ddot{a}_{80}^{f} - \ddot{a}_{75:80} = 9.456 + 8.989 - 6.822 = 11.623 \qquad [1]$$

The assurance factor can be calculated by premium conversion:

$$\bar{A}_{\overline{75:80}} \approx 1.04^{0.5} A_{\overline{75:80}} = 1.04^{0.5}\left(1 - d\,\ddot{a}_{\overline{75:80}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 11.623\right) = 0.56391 \qquad [1]$$

The premium equation becomes:

$$11.623P = 25{,}000 \times 0.56391 + 250 + 0.05P(11.623 - 1)$$

So the premium is:

$$P = \frac{14{,}347.81}{11.09185} = £1{,}293.55 \qquad [1]$$

[Total 4]

(ii) ***Gross premium prospective reserve at time 5***

Just before the sixth premium payment (*ie* at time 5), William is aged 80 and Laura is aged 85. The gross premium prospective reserve is given by:

$${}_5V = 25{,}000\bar{A}_{\overline{80:85}} + 0.05P\ddot{a}_{\overline{80:85}} - P\ddot{a}_{\overline{80:85}} = 25{,}000\bar{A}_{\overline{80:85}} - 0.95P\ddot{a}_{\overline{80:85}} \qquad [1]$$

where $P = £1{,}293.55$ from part (i).

Using the same approach as in part (i):

$$\ddot{a}_{\overline{80:85}} = \ddot{a}_{80}^{m} + \ddot{a}_{85}^{f} - \ddot{a}_{80:85} = 7.506 + 7.220 - 5.161 = 9.565 \qquad [½]$$

$$\bar{A}_{\overline{80:85}} \approx 1.04^{0.5} A_{\overline{80:85}} = 1.04^{0.5}\left(1 - d\,\ddot{a}_{\overline{80:85}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 9.565\right) = 0.64463 \qquad [½]$$

So the gross premium prospective reserve at time 5 is:

$${}_5V = 25{,}000 \times 0.64463 - 0.95 \times 1{,}293.55 \times 9.565 = £4{,}361.68 \qquad [1]$$

[Total 3]

21.7 (i) ***Expected present value***

The expected present value of the annuity is:

$$10{,}000\,a_{\overline{68:65}} = 10{,}000\left(\ddot{a}_{\overline{68:65}} - 1\right) = 10{,}000\left(\ddot{a}^m_{68} + \ddot{a}^f_{65} - \ddot{a}_{68:65} - 1\right) \qquad [1]$$

Looking up the values, the EPV of the annuity is:

$$10{,}000(12.412 + 14.871 - 11.112 - 1) = £151{,}710 \qquad [1]$$

[Total 2]

(ii) ***Variance of the present value***

The present value random variable for the benefits from this contract is:

$$10{,}000\,a_{\overline{K_{\overline{68:65}}}|} \qquad [½]$$

The variance of this random variable is given by:

$$\begin{aligned}
\operatorname{var}\left(10{,}000\,a_{\overline{K_{\overline{68:65}}}|}\right) &= 10{,}000^2 \operatorname{var}\left(\frac{1 - v^{K_{\overline{68:65}}}}{i}\right) \\
&= \frac{10{,}000^2}{i^2} \operatorname{var}\left(v^{K_{\overline{68:65}}}\right) \\
&= \frac{10{,}000^2}{i^2 v^2} \operatorname{var}\left(v^{K_{\overline{68:65}}+1}\right) \\
&= \frac{10{,}000^2}{d^2} \operatorname{var}\left(v^{K_{\overline{68:65}}+1}\right)
\end{aligned} \qquad [1½]$$

Now:

$$\operatorname{var}\left(v^{K_{\overline{68:65}}+1}\right) = E\left[\left(v^{K_{\overline{68:65}}+1}\right)^2\right] - \left[E\left(v^{K_{\overline{68:65}}+1}\right)\right]^2 = {}^2A_{\overline{68:65}} - \left(A_{\overline{68:65}}\right)^2$$

where the pre-superscript of 2 indicates that the function is evaluated at an interest rate of $1.04^2 - 1 = 8.16\%$. [1]

Also:

$$A_{\overline{68:65}} = A_{68} + A_{65} - A_{68:65}$$

and:

$${}^2A_{\overline{68:65}} = {}^2A_{68} + {}^2A_{65} - {}^2A_{68:65}$$

So:

$$\text{var}\left(10{,}000\, a_{\overline{K_{\overline{68:65}}}|}\right) = \frac{10{,}000^2}{d^2}\left[{}^2A_{\overline{68:65}} - \left(A_{\overline{68:65}}\right)^2\right]$$

$$= \frac{10{,}000^2}{d^2}\left[{}^2A_{68} + {}^2A_{65} - {}^2A_{68:65} - \left(A_{68} + A_{65} - A_{68:65}\right)^2\right] \qquad [1]$$

[Total 4]

(iii) ***Probability that the insurance company makes a profit***

The life insurance company charges a premium of £150,000. The present value of the profit is:

$$X = 150{,}000 - 10{,}000\, a_{\overline{K_{\overline{68:65}}}|}$$

So:

$$P(X>0) = P\left(10{,}000\, a_{\overline{K_{\overline{68:65}}}|} < 150{,}000\right) \qquad [½]$$

Now:

$$10{,}000\, a_{\overline{K_{\overline{68:65}}}|} < 150{,}000 \iff a_{\overline{K_{\overline{68:65}}}|} < 15$$

$$\iff \frac{1 - v^{K_{\overline{68:65}}}}{i} < 15$$

$$\iff v^{K_{\overline{68:65}}} > 1 - 15i$$

$$\iff K_{\overline{68:65}} < \frac{\ln(1 - 15 \times 0.04)}{-\ln 1.04} = 23.4 \qquad [1½]$$

Since $K_{\overline{68:65}}$ can take only integer values:

$$P\left(K_{\overline{68:65}} < 23.4\right) = P\left(K_{\overline{68:65}} \le 23\right) = P\left(K_{\overline{68:65}} < 24\right) \qquad [½]$$

Alternatively, we could obtain this result by inspection. Looking up annuity values at 4% in the Tables, we see that $a_{\overline{23}|} = 14.8568$ *but* $a_{\overline{24}|} = 15.2470$*, so* $P\left(a_{\overline{K_{\overline{68:65}}}|} < 15\right) = P\left(K_{\overline{68:65}} < 24\right)$.

We want the probability that the last survivor status fails within 24 years. This will happen if both lives die within 24 years. The required probability, $P(X>0)$, is therefore:

$${}_{24}q_{68} \times {}_{24}q_{65} = \left(1 - \frac{l_{92}}{l_{68}}\right)\left(1 - \frac{l_{89}}{l_{65}}\right)$$

$$= \left(1 - \frac{1{,}911.771}{9{,}440.717}\right)\left(1 - \frac{4{,}533.230}{9{,}703.708}\right) = 0.424935 \qquad [1½]$$

[Total 4]

21.8 (i) ***Present value random variable***

This is:

$$20{,}000a_{\overline{K_{65}}|}+15{,}000a_{\overline{K_{60}}|}-10{,}000a_{\overline{K_{60:65}}|} \qquad \text{[Total 2]}$$

We can consider this situation as a single life annuity of £20,000 pa payable to the woman for the whole of her life, plus a single life annuity of £15,000 pa payable to the man for the whole of his life, minus a joint life annuity of £10,000 pa payable while both lives are alive. Then:

- *when only the woman is alive, she receives £20,000 pa*
- *when only the man is alive, he received £15,000 pa, and*
- *when both lives are alive, the annual payment is* $20{,}000+15{,}000-10{,}000=£25{,}000$

as required.

(ii) ***Expected present value***

The expected present value is:

$$20{,}000a_{65}^{f}+15{,}000a_{60}^{m}-10{,}000a_{60(m):65(f)}$$

$$=20{,}000\left(\ddot{a}_{65}^{f}-1\right)+15{,}000\left(\ddot{a}_{60}^{m}-1\right)-10{,}000\left(\ddot{a}_{60(m):65(f)}-1\right) \qquad [1]$$

Looking up the values:

$$EPV=20{,}000(14.871-1)+15{,}000(15.632-1)-10{,}000(13.101-1)=£375{,}890 \qquad [1]$$

[Total 2]

21.9 *This question is Subject CT5, April 2009, Question 6.*

By analogy with the formula for the expected future lifetime of a single life which is:

$$E\left[T_x\right]=\int_0^{\infty} {}_tp_x\,dt$$

the formula for the expected joint future lifetime is:

$$E\left[T_{xy}\right]=\int_0^{\infty} {}_tp_{xy}\,dt=\int_0^{\infty} {}_tp_x\,{}_tp_y\,dt \qquad [2]$$

Now, since the force of mortality for each life is constant, we have:

$${}_tp_x=e^{-0.02t} \quad \text{and} \quad {}_tp_y=e^{-0.03t} \qquad [1]$$

Therefore:

$$E\left[T_{xy}\right]=\int_0^\infty e^{-0.02t}e^{-0.03t}dt=\int_0^\infty e^{-0.05t}dt=\left[\frac{e^{-0.05t}}{-0.05}\right]_0^\infty=\frac{1}{0.05}=20 \text{ years} \qquad [2]$$

[Total 5]

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Contingent and reversionary benefits

Syllabus objectives

5.1 Define and use assurance and annuity functions involving two lives.

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

5.1.2 Extend the technique of 5.1.1 to deal with functions dependent upon a fixed term as well as age.