Detailed syllabus objectives

1. The basics of modelling (10%)

1.1 Describe the principles of actuarial modelling. (Chapter 1)

1.1.1 Describe why and how models are used including, in general terms, the use of models for pricing, reserving, and capital modelling.

1.1.2 Explain the benefits and limitations of modelling.

1.1.3 Explain the difference between a stochastic and a deterministic model, and identify the advantages/disadvantages of each.

1.1.4 Describe the characteristics, and explain the use of, scenario-based and proxy models.

1.1.5 Describe, in general terms, how to decide whether a model is suitable for any particular application.

1.1.6 Explain the difference between the short-run and long-run properties of a model, and how this may be relevant in deciding whether a model is suitable for any particular application.

1.1.7 Describe, in general terms, how to analyse the potential output from a model, and explain why this is relevant to the choice of model.

1.1.8 Describe the process of sensitivity testing of assumptions, and explain why this forms an important part of the modelling process.

1.1.9 Explain the factors that must be considered when communicating the results following the application of a model.

1.2 Describe how to use a generalised cashflow model to describe financial transactions. (Chapter 2)

1.2.1 State the inflows and outflows in each future time period, and discuss whether the amount or the timing (or both) is fixed or uncertain for a given cashflow process.

1.2.2 Describe in the form of a cashflow model the operation of financial instruments (like a zero-coupon bond, a fixed-interest security, an index-linked security, a current account, cash on deposit, a credit card, an equity, an interest-only loan, a repayment loan, and an annuity-certain) and insurance contracts (like an endowment, a term assurance, a contingent annuity, car insurance and health cash plans).

2. Theory of interest rates (20%)

2.1 Show how interest rates may be expressed in different time periods. (Chapters 3 and 4)

2.1.1 Describe the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

2.1.2 Derive the relationships between the rate of interest payable once per measurement period (effective rate of interest) and the rate of interest payable p (> 1) times per measurement period (nominal rate of interest) and the force of interest.

2.1.3 Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

2.2 Demonstrate a knowledge and understanding of real and money interest rates. (Chapters 5 and 12)

2.3 Describe how to take into account time value of money using the concepts of compound interest and discounting. (Chapter 3)

2.3.1 Accumulate a single investment at a constant rate of interest under the operation of simple and compound interest.

2.3.2 Define the present value of a future payment.

2.3.3 Discount a single investment under the operation of a simple (commercial) discount at a constant rate of discount.

2.4 Calculate the present value and accumulated value for a given stream of cashflows under the following individual or combination of scenarios: (Chapter 6)

2.4.1 Cashflows are equal at each time period.

2.4.2 Cashflows vary with time, which may or may not be a continuous function of time.

2.4.3 Some of the cashflows are deferred for a period of time.

2.4.4 Rate of interest or discount is constant.

2.4.5 Rate of interest or discount varies with time, which may or may not be a continuous function of time.

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$:
(Chapters 7 and 8)

2.5.1 $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$.

2.5.2 ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a^{(p)}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and ${}_{m|}\bar{a}_{\overline{n}|}$.

2.5.3 $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}\bar{a})_{\overline{n}|}$ and the respective deferred annuities.

2.6 Show an understanding of the term structure of interest rates. (Chapter 13)

2.6.1 Describe the main factors influencing the term structure of interest rates.

2.6.2 Explain what is meant by, derive the relationships between and evaluate:

- discrete spot rates and forward rates.
- continuous spot rates and forward rates.

2.6.3 Explain what is meant by the par yield and yield to maturity.

2.7 Show an understanding of duration, convexity and immunisation of cashflows.
(Chapter 13)

2.7.1 Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

2.7.2 Evaluate the duration and convexity of a cashflow sequence.

2.7.3 Explain how duration and convexity are used in the (Redington) immunisation of a portfolio of liabilities.

3. Equation of value and its applications (15%)

3.1 Define an equation of value. (Chapter 9)

3.1.1 Define an equation of value, where payment or receipt is certain.

3.1.2 Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.

3.1.3 Understand the two conditions required for there to be an exact solution to an equation of value.

3.2 Use the concept of equation of value to solve various practical problems. (Chapters 10 and 12)

3.2.1 Apply the equation of value to loans repaid by regular instalments of interest and capital. Obtain repayments, interest and capital components, the effective interest rate (APR) and construct a schedule of repayments.

3.2.2 Calculate the price of, or yield (nominal or real allowing for inflation) from, a bond (fixed-interest or index-linked) where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to deduction of capital gains tax.

3.2.3 Calculate the running yield and the redemption yield for the financial instrument as described in 3.2.2.

3.2.4 Calculate the upper and lower bounds for the present value of the financial instrument as described in 3.2.2, when the redemption date can be a single date within a given range at the option of the borrower.

3.2.5 Calculate the present value or yield (nominal or real allowing for inflation) from an ordinary share or property, given constant or variable rate of growth of dividends or rents.

3.3 Show how discounted cashflow and equation of value techniques can be used in project appraisals. (Chapter 11)

3.3.1 Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.

3.3.2 Calculate the internal rate of return, payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.

4. Single decrement models (10%)

4.1 Define various assurance and annuity contracts. (Chapters 14, 15, 16, 18 and 25)

4.1.1 Define the following terms:

- whole-life assurance
- term assurance
- pure endowment
- endowment assurance
- whole-life level annuity
- temporary level annuity
- guaranteed level annuity
- premium
- benefit

including assurance and annuity contracts where the benefits are deferred.

4.1.2 Describe the operation of conventional with-profits contracts, in which profits are distributed by the use of regular reversionary bonuses, and by terminal bonuses. Describe the benefits payable under the above assurance-type contracts.

4.1.3 Describe the operation of conventional unit-linked contracts, in which death benefits are expressed as combination of absolute amount and relative to a unit fund.

4.1.4 Describe the operation of accumulating with-profits contracts, in which benefits take the form of an accumulating fund of premiums, where either:

- the fund is defined in monetary terms, has no explicit charges, and is increased by the addition of regular guaranteed and bonus interest payments plus a terminal bonus; or
- the fund is defined in terms of the value of a unit fund, is subject to explicit charges, and is increased by regular bonus additions plus a terminal bonus (unitised with-profits).

In the case of unitised with-profits, the regular additions can take the form of (a) unit price increases (guaranteed and/or discretionary) or (b) allocations of additional units.

In either case, a guaranteed minimum monetary death benefit may be applied.

4.2 Develop formulae for the means and variances of the payments under various assurance and annuity contracts, assuming constant deterministic interest rate. (Chapters 14, 15, 16, 17, 18 and 20)

4.2.1 Describe the life table functions l_x and d_x and their select equivalents $l_{[x]+r}$ and $d_{[x]+r}$.

4.2.2 Define the following probabilities: ${}_np_x$, ${}_nq_x$, ${}_{n|m}q_x$, ${}_{n|}q_x$ and their select equivalents ${}_np_{[x]+r}$, ${}_nq_{[x]+r}$, ${}_{n|m}q_{[x]+r}$, ${}_{n|}q_{[x]+r}$.

4.2.3 Express the probabilities defined in 4.2.2 in terms of life table functions defined in 4.2.1.

4.2.4 Define the assurance and annuity factors and their select and continuous equivalents. Extend the annuity factors to allow for the possibility that payments are more frequent than annual but less frequent than continuous.

4.2.5 Understand and use the relations between annuities payable in advance and in arrear, and between temporary, deferred and whole life annuities.

4.2.6 Understand and use the relations between assurance and annuity factors using equation of value, and their select and continuous equivalents.

4.2.7 Obtain expressions in the form of sums/integrals for the mean and variance of the present value of benefit payments under each contract defined in 4.1.1, in terms of the (curtate) random future lifetime, assuming:

- contingent benefits (constant, increasing or decreasing) are payable at the middle or end of the year of contingent event or continuously.
- annuities are paid in advance, in arrear or continuously, and the amount is constant, increases or decreases by a constant monetary amount or by a fixed or time-dependent variable rate.
- premiums are payable in advance, in arrear or continuously; and for the full policy term or for limited period.

Where appropriate, simplify the above expressions into a form suitable for evaluation by table look-up or other means.

4.2.8 Define and evaluate the expected accumulations in terms of expected values for the contracts described in 4.1.1 and contract structures described in 4.2.7.

5. Multiple decrement and multiple life models (10%)

5.1 Define and use assurance and annuity functions involving two lives. (Chapters 21 and 22)

5.1.1 Extend the techniques of objectives 4.2 to deal with cashflows dependent upon the death or survival of either or both of two lives.

5.1.2 Extend the technique of 5.1.1 to deal with functions dependent upon a fixed term as well as age.

5.2 Describe and illustrate methods of valuing cashflows that are contingent upon multiple transition events. (Chapter 24)

5.2.1 Define health insurance, and describe simple health insurance premium and benefit structures.

5.2.2 Explain how a cashflow, contingent upon multiple transition events, may be valued using a multiple state Markov model, in terms of the forces and probabilities of transition.

5.2.3 Construct formulae for the expected present values of cashflows that are contingent upon multiple transition events, including simple health insurance premiums and benefits, and calculate these in simple cases. Regular premiums and sickness benefits are payable continuously and assurance benefits are payable immediately on transition.

5.3 Describe and use methods of projecting and valuing expected cashflows that are contingent upon multiple decrement events. (Chapter 24)

5.3.1 Describe the construction and use of multiple decrement tables.

5.3.2 Define a multiple decrement model as a special case of a multiple state Markov model.

5.3.3 Derive dependent probabilities for a multiple decrement model in terms of given forces of transition, assuming forces of transition are constant over single years of age.

5.3.4 Derive forces of transition from given dependent probabilities, assuming forces of transition are constant over single years of age.

6. Pricing and reserving (35%)

6.1 Define the gross random future loss under an insurance contract, and state the principle of equivalence. (Chapter 19)

6.2 Describe and calculate gross premiums and reserves of assurance and annuity contracts. (Chapters 19 and 20)

6.2.1 Define and calculate gross premiums for the insurance contract benefits as defined in objective 4.1 under various scenarios using the equivalence principle or otherwise:

- contracts may accept only single premium;
- regular premiums and annuity benefits may be payable annually, more frequently than annually, or continuously;
- death benefits (which increase or decrease by a constant compound rate or by a constant monetary amount) may be payable at the end of the year of death, or immediately on death;
- survival benefits (other than annuities) may be payable at defined intervals other than at maturity.

6.2.2 State why an insurance company will set up reserves.

6.2.3 Define and calculate gross prospective and retrospective reserves.

6.2.4 State the conditions under which, in general, the prospective reserve is equal to the retrospective reserve allowing for expenses.

6.2.5 Prove that, under the appropriate conditions, the prospective reserve is equal to the retrospective reserve, with or without allowance for expenses, for all fixed benefit and increasing / decreasing benefit contracts.

6.2.6 Obtain recursive relationships between successive periodic gross premium reserves, and use this relationship to calculate the profit earned from a contract during the period.

6.2.7 Outline the concepts of net premiums and net premium valuation and how they relate to gross premiums and gross premium valuation respectively.

6.3 Define and calculate, for a single policy or a portfolio of policies (as appropriate):

- death strain at risk;
- expected death strain;
- actual death strain; and
- mortality profit

for policies with death benefits payable immediately on death or at the end of the year of death, policies paying annuity benefits at the start of the year or on survival to the end of the year, and policies where single or non-single premiums are payable. (Chapter 23)

6.4 Project expected future cashflows for whole life, endowment and term assurances, annuities, unit-linked contracts, and conventional/unitised with-profits contracts, incorporating multiple decrement models as appropriate. (Chapters 26 and 27)

6.4.1 Profit test life insurance contracts of the types listed above and determine the profit vector, the profit signature, the net present value, and the profit margin.

6.4.2 Show how a profit test may be used to price a product, and use a profit test to calculate a premium for life insurance contracts of the types listed above.

6.4.3 Show how gross premium reserves can be computed using the above cashflow projection model and included as part of profit testing.

6.5 Show how, for unit-linked contracts, non-unit reserves can be established to eliminate ('zeroise') future negative cashflows, using a profit test model. (Chapter 27)

Core Reading

The Subject CM1 Course Notes include the Core Reading in full, integrated throughout the course.

Accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of the material contained in the Core Reading.

Further reading

The exam will be based on the relevant Syllabus and Core Reading and the ActEd course material will be the main source of tuition for students.

However, some students may find it useful to obtain a different viewpoint on a particular topic covered in Subject CM1. A list of suggested further reading for Subject CM1 has been prepared by the Institute and Faculty of Actuaries and can be found on their website. This list is not exhaustive and other useful material may be available.

1.3 Subject CM1 – summary of ActEd products

The following products are available for Subject CM1:

- Course Notes
- Paper B Online Resources (PBOR), including the Y Assignments
- X Assignments – five assignments:
  - X1, X2, X3: 80-mark tests (you are allowed 2¾ hours to complete these)
  - X4, X5: 100-mark tests (you are allowed 3¼ hours to complete these)
- Y Assignments – two assignments:
  - Y1, Y2: 100-mark tests (you are allowed 1¾ hours to complete these)
- Series X Marking
- Series Y Marking
- Online Classroom – over 150 tutorial units
- Flashcards
- Revision Notes – 13 A5 booklets
- ASET (2014-17 papers) – four years of exam papers, *ie* eight sittings, covering the period April 2014 to September 2017
- ASET (2019-21 papers) – three years of exam papers, covering the period April 2019 to September 2021
- Mini ASET – covering the April 2022 exam paper
- Mock Exam – one 100-mark test for the Paper A examination and a separate 100-mark test for the practical Paper B exam
- Additional Mock Pack (AMP) – two additional 100-mark Paper A tests and two additional 100-mark Paper B tests
- Mock Exam Marking
- Marking Vouchers.

Products are generally available in both paper and eBook format. Visit **www.ActEd.co.uk** for full details about available eBooks, software requirements and restrictions.

The following tutorials are typically available for Subject CM1:

- Regular Tutorials (five days)
- Block Tutorials (five days)
- A Preparation Day for the practical exam
- Six-day Bundles in both Regular and Block format, covering the five days for the Paper A exam, plus the Preparation Day for the practical exam.

Full details are set out in our *Tuition Bulletin*, which is available on our website at **www.ActEd.co.uk**.

1.4 Subject CM1 – skills and assessment

Technical skills

Subjects CM1 and CM2 are very mathematical and have relatively few questions requiring wordy answers.

Exam skills

Exam question skill levels

In the CM subjects, the approximate split of assessment across the three skill types is:

- Knowledge – 20%
- Application – 65%
- Higher Order skills – 15%.

Assessment

Assessment consists of a combination of a 3¼-hour examination and a 1¾-hour practical modelling examination.

1.5 Subject CM1 – frequently asked questions

Q: ***What knowledge of earlier subjects should I have and what level of mathematics is required?***

A: Subject CM1 does require some knowledge of statistics. In particular, you need to be familiar with random variables, probabilities, expectations and variances. These topics are prerequisites for studying *any* of the IFoA examinations, and are further developed in Subject CS1, although it is not essential to have studied Subject CS1 before Subject CM1.

The level of maths you need for this course is broadly A-level standard. In particular, you will need a knowledge of calculus, *ie* differentiation and integration. You will find the course much easier if you feel comfortable with the mathematical techniques used and you feel confident in applying them yourself.

If your maths or statistics is a little rusty you may wish to consider purchasing additional material to help you get up to speed. The course 'Pure Maths and Statistics for Actuarial Studies' is available from ActEd and it covers the mathematical techniques that are required for the Core Principles subjects, some of which are beyond A-Level (or Higher) standard. You do not need to work through the whole course in order – you can just refer to it when you need help on a particular topic. An initial assessment to test your mathematical skills and further details regarding the course can be found on our website.

Q: ***What should I do if I discover an error in the course?***

A: If you find an error in the course, please check our website at:

www.ActEd.co.uk/paper_corrections.html

to see if the correction has already been dealt with. Otherwise please send details via email to **CM1@bpp.com**.

Q: ***Who should I send feedback to?***

A: We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses.

If you have any comments on this course in general, please email them to **CM1@bpp.com**.

If you have any comments or concerns about the Syllabus or Core Reading, these can be passed on to the profession via ActEd. Alternatively, you can send them directly to the Institute and Faculty of Actuaries' Examination Team by email to **education.services@actuaries.org.uk**.

1.6 Subject CM1 – Some useful formulae

The formulae below are useful for parts 3 to 5 of the Subject CM1 course. These formulae are explained and developed in the relevant chapters of the Course Notes, but you may find this formula sheet helpful when starting to practise questions. Other useful formulae are given on pages 36 and 37 of the *Tables*.

Assurances:

$A^{1}_{x:\overline{n|}} = A_x - v^n \, {}_n p_x \, A_{x+n} = A_x - \dfrac{D_{x+n}}{D_x} A_{x+n}$ $\qquad$ $A_{x:\overline{n|}}^{\;\;1} = v^n \, {}_n p_x = \dfrac{D_{x+n}}{D_x}$

$A_{x:\overline{n|}} = A^{1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;1}$ $\qquad$ $\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;1}$

${}_{n|}A_x = A_x - A^{1}_{x:\overline{n|}} = v^n \, {}_n p_x \, A_{x+n}$ $\qquad$ $\bar{A}_x \approx (1+i)^{\frac{1}{2}} A_x \text{ or } (1+\tfrac{1}{2}i)A_x \text{ or } \dfrac{i}{\delta} A_x$

$(IA)^{1}_{x:\overline{n|}} = (IA)_x - v^n \dfrac{l_{x+n}}{l_x}\left[(IA)_{x+n} + nA_{x+n}\right]$ $\qquad$ $(IA)_{x:\overline{n|}} = (IA)^{1}_{x:\overline{n|}} + nv^n \dfrac{l_{x+n}}{l_x}$

$A^{1}_{xx} = ½A_{xx}$ $\qquad$ $A^{2}_{xx} = ½A_{\overline{xx}}$

$A_{\overline{xy}} = A_x + A_y - A_{xy} = 1 - d\ddot{a}_{\overline{xy}}$ $\qquad$ $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 1 - \delta\bar{a}_{\overline{xy}} = (1+i)^{\frac{1}{2}}\left(1 - d\ddot{a}_{\overline{xy}}\right)$

$A_{\overline{xy}:\overline{n|}} = A_{x:\overline{n|}} + A_{y:\overline{n|}} - A_{xy:\overline{n|}} = 1 - d\ddot{a}_{\overline{xy}:\overline{n|}}$

Annuities:

$a_x = \ddot{a}_x - 1 = v\,p_x \ddot{a}_{x+1}$ $\qquad$ $\bar{a}_x \approx \ddot{a}_x - \dfrac{1}{2} \text{ or } a_x + \dfrac{1}{2}$

$a_x^{(m)} \approx a_x + \dfrac{m-1}{2m}$ $\qquad$ $\ddot{a}_{x:\overline{n|}} = \ddot{a}_x - v^n \, {}_n p_x \, \ddot{a}_{x+n} = 1 + a_{x:\overline{n-1|}}$

$a_{x:\overline{n|}} = a_x - v^n \, {}_n p_x \, a_{x+n} = v\,p_x \, \ddot{a}_{x+1:\overline{n|}} = \ddot{a}_{x:\overline{n|}} - 1 + v^n \, {}_n p_x$

$\bar{a}_{x:\overline{n|}} \approx \ddot{a}_{x:\overline{n|}} - \dfrac{1}{2}(1 - v^n \, {}_n p_x)$ $\qquad$ ${}_{n|}a_x = a_x - a_{x:\overline{n|}} = v^n \, {}_n p_x a_{x+n}$

$(I\ddot{a})_{x:\overline{n|}} = (I\ddot{a})_x - v^n \dfrac{l_{x+n}}{l_x}\left[n\ddot{a}_{x+n} + (I\ddot{a})_{x+n}\right]$ $\qquad$ $(Ia)_x = (I\ddot{a})_x - \ddot{a}_x$

$a_{\overline{xy}} = a_x + a_y - a_{xy}$ $\qquad$ $a_{\overline{xy}:\overline{n|}} = a_{x:\overline{n|}} + a_{y:\overline{n|}} - a_{xy:\overline{n|}}$

$a_{x|y} = a_y - a_{xy} = \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}_{x|y}$ $\qquad$ $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \dfrac{\bar{A}_{xy} - \bar{A}_y}{\delta} = \displaystyle\int_{t=0}^{t=\infty} v^t \, \bar{a}_{y+t} \, {}_t p_{xy} \, \mu_{x+t} \, dt$

2.1 Before you start

When studying for the Institute and Faculty of Actuaries' exams, you will need:

- a copy of the **Formulae and Tables for Examinations of the Faculty of Actuaries and the Institute of Actuaries, 2nd Edition (2002)** – these are referred to simply as the *Tables*
- an **scientific calculator** or **Excel** – you will find the list of permitted calculators on the profession's website. Please check the list carefully, since it is reviewed each year.

The *Tables* are available from the Institute and Faculty of Actuaries' eShop. Please visit **www.actuaries.org.uk**.

2.2 Core study material

This section explains the role of the Syllabus, Core Reading and supplementary ActEd text. It also gives guidance on how to use these materials most effectively in order to pass the exam.

Some of the information below is also contained in the introduction to the Core Reading produced by the Institute and Faculty of Actuaries.

Syllabus

The Syllabus for Subject CM1 has been produced by the Institute and Faculty of Actuaries. The relevant individual syllabus objectives are included at the start of each course chapter and a complete copy of the Syllabus is included in Section 1.2 of this Study Guide. We recommend that you use the Syllabus as an important part of your study.

Core Reading

The Core Reading has been produced by the Institute and Faculty of Actuaries. The purpose of the Core Reading is to ensure that tutors, students and examiners understand the requirements of the Syllabus for the qualification examinations for Fellowship of the Institute and Faculty of Actuaries.

The Core Reading supports coverage of the Syllabus in helping to ensure that both depth and breadth are re-enforced. It is therefore important that students have a good understanding of the concepts covered by the Core Reading.

The examinations require students to demonstrate their understanding of the concepts given in the Syllabus and described in the Core Reading; this will be based on the legislation, professional guidance, *etc* that are in force when the Core Reading is published, *ie* on 31 May in the year preceding the examinations.

Therefore the exams in April and September 2022 will be based on the Syllabus and Core Reading as at 31 May 2021. We recommend that you always use the up-to-date Core Reading to prepare for the exams.

Examiners will have this Core Reading when setting the papers. In preparing for examinations, students are advised to work through past examination questions and will find additional tuition helpful. The Core Reading will be updated each year to reflect changes in the Syllabus, to reflect current practice, and in the interest of clarity.

Accreditation

The Institute and Faculty of Actuaries would like to thank the numerous people who have helped in the development of the material contained in this Core Reading.

ActEd text

Core Reading deals with each syllabus objective and covers what is needed to pass the exam. However, the tuition material that has been written by ActEd enhances it by giving examples and further explanation of key points. Here is an excerpt from some ActEd Course Notes to show you how to identify Core Reading and the ActEd material. **Core Reading is shown in this bold font.**

In the example given above, the index *will* fall if the actual share price goes below the theoretical ex-rights share price. Again, this is consistent with what would happen to an underlying portfolio.

This is ActEd text

After allowing for chain-linking, **the formula for the investment index then becomes:**

$$I(t) = \frac{\sum_i N_{i,t} P_{i,t}}{B(t)}$$

This is Core Reading

where $N_{i,t}$ is the number of shares issued for the *i*th constituent at time *t*;

$B(t)$ is the base value, or divisor, at time *t*.

2.3 ActEd study support

This section gives a description of the products offered by ActEd.

Successful students tend to undertake three main study activities:

1. *Learning* – initial study and understanding of subject material
2. *Revision* – learning subject material and preparing to tackle exam-style questions
3. *Rehearsal* – answering exam-style questions, culminating in answering questions at exam speed.

Different approaches suit different people. For example, you may like to revise material gradually over the months running up to the exams or you may do your revision in a shorter period just before the exams. Also, these three activities will almost certainly overlap.

We offer a flexible range of products to suit you and let you control your own learning and exam preparation. The following table shows the products that we produce. Not all products are available for all subjects.

LEARNING	**LEARNING & REVISION**	**REVISION**	**REVISION & REHEARSAL**	**REHEARSAL**
Course Notes	Assignments	Flashcards	Revision Notes	Mock Exam
PBOR	Combined Materials Pack (CMP)	Sound revision	ASET	Additional Mock Pack (AMP)
	Assignment Marking			Mock Marking
	Tutorials			
	Online Classroom			

The products and services are described in more detail below.

‘Learning’ products

Course Notes

The Course Notes will help you develop the basic knowledge and understanding of principles needed to pass the exam. They incorporate the complete Core Reading and include full explanation of all the syllabus objectives, with worked examples and questions (including some past exam questions) to test your understanding.

Each chapter includes:

- the relevant syllabus objectives
- a chapter summary
- a page of important formulae or definitions (where appropriate)
- practice questions with full solutions.

Paper B Online Resources (PBOR)

The Paper B Online Resources (PBOR) will help you prepare for the practical paper. Delivered through a virtual learning environment (VLE), you will have access to worked examples and practice questions. PBOR will also include the Y Assignments, which are two exam-style assessments.

‘Learning & revision’ products

X Assignments

The Series X Assignments are assessments that cover the material in each part of the course in turn. They can be used to develop and test your understanding of the material.

Y Assignments

The Series Y Assignments are exam-style assessments that cover material across the whole course.

Combined Materials Pack (CMP)

The Combined Materials Pack (CMP) comprises the Course Notes, PBOR and the Series X Assignments.

CMP Upgrade

The purpose of the CMP Upgrade is to enable you to amend last year’s study material to make it suitable for study for this year.

Wherever possible, it lists the changes to the syllabus objectives, Core Reading, the Course Notes and the X / Y Assignments since last year that might realistically affect your chance of success in the exam. It is produced so that you can manually amend your notes. The upgrade includes replacement pages and additional pages where appropriate.

However, if a large number of changes have been made to the Course Notes and X / Y Assignments, it is not practical to produce a full upgrade, and the upgrade will only *outline* the most significant changes. In this case, we recommend that you purchase a replacement CMP (printed copy or eBook) or Course Notes at a significantly reduced price.

The CMP Upgrade can be downloaded free of charge on our website at **www.ActEd.co.uk**.

A separate upgrade for eBooks is not produced but a significant discount is available for retakers wishing to re-purchase the latest eBook.

X / Y Assignment Marking

We are happy to mark your attempts at the X and/or Y assignments. Marking is not included with the Assignments or the CMP and you need to order both Series X and Series Y Marking separately. You should submit your script as an attachment to an email, in the format detailed in your assignment instructions. You will be able to download your marker's feedback via a secure link.

Don't underestimate the benefits of attempting and submitting assignments:

- Question practice during this phase of your study gives an early focus on the end goal of answering exam-style questions.
- You're incentivised to keep up with your study plan and get a regular, realistic assessment of your progress.
- Objective, personalised feedback from a high quality marker will highlight areas on which to work and help with exam technique.

In a recent study, we found that students who attempt more than half the assignments have significantly higher pass rates.

There are two different types of marking product: Series Marking and Marking Vouchers.

Series Marking

Series Marking applies to a specified subject, session and student. If you purchase Series Marking, you will **not** be able to defer the marking to a future exam sitting or transfer it to a different subject or student.

We typically provide full solutions with the Series Assignments. However, if you order Series Marking at the same time as you order the Series Assignments, you can choose whether or not to receive a copy of the solutions in advance. If you choose not to receive them with the study material, you will be able to download the solutions via a secure link when your marked script is returned (or following the final deadline date if you do not submit a script).

If you are having your attempts at the assignments marked by ActEd, you should submit your scripts regularly throughout the session, in accordance with the schedule of recommended dates set out on our website at **www.ActEd.co.uk**. This will help you to pace your study throughout the session and leave an adequate amount of time for revision and question practice.

The recommended submission dates are realistic targets for the majority of students. Your scripts will be returned more quickly if you submit them well before the final deadline dates.

Any script submitted *after* the relevant final deadline date will not be marked. It is your responsibility to ensure that we receive scripts in good time.

Marking Vouchers

Marking Vouchers give the holder the right to submit a script for marking at any time, irrespective of the individual assignment deadlines, study session, subject or person.

Marking Vouchers can be used for any assignment. They are valid for four years from the date of purchase and can be refunded at any time up to the expiry date.

Although you may submit your script with a Marking Voucher at any time, you will need to adhere to the explicit Marking Voucher deadline dates to ensure that your script is returned before the date of the exam. The deadline dates are provided on our website at **www.ActEd.co.uk**.

Tutorials

Our tutorials are specifically designed to develop the knowledge that you will acquire from the course material into the higher-level understanding that is needed to pass the exam.

We run a range of different tutorials including face-to-face tutorials at various locations, and Live Online tutorials. Full details are set out in our *Tuition Bulletin*, which is available on our website at **www.ActEd.co.uk**.

Regular and Block Tutorials

In preparation for these tutorials, we expect you to have read the relevant part(s) of the Course Notes before attending the tutorial so that the group can spend time on exam questions and discussion to develop understanding rather than basic bookwork.

You can choose *one* of the following types of tutorial:

- **Regular Tutorials** spread over the session
- a **Block Tutorial** held two to eight weeks before the exam.

The tutorials outlined above will focus on and develop the skills required for the Paper A examination. Students wishing for some additional tutor support working through exam-style questions for Paper B may wish to attend a Preparation Day. These will be available Live Online or face-to-face, where students will need to provide their own device capable of running Excel.

Online Classroom

The Online Classroom acts as either a valuable add-on or a great alternative to a face-to-face or Live Online tutorial, focussing on the Paper A examination.

At the heart of the Online Classroom in each subject is a comprehensive, easily-searched collection of tutorial units. These are a mix of:

- teaching units, helping you to really get to grips with the course material, and
- guided questions, enabling you to learn the most efficient ways to answer questions and avoid common exam pitfalls.

The best way to discover the Online Classroom is to see it in action. You can watch a sample of the Online Classroom tutorial units on our website at **www.ActEd.co.uk**.

'Revision' products

For most subjects, there is ***a lot of material*** to revise. Finding a way to fit revision into your routine as painlessly as possible has got to be a good strategy. Flashcards and Sound Revision are inexpensive options that can provide a massive boost. They can also provide a variation in activities during a study day, and so help you to maintain concentration and effectiveness.

Flashcards

Flashcards are a set of A6-sized cards that cover the key points of the subject that most students want to commit to memory. Each flashcard has questions on one side and the answers on the reverse. We recommend that you use the cards actively and test yourself as you go.

Sound revision

It is reported that only 30% of information that is read is retained but this rises to 50% if the information is also heard. Sound Revision is a set of audio files, designed to help you remember the most important aspects of the Core Reading.

The files cover the majority of the course, split into a number of manageable topics based on the chapters in the Course Notes. Each section lasts no longer than a few minutes.

Choice of revision product

Different students will have preferences for different revision products.

So, what might influence your choice between these study aids? The following questions and comments might help you to choose the revision products that are most suitable for you:

- Do you have a regular train or bus journey?

  Flashcards are ideal for regular bursts of revision on the move.

- Do you want to fit more study into your routine?

  Flashcards are a good option for 'dead time', eg using flashcards on your phone or sticking them on the wall in your study.

- Do you find yourself cramming for exams (even if that's not your original plan)?

  Flashcards are an extremely efficient way to do your pre-exam preparation.

- Do you have some regular time where carrying other materials isn't practical, *eg* commuting, at the gym, walking the dog?

  Sound Revision is an ideal 'hands-free' revision tool.

- Do you have a preference for auditory learning, *eg* do you remember conversations more easily than emails?

  Sound Revision will suit your preferred style and be especially effective for you.

Choosing more than one revision product

As there is some degree of overlap between revision products, we do not necessarily recommend using them all simultaneously. However, if you are retaking a subject, then you might consider using a different product than on a previous attempt to keep your revision fresh and effective.

'Revision & rehearsal' products

Revision Notes

Our Revision Notes have been designed with input from students to help you revise efficiently. They are suitable for first-time sitters who have worked through the ActEd Course Notes or for retakers (who should find them much more useful and challenging than simply reading through the course again).

The Revision Notes are a set of A5 booklets – perfect for revising on the train or tube to work. Each booklet covers one main theme or a set of related topics from the course and includes:

- Core Reading to develop your bookwork knowledge
- relevant past exam questions with concise solutions from the last ten years
- other useful revision aids.

ActEd Solutions with Exam Technique (ASET)

The ActEd Solutions with Exam Technique (ASET) contains our solutions to a number of past exam papers, plus comment and explanation. In particular, it highlights how questions might have been analysed and interpreted so as to produce a good solution with a wide range of relevant points. This will be valuable in approaching questions in subsequent examinations.

'Rehearsal' products

Mock Exam

The Mock Exam consists of two papers. There is a 100-mark mock exam for the Paper A examination and a separate mock exam for the practical Paper B exam. These provide a realistic test of your exam readiness.

It is based on the Mock Exam from last year but it has been updated to reflect any changes to the Syllabus and Core Reading.

Additional Mock Pack (AMP)

The Additional Mock Pack (AMP) consists of four further 100-mark mock exam papers – Mock Exam 2 (Papers A and B) and Mock Exam 3 (Papers A and B). This is ideal if you are retaking and have already sat the Mock Exam, or if you just want some extra question practice.

Mock Marking

We are happy to mark your attempts at the mock exams. The same general principles apply as for the Assignment Marking. In particular:

- Mock Exam Marking applies to a specified subject, session and student. In this subject it covers the marking of both Paper A and Paper B.
- Marking Vouchers can be used for each mock exam paper. You will need two marking vouchers in order to have both Paper A and Paper B marked. Marking vouchers have to be used for marking the AMP mocks and can be used for marking the Mock Exam.

Recall that:

- marking is not included with the products themselves and you need to order it separately
- you should submit your script via email in the format detailed in the mock exam instructions
- you will be able to download the feedback on your marked script via a secure link.

2.4 Study skills and assessment

Technical skills

The Core Reading and exam papers for these subjects tend to be very technical. The exams themselves have many calculation and manipulation questions. The emphasis in the exam will therefore be on *understanding* the mathematical techniques and applying them to various, frequently unfamiliar, situations. It is important to have a feel for what the numerical answer should be by having a deep understanding of the material and by doing reasonableness checks.

As a high level of pure mathematics and statistics is generally required for the Core Principles subjects, it is important that your mathematical skills are extremely good. If you are a little rusty you may wish to consider purchasing additional material to help you get up to speed. The course 'Pure Maths and Statistics for Actuarial Studies' is available from ActEd and it covers the mathematical techniques that are required for the Core Principles subjects, some of which are beyond A-Level (or Higher) standard. You do not need to work through the whole course in order – you can just refer to it when you need help on a particular topic. An initial assessment to test your mathematical skills and further details regarding the course can be found on our website at **www.ActEd.co.uk**.

Study skills

Overall study plan

We suggest that you develop a realistic study plan, building in time for relaxation and allowing some time for contingencies. Be aware of busy times at work, when you may not be able to take as much study leave as you would like. Once you have set your plan, be determined to stick to it. You don't have to be too prescriptive at this stage about what precisely you do on each study day. The main thing is to be clear that you will cover all the important activities in an appropriate manner and leave plenty of time for revision and question practice.

Aim to manage your study so as to allow plenty of time for the concepts you meet in these courses to 'bed down' in your mind. Most successful students will probably aim to complete the courses at least a month before the exam, thereby leaving a sufficient amount of time for revision. By finishing the courses as quickly as possible, you will have a much clearer view of the big picture. It will also allow you to structure your revision so that you can concentrate on the important and difficult areas.

You can also try looking at our discussion forum, which can be accessed at **www.ActEd.co.uk/forums** (or use the link from our home page at **www.ActEd.co.uk**). There are some good suggestions from students on how to study.

Study sessions

Only do activities that will increase your chance of passing. Try to avoid including activities for the sake of it and don't spend time reviewing material that you already understand. You will only improve your chances of passing the exam by getting on top of the material that you currently find difficult.

Ideally, each study session should have a specific purpose and be based on a specific task, *eg 'Finish reading Chapter 3 and attempt Practice Questions 3.4, 3.7 and 3.12'*, as opposed to a specific amount of time, *eg 'Three hours studying the material in Chapter 3'*.

Try to study somewhere quiet and free from distractions (*eg* an area at home dedicated to study). Find out when you operate at your peak, and endeavour to study at those times of the day. This might be between 8*am* and 10*am* or could be in the evening. Take short breaks during your study to remain focused – it's definitely time for a short break if you find that your brain is tired and that your concentration has started to drift from the information in front of you.

Order of study

We suggest that you work through each of the chapters in turn. To get the maximum benefit from each chapter you should proceed in the following order:

1. Read the syllabus objectives. These are set out in the box at the start of each chapter.

2. Read the Chapter Summary at the end of each chapter. This will give you a useful overview of the material that you are about to study and help you to appreciate the context of the ideas that you meet.

3. Study the Course Notes in detail, annotating them and possibly making your own notes. Try the self-assessment questions as you come to them. As you study, pay particular attention to the listing of the syllabus objectives and to the Core Reading.

4. Read the Chapter Summary again carefully. If there are any ideas that you can't remember covering in the Course Notes, read the relevant section of the notes again to refresh your memory.

5. Attempt (at least some of) the Practice Questions that appear at the end of the chapter.

6. Where relevant, work through the relevant Paper B Online Resources for the chapter(s). You will need to have a good understanding of the relevant section of the course before you attempt the corresponding section of PBOR.

7. Think about what specifically you might want to include from that chapter in the reference materials that you choose to have to hand during the exam. For example, you might want to put together some easy-reference lists of key concepts or formulae that can be referred to quickly and conveniently.

It's a fact that people are more likely to absorb something if they review it several times. So, do look over the chapters you have studied so far from time to time. It is useful to re-read the Chapter Summaries or to try the Practice Questions again a few days after reading the chapter itself. It's a good idea to annotate the questions with details of when you attempted each one. This makes it easier to ensure that you try all of the questions as part of your revision without repeating any that you got right first time.

Once you've read the relevant part of the notes and tried a selection of questions from the Practice Questions (and attended a tutorial, if appropriate) you should attempt the corresponding assignment. If you submit your assignment for marking, spend some time looking through it carefully when it is returned. It can seem a bit depressing to analyse the errors you made, but you will increase your chances of passing the exam by learning from your mistakes. The markers will try their best to provide practical comments to help you to improve.

To be really prepared for the exam, you should not only know and understand the Core Reading but also be aware of what the examiners will expect. Your revision programme should include plenty of question practice so that you are aware of the typical style, content and marking structure of exam questions. You should attempt as many past exam questions as you can.

Active study

Here are some techniques that may help you to study actively.

1. Don't believe everything you read. Good students tend to question everything that they read. They will ask 'why, how, what for, when?' when confronted with a new concept, and they will apply their own judgement. This contrasts with those who unquestioningly believe what they are told, learn it thoroughly, and reproduce it (unquestioningly?) in response to exam questions.

2. Another useful technique as you read the Course Notes is to think of possible questions that the examiners could ask. This will help you to understand the examiners' point of view and should mean that there are fewer nasty surprises in the exam room. Use the Syllabus to help you make up questions.

3. Annotate your notes with your own ideas and questions. This will make you study more actively and will help when you come to review and revise the material. These notes may also be useful to refer to in the exam. Do not simply copy out the notes without thinking about the issues.

4. Attempt the questions in the notes as you work through the course. Produce your answer before you refer to the solution.

5. Attempt other questions and assignments on a similar basis, *ie* produce your answer before looking at the solution provided. Attempting the assignments under exam conditions has some particular benefits:

   - It forces you to think and act in a way that is similar to how you will behave in the exam.
   - When you have your assignments marked it is *much* more useful if the marker's comments can show you how to improve your performance under timed conditions than your performance when you have access to the notes and are under no time pressure.
   - The knowledge that you are going to do an assignment under timed conditions and then submit it (however good or bad) for marking can act as a powerful incentive to make you study each part as well as possible.
   - It is also quicker than trying to produce perfect answers.

6. Sit a mock exam four to six weeks before the real exam to identify your weaknesses and work to improve them. You could use a mock exam written by ActEd or a past exam paper. Ensure that you have your reference materials handy, as you plan to in the actual exam, so that you can practise finding what you need in them quickly and efficiently. (You might even be able to add to / modify your reference materials to increase their usefulness.)

You can find further information on how to study in the profession's Student Handbook, which you can download from their website at **www.actuaries.org.uk/studying.**

Revision and exam skills

Revision skills

You will have sat many exams before and will have mastered the exam and revision techniques that suit you. However it is important to note that due to the high volume of work involved in the Core Principles subjects it is not possible to leave all your revision to the last minute. Students who prepare well in advance have a better chance of passing their exams on the first sitting.

Unprepared students find that they are under time pressure in the exam. Therefore it is important to find ways of maximising your score in the shortest possible time. Part of your preparation should be to practise a large number of exam-style questions under timed conditions as soon as possible. This will:

- help you to develop the necessary understanding of the techniques required
- highlight the key topics, which crop up regularly in many different contexts and questions
- help you to practise the specific skills that you will need to pass the exam.

There are many sources of exam-style questions. You can use past exam papers, the Practice Questions at the end of each chapter (which include many past exam questions), assignments, mock exams, the Revision Notes and ASET.

Exam question skill levels

Exam questions are not designed to be of similar difficulty. The Institute and Faculty of Actuaries specifies different skill levels at which questions may be set.

Questions may be set at any skill level:

- Knowledge – demonstration of a detailed knowledge and understanding of the topic
- Application – demonstration of an ability to apply the principles underlying the topic within a given context
- Higher Order – demonstration of an ability to perform deeper analysis and assessment of situations, including forming judgements, taking into account different points of view, comparing and contrasting situations, suggesting possible solutions and actions, and making recommendations.

Command verbs

The Institute and Faculty of Actuaries use command verbs (such as 'Define', 'Discuss' and 'Explain') to help students to identify what the question requires. The profession has produced a document, 'Command verbs used in the Associate and Fellowship examinations', to help students to understand what each command verb is asking them to do.

It also gives the following advice:

- The use of a specific command verb within a syllabus objective does not indicate that this is the only form of question which can be asked on the topic covered by that objective.
- The examiners may ask a question on any syllabus topic using any of the agreed command verbs, as are defined in the document.

You can find the relevant document on the profession's website at:

www.actuaries.org.uk/studying/prepare-your-exams

Past exam papers

You can download some past exam papers and Examiners' Reports from the profession's website at **www.actuaries.org.uk**. However, please be aware that the majority of these exam papers are for the pre-2019 syllabus and so not all questions will be relevant.

The examination

IMPORTANT NOTE: The following information was correct at the time of publication, however it is important to keep up-to-date with any changes. See the IFOA's website for the latest guidance.

There is a lot of useful information about the exams at:

www.actuaries.org.uk/studying/my-exams/ifoa-exams

including an Examinations Handbook that gives guidance specific to sitting exams online.

For the exam, ensure you have ready:

- your reference materials, with helpful bookmarks
- rough paper and a pen / pencil
- a calculator / Excel (or equivalent)
- a printer (if you wish to print out the exam paper)
- a copy of the Tables.

Please also refer to the profession's website and your examination instructions for details about what you will need for the practical Paper B exam.

2.5 Queries and feedback

Questions and queries

From time to time you may come across something in the study material that is unclear to you. The easiest way to solve such problems is often through discussion with friends, colleagues and peers – they will probably have had similar experiences whilst studying. If there's no-one at work to talk to then use our discussion forum at **www.ActEd.co.uk/forums** (or use the link from our home page at **www.ActEd.co.uk**).

Our online forum is dedicated to actuarial students so that you can get help from fellow students on any aspect of your studies from technical issues to study advice. You could also use it to get ideas for revision or for further reading around the subject that you are studying. ActEd tutors will visit the site from time to time to ensure that you are not being led astray and we also post other frequently asked questions from students on the forum as they arise.

If you are still stuck, then you can send queries by email to the relevant subject email address (see Section 1.5), but we recommend that you try the forum first. We will endeavour to contact you as soon as possible after receiving your query but you should be aware that it may take some time to reply to queries, particularly when tutors are away from the office running tutorials. At the busiest teaching times of year, it may take us more than a week to get back to you.

If you have many queries on the course material, you should raise them at a tutorial or book a personal tuition session with an ActEd tutor. Information about personal tuition is set out in our current brochure. Please email **ActEd@bpp.com** for more details.

Feedback

If you find an error in the course, please check the corrections page of our website (**www.ActEd.co.uk/paper_corrections.html**) to see if the correction has already been dealt with. Otherwise please send details via email to the relevant subject email address (see Section 1.5).

Each year our tutors work hard to improve the quality of the study material and to ensure that the courses are as clear as possible and free from errors. We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses. If you have any comments on this course please email them to the relevant subject email address (see Section 1.5).

Our tutors also work with the profession to suggest developments and improvements to the Syllabus and Core Reading. If you have any comments or concerns about the Syllabus or Core Reading, these can be passed on via ActEd. Alternatively, you can send them directly to the Institute and Faculty of Actuaries' Examination Team by email to **education.services@actuaries.org.uk**.

1

Principles of actuarial modelling

Syllabus objectives

1.1 Describe the principles of actuarial modelling.

- 1.1.1 Describe why and how models are used including, in general terms, the use of models for pricing, reserving and capital modelling.
- 1.1.2 Explain the benefits and limitations of modelling.
- 1.1.3 Explain the difference between a stochastic and a deterministic model, and identify the advantages/disadvantages of each.
- 1.1.4 Describe the characteristics, and explain the use of, scenario-based and proxy models.
- 1.1.5 Describe, in general terms, how to decide whether a model is suitable for any particular application.
- 1.1.6 Explain the difference between the short-run and long-run properties of a model, and how this may be relevant in deciding whether a model is suitable for any particular application.
- 1.1.7 Describe, in general terms, how to analyse the potential output from a model, and explain why this is relevant to the choice of model.

Syllabus objectives continued

1.1.8 Describe the process of sensitivity testing of assumptions, and explain why this forms an important part of the modelling process.

1.1.9 Explain the factors that must be considered when communicating the results following the application of a model.

0 Introduction

This chapter provides an introduction to modelling, in particular within an actuarial context. The following general principles are covered:

- why we want to model
- how to model
- the benefits and limitations of modelling
- testing the suitability of the model
- analysing the output
- communicating the results.

1 Models

1.1 Why models are used

A model is an imitation of a real-world system or process. Models of many activities can be developed, for example:

- **the economy of a country**
- **the workings of the human heart, and**
- **the future cashflows of the broker distribution channel of a life insurance company.**

The expression 'broker distribution channel' refers to the fact that in life insurance it is common for payments to go through a broker who acts as an intermediary between the policyholder and the insurance company. This is also true of general insurance.

Suppose we wished to 'predict' the effect that a real-world change would have on these three models. In some cases it might be too risky, or too expensive or too slow, to try a proposed change in the real world even on a sample basis. Trying out the change first without the benefit of a model could have serious consequences. The economy might go into recession costing a government the next election, the patient might die and the life office could suffer a surge in new business but at highly unprofitable premium rates.

Parameters

A model enables the possible consequences to be investigated. The effect of changing certain input parameters can be studied before a decision is made to implement the plans in the real world.

Suppose, for example, we want to buy shares in a particular company. In order to calculate what price we should pay, we would like to have some idea of the future price of these shares, so we could develop a model relating the price of the shares to various factors. In our model we *might* assume (among other things) a fixed value for future interest rates, say 6% *pa*. This value is an input parameter. It is fixed within the model, but we can vary it to obtain different models. Using the figure 6% will give us one value for the price we should pay, but we cannot be certain that 6% is the correct input. What if interest rates rise to 10% *pa*, or fall to 3% *pa*, for example? By varying the parameters in the model we can answer questions such as these.

If we assume that the interest rate is a lognormal random variable with parameters μ and σ, then although the interest rate itself is not fixed, its distribution is. The input parameters are then μ and σ.

To build a model of a system or process, a set of mathematical or logical assumptions about how it works needs to be developed. The complexity of a model is determined by the complexity of the relationships between the various model parameters. For example, in modelling a life office, consideration must be given to issues such as regulations, taxation and cancellation terms. Future events affecting investment returns, inflation, new business, lapses, mortality and expenses also affect these relationships.

In insurance terms, a cancellation occurs when a policyholder terminates their insurance cover, *eg* when a driver changes their car insurance to a different insurer mid-year. A lapse occurs when a policyholder doesn't pay a premium that is due, *eg* when a policyholder changes to a new insurer when their existing policy is up for renewal.

Data

In order to produce the model and determine suitable parameters, data need to be considered and judgements need to be made as to the relevance of the observed data to the future environment. Such data may result from past observations, from current observations (such as the rate of inflation) or from expectations of future changes.

An example of a future change in an insurance context would be if the government decided to increase the rate of insurance tax.

The analysis of data is covered in more detail in Subject CS1.

Where observed data are considered to be suitable for producing the parameters for a chosen model, statistical methods can be used to fit the data.

For example, if we were to model the daily changes in the price of shares in a particular company using a normal distribution, then the parameters would be the mean and the variance. We could estimate these by looking at past data and calculating the sample mean and sample variance. These would be sensible estimates.

Objectives

Before finalising the choice of model and parameters, it is important to consider the objectives for creation and use of the model. For example, in many cases there may not be a desire to create the most accurate model, but instead to create a model that will not understate costs or other risks that may be involved.

Moreover, we need to decide what the 'best' model is, within the context of its intended use. For example, we might wish to model the number of claims made to an insurance company for various amounts. To simplify matters, suppose there are only two possible probability distributions that we are choosing between. If most claims are for small amounts and one distribution models these small amounts better than the other, then this might seem a natural choice. However, if this same distribution models the large claims relatively poorly, then although there are not many of these large claims, the fact that they are for large amounts is significant. The decisions resulting from a poor model of the higher claims may be more costly to the company than those based on a model that is slightly inaccurate on a lot of small claims.

1.2 How models are used

While in reality a modelling process does not follow a rigid pattern of prescribed steps, it is helpful in introducing the topic to imagine a set of key steps. In practice, actuaries who build and use models move back and forth between these key steps continuously to improve the model.

The key steps in a modelling process can be described as follows:

1. **Develop a well-defined set of objectives that need to be met by the modelling process.**

   Continuing our last example on modelling the size of insurance claims, in addition to the basic purpose of being able to predict the number of claims of different sizes, one objective might be to give as accurate a prediction as possible for 95% of the claims.

2. **Plan the modelling process and how the model will be validated.**

   The validation of the model will involve a series of diagnostic tests to ensure that the model meets the objectives we want.

3. **Collect and analyse the necessary data for the model.**

4. **Define the parameters for the model and consider appropriate parameter values.**

5. **Define the model initially by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage.**

6. **Involve experts on the real-world system you are trying to imitate to get feedback on the validity of the conceptual model.**

7. **Decide on whether a simulation package or a general-purpose language is appropriate for the implementation of the model. Choose a statistically reliable random number generator that will perform adequately in the context of the complexity of the model.**

   Models based on a deterministic approach would not need this.

8. **Write the computer program for the model.**

   After this stage we can run the model.

9. **Debug the program to make sure it performs the intended operations in the model definition.**

10. **Test the reasonableness of the output from the model.**

11. **Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.**

    Suppose, for example, that small changes in the input parameters lead to large changes in the output. If these parameters cannot be known with great accuracy, then we cannot be sure that our predictions will be valid. However, we could still use the model to come up with a range of possible outputs by assuming a range of values for the input parameters. This procedure is called sensitivity testing and is discussed later in this chapter.

12. **Analyse the output from the model.**

13. **Ensure that any relevant professional guidance has been complied with. For example, the Financial Reporting Council has issued a Technical Actuarial Standard on the principles for Technical Actuarial Work (TAS100), which includes principles for models used in technical actuarial work. (Knowledge of the detail of this TAS is not required for CM1.)**

- **In a stochastic model, for any given set of inputs each run gives only estimates of a model's outputs. So, to study the outputs for any given set of inputs, several independent runs of the model are needed.**

  A stochastic model allows for randomness and each computer 'run' would give the figures for one possible outcome.

- **As a rule, models are more useful for comparing the results of input variations than for optimising outputs.**

  In other words, it's easier to use a model to predict what *might* happen than it is to determine what inputs would be required to obtain a particular outcome.

- **Models can look impressive when run on a computer so that there is a danger that one gets lulled into a false sense of confidence. If a model has not passed the tests of validity and verification, its impressive output is a poor substitute for its ability to imitate its corresponding real-world system.**

- **Models rely heavily on the data input. If the data quality is poor or lacks credibility, then the output from the model is likely to be flawed.**

  In actuarial terminology, 'credibility' refers to the extent to which data can be relied on.

- **It is important that the users of the model understand the model and the uses to which it can be safely put. There is a danger of using a model as a 'black box' from which it is assumed that all results are valid without considering the appropriateness of using that model for the data input and the output expected.**

- **It is not possible to include all future events in a model. For example, a change in legislation could invalidate the results of a model, but may be impossible to predict when the model is constructed.**

- **It may be difficult to interpret some of the outputs of the model. They may only be valid in relative rather than absolute terms, as when, for example, comparing the level of risk of the outputs associated with different inputs.**

  'Risk' refers to the level of uncertainty associated with an outcome.

3 Stochastic and deterministic models

If it is desired to represent reality as accurately as possible, the model needs to imitate the random nature of the variables. A *stochastic* model is one that recognises the random nature of the input components. A model that does not contain any random component is *deterministic* in nature.

In a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined. By contrast, in a stochastic model the output is random in nature – like the inputs, which are random variables. The output is only a snapshot or an estimate of the characteristics of the model for a given set of inputs. Several independent runs are required for each set of inputs so that statistical theory can be used to help in the study of the implications of the set of inputs.

A deterministic model is really just a special (simplified) case of a stochastic model.

The following example illustrates the difference between the two approaches.

Question

An investor has bought shares worth £5,000 and wants to estimate how much they will be worth in a year's time. Describe both a deterministic and stochastic model based on an expected growth rate of 7% over the year.

Solution

Deterministic model

The outcome from a deterministic model is the prediction that the value in a year's time would be:

$$5,000 \times 1.07 = £5,350$$

Stochastic model

A stochastic model allows for randomness in the growth rate. For example, it might be decided (based on past performance of the company and prospects for the company, the investment sector and the economy in general) that the probabilities of different growth rates for the shares are:

$$\text{growth} = \begin{cases} 20\% & \text{with probability } 0.1 \\ 10\% & \text{with probability } 0.6 \\ 0\% & \text{with probability } 0.2 \\ -10\% & \text{with probability } 0.1 \end{cases}$$

These probabilities add up to 1 and the expected growth rate is 7% for this model since:

$$0.1 \times 20\% + 0.6 \times 10\% + 0.2 \times 0\% + 0.1 \times (-10\%) = 7\%$$

The outcome from this model, if it is run 100 times, is a list of 100 predicted values, which might look like this:

£5500, £5000, £5000, £6000, £5000, … , £4500, £6000

Whether one wishes to use a deterministic or a stochastic model depends on whether one is interested in the results of a single 'scenario' or in the distribution of results of possible 'scenarios'. A deterministic model will give one the results of the relevant calculations for a single scenario; a stochastic model gives distributions of the relevant results for a distribution of scenarios. If the stochastic model is investigated by using 'Monte Carlo' simulation, then this provides a collection of a suitably large number of different deterministic models, each of which is considered equally likely.

'Monte Carlo' simulation is where a computer is set up to run a stochastic model a number of times, *eg* 10,000 times, using pseudo-random numbers generated by the computer. The results look like the list of numbers in the stochastic model in the example above. The numbers generated by the computer are pseudo-random rather than truly random because they are generated using a prescribed method.

As stochastic models provide a distribution of outputs, they can be used to estimate the probability that a particular event occurs, or to calculate approximate confidence intervals. This wouldn't be possible using a deterministic model.

The results for a deterministic model can often be obtained by direct calculation, but sometimes it is necessary to use numerical approximations, either to integrate functions or to solve differential equations.

If a stochastic model is sufficiently tractable, it may be possible to derive the results one wishes by analytical methods. If this can be done it is often preferable to, and also often much quicker than, Monte Carlo simulation. One gets precise results and can analyse the effect of changes in the assumptions more readily. Monte Carlo simulation is covered in CS1.

Many practical problems, however, are too complicated for analytical methods to be used easily, and Monte Carlo simulation is an extremely powerful method for solving complicated problems. But if even part of a model can be treated analytically, it may provide a check on any simulation method used. It may be possible to use a deterministic method to calculate the expected values, or possibly the median values, for a complicated problem, where the distributions around these central values are estimated by simulation.

One also needs to recognise that a simulation method generally provides 'what if?' answers; what the results are on the basis of the assumptions that have been made. It is much harder to use simulation to provide the optimum solution; in other words to find the set of assumptions that maximises or minimises some desired result.

A further limitation is that the precision of a simulated result depends on the number of simulations that are carried out. This is covered in more detail in CS1.

4 Discrete and continuous state spaces and time sets

The state of a model is the set of variables that describe the system at a particular point in time taking into account the goals of the study. It is possible to represent any future scenarios as states, as will be developed in later chapters.

Consider an insurance policy that will provide an income if the policyholder is sick and unable to work. We might model the future state of the policyholder as:

- healthy (*ie* not receiving a benefit)
- sick (*ie* receiving a benefit)
- dead.

Discrete states are where the variables exhibit step function changes in time. For example, from a state of alive to dead, or an increase in the number of policies for an insurer. Continuous states are where the variables change continuously with respect to time. For example, real time changes in values of investments.

The decision to use a discrete or continuous state model for a particular system is driven by the objectives of the study, rather than whether or not the system itself is of a discrete or continuous nature.

A model may also consider time in a discrete or a continuous way. This may reflect the fact that outputs from the model are only required at discrete points in time, or may be to satisfy the objectives of the modelling.

For example, an insurance company may wish to model the running total of the number of claims it has received since the start of the year (*ie* in continuous time), or it may simply model the number of claims it has received at the end of each month (*ie* in discrete time).

One cannot in practice use Monte Carlo simulation for a continuous time problem; one has to discretise the time step. This can be done with whatever precision one likes, but the higher the precision the longer the time taken to process any particular model. This may or may not be a limitation in practice. And it should be remembered that some results for continuous time, continuous space models cannot be obtained by discrete simulation at all.

Processes with continuous state spaces are discussed in Subjects CS2 and CM2.

5 Scenario-based and proxy models

Most models depend on many input parameters. A *scenario-based* model would take into consideration a particular scenario; that is a series of input parameters based on this scenario. Different scenarios would be useful in decision analysis as one can evaluate the expected impact of a course of action.

For example, we could model the financial performance of a company under different future scenarios, such as 'recession' or 'economic boom'. The values of any input parameters (*eg* rate of inflation, interest rate, level of taxation) would be selected to be appropriate to the specific scenario under consideration.

The projected valuation of assets and liabilities for an insurer can be very complex, requiring significant runs of Monte Carlo simulation. Therefore, *proxy* models are used to replace Monte Carlo simulations. These are expected to be faster but less accurate.

For example, a Monte Carlo simulation evaluating the total claims paid each year of a car insurer may depend on the number of cars insured and written premium (this is too simplistic for a real-life example). A Monte-Carlo simulation would provide a range of results, including the mean for different input scenarios. However, given a range of inputs and outputs, a regression function can be set up based on these two parameters and their derivatives (such as squared value). This would then act as a proxy model for the actual expected total amount of claims paid for other intermediate values.

Performing a Monte Carlo simulation is a time-consuming process. If it is possible to develop a simplified formula, based on the same inputs, that we believe predicts the result with reasonable accuracy, we may use this instead to save time. The simplified formula is then used as a substitute (or proxy) for running the full model.

So, using the example given above, if N denotes the number of cars insured and P denotes the written premium, then a proxy model may take the form:

$$\text{Total claims paid each year} = a + bN + cP + dP^2$$

where a, b, c and d are constants.

6 Suitability of a model

In assessing the suitability of a model for a particular exercise it is important to consider the following:

- **The objectives of the modelling exercise.**
- **The validity of the model for the purpose to which it is to be put.**
- **The validity of the data to be used.**
- **The validity of the assumptions.**
- **The possible errors associated with the model or parameters used not being a perfect representation of the real-world situation being modelled.**
- **The impact of correlations between the random variables that 'drive' the model.**
- **The extent of correlations between the various results produced from the model.**
- **The current relevance of models written and used in the past.**
- **The credibility of the data input.**
- **The credibility of the results output.**
- **The dangers of spurious accuracy.**
- **The ease with which the model and its results can be communicated.**
- **Regulatory requirements.**

The important actuarial/investment concept of 'matching' of assets and liabilities relies on the fact that the value of the matched assets and liabilities will tend to move together, *ie* they are positively correlated. In models of such situations, it would be essential to incorporate this correlation.

Similarly, when modelling the value of a portfolio invested in a variety of different shares, it would be important to allow for the fact that the different share prices are likely to be positively correlated. This is because there is a tendency for the prices of different shares to move together in response to general factors affecting the whole market, *eg* changes in interest rates.

An example of 'spurious accuracy' is the statement: 'The value of our company pension fund's assets is £46,279,312.86.' This is spurious accuracy because the market value of the investments will change by the minute and this level of accuracy cannot be justified. A more appropriate figure to give is £46.3 million.

7 Short-term and long-term properties of a model

The stability of the relationships incorporated in the model may not be realistic in the longer term. For example, exponential growth can appear linear if surveyed over a short period of time. If changes can be predicted, they can be incorporated in the model, but often it must be accepted that longer-term models are suspect.

Models are, by definition, simplified versions of the real world. They may, therefore, ignore 'higher order' relationships which are of little importance in the short term, but which may accumulate in the longer term.

There is an analogy here with series. For small values of time t, the exponential function e^t appears linear, since:

$$e^t = 1 + t + o(t)$$

But if t is greater, we need to include the higher order terms:

$$e^t = 1 + t + \tfrac{1}{2}t^2 + \cdots$$

8 Analysing the output of a model

Statistical sampling techniques are needed to analyse the output of a model, as a simulation is just a computer-aided statistical sampling project. The actuary must exercise great care and judgement at this stage of the modelling process as the observations in the process are correlated with each other and the distributions of the successive observations change over time. The useless and fatally attractive temptation of assuming that the observations are independent and identically distributed is to be avoided.

If there is a real-world system against which results can be compared, a 'Turing' test should be used. In a Turing test, experts on the real-world system are asked to compare several sets of real-world and model data without being told which are which. If these experts can differentiate between the real-world and model data, their techniques for doing so could be used to improve the model.

This is an extension of the original meaning of a Turing test named after the British mathematician and early computer pioneer, Alan Turing. The original Turing test relates to one objective of artificial intelligence researchers, which is to write a computer program that cannot be distinguished from a real person by a user asking questions over a computer link.

9 Sensitivity testing

Where possible, it is important to test the reasonableness of the output from the model against the real world. To do this, an examination of the sensitivity of the outputs to small changes in the inputs or their statistical distributions should be carried out. The appropriateness of the model should then be reviewed, particularly if small changes in inputs or their statistical distributions give rise to large changes in the outputs. In this way, the key inputs and relationships to which particular attention should be given in designing and using the model can be determined.

If small changes in the inputs give rise to large changes in the outputs, then our initial choices are more crucial. How confident are we in our choices of input? If the resulting changes in output are small, then our initial choices are less important in this respect.

Sensitivity testing also refers to the approach of using a deterministic model with changes in one or more of the assumptions to see the range of possible outcomes that might occur. For example, an insurance company providing personal pension plans might give illustrations to policyholders showing how much pension they would get if growth rates were 2%, 5% and 8% *pa* in the period before retirement. This allows the policyholder to gauge the extent to which the resulting pension will be affected by changes in the growth rate.

The model should be tested by designing appropriate simulation experiments. It is through this process that the model can be refined.

An approach that has been traditionally used by actuaries in the fields of insurance, pensions and investment is to carry out a set of deterministic calculations using different actuarial bases, *ie* under different sets of assumptions. By varying the assumptions, the actuary could use the model to arrive at figures that are 'best estimates' (the most likely, or median, result) or 'optimistic' (if things work out favourably) or 'cautious' (if things work out badly). This is an example of a scenario-based approach to modelling.

10 Communication of the results

The final step in the modelling process is the communication and documentation of the results and the model itself to others. The communication must be such that it takes account of the knowledge of the target audience and their viewpoint. A key issue here is to make sure that the client accepts the model as being valid and a useful tool in decision making. It is important to ensure that any limitations on the use and validity of the model are fully appreciated.

The following example illustrates one possible limitation of a model that would need to be explained to a client.

Example

An actuarial student working at a consultancy is asked by a client how expensive it would be for the client's company pension scheme to offer more generous benefits to members who leave the company before reaching retirement age.

As the company has historically had a very stable workforce with very little turnover of staff, the current model assumes that everyone stays with the company until retirement age. So the client's question could not be answered using the existing model.

To answer the client's question, the model could be extended to allow for an appropriate rate of withdrawal of staff before retirement age. This amended model could then be used to compare the costs of the scheme with the existing withdrawal benefit with the costs of the 'improved' one.

Chapter 1 Summary

A model is an imitation of a real-world system or process.

It enables possible consequences to be investigated without carrying out the actions themselves.

There are benefits to modelling, such as the possibility of looking at long-term phenomena in an accelerated time frame, but there are also limitations that must be considered such as the time and expertise required to develop and run a model.

The following 14 key steps can be considered in the construction and use of a model:

1. Develop a well-defined set of objectives that need to be met by the modelling process.
2. Plan the modelling process and how the model will be validated.
3. Collect and analyse the necessary data for the model.
4. Define the parameters for the model and consider appropriate parameter values.
5. Define the model initially by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage.
6. Involve the experts on the real-world system you are trying to model in order to get feedback on the validity of the conceptual model.
7. Decide on whether a simulation package or general-purpose language is appropriate for the implementation of the model. If necessary, choose a statistically reliable random number generator that will perform adequately in the context and complexity of the model.
8. Write the computer program for the model.
9. Debug the program to make sure it performs the intended operations in the model definition.
10. Test the reasonableness of the output from the model.
11. Review and carefully consider the appropriateness of the model in the light of small changes to the input parameters.
12. Analyse the output from the model.
13. Ensure that any relevant professional guidance has been complied with. For example, the Financial Reporting Council has issued a Technical Actuarial Standard on the principles for Technical Actuarial Work (TAS100), which includes principles for models used in technical actuarial work.
14. Communicate and document the results and the model.

When building a model, the suitability of the model to the objectives should be borne in mind. Some mathematical or logical assumptions about the workings of the real-world system must be made. Input parameters need to be chosen, possibly by statistical analysis of past data. Sensitivity analysis of the dependence of the output on these parameters can be carried out.

A model may be stochastic or deterministic. For stochastic models it is better to use direct calculation if possible, but in complex situations it may be necessary to use Monte Carlo simulation on a computer.

Models may also be constructed in discrete or continuous time and with discrete or continuous state spaces. Monte Carlo simulations have to be run in discrete time.

A scenario-based model would take into consideration a particular scenario, with the values of any input parameters being selected to be appropriate to that specific scenario.

A proxy model may be used to replace Monte Carlo simulation, providing faster but less accurate results. A simplified formula is developed that we believe predicts the result with reasonable accuracy, and this is then used as a substitute for running the full model.

The output from any model needs to be analysed. This can be done using the idea of a Turing test – can an expert tell the difference between a set of simulated outputs and actual occurrences?

The results need to be communicated to other people. The key question when framing the level of communication is 'to whom?'

Chapter 1 Practice Questions

1.1 Explain what is meant by a 'stochastic model' and state two advantages these have over deterministic models.

1.2 Describe the role of simulation in sensitivity analysis.

1.3 The government of a small island state intends to set up a model to analyse the mortality of the island's population over the past 50 years.

Exam style

Describe the process that would be followed to carry out the analysis. [6]

1.4 (i) List the advantages and disadvantages of using models in actuarial work. [4]

Exam style

A new town is planned in a currently rural area. A model is to be developed to recommend the number and size of schools required in the new town. The proposed modelling approach is as follows:

- The current age distribution of the population in the area is multiplied by the planned population of the new town to produce an initial population distribution.
- Current national fertility and mortality rates by age are used to estimate births and deaths.
- The births and deaths are applied to the initial population distribution to generate a projected distribution of the town's population by age for each future year, and hence the number of school age children.

(ii) Discuss the appropriateness of the proposed modelling approach. [5]
[Total 9]

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 1 Solutions

1.1 A stochastic model is one that recognises the random nature of the input components.

Two advantages of stochastic models over deterministic models are:

1. To reflect reality as accurately as possible, the model should imitate the random nature of the variables involved.
2. A stochastic model can provide information about the distribution of the results (*eg* probabilities, variances *etc*), not just a single best estimate figure.

1.2 Sensitivity analysis involves testing the robustness of the model by making small changes to the input parameters. This should result in small changes to the output from the model that are consistent with the real-world behaviour of the situation we are modelling.

The usual method of carrying out a sensitivity analysis is to run a large number of computer simulations based on the original parameter values, then to repeat these using several sets of slightly different parameter values.

For consistency, the same set of pseudo-random numbers should be used for each set of simulations. This removes the effect of the additional source of randomness that would be introduced by using different pseudo-random numbers.

1.3 *This question is Subject CT4, October 2010, Question 3.*

The process that would be followed to carry out the analysis is as follows:

1. Develop a well-defined set of objectives that need to be met by the modelling process, *ie* state the point of the exercise. In this case we should think about what aspects of mortality are to be analysed, *eg* average mortality rates, split of males/females, trends in mortality over the last 50 years.
2. Plan the modelling process and how the model will be validated.
3. Collect and analyse the necessary data for the model. In this case we would need the numbers of deaths over the last 50 years and any available census data. Problems may arise as some of the data may be missing or inaccurate, and recording practices may have changed over the last 50 years.
4. Define the parameters for the model and consider appropriate parameter values.
5. Define the model firstly by capturing the essence of the real-world system. Refining the level of detail in the model can come at a later stage. For this model, this means we should identify the main features of mortality.
6. Involve people with expert knowledge of the real-world system you are trying to imitate so as to get feedback on the validity of the conceptual model. For example, there may be a national census office or government department that can help.
7. Decide on whether a simulation package or a general-purpose language is appropriate for the implementation of the model.
8. Write the computer program for the model.

9. Debug the program to make sure it performs the intended operations defined in the initial modelling process.

10. Test the reasonableness of the output from the model. For example, we could check that it is a good fit to the actual mortality experience of the island over subsets of the last 50 years.

11. Review and carefully consider the appropriateness of the model in the light of small changes in input parameters.

12. Analyse the output from the model.

13. Ensure that any relevant professional guidance has been complied with. This may include standards on data, modelling and reporting.

14. Document the results of the model and communicate these to the government.

[½ for each point, up to a maximum of 6]

1.4 *This question is Subject CT4, April 2012, Question 6.*

(i) ***Advantages / disadvantages of using models in actuarial work***

Advantages

Models allow us to investigate the future behaviour of a process in compressed time.

Models allow us to study the stochastic nature of the results, *eg* using Monte Carlo simulation.

Models allow us to compare a large number of different scenarios (in a short space of time) to determine the best strategy.

Models allow us to consider scenarios that would not be feasible in practice, *eg* because of cost or other business considerations.

Models allow control over experimental conditions, so that we can reduce the variance of the results output, without affecting the mean values.

[½ for each advantage, up to a maximum of 2]

Disadvantages

Models can be time-consuming and expensive to set up.

Stochastic models require a large number of simulations to be carried out to get accurate results.

In general, models are more useful for comparing the results of input variations than for optimising outputs.

Models can give an impression of greater accuracy and reliability than they actually have, and so may create a sense of false security.

The results from models are dependent on the data used, which may be inaccurate or unreliable.

Users of the model may not understand fully how it works and its limitations.

Models may not capture sufficiently accurately the real-world situation.

Models require simplifying assumptions that might turn out to be wrong, *eg* they may ignore certain features (*eg* inflation) or certain types of events that could occur (*eg* catastrophes).

Some models may be difficult to explain to clients.

[½ for each disadvantage, up to a maximum of 2]
[Total 4]

(ii) ***Appropriateness of the proposed modelling approach***

The model uses an established method that is straightforward to apply, understand and explain to the town planners. [1]

If the rural area is in a developed country, mortality will probably have little effect (since the mortality of children and those of child-bearing age will be relatively low) and so the model could be simplified by ignoring this factor. [1]

The results will probably be accurate over the short term (*ie* for the next few years), but may become less reliable if applied over longer periods. [½]

The model makes no allowance for migration, *ie* people moving in or out of the town. This could be an important factor that could increase or reduce the size of the population. [½]

It should be possible to obtain sufficiently accurate data about the current age distribution. [½]

However, the model assumes that the initial population profile of the town will be the same as for the rural area in which it is located, which may not be true for a new town. [1]

The initial population profile will depend on the type of housing (*eg* family homes, bungalows for the elderly) and the type of employment opportunities available in the town. [1]

It should be possible to obtain reliable estimates for fertility rates and mortality rates, but these may change in the future, *eg* people may have fewer children during an economic recession. [½]

Also the fertility rates may be different in different areas, so that the national rates would not be appropriate. [½]

There could be changes to government education policy that would affect the number of school places required. For example, the ages when school attendance is compulsory could be changed or new types of school could be introduced (*eg* boarding schools, where children are schooled outside the area). [½]

[Maximum 5]

2

Cashflow models

Syllabus objectives

1.2 Describe how to use a generalised cashflow model to describe financial transactions.

1.2.1 State the inflows and outflows in each future time period, and discuss whether the amount or the timing (or both) is fixed or uncertain for a given cashflow process.

1.2.2 Describe in the form of a cashflow model the operation of financial instruments (like a zero-coupon bond, a fixed-interest security, an index-linked security, a current account, cash on deposit, a credit card, an equity, an interest-only loan, a repayment loan and an annuity-certain) and an insurance contract (like endowment, term assurance, contingent annuity, car insurance and health cash plans).

0 Introduction

A cashflow model is a mathematical projection of the payments arising from a financial transaction, *eg* a loan, a share or a capital project. Payments received are referred to as *income* and are shown as *positive* cashflows. Payments made are referred to as *outgo* and are shown as *negative* cashflows. The difference at a single point in time (income less outgo) is called the *net cashflow* at that point in time.

This chapter considers the cashflows that emerge in a number of practical situations that you will come across in the actuarial field.

1 Cashflow process

The practical work of the actuary often involves the management of various *cashflows*. These are simply sums of money, which are paid or received at different times. The timing of the cashflows may be known or uncertain. The amount of the individual cashflows may also be known or unknown in advance.

For example, a company operating a privately owned bridge, road or tunnel will receive toll payments. The company will pay out money for maintenance, debt repayment and for other management expenses. From the company's viewpoint the toll payments are positive cashflows (*ie* money received) while the maintenance, debt repayments and other expenses are negative cashflows (*ie* money paid out). Similar cashflows arise in all businesses.

From a theoretical viewpoint one may also consider a *continuously* payable cashflow.

Continuously payable cashflows are often used when payments are made very frequently, *eg* daily or weekly. This is because the mathematics is sometimes easier if we assume that the payments are made continuously rather that at regular intervals. This will become clearer when this mathematics is introduced to you later in the course.

Question

Outline some cashflows, both positive and negative, that will occur in the next month where one of the parties involved in the cashflow is (i) you, (ii) your employer.

Solution

(i) ***Cashflows involving you***

Positive: receiving your salary, borrowing some money from a friend.

Negative: repaying some borrowed money, buying something from a shop using cash.

(ii) ***Cashflows involving your employer***

Positive: receiving payments for supplying products or services, *eg* receiving premiums.

Negative: paying salaries, paying property expenses, *eg* electricity bills.

In some businesses, such as insurance companies, investment income will be received in relation to positive cashflows (premiums) received before the negative cashflows (claims and expenses).

For example, consider a premium received by an insurance company from a policyholder. Some of the premium might be used to cover the costs associated with setting up the insurance policy. The remainder of the premium could be put into a bank account. Investment income, in this case interest, will be earned on the money in the bank account until the money is needed for further expenses or payments back to the policyholder.

Where there is uncertainty about the amount or timing of cashflows, an actuary can assign probabilities to both the amount and the existence of a cashflow.

The amount and timing of some cashflows will be known with great certainty. For example, an employed person who gets paid on the last Friday of every month will be almost certain to receive a payment on the last Friday of this month. The amount of the payment is also likely to be known.

Question

List reasons why this person is not *completely certain* of receiving a payment on the last Friday of this month.

Solution

The payment might not be made if:

- the employer makes an error or goes bankrupt
- the bank makes an error and doesn't make the payment on the due date
- the person leaves the company before the end of the month
- the last Friday is a public holiday and other arrangements are in place.

Other cashflows are not so certain. If you buy a lottery ticket every week, you don't know when, or if, you will win or how much you may win.

The probability that the payment will take place could be estimated by looking at past results. If there is no past information relating to the event being considered, then data from similar events would be used.

2 Examples of cashflow scenarios

In this section, we provide examples of practical situations with cashflows that are assumed to be certain. In reality, this may not be the case, as the counterparty of a particular cashflow may not be able to pay out. For example, a company may fail and not be able to pay out interest on issued bonds.

If a company fails to pay the interest it owes on its bonds, this is called a default event.

The first few examples considered below are types of security or investment. A security is a tradeable financial instrument, *ie* a financial contract that can be bought and sold.

2.1 A zero-coupon bond

The term *zero-coupon bond* is used to describe a security that is simply a contract to provide a specified lump sum at some specified future date. For the investor there is a negative cashflow at the point of investment and a single known positive cashflow on the specified future date.

For example, the investor may give the issuer of the zero-coupon bond £400,000, and in return the investor will receive £500,000 from the issuer in exactly 5 years' time. The issuer may be a government or a large company.

The positive cashflow is paid on a set future date and is of a set amount, but, as mentioned above, it is not certain that the payment will be made. There is a chance that the issuing organisation will not make the payment, *ie* that it will default. This risk is usually negligible for bonds issued by governments of developed countries, since the government can always raise taxes. The risk of default is greater for issuing organisations that may go bust, *eg* companies.

A zero-coupon bond is a form of loan from the investor to the issuer. The loan is repaid by one single payment of a fixed amount at a fixed date in the future. It is a special case of a fixed-interest security with no interest payments before redemption. We will study fixed-interest securities in the next section.

We can plot the cashflows of the investor on a timeline:

Cashflows	– £400,000	+£500,000
Time	0	5

For the issuing organisation, there is a positive cashflow at the point of investment and a single known negative cashflow on a specified future date. These cashflows are the opposite of those experienced by the investor, and can be shown on a timeline as:

Cashflows	£400,000	– £500,000
Time	0	5

The investor may also be referred to as the lender, and the issuer may be referred to as the borrower.

2.2 A fixed-interest security

A body such as an industrial company, a local authority, or the government of a country may raise money by floating a loan on the stock exchange.

This means that the organisation borrows money by issuing a loan to investors. The loan is simultaneously listed on the stock exchange so that after issue the securities can be traded on the stock exchange. This means that investors can sell their right to receive the future cashflows.

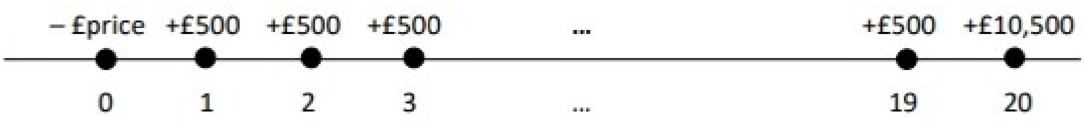
In many instances such a loan takes the form of a *fixed-interest security*, which is issued in bonds of a stated nominal amount. The characteristic feature of such a security in its simplest form is that the holder of a bond will receive a lump sum of specified amount at some specified future time together with a series of regular level interest payments until the repayment (or 'redemption') of the lump sum.

The regular level interest payments are referred to as coupons. Thus a zero-coupon bond has no interest payments.

The investor has an initial negative cashflow, a single known positive cashflow on the specified future date, and a series of smaller known positive cashflows on a regular set of specified future dates.

An investor might buy a 20-year fixed-interest security of nominal amount £10,000. This means that the face value of the loan is £10,000. The investor is unlikely to pay exactly £10,000 for this security but will pay a price that is acceptable to both parties. This may be higher or lower than £10,000. The investor will then receive a lump sum payment in 20 years' time. This lump sum is most commonly equal to the nominal amount, in this case £10,000, but could be a pre-specified amount higher or lower than this. The investor will also receive regular payments throughout the 20 years of, say, £500 *pa*. These regular payments could be made at the end of each year or half-year or at different intervals.

In the case where the regular payments are made at the end of each year, the cashflows of the investor can be represented on a timeline, as follows:



The last payment is made up of the final regular payment (£500) and the lump sum payment (£10,000).

2.3 An index-linked security

Inflation is a measure of the rate of change of the price of goods and services, including salaries. High inflation implies that prices are rising quickly and low inflation implies that prices are rising slowly.

If a pair of socks costs £1, then £5 could be used to buy 5 pairs of socks. However, if inflation is high, then the cost of socks in 1 year's time might be £1.25. £5 would then only buy 4 pairs of socks.

This simple example shows how the 'purchasing power' of a given sum of money, *ie* the quantity of goods that can be bought with the money, can diminish if inflation is high. In the socks example, the annual rate of inflation is 25%.

With a conventional fixed-interest security, the interest payments are all of the same amount. If inflationary pressures in the economy are not kept under control, the purchasing power of a given sum of money diminishes with the passage of time, significantly so when the rate of inflation is high. For this reason some investors are attracted by a security for which the actual cash amount of interest payments and of the final capital repayment are linked to an 'index' which reflects the effects of inflation.

Here the initial negative cashflow is followed by a series of unknown positive cashflows and a single larger unknown positive cashflow, all on specified dates. However, it is known that the amounts of the future cashflows relate to the inflation index. Hence these cashflows are said to be known in 'real' terms.

Real terms means taking into account inflation, whereas *nominal* (or *money*) *terms* means ignoring inflation. For example, if your salary is rising at 5% *pa* and inflation is 7% *pa*, your salary is *falling* in real terms (as you will be able to buy less with your 'higher salary'), even though your salary is *rising* in nominal (or money) terms.

So for an index-linked bond, the cashflows are known in real terms, but are unknown in nominal (or money) terms. For a fixed-interest bond, on the other hand, the cashflows are known in nominal (or money) terms, but are unknown in real terms.

Both the regular interest payments and the final capital payment from an index-linked security are linked to the inflation index. If inflation is high, then the regular payments will rise by larger amounts than if inflation is low.

If inflation is 10% per time period and the regular interest payment after one time period is £500, then the payment after two time periods will be £550 ($= 500 \times 1.1$), and the payment after three time periods will be £605 ($= 500 \times 1.1^2$) *etc*.

Inflation is often measured by reference to an index. For example an inflation index might take values as set out in the table below.

Date	1.1.2015	1.1.2016	1.1.2017	1.1.2018
Index	100.00	105.00	108.00	113.00

Based on this, the rate of inflation during 2016 is 2.86% *pa* (*ie* $108/105 - 1$).

Question

An investor purchased a three-year index-linked security on 1.1.2015. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

Solution

The payments received by the investor are calculated in the table below:

Payment date	Payment amount
1.1.2016	$600\times\frac{105}{100}=£630$
1.1.2017	$600\times\frac{108}{100}=£648$
1.1.2018	$(11{,}000+600)\times\frac{113}{100}=£13{,}108$

Note that in practice the operation of an index-linked security will be such that the cashflows do not relate to the inflation index at the time of payment, due to delays in calculating the index. It is also possible that the need of the borrower (or perhaps the investors) to know the amounts of the payments in advance may lead to the use of an index from an earlier period.

Question

An investor purchased a two-year index-linked security on 1.1.2016. In return, the investor received payments at the end of each year plus a final redemption payment, all of which were increased in line with the index given in the table above, with a one-year indexation lag, *ie* the index value one year before each payment is used. The payments would have been £600 each year and £11,000 on redemption if there had been no inflation.

Calculate the payments actually received by the investor.

Solution

The payments received by the investor are calculated in the table below:

Payment date	Payment amount
1.1.2017	$600\times\frac{105}{100}=£630$
1.1.2018	$(11{,}000+600)\times\frac{108}{100}=£12{,}528$

The one-year indexation lag means that the payment on 1 January 2017 is calculated using the index values on 1 January 2015 (one year before the date of issue of the bond) and on 1 January 2016 (one year before the date of payment).

2.4 Cash on deposit

If cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider. These additions may only be known on a day-to-day basis. The amounts and timing of cashflows will therefore be unknown.

This is describing a bank account that pays interest and allows instant access. Consider your own bank account. You can choose when to invest money, *ie* pay money in, and when to disinvest money, *ie* withdraw money. The interest you receive on your money will depend on the current interest rate and this may change with little or no notice.

2.5 An equity

Equity shares (also known as *shares* or *equities* in the UK and as *common stock* in the USA) are securities that are held by the owners of an organisation. Equity shareholders own the company that issued the shares. For example, if a company issues 4,000 shares and an investor buys 1,000, the investor owns 25% of the company. In a small company all the equity shares may be held by a few individuals or institutions. In a large organisation there may be many thousands of shareholders.

Equity shares do not earn a fixed rate of interest as fixed-interest securities do. Instead the shareholders are entitled to a share in the company's profits, in proportion to the number of shares owned.

The distribution of profits to shareholders takes the form of regular payments of *dividends*. Since they are related to the company profits that are not known in advance, dividend rates are variable. It is expected that company profits will increase over time, and also, therefore, expected that dividends per share will increase – though there are likely to be fluctuations. This means that, in order to construct a cashflow schedule for an equity, it is necessary first to make an assumption about the growth of future dividends. It also means that the entries in the cashflow schedule are uncertain – they are estimates rather than known quantities.

In practice, the relationship between dividends and profits is not a simple one. Companies will, from time to time, need to hold back some profits to provide funds for new projects or expansion. They may also hold back profits in good years to subsidise dividends in years with poorer profits. Additionally, companies may be able to distribute profits in a manner other than dividends, such as by buying back the shares issued to some investors.

Share buy-backs will result in some investors having to sell their shares back to the company. The remaining shareholders will subsequently own a greater percentage of the company and should expect greater future profits.

The following table shows the projected future cashflows for a shareholder who has just purchased a block of shares for £6,000 and expects the dividends in each year to be 5% higher than the corresponding amounts in the previous year. Dividends are paid twice yearly. This shareholder expects the two dividends in the first year to be £100 each and intends to sell all the shares after 2 years.

In the table, time is measured from the date of purchase.

Time (years)	Purchase price (£)	Dividends (£)	Sale proceeds (£)
0	– 6,000		
½		+100	
1		+100	
1½		+105	
2		+105	+6,615

In this example we have assumed that the price also grows at 5% *pa* to calculate the sale proceeds.

Since equities do not have a fixed redemption date, but can be held in perpetuity, we may assume that dividends continue indefinitely (unless the investor sells the shares or the company buys them back), but it is important to bear in mind the risk that the company will fail, in which case the dividend income will cease and the shareholders would only be entitled to any assets which remain after creditors are paid. The future positive cashflows for the investor are therefore uncertain in amount and may even be lower, in total, than the initial negative cashflow.

'In perpetuity' means that the payments continue forever.

2.6 An annuity-certain

An *annuity-certain* provides a series of regular payments in return for a single premium (*ie* a lump sum) paid at the outset. The precise conditions under which the annuity payments will be made will be clearly specified. In particular, the number of years for which the annuity is payable, and the frequency of payment, will be specified. Also, the payment amounts may be level or might be specified to vary – for example in line with an inflation index, or at a constant rate.

The cashflows for the investor will be an initial negative cashflow followed by a series of smaller regular positive cashflows throughout the specified term of payment. In the case of level annuity payments, the cashflows are similar to those for a fixed-interest security.

However there will not be a redemption payment as there normally is for a fixed-interest security.

From the perspective of the annuity provider, there is an initial positive cashflow followed by a known number of regular negative cashflows.

A key characteristic of an annuity-certain is that a known number of payments will be made, *eg* payments of £1,000 at the end of each of the next 10 years.

The theory can be extended to deal with annuities where the payment term is uncertain, that is, for which payments are made only so long as the annuity policyholder survives – see Section 3.4 below.

An example of this is a *whole life annuity* where a policyholder receives a fixed amount of money per month until they die. In more common language, this type of policy is called a pension.

2.7 A credit card

Credit cards allow flexible borrowing, generally known as revolving credit. Credit card holders can spend up to their agreed credit limit and must pay back a minimum amount each month. Credit card holders are divided between 'transactors', who repay in full each month, and 'revolvers' who take advantage of flexible borrowing and repayments.

Credit cards charge interest on amounts borrowed and fees for late payments and other services. Many 'transactors' pay no charges on their credit cards.

Cashflows on credit cards are uncertain and are hard to model because they depend on customer behaviour which can change over time. For example, customers who are getting into financial difficulties may increase their borrowing up to their agreed credit limit and then default.

2.8 A current account

Current accounts are 'bundled' products which allow both savings and borrowing (through overdrafts) and enable payments by various methods including cash withdrawals, debit cards, direct debits, online and mobile payments and cheques.

Current accounts typically charge interest on overdrafts and various fees, which may include regular monthly charges and/or fees for certain transactions. In the UK, banks typically do not make monthly charges (except for packaged current accounts with loyalty benefits) and many current account customers enjoy 'free-if-in-credit' banking.

Indeed, any savings within a current account may accrue interest and so current accounts and cash on deposit are very similar. The main difference is the ability to borrow through a current account.

As for credit cards, cashflows on current accounts are uncertain and, because they depend on customer behaviour, are hard to model.

2.9 An 'interest-only' loan

An 'interest-only' loan is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.

The regular repayments only cover the interest owed, so the full capital amount borrowed remains outstanding throughout the term of the loan.

In the simplest of cases, the cashflows are the reverse of those for a fixed-interest security. The provider of the loan effectively buys a fixed-interest security from the borrower.

In practice, however, the interest rate need not be fixed in advance. The regular cashflows may therefore be of unknown amounts.

It may also be possible for the loan to be repaid early. The number of cashflows and the timing of the final cashflows may therefore be uncertain.

2.10 A repayment loan (or mortgage)

A repayment loan is a loan that is repayable by a series of payments that include partial repayment of the loan capital in addition to the interest payments.

In its simplest form, the interest rate will be fixed and the payments will be of fixed equal amounts, paid at regular known times.

The cashflows are similar to those for an annuity-certain.

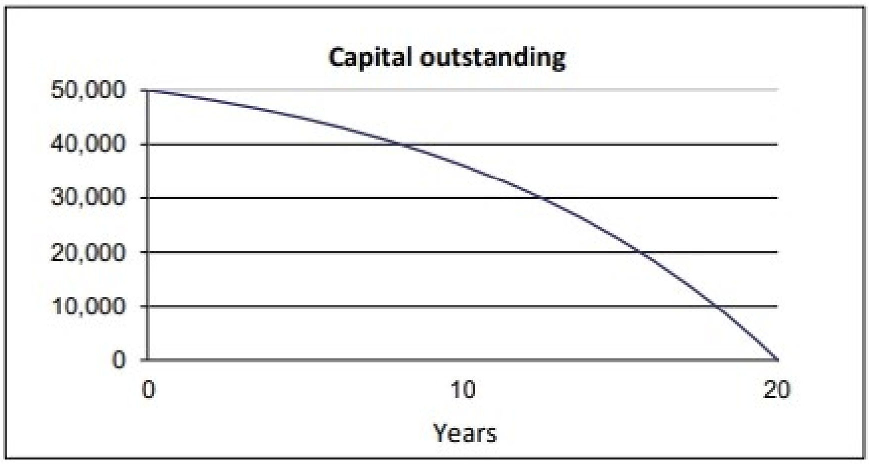
As for the 'interest-only' loan, complications may be added by allowing the interest rate to vary or the loan to be repaid early. Additionally, it is possible that the regular repayments could be specified to increase (or decrease) with time. Such changes could be smooth or discrete.

Each payment can be considered to consist of an interest payment and a capital repayment. The interest payment covers the interest that will be charged over the period since the previous payment. The interest payment will be calculated by reference to the amount of the loan outstanding just after the last payment. The remainder of each payment is the capital repayment, which is used to reduce the amount of the loan outstanding.

It is important to appreciate that with a repayment loan the breakdown of each payment into 'interest' and 'capital' changes significantly over the period of the loan. The first repayment will consist almost entirely of interest and will provide only a very small capital repayment. In contrast, the final repayment will consist almost entirely of capital and will have a small interest content.

This is because the amount of the loan outstanding will reduce throughout the term of the loan. At the start of the contract, the entire loan will be outstanding and so the interest portion of each payment will be large. The remainder, the capital portion, will therefore be relatively small. At the end of the contract, the amount of the loan outstanding will be small and so the interest due will also be small. The capital repayment will then be much larger.

The diagram below shows how the capital outstanding reduces for a repayment loan – relatively slowly at first, but then more quickly towards the end of the term of the loan. The graph is based on a loan of £50,000 being repaid by monthly instalments over 20 years.



We'll look at numerical examples of repayment loans in Chapter 10.

3 Insurance contracts

In the previous section, we looked at the cashflows arising from different financial securities. We now turn our attention to the cashflows arising from different policies sold by insurance companies.

The cashflows for the examples covered in this section differ from those in the previous section in that the frequency, severity, and/or timing of the cashflows may be unknown. For example, a typical life cover may have a specified date on which a pre-agreed amount is paid on survival (Section 3.1) – but the benefit payment may not be paid if the individual does not survive. Similarly a pension pays out a known amount at a specified time per month, but only if the individual is alive (Section 3.4). Typically the severity is known and pre-specified in life insurance contracts.

Here, 'severity' is referring to the benefit or claim amount paid out by the insurer.

On the other hand, a non-life (general) insurance cover tends not to have known severities. For example, the cost of a car accident may range from a few pounds in the case of a small collision to millions in the case of a major accident that caused death.

A life insurer sells policies related to events that might be experienced by an individual in the course of their future life, *eg* policies may provide a payment if an individual survives, dies or falls sick. A general insurer sells policies related to specific items, *eg* car insurance, home insurance.

3.1 A pure endowment

A pure endowment is an insurance policy which provides a lump sum benefit on survival to the end of a specified term usually in return for a series of regular premiums.

For example, a payment of £50,000 is made if a life now aged 30 survives to age 60, but no payment is made if this life dies before age 60.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier. A large, positive cashflow occurs at the end of the term, only if the policyholder has survived. If the policyholder dies before the end of the term there is no positive cashflow.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder survival.

The cashflows experienced by the insurer are the exact opposite to those experienced by the policyholder.

In some cases a lump sum premium is paid. In this case, the cashflows for the policyholder will be a negative cashflow at inception and, if the policyholder has survived, a positive cashflow at the end of the term.

In general, insurance policies can be paid for by either:

- a one-off, lump sum premium payable at outset, or
- regular premiums payable during the term of the contract, while the policyholder is alive.

3.2 An endowment assurance

An endowment assurance is similar to a pure endowment **in that it provides a survival benefit at the end of the term, but it also provides a lump sum benefit on death before the end of the term. The benefits are provided in return for a series of regular premiums.**

So, under an endowment assurance, a payment is made whether the policyholder dies during the term or survives to the end of the term.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier, followed by a large positive cashflow at the end of the term (or death, if earlier). Depending on the terms of the policy, the amount payable on death may not be the same as that payable on survival.

For example, the payment may be £10,000 if the policyholder dies during the term, and £20,000 if the policyholder survives to the end of the term.

From the perspective of the insurer, there is a stream of regular positive cashflows which ceases at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow. The negative cashflow is certain to be paid, but the timing of that payment depends on whether/when the policyholder dies.

The proceeds from an endowment assurance could be used to repay the loan amount under an 'interest-only' loan, by ensuring that the payment made from the endowment assurance matches the loan amount. In this way, the loan will be repaid whether the policyholder dies during the term or survives to the end of the term.

3.3 A term assurance

A term assurance is an insurance policy which provides a lump sum benefit on death before the end of a specified term usually in return for a series of regular premiums.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier, (or one negative cashflow at inception if paid on a lump-sum basis), followed by a large positive cashflow payable on death, if death occurs before the end of the term. If the policyholder survives to the end of the term, there is no positive cashflow.

If a policyholder purchases a term assurance and a pure endowment, this is equivalent to purchasing an endowment assurance (as a payment will be made from the term assurance if the policyholder dies, or from the pure endowment if the policyholder survives).

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder death during the term.

Generally, the negative cashflow (death benefit), if it occurs, is significantly higher than the positive cashflow (premiums), when compared to, say, a pure endowment. This is because, for each individual policy, the probability of the benefit being paid is generally lower than for endowments, because it is contingent on death, rather than on survival.

Question

Consider a 10-year term assurance and a 10-year pure endowment, both providing a lump sum benefit of £50,000.

Explain which policy would have higher premiums, if both policies are sold to an individual currently aged:

(a) 40

(b) 90.

Solution

The premiums charged under any insurance policy reflect the risk covered. So, the more likely the insurance company is to have to make a payment, the higher the premium will be.

(a) A 40-year-old is more likely to survive for the next 10 years than to die during the next 10 years. So the premiums charged for the pure endowment will be higher than those charged for the term assurance, as the payment on survival is more likely to be made than the payment on death.

(b) On the other hand, a 90-year-old is more likely to die during the next 10 years than to survive to age 100. So, in this case, the premiums charged for the term assurance will be higher than those charged for the pure endowment.

3.4 A contingent annuity

This is a similar contract to the annuity-certain (see Section 2.6 above) but the payments are contingent upon certain events, such as survival, hence the payment term for the regular cashflows (which will be negative from the perspective of the annuity provider) is uncertain.

As mentioned before, a pension payable after retirement is an example of a contingent annuity.

Typical examples of contingent annuities include:

- **a single life annuity – where the regular payments made to the annuitant are contingent on the survival of that annuitant**
- **a joint life annuity – which covers two lives, where the regular payments are contingent on the survival of one or both of those lives**
- **a reversionary annuity – which is based on two lives, where the regular payments start on the death of the first life if, and only if, the second life is alive at the time. Payments then continue until the death of the second life.**

A reversionary annuity is one which would be paid, for example, to a wife after her husband has died, or to a husband after his wife has died. It provides the surviving partner with an income, and hence some financial security, for their remaining lifetime. In the context of pensions, a reversionary annuity is sometimes called a spouse's pension or a dependant's pension.

3.5 A car insurance policy

A typical car insurance contract lasts for one year. In return for a premium which can be paid as a single lump sum or at monthly intervals, the insurer will provide cover to pay for damage to the insured vehicle or fire or theft of the vehicle, known as 'property cover'. In many countries, such as the UK, the contract also provides cover for compensation payable to third parties for death, injury or damage to their property, known as 'liability cover'.

So, property cover relates to damaged caused *to* the car, and liability cover relates to damaged caused *by* the car.

Depending on the terms of the policy, the insurance company may settle claims directly with the policyholder or with another party. For example, in the case of theft or total loss, the insurance company may pay a lump sum to the policyholder in lieu of that loss. In the case of damage to the insured vehicle, the insurance company may settle the claim directly with the party undertaking the repairs without involving the policyholder. In the case of third party liability claims, the insurance company may settle the claims directly with the third party.

The 'third party' here is the person whose car or property has been damaged by the insured vehicle.

In some cases, the policyholder may be required to cover the cost of damage or repairs first before the insurance company settles the claim, in which case the insurance company will pay the policyholder directly.

The cashflows for the policyholder will usually be a single negative cashflow at the beginning of the year. Further cashflows only take place in the event of a claim. If the policyholder has to pay for repairs or compensation, this will incur a further negative cashflow, followed by a positive cashflow when the insurance company settles the claim. If the insurance company settles the claim directly with the repair company or third party, the policyholder may not experience further cashflows.

From the insurer's perspective, there will be a positive cashflow at the beginning of the policy, followed by a negative cashflow when the claim is settled.

The timing of the cashflows will depend on how long the claim takes to be reported and settled. Typically, property claims take less time to settle than liability claims. Where liability claims involve disputes, for example necessitating court judgements, they can take years to settle and the amounts are less certain.

For example, it may be difficult to establish precisely which vehicle was responsible for a major car accident.

Cashflows tend to be short-term and are payable within the year.

3.6 A health cash plan

A typical health insurance contract lasts for one year. In return for a premium, the policyholder is entitled to benefits which may include hospital treatment either paid for in full or in part, and/or cash benefits in lieu of treatment, such as a fixed sum per day spent in hospital as an in-patient.

Typical benefits provided by a health cash plan might include:

- £150 towards any dentist or optician fees
- £75 per day or overnight stay in hospital up to a maximum of 25 stays
- £350 towards the cost of a specialist consultation or procedure, such as an MRI scan.

From the policyholder's perspective, the cashflows will include a negative cashflow at the beginning of the year followed by positive cashflows in the event of a claim in the case of a cash benefit. Where the insurance company pays for hospital treatment directly, the policyholder may experience no more cashflows after paying the initial premium.

From the perspective of the insurer, there will be an initial positive cashflow at the start of the policy followed by negative cashflows in the event of a claim, when those claims are settled.

Cashflows tend to be short-term and are payable within the year.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

A *car insurance policy* usually lasts for one year, and provides payments to cover any damage to the insured vehicle (property cover) or any damage caused by the insured vehicle (liability cover).

A *health cash plan* usually lasts for one year, and provides payments to cover medical expenses, such as hospital treatment.

A *health cash plan* usually lasts for one year, and provides payments to cover medical expenses, such as hospital treatment.

Chapter 2 Practice Questions

2.1 Complete the table below using the symbols: ✓ (= yes) or × (= no).

Security	Absolute amount of payments known in advance?	Timing of payments known in advance?
Zero-coupon bond		
Fixed-interest security		
Index-linked security		
Cash on deposit		
Equity		
Credit cards		
Current account		

2.2 Describe the characteristics of:

Exam style

(a) an interest-only loan (or mortgage); and

(b) a repayment loan (or mortgage). [4]

2.3 Complete the table below using the symbols: ✓ (= yes) or × (= no).

Insurance contract	Absolute amount of payments made by insurer known in advance?	Timing of payments made by insurer known in advance?
Pure endowment		
Term assurance		
Endowment assurance		
Car insurance		
Health cash plan		

2.4 Outline the similarities and differences between an annuity-certain and a contingent annuity. [3]

Exam style

The solutions start on the next page so that you can separate the questions and solutions.

Chapter 2 Solutions

2.1 The completed table is as follows:

Contract	Absolute amount of payments known in advance?	Timing of payments known in advance?
Zero-coupon bond	✓	✓
Fixed-interest security	✓	✓
Index-linked security	×	✓
Cash on deposit	×	×
Equity	×	×
Credit card	×	×
Current account	×	×

2.2 *This question is Subject CT1, April 2009, Question 2.*

(a) ***Interest-only loan***

An interest-only loan is a loan repayable by a series of interest payments during the term of the loan, followed by repayment of the full capital amount at the end of the loan. The amount of capital outstanding therefore remains fixed over the term of the loan. [1]

If the interest rate is fixed, the amount of each interest cashflow is known in advance. If the interest rate is variable, the interest payments will be unknown at outset. [1]

(b) ***Repayment loan***

A repayment loan is a loan repayable by a series of payments, each of which includes partial repayment of the loan capital in addition to interest. This means the amount of capital outstanding reduces over the term of the loan. [1]

If the interest rate is fixed over the term of the loan, the repayments will all be for fixed amounts. If the interest rate varies, so will the repayment amounts. [1]

[Total 4]

2.3 The completed table is as follows:

Insurance contract	**Absolute amount of payments made by insurer known in advance?**	**Timing of payments made by insurer known in advance?**
Pure endowment	✓	✓
Term assurance	✓	×
Endowment assurance	✓	×
Car insurance	×	×
Health cash plan	×	×

2.4 Both an annuity-certain and a contingent annuity provide a regular series of payments in return for a single premium (*ie* a lump sum) paid at the outset. [1]

For both contracts, the frequency of payments will be specified, as will the payment amount, *eg* payments could be level, increasing at a constant rate, or increasing in line with an inflation index. [1]

The difference between these contracts is that under an annuity-certain the number of annuity payments is fixed in advance, whereas for a contingent annuity the payments are dependent upon certain events, usually the survival of the policyholder. So for a contingent annuity the number of payments made will not be known in advance. [1]

[Total 3]

3

The time value of money

Syllabus objectives

2.1 Show how interest rates may be expressed in different time periods.

- 2.1.1 Describe the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

2.3 Describe how to take into account the time value of money using the concepts of compound interest and discounting.

- 2.3.1 Accumulate a single investment at a constant rate of interest under the operation of simple and compound interest.
- 2.3.2 Define the present value of a future payment.
- 2.3.3 Discount a single investment under the operation of a simple (commercial) discount at a constant rate of discount.

0 Introduction

***Interest* may be regarded as a reward paid by one person or organisation (the *borrower*) for the use of an asset, referred to as *capital*, belonging to another person or organisation (the *lender*).**

In return for the use of the investor's capital, the borrower will be expected to pay interest to the lender. For example:

- a bank will be expected to pay interest to its customers on money held in their savings accounts
- a company will be expected to pay interest to a bank on money lent to it for a business project.

Interest may be fixed or variable. In early chapters we assume that interest is fixed. In Chapter 12 Section 2 we consider various types of securities where interest is variable.

In this chapter we will look at the basic ideas underlying the theory of interest rates. Interest rates are a fundamental part of actuarial work.

Question

List some common situations in which a bank will act as:

(i) a lender

(ii) a borrower.

Solution

(i) A bank acts as a lender when:

- it offers a mortgage to a person wanting to buy a house
- it makes a business loan to a company
- it buys fixed-interest securities.

(ii) A bank acts as a borrower when:

- it accepts money from savers
- it issues shares (*ie* the bank's own shares) to investors
- it sells fixed interest securities.

When the capital and interest are expressed in monetary terms, capital is also referred to as *principal*. The total received by the lender after a period of time is called the *accumulated value*. The difference between the principal and the accumulated value is called the *interest*. Note that we are assuming here that no other payments are made or incurred (*eg* charges, expenses).

For example, consider the situation where a bank lends an individual £1,000, and in return the individual has to pay £1,200 to the bank in 1 year's time. The *capital* (or *principal*) is the £1,000 lent to the individual. The *accumulated value* is the £1,200 paid back. Since the individual has to pay back £200 more than was borrowed, this is the *interest*.

The above example is a very simple case where there is just one payment. Things will get messier when payments are more frequent.

If there is some risk of default (*ie* loss of capital or non-payment of interest), a lender would expect to be paid a higher rate of interest than would otherwise be the case.

If an investor lends money both to the US government (by purchasing a fixed-interest security) and to a property developer, the investor will probably demand a higher rate of interest from the property developer. This is because the property developer is more likely not to repay the loan or not to pay the interest due on the loan.

Another factor which may influence the rate of interest on any transaction is an allowance for the possible depreciation or appreciation in the value of the currency in which the transaction is carried out. This factor is very important in times of high inflation.

If the rate of inflation over the period of a loan is expected to be high, then the lender may demand a higher rate of interest to compensate for the lower *real* value of the capital when it is returned at the end of the loan period. We will consider real interest rates in more detail later in the course.

We will now consider two types of interest within the framework of a savings account.

1 Interest

1.1 Simple interest

The essential feature of *simple interest* is that interest, once credited to an account, does not itself earn further interest.

Suppose an amount C is deposited in an account that pays simple interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1+ni) \tag{1.1}$$

Question

An investor deposits £10,000 in a bank account that pays simple interest at a rate of 5% *pa*. Calculate the accumulated value of the deposit after 3 years.

Solution

After 1 year, the investor will have earned interest of $0.05 \times £10{,}000 = £500$.

Whilst the investor's bank balance is now £10,500, we know that interest is not earned on interest. So in the second year, the investor will again earn interest of $0.05 \times £10{,}000 = £500$.

There is now £11,000 in the bank, and in the third year the investor will again earn interest of $0.05 \times £10{,}000 = £500$. So at the end of the third year the investor's deposit will have accumulated to £11,500.

Essentially the investor is only earning interest on the original capital of £10,000. Since this is the same every year, the total interest after 3 years is $3 \times 0.05 \times £10{,}000 = £1{,}500$. So the final accumulated value is the original capital plus the interest earned:

$$£10{,}000 + 3 \times 0.05 \times £10{,}000 = £11{,}500$$

Taking out the common factor of £10,000, we have:

$$£10{,}000(1 + 3 \times 0.05) = £11{,}500$$

This figure can also be obtained using the formula $C(1+ni)$.

When n is not an integer, interest is paid on a pro-rata basis.

This is normal commercial practice so equation (1.1) could be applied to an accumulation over 3.6 years, say.

However, the situation where further interest is not earned on earlier interest can lead to problems, as illustrated in the following question.

Question

An investor deposits £5,000 into a savings account that pays 10% simple interest at the end of each year. Compare how much the investor would have after 6 years if the money were:

(i) invested for 6 years

(ii) invested for 3 years, then immediately reinvested for a further 3 years.

Solution

(i) At the end of 6 years, the investor will have $5{,}000(1+6\times 0.1)=£8{,}000$.

(ii) At the end of 3 years, the investor will have $5{,}000(1+3\times 0.1)=£6{,}500$.

At the end of 6 years, the investor will have $6{,}500(1+3\times 0.1)=£8{,}450$.

So under a simple interest arrangement, an investor can earn more interest by withdrawing the capital and then immediately reinvesting it.

The situation illustrated above is undesirable, since it encourages unnecessary additional transactions. Hence we will now consider the case where interest does itself earn interest. This is called *compound interest*.

1.2 Compound (effective) interest

The essential feature of compound interest is that interest itself earns interest.

Suppose an amount C is deposited in an account that pays compound interest at the rate of $i\times 100\%$ per annum. Then after n years the deposit will have accumulated to:

$$C(1+i)^n \qquad \textbf{(1.2)}$$

Question

An investor deposits £10,000 in a bank account that pays compound interest at a rate of 5% *pa*. Calculate the accumulated value of the deposit after 3 years.

Solution

After 1 year, the investor will earn interest of $0.05\times £10{,}000=£500$. So there will be £10,500 in the bank.

In the second year, the investor will earn interest of $0.05\times £10{,}500=£525$. So there will then be £11,025 in the bank.

In the third year, the investor will earn interest of $0.05\times £11{,}025=£551.25$. So at the end of the third year the investor's deposit will have accumulated to £11,576.25.

Every year interest of 5% is added to the balance – so the new balance is 105% of the old balance, which we can calculate by multiplying by 1.05. So after 3 years the accumulation is:

$$£10,000 \times 1.05 \times 1.05 \times 1.05 = £11,576.25$$

This is the same as:

$$£10,000 \times 1.05^3 = £11,576.25$$

which is given directly by the formula $C(1+i)^n$.

Unlike simple interest, with compound interest, an investor cannot earn more interest by withdrawing the capital and then immediately reinvesting it.

Question

An investor deposits £5,000 into a savings account that pays 10% compound interest at the end of each year. Compare how much the investor would have after 6 years if the money were:

(i) invested for 6 years

(ii) invested for 3 years, then immediately reinvested for a further 3 years.

Solution

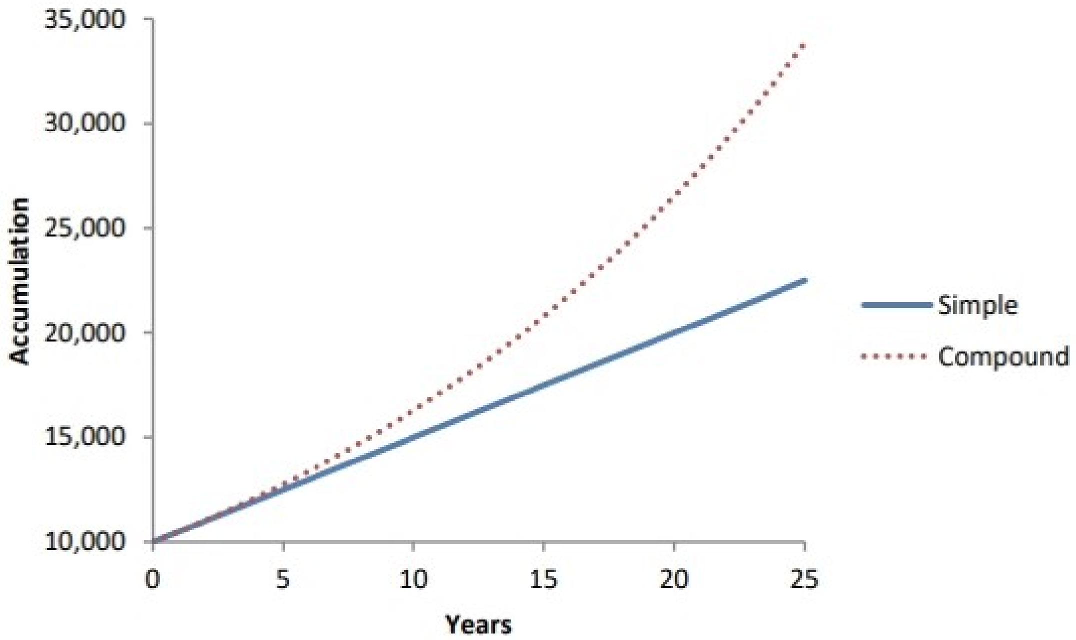
(i) At the end of 6 years the investor will have $5,000 \times 1.1^6 = £8,858$.

(ii) At the end of 3 years the investor has $5,000 \times 1.1^3 = £6,655$.

At the end of 6 years the investor will have $6,655 \times 1.1^3 = £8,858$.

The accumulated value is the same with and without the reinvestment.

The following graph shows how £10,000 invested in a savings account would grow over the next few years if the account paid either 5% *pa* simple interest (solid line) or 5% *pa* compound interest (dotted line):



We see that the simple interest account produces a straight-line graph, whereas the graph of the compound interest account is exponentially shaped.

1.3 Accumulation factors

For $t_1 < t_2$ we define $A(t_1,t_2)$ to be the accumulation at time t_2 of an investment of 1 at time t_1.

The number $A(t_1,t_2)$ is often called an *accumulation factor*, since the accumulation at time t_2 of an investment of C at time t_1 is, by proportion:

$$CA(t_1,t_2) \tag{1.3}$$

$A(n)$ is often used as an abbreviation for the accumulation factor $A(0,n)$.

For example, if an investor deposits £10,000 in a bank account that pays simple interest at a rate of 5% *pa*, after 3 years the accumulated value of the deposit will be £11,500. So the accumulation factor from time 0 to time 3 is:

$$A(0,3) = \frac{11,500}{10,000} = 1.15$$

So here the accumulation factor is of the form $1+ni$, as in equation (1.1).

In summary, using equation (1.1) the accumulation factor for simple interest is:

$$A(n) = A(0,n) = 1 + ni$$

and using equation (1.2) the accumulation factor for compound interest is:

$$A(n) = A(0,n) = (1+i)^n$$

Question

An investment of £1,000 accumulates to £2,500 after 5 years.

(i) Calculate the accumulation factor $A(0,5)$.

(ii) (a) Calculate the simple annual interest rate that would give the accumulation factor in part (i).

(b) Calculate the annual compound interest rate that would give the accumulation factor in part (i).

Solution

(i) $A(0,5) = \dfrac{2,500}{1,000} = 2.5$

(ii)(a) $A(0,5) = (1+5i) = 2.5 \Rightarrow i = \dfrac{2.5-1}{5} = 30\%$

(ii)(b) $A(0,5) = (1+i)^5 = 2.5 \Rightarrow i = (2.5)^{1/5} - 1 = 20.1\%$

1.4 The principle of consistency

Now let $t_0 \le t_1 \le t_2$ and consider an investment of 1 at time t_0. The proceeds at time t_2 will be $A(t_0,t_2)$ if one invests at time t_0 for term $t_2 - t_0$, or $A(t_0,t_1)A(t_1,t_2)$ if one invests at time t_0 for term $t_1 - t_0$ and then, at time t_1, reinvests the proceeds for term $t_2 - t_1$. In a consistent market these proceeds should not depend on the course of action taken by the investor. Accordingly, we say that under the *principle of consistency*:

$$A(t_0,t_n) = A(t_0,t_1)A(t_1,t_2) \dots A(t_{n-1},t_n) \qquad \textbf{(1.4)}$$

If we invest £1,000 for 8 years in a bank account that pays compound interest of 10% *pa*, the balance at the end of this period would be $£1,000A(0,8) = £1,000 \times 1.1^8 = £2,143.59$.

Similarly, the balance after 3 years would be $£1,000A(0,3) = £1,000 \times 1.1^3 = £1,331.00$. If we then reinvest this in the same account for a further 5 years, the balance would be $£1,331.00A(3,8) = £1,331.00 \times 1.1^5 = £2,143.59$, as before.

Hence $A(0,8) = A(0,3)A(3,8)$.

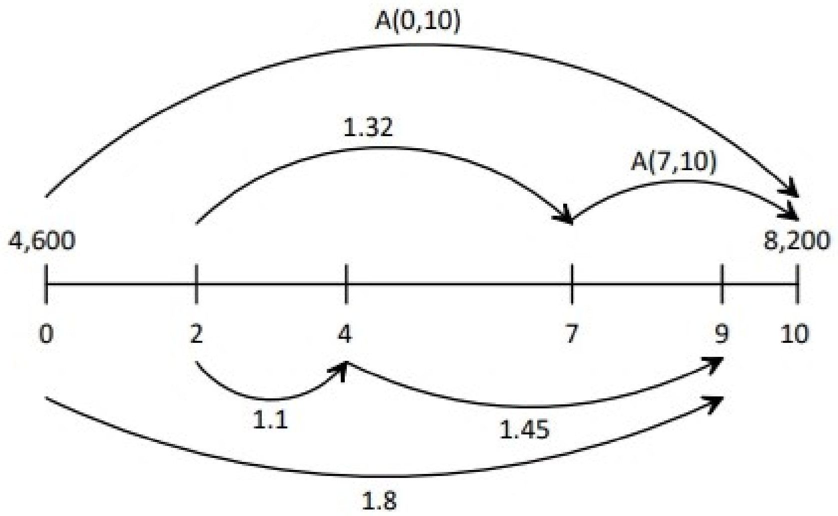
Question

£4,600 is invested at time 0 and the proceeds at time 10 are £8,200.

Calculate $A(7,10)$ if $A(0,9)=1.8$, $A(2,4)=1.1$, $A(2,7)=1.32$, $A(4,9)=1.45$.

Solution

Representing the initial investment, proceeds, and accumulation factors on a diagram, we have:



Since £4,600 accumulates to £8,200 in 10 years, we know:

$$A(0,10)=\frac{8,200}{4,600}=1.7826$$

Using the principle of consistency:

$$A(0,10)=1.7826=A(0,2)A(2,7)A(7,10) \quad\Rightarrow\quad A(7,10)=\frac{1.7826}{A(0,2)\times 1.32}$$

where $A(2,7)=1.32$. We can find $A(0,2)$ using the principle of consistency again:

$$A(0,9)=A(0,2)A(2,4)A(4,9) \quad\Rightarrow\quad A(0,2)=\frac{A(0,9)}{A(2,4)A(4,9)}=\frac{1.8}{1.1\times 1.45}=1.1285$$

So:

$$A(7,10)=\frac{1.7826}{1.1285\times 1.32}=1.1967$$

2 Present values

In the previous section we saw how to answer the question: how much will a single payment accumulate to at a later time? In actuarial work, we are usually aiming to make payments at certain future dates, *eg* making pension payments to a worker after retirement or making a life assurance payment when an individual dies. So actuaries are usually more interested in answering the question: how much do we need to invest now to provide payments at a later time? This amount is called the *present value* (PV) or *discounted value* of the payments.

Question

An investor must make a payment of £5,000 in 5 years' time. The investor wishes to make provision for this payment by investing a single sum now in a deposit account that pays 10% per annum compound interest.

Calculate the initial investment required to meet the payment of £5,000 in 5 years' time.

Solution

By the end of 5 years an initial payment of $£X$ will accumulate to $£X \times 1.1^5$.

So:

$$X \times 1.1^5 = 5{,}000 \;\Rightarrow\; X = \frac{5{,}000}{1.1^5} = £3{,}104.61$$

It follows by formula (1.2) that an investment of:

$$C/(1+i)^n \tag{2.1}$$

at time 0 (the present time) will give C at time $n \geq 0$.

This is called the *discounted present value* (or, more briefly, the *present value*) of C due at time $n \geq 0$.

It is the amount that needs to be invested at time 0 at compound interest rate i to accumulate to C at time n.

We can define the function:

$$v = \frac{1}{1+i} \tag{2.2}$$

It follows by formulae (2.1) and (2.2) that the discounted present value of C due at time $n \geq 0$ is:

$$Cv^n \tag{2.3}$$

Using this notation, we could have found the answer to the previous question by first calculating:

$$v = \frac{1}{1+i} = \frac{1}{1.1} = 0.9090909$$

and then the amount of the initial investment required, *ie* the present value of £5,000 due at time 5 years, is:

$$\Rightarrow \quad PV = 5{,}000v^5 = 5{,}000 \times 0.9090909^5 = £3{,}104.61$$

Although the above approach is correct, it's quicker to cut out the middle step of calculating the value of v, and instead work with negative powers:

$$PV = 5{,}000v^5 = 5{,}000 \times 1.1^{-5} = £3{,}104.61$$

Formulae and tables for actuarial examinations

Values of v^n at various interest rates are tabulated in 'Formulae and Tables for Examinations', which are provided in the exams. They are available to purchase from the Institute and Faculty of Actuaries. From now on we will refer to this book as simply the *Tables*.

Question

Calculate v, assuming an annual compound rate of interest of 4%.

Solution

Using the definition of v:

$$v = \frac{1}{1+i} = \frac{1}{1.04} = 0.96154$$

Alternatively, we can find this on page 56 of the *Tables*, either at the top of the v^n column or among the 'constants' listed on the left-hand side.

Note on rounding

How to round your answers is an important factor that you need to consider in your calculations. The rounding used should be appropriate for the level of accuracy used in your calculations. For example, if you are using numbers from the *Tables* then you shouldn't quote an answer to more significant figures than given in the *Tables*. The number of significant figures that you quote is usually more important than the number of decimal places. When using your calculator you can keep the accurate values of intermediate calculations in your memories. Your final answer will then be more accurate. In an exam you need not write down the fully accurate intermediate figures.

Remember that 104.27 is rounded to 5 significant figures, but 2 decimal places!

3 Discount rates

In Section 1, we were given an interest rate, which we used to obtain the appropriate accumulation factor. Multiplying the amount invested by the accumulation factor gives the accumulated value of the investment. For example, with compound interest rate i the *accumulation factor* is $(1+i)^n$.

In Section 2, we looked at discounting – that is, given the accumulated value, how much was the initial investment? We also developed the compound *discount factor* v^n. Multiplying the accumulated value at time n by the discount factor gives the present value of the investment.

We now want to calculate the corresponding compound *discount rate* that goes with this discount factor. In addition, we'll also need to obtain *simple* discount rates and factors.

An alternative way of obtaining the discounted value of a payment is to use discount rates.

3.1 Simple discount

As has been seen with simple interest, the interest earned is not itself subject to further interest. The same is true of simple discount, which is defined below.

Suppose an amount C is due after n years and a rate of *simple discount* of d per annum applies. Then the sum of money required to be invested now to amount to C after n years (*ie* the present value of C) is:

$$C(1-nd) \tag{3.1}$$

Question

Calculate the present value of £10,000 due at time 3 years, using a simple discount rate of 5% *pa*.

Solution

After 1 year the discount will be $0.05 \times £10,000 = £500$. This amount is deducted from the payment of £10,000, leaving £9,500.

Whilst we've now got £9,500 left, we know that simple discount does not discount the discount. So in the second year the discount will again be $0.05 \times £10,000 = £500$.

We've now got £9,000 left. In the third year the discount will again be $0.05 \times £10,000 = £500$. So at the end of the third year we will have £8,500.

Essentially we are only discounting the original £10,000. Since this is the same every year, the total discount after 3 years is $3 \times 0.05 \times £10,000 = £1,500$. So the final discounted value is the original amount minus the discount:

$$£10,000 - 3 \times 0.05 \times £10,000 = £8,500$$

Taking out the common factor of £10,000, we have:

$$£10,000(1-3\times 0.05)=£8,500$$

This is the present value of £10,000 due in 3 years and can be obtained directly using the formula $C(1-nd)$.

In normal commercial practice, d is usually encountered only for periods of less than a year. If a lender bases his short-term transactions on a simple rate of discount d then, in return for a repayment of X after a period t $(t<1)$ he will lend $X(1-td)$ at the start of the period. In this situation, d is also known as a rate of *commercial discount*.

Simple discount is the rate quoted for treasury bills, which are short-term loans made by the government. Rather than quoting the amount it wishes to borrow, the government quotes the amount of the repayment. The rate of simple discount is then used to calculate the purchase price of the treasury bill, *ie* the amount actually lent to the government.

Question

An 8-month loan is repayable by a single payment of £100,000. If the loan is issued at a rate of commercial discount of 15% *pa*, calculate how much is initially lent to the borrower.

Solution

The amount lent is:

$$£100,000\left(1-\frac{8}{12}\times 0.15\right)=£90,000$$

Since 15% is the annual rate of discount, we use $\frac{8}{12}$ as the length of time, ie 8 months expressed in years.

3.2 Compound (effective) discount

As has been seen with compound interest, the interest earned is subject to further interest. The same is true of compound discount, which is defined below.

Suppose an amount C is due after n years and a rate of compound (or effective) discount of d per annum applies. Then the sum of money required to be invested now to accumulate to C after n years (*ie* the present value of C) is:

$$C(1-d)^n \qquad \textbf{(3.2)}$$

Question

Calculate the present value of £10,000 due at time 3 years, using a compound discount rate of 5% *pa*.

Solution

After 1 year, the discount will be $0.05 \times £10,000 = £500$. This amount is deducted from the payment of £10,000, leaving £9,500.

In the second year, we discount the discounted amount, so the discount will be $0.05 \times £9,500 = £475$. So we will now have £9,025 left.

In the third year, the discount will be $0.05 \times £9,025 = £451.25$. So after discounting for the third year, there will be £8,573.75 left.

Every year a 5% discount is subtracted from the amount – so the new amount is 95% of the old amount, which we can calculate by multiplying by 0.95. So after 3 years the final discounted value is:

$$£10,000 \times 0.95 \times 0.95 \times 0.95 = £8,573.75$$

This can be written as:

$$£10,000 \times 0.95^3 = £8,573.75$$

This is the present value of £10,000 due in 3 years and can be obtained directly using the formula $C(1-d)^n$.

3.3 Discount factors

In the same way that the accumulation factor $A(n)$ gives the accumulation at time n of an investment of 1 at time 0, we define $v(n)$ to be the present value of a payment of 1 due at time n. Hence:

$$\mathbf{v(n) = \frac{1}{A(n)}} \qquad \textbf{(3.3)}$$

So using equation (3.2) the discount factor for compound discount is:

$$v(n) = (1-d)^n$$

However, using equation (3.3) and equation (1.2), we could also give a discount factor in terms of compound interest:

$$v(n) = \frac{1}{A(n)} = \frac{1}{(1+i)^n} \qquad (3.4)$$

Using the definition of v from (2.2) in (3.4) we get:

$$v(n) = \frac{1}{A(n)} = \frac{1}{(1+i)^n} = v^n$$

This is the discount factor that we used in (2.3) to calculate the present value.

We could also use (3.3) and (3.2) to give an *accumulation* factor in terms of compound discount:

$$A(n)=\frac{1}{v(n)}=\frac{1}{(1-d)^n}=(1-d)^{-n}$$

So regardless of whether we're given interest or discount rates, we can calculate both accumulations and present values. Of course this does mean that we have to read questions carefully to see what rates we are given and what we are asked to do with them!

Question

(i) Given an investment of €1,000, calculate the accumulation after 5 years using:

(a) simple discount of 8% *pa*

(b) compound discount of 8% *pa*

(c) compound interest of 8% *pa*.

(ii) A payment of €2,000 is due in 4 years' time. Calculate the present value using:

(a) simple interest of 3% *pa*

(b) simple discount of 3% *pa*

(c) compound interest of 3% *pa*.

Solution

(i)(a) Using (3.3) and (3.1), the accumulation is:

$$1{,}000A(5)=1{,}000\times\frac{1}{v(5)}=1{,}000\times\frac{1}{1-5\times0.08}=€1{,}666.67$$

(i)(b) Using (3.3) and (3.2), the accumulation is:

$$1{,}000A(5)=1{,}000\times\frac{1}{v(5)}=1{,}000\times\frac{1}{(1-0.08)^5}=€1{,}517.26$$

(i)(c) Just using (1.2), the accumulation is:

$$1{,}000A(5)=1{,}000\times1.08^5=€1{,}469.33$$

(ii)(a) Using (3.3) and (1.1), the present value is:

$$2{,}000v(4)=2{,}000\times\frac{1}{A(4)}=2{,}000\times\frac{1}{1+4\times0.03}=€1{,}785.71$$

(ii)(b) Just using (3.1), the present value is:

$$2{,}000v(4) = 2{,}000 \times (1 - 4 \times 0.03) = €1{,}760$$

(ii)(c) Using (3.3) and (1.2) the present value is:

$$2{,}000v(4) = 2{,}000 \times \frac{1}{A(4)} = 2{,}000 \times \frac{1}{(1.03)^4} = €1{,}776.97$$

4 Effective rates of interest and discount

Effective rates are compound rates that have interest paid *once* per unit time either at the end of the period (effective interest) or at the beginning of the period (effective discount). This distinguishes them from nominal rates where interest is paid more frequently than once per unit time.

Bank accounts sometimes use nominal rates of interest. They might quote the annual interest rate (and so the unit time is one year) but interest is actually added at the end of each month (so interest is paid more frequently than once per unit year). We will meet nominal rates in the next chapter.

We can demonstrate the equivalence of compound and effective rates by an alternative way of considering effective rates.

4.1 Effective rate of interest

An investor will lend an amount 1 at time 0 in return for a repayment of $(1+i)$ at time 1. Hence we can consider i to be the interest paid at the *end* of the year. Accordingly i is called the *rate of interest* (or the *effective rate of interest*) per unit time.

Now we'll consider the general case. Investing 1 at time 0, the accumulation at time $n-1$ will be $A(n-1)$ and the accumulation at time n will be $A(n)$.

So denoting the effective rate of interest during the nth period (*ie* between time $n-1$ and time n) **by i_n, we have:**

$$A(n)=(1+i_n)A(n-1)$$

Rearranging gives:

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} \tag{4.1}$$

This formula should be intuitive – subtracting the accumulations in the numerator gives the interest earned during the nth period. By dividing by the amount we started with, we will obtain the interest rate.

Question

An investor's bank balance at various times is as follows:

1 Jan 2017	1 Jul 2017	1 Jan 2018
£3,000	£3,100	£3,300

Calculate the:

(i) effective six-monthly rate between 1 January 2017 and 1 July 2017

(ii) effective annual rate between 1 January 2017 and 1 January 2018.

Solution

(i) $i = \dfrac{£3{,}100 - £3{,}000}{£3{,}000} = 3.33\%$ per six months

(ii) $i = \dfrac{£3{,}300 - £3{,}000}{£3{,}000} = 10\%\ pa$

If *i* is the compound rate of interest, we have:

$$A(n) = (1+i)^n \text{ and } A(n-1) = (1+i)^{n-1}$$

Hence substituting in (4.1) we have:

$$i_n = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = (1+i) - 1 = i \qquad \textbf{(4.2)}$$

Since this is independent of n, we see that the effective rate of interest is identical to the compound rate of interest we met earlier.

Hence, hereafter, we shall use the terms compound interest and effective interest interchangeably.

Question

Show that the effective rate of interest, when accumulating using a constant simple interest rate, decreases over time.

Solution

For simple interest i, the accumulations are:

$$A(n) = (1+ni) \text{ and } A(n-1) = (1+(n-1)i)$$

Hence substituting in (4.1) we have:

$$i_n = \frac{(1+ni) - (1+(n-1)i)}{(1+(n-1)i)} = \frac{i}{1+(n-1)i}$$

So the effective interest rate decreases as n gets larger. This makes sense as simple interest pays a constant amount each year (as interest doesn't earn interest) but the accumulation is getting larger each year.

4.2 Effective rate of discount

We can think of compound discount as an investor lending an amount $(1-d)$ at time 0 in return for a repayment of 1 at time 1. The sum of $(1-d)$ may be considered as a loan of 1 (to be repaid after 1 unit of time) on which interest of amount d is payable in *advance*. Accordingly d is called the *rate of discount* (or the *effective rate of discount*) per unit time.

Consider the situation where an individual borrows a sum of £5,000 and agrees to pay this back at the end of 1 year with interest calculated at an effective rate of 10% per annum. The total amount repayable will therefore be £5,500.

An alternative way of looking at this arrangement would be to say that the individual has borrowed £5,500 (the amount to be repaid) but the lender has deducted the interest payment of £500 at the time the money was lent. Presented in this way, it would seem logical to express the interest rate as 9.09% (*ie* $500/5{,}500$) of the amount borrowed, where the interest is payable at the *beginning* of the year.

Therefore the effective rate of discount over a given time period is the amount of interest payable at the beginning of the time period, expressed as a proportion of the total amount paid at the end of the period.

In symbols, denoting the effective rate of discount during the nth period (*ie* between time $n-1$ and time n) by d_n, we have:

$$d_n = \frac{A(n) - A(n-1)}{A(n)}$$

We can also show that the effective rate of discount is identical to the compound rate of discount we met earlier.

For compound discount, d, we have:

$$A(n) = \frac{1}{v(n)} = \frac{1}{(1-d)^n} = (1-d)^{-n} \quad \text{and} \quad A(n-1) = \frac{1}{v(n-1)} = \frac{1}{(1-d)^{n-1}} = (1-d)^{-(n-1)}$$

Hence:

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{(1-d)^{-n} - (1-d)^{-(n-1)}}{(1-d)^{-n}} = 1 - (1-d) = d$$

Hence, hereafter, we shall use the terms compound discount and effective discount interchangeably.

5 Equivalent rates

Two rates of interest and/or discount are equivalent if a given amount of principal invested for the same length of time produces the same accumulated value under each of the rates.

Converting between interest rates efficiently is an important skill. The key is to equate the accumulation factors or to equate the discount factors over the same time period.

Question

Calculate the effective annual interest rate that is equivalent to a simple interest rate of 3% *pa* over 4 years.

Solution

The accumulation factor for 3% *pa* simple interest over 4 years is:

$$A(4) = 1 + 4 \times 0.03 = 1.12$$

The accumulation factor for effective interest over 4 years is:

$$A(4) = (1+i)^4$$

If two rates are equivalent, then they will result in the same accumulation factors:

$$(1+i)^4 = 1.12 \quad \Rightarrow \quad i = 2.87\%\ pa$$

Comparing formulae (2.3) and (3.2), we see that the present value of C due at time n can be expressed as Cv^n or as $C(1-d)^n$. So equating the discount factors we see that:

$$\boldsymbol{v = 1 - d} \qquad \textbf{(5.1)}$$

And from (2.2) and (5.1) we obtain the rearrangements:

$$\boldsymbol{d} = 1 - v = 1 - \frac{1}{1+i} = \frac{\boldsymbol{i}}{\boldsymbol{1+i}} \qquad \textbf{(5.2)}$$

or: $$\boldsymbol{d = iv} \qquad \textbf{(5.3)}$$

Recall that d is the interest paid at time 0 on a loan of 1, whereas i is the interest paid at time 1 on the same loan. If the rates are equivalent then if we discount i from time 1 to time 0 we will obtain d. This is the interpretation of equations (5.2) and (5.3).

Note that formulae (5.1), (5.2) and (5.3) apply for effective rates. They *cannot* be used to convert between simple interest and discount.

Chapter 3 Summary

Many financial arrangements involve a borrower and a lender. Borrowers reward lenders by paying interest to them.

Two factors that might influence the level of interest rates are the likelihood of default on payments and the possible appreciation or depreciation of currency.

The calculation of the amount of interest payable under a financial arrangement can be expressed in terms of compound (effective) interest or simple interest.

The essential feature of *simple interest* is that interest, once credited to an account, does not itself earn further interest. In the case of simple interest, the formula for the n-year accumulation factor is:

$$A(n)=1+ni$$

The essential feature of *compound (effective) interest* is that interest itself earns interest. In the case of compound interest, the formula for the n-year accumulation factor is:

$$A(n)=(1+i)^n$$

The principle of consistency says that the accumulated proceeds of an investment in a consistent market will not depend on the action of the investor.

The *present value* of a series of payments is the amount that is invested now in order to meet those payments. We can obtain present values using discount factors. In terms of effective interest the n-year discount factor is:

$$v(n)=v^n=\frac{1}{(1+i)^n}$$

The n-year discount factor in terms of the effective discount rate is: $v(n)=(1-d)^n$

The n-year discount factor in terms of the simple discount rate is: $v(n)=1-nd$

The relationship between discount and accumulation factors is: $v(n)=\dfrac{1}{A(n)}$

The *effective rate of interest* over a given time period is the amount of interest a single initial investment will earn at the end of the time period, expressed as a proportion of the initial amount.

The *effective rate of discount* over a given time period is the amount of interest a single initial investment will earn at the start of the time period, expressed as a proportion of the final amount.

Useful relationships between effective rates are: $v=1-d$ $\quad d=\dfrac{i}{1+i}$ $\quad d=iv$

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 3 Practice Questions

3.1 An investor pays £100 into an account today. The account pays simple interest at a rate of 4% *pa*. Calculate the amount in the account in five years' time.

3.2 A company is due to receive a payment of £500,000 from a customer in 6 months' time. To smooth its cashflows, the company would prefer to receive the payment immediately, and has agreed to transfer its entitlement to this payment to a third party (a discount house) in return for an immediate payment calculated using a rate of commercial discount of 16% per annum.

Calculate the amount of the immediate payment made by the discount house.

3.3 A bank account pays an effective annual interest rate of 10% over 5 years. Calculate the equivalent:

(i) simple annual interest rate

(ii) effective monthly interest rate

(iii) effective two-yearly interest rate

(iv) effective annual discount rate

(v) simple annual discount rate.

3.4 A woman who has won a prize is offered either:

- a lump sum of £100,000 to invest now, or
- £55,000 to invest in one year's time and another £55,000 to invest in two years' time.

If all investments are assumed to earn interest at a rate of 7% *pa* effective, determine which option she should choose if she intends to withdraw the money after:

(i) 4 years

(ii) 2 years.

3.5 Calculate the effective annual rate of interest for:

(i) a transaction in which £200 is invested for 18 months to give £350.

(ii) a transaction in which £100 is invested for 24 months and another £100 is invested for 12 months (starting 12 months after the first investment) to give a total of £350.

3.6 A 182-day treasury bill, redeemable at \$100, was purchased for \$96.50 at the time of issue and later sold to another investor for \$98 who held the bill to maturity. The rate of return received by the initial purchaser was 4% per annum effective.

Exam style

(i) Calculate the length of time in days for which the initial purchaser held the bill. [2]

(ii) Calculate the annual simple rate of return achieved by the second investor. [2]

(iii) Calculate the annual effective rate of return achieved by the second investor. [2]

[Total 6]

Chapter 3 Solutions

3.1 The amount in the account will be $100(1+0.04\times 5)=£120$.

3.2 The amount of the immediate payment will be:

$$500,000\,(1-\tfrac{6}{12}\times 0.16)=£460,000$$

3.3 (i) Equating the accumulation factors for simple and effective interest over 5 years:

$$A(5)=(1+5i)=1.10^5 \;\Rightarrow\; i=12.2\%\ pa$$

(ii) Equating the accumulation factors for monthly and annual effective interest over 5 years gives:

$$A(5)=(1+i)^{5\times 12}=1.1^5 \;\Rightarrow\; (1+i)^{12}=1.1 \;\Rightarrow\; i=0.797\%\ \text{per month}$$

Note that because we are changing between two effective rates the 5 years is not actually needed. (The same would be true if we were changing between two simple rates.)

(iii) Equating the accumulation factors for two-yearly and annual effective interest over 5 years gives:

$$A(5)=(1+i)^{2.5}=1.1^5 \;\Rightarrow\; (1+i)^{½}=1.1 \;\Rightarrow\; i=21\%\ \text{per two-years}$$

Again because we are changing between two effective rates the 5 years is not actually needed.

(iv) Equating discount factors for effective annual interest and effective annual discount over the 5 years gives:

$$v(5)=v^5=1.1^{-5}=(1-d)^5 \;\Rightarrow\; 1.1^{-1}=1-d \;\Rightarrow\; d=9.09\%\ pa$$

Again because we are changing between two effective rates the 5 years is not actually needed.

(v) Equating discount factors for effective annual interest and simple annual discount over the 5 years gives:

$$v(5)=v^5=1.1^{-5}=(1-5d) \;\Rightarrow\; d=7.58\%\ pa$$

3.4 The time at which she intends to withdraw the money is irrelevant – she should choose the option that maximises the present value of the payments. This is because the higher the present value, the higher the accumulated value will be, irrespective of the time of withdrawal.

The PV of the lump sum option is £100,000. The PV of the two-payment option is:

$$55,000(v+v^2)=55,000(1.07^{-1}+1.07^{-2})=£99,441\,.$$

So she should choose the lump sum in either case.

3.5 (i) The effective annual interest rate i satisfies:

$$200(1+i)^{\frac{18}{12}}=350$$

Solving for i gives:

$$i=\left(\frac{350}{200}\right)^{\frac{12}{18}}-1=0.4522 \quad ie\ 45.22\%$$

(ii) The effective annual interest rate i satisfies:

$$100(1+i)^2+100(1+i)=350$$

Dividing by 100 and rearranging:

$$(1+i)^2+(1+i)-3.5=0$$

This is a quadratic equation in $(1+i)$. Applying the quadratic formula:

$$1+i=\frac{-1\pm\sqrt{1-4(-3.5)}}{2}=1.4365 \text{ (or } -2.4365)$$

Therefore $i=43.65\%$.

3.6 *This question is Subject CT1, April 2015, Question 3.*

(i) ***Length of time held***

We have:

$$96.5\times1.04^t=98$$ [1]

Taking logs of both sides of this equation gives:

$$\log 96.5+t\log 1.04=\log 98$$

Rearranging this, we find that:

$$t=\frac{\log 98-\log 96.5}{\log 1.04}=0.39327$$

Converting this to days, we find that the length of time is:

$$t=0.39327\times365=143.545$$

or about 144 days. [1]

[Total 2]

(ii) ***Annual simple rate of return***

Since the original investor held the bill for 144 days, the bill was held by the second investor for $182-144=38$ days. So the simple rate of return experienced by the second investor is the solution of the equation:

$$98\left(1+\frac{38}{365}\times i\right)=100$$ [1]

Solving this equation, we find that:

$$i=\left(\frac{100}{98}-1\right)\times\frac{365}{38}=0.19603 \quad \textit{ie} \ \ 19.60\% \ \textit{pa}$$ [1]

[Total 2]

(iii) ***Annual effective rate of return***

The equation of value is now:

$$98(1+i)^{38/365}=100$$ [1]

Solving this equation, we find that:

$$i=\left(\frac{100}{98}\right)^{365/38}-1=0.21416 \quad \textit{ie} \ \ 21.42\% \ \textit{pa}$$ [1]

[Total 2]

4

Interest rates

Syllabus objectives

2.1 Show how interest rates may be expressed in different time periods.

2.1.2 Derive the relationships between the rate of interest payable once per measurement period (effective rate of interest) and the rate of interest payable p (>1) times per measurement period (nominal rate of interest) and the force of interest.

2.1.3 Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

0 Introduction

The previous chapter considered the accumulation and present value of a single payment and introduced the ideas of effective interest rates and effective discount rates. This chapter describes the alternative ways of expressing interest rates and shows the relationships between them.

1 Nominal rates of interest and discount

Recall from Chapter 3 that 'effective' rates of interest and discount have interest paid once per measurement period, either at the end of the period or at the beginning of the period.

The effective annual rate of interest tells us the amount of interest to be paid at the *end* of each year. The effective annual rate of discount tells us the amount of interest to be paid at the *start* of each year.

'Nominal' is used where interest is paid more (or less) frequently than once per measurement period.

The example we gave in the previous chapter was that bank accounts sometimes use nominal rates. They might quote the annual interest rate (and so the time unit or measurement period is one year), but interest might actually be added at the end of each month (so interest is paid more frequently than once per unit year).

1.1 Nominal rates of interest

We denote the nominal rate of interest *payable p* times per period by $i^{(p)}$. This is also referred to as the rate of interest *convertible p*thly or *compounded p*thly.

Therefore, working in years, $i^{(12)}$ is referred to as a nominal interest rate convertible monthly and $i^{(4)}$ as a nominal interest rate convertible quarterly, *etc.*

A nominal rate of interest per period, payable *p*thly, $i^{(p)}$, is defined to be a rate of interest of $i^{(p)}/p$ applied for each *p*th of a period. For example, a nominal rate of interest of 6% *pa* convertible quarterly means an interest rate of $6/4 = 1.5\%$ per quarter.

Essentially what we are doing is 'annualising' a *p*thly effective interest rate. That is, we are converting a non-annual rate to an annual rate by multiplying. For example, suppose interest is 3% effective per half-year. We could annualise this rate by doubling it, which would give 6% *pa*. However, this is clearly not the correct annual effective rate as it has ignored the effect of compounding. The true effective annual rate is $1.03^2 - 1 = 6.09\%$ *pa*. We call 6% a nominal rate and give the period it actually refers to. Dividing by p gives us the correct *p*thly effective rate. So, in this case, we would say that the nominal rate of interest is 6% *pa* convertible half-yearly and we denote this by $i^{(2)}$.

Hence, by definition, $i^{(p)}$ is equivalent to a *p*thly *effective* rate of interest of $i^{(p)}/p$.

Question

(i) Express a monthly effective interest rate of 2% as a nominal annual interest rate convertible monthly.

(ii) (a) State the two-monthly effective interest rate that corresponds to a nominal interest rate of 3% *pa* convertible two-monthly.

(b) Hence, calculate the equivalent annual effective interest rate.

Solution

(i) A monthly effective interest rate of 2% is equivalent to a nominal annual interest rate of $2\% \times 12 = 24\%$ *pa* convertible monthly. In symbols, this is $i^{(12)} = 24\%$.

(ii) (a) There are six two-monthly periods in a year, so we are given $i^{(6)} = 3\%$. Therefore the two-monthly effective interest rate is $\frac{3\%}{6} = 0.5\%$.

(b) Using the two-monthly effective interest rate of 0.5%, the accumulation factor for one year is $1.005^6 = 1.030378$. Hence, the annual effective rate is 3.0378% *pa*.

(The effective rate is greater than the nominal rate as the effective rate takes account of the effect of compounding.)

In part (ii) of the above question, we have a nominal rate of interest of 3% *pa* convertible two-monthly (*ie* $i^{(6)} = 3\%$), which is equivalent to an effective interest rate of $\frac{i^{(6)}}{6} = 0.5\%$ per two-months. So, an initial investment of 1 unit will amount to 1.005^6 *ie* $\left(1+\frac{i^{(6)}}{6}\right)^6$ by the end of one year. If the equivalent effective annual rate of interest is i, an initial investment of 1 unit will amount to $1+i$. So we must have $1.005^6 = 1+i$, *ie* $\left(1+\frac{i^{(6)}}{6}\right)^6 = 1+i$. Hence, in the question, the equivalent effective annual rate of interest is $i = 1.005^6 - 1 = 0.030378$, *ie* 3.0378%.

In general, if we are given $i^{(p)}$, then the *p*thly effective interest rate is $\frac{i^{(p)}}{p}$, *ie* we have an effective rate of interest of $\frac{i^{(p)}}{p}$ for each period of length $\frac{1}{p}$. Hence, the accumulation factor over one time period is $\left(1+\frac{i^{(p)}}{p}\right)^p$.

Therefore the effective interest rate i is obtained from:

$$1+i = \left(1+\frac{i^{(p)}}{p}\right)^p \tag{1.1}$$

Note that $i^{(1)} = i$.

This can be seen by substituting $p=1$ into equation (1.1). If $p=1$ we don't bother to write the superscript.

Rearranging equation (1.1) gives:

$$i^{(p)} = p\left[(1+i)^{1/p} - 1\right] \tag{1.2}$$

This equation appears on page 120 of the *Tables*.

We can use equations (1.1) and (1.2) to convert between nominal and effective interest rates.

Question

(i) Calculate the nominal annual interest rate convertible quarterly that is equivalent to an interest rate of 5% *pa* effective.

(ii) Calculate the annual effective interest rate that is equivalent to a nominal interest rate of 12% *pa* convertible four-monthly.

Solution

(i) Using equation (1.2) we have:

$$i^{(4)} = 4(1.05^{1/4} - 1) = 4.90889\%$$

(ii) There are three four-month periods in a year, so we are given $i^{(3)} = 12\%$. Using equation (1.1) we have:

$$1+i = \left(1+\frac{0.12}{3}\right)^3 = 1.124864 \quad \Rightarrow \quad i = 12.4864\% \text{ } pa \text{ effective}$$

1.2 Accumulating and discounting using nominal interest rates

Since a nominal interest rate is a multiple of an effective interest rate for a period, we cannot directly use a nominal interest rate to accumulate or discount sums of money. However, we could use equation (1.1) to convert the nominal rate into an effective rate and then use the formulae from the previous chapter:

$$A(n) = (1+i)^n$$

$$v(n) = v^n = (1+i)^{-n}$$

Question

(i) €500 is invested in an account that pays nominal interest of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of \$800 is due in 5 years' time. Calculate the present value of this payment using an interest rate of 9% *pa* convertible monthly.

Solution

(i) We are given $i^{(2)}=8\%$. Using equation (1.1), this is equivalent to an annual effective rate of $i=\left(1+\frac{0.08}{2}\right)^2-1=8.16\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500\times 1.0816^3=€632.66$$

(ii) We are given $i^{(12)}=9\%$. Using equation (1.1), this is equivalent to an annual effective rate of $i=\left(1+\frac{0.09}{12}\right)^{12}-1=9.3806898\%$. Discounting the \$800 for 5 years at this rate gives:

$$\$800\times 1.093806898^{-5}=\$510.96$$

Instead of converting the nominal interest rate convertible *p*thly into an annual effective rate, we could convert it to a *p*thly effective rate.

The treatment of problems involving nominal rates of interest (or discount) is almost always considerably simplified by an appropriate choice of the time unit.

By choosing the basic time unit to be the period corresponding to the frequency with which the nominal rate of interest is convertible, we can use $i^{(p)}/p$ as the effective rate of interest per unit time. For example, if we have a nominal rate of interest of 18% per annum convertible monthly, we should take one month as the unit of time and 1½% as the rate of interest per unit time.

We'll now repeat the previous question using this method.

Question

(i) €500 is invested in an account that pays nominal interest of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of \$800 is due in 5 years' time. Calculate the present value of this payment using an interest rate of 9% *pa* convertible monthly.

Solution

(i) We are given $i^{(2)} = 8\%$. This is equivalent to a half-yearly effective rate of $\frac{8\%}{2} = 4\%$.

Accumulating the €500 for 3 years (= 6 half-years) at this rate gives:

$$€500 \times 1.04^6 = €632.66$$

(ii) We are given $i^{(12)} = 9\%$. This is equivalent to a monthly effective rate of $\frac{9\%}{12} = 0.75\%$.

Discounting the \$800 for 5 years (= 60 months) at this rate gives:

$$\$800 \times 1.0075^{-60} = \$510.96$$

1.3 Nominal rates of discount

We denote the nominal rate of discount *payable p* times per period by $d^{(p)}$. This is also referred to as the rate of discount *convertible* pthly or *compounded* pthly.

Therefore, working in years, $d^{(12)}$ is referred to as a nominal discount rate convertible monthly and $d^{(4)}$ as a nominal discount rate convertible quarterly, *etc.*

A nominal rate of discount per period payable *p*thly, $d^{(p)}$, is defined as a rate of discount of $d^{(p)}/p$ applied for each *p*th of a period.

Again, we are 'annualising' a *p*thly effective discount rate. For example, to 'annualise' an effective discount rate of 3% per half-year, we would double it to get 6% *pa*. This is not the correct annual effective discount rate (as it has ignored the effect of compounding). The 6% we have obtained is referred to as a nominal discount rate convertible half-yearly and is denoted by $d^{(2)}$.

Hence, by definition, $d^{(p)}$ is equivalent to a *p*thly *effective* rate of discount of $d^{(p)}/p$.

Recall from the previous chapter that given an effective discount *rate*, d, the discount *factor* for n periods is:

$$v(n) = (1-d)^n$$

Question

(i) Express a monthly effective discount rate of 2% as a nominal annual discount rate convertible monthly.

(ii) Calculate the effective annual discount rate that is equivalent to a nominal discount rate of 3% *pa* convertible two-monthly.

Solution

(i) A monthly effective discount rate of 2% is equivalent to a nominal annual discount rate of $2\% \times 12 = 24\%$ *pa* convertible monthly. In symbols this is $d^{(12)} = 24\%$.

(ii) There are six two-monthly periods in a year, so we are given $d^{(6)} = 3\%$. Therefore the two-monthly effective discount rate is $\frac{3\%}{6} = 0.5\%$.

The discount factor for one year is $(1-0.005)^6 = 0.970373$. Hence, the annual effective discount rate is $1-0.970373 = 0.029627$, *ie* 2.9627%.

In this question, we have a nominal rate of discount of 3% *pa* convertible two-monthly (*ie* $d^{(6)} = 3\%$), which is equivalent to an effective discount rate of $\frac{d^{(6)}}{6} = 0.5\%$ per two-months. So, the present value of 1 unit due in one year's time is 0.995^6, *ie* $\left(1-\frac{d^{(6)}}{6}\right)^6$. If the equivalent effective annual rate of discount is d, the present value of 1 unit in one year's time is $1-d$. So we must have $0.995^6 = 1-d$, *ie* $\left(1-\frac{d^{(6)}}{6}\right)^6 = 1-d$. Hence, in the example, the equivalent effective annual rate of discount is $d = 1-0.995^6 = 0.029627$ *ie* 2.9627%.

In general, if we are given $d^{(p)}$, then the *p*thly effective discount rate is $\frac{d^{(p)}}{p}$, *ie* we have an effective rate of discount of $\frac{d^{(p)}}{p}$ for each period of length $\frac{1}{p}$. Hence, the discount factor over one time period is $\left(1-\frac{d^{(p)}}{p}\right)^p$.

Therefore the effective discount rate d is obtained from:

$$1-d = \left(1-\frac{d^{(p)}}{p}\right)^p \tag{1.3}$$

Note that $d^{(1)} = d$.

This can be seen by substituting $p=1$ into equation (1.3). If $p=1$ we don't bother to write the superscript.

Rearranging equation (1.3) gives:

$$d^{(p)} = p\left[1-(1-d)^{1/p}\right] \tag{1.4}$$

We can use equations (1.3) and (1.4) to convert quickly between nominal and effective discount rates.

Question

(i) Calculate the nominal annual discount rate convertible quarterly that is equivalent to an effective discount rate of 5% *pa*.

(ii) Calculate the annual effective discount rate that is equivalent to a nominal discount rate of 12% *pa* convertible four-monthly.

Solution

(i) Using equation (1.4) we have:

$$d^{(4)} = 4(1-(1-0.05)^{1/4}) = 5.09658\%$$

(ii) There are three four-month periods in a year, so we are given $d^{(3)} = 12\%$. Using equation (1.3) we have:

$$1-d = \left(1-\frac{0.12}{3}\right)^3 = 0.884736 \quad \Rightarrow \quad d = 11.5264\%$$

However, it's often far more convenient to convert a nominal discount rate into an effective *interest* rate and *vice versa*.

Since $v = 1-d$, we could rewrite equation (1.4) as:

$$d^{(p)} = p\left[1-v^{1/p}\right]$$

Recalling that $v = \frac{1}{1+i} = (1+i)^{-1}$, we then have:

$$d^{(p)} = p\left[1-(1+i)^{-1/p}\right] \tag{1.5}$$

Rearranging (1.5) gives:

$$1+i = \left(1-\frac{d^{(p)}}{p}\right)^{-p} \tag{1.6}$$

This allows us to calculate the effective interest rate given a nominal discount rate convertible *p*thly.

Question

(i) Calculate the nominal annual discount rate convertible monthly that is equivalent to an effective annual interest rate of 10%.

(ii) Calculate the annual effective interest rate that is equivalent to a discount rate of 8% *pa* convertible quarterly.

Solution

(i) Using equation (1.5) we have:

$$d^{(12)} = 12\left[1 - 1.1^{-1/12}\right] = 9.49327\%$$

(ii) Using equation (1.6) we have:

$$1 + i = \left(1 - \frac{0.08}{4}\right)^{-4} = 1.0841658 \quad \Rightarrow \quad i = 8.41658\%\ pa$$

1.4 Accumulating and discounting using nominal discount rates

Since a nominal discount rate is a multiple of an effective discount rate for a period, we cannot directly use a nominal discount rate to accumulate or discount sums of money. However, we could use equation (1.3) to convert it to an effective discount rate and then use the following formulae from the previous chapter:

$$v(n) = (1-d)^n$$

$$A(n) = \frac{1}{v(n)} = (1-d)^{-n}$$

Alternatively, we could use equation (1.6) to convert it to an effective interest rate and then use the following formulae from the previous chapter:

$$A(n) = (1+i)^n$$

$$v(n) = v^n = (1+i)^{-n}$$

We will use the second method in the next question.

Question

(i) €500 is invested in an account that pays interest equivalent to a nominal discount rate of 8% *pa* convertible half-yearly. Calculate the accumulated amount in the account after 3 years.

(ii) A payment of \$800 is due in 5 years' time. Calculate the present value of this payment using a discount rate of 9% *pa* convertible monthly.

Solution

(i) We are given $d^{(2)} = 8\%$. Using equation (1.6), this is equivalent to an annual effective interest rate of $i = \left(1 - \frac{0.08}{2}\right)^{-2} - 1 = 8.50694\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500 \times 1.0850694^3 = €638.77$$

(ii) We are given $d^{(12)} = 9\%$. Using equation (1.6), this is equivalent to an annual effective interest rate of $i = \left(1 - \frac{0.09}{12}\right)^{-12} - 1 = 9.4545487\%$. Discounting the \$800 for 5 years at this rate gives:

$$\$800 \times 1.094545487^{-5} = \$509.24$$

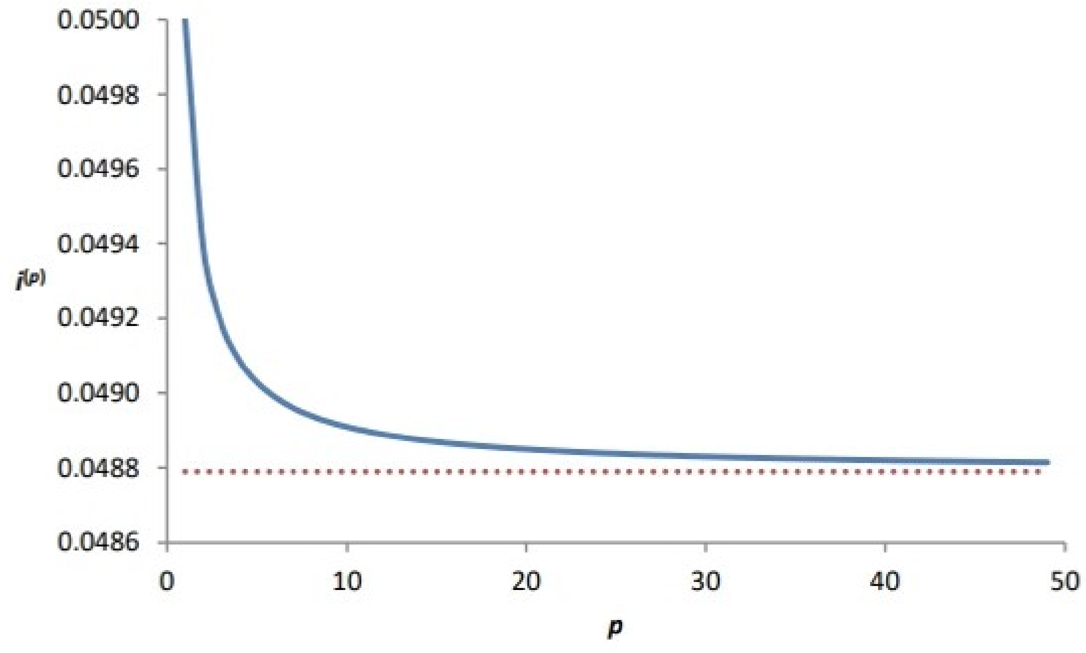
2 The force of interest

An effective rate of interest is the rate of interest a single initial investment will earn at the end of the time period. We now move on to the case where the interest is paid continuously throughout the time period.

If we consider a nominal interest rate convertible very frequently (*eg* every second), we are no longer thinking of a fund that suddenly acquires an interest payment at the end of each interval, but of a fund that steadily accumulates over the period as interest is earned and added. In the limiting case, the amount of the fund can be considered to be subject to a constant 'force' causing it to grow. This leads us to the concept of a *force of interest*, which is the easiest way to model continuously paid interest rates mathematically.

2.1 Derivation from nominal interest convertible *p*thly

Using $i^{(p)} = p\left[(1+i)^{1/p} - 1\right]$ with an effective rate of interest of 5% *pa*, we can obtain the equivalent nominal rates of interest convertible *p*thly (*eg* $i^{(2)} = 0.04949$, $i^{(4)} = 0.04909$, *etc*). If we let the value of *p* increase, we obtain the following graph:



We can see that $i^{(p)}$ is approaching a limit as $p \to \infty$. This is the nominal rate of interest convertible *continuously* (*ie* every instant) and is the called the *force of interest*. We denote the force of interest by δ.

We assume that for each value of i there is number, δ, such that:

$$\lim_{p \to \infty} i^{(p)} = \delta$$

δ is the nominal rate of interest per unit time convertible continuously (or momently). This is also referred to as the rate continuously compounded. We call it the *force of interest*.

Whilst this is all very interesting, it gives us no practical way of *calculating* the value of δ. So what we're going to do now is derive a relationship between the force of interest and the effective rate of interest.

Euler's rule states that:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Applying this to the right-hand-side of (1.1), which states that:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^p$$

gives:

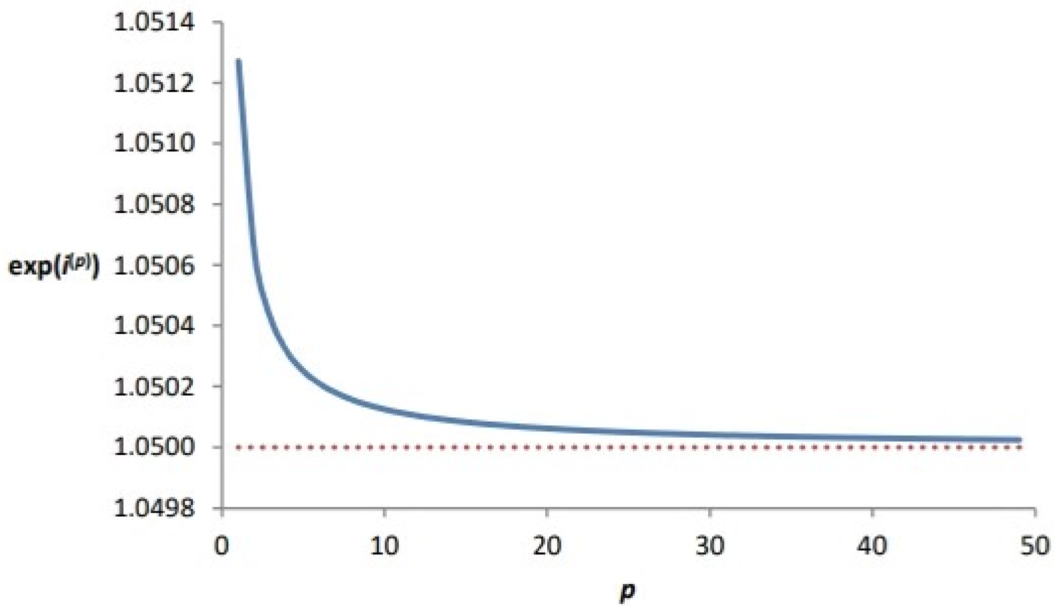
$$1+i=\lim_{p\to\infty}\left(1+\frac{i^{(p)}}{p}\right)^p = e^{i^{(\infty)}}$$

But we defined $i^{(\infty)}$ to be δ. **Hence:**

$$1+i=e^{\delta} \qquad \textbf{(2.1)}$$

This gives us our connection between the effective interest rate, i, and the force of interest, δ.

Taking the exponential of the $i^{(p)}$'s that we calculated using an effective rate of interest of 5% *pa* in our previous graph we obtain the following:



We can see that $\lim_{p\to\infty} e^{i^{(p)}} = e^{\delta} = 1.05$. This is the value of $1+i$, as seen in equation (2.1).

Rearranging the equation $e^{\delta} = 1+i$ gives:

$$\delta = \ln(1+i) \qquad (2.2)$$

In our first graph the limiting value of $i^{(p)}$ is $\delta = \ln 1.05 \approx 0.048790$.

Since we will only ever use natural log (*ie* $\ln = \log_e$) in this course, the Core Reading, the examiners and ActEd materials use $\log_e$, log and ln interchangeably.

2.2 Accumulating and discounting using the force of interest

Since the force of interest is also an annualised rate, we could convert it to an effective rate before we accumulate or discount.

Question

€500 is invested in an account that pays a force of interest of 8% *pa*. Calculate the accumulated amount in the account after 3 years.

Solution

We are given $\delta = 8\%$. Using equation (2.1), this is equivalent to an annual effective interest rate of $i = e^{0.08} - 1 = 8.3287068\%$. Accumulating the €500 for 3 years at this rate gives:

$$€500 \times 1.083287068^3 = €635.62$$

However, we can actually obtain the accumulation factor without first converting to an effective interest rate. From the previous chapter, the accumulation factor for effective interest is:

$$A(n) = (1+i)^n$$

Now (2.1) tells us that $1+i = e^{\delta}$. Substituting this in gives:

$$A(n) = (e^{\delta})^n = e^{\delta n} \qquad (2.3)$$

So, in the previous question, we could have calculated the accumulation over 3 years as:

$$€500 \times e^{0.08\times 3} = €635.62$$

Similarly we can develop a formula for a discount factor using the force of interest.

Since $v = (1+i)^{-1}$, and (2.1) tells us that $1+i = e^{\delta}$, **we have:**

$$\boldsymbol{v = e^{-\delta}} \qquad \textbf{(2.4)}$$

From equation (2.4) we have:

$$v^t = (e^{-\delta})^t = e^{-\delta t}$$

In the previous chapter, we saw that the discount factor over n years is:

$$v(n) = v^n = (1+i)^{-n}$$

Hence, the discount factor for a force of interest δ is:

$$v(n) = e^{-\delta n} \tag{2.5}$$

Question

A payment of \$800 is due in 5 years' time. Calculate the present value of this payment using a force of interest of 9% *pa*.

Solution

We are given $\delta = 9\%$. Using equation (2.5), the present value is:

$$\$800 \times e^{-0.09 \times 5} = \$510.10$$

Alternatively, we can first calculate the annual effective interest rate, i, using equation (2.1):

$$i = e^{0.09} - 1 = 9.4174284\% .$$

Then we can discount the \$800 for 5 years at this rate to give:

$$\$800 \times 1.094174284^{-5} = \$510.10$$

2.3 Derivation from nominal discount convertible *p*thly

In Section 2.1 we defined the force of interest, δ, as $\lim_{p\to\infty} i^{(p)}$.

It can also be shown that:

$$\lim_{p\to\infty} d^{(p)} = \delta \tag{2.6}$$

The proof is very similar to before. We apply Euler's rule, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$, to the RHS of (1.6):

$$1+i = \lim_{p\to\infty}\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

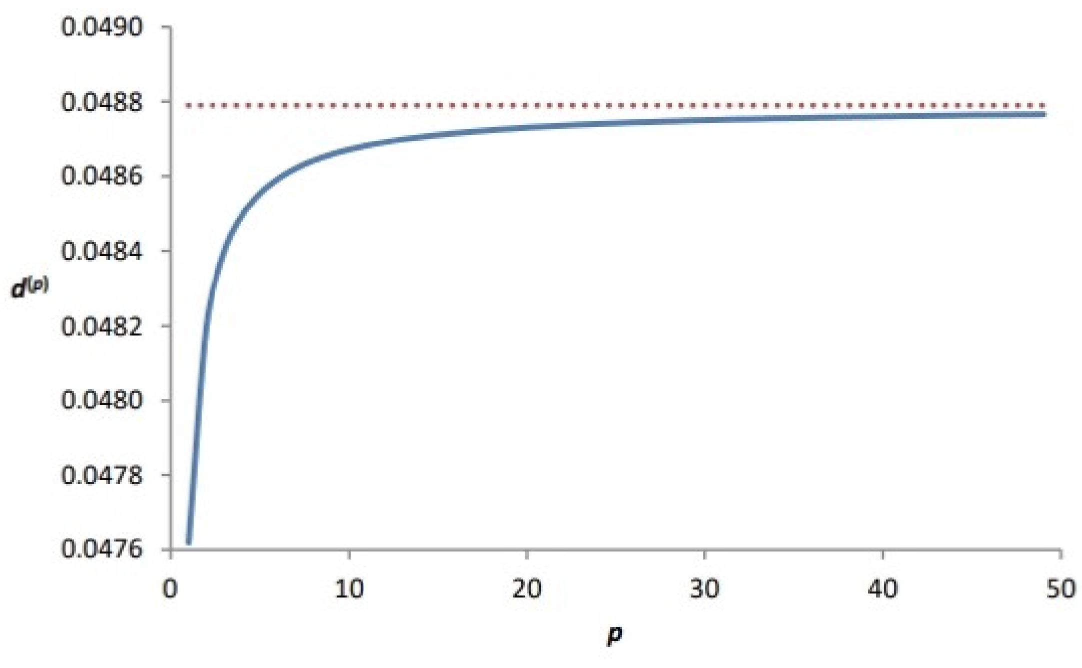
We then use the fact that $\lim \frac{1}{f(x)} = \frac{1}{\lim f(x)}$. Hence:

$$1+i = \frac{1}{\lim\limits_{p\to\infty}\left(1-\frac{d^{(p)}}{p}\right)^p} = \frac{1}{e^{-d^{(\infty)}}} = e^{d^{(\infty)}}$$

We showed that $1+i=e^{\delta}$ in (2.1) and here we have seen that $1+i=e^{d^{(\infty)}}$. Therefore $d^{(\infty)}=\delta$, or more correctly $\lim\limits_{p\to\infty} d^{(p)} = \delta$.

Let's have a look at an example of this.

Using $d^{(p)} = p\left[1-(1+i)^{-1/p}\right]$ with an effective rate of interest of 5% *pa* we can obtain the equivalent nominal rates of discount convertible *p*thly. If we let the value of *p* increase, we obtain the graph below:



We can see that $d^{(p)}$ is approaching the same limit of $\delta = \ln 1.05 = 0.048790$ as before.

However, $d^{(p)}$ tends to this limit from below whereas $i^{(p)}$ tends to this limit from above.

Hence, we have:

$$\boldsymbol{d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i}$$

This order reflects how late interest is paid. For example, d corresponds to interest paid immediately, and so requires a smaller payment amount than i , which corresponds to interest paid in one year's time.

To demonstrate this further, pick an interest rate (say $i = 8\%$), go to page 60 of the *Tables* and look up some values.

Being able to convert quickly between the different rates of interest and discount is a key skill needed for the exam.

Question

(i) Given $\delta = 8\%$, calculate i, $i^{(4)}$ and $d^{(12)}$.

(ii) Given $i = 7\%$, calculate d, $d^{(4)}$, $i^{(2)}$ and δ.

(iii) Given $d = 9\%$, calculate i, $d^{(2)}$, $i^{(12)}$ and δ.

Solution

(i) $1+i = e^{\delta} = e^{0.08} \;\Rightarrow\; i = e^{0.08} - 1 = 8.33\%$

$$i^{(4)} = 4\left[(1+i)^{1/4} - 1\right] = 4\left[(e^{0.08})^{1/4} - 1\right] = 4\left[e^{0.02} - 1\right] = 8.08\%$$

$$d^{(12)} = 12\left[1 - (1+i)^{-1/12}\right] = 12\left[1 - (e^{0.08})^{-1/12}\right] = 7.97\%$$

(ii) $d = 1 - v = 1 - 1.07^{-1} = 6.54\%$

$$d^{(4)} = 4\left[1 - 1.07^{-1/4}\right] = 6.71\%$$

$$i^{(2)} = 2\left[1.07^{1/2} - 1\right] = 6.88\%$$

$$\delta = \ln(1.07) = 6.77\%$$

(iii) $v = 1 - d = 0.91 \;\Rightarrow\; 1+i = \dfrac{1}{0.91} \;\Rightarrow\; i = 9.89\%$

$$d^{(2)} = 2\left[1 - (1-d)^{1/2}\right] = 2\left[1 - (1-0.09)^{1/2}\right] = 9.21\%$$

$$i^{(12)} = 12\left[(1+i)^{1/12} - 1\right] = 12\left[0.91^{-1/12} - 1\right] = 9.47\%$$

$$\delta = \ln(1+i) = \ln 1.0989 = 9.43\%$$

The following table summarises the formulae we've met that link the effective annual interest rate i, the effective annual discount rate d, the force of interest δ, and the one-year discount factor v.

		Value of..			
		δ	i	v	d
In terms of..	δ		$e^{\delta}-1$	$e^{-\delta}$	$1-e^{-\delta}$
	i	$\ln(1+i)$		$(1+i)^{-1}$	$i/(1+i)$
	v	$-\ln v$	$v^{-1}-1$		$1-v$
	d	$-\ln(1-d)$	$(1-d)^{-1}-1$	$1-d$	

These can all be obtained by manipulating the basic relationships:

- $v=\dfrac{1}{1+i}$
- $\delta=\ln(1+i)$
- $d=iv=1-v$

3 Relationships between effective, nominal and force of interest

3.1 An alternative way of considering nominal interest convertible *p*thly

Recall that effective interest i can be thought of as interest paid at the *end* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of $(1+i)$ at time 1.

Similarly, nominal interest convertible *p*thly can be thought of as the total interest per unit of time paid on a loan of amount 1 at time 0, where interest is paid in p equal instalments at the *end* of each p th subinterval (*ie* at times $1/p, 2/p, 3/p, \ldots, 1$).

Instead of paying one instalment of interest at time 1, we are paying p instalments throughout the time period. Therefore, the accumulated value of the p interest payments, each of amount $i^{(p)}/p$, is equal to i.

Since $i^{(p)}$ is the *total* interest paid and each interest payment is of amount $i^{(p)}/p$, the accumulated value at time 1 of the interest payments is:

$$\frac{i^{(p)}}{p}(1+i)^{(p-1)/p} + \frac{i^{(p)}}{p}(1+i)^{(p-2)/p} + \cdots + \frac{i^{(p)}}{p} = i$$

The terms on the left-hand side of the above equation form a geometric progression, so we can use the formula for the sum of the first n terms of a geometric progression:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r}$$

where: $a = \text{first term} = \dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}$

$r = \text{common ratio} = (1+i)^{-1/p}$

and: $n = \text{number of terms in the sum} = p$.

So, provided that $i \neq 0$, the left-hand side is:

$$\frac{\dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}\left(1-\left[(1+i)^{-1/p}\right]^p\right)}{1-(1+i)^{-1/p}} = \frac{\dfrac{i^{(p)}}{p}(1+i)^{(p-1)/p}\left(1-(1+i)^{-1}\right)}{1-(1+i)^{-1/p}}$$

Multiplying the numerator and denominator by $(1+i)^{1/p}$ gives:

$$\frac{\dfrac{i^{(p)}}{p}(1+i)^1\left(1-(1+i)^{-1}\right)}{(1+i)^{1/p}-1} = i^{(p)} \times \frac{(1+i)^1-1}{p\left[(1+i)^{1/p}-1\right]} = i^{(p)} \times \frac{i}{p\left[(1+i)^{1/p}-1\right]}$$

Now recall that this sum is equal to i, so:

$$i^{(p)} \times \frac{i}{p\left[(1+i)^{1/p}-1\right]} = i$$

Hence, cancelling the i's and rearranging:

$$\boldsymbol{i^{(p)} = p\left[(1+i)^{1/p}-1\right]}$$

This is the result we obtained in equation (1.2), which confirms that our alternative way of thinking about nominal interest is correct. The fact that $i^{(p)}$ can be used to represent *p*thly payments of interest will be useful to us later when we introduce *p*thly annuities.

3.2 An alternative way of considering nominal discount convertible *p*thly

Recall that effective discount d can be thought of as interest paid at the *start* of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of 1 at time 1, but d is paid at the start so a sum of $(1-d)$ is lent at time 0.

Similarly, $d^{(p)}$ is the total amount of interest per unit of time payable in equal instalments at the *start* of each p th subinterval (*ie* at times $0, 1/p, 2/p, \ldots, (p-1)/p$).

Instead of paying one instalment of interest of d at time 0, we are paying p instalments, each of amount $d^{(p)}/p$, throughout the time period.

As a consequence the present value at time 0 of the interest payments is:

$$\boldsymbol{\frac{d^{(p)}}{p} + \frac{d^{(p)}}{p}(1-d)^{1/p} + \cdots + \frac{d^{(p)}}{p}(1-d)^{(p-1)/p} = d}$$

The terms on the left-hand side of the above equation form a geometric progression, with first term $\dfrac{d^{(p)}}{p}$, common ratio $(1-d)^{1/p}$ and p terms. Summing this (provided that $d \neq 0$) gives:

$$\frac{\dfrac{d^{(p)}}{p}\left(1-\left[(1-d)^{1/p}\right]^{p}\right)}{1-(1-d)^{1/p}}$$

Simplifying this gives:

$$\frac{d^{(p)}}{p} \times \frac{(1-(1-d))}{1-(1-d)^{1/p}} = d^{(p)} \times \frac{d}{p\left[1-(1-d)^{1/p}\right]}$$

Now recall that this sum is equal to d, so:

$$d^{(p)} \times \frac{d}{p\left[1-(1-d)^{1/p}\right]} = d$$

Hence, cancelling the d's and rearranging:

$$\boldsymbol{d^{(p)} = p\left[1-(1-d)^{1/p}\right]}$$

This is the result we obtained in equation (1.4), which confirms that our alternative way of thinking about nominal discount is correct. The fact that $d^{(p)}$ can be used to represent *p*thly payments of interest in advance will also be useful to us later.

3.3 An alternative way of considering force of interest

Now δ is the total amount of interest payable as a continuous payment stream, *ie* an amount δdt is paid over an infinitesimally small period dt at time t.

So the accumulated value at time 1 of a single payment of δdt at time t is:

$$\delta(1+i)^{1-t}\,dt$$

If t can take any value between 0 and 1, then the total accumulated value (allowing for all values of t) can be calculated by integration. (Integration can be thought of as summation in a continuous sense.)

As a consequence the accumulated value at time 1 of these interest payments is:

$$\boldsymbol{\int_0^1 \delta(1+i)^{1-t}\,dt}$$

The total of all these interest payments is again equal to a single payment of i at time 1, *ie*:

$$\int_0^1 \delta(1+i)^{1-t}\,dt = i$$

Since we are integrating $\delta(1+i)^{1-t}$ from $t=0$ to 1, the integrand ranges from $\delta(1+i)$ to δ. Equally we could have integrated $\delta(1+i)^t$ from $t=1$ to 0 to give the same answer.

So the integral, **by symmetry, is equal to:**

$$\boldsymbol{\int_0^1 \delta(1+i)^t\,dt = i}$$

This can also be seen by making the substitution $u=1-t$ in the original integral above.

To evaluate the integral above, note that:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a} = e^{x\ln a} \times \ln a = a^x \times \ln a$$

So $\int a^x \, dx = \frac{a^x}{\ln a}$ (ignoring the constant of integration). Here, a is $(1+i)$, so:

$$\int_0^1 \delta(1+i)^t \, dt = \delta\left[\frac{(1+i)^t}{\ln(1+i)}\right]_0^1 = \delta\frac{(1+i)-1}{\ln(1+i)} = \delta\frac{i}{\ln(1+i)}$$

We now set this equal to i and rearrange. **Hence:**

$$\boldsymbol{\delta = \ln(1+i) \qquad \text{or} \qquad e^{\delta} = 1+i}$$

This is the result we obtained in equation (2.1), which confirms that our alternative way of thinking about the force of interest is correct. The fact that δ can be used to represent continuous interest payments will also be useful to us later.

We have shown several different but equivalent ways of interest being paid over time. These are summarised below.

It is essential to appreciate that, at force of interest δ per unit time, the five series of payments illustrated in Figure 1 below all have the same value.

	0	$\frac{1}{p}$	$\frac{2}{p}$	$\frac{3}{p}$	...	$\frac{p-1}{p}$	1	time
(1)	d							
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	...	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	...	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	**equivalent payments**
(4)							i	
(5)	$\longleftarrow$			δ			$\longrightarrow$	

Figure 1 Equivalent payments

4 Force of interest as a function of time

In Section 2, we introduced the constant force of interest to describe the value of payments that are made continuously. In this section, we generalise this idea to consider the case when the force of interest is a function of time.

4.1 Formal definition

The force of interest is the instantaneous change in the fund value, expressed as an annualised percentage of the current fund value.

So the force of interest at time t is defined to be:

$$\boldsymbol{\delta(t) = \frac{V_t'}{V_t}} \tag{4.1}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

How does this relate back to our previous definition of the (constant) force of interest? Suppose that we have a constant effective rate i and we invest C at time zero. Then the value at the fund at time t is:

$$V_t = C(1+i)^t$$

We now differentiate this with respect to t. Recall that $\frac{d}{dx}a^x = a^x \ln a$. So we have:

$$V_t' = C(1+i)^t \ln(1+i)$$

Substituting these into our definition (4.1) gives a constant force of interest of:

$$\delta(t) = \frac{C(1+i)^t \ln(1+i)}{C(1+i)^t} = \ln(1+i)$$

This is what we saw in equation (2.2). So our previous definition of a constant force of interest is actually a special case of definition (4.1).

Let's now obtain an expression for the accumulation factor using this variable force of interest.

Using the chain rule for differentiation, $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$. Applying this result to our definition $\delta(t) = \frac{V_t'}{V_t}$, we see that:

$$\boldsymbol{\delta(t) = \frac{d}{dt}\ln V_t}$$

Integrating this from t_1 to t_2 gives:

$$\int_{t_1}^{t_2} \delta(t)dt = \left[\ln V_t\right]_{t_1}^{t_2} = \ln V_{t_2} - \ln V_{t_1} = \ln\left(\frac{V_{t_2}}{V_{t_1}}\right)$$

Taking exponentials of both sides gives:

$$\Rightarrow \quad \frac{V_{t_2}}{V_{t_1}} = \exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right)$$

Hence:

$$A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right) \qquad \textbf{(4.2)}$$

This formula gives us an accumulation factor between times t_1 and t_2 when we have a variable force of interest, and appears on page 31 of the *Tables*. We can also write:

$$A(n) = A(0,n) = \exp\left(\int_0^n \delta(t)\,dt\right)$$

Let's apply this formula.

Question

The force of interest at time t is $\delta(t) = 0.02 + 0.01t$. Calculate the accumulated value at time 8 of an investment of £1,000 at time:

(i) 0

(ii) 5

Solution

(i) The accumulated value is:

$$V_8 = 1{,}000A(0,8)$$

Using (4.2) we have:

$$V_8 = 1{,}000\exp\left(\int_0^8 0.02 + 0.01t\,dt\right) = 1{,}000\exp\left(\left[0.02t + 0.005t^2\right]_0^8\right)$$

$$= 1{,}000e^{0.48} = £1{,}616.07$$

(ii) The accumulated value is:

$$V_8 = 1{,}000A(5,8)$$

Using (4.2) we have:

$$V_8 = 1{,}000\exp\left(\int_5^8 0.02 + 0.01t\,dt\right) = 1{,}000\exp\left(\left[0.02t + 0.005t^2\right]_5^8\right)$$

$$= 1{,}000e^{0.48-0.225} = £1{,}290.46$$

The force of interest can also be a stepwise function of time. In that case, we can apply (4.2) repeatedly, once for each of the different time periods covered by the accumulation.

Question

The force of interest at time t is given by:

$$\delta(t) = \begin{cases} 0.08 & 0 \le t < 5 \\ 0.13 - 0.01t & 5 \le t \end{cases}$$

Calculate the accumulated value at time 10 of an investment of \$500 at time 2.

Solution

The accumulated value is:

$$\begin{aligned} V_{10} &= 500A(2,10) \\ &= 500A(2,5)A(5,10) \\ &= 500\exp\left(\int_2^5 0.08\,dt\right)\exp\left(\int_5^{10} 0.13 - 0.01t\,dt\right) \\ &= 500\exp\left(\left[0.08t\right]_2^5\right)\exp\left(\left[0.13t - 0.005t^2\right]_5^{10}\right) \\ &= 500e^{0.4-0.16}e^{0.8-0.525} \\ &= 500e^{0.515} \\ &= \$836.82 \end{aligned}$$

4.2 Relationship to constant force of interest

For the case when the force of interest is constant, δ, between time 0 and time n, we have:

$$A(n) = \mathbf{A(0,n)} = e^{\int_0^n \delta\, dt} = e^{\delta n}$$

This is equation (2.3) that we obtained earlier.

Equating this accumulation factor to the effective interest accumulation factor, we can obtain the relationship between constant force of interest and effective interest.

Hence:

$$(1+i)^n = e^{\delta n}$$

Therefore:

$$(1+i) = e^{\delta}$$

as before.

This is equation (2.1), which we obtained from our definition of the force of interest in terms of nominal rates of interest convertible *p*thly. Here we have reached it from a different starting point.

4.3 Present values

We can also obtain present values when we have a variable force of interest. The accumulation at time t_2 of an investment of C at time t_1 is given by:

$$C\,A(t_1,t_2) = C\exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right)$$

Therefore an investment of:

$$\frac{C}{A(t_1,t_2)} = \frac{C}{\exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right)} = C\exp\left(-\int_{t_1}^{t_2} \delta(t)\,dt\right)$$

at time t_1 will give C at time t_2. This is the discounted value at time t_1 of C due at time t_2. We simply have a negative power for discounting, which is similar to using $(1+i)^{-n}$ to discount for n years at a constant effective rate of interest.

In particular, the discounted value at time 0 (the present time) of C due at time $n \geq 0$ is called its discounted *present* value (or, more briefly, its present value). It is equal to:

$$C\exp\left(-\int_0^n \delta(t)\,dt\right)$$

Using the notation of $v(n)$ to represent the present value of a payment of 1 due at time n as defined in the previous chapter, we have:

$$v(n) = \exp\left(-\int_0^n \delta(t)\,dt\right)$$

We'll now look at some examples of calculating present values when the force of interest changes over time. We'll apply the same principles as when we were accumulating, but just put a negative in the power of the exponential.

Question

The force of interest at time t is given by:

$$\delta(t) = \begin{cases} 0.08 & 0 \leq t < 5 \\ 0.13 - 0.01t & 5 \leq t \end{cases}$$

(i) Calculate the present value at time 3 of a payment of \$500 at time 10.

(ii) Calculate the annual effective rate of interest that is equivalent to this variable force of interest from time 3 to time 10.

Solution

(i) The present value is:

$$\begin{aligned} V_3 &= 500\frac{1}{A(3,10)} \\ &= 500\frac{1}{A(3,5)}\frac{1}{A(5,10)} \\ &= 500\exp\left(-\int_3^5 0.08\,dt\right)\exp\left(-\int_5^{10} 0.13 - 0.01t\,dt\right) \\ &= 500\exp\left(-\left[0.08t\right]_3^5\right)\ \exp\left(-\left[0.13t - 0.005t^2\right]_5^{10}\right) \\ &= 500e^{-(0.4-0.24)}e^{-(0.8-0.525)} = 500e^{-0.435} \\ &= \$323.63 \end{aligned}$$

(ii) In terms of an annual effective rate of interest i, the discount factor for 7 years, from time 10 to time 3 is:

$$v^7 = (1+i)^{-7}$$

Setting this equal to the discount factor from time 10 to time 3 based on the variable force of interest calculated in part (i), we find:

$$(1+i)^{-7} = e^{-0.435} \quad \Rightarrow \quad i = \left(e^{-0.435}\right)^{-1/7} - 1 = 0.06411 \quad ie \quad 6.411\%$$

Finally, we'll obtain some general expressions for accumulation factors where there is a variable force of interest. We can use the same approach to obtain general expressions for discount factors.

Question

The force of interest at time t is:

$$\delta(t) = \begin{cases} 0.08 & 0 \le t < 5 \\ 0.13 - 0.01t & 5 \le t \end{cases}$$

Determine expressions for the accumulation factor from time 0 to time t.

Solution

We need separate expressions here for the accumulation factor depending on whether t is less than 5 or greater than or equal to 5. When $0 \le t < 5$ we have:

$$A(0,t) = \exp\left(\int_0^t 0.08\,ds\right) = \exp\left(\left[0.08s\right]_0^t\right) = e^{0.08t}$$

We should not use t as both the variable of integration and the upper limit on the integral. So we change the variable to another letter, say s.

When $5 \le t$, we need to determine the product of two accumulation factors:

$$A(0,t) = A(0,5)A(5,t) = \exp\left(\int_0^5 0.08\,dt\right)\exp\left(\int_5^t 0.13 - 0.01s\,ds\right)$$

$$= \exp\left(\left[0.08t\right]_0^5\right)\exp\left(\left[0.13s - 0.005s^2\right]_5^t\right)$$

$$= e^{0.4} e^{0.13t - 0.005t^2 - 0.525}$$

$$= e^{0.13t - 0.005t^2 - 0.125}$$

We could also have obtained $A(0,5)$ *by substituting* $t=5$ *into our first expression for* $A(0,t)$.

So, in summary:

$$A(0,t)=\begin{cases} e^{0.08t} & 0\le t<5 \\ e^{0.13t-0.005t^2-0.125} & 5\le t \end{cases}$$

4.4 Applications of force of interest

Although the force of interest is a theoretical measure, it is the most fundamental measure of interest (as all other interest rates can be derived from it). However, since the majority of transactions involve discrete processes we tend to use other interest rates in practice.

It still remains a useful conceptual and analytical tool and can be used as an approximation to interest paid very frequently, *eg* daily.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 4 Summary

Effective rates have interest paid *once* per measurement period. Effective *interest*, i, refers to the amount of interest that is paid at the *end* of the year. Effective *discount*, d, refers to the amount of interest that is paid at the *start* of the year.

Nominal rates are paid *more frequently* than once per measurement period. A nominal rate of *interest* convertible $p$thly, $i^{(p)}$, refers to the total amount of interest that is paid in p equal instalments at the *end* of each p th subinterval.

The relationships between nominal interest and effective interest are:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^{p} \qquad i^{(p)}=p\left[(1+i)^{1/p}-1\right]$$

$i^{(p)}/p$ is the effective p thly rate of interest (*ie* the effective rate of interest for a time period of length $1/p$).

A nominal rate of *discount* convertible p thly, $d^{(p)}$, refers to the total amount of interest that is paid in p equal instalments at the *start* of each p th subinterval.

The relationships between nominal discount and effective interest and discount are:

$$1-d=\left(1-\frac{d^{(p)}}{p}\right)^{p} \qquad d^{(p)}=p\left[1-(1-d)^{1/p}\right]$$

$$1+i=\left(1-\frac{d^{(p)}}{p}\right)^{-p} \qquad d^{(p)}=p\left[1-(1+i)^{-1/p}\right]$$

$d^{(p)}/p$ is the effective p thly rate of discount (*ie* the effective rate of discount for a time period of length $1/p$).

To accumulate or discount with nominal rates requires that they are first converted to effective rates.

The **force of interest**, δ, is the amount of interest that is paid continuously over a time period. It is defined as $\delta=\lim_{p\to\infty} i^{(p)}=\lim_{p\to\infty} d^{(p)}$.

The relationships between the force of interest and effective interest are:

$$1+i=e^{\delta} \qquad \delta=\ln(1+i)$$

To accumulate or discount at a (constant) force of interest δ, we use the following:

$$A(n) = e^{\delta n} \qquad v(n) = e^{-\delta n}$$

More generally, we can define the force of interest as the instantaneous change in the fund value, expressed as an annualised percentage of the current fund value:

$$\delta(t) = \frac{V_t'}{V_t}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

Using this, we obtain the following accumulation factor:

$$A(t_1, t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)\,dt\right)$$

Chapter 4 Practice Questions

4.1 Given that $i = 0.07$, calculate:

(i) $i^{(6)}$

(ii) $d^{(6)}$

(iii) $i^{(4)}$

(iv) $d^{(2)}$

4.2 Calculate the annual effective rate of discount that is equivalent to a rate of interest of 4% *pa* convertible monthly.

4.3 (i) Calculate the effective annual rate of interest corresponding to:

Exam style

(a) a nominal rate of interest of 11% *pa* convertible half-yearly

(b) a nominal rate of interest of 12% *pa* convertible monthly. [2]

(ii) Calculate the rate of interest convertible monthly corresponding to:

(a) an effective rate of interest of 14.2% *pa*

(b) a nominal rate of interest of 11% *pa* convertible three times a year. [3]

[Total 5]

4.4 The constant nominal rate of interest convertible quarterly is 15% *pa*. Calculate the accumulated value after 7 years of a payment of £300.

4.5 £250 is invested at a discount rate of 18% *pa* convertible monthly for the first 3 months followed by an interest rate of 20% *pa* convertible quarterly for the next 9 months. Calculate the accumulated sum at the end of the year.

4.6 Assuming a force of interest of 9% *pa*, calculate the accumulated value of £6.34 after:

(i) 3 months

(ii) 3 years

(iii) 7 years and 5 days.

4.7 (i) Calculate the accumulated value after 6 months of an investment of £100 at time 0 using the following rates of interest:

Exam style

(a) a force of interest of 5% *pa*

(b) a rate of interest of 5% *pa* convertible monthly

(c) an effective rate of interest of 5% *pa*. [3]

(ii) Explain why the answer obtained in (i)(a) is higher than the answer obtained in (i)(c). [2]

[Total 5]

4.8 The force of interest at time t is:

$$\delta(t) = \begin{cases} 0.04 & 0 \le t < 6 \\ 0.2 - 0.02t & 6 \le t \end{cases}$$

Calculate the accumulated value at time 8 of a payment of $400 at time 3.

4.9 The force of interest at time t is given by $\delta(t) = 0.01t + 0.04$. Calculate the corresponding nominal rate of discount convertible half-yearly for the period $t = 1$ to $t = 2$.

4.10 The force of interest at time t is given by:

Exam style

$$\delta(t) = \begin{cases} 0.08 - 0.001t & 0 \le t < 3 \\ 0.025t - 0.04 & 3 \le t < 5 \\ 0.03 & 5 \le t \end{cases}$$

(i) Calculate the present value at time 2 of a payment of £1,000 at time 10. [5]

(ii) Calculate the annual effective rate of interest that is equivalent to this variable force of interest from time 2 to time 10. [2]

[Total 7]

4.11 The force of interest at time t is given by:

Exam style

$$\delta(t) = \begin{cases} 0.04 + 0.002t & 0 \le t < 10 \\ 0.015t - 0.08 & 10 \le t < 12 \\ 0.07 & 12 \le t \end{cases}$$

Determine expressions for the present value at time 0 of 1 unit of money due at time t. [5]

Chapter 4 Solutions

4.1 (i) $i^{(6)} = 6\left((1+i)^{1/6} - 1\right) = 6\left((1.07)^{1/6} - 1\right) = 0.068042 \quad \textit{ie} \quad 6.8042\%$

(ii) $d^{(6)} = 6\left(1 - (1+i)^{-1/6}\right) = 6\left(1 - (1.07)^{-1/6}\right) = 0.067279 \quad \textit{ie} \quad 6.7279\%$

The relationship $d = iv$ only holds for effective rates of interest and discount, so we cannot calculate $d^{(6)}$ from $i^{(6)}$ using $d^{(6)} = \dfrac{i^{(6)}}{1+i^{(6)}}$. However, we can calculate it using:

$$\frac{d^{(6)}}{6} = \frac{\frac{i^{(6)}}{6}}{1+\frac{i^{(6)}}{6}}$$

(iii) $i^{(4)} = 4\left((1+i)^{1/4} - 1\right) = 4\left((1.07)^{1/4} - 1\right) = 0.068234 \quad \textit{ie} \quad 6.8234\%$

Alternatively, this value can be taken directly from the Tables.

(iv) $d^{(2)} = 2\left(1 - (1+i)^{-1/2}\right) = 2\left(1 - (1.07)^{-1/2}\right) = 0.066527 \quad \textit{ie} \quad 6.6527\%$

Alternatively, this value can be taken directly from the Tables.

4.2 Given $i^{(12)} = 4\%$, we can calculate the annual effective interest rate as follows:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.04}{12}\right)^{12} - 1 = 0.040742$$

So the annual effective discount rate is:

$$d = 1 - v = 1 - \frac{1}{1+i} = 1 - \frac{1}{1.040742} = 0.039147 \quad \textit{ie} \quad 3.9147\%$$

Alternatively, we can calculate d directly, using different forms of discount factor:

$$d = 1 - v = 1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-12} = 1 - \left(1 + \frac{0.04}{12}\right)^{-12} = 0.039147$$

4.3 (i) (a) $i = \left(1 + \frac{i^{(2)}}{2}\right)^{2} - 1 = \left(1 + \frac{0.11}{2}\right)^{2} - 1 = 0.113025 \quad \textit{ie} \quad i = 11.3025\%$ [1]

(b) $i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 0.126825 \quad \textit{ie} \quad i = 12.6825\%$ [1]

[Total 2]

(ii) (a) $i^{(12)} = 12\left((1+i)^{1/12} - 1\right) = 12\left(1.142^{1/12} - 1\right) = 0.133518$ *ie* $i^{(12)} = 13.3518\%$ [1]

(b) If $i^{(3)} = 11\%$, then:

$$i = \left(1 + \frac{i^{(3)}}{3}\right)^3 - 1 = \left(1 + \frac{0.11}{3}\right)^3 - 1 = 0.114083$$ [1]

So:

$$i^{(12)} = 12\left((1+i)^{1/12} - 1\right) = 12\left[1.114083^{1/12} - 1\right] = 0.108519$$

ie $i^{(12)} = 10.8519\%$. [1]

[Total 3]

4.4 We are given $i^{(4)} = 15\%$. This is equivalent to an annual effective rate of:

$$i = \left(1 + \tfrac{0.15}{4}\right)^4 - 1 = 15.8650\%$$

Accumulating £300 for 7 years at this rate gives:

$$£300 \times 1.158650^7 = £840.98$$

Alternatively, we could work in quarters. We are given $i^{(4)} = 15\%$. *This is equivalent to a quarterly effective interest rate of* $\tfrac{15\%}{4} = 3.75\%$. *Accumulating £300 for 7 years (= 28 quarters) at this rate gives:*

$$£300 \times 1.0375^{28} = £840.98$$

4.5 For the first 3 months, we have $d^{(12)} = 18\%$. So the annual effective interest rate is:

$$i = \left(1 - \tfrac{0.18}{12}\right)^{-12} - 1 = 19.88511\%$$

Accumulating £250 for 3 months at this rate gives:

$$£250 \times 1.1988511^{3/12} = £261.60$$

For the next 9 months, we have $i^{(4)} = 20\%$. So the annual effective interest rate is:

$$i = \left(1 + \tfrac{0.2}{4}\right)^4 - 1 = 21.550625\%$$

Accumulating £261.60 for 9 months at this rate gives:

$$£260.61 \times 1.21550625^{9/12} = £302.83$$

Alternatively, we could work in months for the first 3 months. We are given $d^{(12)} = 18\%$. *This is equivalent to a monthly effective discount rate of* $\frac{18\%}{12} = 1.5\%$. *Accumulating £250 for 3 months at this rate gives:*

$$£250 \times (1-0.015)^{-3} = £261.60$$

Then we could work in quarters for the next 9 months. We have $i^{(4)} = 20\%$, *which is equivalent to an effective quarterly interest rate of* $\frac{20\%}{4} = 5\%$. *Accumulating £261.60 for 9 months (= 3 quarters) at this rate gives:*

$$£261.60 \times 1.05^3 = £302.83$$

4.6 (i) $6.34e^{\frac{3}{12}\times 0.09} = £6.48$

(ii) $6.34e^{3\times 0.09} = £8.31$

(iii) $6.34e^{\left(7+\frac{5}{365}\right)\times 0.09} = £11.92$

4.7 (i) The accumulated values after 6 months are:

(a) $100e^{0.5\delta} = 100e^{0.5\times 0.05} = £102.53$ [1]

(b) $100\left(1+\frac{i^{(12)}}{12}\right)^6 = 100\left(1+\frac{0.05}{12}\right)^6 = £102.53$ [1]

(c) $100(1+i)^{0.5} = 100 \times 1.05^{0.5} = £102.47$ [1]

[Total 3]

(ii) The force of interest in (i)(a) is 5% *pa* and the effective annual rate of interest in (i)(c) is 5% *pa*. The force of interest relates to interest being paid continuously over the period, whereas the effective annual rate of interest relates to interest being paid at the end of the period. [1]

Since the interest received early in the period under the force of interest will earn further interest, the accumulated value is higher than when the interest is all paid at the end of the period (and can therefore not earn further interest). [1]

A force of interest of 5% pa is equivalent to an annual effective interest rate of $i = e^{0.05} - 1 = 5.127\%$, *which is greater than the 5% figure used in (i)(c).*

[Total 2]

4.8 The accumulated value at time 8 is:

$$\begin{aligned}400A(3,8) &= 400A(3,6)A(6,8)\\ &= 400\exp\left(\int_3^6 0.04\,dt\right)\exp\left(\int_6^8 0.2-0.02t\,dt\right)\\ &= 400\exp\left([0.04t]_3^6\right)\exp\left(\left[0.2t-0.01t^2\right]_6^8\right)\\ &= 400e^{0.24-0.12}e^{0.96-0.84}\\ &= 400e^{0.24}\\ &= \$508.50\end{aligned}$$

4.9 The accumulation factor from time 1 to time 2 is:

$$\begin{aligned}A(1,2) &= \exp\left(\int_1^2 (0.01t+0.04)dt\right)\\ &= \exp\left(\left[0.005t^2+0.04t\right]_1^2\right)\\ &= \exp[(0.02+0.08)-(0.005+0.04)]\\ &= e^{0.055}\end{aligned}$$

Equating this to a one-year accumulation factor in terms of $d^{(2)}$, the nominal rate of discount convertible half-yearly, we find that:

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2} = e^{0.055} \quad \Rightarrow \quad d^{(2)} = 2\left(1-\left(e^{0.055}\right)^{-1/2}\right) = 0.054251 \quad ie \quad 5.4251\%$$

4.10 (i) ***Present value***

The present value at time 2 is given by the expression:

$$PV_{t=2} = 1{,}000\exp\left(-\int_2^3 (0.08-0.001t)\,dt\right)\exp\left(-\int_3^5 (0.025t-0.04)\,dt\right)\exp\left(-\int_5^{10} 0.03\,dt\right) \qquad [2]$$

Evaluating the integrals, we find that:

$$PV_{t=2} = 1{,}000\exp\left(-\left[0.08t - 0.0005t^2\right]_2^3\right)\exp\left(-\left[0.0125t^2 - 0.04t\right]_3^5\right)\exp\left(-\left[0.03t\right]_5^{10}\right)$$

$$= 1{,}000e^{-(0.2355-0.158)}e^{-(0.1125+0.0075)}e^{-(0.3-0.15)}$$

$$= 1{,}000e^{-0.3475}$$

$$= £706.45$$ [3]

[Total 5]

(ii) ***Equivalent annual effective interest rate***

If the annual effective rate of interest is i, we have:

$$706.45(1+i)^8 = 1{,}000$$ [1]

Solving for i gives:

$$i = \left(\frac{1{,}000}{706.45}\right)^{1/8} - 1 = 4.44\%\ pa$$ [1]

[Total 2]

4.11 We need separate expressions here for the present values depending on whether t is less than 10, between 10 and 12 or greater than or equal to 12.

When $0 \le t < 10$:

$$v(t) = \exp\left(-\int_0^t 0.04 + 0.002s\, ds\right)$$

$$= \exp\left(-\left[0.04s + 0.001s^2\right]_0^t\right)$$

$$= e^{-0.04t-0.001t^2}$$ [1]

When $10 \le t < 12$, we need the product of two discount factors:

$$v(t) = \exp\left(-\int_0^{10} (0.04 + 0.002t)\, dt\right)\exp\left(-\int_{10}^t (0.015s - 0.08)\, ds\right)$$

$$= \exp\left(-\left[0.04t + 0.001t^2\right]_0^{10}\right)\exp\left(-\left[0.0075s^2 - 0.08s\right]_{10}^t\right)$$

$$= e^{-0.5}e^{-0.0075t^2+0.08t-0.05}$$

$$= e^{-0.0075t^2+0.08t-0.55}$$ [2]

Alternatively, we could have set $t = 10$ in the formula for $0 \le t < 10$ to obtain the factor of $e^{-0.5}$.

When $12 \le t$, we need to the product of three discount factors:

$$v(t) = \exp\left(-\int_0^{10} (0.04 + 0.002t)\,dt\right)\exp\left(-\int_{10}^{12} (0.015s - 0.08)\,ds\right)\exp\left(-\int_{12}^{t} 0.07\,ds\right)$$

$$= \exp\left(-\left[0.04t + 0.001t^2\right]_0^{10}\right)\exp\left(-\left[0.0075s^2 - 0.08s\right]_{10}^{12}\right)\exp\left(-\left[0.07s\right]_{12}^{t}\right)$$

$$= e^{-0.5}e^{-(0.12+0.05)}e^{-0.07t+0.84}$$

$$= e^{-0.67}e^{-0.07t+0.84}$$

$$= e^{-0.07t+0.17}$$

[2]

Alternatively, we could have set $t = 12$ in the formula for $10 \le t < 12$ to obtain the factor $e^{-0.67}$.

In summary:

$$v(t) = \begin{cases} e^{-0.04t-0.001t^2} & 0 \le t < 10 \\ e^{-0.0075t^2+0.08t-0.55} & 10 \le t < 12 \\ e^{-0.07t+0.17} & 12 \le t \end{cases}$$

[Total 5]

1 Definition of real and money interest rates

Accumulating an investment of 1 for a period of time t from time 0 produces a new total accumulated value $A(0,t)$, say.

Typically the investment of 1 will be a sum of money, say £1 or \$1 or €1.

Question

Calculate the accumulated value if \$1 is invested for 7 years at an interest rate of 6.5% *pa* effective.

Solution

The accumulated value at time 7 years is:

$$\$1 \times 1.065^7 = \$1.55$$

In this case, if we are given the information on the initial investment of 1 in the specified currency, the period of the investment, and the cash amount of money accumulated, then the underlying interest rate is termed a 'money rate of interest'.

In the above question, we have a money rate of interest of 6.5% *pa* effective.

More generally, given any series of monetary payments accumulated over a period, a money rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation.

In practice, most such accumulations will take place in economies subject to inflation, where a given sum of money in the future will have less purchasing power than at the present day. It is often useful, therefore, to reconsider what the accumulated value is worth allowing for the eroding effects of inflation.

Inflation is a measure of the increase in the cost of goods and services, for example the price of a loaf of bread or a litre of petrol.

Purchasing power refers to the goods that a given amount of money can buy. When inflation occurs, we can buy less goods with the same amount of money.

Question

A bunch of flowers costs £13 on 1/1/18 and £14 on 31/12/18.

Calculate the annual rate of inflation on bunches of flowers during 2018.

Solution

The annual rate of inflation, j, is given by:

$$1+j=\frac{14}{13} \quad \Rightarrow \quad j=7.69\%$$

In this question, since there has been inflation during 2018, the same amount of money buys fewer bunches of flowers at the end of the year. For example, £182 would buy $\frac{182}{13}=14$ bunches of flowers on 1/1/18, but only $\frac{182}{14}=13$ bunches of flowers on 31/12/18. So, the purchasing power of £182 is lower at the end of the year than at the start of the year.

Question

Based on the prices of flowers given above, calculate the amount of money at the start of the year that is equivalent to £182 at the end of the year.

Solution

Since £182 can only buy 13 bunches of flowers at the end of the year, it is equivalent in value to $13\times £13=£169$ at the start of the year.

Notice that $\frac{182}{1.0769}=169$*, ie we divide by (1 + the rate of inflation) to calculate how much money at the end of the year is worth.*

Returning to the initial Core Reading example above, suppose the accumulation took place in an economy subject to inflation so that the cash $A(0,t)$ is effectively worth only $A^{\bullet}(0,t)$ after allowing for inflation, where $A^{\bullet}(0,t)<A(0,t)$. In this case, the rate of interest at which the original sum of 1 would have to be accumulated to produce the sum $A^{\bullet}$ is lower than the money rate of interest.

The sum $A^{\bullet}(0,t)$ is referred to as the real amount accumulated, and the underlying interest rate, reduced for the effects of inflation, is termed a 'real rate of interest'.

Question

A bank offers an effective annual rate of interest on one of its accounts of 4.2%. The rate of inflation is 3% *pa* effective. Calculate the real rate of interest.

Solution

£1 accumulates in the bank account over one year to £1.042.

Since the rate of inflation is 3% *pa*, goods that cost £1 at the start of the year, cost £1.03 at the end of the year. So the quantity of goods that can be purchased at the end of the year is:

$$\frac{1.042}{1.03}=1.0117$$

times the quantity that can be purchased at the start of the year.

Therefore the real rate of interest is 1.17% *pa* effective.

We will look at these calculations in more detail later in this course.

More generally, given any series of monetary payments accumulated over a period, a real rate of interest is that rate which will have been earned so as to produce the total amount of cash in hand at the end of the period of accumulation reduced for the effects of inflation.

Chapter 12 of this subject describes ways of calculating real rates of interest given money rates of interest (and vice versa).

2 Deflationary conditions

The above descriptions assume that the inflation rate is positive. Where the inflation rate is negative, termed 'deflation', the above theory still applies and $A^{\bullet}(0,t) > A(0,t)$, giving rise to the conclusion that the real rate of interest in such circumstances would be higher than the money rate of interest.

Negative inflation rates do occur in some countries from time to time, *eg* the UK had a negative inflation rate during parts of 2015.

Question

A bank offers an annual effective rate of interest on one of its accounts of 4.2%. The rate of inflation is -2% *pa* effective. Calculate the real rate of interest.

Solution

£1 accumulates in the bank account over one year to £1.042.

Since the rate of inflation is -2% *pa*, goods that cost £1 at the start of the year, cost £0.98 at the end of the year. So the quantity of goods that can be purchased at the end of the year is:

$$\frac{1.042}{0.98} = 1.0633$$

times the quantity that can be purchased at the start of the year.

Therefore the real rate of interest is 6.33% *pa* effective.

Notice here that (1 + the rate of inflation) < 1 since inflation is negative.

As might be expected, where there is no inflation $A^{\bullet}(0,t) = A(0,t)$, and the real and money rates of interest are the same.

This is because any amount of money can buy the same amount of goods at both the start and the end of the year.

3 Usefulness of real and money interest rates

We assume here that we have a positive inflation rate.

Which of the two rates of interest, real or money, is the more useful will depend on two main factors:

- **the purpose to which the rate will be put**
- **whether the underlying data have or have not already been adjusted for inflation.**

The purpose to which the rate will be put

Generally, where the actuary is performing calculations to determine how much should be invested to provide for future outgo, the first step will be to determine whether the future outgo is real or monetary in nature. The type of interest rate to be assumed would then be, respectively, a real or a monetary rate.

For example, first suppose that an actuary was asked to calculate the sum to be invested by a person aged 40 to provide for a round-the-world cruise when the person reaches 60, and where the person says the cruise costs £25,000.

Unless the person has, for some reason, already made an allowance for inflation in suggesting a figure of £25,000 then that amount is probably today's cost of the cruise. In this case, the actuary would be wise to assume (checking his understanding with the person) an inflation rate and this could be achieved by assuming a real rate of interest.

Here the future outgo of £25,000 is real in nature in that it is likely to be more than £25,000 in 20 years' time.

As an alternative example, suppose that a person has a mortgage of £50,000 to be paid off in twenty years' time. Here, the party that granted the mortgage would contractually be entitled to only £50,000 in twenty years' time. Accordingly, in working out how much should be invested to repay the outgo in this case, a money rate of interest would be assumed.

Here the future outgo of £50,000 is monetary in nature in that it is going to be exactly £50,000 in 20 years' time.

Whether the underlying data have or have not already been adjusted for inflation

In the first example above, we see that the data may already have been adjusted for inflation and in that case it would not be appropriate to allow for inflation again. A money rate would then be assumed.

More generally in actuarial work, the nature of the data provided must be understood before choosing the type and amount of assumptions to be made.

The chapter summary starts on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 5 Summary

Real rates of interest allow for future inflation. Money rates of interest ignore the effects of inflation.

In periods of positive inflation, the real rate of interest is lower than the money rate of interest.

In periods of negative inflation, the real rate of interest is higher than the money rate of interest.

In periods of zero inflation, the real rate of interest is equal to the money rate of interest.

Which of the two rates of interest, real or money, is the more useful will depend on two main factors:

- the purpose to which the rate will be put
- whether the underlying data values have or have not been adjusted for inflation.

The practice questions start on the next page so that you can keep the chapter summaries together for revision purposes.

Chapter 5 Practice Questions

5.1 (i) Define a real rate of interest.

(ii) Define a money rate of interest.

(iii) State the circumstances in which the real rate of interest would be lower than the money rate of interest.

5.2 An inflation index $Q(t)$ is such that $Q(1/1/13) = 724$ and $Q(1/1/18) = 913$. Calculate the average annual rate of inflation over the period from 1 January 2013 to 1 January 2018.

5.3 In each of the following circumstances, state whether the calculations should use a money or real rate of interest.

(i) An actuarial student wants to invest an amount of money now to buy a new car in one year's time. Today's list price of the car is available.

(ii) A woman wants to invest a lump sum today in order to provide her with a fixed income of £25,000 *pa* for the rest of her life.

(iii) A man buys a zero-coupon bond that will provide him with £100,000 in 10 years' time. He is trying to calculate an appropriate purchase price.

The solutions start on the next page so that you can separate the questions and solutions.

1 Present values of cashflows

In many compound interest problems, one must find the discounted present value of cashflows due in the future. It is important to distinguish between (a) discrete and (b) continuous payments.

In Section 1.1 we will consider discrete payments before looking at continuous payments in Section 1.2.

1.1 Discrete cashflows

We have already seen that the present value of a cashflow, C, due at time t years is Cv^t where $v=1/(1+i)$ and i is the effective rate of interest per annum. Here we are assuming that the effective rate of interest is constant over the period.

What if we have two payments, C_1 due at time t_1, and C_2 at time t_2? The present value of these payments is the amount we would have to invest in a bank account to be able to pay each of the payments at the times they are required. Rather than investing a single sum into a single bank account to provide for the payments, we could have set up a *separate* bank account to cater for each payment and invested the present value of each payment in the corresponding account.

This alternative arrangement would have exactly the same result. So, we see that the present value of the two payments is just the sum of the individual present values.

More generally, the present value of a series of payments of $c_{t_1}, c_{t_2}, \ldots, c_{t_n}$ due at times $t_1, t_2, \ldots, t_n$ is given by:

$$\sum_{j=1}^{n} c_{t_j} v^{t_j}$$

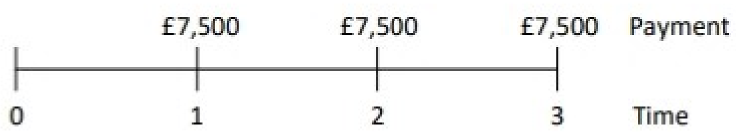
Question

Under its current rental agreement, a company is obliged to make payments of £7,500 at the end of each of the next three years for the building it occupies. The company wishes to provide for these payments by investing a single sum in its bank account that pays interest at a rate of 7.5% *pa* effective.

Calculate the sum that must be invested today.

Solution

The payments can be represented on a timeline as follows:



Here: $v = \frac{1}{1+i} = 1.075^{-1}$.

So: PV of payment due at time 1 $= 7{,}500v = 7{,}500 \times 1.075^{-1} = 6{,}976.74$

PV of payment due at time 2 $= 7{,}500v^2 = 7{,}500 \times 1.075^{-2} = 6{,}489.99$

PV of payment due at time 3 $= 7{,}500v^3 = 7{,}500 \times 1.075^{-3} = 6{,}037.20$

So the amount that must be invested in the account today is:

PV all payments $= 6{,}976.74 + 6{,}489.99 + 6{,}037.20 = £19{,}504$

If the effective interest rate is not constant, then we could write the present value in terms of the function $v(t)$, where $v(t)$ is the (discounted) present value of 1 due at time t.

The present value of the sums $c_{t_1}, c_{t_2}, \ldots, c_{t_n}$ due at times $t_1, t_2, \ldots, t_n$ (where $0 \le t_1 < t_2 < \ldots < t_n$) is:

$$c_{t_1} v(t_1) + c_{t_2} v(t_2) + \cdots + c_{t_n} v(t_n) = \sum_{j=1}^{n} c_{t_j} v(t_j)$$

If the number of payments is infinite, the present value is defined to be:

$$\sum_{j=1}^{\infty} c_{t_j} v(t_j)$$

provided that this series converges. It usually will in practical problems.

We could also express the present value in terms of a force of interest that varies over time. If the force of interest at time t is $\delta(t)$, then:

$$v(t) = \exp\left(-\int_0^t \delta(s)\,ds \right)$$

and the present value of the sums $c_{t_1}, c_{t_2}, \ldots, c_{t_n}$ due at times $t_1, t_2, \ldots, t_n$ is:

$$\sum_{j=1}^{n} c_{t_j} \exp\left(-\int_0^{t_j} \delta(s)\,ds \right)$$

Question

Calculate the value at time $t=0$ of \$250 due at time $t=6$ and \$600 due at time $t=8$ if $\delta(t)=3\%$ *pa* for all t.

Solution

If the force of interest, δ, is constant, then the present value of a payment of C due at time t is:

$$Ce^{-\delta t}$$

Therefore the present value of these two payments is:

$$250e^{-6\times 0.03}+600e^{-8\times 0.03}=\$680.79$$

1.2 Continuously payable cashflows (payment streams)

Suppose that $T>0$ and that between times 0 and T an investor will be paid money continuously, the rate of payment at time t being £ $\rho(t)$ per unit time. What is the present value of this cashflow?

In order to answer this question, it is essential to understand what is meant by the 'rate of payment' of the cashflow at time *t*. If $M(t)$ denotes the total payment made between time 0 and time *t*, then by definition:

$$\rho(t)=M'(t) \qquad \text{for all } t$$

Then, if $0\le\alpha<\beta\le T$, the total payment received between time α and time β is:

$$\begin{aligned} M(\beta)-M(\alpha) &= \int_\alpha^\beta M'(t)dt \\ &= \int_\alpha^\beta \rho(t)dt \end{aligned} \qquad \textbf{(1.1)}$$

Thus the rate of payment at any time is simply the derivative of the *total* amount paid up to that time, and the total amount paid between any two times is the integral of the rate of payments over the appropriate time interval.

Intuitively, we can think of the integral in (1.1) as the sum of lots of small payments, each of amount $\rho(t)dt$. It may help to consider the following simple example.

If the rate of payment is a constant £24 *pa*, then in any one year the total amount paid is £24, but this payment is spread evenly over the year. In half a year, the total paid is $24\times\frac{1}{2}$, *ie* £12. In one month, the total paid is $24\times\frac{1}{12}$, *ie* £2. So, in a small time period *dt*, the total paid is £$24dt$.

Question

A life insurer starts issuing a new type of 10-year savings policy to young investors who pay weekly premiums of £10. The insurer assumes that it will sell 10,000 policies evenly over each year and that no policyholders will stop paying premiums after taking out the policy.

Calculate the total premium income that will be received during the first 3 years, assuming that there are 52.18 ($=365.25/7$) weeks in each year.

Solution

Let t measure the time in years from when the first policy is sold. By time t, the insurer will have sold $10,000t$ policies.

For example, at time 0, the insurer will have sold 0 policies, at time 1 it will have sold 10,000 policies, and at time 0.5 it will have sold 5,000 (as the policies are assumed to be sold evenly over each year).

So the weekly premium income at time t will be $10,000t \times £10 = £100,000t$, and this corresponds to an annual rate of premium income at time t of:

$$52.18 \times £100,000t = £5,218,000t$$

The total premium income received in the first three years is the integral of the rate of payment of premium income over the time interval:

$$\int_0^3 5,218,000t\,dt = \left[\frac{5,218,000t^2}{2}\right]_0^3 = 2,609,000 \times 9 = £23,481,000$$

We can check this answer by reasoning in a different way. Since the policies are sold evenly over each year, in the first year there are on average 5,000 policies paying premiums (as no policies have been sold at the start of the year and 10,000 have been sold by the end of the year). These will generate income during the first year of:

$$5,000 \times 10 \times 52.18 = £2,609,000$$

In the second year, there are on average 15,000 policies paying premiums (as 10,000 policies have been sold at the start of the year and 20,000 have been sold by the end of the year). These will generate income during the second year of:

$$15,000 \times 10 \times 52.18 = £7,827,000$$

In the third year, there are on average 25,000 policies paying premiums. These will generate income during the third year of:

$$25,000 \times 10 \times 52.18 = £13,045,000$$

So the total premium income received during the first three years is, as before:

$$2,609,000 + 7,827,000 + 13,045,000 = £23,481,000$$

(This alternative approach only works because the number of policies sold increases linearly over time.)

Between times t and $t + dt$ the total payment received is $M(t + dt) - M(t)$. If dt is very small this is approximately $M'(t)dt$ or $\rho(t)dt$. Theoretically, therefore, we may consider the present value of the money received between times t and $t + dt$ as $v(t)\rho(t)dt$. The present value of the entire cashflow is obtained by integration as:

$$\int_0^T v(t)\rho(t)dt$$

where T is the time at which the payment stream ends.

Intuitively, we can view this integral as summing, between times $t = 0$ and $t = T$, the present values of an infinite number of payments. At each time point, t, the payment made (in the time interval of length dt) is $\rho(t)dt$. We use $v(t)$ to discount the payment from time t to time 0, and the integral 'sums' the infinite number of payments.

If T is infinite we obtain, by a similar argument, the present value:

$$\int_0^\infty v(t)\rho(t)dt$$

By combining the results for discrete and continuous cashflows, we obtain the formula:

$$\sum c_t v(t) + \int_0^\infty v(t)\rho(t)dt \qquad (1.2)$$

for the present value of a general cashflow (the summation being over those values of t for which c_t, the discrete cashflow at time t, is non-zero).

Assuming a constant interest rate this simplifies slightly to the important result for the present value of a series of discrete cashflows and a continuous cashflow:

$$\sum c_t v^t + \int_0^\infty v^t \rho(t)dt$$

Question

A company expects to receive a continuous cashflow for the next five years, where the rate of payment is 100×0.8^t at time t years.

Calculate the present value of this cashflow assuming a constant force of interest of 8% *pa* throughout the period.

Solution

The present value can be calculated using the formula:

$$\int_0^5 v(t)\rho(t)dt$$

with $\rho(t) = 100 \times 0.8^t$ and $v(t) = e^{-0.08t}$. Using the properties of exponentials and logs, we can write $0.8^t = e^{\ln 0.8^t} = e^{t \ln 0.8}$, so the present value is:

$$\begin{aligned}\int_0^5 e^{-0.08t} \times 100 \times e^{t\ln 0.8}\,dt &= 100\int_0^5 e^{(-0.08+\ln 0.8)t}dt \\ &= 100\left[\frac{e^{(-0.08+\ln 0.8)t}}{-0.08+\ln 0.8}\right]_0^5 \\ &= 100\left[\frac{e^{(-0.08+\ln 0.8)\times 5}-1}{-0.08+\ln 0.8}\right] \\ &= £257.42\end{aligned}$$

So far we have assumed that all payments, whether discrete or continuous, are positive. If one has a series of income payments (which may be regarded as positive) and a series of outgoings (which may be regarded as negative) their *net present value* is defined as the difference between the value of the positive cashflow and the value of the negative cashflow.

The net present value will often be abbreviated to NPV. We will study net present values in Chapter 11.

Question

A company expects to receive a continuous cashflow of £350 *pa* for the next five years. It also expects to have to pay out £600 at the end of the first year and £400 at the end of the third year.

Calculate the net present value of these cashflows if $v(t) = 1 - 0.01t$ for $0 \le t \le 5$.

Solution

The present value of the income can be calculated using the formula:

$$\int_0^5 v(t)\rho(t)dt$$

with $\rho(t) = 350$ and $v(t) = 1 - 0.01t$.

This gives:

$$\int_0^5 350(1-0.01t)dt = \left[350t - 1.75t^2\right]_0^5 = £1,706.25$$

The present value of the outgo is:

$$600v(1) + 400v(3) = 600 \times 0.99 + 400 \times 0.97 = £982$$

So the net present value is:

$$\text{NPV} = 1,706.25 - 982 = £724.25$$

2 Valuing cashflows

Consider times t_1 and t_2, where t_2 is not necessarily greater than t_1. *The value at time t_1 of the sum C due at time t_2* is defined as:

(a) If $t_1 \geq t_2$, the accumulation of *C* from time t_2 until time t_1; or

(b) If $t_1 < t_2$, the discounted value at time t_1 of *C* due at time t_2.

In both cases the value at time t_1 of C due at time t_2 is:

$$C\exp\left[-\int_{t_1}^{t_2}\delta(t)dt\right] \tag{2.1}$$

(Note the convention that, if $t_1 > t_2$, $\int_{t_1}^{t_2}\delta(t)dt = -\int_{t_2}^{t_1}\delta(t)dt$.)

This result was derived in the last chapter. If there is a constant force of interest, *ie* $\delta(t)=\delta$ for all *t*, then this result becomes:

$$Ce^{-(t_2-t_1)\delta} = Ce^{(t_1-t_2)\delta}$$

Since:

$$\int_{t_1}^{t_2}\delta(t)dt = \int_0^{t_2}\delta(t)dt - \int_0^{t_1}\delta(t)dt$$

it follows immediately from Equation (2.1) that the value at time t_1 of *C* due at time t_2 is:

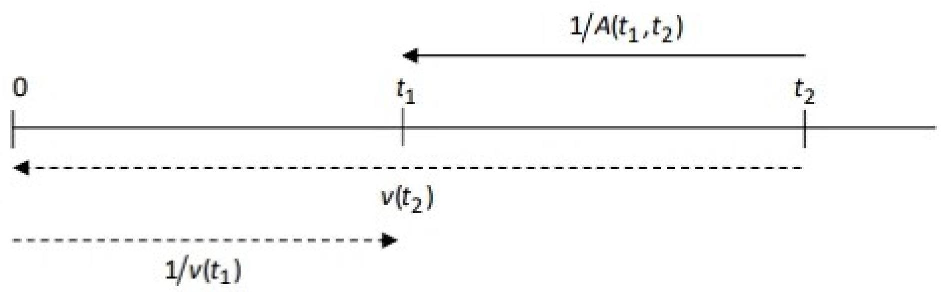
$$C\frac{v(t_2)}{v(t_1)} \tag{2.2}$$

Remembering that $v(t)=1/A(0,t)$, this result can be written in the form $Cv(t_2)A(0,t_1)$.

This tells us that in order to find the value at time t_1 of *C* due at time t_2, we could first discount back to time 0 by multiplying by $v(t_2)$, and then accumulate to time t_1 by multiplying by $A(0,t_1)$. The last step is equivalent to dividing by $v(t_1)$.

Alternatively, this calculation could be performed directly using the expression $C/A(t_1,t_2)$.

These two approaches are represented diagrammatically below:



Question

Calculate the value at time 4 of a payment of 860 at time 10 if $v(10)=0.76$ and $v(4)=0.91$.

Solution

The value at time 4 is:

$$860\times\frac{v(10)}{v(4)}=860\times\frac{0.76}{0.91}=718.24$$

The value at a general time t_1 of a discrete cashflow of c_t at time t (for various values of t) and a continuous payment stream at rate $\rho(t)$ per time unit may now be found, by the methods given in Section 1, as:

$$\sum c_t\frac{v(t)}{v(t_1)}+\int_{-\infty}^{\infty}\rho(t)\frac{v(t)}{v(t_1)}dt \tag{2.3}$$

where the summation is over those values of t for which $c_t\neq 0$.

We note that in the special case when $t_1=0$ (the present time), the value of the cashflow is:

$$\sum c_t v(t)+\int_{-\infty}^{\infty}\rho(t)v(t)dt$$

where the summation is over those values of t for which $c_t\neq 0$. This is a generalisation of formula (1.2) to cover the past as well as present or future payments. If there are incoming and outgoing payments, the corresponding *net value* may be defined, as in Section 1, as the difference between the value of the *positive* and the *negative* cashflows. If all the payments are due at or after time t_1, their value at time t_1 may also be called their 'discounted value', and if they are due at or before time t_1, their value may be referred to as their 'accumulation'.

It follows that any value may be expressed as the sum of a discounted value and an accumulation. This fact is helpful in certain problems. Also, if $t_1=0$ and all the payments are due at or after the present time, their value may also be described as their '(discounted) present value', as defined by formula (1.2).

It follows from formula (2.2) that the value at any time t_1 of a cashflow may be obtained from its value at another time t_2 by applying the factor $v(t_2)/v(t_1)$, *ie*:

$$\begin{bmatrix}\text{Value at time } t_1\\ \text{of cashflow}\end{bmatrix}=\begin{bmatrix}\text{Value at time } t_2\\ \text{of cashflow}\end{bmatrix}\begin{bmatrix}\dfrac{v(t_2)}{v(t_1)}\end{bmatrix}$$

or:

$$\begin{bmatrix}\text{Value at time } t_1\\ \text{of cashflow}\end{bmatrix}\left[v(t_1)\right]=\begin{bmatrix}\text{Value at time } t_2\\ \text{of cashflow}\end{bmatrix}\left[v(t_2)\right] \tag{2.4}$$

Each side of Equation (2.4) is the value of the cashflow at the present time (time 0).

In particular, by choosing time t_2 as the present time and letting $t_1 = t$, we obtain the result:

$$\begin{bmatrix}\text{Value at time } t \\ \text{of cashflow}\end{bmatrix} = \begin{bmatrix}\text{Value at the present} \\ \text{time of cashflow}\end{bmatrix} \left[\frac{1}{v(t)}\right]$$

These results are useful in many practical examples. The time 0 and the unit of time may be chosen so as to simplify the calculations.

Question

Consider the following four payments: £100 on 1 January 2017, £130 on 1 January 2018, £150 on 1 January 2020 and £160 on 1 January 2021.

If $t=0$ on 1 January 2016 and $v(t)=0.92-\frac{(t-2)^3}{100}$, calculate the value of these payments on 1 January 2019.

Solution

From the formula $v(t)=0.92-\frac{(t-2)^3}{100}$, we can calculate $v(1)=0.93$, $v(2)=0.92$, $v(3)=0.91$, $v(4)=0.84$, $v(5)=0.65$.

The present value of these cashflows at $t=0$ (*ie* 1 January 2016) is:

$$100v(1)+130v(2)+150v(4)+160v(5)=£442.60$$

The present value of these cashflows at $t=3$ (*ie* 1 January 2019) is therefore:

$$\frac{1}{v(3)}\times 442.60=£486.37$$

The first two payments occur before 1 January 2019, so these have been accumulated. The last two payments occur after 1 January 2019, so these have been discounted.

2.1 Constant interest rate

The special case when we assume that interest rates remain constant is of particular importance. Using this assumption $v(t)=v^t$ for all t. Remember also that $v=1/(1+i)$ and so $1/v^t=(1+i)^t$.

In the diagram below, each of the three payments P_1, P_2 and P_3 has the same present value.

Payments	$P_1 = Xv^n$ ↓	$P_2 = X$ ↓	$P_3 = X(1+i)^n$ ↓
Time	$-n$	0	n

This shows that we can think of the factors $(1+i)^n$ and v^n as a way of adjusting payments to a different point on the timeline.

If the present value of a series of definite payments at a particular date is X, then:

- the accumulated value at a date n years later is $X(1+i)^n$
- the present value at a date n years earlier is Xv^n.

Note that n does not have to be a whole number in these formulae.

Question

Under its current rental agreement, a company is obliged to make annual payments of £7,500 for the building it occupies. Payments are due on 1 January 2020, 1 January 2021 and 1 January 2022. The nominal rate of interest is 8% per annum, convertible quarterly.

Calculate the value of these rental payments on:

(i) 1 January 2019

(ii) 1 January 2018

(iii) 1 July 2033

Solution

A nominal rate of interest of 8% *pa* convertible quarterly is equivalent to a quarterly effective interest rate of $\frac{8\%}{4} = 2\%$.

(i) Working in quarters, the value of the rental payments on 1 January 2019 is:

$$7,500(v^4 + v^8 + v^{12}) = 7,500(1.02^{-4} + 1.02^{-8} + 1.02^{-12}) = £19,243.72$$

(ii) Since 1 January 2018 is 1 year (= 4 quarters) before 1 January 2019, the value of the rental payments on 1 January 2018 is:

$$19,243.72 \times 1.02^{-4} = £17,778.22$$

(iii) Since 1 July 2033 is 14.5 years (= 58 quarters) after 1 January 2019, the value of the rental payments on 1 July 2033 is:

$$19,243.72 \times 1.02^{58} = £60,687.46$$

Alternatively, we could first calculate the effective annual rate, i, as $1.02^4 - 1 = 8.243216\%$, *and then work in years. So, for example, the value in part (i) is:*

$$7,500(v + v^2 + v^3) = £19,243.72 \text{ where } v = 1/1.08243216$$

2.2 Payment streams

In Section 1.2, we saw that the present value at time 0 of a continuous payment stream received from time 0 to time T, where the rate of payment at time t is $\rho(t)$, is given by:

$$PV_{t=0} = \int_0^T v(t)\,\rho(t)\,dt$$

If the continuous payment stream is received from time a to time b, then this formula becomes:

$$PV_{t=0} = \int_a^b v(t)\,\rho(t)\,dt$$

Intuitively, we can obtain this formula by first considering the payment at time t, which is at a rate of $\rho(t)$. This payment needs to be discounted back to time 0 and the appropriate discount factor is $v(t)$. Finally, we need to add together all the present values of the payments at the different times. Since we are receiving payments continuously, we integrate these present values between the limits a and b, *ie* the times between which the payment comes in.

We can also work out the present value of a continuous payment stream at a time other than time 0. Suppose we wanted to calculate the value of the same payment stream as at the start of the payment period, *ie* at time a rather than at time 0. If the force of interest at time s is $\delta(s)$, this present value is:

$$PV_{t=a} = \int_a^b \rho(t)\exp\left(-\int_a^t \delta(s)\,ds\right)dt$$

We can think of this formula in the same way as that given above, except that instead of discounting payments back to time 0 using $v(t)$, we need to discount them back to time a, which can be done using the discount factor $\exp\left(-\int_a^t \delta(s)\,ds\right)$.

Question

A continuous payment stream is paid at rate $e^{-0.03t}$ from time $t=0$ to time $t=10$.

Calculate the present value of this payment stream at time $t=0$, given that the force of interest over this time period is 0.04 *pa*.

Solution

The payment stream starts at time 0 and finishes at time 10, so we can set $a=0$ and $b=10$. Using these values along with $\rho(t)=e^{-0.03t}$ and $\delta(t)=0.04$ gives:

$$PV_{t=0}=\int_0^{10} e^{-0.03t}e^{-\int_0^t 0.04ds}\,dt=\int_0^{10} e^{-0.03t}e^{-[0.04s]_0^t}\,dt=\int_0^{10} e^{-0.03t}e^{-0.04t}\,dt$$

This integral simplifies and is evaluated as follows:

$$PV_{t=0}=\int_0^{10} e^{-0.07t}\,dt=\left[-\frac{1}{0.07}e^{-0.07t}\right]_0^{10}=\frac{1}{0.07}(1-e^{-0.7})=7.192$$

We can also consider the accumulated value of the continuous payment stream paid at a rate of $\rho(t)$ from time *a* to time *b*, during which time the force of interest is $\delta(t)$. The accumulated value at time *b* of this payment stream is:

$$AV_{t=b}=\int_a^b \rho(t)\exp\left(\int_t^b \delta(s)ds\right)dt$$

Intuitively, we can obtain this formula by first considering the payment at time *t*, which is at a rate of $\rho(t)$. This payment needs to be accumulated to time *b* and the accumulation factor is $\exp\left(\int_t^b \delta(s)ds\right)$ in terms of the force of interest. Finally, we need to add together all the accumulated values of the payments at the different times. Since we are receiving payments continuously, we integrate these accumulated values between the limits *a* and *b*, *ie* the times between which the payment comes in.

Question

The force of interest at time t, where $0\le t\le 10$, is given by $\delta(t)=0.07$.

Calculate the accumulated value at time 10 of a payment stream, paid continuously from time 5 to time 10, under which the rate of payment at time t is $10e^{0.05t}$.

Solution

The payment stream starts at time 5 and finishes at time 10, so we can set $a=5$ and $b=10$. Using these values along with $\rho(t)=10e^{0.05t}$ and $\delta(t)=0.07$ gives:

$$AV_{t=10}=\int_5^{10} 10e^{0.05t}e^{\int_t^{10} 0.07ds}dt=\int_5^{10} 10e^{0.05t}e^{[0.07s]_t^{10}}\,dt=\int_5^{10} 10e^{0.05t}\,e^{0.7-0.07t}dt$$

This integral simplifies and is evaluated as follows:

$$AV_{t=10} = 10e^{0.7}\int_5^{10} e^{-0.02t}\,dt = 10e^{0.7}\left[\frac{e^{-0.02t}}{-0.02}\right]_5^{10} = 10e^{0.7}\left(\frac{e^{-0.2}-e^{-0.1}}{-0.02}\right) = 86.699$$

One result that can be useful here is the chain rule for differentiation expressed in integral form. The chain rule applied to the exponential of a function tells us that:

$$\frac{d}{dt}e^{f(t)} = f'(t)e^{f(t)}$$

Integrating both sides, we have:

$$\int_a^b f'(t)e^{f(t)}\,dt = \left[e^{f(t)}\right]_a^b$$

Question

The force of interest at time t is given by:

$$\delta(t) = 0.01t + 0.04 \qquad 0 \le t \le 5$$

Calculate the present value at time 0 of a payment stream, received continuously from time 0 to time 5, under which the rate of payment at time t is $0.5t+2$.

Solution

Here $a=0$, $b=5$ and $\rho(t)=0.5t+2$, so we have:

$$\begin{aligned} PV_{t=0} &= \int_0^5 (0.5t+2)\exp\left(-\int_0^t (0.01s+0.04)\,ds\right)dt \\ &= \int_0^5 (0.5t+2)\exp\left(-\left[0.005s^2+0.04s\right]_0^t\right)dt \\ &= \int_0^5 (0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt \end{aligned}$$

Now, using the general result $\int_a^b f'(t)e^{f(t)}\,dt = \left[e^{f(t)}\right]_a^b$, with $f(t) = -\left[0.005t^2+0.04t\right]$, so that $f'(t) = -(0.01t+0.04)$, we know that:

$$\int_0^5 -(0.01t+0.04)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt = \left[\exp\left(-\left[0.005t^2+0.04t\right]\right)\right]_0^5$$

Now, since $0.5t+2=-50\times-(0.01t+0.04)$, we have:

$$\begin{aligned}PV_{t=0}&=\int_0^5(0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt\\&=-50\int_0^5-(0.01t+0.04)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt\\&=-50\left[\exp\left(-\left[0.005t^2+0.04t\right]\right)\right]_0^5\\&=-50\left(\exp(-0.125-0.2)-\exp(0)\right)\\&=13.87\end{aligned}$$

Alternatively, we can evaluate the integral:

$$PV_{t=0}=\int_0^5(0.5t+2)\exp\left(-\left[0.005t^2+0.04t\right]\right)dt$$

using the substitution $u=-0.005t^2-0.04t$*, so that:*

$$\frac{du}{dt}=-0.01t-0.04\;\Rightarrow\;du=(-0.01t-0.04)dt\;\Rightarrow\;-50du=(0.5t+2)dt$$

Also, when $t=0$*,* $u=0$ *and when* $t=5$*,* $u=-0.325$*, so the integral becomes:*

$$PV_{t=0}=\int_0^{-0.325}-50e^u\,du=-50\left[e^u\right]_0^{-0.325}=-50\left(e^{-0.325}-1\right)=13.87$$

2.3 Sudden changes in interest rates

In Chapter 4, we saw that when the force of interest is not a continuous function of time, it is necessary to break up the calculations at the points when the force changes. The same idea applies when we are given a rate of interest that changes at certain points in time. We need to take care to ensure that we use the correct interest rate for each period.

Question

Calculate the present value as at 1 January 2020 of the following payments:

(i) a single payment of £2,000 payable on 1 July 2024

(ii) a single payment of £5,000 payable on 31 December 2031.

Assume effective rates of interest of 8% *pa* until 31 December 2026 and 6% *pa* thereafter.

Solution

(i) Here, the interest rate is constant throughout the relevant period, so the present value is just:

$$2{,}000v^{4\frac{1}{2}@8\%} = 2{,}000 \times 1.08^{-4.5} = £1{,}415$$

(ii) Here, we need to break the calculation up at 31 December 2026 when the interest rate changes. So, we discount the payment for 5 years (from 31 December 2031 to 31 December 2026) at 6% *pa*, and then for a further 7 years (from 31 December 2026 to 1 January 2020) at 8% *pa*:

$$5{,}000v^{7@8\%} \times v^{5@6\%} = 5{,}000 \times 1.08^{-7} \times 1.06^{-5} = £2{,}180$$

3 Interest income

Consider now an investor who wishes not to accumulate money but to receive an income while keeping his capital fixed at C. If the rate of interest is fixed at i per time unit, and if the investor wishes to receive income at the end of each time unit, it is clear that the income will be iC per time unit, payable in arrears, until such time as the capital is withdrawn.

This is because the effective rate of interest, i, is defined to be the amount of interest a single initial investment will earn at the end of the time period.

For example, consider an investor who wishes to receive income at the end of each year. If the investor deposits £1,000 in a bank account that pays an effective rate of interest of 8% per annum, the amount of each payment will be £80 (8% of 1,000).

However, if interest is paid continuously with force of interest $\delta(t)$ at time t then the income received between times t and $t+dt$ will be $C\delta(t)dt$. So the total interest income from time 0 to time T will be:

$$I(T)=\int_0^T C\delta(t)\,dt \tag{3.1}$$

If interest is paid continuously to the investor, then we are just considering a continuous cashflow with a rate of payment of $C\delta(t)$. The total amount of interest received can therefore be calculated by applying formula (1.1), which gives formula (3.1). The total amount of interest received is the sum, between 0 and T, of lots of small interest payments, each of amount $C\delta(t)dt$.

If the investor withdraws the capital at time T, the present values of the income and capital at time 0 are:

$$C\int_0^T \delta(t)v(t)dt \tag{3.2}$$

and:

$$Cv(T) \tag{3.3}$$

respectively.

Since:

$$\int_0^T \delta(t)v(t)dt=\int_0^T \delta(t)\exp\left[-\int_0^t \delta(s)ds\right]dt=\left[-\exp\left(-\int_0^t \delta(s)ds\right)\right]_0^T=1-v(T)$$

we obtain:

$$C=C\int_0^T \delta(t)v(t)dt+Cv(T) \tag{3.4}$$

as one would expect by general reasoning.

If we invest an amount of capital *C*, then the present value of the proceeds we receive from this investment should equal our original amount of capital.

Question

An investor deposits £2,000 in a bank account and receives income at the end of each of the next three years. The rate of interest is 4% *pa* effective. The investor withdraws the capital after three years.

Show that the present value of the proceeds from this arrangement is equal to the initial amount deposited.

Solution

At the end of each year the investor receives $0.04 \times 2{,}000 = £80$. The present value of the interest received is:

$$80(v + v^2 + v^3) = 80(1.04^{-1} + 1.04^{-2} + 1.04^{-3}) = £222.01$$

The present value of the capital received after three years is:

$$2{,}000v^3 = 2{,}000 \times 1.04^{-3} = £1{,}777.99$$

The present value of the capital plus the present value of the interest equals the initial investment, *ie*:

$$1{,}777.99 + 222.01 = £2{,}000$$

So far we have described the difference between money returned at the end of the term and the cash originally invested as 'interest'. In practice, however, this quantity may be divided into *interest income* and *capital gains*, the term *capital loss* being used for a negative capital gain.

In return for an investment of capital, an investor will expect to receive interest payments. In addition, the value of the capital may also increase (or decrease). Equities or shares are a good example of this. If an investor buys some shares in a company, then the investor should receive dividends (*ie* interest) from the company. However the capital value that the investor receives back will depend upon the market price of the shares at the time they are sold. We will consider this in more detail later in the course.

Chapter 6 Summary

In many compound interest problems, we may need to determine the discounted present value of cashflows due in the future. It is important to distinguish between discrete and continuous payments.

The present value of a series of discrete payments is the sum of the individual present values. The present value of continuous payments is calculated by integrating the rate of payment multiplied by a discount factor. The formula for the present value is:

$$\sum c_t v(t) + \int_0^\infty v(t)\rho(t)\,dt$$

The *net present value* is defined as the difference between the value of the positive cashflows and the value of the negative cashflows.

The value of payments that are due after the time of valuation is called a *discounted* value.

The value of payments that are due before the time of valuation is called an *accumulated* value.

The value of a cashflow at one particular time can easily be found from the value of the cashflow at a different time. The formula for moving along the timeline is:

$$\begin{bmatrix}\text{Value at time } t_1 \\ \text{of cashflow}\end{bmatrix} = \begin{bmatrix}\text{Value at time } t_2 \\ \text{of cashflow}\end{bmatrix}\left[\frac{v(t_2)}{v(t_1)}\right]$$

An investor may wish to receive an income while keeping the amount of capital fixed. The present value of the income plus the present value of the returned capital equals the initial capital invested, *ie*:

$$C = C\int_0^T \delta(t)v(t)\,dt + Cv(T)$$

The practice questions start on the next page so that you can keep all the chapter summaries together for revision purposes.

Chapter 6 Practice Questions

6.1 Calculate the total present value as at 1 September 2022 of payments of £280 due on 1 September 2024 and £360 due on 1 March 2025, assuming the interest rate is 15% *pa* effective.

6.2 Calculate the total present value as at 1 June 2019 of payments of £100 on 1 January 2020 and £200 on 1 May 2020, assuming a rate of discount of 12% *pa* convertible quarterly.

6.3 An investment of £1,000 made at time 0 is accumulated at the following rates: 8% per annum simple for two years, followed by a rate of discount of 6% per annum convertible monthly for two years. Calculate the accumulated amount of the investment after 4 years.

6.4 A company is contracted to make payments of £1,500 at time 3, £4,000 at time 7 and £5,500 at time 10. The effective annual interest rate is assumed to be:

- 3% from time 0 to time 4
- 5% from time 4 to time 8
- 8% from time 8 onwards.

Calculate the value of the payments as at:

(i) time 0

(ii) time 5.

6.5 A woman deposits £200 in a special bank account. Interest is paid to the woman every year on her birthday for five years. The capital is returned after exactly five years, along with any interest accrued since her last birthday. Interest is calculated at an effective rate of 6% *pa*.

Calculate the present value of the interest received by the woman.

6.6 The force of interest at time t, where t is measured in years, is given by:

$$\delta(t)=\begin{cases} 0.002+0.01t+0.0004t^2 & 0\le t<6 \\ 0.01+0.003t & 6\le t<10 \\ 0.04 & t\ge 10 \end{cases}$$

Calculate the present value at time 0 of a continuous payment stream of £120 per annum payable from time 10 to time 15.

6.7 The force of interest at time t is given by:

$$\delta(t)=0.01+0.05t \qquad 0\le t\le 10$$

Calculate the accumulated value at time 10 of a continuous payment stream that is received from time 4 to time 8, under which the rate of payment at time t is $0.3+1.5t$.

6.8 Exam style

The force of interest at time t is given by:

$$\delta(t)=\begin{cases} 0.04 & 0<t\le 1 \\ 0.05t-0.01 & 1<t\le 5 \\ 0.24 & t>5 \end{cases}$$

(i) Derive and simplify as far as possible expressions for $A(t)$, where $A(t)$ is the total accumulated value at time t (>0) of an investment of 1 at time 0. [5]

(ii) A continuous payment stream is received at a rate of $25e^{-0.02t}$ units per annum between time 5 and time 10. Calculate the present value (at time 0) of this payment stream. [4]

[Total 9]

6.9 Exam style

The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:

$$\delta(t)=\begin{cases} 0.03 & \text{for} \quad 0<t\le 10 \\ 0.003t & \text{for} \quad 10<t\le 20 \\ 0.0001t^2 & \text{for} \quad t>20 \end{cases}$$

(i) Calculate the present value of a unit sum of money due at time $t=28$. [7]

(ii) (a) Calculate the equivalent constant force of interest from $t=0$ to $t=28$.

(b) Calculate the equivalent annual effective rate of discount from $t=0$ to $t=28$. [3]

A continuous payment stream is paid at the rate of $e^{-0.04t}$ per unit time between $t=3$ and $t=7$.

(iii) Calculate the present value of the payment stream. [4]

[Total 14]

Chapter 6 Solutions

6.1 The payment of £280 is made in two years' time, and the payment of £360 is made in 2.5 years' time, so the present value is:

$$280v^2 + 360v^{2.5} = 280 \times 1.15^{-2} + 360 \times 1.15^{-2.5} = £465.56$$

6.2 We are given $d^{(4)} = 12\%$. This is equivalent to:

$$i = \left(1 - \frac{d^{(4)}}{4}\right)^{-4} - 1 = \left(1 - \frac{0.12}{4}\right)^{-4} - 1 = 12.95698\%\ pa$$

The payment of £100 is made in 7 months' time and the payment of £200 is made in 11 months' time, so working in years, the present value is:

$$100v^{7/12} + 200v^{11/12} = 100(1.1295698)^{-7/12} + 200(1.1295698)^{-11/12} = £272.00$$

6.3 The accumulation factor for 2 years in terms of a simple rate of interest i is $(1+2i)$.

Since $1+i = \left(1 - \frac{d^{(12)}}{12}\right)^{-12}$, the accumulation factor for 2 years in terms of $d^{(12)}$ is $\left(1 - \frac{d^{(12)}}{12}\right)^{-24}$.

So the accumulated value of the investment at time 4 years is:

$$1{,}000(1 + 2 \times 0.08)\left(1 - \tfrac{0.06}{12}\right)^{-24} = £1{,}308.29$$

6.4 (i) To calculate the value of the payments at time 0:

- the payment of £1,500 at time 3 needs to be discounted for 3 years at 3% *pa*
- the payment of £4,000 at time 7 needs to be discounted for 7 years in total – 3 years at 5% *pa* (from time 7 back to time 4) and 4 years at 3% *pa* (from time 4 back to time 0)
- the payment of £5,500 at time 10 needs to be discounted for 10 years in total – 2 years at 8% *pa* (from time 10 back to time 8), 4 years at 5% *pa* (from time 8 back to time 4) and 4 years at 3% *pa* (from time 4 back to time 0).

So, the present value at time 0 is:

$$1{,}500v^{3@3\%} + 4{,}000v^{4@3\%}v^{3@5\%} + 5{,}500v^{4@3\%}v^{4@5\%}v^{2@8\%}$$

$$= 1{,}500(1.03)^{-3} + 4{,}000(1.03)^{-4}(1.05)^{-3} + 5{,}500(1.03)^{-4}(1.05)^{-4}(1.08)^{-2}$$

$$= £7{,}889.49$$

(ii) The value at time 5 can be found by accumulating the value at time 0 for 4 years at 3% *pa* and 1 year at 5% *pa*. This gives:

$$7{,}889.49(1.03)^4(1.05) = £9{,}323.68$$

6.5 We know that the present value of the interest received plus the present value of the capital returned will equal the initial deposit. Therefore:

$$200 = PV(\text{interest}) + 200v^5$$

$$\Rightarrow \quad PV(\text{interest}) = 200 - 200v^5 = 200(1 - 1.06^{-5}) = £50.55$$

This answer does not depend on where the woman's birthday falls during the year.

6.6 We can calculate the present value at time 10 (when the payment stream begins) and then discount that value back to time 0. Using $a = 10$, $b = 15$ and $\rho(t) = 120$:

$$PV_{t=10} = \int_{10}^{15} 120\exp\left[-\int_{10}^{t} 0.04\,ds\right] dt$$

$$= \int_{10}^{15} 120\exp\left(-\left[0.04s\right]_{10}^{t}\right) dt$$

$$= \int_{10}^{15} 120\exp\left(-0.04(t-10)\right) dt$$

$$= 120e^{0.4}\int_{10}^{15} e^{-0.04t}\,dt$$

Carrying out the integration, we have:

$$PV_{t=10} = 120e^{0.4}\left[\frac{e^{-0.04t}}{-0.04}\right]_{10}^{15} = 120e^{0.4}\left(\frac{e^{-0.04\times15} - e^{-0.04\times10}}{-0.04}\right) = £543.81$$

We then need to discount the present value at time 10 back to time 0 using the appropriate values of $\delta(t)$:

$$PV_{t=0} = 543.81\exp\left[-\int_{6}^{10}(0.01+0.003t)dt\right]\exp\left[-\int_{0}^{6}(0.002+0.01t+0.0004t^2)dt\right]$$

Now:

$$\exp\left[-\int_{6}^{10}(0.01+0.003t)dt\right] = \exp\left(-\left[0.01t+0.0015t^2\right]_{6}^{10}\right) = e^{-0.136}$$

Also:

$$\exp\left[-\int_0^6 (0.002+0.01t+0.0004t^2)dt\right]=\exp\left(-\left[0.002t+0.005t^2+\frac{0.0004}{3}t^3\right]_0^6\right)=e^{-0.2208}$$

So:

$$PV_{t=0}=543.81\times e^{-0.136}\times e^{-0.2208}=£380.62$$

Alternatively, we could calculate the present value at time 0 all in one go. The discount factor from time t (>10) to time 0 is:

$$v(t)=\exp\left[-\int_0^6 (0.002+0.01t+0.0004t^2)dt\right]\times\exp\left[-\int_6^{10} (0.01+0.003t)dt\right]\times\exp\left[-\int_{10}^{t} 0.04\,ds\right]$$

$$=e^{-0.2208}\times e^{-0.136}\times e^{-0.04(t-10)}$$

$$=e^{0.0432-0.04t}$$

using results obtained earlier. Then the present value at time 0 is obtained from the integral:

$$PV_{t=0}=\int_{10}^{15}\rho(t)v(t)\,dt=\int_{10}^{15}120e^{0.0432-0.04t}\,dt=120e^{0.0432}\int_{10}^{15}e^{-0.04t}\,dt$$

6.7 We can calculate the accumulated value at time 8 first and then accumulate that value to time 10. Using the general formula for the accumulated value of a payment stream with $a=4$, $b=8$ and $\rho(t)=0.3+1.5t$:

$$AV_{t=8}=\int_4^8 (0.3+1.5t)\exp\left(\int_t^8 (0.01+0.05s)ds\right)dt$$

$$=\int_4^8 (0.3+1.5t)\exp\left(\left[0.01s+0.025s^2\right]_t^8\right)dt$$

$$=\int_4^8 (0.3+1.5t)\exp\left(\left[1.68-0.01t-0.025t^2\right]\right)dt$$

Now, using the general result that $\int_a^b f'(t)e^{f(t)}\,dt=\left[e^{f(t)}\right]_a^b$, we know:

$$\int_4^8 -(0.01+0.05t)\exp\left(1.68-0.01t-0.025t^2\right)dt=\left[\exp\left(1.68-0.01t-0.025t^2\right)\right]_4^8$$

So, since $0.3+1.5t=-30\times-(0.01+0.05t)$, we have:

$$AV_{t=8}=-30\int_4^8 -(0.01+0.05t)\exp\left(1.68-0.01t-0.025t^2\right)dt$$

$$=-30\left[\exp\left(1.68-0.01t-0.025t^2\right)\right]_4^8$$

Evaluating this gives:

$$AV_{t=8}=-30\left(\exp(1.68-0.08-1.6)-\exp(1.68-0.04-0.4)\right)$$

$$=-30\left(e^0-e^{1.24}\right)$$

$$=73.6684$$

We now need to accumulate this value at time 8 forward to time 10:

$$AV_{t=10}=AV_{t=8}\times A(8,10)$$

$$=73.6684\times\exp\left(\int_8^{10}(0.01+0.05t)\,dt\right)$$

$$=73.6684\times\exp\left(\left[0.01t+0.025t^2\right]_8^{10}\right)$$

$$=73.6684\times\exp\left((0.1+2.5)-(0.08+1.6)\right)$$

$$=73.6684\,e^{0.92}$$

$$=184.855$$

Alternatively, we can carry out the integral:

$$AV_{t=8}=\int_4^8(0.3+1.5t)\exp\left(\left[1.68-0.01t-0.025t^2\right]\right)dt$$

using the substitution $u=1.68-0.01t-0.025t^2$*, so that:*

$$\frac{du}{dt}=-0.01-0.05t \;\Rightarrow\; du=\left(-0.01-0.05t\right)dt \;\Rightarrow\; -30du=\left(0.3+1.5t\right)dt$$

Also, when $t=4$, $u=1.24$ *and when* $t=8$, $u=0$, *so the integral becomes:*

$$AV_{t=8}=\int_{1.24}^{0}-30e^u\,du=-30\left[e^u\right]_{1.24}^{0}=-30\left(1-e^{1.24}\right)=73.6684$$

6.8 (i) We have to break down the expression for $A(t)$ at times when the force of interest changes. Since $A(t)$ represents the accumulated value at time t of an investment of 1 at time 0, we have:

$0 < t \le 1$:

$$A(t) = e^{0.04t}$$ [1]

$1 < t \le 5$:

$$A(t) = A(1) \times \exp\left[\int_1^t (0.05s - 0.01)\,ds\right]$$

$$= e^{0.04} \times \exp\left[\int_1^t (0.05s - 0.01)\,ds\right]$$

$$= e^{0.04} \times \exp\Big[0.025s^2 - 0.01s\Big]_1^t$$

$$= e^{0.04} \times e^{0.025t^2 - 0.01t - 0.025 + 0.01}$$

$$= e^{0.025t^2 - 0.01t + 0.025}$$ [2]

$5 < t$:

$$A(t) = A(5) \times \exp\left[\int_5^t 0.24\,ds\right]$$

$$= e^{0.025 \times 5^2 - 0.01 \times 5 + 0.025} \times \exp\left[\int_5^t 0.24\,ds\right]$$

$$= e^{0.6} e^{0.24t - 1.2}$$

$$= e^{0.24t - 0.6}$$ [2]

[Total 5]

(ii) We will calculate the present value at time 5 first and then discount it back to time 0. The present value at time 5 is:

$$PV_{t=5} = \int_5^{10} 25e^{-0.02t} \exp\left[-\int_5^t 0.24\,ds\right] dt$$ [1]

Carrying out the integration:

$$PV_{t=5} = \int_5^{10} 25e^{-0.02t} \exp\left(-\left[0.24s\right]_5^t\right) dt$$

$$= \int_5^{10} 25e^{-0.02t} e^{-0.24t+1.2} dt$$

$$= 25e^{1.2} \int_5^{10} e^{-0.26t} dt$$

$$= 25e^{1.2} \left[\frac{e^{-0.26t}}{-0.26}\right]_5^{10}$$

$$= \frac{25e^{1.2}}{-0.26}\left[e^{-2.6} - e^{-1.3}\right]$$

$$= 63.2924$$

[2]

We then need to discount this back to time 0. We know from part (i) that the accumulation factor from time 0 to time 5 is:

$$A(5) = e^{0.025\times 5^2 - 0.01\times 5 + 0.025} = e^{0.6}$$

So the discount factor from time 5 to time 0 is $e^{-0.6}$. The present value at time 0 of the payment stream is therefore:

$$PV_{t=0} = 63.2924e^{-0.6} = 34.7356$$

[1]

[Total 4]

Alternatively, we could calculate the present value of the payment stream at time 0 all in one go. From part (i) we know that for $t > 5$:

$$A(t) = e^{0.24t-0.6}$$

So the discount factor from time t (>5) to time 0 is:

$$v(t) = \frac{1}{A(t)} = e^{-0.24t+0.6}$$

and the present value of the payment stream at time 0 is obtained from the integral:

$$PV_{t=0} = \int_5^{10} 25e^{-0.02t} v(t)\,dt = \int_5^{10} 25e^{-0.02t} e^{-0.24t+0.6} dt = 25e^{0.6} \int_5^{10} e^{-0.26t} dt$$

6.9 (i) ***Present value of a unit sum due at time 28***

The present value is:

$$PV = e^{-\int_0^{10} 0.03\,dt} \times e^{-\int_{10}^{20} 0.003t\,dt} \times e^{-\int_{20}^{28} 0.0001t^2\,dt} \qquad [3]$$

Evaluating the integrals:

$$\int_0^{10} 0.03\,dt = \left[0.03t\right]_0^{10} = 0.3 \qquad [1]$$

$$\int_{10}^{20} 0.003t\,dt = \left[0.0015t^2\right]_{10}^{20} = 0.0015\left(20^2 - 10^2\right) = 0.45 \qquad [1]$$

and:

$$\int_{20}^{28} 0.0001t^2\,dt = \left[0.0001\frac{t^3}{3}\right]_{20}^{28} = \frac{0.0001}{3}\left(28^3 - 20^3\right) = 0.46507 \qquad [1]$$

So the present value is:

$$PV = e^{-0.3} \times e^{-0.45} \times e^{-0.46507} = e^{-1.21507} = 0.29669 \qquad [1]$$

[Total 7]

(ii)(a) ***Equivalent constant force of interest***

We want the value of δ that satisfies the equation:

$$e^{-28\delta} = 0.29669 \qquad [1]$$

Solving this equation, we find that $\delta = \dfrac{\ln 0.29669}{-28} = 0.04340$, or 4.340% *pa*. [1]

(ii)(b) ***Equivalent annual effective rate of discount***

We have:

$$d = 1 - v = 1 - e^{-\delta} = 1 - e^{-0.04340} = 0.04247$$

The rate of discount is 4.247% *pa*. [1]

[Total 3]

(iii) ***Present value of the payment stream***

The present value of the payment stream at time 0 is obtained from the integral:

$$PV_{t=0}=\int_3^7 e^{-0.04t}\,v(t)\,dt \qquad [1]$$

The discount factor from time t (<10) to time 0 is:

$$v(t)=e^{-0.03t} \qquad [1]$$

So the present value is:

$$PV_{t=0}=\int_3^7 e^{-0.04t}\,e^{-0.03t}\,dt=\int_3^7 e^{-0.07t}\,dt$$

$$=\left[\frac{e^{-0.07t}}{-0.07}\right]_3^7=\frac{1}{0.07}\left(e^{-0.21}-e^{-0.49}\right)=2.82797 \qquad [2]$$

[Total 4]

Alternatively, we could calculate the present value of the payment stream at time 3 first and then discount it back to time 0. The present value at time 3 is:

$$PV_{t=3}=\int_3^7 e^{-0.04t}\exp\left(-\int_3^t 0.03\,ds\right)dt$$

$$=\int_3^7 e^{-0.04t}\exp\left(-\left[0.03s\right]_3^t\right)dt$$

$$=\int_3^7 e^{-0.04t}e^{-0.03(t-3)}\,dt$$

$$=e^{0.09}\int_3^7 e^{-0.07t}\,dt$$

Evaluating the integral gives:

$$PV_{t=3}=e^{0.09}\left[\frac{e^{-0.07t}}{-0.07}\right]_3^7=\frac{e^{0.09}}{0.07}\left(e^{-0.21}-e^{-0.49}\right)=3.09429$$

Then we can discount this value at time 3 back to time 0, to obtain the present value at time 0:

$$PV_{t=0}=3.09429\times e^{-0.03\times 3}=2.82797$$

7

Level annuities

Syllabus objectives

2.5 Define and derive the following compound interest functions (where payments can be in advance or in arrears) in terms of i , v , n , d , δ , $i^{(p)}$ and $d^{(p)}$:

2.5.1 $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$

2.5.2 ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a^{(p)}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and ${}_{m|}\bar{a}_{\overline{n}|}$

0 Introduction

In the next two chapters we will learn how to calculate the present value of a series of payments. We will also meet some actuarial symbols representing compound interest functions, which we will use very frequently in this course.

Terminology

Here is a brief summary of the key terms that we will use:

An *annuity* is a regular series of payments. An *annuity-certain* is an annuity payable for a definite period of time: the payments do not depend on some factor, such as whether a person is alive or not.

If payments are made at the *end* of each time period, they are paid *in arrears*. If they are made at the *beginning* of each time period, they are paid *in advance*. An annuity paid in advance is also known as an *annuity-due*.

Where the first payment is made during the first time period, this is an *immediate* annuity. Where no payments are made during the first time period, this is a *deferred* annuity.

If each payment is for the same amount, this is a *level* annuity. If payments increase (decrease) each time by the same amount, this is a *simple increasing (decreasing)* annuity.

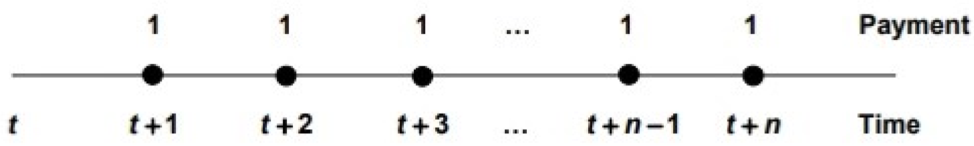
So, for example, payments of £2 made at the start and halfway through each of the next five years can be described as a level immediate annuity payable half-yearly in advance for five years

This chapter covers level annuities – both immediate and deferred. Chapter 8 then covers increasing and decreasing annuities, and ultimately deals with variable payments that can be evaluated using similar techniques to the ones detailed here.

1 Present values

1.1 Payments made in arrears

Consider a series of n payments, each of amount 1, to be made at time intervals of one unit, the first payment being made at time $t+1$.



Such a sequence of payments is illustrated in the diagram above, in which the rth payment is made at time $t+r$.

The value of this series of payments *one unit of time before the first payment is made* is denoted by $a_{\overline{n}|}$.

So, in the above example, $a_{\overline{n}|}$ is the value at time t of the payments shown.

The symbol $a_{\overline{n}|}$ (pronounced 'A.N.') represents the present value of an annuity consisting of n payments of 1 unit made at the end of each of the next n time periods. This is called an annuity paid in arrears.

Clearly, if $i=0$, then $a_{\overline{n}|}=n$; otherwise:

$$\begin{aligned}a_{\overline{n}|} &= v+v^2+v^3+\cdots+v^n\\ &= \frac{v(1-v^n)}{1-v}\\ &= \frac{1-v^n}{v^{-1}-1}\\ &= \frac{1-v^n}{i} \qquad \textbf{(1.1)}\end{aligned}$$

The derivation above uses the fact that the terms on the right-hand side of the first equation form a geometric progression, so we can use the formula for the sum of the first n terms of a geometric progression:

$$a+ar+ar^2+\cdots+ar^{n-1}=\frac{a\left(1-r^n\right)}{1-r}$$

where: $a=\text{first term}=v$

and: $r=\text{common ratio}=v$.

An alternative way is prove result (1.1) is to multiply the first equation through by $(1+i)$, so that v^3 becomes v^2 *etc*. This gives a series very similar to the original series. We can then obtain a formula for $a_{\overline{n|}}$ by subtracting the two equations:

$$a_{\overline{n|}} = \quad v + v^2 + v^3 + \cdots + v^{n-1} + v^n$$

$$(1+i)a_{\overline{n|}} = 1 + v + v^2 + v^3 + \cdots + v^{n-1}$$

$$\therefore \quad i a_{\overline{n|}} = 1 - v^n$$

So:

$$a_{\overline{n|}} = \frac{1-v^n}{i}$$

If $n=0$, $a_{\overline{n|}}$ is defined to be zero.

Thus $a_{\overline{n|}}$ is the value at the start of any period of length *n* of a series of *n* payments, each of amount 1, to be made *in arrears* at unit time intervals over the period. It is common to refer to such a series of payments, made in arrears, as an *immediate annuity-certain* and to call $a_{\overline{n|}}$ the present value of the immediate annuity-certain. When there is no possibility of confusion with a life annuity (*ie* a series of payments dependent on the survival of one or more human lives), the term *annuity* may be used as an alternative to annuity-certain.

$a_{\overline{n|}}$ is called an immediate annuity even though the first payment is made at the end of the first time period, and hence not 'immediately'. Immediate annuities are annuities where the first payment is made during the first time period, including at the end of the period. If no payments are made during the first time period then the annuity is *deferred*. Deferred annuities are covered in Section 7.

In the *Tables*, values of $a_{\overline{n|}}$ are tabulated at various interest rates from ½% to 25%. For example, the value of $a_{\overline{10|}}$ calculated at 4% interest is given on page 56 of the *Tables* as 8.1109. We can verify this using the formula obtained above:

$$a_{\overline{10|}} = \frac{1-v^{10}}{i} = \frac{1-1.04^{-10}}{0.04} = 8.1109$$

We can denote $a_{\overline{n|}}$ calculated at a rate of interest of i% effective as $a_{\overline{n|}}^{@i\%}$, $a_{\overline{n|}@i\%}$ or $a_{\overline{n|},i\%}$.

Question

Calculate the present value as at 1 March 2020 of a series of payments of £1,000 payable on the first day of each month from April 2020 to December 2020 inclusive, assuming a rate of interest of 6% *pa* convertible monthly.

Solution

An interest rate of 6% *pa* convertible monthly is equivalent to an effective monthly interest rate of ½%. There are 9 payments of £1,000 each, starting in one month's time. So, working in months, the PV of the payments is:

$$1{,}000a_{\overline{9|}}^{@\frac{1}{2}\%}$$

We can look up $a_{\overline{9|}}^{@\frac{1}{2}\%}$ in the *Tables* or alternatively apply the formula:

$$1{,}000a_{\overline{9|}}^{@\frac{1}{2}\%} = 1000 \times \frac{1-1.005^{-9}}{0.005} = 1000 \times 8.7791 = £8{,}779$$

1.2 Payments made in advance

Now consider a series of n payments made at the start of each time period as represented on the timeline below. The first payment is made at time zero and the last at time $n-1$.

1	1	1	1	...	1	Payment
0	1	2	3	...	n–1	Time

The value of this series of payments *at the time the first payment is made* is denoted by $\ddot{a}_{\overline{n|}}$.

The symbol $\ddot{a}_{\overline{n|}}$ is pronounced 'A due n'.

If $i=0$, then $\ddot{a}_{\overline{n|}} = n$; otherwise:

$$\begin{aligned} \ddot{a}_{\overline{n|}} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \frac{1-v^n}{1-v} \\ &= \frac{1-v^n}{d} \end{aligned} \qquad \textbf{(1.2)}$$

Again, the derivation of the formula above uses the fact that the n terms on the right-hand side form a geometric progression with first term 1, and common ratio v.

Thus $\ddot{a}_{\overline{n}|}$ is the value at the start of any given period of length *n* of a series of *n* payments, each of amount 1, to be made *in advance* at unit time intervals over the period. It is common to refer to such a series of payments, made in advance, as an *annuity-due* and to call $\ddot{a}_{\overline{n}|}$ the present value of the annuity-due.

It follows directly from the above definitions that:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$

and that, for $n \geq 2$, $\quad \ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$ **(1.3)**

The first of the equations can be obtained by reasoning that the payments for $\ddot{a}_{\overline{n}|}$ correspond exactly with those for $a_{\overline{n}|}$, except that each payment is made one year earlier, *ie* each payment has a present value that is greater by a factor of $(1+i)$. So $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

Alternatively, since $(1+i)v = 1$, we can obtain this algebraically:

$$\begin{aligned}\ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= (1+i)(v + v^2 + \cdots + v^{n-1} + v^n) \\ &= (1+i)a_{\overline{n}|}\end{aligned}$$

The second of the equations (1.3) can be obtained by noting that $\ddot{a}_{\overline{n}|}$, which is the PV at time $t=0$ of a series of payments of 1 payable at times $t = 0, 1, 2, \ldots, n-1$, is the same as:

- an initial payment of 1 (which has a PV of 1) plus
- a series of payments of 1, payable at times $t = 1, 2, \ldots, n-1$ (which has a PV of $a_{\overline{n-1}|}$).

So $\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1$.

Alternatively, algebraically, we have:

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1} = 1 + (v + v^2 + \cdots + v^{n-1}) = 1 + a_{\overline{n-1}|}$$

Question

Calculate $a_{\overline{25}|}$ and $\ddot{a}_{\overline{15}|}$ using an interest rate of 13½% *pa* effective.

Solution

$$a_{\overline{25}|} = \frac{1-v^{25}}{i} = \frac{1-1.135^{-25}}{0.135} = 7.095 \quad \text{and} \quad \ddot{a}_{\overline{15}|} = \frac{1-v^{15}}{d} = \frac{1-1.135^{-15}}{0.135/1.135} = 7.149$$

2 Accumulations

The value of the series of payments *at the time the last payment is made* is denoted by $s_{\overline{n|}}$. The value *one unit of time after the last payment is made* is denoted by $\ddot{s}_{\overline{n|}}$.

In other words, $s_{\overline{n|}}$ considers the same series of payments as $a_{\overline{n|}}$ but it is the accumulated value at time n, as opposed to the present value at time 0. Similarly, $\ddot{s}_{\overline{n|}}$ is the accumulated value at time n of the annuity we looked at above when defining $\ddot{a}_{\overline{n|}}$, as shown on the timeline:

If $i = 0$ then $s_{\overline{n|}} = \ddot{s}_{\overline{n|}} = n$; otherwise

$$\begin{aligned} s_{\overline{n|}} &= (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \cdots + 1 \\ &= (1+i)^n a_{\overline{n|}} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned} \qquad \textbf{(2.1)}$$

and:

$$\begin{aligned} \ddot{s}_{\overline{n|}} &= (1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) \\ &= (1+i)^n \ddot{a}_{\overline{n|}} \\ &= \frac{(1+i)^n - 1}{d} \end{aligned} \qquad \textbf{(2.2)}$$

Thus $s_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ are the values at the end of any period of length n of a series of n payments, each of amount 1, made at unit time intervals over the period, where the payments are made in arrears and in advance respectively. Sometimes $s_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ are called the 'accumulation' (or the 'accumulated amount') of an immediate annuity and an annuity-due respectively. When $n = 0$, $s_{\overline{n|}}$ and $\ddot{s}_{\overline{n|}}$ are defined to be zero. It is an immediate consequence of the above definition that:

$$\ddot{s}_{\overline{n|}} = (1+i)s_{\overline{n|}}$$

and that $$s_{\overline{n+1|}} = 1 + \ddot{s}_{\overline{n|}}$$

or $$\ddot{s}_{\overline{n|}} = s_{\overline{n+1|}} - 1$$

Question

Calculate $s_{\overline{10}|}$ and $\ddot{s}_{\overline{13}|}$ using an interest rate of 3½% *pa* effective.

Solution

$$s_{\overline{10}|} = \frac{(1+i)^{10} - 1}{i} = \frac{1.035^{10} - 1}{0.035} = 11.731 \quad \text{and} \quad \ddot{s}_{\overline{13}|} = \frac{(1+i)^{13} - 1}{d} = \frac{1.035^{13} - 1}{0.035/1.035} = 16.677$$

3 Continuously payable annuities

Let *n* be a non-negative number. The value at time 0 of an annuity payable continuously between time 0 and time *n*, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n|}}$.

This is a mathematical idealisation, which makes calculations easier. With a continuously payable annuity, the payments are considered as a continuous cashflow that is payable at a given *rate* over a given period of time. The symbol $\bar{a}_{\overline{n|}}$ is pronounced 'A bar n'.

We can obtain a formula for $\bar{a}_{\overline{n|}}$ using the formula for the present value of a continuous payment stream we met in Chapter 6.

Clearly:

$$\bar{a}_{\overline{n|}} = \int_0^n 1.v^t\,dt = \int_0^n e^{-\delta t}\,dt = \left[-\frac{1}{\delta}e^{-\delta t}\right]_0^n$$

$$= \frac{1-e^{-\delta n}}{\delta}$$

$$= \frac{1-v^n}{\delta} \qquad (\text{if } \delta \neq 0) \qquad (3.1)$$

Note that $\bar{a}_{\overline{n|}}$ is defined even for non-integral values of *n*. If $\delta = 0$ (or, equivalently, $i = 0$), $\bar{a}_{\overline{n|}}$ is of course equal to *n*.

Since equation (3.1) may be written as:

$$\bar{a}_{\overline{n|}} = \frac{i}{\delta}\left(\frac{1-v^n}{i}\right)$$

it follows immediately that, if *n* is an integer:

$$\bar{a}_{\overline{n|}} = \frac{i}{\delta}a_{\overline{n|}} \qquad (\text{if } \delta \neq 0)$$

Note that this relationship can be very useful, especially when using the *Tables*. Although values of $\bar{a}_{\overline{n|}}$ are not tabulated, values of $a_{\overline{n|}}$ and i/δ are.

The *accumulated amount* of such an annuity at the time the payments cease is denoted by $\bar{s}_{\overline{n|}}$. By definition, therefore:

$$\bar{s}_{\overline{n|}} = \int_0^n e^{\delta(n-t)}\,dt$$

Hence:

$$\bar{s}_{\overline{n|}} = (1+i)^n\,\bar{a}_{\overline{n|}}$$

If the rate of interest is non-zero:

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

$$= \frac{i}{\delta} s_{\overline{n}|}$$

Question

A payment of £1 is made at the beginning of each week for 1 year.

By approximating these weekly payments as a continuously payable annuity, and assuming that there are 52.18 weeks in a year, calculate the present value of these payments, using an effective rate of interest of 8% *pa*.

Solution

Using the assumption given, £52.18 will be paid in total over the year. The PV of the weekly payments is then:

$$52.18\bar{a}_{\overline{1}|}^{@8\%} = 52.18\left(\frac{1-v}{\delta}\right) = 52.18 \times \frac{1-1.08^{-1}}{\ln(1.08)} = 52.18 \times 0.96244 = £50.22$$

Using this approximation, we would have arrived at exactly the same answer if the payments were made at the ***end*** *of each week.*

Notice the similarity between the formulae for the present and accumulated values of annuities:

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \qquad \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

The numerators are always consistent – the only difference between the formulae is in the denominators.

4 Annuities payable *p*thly

4.1 Present values

Where annuity payments are made p times a year (*eg* $p=12$ for a monthly annuity), a superscript (p) is added in the top right-hand corner of the symbol. The annuity is still payable for n years and still refers to a total *annual* amount of 1 unit, *ie* the annuity consists of np payments, each of amount $1/p$ units.

If p and n are positive integers, the notation $a_{\overline{n}|}^{(p)}$ is used to denote the value at time 0 of a level annuity payable $p$thly in arrears at the rate of 1 per unit time over the time interval $[0,n]$. For this annuity the payments are made at times $1/p, 2/p, 3/p, \ldots, n$ and the amount of each payment is $1/p$.

By definition, a series of p payments, each of amount $i^{(p)}/p$ in arrears at $p$thly subintervals over any unit time interval, has the same value as a single payment of amount i at the end of the interval. By proportion, p payments, each of amount $1/p$ in arrears at $p$thly subintervals over any unit time interval, have the same value as a single payment of amount $i/i^{(p)}$ at the end of the interval.

Consider now that annuity for which the present value is $a_{\overline{n}|}^{(p)}$. The remarks in the preceding paragraph show that the p payments after time $r-1$ and not later than time r have the same value as a single payment of amount $i/i^{(p)}$ at time r. This is true for $r = 1, 2, \ldots, n$, so the annuity has the same value as a series of n payments, each of amount $i/i^{(p)}$, at times $1, 2, \ldots, n$. This means that:

$$a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}} a_{\overline{n}|} \tag{4.1}$$

Note that $i/i^{(p)}$ is tabulated in the *Tables* and so $a_{\overline{n}|}^{(p)}$ can quickly be calculated for some interest rates by looking up the values of $i/i^{(p)}$ and $a_{\overline{n}|}$.

An alternative approach, from first principles, is to write:

$$a_{\overline{n}|}^{(p)} = \sum_{t=1}^{np} \frac{1}{p} v^{t/p} = \frac{1}{p} \frac{v^{1/p}(1-v^n)}{1-v^{1/p}}$$

$$= \frac{1-v^n}{p\left[(1+i)^{1/p} - 1\right]}$$

$$= \frac{1-v^n}{i^{(p)}} \tag{4.2}$$

which confirms equation (4.1).

The first line in the above derivation uses the formula for the sum of a geometric progression of np terms, with first term $\frac{1}{p}v^{1/p}$ and common ratio $v^{1/p}$.

Question

Calculate $a_{\overline{6}|}^{(4)}$ at 1½% *pa*, both with and without using the *Tables*.

Solution

Using the *Tables*:

$$a_{\overline{6}|}^{(4)} = \frac{i}{i^{(4)}} a_{\overline{6}|} = 1.005608 \times 5.6972 = 5.729$$

Without using the *Tables*:

$$a_{\overline{6}|}^{(4)} = \frac{1-v^6}{i^{(4)}} = \frac{1-1.015^{-6}}{4(1.015^{0.25}-1)} = 5.729$$

Likewise, we define $\ddot{a}_{\overline{n}|}^{(p)}$ to be the present value of a level annuity-due payable *p*thly at the rate of 1 per unit time over the time interval $[0,n]$. (The annuity payments, each of amount $1/p$, are made at times $0, 1/p, 2/p, \ldots, n-(1/p)$.)

A series of *p* payments, each of amount $d^{(p)}/p$, in advance at *p*thly subintervals over any unit time interval has the same value as a single payment of amount *i* at end of the interval. Hence, by proportion, *p* payments, each of amount $1/p$ in advance at *p*thly subintervals, have the same value as a single payment of amount $i/d^{(p)}$ at the end of the interval. This means (by an identical argument to that above) that:

$$\ddot{a}_{\overline{n}|}^{(p)} = \frac{i}{d^{(p)}} a_{\overline{n}|} \qquad (4.3)$$

Alternatively, from first principles, we may write:

$$\ddot{a}_{\overline{n}|}^{(p)} = \sum_{t=1}^{np} \frac{1}{p} v^{(t-1)/p}$$

$$= \frac{1-v^n}{d^{(p)}} \qquad (4.4)$$

(on simplification), which confirms equation (4.3).

The simplification referred to here involves using the formula for the sum of a geometric progression of np terms, with first term $\frac{1}{p}$ and common ratio $v^{1/p}$.

Note that:

$$a_{\overline{n|}}^{(p)} = v^{1/p}\ddot{a}_{\overline{n|}}^{(p)} \qquad \textbf{(4.5)}$$

each expression being equal to $\dfrac{(1-v^n)}{i^{(p)}}$**.**

Equation (4.5) can be derived by general reasoning. The payments for $\ddot{a}_{\overline{n|}}^{(p)}$ correspond exactly with those for $a_{\overline{n|}}^{(p)}$, except that each payment is made a period of length $1/p$ earlier, *ie* each payment has a present value that is greater by a factor of $(1+i)^{1/p}$. So $\ddot{a}_{\overline{n|}}^{(p)} = (1+i)^{1/p} a_{\overline{n|}}^{(p)}$, which is equivalent to equation (4.5).

Note that, since:

$$\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} d^{(p)} = \delta$$

it follows immediately from equation (4.2) and (4.4) that:

$$\lim_{p\to\infty} a_{\overline{n|}}^{(p)} = \lim_{p\to\infty} \ddot{a}_{\overline{n|}}^{(p)} = \bar{a}_{\overline{n|}}$$

Question

Calculate the present value as at 1 January 2019 of a series of payments of £100 payable on the first day of each month during 2020, 2021 and 2022, assuming an effective rate of interest of 8% per annum.

Solution

The present value of the payments as at 1 January 2020 is $1{,}200\ddot{a}_{\overline{3|}}^{(12)}$.

So the present value as at 1 January 2019 is:

$$1{,}200v\ddot{a}_{\overline{3|}}^{(12)} = 1{,}200v\frac{1-v^3}{d^{(12)}} = 1{,}200 \times 0.92593 \times \frac{1-0.79383}{0.076714} = £2{,}986$$

using the values given in the *Tables*.

Since no payments are received during 2019, this is an example of a deferred annuity. These are dealt with in more detail in Section 7 of this chapter.