Basics of strong-field physics
(and its application to heavy-ion physics)

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Plan

1. General introduction.
   - Why strong-field physics interesting?
   - Where?
   - An example in heavy-ion phys
     → early-stage dynamics of HIC.
2. Schwinger effect.
   - Overview
   - Theory
     - Setup

- Recap: "usual" canonical quantization in QFT.

- Bogoliubov-transformation approach to the Schwinger effect

- Realtime dynamics and the backreaction problem.

$\boxed{\text{Useful references}}$

← motivations
- old theory but incomplete (scattering, non-eq. QFT, ...)
- App. to hadron phys (Lund model, early-time dynamics of QGP, ...)
- App to other areas (early Universe, Hawking rad., ...)
- Instructive for QFT (particle picture, regularization, ...)

For strong-field QED:

- Piazza-Muller-Hatsagortsyan-Keitel 1111.3886
- Fedotov et al. 2203.00019.
- Hattori-Itakura-Ozaki 2305.03865

For Schwinger effect

- Dunne hep-th/0406216
- Gelis-Tanji 1510.05451
- Tanji's note (in Japanese): see my webpage

For backreaction

- Kluger et. al. PRD 45, 4659 (1992)
- Tanji 0810.4429
- Tanya's thesis : see my webpage
- Textbooks on QFT in curved spacetime
  e.g., Birrel-Davies, Parker-Toms, ...

§1. General introduction.

Why strong-field interesting

Strong field = So many particles

such that $\langle\phi\rangle \gg 1$.

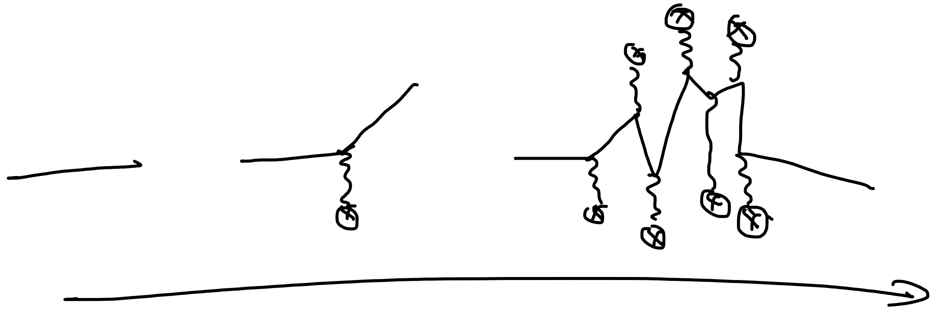
Field: can be anything

- EM field. ← main focus
- Gravitational field
- Gluon field
- Condensate ...

Strong vs. weak field

L5

Example : propagation of a particle



vacuum

weak field

$gF/m^2 \lesssim 1$

strong field

$gF/m^2 \gtrsim 1.$

two dim.ful parameters in the problem.

field → gF
↑ coupling

particle → m.

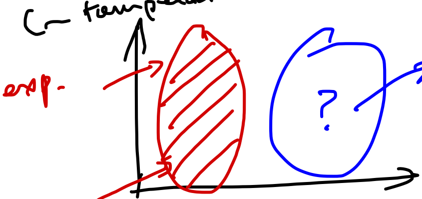
↳ one dim.less. parameter.

no other param.

Small change
‖
perturbative
↓
well understood

e.g., anomalous magnetic moment.

BIG change
‖
NON perturbative
↓
- less understood
← smtg "new" beyond pert. picture happen
- "new" regime for particle physics



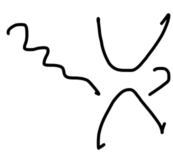
Where can we find strong field? L6

Short answer →

- Impossible in the 20th century
- Now, the situation is gradually changing

→ Timely

Typical order of magnitude : EM field.

daily life	industry	Science	
LED light.	Laser welding	Cond-mat. (THz laser)	Guiness WR. HERCULES 2008
		 $O(1\sim 10\,eV)$	
$I \sim 10^{-5}\,W/cm^2$	$10^{6}\,W/cm^2$	$10^{10}\,W/cm^2$	$10^{22}\,W/cm^2$
$eE \sim (10^{-3}\,eV)^2$	$(10^{-1}\,eV)^2$	$(1\,eV)^2$	$(1\,keV)^2$

Much weaker than the electron mass $m_e = 511\,\mathrm{keV}$

Technology development → Availability of strong field (observability)

EM field:

- Intense laser (e.g., ELI, SULF, ...) → 10 keV
  cf. CPA technique 2018 Nobel Prize

  Go beyond $eE > m_e^2$ in the boosted frame

- Collider exp. (e.g., ILC)
  ($\mathcal{O}(100\,\mathrm{GeV})$ electron beam)
- Collider + laser (e.g., FACET-II (SLAC), LUXE (DESY), ...)

- Heavy-ion collisions (e.g., LHC, RHIC) → $\gtrsim m_\pi^2$
- Magnetars (e.g., IXPE, XL-Calibur (SUZAKU)) → $\gtrsim m_e^2$

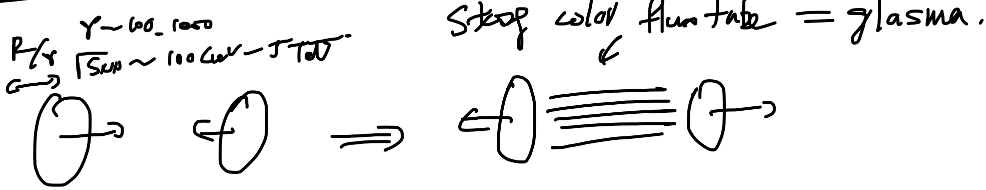
For other fields:

- Glasma in HIC → strong color field (gluon)
- Blackhole → strong gravitational field
- (P)reheating in the early Universe → strong inflaton field
- Electrical breakdown (non-linear optics) → strong EM field in material

A bit more about the strong color field in HIC

L9

Spacetime evolution of HIC



QGP.
= "thermalized" matter composed of deconfined quarks and gluons.

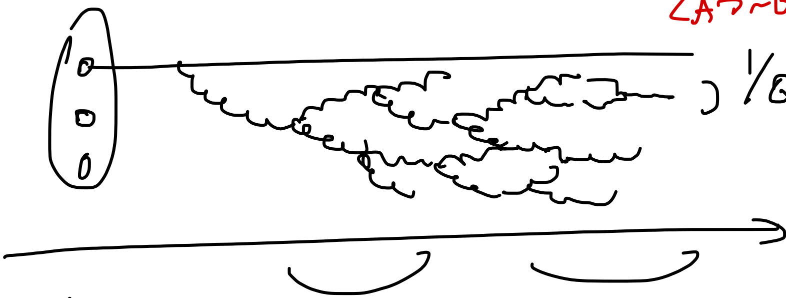
The property of glasma

- How appear? → formation of "color" capacitor.
  - ∵ Incident high-energy ion ≈ dense "color" plate

· Gluon saturation (color glass condensate).

unique scale!

$\langle A \rangle \sim Q_s$ saturation scale $\sim$ a few GeV.



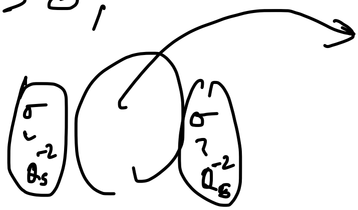
Energy $\left(\sim 1/\text{Bjorken } x\right)$

quark model — more gluons — saturation

$g \to gg$ — $g \to gg$ vs $gg \to g$

$H = 3q$.

· So,



have (color) source $\to$ (color) EM field

$$\mathrm{div}\,\mathbf{E} = \rho + \underbrace{[\cdots]}$$

$$\mathrm{div}\,\mathbf{B} = 0 + \underbrace{[\cdots]}$$

non-Abelian feature

$$D_\mu \overset{(\sim)}{F}{}^{\mu\nu} = \partial_\mu \overset{(\sim)}{F}{}^{\mu\nu} + ig\left[A_\mu, \overset{(\sim)}{F}{}^{\mu\nu}\right]$$

$\Rightarrow$ ✓ $E, B \sim O(Q_s^2) \gg m_q^2$ $\to$ strong.

✓ $E /\!/ B$ realized

Open question

How glasma decays into QGP ?

→ NO established understanding !..

← Essentially, a backreaction prob. initiated by strong color EM field.

→ Today's lecture will be about the QED formulation of this.

§2. Schwinger effect

Overview

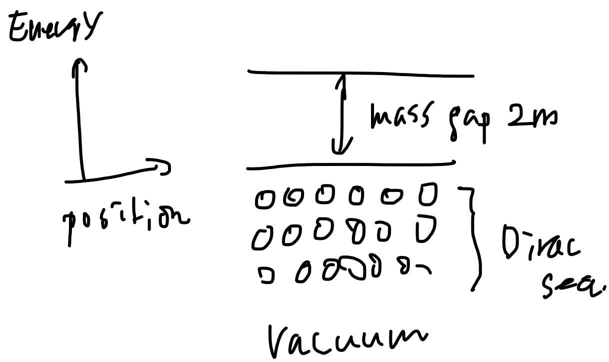
What is it?

A novel phenomenon due to strong EM field.

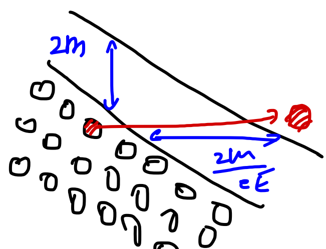
Strong E field → Vacuum decays against pair production.

Intuitive picture

$$e\phi = -eEx$$



E field →



If you remember WKB treatment of tunneling (Gamow theory)   cf. Glendenning–Matsui (1983)

$$P_{\text{tunnel}} \propto \exp\left[-\int_{\substack{\text{forbidden}\\ \text{band.}}} v\, dx\right].$$

$$\sim \exp\left[-(\text{area of the gap})\right]$$

$$\sim \exp\left[-\# \; m \times \frac{m}{eE}\right]$$

$$= \exp\left[-\# \frac{m^2}{eE}\right].$$

If you do a QFT calculation (Schwinger 1951)

$$N \propto \exp\left[-\pi \frac{m^2}{eE}\right].$$

- Non-perturbative $\propto e^{-\#/e}$
  → cannot be captured by pert. theory
- Needs strong field $eE \gtrsim m^2$.
  → cannot be realized by weak fields

Why important?

- Physics of the vacuum.
  → the most fundamental process, since everything happens on top of the vacuum

- Important (as a toy model) to understand some physical process under extreme conditions

  e.g., Early-time dynamics of HIC.
  Hawking radiation.
  (P)reheating, ...

  in particular baryocreation.

- Timely
  → Maybe testable in the near future with intense lasers (hopefully).

Theory of the Schwinger effect

Setup

will consider later.
w/ → backreaction
w/o → no backreact.

- Scalar QED

$$\mathcal{L} = \overbrace{|\underbrace{D_\mu}_{= \partial_\mu + ieA_\mu}\phi|^2 - m^2|\phi|^2}^{\mathcal{L}_{mat}} + \overbrace{\frac{-1}{4}F_{\mu\nu}F^{\mu\nu}}^{\mathcal{L}_{Maxwell}} + \underbrace{J_{ext}^\mu A_\mu}_{\mathcal{L}_{source}}$$

$\mathcal{L}_{source}$: ext. source

- Strong-field approximation. ($J_{ext} \gg 1$)

$$A_\mu = \underbrace{\langle A_\mu \rangle}_{= \bar{A}_\mu} + \underbrace{\left(A_\mu - \langle A_\mu \rangle\right)}_{= a_\mu},$$

$\bar{A}_\mu$: classical coherent macroscopic field.

a_μ: quantum flct.

Assume

$$\bar{A}_\mu \gg \| a_\mu \| \iff A_\mu \approx \bar{A}_\mu .$$

$$\Downarrow$$

cf. coherent state.

drop all correlations

e.g., $\langle A_\mu A_\nu \rangle \approx \bar{A}_\mu \bar{A}_\nu$.

Then,

$$\mathcal{L}_{\text{mat}} = \left| \left(\overset{\partial_\mu + ie\bar{A}_\mu}{\bar{D}_\mu} + iea_\mu \right) \phi \right|^2 - m^2 |\phi|^2$$

$\mathcal{L}_{\text{Maxwell}}$, $\mathcal{L}_{\text{source}}$ do not couple to ϕ directly

$$= \underbrace{|\bar{D}_\mu \phi|^2 - m^2 |\phi|^2}_{\to \mathcal{L}_0}$$

$$\underbrace{+ \left\{ -iea^\mu \phi^\dagger \bar{D}_\mu \phi + (\text{h.c.}) \right\} + e^2 a^2 |\phi|^2}_{\to \mathcal{L}_{\text{int}}}$$

$$\approx \mathcal{L}_0 .$$

- Remark : You can include $\mathcal{L}_{int}$ perturbatively by using a dressed propagator set by $\mathcal{L}_0$

→ Furry-picture perturbation theory. (Furry 1951)

cf. called differently in other areas.

DBWA in nuclear phys.

dressed-state formalism in optics.

Floquet theory in cond. mat (for periodic driving)

→ Give "new" physical processes due to the dressing.

Examples.

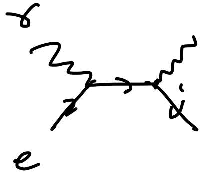
Compton.

$e\gamma \to e\gamma$

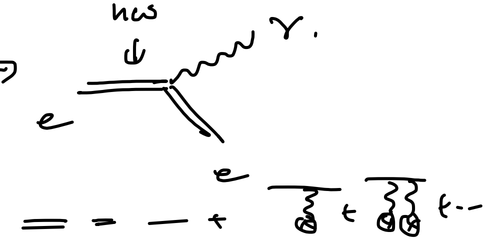
Non-linear compton.

$e + n\omega \to e\gamma$ ← strong field

Rather trivial



→

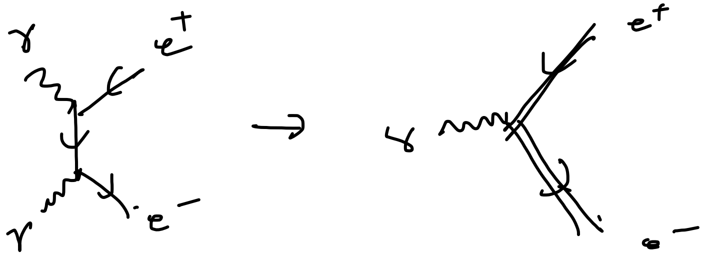


Breit-Wheeler.

$\gamma\gamma \to e^+e^-$

Non-linear BW

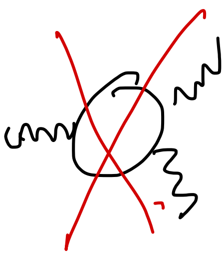
$\gamma + n\gamma \to e^+e^-$



(1st obs.
RHIC. 2021).

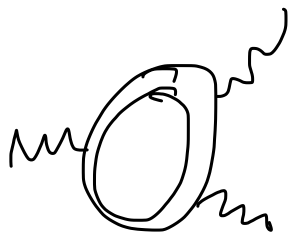
A bit non-trivial

Photon splitting.



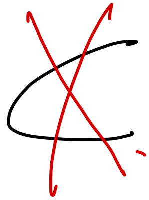
Furry thm.
(Gauge inv.)

possible!



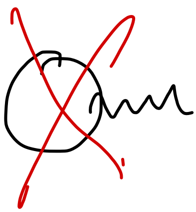
Schwinger effect

possible



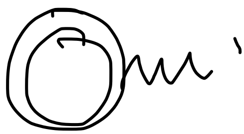
Energy consv.

Vacuum photon emission



Furry thm.

Energy consv.



also important for HHG (high-harmonic generation).

Recap: "usual" canonical quantization in QFT

Starting point: Field equation (Klein-Gordon)

$$0 = \left[\partial^2 + m^2\right]\phi .$$

1. Mode expansion

KG eq. is spatially invariant.

→ convenient to work in the Fourier space

$$\phi(t, x) = \int d^3p \, \frac{e^{ip\cdot x}}{(2\pi)^{3/2}} \left[\phi_p(t)\, a_p + \phi_p^*(t)\, b_{-p}^\dagger \right]$$

where the mode function ϕ_p satisfies

$$0 = \left[\partial_t^2 + \underbrace{m^2 + p^2}_{= \omega_p^2}\right] \phi_p$$

→ It is "natural" to take

$$\phi_p = \# \times e^{-i\omega_p t} .$$

$$= \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\omega_{\mathbf{p}} t}$$

normalized as

$$i\, \phi_{\mathbf{p}}^* \overleftrightarrow{\partial_t} \phi_{\mathbf{p}} = 1.$$

(i.e., unit charge is normalized to be 1 (one))

Note: The coefficients $\begin{pmatrix} a_{\mathbf{p}} \\ b_{-\mathbf{p}}^\dagger \end{pmatrix} \longleftrightarrow$ mode func. $\phi_{\mathbf{p}}$

$$\because \begin{pmatrix} a_{\mathbf{p}} \\ b_{-\mathbf{p}}^\dagger \end{pmatrix} = i \int d^3x \begin{pmatrix} \left(\phi_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\right)^* \\ -\left(\phi_{\mathbf{p}}^* \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\right)^* \end{pmatrix} \overleftrightarrow{\partial_t} \phi.$$

2. Imposing canonical commutation relation

$$\begin{cases} [\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] = i\,\delta^3(\mathbf{x}-\mathbf{x}') \qquad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \dot{\phi} \\ (\text{other commutations}) = 0 \end{cases}$$

$\rightarrow \begin{pmatrix} a_{\mathbf{p}} \\ b_{-\mathbf{p}}^\dagger \end{pmatrix}$ must be promoted to be operators.

$$\begin{cases} [a_{\mathbb{p}}, a^{\dagger}_{\mathbb{p}'}] = [b_{\mathbb{p}}, b^{\dagger}_{\mathbb{p}'}] = \delta^3(\mathbb{p}-\mathbb{p}') \\ (\text{others}) = 0 \end{cases}$$

3. Interpretation of $a_{\mathbb{p}}$ and $b_{\mathbb{p}}$

$a^{(\dagger)}_{\mathbb{p}}$: annihilation (creation) operator of a particle with energy $\omega_{\mathbb{k}}$, momentum $\mathbb{p}$, charge $+e$

$b^{(\dagger)}_{\mathbb{p}}$: same for an anti-particle

Why → If we calculate relevant physical observables, they take such forms.

That is, if we define

$$\begin{cases} \text{Vacuum } |0\rangle & \text{as a state s.t. } a_{\mathbb{p}}|0\rangle = b_{\mathbb{p}}|0\rangle = 0 \\ n\text{-particle state } |n\rangle \propto a^{\dagger}_{\mathbb{p}_1} a^{\dagger}_{\mathbb{p}_2} \cdots a^{\dagger}_{\mathbb{p}_n} |0\rangle \end{cases}$$

(similar for anti-particle)

$$\Rightarrow \langle \hat{O} \rangle \sim O_{1\text{-particle}} \times (\#\text{ of particle}).$$

Let us examine this more carefully

→ Consider a two-point function.

$$\hat{O} = \phi^\dagger \Gamma \phi$$

e.g.) $\hat{T}^{\mu\nu} \Rightarrow \Gamma = \overleftarrow{\partial^\mu}\overrightarrow{\partial^\nu} + \overleftarrow{\partial^\nu}\overrightarrow{\partial^\mu} - g^{\mu\nu}(\overleftarrow{\partial_\lambda}\overrightarrow{\partial^\lambda} - M^2)$

And calculate

$$O = \langle \hat{O} \rangle = \langle \phi^\dagger \Gamma \phi \rangle.$$

Four important points:

(i). O is finite even for $|0\rangle$ and can be UV divergent because of this vacuum contribution.

→ Need subtraction : Normal ordering.

$$\int d^3\mathbb{p} \frac{e^{i\mathbb{p}\cdot x}}{(2\pi)^{3/2}} [\cdots + \phi^*_{\mathbb{p}} b^\dagger_{-\mathbb{p}}]$$

$$\langle 0 | \phi^\dagger \Gamma \phi | 0 \rangle$$

$$= \int d^3\mathbb{p}\, d^3\mathbb{p}' \left(\phi^*_{\mathbb{p}} \frac{e^{i\mathbb{p}\cdot x}}{(2\pi)^{3/2}}\right)^* \Gamma \left(\phi_{\mathbb{p}'} \frac{e^{i\mathbb{p}'\cdot x}}{(2\pi)^{3/2}}\right)$$

$$\times \underbrace{\langle 0 | b_{-\mathbb{p}}\, b^\dagger_{-\mathbb{p}'} | 0 \rangle}_{= \delta^3(\mathbb{p}-\mathbb{p}')}$$

$$= \int d^3p \left(\phi_p \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\right) \Gamma \left(\phi_p^* \frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\right)$$

$$= (\text{finite})$$

$$\phi^+ \left(2 \overleftarrow{\partial_0} \overrightarrow{\partial_0} - \left[\overleftarrow{\partial_\mu} \overrightarrow{\partial^\mu} - m^2\right]\right)\phi$$
$$= \phi^+ \left(\overleftarrow{\partial_0} \overrightarrow{\partial_0} + \overleftarrow{\nabla} \cdot \overrightarrow{\nabla} + m^2\right)\phi.$$

e.g.) $\varepsilon = \langle \frac{\hat{\Lambda}^{00}}{V} \rangle$

$$= \int d^3p \left(\phi_p \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\right) \left(\cdots \right) \left(\phi_p^* \frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\right)$$

!!

$$\begin{cases} \partial_0 \phi_p = -i\omega_p \phi_p \\ \nabla e^{ip\cdot x} = ip \\ (\phi_p)^2 = \frac{1}{2\omega_p} \end{cases}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p.$$

Used $\partial_\mu \phi = ip_\mu \phi$

i.e., mode func. is an eigen func. of the translation.

$$\longrightarrow \Lambda^\varphi = \infty.$$

This should be subtracted

, vacuum value is just a reference and the deviation from it has the physical meaning

. divergent value is ill-defined.

So, introduce normal ordering

$$\langle : \hat{O} : \rangle \equiv \langle \hat{O} \rangle - \langle 0 | \hat{O} | 0 \rangle.$$

Note: This is equivalent to normal-order the operators $:O_1 O_2^\dagger: = O_2^\dagger O_1$.

In fact,

$$\langle \phi^\dagger \Gamma \phi \rangle$$

$$= \int d^3\mathbf{p}\, d^3\mathbf{p}' \Big[\Big(\phi_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot x}}{(2\pi)^{3/2}}\Big)^* \Gamma \big(\phi_{\mathbf{p}'} \cdots\big) \langle a_{\mathbf{p}}^\dagger a_{\mathbf{p}'} \rangle$$

$$+ \big(\phi_{\mathbf{p}}^* \cdots\big)^* \Gamma \big(\phi_{\mathbf{p}'}^* \cdots\big) \langle b_{-\mathbf{p}} b_{-\mathbf{p}'}^\dagger \rangle$$

$$+ (\text{interference}) \Big].$$

$$\langle b_{-\mathbf{p}'}^\dagger b_{-\mathbf{p}} \rangle + \langle [\ ,\ ] \rangle$$

$$= \underbrace{\langle b_{-\mathbf{p}'}^\dagger b_{-\mathbf{p}} \rangle}_{\langle :\phi^\dagger \Gamma \phi: \rangle} + \underbrace{\delta^3(\mathbf{p}-\mathbf{p}')}_{\langle 0|\phi^\dagger \Gamma \phi|0\rangle}$$

$$= \langle :\phi^\dagger \Gamma \phi: \rangle + \langle 0|\phi^\dagger \Gamma \phi|0\rangle.$$

(ii) Then, $\langle :\hat{O}: \rangle$ has the form

$$\langle :\hat{O}: \rangle \sim O_{1\text{-particle}} \times (\#\text{ of particles})$$

$\Rightarrow$ justify the physical meaning of $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$, provided $\langle a^\dagger a \rangle$ can be interpreted as the mean number.

(iii) To justify (ii), it's implicitly assumed $\ulcorner \partial_t \phi_{\mathbf{p}} = -i\omega_{\mathbf{p}} \phi_{\mathbf{p}} \lrcorner$

This is important because,

$$\tilde{\phi}_{\mathbf{p}} = \alpha_{\mathbf{p}} \phi_{\mathbf{p}} + \beta_{\mathbf{p}} \phi_{\mathbf{p}}^*$$

also satisfies the KG equation.

So, it's no problem to expand ϕ as

$$\phi = \int d^3\mathbf{p} \frac{e^{i\mathbf{p}x}}{(2\pi)^{3/2}} \left[\tilde{\phi}_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \tilde{\phi}_{\mathbf{p}}^* \hat{b}_{-\mathbf{p}}^\dagger \right]$$

Notice

$$\begin{pmatrix} \tilde{a}_{\mathbf{p}} \\ \tilde{b}_{\mathbf{p}}^\dagger \end{pmatrix} \neq \begin{pmatrix} a_{\mathbf{p}} \\ b_{\mathbf{p}}^\dagger \end{pmatrix} \quad \because \tilde{\phi}_{\mathbf{p}} \neq \phi_{\mathbf{p}}$$

$$\begin{pmatrix} \tilde{a}_{\mathbf{p}} \\ \tilde{b}_{-\mathbf{p}}^\dagger \end{pmatrix} = i \int d^3x \begin{pmatrix} (\tilde{\cdots})^* \\ -(\tilde{\cdots})^\dagger \end{pmatrix} \overleftrightarrow{\partial_t} \phi.$$

$$\parallel \quad \begin{pmatrix} \alpha_{\mathbf{p}} & \beta_{\mathbf{p}}^* \\ \beta_{\mathbf{p}} & \alpha_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{p}} \\ b_{-\mathbf{p}}^\dagger \end{pmatrix} \longrightarrow \text{Bogoliubov transformation}$$

$\Rightarrow$ Different particle picture.

(e.g., $\tilde{a}_{\mathbf{p}}$ does not have energy $\omega_{\mathbf{p}}$).

Why can we set ... ?

→ Because time translation is a good symmetry

⇒ The corresponding eigenvalue, energy, is conserved and serves as a good label to characterize a particle.

(iv) Conversely, if there's no such symmetry (e.g., external field), there's no guiding principle to define a particle

→ main issue of the Schwinger effect.

Bogoliubov-transformation approach to the Schwinger effect

- Basic steps are the same as before $\quad \Big[\partial_t^2 + \underbrace{(\mathbf{p} - e\mathbf{A}(t))^2 + m^2}_{\omega_{\mathbf{p}}^2(t)}\Big]\phi_{\mathbf{p}}$

$$0 = [\bar{D}^2 + m^2]\phi \to \text{Mode expand \& solve} \to \text{Impose CCR.}$$

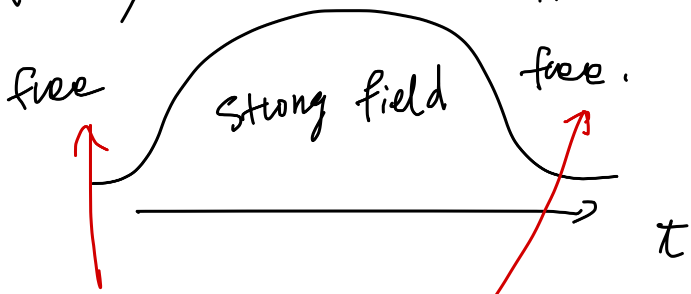
$\hookrightarrow \bar{D}_\mu = \partial_\mu + ie\bar{A}_\mu$; for simplicity $\bar{A}_\mu = (0, -\mathbf{A}(t))$ = spatial homogeneity

- Issue : How to define a particle in the presence of strong field ? $\leftrightarrow$ choice of mode function $\phi_{\mathbf{p}}$

① Give up and consider asymptotic states where $\bar{A}_\mu \to 0$ (adiabatic hypothesis).
$\to$ Use free field at $|t| \to \infty$

② Do it anyway by introducing a "natural" mode function to define "your" particle
$\to$ inevitably ambiguous but useful for physics

- Here, consider the approach ①



translation symmetry is restored

→ "natural" to take plane wave $e^{\mp ip.x}$ to define a particle.

⟹ We can construct two mode functions:

$$\begin{cases} \phi^{in}_{\mathbb{P}} \ \text{s.t.} \ \lim\limits_{t\to-\infty} \phi^{in}_{\mathbb{P}} \propto e^{\mp ip.x} \\ \phi^{out}_{\mathbb{P}} \ \text{s.t.} \ \lim\limits_{t\to+\infty} \phi^{out}_{\mathbb{P}} \propto e^{\mp ip.x} \end{cases}$$

(Both satisfy $[D^2+m^2]\,\phi^{as}_{\mathbb{P}} = 0$)

For $A_\mu = 0$, $\phi^{in}_{\mathbb{P}} = \phi^{out}_{\mathbb{P}}$ but in general it's not,

(cf. analogy to 1-dim scattering )

$\phi^{in}_{\mathbb{p}} \neq \phi^{out}_{\mathbb{p}} \longleftrightarrow \begin{pmatrix} a^{in}_{\mathbb{p}} \\ b^{in}_{\mathbb{p}} \end{pmatrix} \neq \begin{pmatrix} a^{out}_{\mathbb{p}} \\ b^{out}_{\mathbb{p}} \end{pmatrix}$

↳ expressed by the Bogoliubov transformation.

$$\begin{pmatrix} a^{out}_{\mathbb{p}} \\ b^{out\dagger}_{-\mathbb{p}} \end{pmatrix} = \begin{pmatrix} \alpha_{\mathbb{p}} & \beta^{*}_{\mathbb{p}} \\ \beta_{\mathbb{p}} & \alpha^{*}_{\mathbb{p}} \end{pmatrix} \begin{pmatrix} a^{in}_{\mathbb{p}} \\ b^{in\dagger}_{-\mathbb{p}} \end{pmatrix}$$

Use
$$\begin{cases} \begin{pmatrix} a^{as}_{\mathbb{p}} \\ b^{as\dagger}_{-\mathbb{p}} \end{pmatrix} = +i \int d^3x \begin{pmatrix} \left(\phi^{as}_{\mathbb{p}} \frac{e^{i\mathbb{p}\cdot x}}{(2\pi)^{3/2}}\right)^* \\ \left(\phi^{as*}_{\mathbb{p}} \frac{e^{-i\mathbb{p}\cdot x}}{(2\pi)^{3/2}}\right)^* \end{pmatrix} \overleftrightarrow{\partial_t} \phi \\ \phi = \int d^3\mathbb{p} \frac{e^{i\mathbb{p}\cdot x}}{(2\pi)^{3/2}} \left[\phi^{as}_{\mathbb{p}} a^{as}_{\mathbb{p}} + \cdots \right] \end{cases}$$

where
$$\begin{cases} \alpha_{\mathbb{p}} = +i\, \phi^{out*}_{\mathbb{p}} \overleftrightarrow{\partial_t} \phi^{in}_{\mathbb{p}} \\ \beta_{\mathbb{p}} = -i\, \phi^{out}_{\mathbb{p}} \overleftrightarrow{\partial_t} \phi^{in}_{\mathbb{p}} . \end{cases}$$

Note: normalization of $\phi^{as}_{\mathbb{p}}$ → normalize α_p and β_p as $|\alpha_{\mathbb{p}}|^2 - |\beta_{\mathbb{p}}|^2 = 1$.

Point: Solve KG eq. → Get $\phi^{as}_{\mathbb{p}}$
→ Get $\alpha_{\mathbb{p}}$ and $\beta_{\mathbb{p}}$

31)

$\longrightarrow$ Quantify the diff b/w $\begin{pmatrix} a^{in} \\ b^{in} \end{pmatrix}$ and $\begin{pmatrix} a^{out} \\ b^{out} \end{pmatrix}$.

- The consequences of $\begin{pmatrix} a^{in} \\ b^{in} \end{pmatrix} \neq \begin{pmatrix} a^{out} \\ b^{out} \end{pmatrix}$.

(i) The corresponding vacua are different.

Let

$$\begin{pmatrix} a^{as}_{\mathbf{p}} \\ b^{as}_{\mathbf{p}} \end{pmatrix} |0;as\rangle \equiv 0.$$

$\Rightarrow$ $|0;in\rangle$ is no longer a vacuum at $t \to \infty$

because

$$\begin{pmatrix} a^{out}_{\mathbf{p}} \\ b^{out}_{\mathbf{p}} \end{pmatrix} |0;in\rangle = \begin{pmatrix} \alpha_{\mathbf{p}} a^{in}_{\mathbf{p}} + \beta^*_{\mathbf{p}} b^{in\dagger}_{-\mathbf{p}} \\ \beta_{\mathbf{p}} a^{in}_{\mathbf{p}} + \alpha^*_{\mathbf{p}} b^{in\dagger}_{-\mathbf{p}} \end{pmatrix} |0;in\rangle \neq 0.$$

(ii). Not vacuum = should contain particles.

$\Rightarrow$ Particle production occurs!

$$\frac{d^3N^{out}}{dp^3} = \langle 0;in|\, a^{out\,\dagger}_{\mathbf{p}}\, a^{out}_{\mathbf{p}}\, |0;in\rangle \qquad a^{out}_{\mathbf{p}} = \alpha_{\mathbf{p}} a^{in}_{\mathbf{p}} + \beta^*_{\mathbf{p}} b^{in\,\dagger}_{-\mathbf{p}}$$

$$= |\beta_{\mathbf{p}}|^2 \underbrace{\langle 0;in|\, b^{in}_{-\mathbf{p}} b^{in\,\dagger}_{-\mathbf{p}}\, |0;in\rangle}_{=\ \delta^3(\mathbf{p}=0) = \frac{V}{(2\pi)^3}}$$

$$= \frac{V}{(2\pi)^3} |\beta_{\mathbf{p}}|^2$$

$$\lim_{\mathbf{p}\to 0} \frac{1}{(2\pi)^3}\int d^3x\, e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\Rightarrow f^{out} = (2\pi)^3 \frac{d^6N^{out}}{d^3x\, dp^3} = |\beta_{\mathbf{p}}|^2.$$

Similarly, $\bar{f}^{out} = |\beta_{-\mathbf{p}}|^2$ for anti-particle. Vacuum = charge-less, momentum-less $\Downarrow$ pair production

Note: $\beta_{\mathbf{p}}$ is calculable from $\phi^{as}_{\mathbf{p}}$

$\phi^{as}_{\mathbf{p}}$ is known exactly for a few cases

- constant $E \Rightarrow |\beta_{\mathbf{p}}| = e^{-\pi\frac{m^2}{2eE}}$

  ↳ Very original Schwinger effect (Nikishov 1969)

- pulsed E (Sauter field) $\Rightarrow$ complicated $\beta_{\mathbf{p}}$.

Other cases, numerical or approximate methods are used.

- semi-classical approx. (~ gradient expansion of $\bar{A}_\mu$)
- locally-constant-field approx. (LCFA)

– perturbative expansion in eA
$\Leftarrow$ useful but obviously NP information is lost, so not interesting.

(iii). $|0;in\rangle \neq |0;out\rangle \Rightarrow$ Vacuum decay.

That pair production occurs implies

$$|0;out\rangle = \sum_n C_n \, |n \text{ pairs}; in\rangle,$$

i.e, out vacuum is a superposition of multi-particle in-state

Since $\begin{pmatrix} a_{\mathbf{p}}^{as} \\ b_{\mathbf{p}}^{as} \end{pmatrix}$ is known, one can determin the constants C_n [up to unimportant phase factor to get. (after a bit of calculations)

$$|0;out\rangle = \exp\left[-\frac{V}{(2\pi)^3}\int d^3p \, \ln|\alpha_{\mathbf{p}}|\right]$$

$$\times \prod_{\mathbf{p}} \exp\left[\frac{(2\pi)^3}{V}\frac{\beta^*_{\mathbf{p}}}{\alpha_{\mathbf{p}}} a^{in\,\dagger}_{\mathbf{p}} b^{in\,\dagger}_{-\mathbf{p}}\right]$$

$$\times |0;in\rangle.$$

$\Rightarrow$ vacuum is not stable

$\because$ Vacuum persistence probability

$$P = |\langle 0;out|0;in\rangle|^2$$

$$= \exp\left[-\boxed{\frac{V}{(2\pi)^3}\int d^3\mathbf{p}\,\ln|\alpha_{\mathbf{p}}|^2}\right]$$

$VT\underset{\uparrow}{w}$

vacuum decay rate

$$w = \frac{1}{T}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\,\ln|\alpha_{\mathbf{p}}|^2 \qquad \int dP_z \simeq eET.$$

$$= \frac{1}{T}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\,\ln(1+|\beta_{\mathbf{p}}|^2)$$

$$= \frac{1}{T}\frac{1}{(2\pi)^3}\int d^3\mathbf{p}\sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{n}|\beta_{\mathbf{p}}|^{2n}$$

(iv) Heisenberg–Euler effective Lagrangian.

The vacuum persistence prob. P is related to the effective Hamiltonian in a strong field H_{eff}, called Heisenberg–Euler effective Hamiltonian (Heisenberg–Euler (1935), Weisskopf (1936))

(To be strict, HE is originally for a constant EM field but is sometimes used in more general cases like inhomogeneous fields)

vacuum decays
That $w \neq 0$ means H_{eff} has an imaginary part.

$|out\rangle = e^{-iHT}|in\rangle$

$\Rightarrow \langle out|in\rangle = e^{+iHT}$

Namely, let

$$e^{+iH_{eff}T} \equiv \langle 0;out|0;in\rangle$$

$$\Rightarrow H_{eff} = \frac{-i}{T}\ln\langle 0;out|0;in\rangle$$

$$\Rightarrow \mathrm{Im}\,H_{eff} = \frac{-1}{T}\mathrm{Re}\ln\langle 0;out|0;in\rangle$$

$$= V\frac{w}{2}$$

$$\Rightarrow \mathrm{Im}\,\mathcal{H}_{eff} = \frac{w}{2}$$

$(\text{phase})\times e^{-VT\frac{w}{2}}$

Why imaginary part?

$\rightarrow$ Open quantum system.

Once $\mathcal{H}_{eff}$ is obtained, $\mathcal{L}_{eff}$ can be derived by the Legendre transform

$$\begin{cases} d\mathcal{H} = \mathbb{E}\cdot d\mathbb{D} + \mathbb{H}\cdot d\mathbb{B} \\ d\mathcal{L} = \mathbb{D}\cdot d\mathbb{E} - \mathbb{H}\cdot d\mathbb{B} \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{L} = \mathcal{L}(\mathbb{E}, \mathbb{B}) = \dfrac{\partial \mathcal{H}}{\partial \mathbb{D}} \cdot \mathbb{D} - \mathcal{H} \\ \mathcal{H} = \mathcal{H}(\mathbb{D}, \mathbb{B}) = \dfrac{\partial \mathcal{L}}{\partial \mathbb{E}} \cdot \mathbb{E} - \mathcal{L} \end{cases}$$

with $\mathbb{D} = \dfrac{\partial \mathcal{L}}{\partial \mathbb{E}}$, $\mathbb{H} = \dfrac{\partial \mathcal{L}}{\partial \mathbb{B}}$.

The real part may not be obtained from the above argument because of the sloppy treatment of the phases of the states.

So, let me just display the known result for a constant EM field (obtained, e.g., by the proper-time method)

$$\mathcal{L}_{\text{eff}} = \underbrace{\mathcal{L}_{\text{Maxwell}}}_{\substack{\shortparallel \\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbb{E}^2 - \mathbb{B}^2) \equiv -\mathcal{f}}} + \underbrace{\delta \mathcal{L}}_{}$$

strong field modification to the Maxwell theory

where

$$\delta \mathcal{L} = \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left[+ \frac{(es)^2 \mathcal{g}}{\operatorname{Im} \cosh\left(es\sqrt{2(\mathcal{f} + i\mathcal{g})}\right)} + \frac{1}{3}(es)^2 \mathcal{f} - 1 \right]$$

$$= \frac{\alpha^2}{90 m^4}\left(7\mathcal{f}^2 + \mathcal{g}^2\right) + \cdots$$

where $\alpha = \dfrac{e^2}{4\pi}$

$\mathcal{g} = -\frac{1}{4} F_{\mu\nu} \underbrace{\tilde{F}^{\mu\nu}}_{\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}} = \mathbb{E}\cdot\mathbb{B}$

Note: For spinor QED

$$\delta\mathcal{L} = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-m^2 s}\left[(es)^2\,\frac{\mathrm{Re}\cosh\left(es\sqrt{2(\mathcal{F}+i\mathcal{G})}\right)}{\mathrm{Im}\cosh\left(es\sqrt{2(\mathcal{F}+i\mathcal{G})}\right)}\,\mathcal{G} - \frac{2}{3}(es)^2\mathcal{F} - 1\right]$$

$$= \frac{2\alpha^2}{45 m^4}\left(4\mathcal{F}^2 + 7\mathcal{G}^2\right) + \underbrace{O\left(\left(\frac{\alpha\mathcal{F}}{m^2}\right)^4, \left(\frac{\alpha\mathcal{G}}{m^2}\right)^4\right)}$$

Asymptotic series $\sim n!$ (cf. Dyson (1952))

- optimal order $\left|\frac{C_n}{C_{n-1}}\right| \sim n\frac{\alpha\mathcal{F}}{m^2} \overset{!}{<} 1$ $\Rightarrow n_{opt} \sim \frac{m^2}{\alpha\mathcal{F}} \sim \alpha^{-1}$
- Borel resummation $f = \sum_n c_n g^n \rightarrow B[f] = \sum_n \frac{c_n}{n!} t^n$, $\hat{f} = \int_0^\infty dt\, e^{-t/g} B$

Consequences of $\delta\mathcal{L}$

- The equation of motion for $\mathbb{E}$ and $\mathbb{B}$ acquire additional "vacuum polarization contribution"

$$\underbrace{\partial_\mu F^{\mu\nu}}_{\text{from } \mathcal{L}_{Maxwell}} = \underbrace{J^\nu_{vac}[\delta\mathcal{L}]}_{} \quad \nabla\cdot\mathbb{P}$$

$$J^0 = \nabla\cdot\frac{\partial\delta\mathcal{L}}{\partial\mathbb{E}} \qquad \mathbb{J} = \dot{\mathbb{P}} + \underbrace{\nabla\times\frac{\partial\delta\mathcal{L}}{\partial\mathbb{B}}}_{\nabla\times\mathbb{M}}$$

$$\Longleftrightarrow \begin{cases} \operatorname{div}\mathbb{E} = J^0 \\ \operatorname{rot}\mathbb{B} = \frac{\partial\mathbb{E}}{\partial t} + \mathbb{J} \end{cases}$$

Bianchi id. is unmodified.

(∵ it follows from $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$

$\Rightarrow \partial_\mu \tilde{F}^{\mu\nu} \propto \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\rho\sigma}$
$\propto \epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma$
$= 0$)

$$+ \text{ Bianchi id. } \begin{cases} \operatorname{div}\mathbb{B} = 0 \\ \operatorname{rot}\mathbb{E} = -\frac{\partial\mathbb{B}}{\partial t} \end{cases}$$

Analogous to electromagnetism in material, one may absorb J^μ to $\mathbb{E}$ and $\mathbb{B}$ to define macroscopic fields $\mathbb{D}$ and $\mathbb{H}$ as

$$\mathbb{D} = \mathbb{E} + \mathbb{P} = \mathbb{E} + \frac{\partial\, \delta\mathcal{L}}{\partial \mathbb{E}}$$

$$\mathbb{H} = \mathbb{B} - \mathbb{M}. = \mathbb{B} - \frac{\partial\, \delta\mathcal{L}}{\partial \mathbb{B}}.$$

- The vacuum birefringence due to the vacuum current J^μ_{vac}. Consider a propagation of light on top of a strong field:

$$A_\mu = \bar{A}_\mu + a_\mu \qquad (\bar{A} \gg a).$$

The wave equation read ($\partial_\mu A^\mu = 0$) Lorentz gauge

$$\partial^2 A^\mu = J^\mu_{vac}(A) \quad (+\, J^\mu_{ext} \text{ to be precise})$$

$$\partial^2 \bar{A}^\mu + \partial^2 a^\mu = J^\mu_{vac}(\bar{A}) + \frac{\partial J^\mu_{vac}(\bar{A})}{\partial \bar{A}^\nu} a^\nu + \mathcal{O}(a^2)$$

$$\Rightarrow \quad \partial^2 a^\mu = \underbrace{\frac{\partial J^\mu(\bar{A})}{\partial \bar{A}^\nu} a^\nu}_{\not\propto\, a^\mu} \quad \Rightarrow \text{birefringence.}$$

(pol. dep. propagation)

$$\left(\because \begin{cases} J^0(\bar{A}) = \nabla \cdot \dfrac{\partial\, \delta\mathcal{L}}{\partial \bar{\mathbb{E}}} \\ \mathbb{J}(\bar{A}) = \partial_t \dfrac{\partial\, \delta\mathcal{L}}{\partial \bar{\mathbb{E}}} + \nabla \times \dfrac{\partial\, \delta\mathcal{L}}{\partial \bar{\mathbb{B}}} \end{cases} \right.$$

so they care about the directions of $\bar{\mathbb{E}}$ and $\bar{\mathbb{B}}$)

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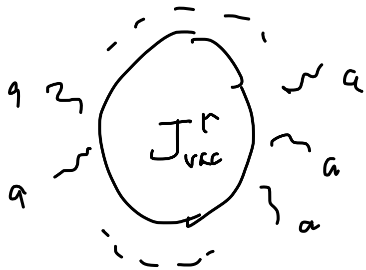
- The dropped $\mathcal{O}(a^n)$ terms are responsible for n-photon interactions, which are prohibited in the usual Maxwell theory



Real time dynamics and backreaction prob.

Adiabatic particle picture

Let us discuss the approach ① to study the real time dynamics of the Schwinger effect

- Reminder: This approach must be ambiguous because there's no rigorous principle to define a particle at intermediate times $\Rightarrow$ what would be "natural"?

- A widely-used approach: (0th order) Adiabatic particle picture

Good points ("naturalness")

- Conserve energy (in general, can be incompatible).
- Smoothly connected to the asymptotic particle picture
  in general, 0th can be insufficient
- can remove UV divergence via normal ordering (in QED)

- no singular behaviors during the real time evo. (in general, can be singular)

- easy to implement in numerics

• Idea : Introduce mode function

$$\phi^{ad}_{\mathbf{p}}(t) \equiv \frac{e^{-i\int dt\, \omega_{\mathbf{p}}(t)}}{\sqrt{2\omega_{\mathbf{p}}(t)}} \quad \xleftarrow{\text{generalization of}} \quad \frac{e^{-i\omega_{\mathbf{p}}t}}{\sqrt{2\omega_{\mathbf{p}}}}$$

and expand the field operator at each instant time as

$$\phi(t,x) = \int d^3p \frac{e^{i\mathbf{p}\cdot x}}{(2\pi)^{3/2}} \left[\phi^{ad}_{\mathbf{p}}(t)\, a^{ad}_{\mathbf{p}}(t) + \phi^{ad*}_{\mathbf{p}}(t)\, b^{ad\dagger}_{-\mathbf{p}}(t) \right]$$

To be precise, I also have to impose a cond. for the 1st order derivative

$$\dot\phi = \int_{\mathbf{p}} \left[-i\omega_{\mathbf{p}} \phi^{ad}_{\mathbf{p}} a_{\mathbf{p}} + \cdots \right]$$

because ϕ obeys the 2nd order ODE, so ϕ and $\dot\phi$ are independent.

must be time dependent since $\phi^{ad}_{\mathbf{p}}$ is not a solution to the KG eq.

• Point : * $\phi^{ad}_{\mathbf{p}}$ is an "approximate" eigenfunction of ∂_t, i.e.,

$$i\partial_t \phi^{ad}_{\mathbf{p}}(t) = \omega_{\mathbf{p}}(t)\, \phi^{ad}_{\mathbf{p}} + \mathcal{O}(\underbrace{\partial_t}_{\text{equivalent to } \hbar})$$

So, as long as the spacetime variation of the EM field is sufficiently slow $\partial_t \sim 0$ (or $T \to \infty$), the quantum produced by $a_{\mathbf{p}}^{\dagger}$ can be interpreted as a particle with energy $\omega_{\mathbf{p}}(t)$, similarly to the asymptotic particle picture.

* And

$$\phi_{\mathbf{p}}^{ad} \xrightarrow{|A| \to \infty} \phi_{\mathbf{p}}^{\text{plane wave}} \Leftrightarrow a_{\mathbf{p}}^{ad} \to a_{\mathbf{p}}^{as}$$

i.e., the adiabatic particle picture recovers the asymptotic particle picture

* Note:
  - Go to higher-order in the derivative expansion to get a "better" mode function → n-th order adiabatic picture

    ⇒ BST, not necessarily good

    → can break conservation law, singular behaviors. etc.
  - Resum high order → superadiabatic particle picture (Dabrowski-Dunne 2016)

- Formulation :

can be done w/ the Bogoliubov-trans. technique.

Namely, use the normalization of ϕ^{ad} to get

$$\begin{pmatrix} a^{ad}_{\mathbf{p}}(t) \\ b^{ad\,\dagger}_{-\mathbf{p}}(t) \end{pmatrix} = i\int d^3x \begin{pmatrix} \left[\phi^{ad}_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\right]^* \\ \left(\ast \right)^* \end{pmatrix} \overleftrightarrow{\partial_t}\,\phi$$

useful to connect $a^{ad}_{\mathbf{p}}$ & $a^{in}_{\mathbf{p}}$ by using $\phi = (a^{in} \ldots)$

$$= i\int d^3x \left(\ \ldots\ \right) \overleftrightarrow{\partial_t} \int d^3p' \frac{e^{i\mathbf{p}'\cdot\mathbf{x}}}{(2\pi)^{3/2}} \left[\phi^{in}_{\mathbf{p}'}\, a^{in}_{\mathbf{p}'} + \phi^{in\,*}_{\mathbf{p}'}\, b^{in\,\dagger}_{-\mathbf{p}'}\right]$$

$$= \begin{pmatrix} \alpha_{\mathbf{p}}(t) & \beta^*_{\mathbf{p}}(t) \\ \beta_{\mathbf{p}}(t) & \alpha_{\mathbf{p}}(t) \end{pmatrix} \begin{pmatrix} a^{in}_{\mathbf{p}} \\ b^{in\,\dagger}_{-\mathbf{p}} \end{pmatrix}$$

where

$$\begin{cases} \alpha_{\mathbf{p}}(t) = +i\, \phi^{ad\,*}_{\mathbf{p}} \overleftrightarrow{\partial_t}\, \phi^{in}_{\mathbf{p}} \\ \beta_{\mathbf{p}}(t) = -i\, \phi^{ad}_{\mathbf{p}} \overleftrightarrow{\partial_t}\, \phi^{in}_{\mathbf{p}} \end{cases}$$

known   fixed by solving KG eq.

$\Rightarrow \begin{pmatrix} \alpha_{\mathbf{p}} \\ \beta_{\mathbf{p}} \end{pmatrix}$ are determined by solving KG eq. $\begin{cases} \text{analytically for some } \bar{A} \\ \text{numerically} \\ \text{approximately} \end{cases}$

Then,

(i) Vacuum at time t

$$\begin{pmatrix} a^{ad}_{\mathbf{p}} \\ b^{ad}_{\mathbf{p}} \end{pmatrix} |0;ad\rangle = 0.$$

which is unequal to $|0;in\rangle$ or $|0;out\rangle$.

(ii) Realtime particle production.

$$\cdots + \beta^*_{\mathbf{p}} b^{in\dagger}_{-\mathbf{p}}$$

$$\frac{d^3N(t)}{d\mathbf{p}^3} = \langle 0;in|\, a^{ad}_{\mathbf{p}}{}^\dagger(t)\; a^{ad}_{\mathbf{p}}(t)\, |0;in\rangle$$

$$= \frac{V}{(2\pi)^3} |\beta_{\mathbf{p}}(t)|^2$$

$$\Rightarrow \quad f(t) = (2\pi)^3 \frac{d^6N}{dx^3 d\mathbf{p}^3} = |\beta_{\mathbf{p}}(t)|^2 .$$

Note: Yields a kinetic eq. w/ a source term

$$\frac{d f(t)}{dt} = S(t) \quad \text{where} \quad S = \frac{d|\beta_{\mathbf{p}}(t)|^2}{dt}$$

The source term doesn't have a simple form but can be approximated with the Schwinger formula if the E field is slow enough

$$S(t) \simeq e^{-\pi\frac{m_\perp^2}{eE(t)}} \delta(p_z)$$

locally-constant-field approximation

sometimes used in the literature for phenomenological analysis of the Schwinger effect.

(iii) Expectation value

Normal ordering w.r.t. adiabatic operators

$$\langle :O: \rangle = \langle O \rangle - \langle 0;ad|\, O\, |0;ad\rangle$$

For $\begin{cases} \mathcal{O} = \phi^\dagger \Gamma \phi , \\ |-\rangle = |0;\text{in}\rangle \end{cases}$

$$\phi = \int d^3p \, \frac{e^{i\mathbf{p}\cdot x}}{(2\pi)^{3/2}} \left[\phi_{\mathbf{p}}^{ad} a_{\mathbf{p}}^{ad} + \cdots \right] = \cdots \left[\phi_{\mathbf{p}}^{in} a_{\mathbf{p}}^{in} + \cdots \right]$$

$$\langle 0;\text{in}| : \mathcal{O} : |0;\text{in}\rangle \equiv \mathcal{O}(t)$$

$$= \int d^3\mathbf{p} \left[\phi_{\mathbf{p}}^{in} \Gamma_{\mathbf{p}} \phi_{\mathbf{p}}^{in*} - \phi_{\mathbf{p}}^{ad} \Gamma_{\mathbf{p}} \phi_{\mathbf{p}}^{ad*} \right]$$

It's clear that

$$\mathcal{O}(t \to -\infty) = 0 \quad \because \ \phi_{\mathbf{p}}^{ad} \to \phi_{\mathbf{p}}^{in}$$

but

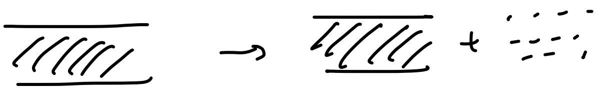
$$\mathcal{O}(t > -\infty) \not\equiv 0 \quad \because \ \phi_{\mathbf{p}}^{ad} \neq \phi_{\mathbf{p}}^{in}.$$

meaning particle prod gives finite contr.

Backreaction

- What is it?

So far, E field is fixed → violates energy conservation.



$\mathcal{E}_{field}$ $\qquad \mathcal{E}_{field} \quad \mathcal{E}_{particle} > \mathcal{E}_{field}$.

$\Longrightarrow$ E field must decay.

(cf. early-time dynamics of HIC, (p)reheating, Hawking rad ...)

- What I show from now.
  - ~~How a strong E field decays spontaneously~~ WRONG.
  - EM field dynamics → Maxwell eq.

$$\partial_\mu F^{\mu\nu} = \underbrace{J^\nu}_{\text{produced by particle production}}$$

Naively, flow of particles (= conduction current)

$$J^\mu \overset{?}{=} J^\mu_{\rm cond}$$

$$= \int \frac{d^3p}{(2\pi)^3} (+e) \underbrace{v_p}_{\frac{p}{\omega_p}} \underbrace{f_p}_{|\beta_p|^2} + \int \frac{d^3p}{(2\pi)^3} (-e) \underbrace{v_p}_{\frac{p}{\omega_p} = -\frac{-p}{\omega_{-p}}} \underbrace{\bar{f}_p}_{f_{-p}}$$

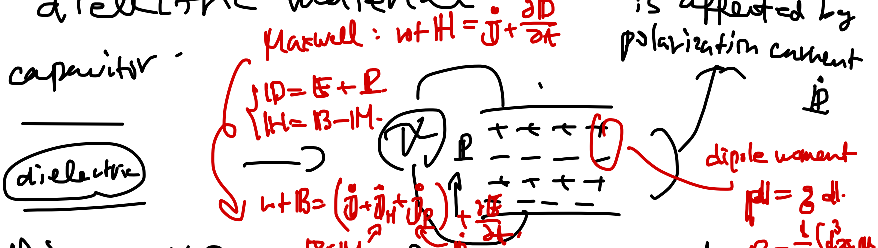
$$= 2\int \frac{d^3p}{(2\pi)^3} e \frac{p}{\omega_p} |\beta_p|^2 .$$

This is WRONG (Gatoff-Kerman-Matsui (1987))

→ Need polarization current

⟺ polarization of the vacuum must be considered!

cf. dielectric material



⟹ Show this within mean-field approach

And clarify how it is related to the Schwinger effect.

48] • Mean-field approach to the backreaction prob.

(cf. same as the Bogoliubov-de Gennes method in cond.-mat.)

Kluger et al (1992)

Starting point: Scalar QED w/ Maxwell term

$$\mathcal{L} = \mathcal{L}_{mat} + \mathcal{L}_{Maxwell}$$

$$= |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

strong-field approx. $A_\mu \simeq \langle A_\mu \rangle = \bar{A}_\mu$

$\Rightarrow$ EoMs.

mat: $0 = (D^2 + m^2)\phi \longrightarrow (\bar{D}^2 + m^2)\phi$.

$$\phi^\dagger \overleftarrow{D}^\mu = (D^\mu \phi)^\dagger = ((\partial^\mu + ieA^\mu)\phi)^\dagger$$

field: $\partial_\mu F^{\mu\nu} = ie\phi^\dagger \overleftrightarrow{D}^\nu \phi$.

SFA. SFA

$\partial_\mu \bar{F}^{\mu\nu}$. ✗ : $ie\phi^\dagger \overleftrightarrow{\bar{D}}^\nu \phi$? $\longrightarrow$ ϕ is operator...

✓ : $ie\langle :\phi^\dagger \overleftrightarrow{\bar{D}}^\nu \phi: \rangle$. $\longrightarrow$ MFA

$\therefore$ Coupled EoMs.

$$\begin{cases} 0 = (\bar{D}^2 + m^2)\phi \\ \partial_\mu \bar{F}^{\mu\nu} = J^\nu_{mat} \end{cases}$$

where

$$J^\nu_{mat} = \langle :ie\, \phi^\dagger \overleftrightarrow{\bar{D}}^\nu \phi :\rangle$$

$$= \langle 0;in| :\phi^\dagger \overleftrightarrow{\bar{D}}^\nu \phi: |0;in\rangle$$

or choose $|\cdots\rangle = |0;in\rangle$

- Numerically, you can just solve this consistently $\longrightarrow$ easy task.

- Let's think about the physics meaning of J^μ_{mat} and see how it differs from J^μ_{cond}

From the def.

$$J^\mu_{mat} = \langle 0;in| :\phi^\dagger \underset{ie\overleftrightarrow{D}^\mu}{\underset{\|}{\Gamma}} \phi: |0;in\rangle$$

$$= \langle 0;in| \phi^\dagger \Gamma \phi |0;in\rangle - \langle 0;ad| \phi^\dagger \Gamma \phi |0;ad\rangle$$

$$= \int d^3\mathbb{p} \left[\phi^{in}_{\mathbb{p}} \Gamma_{\mathbb{p}} \phi^{in*}_{\mathbb{p}} - \phi^{ad}_{\mathbb{p}} \Gamma_{\mathbb{p}} \phi^{ad*}_{\mathbb{p}} \right].$$

$$\Gamma_{\mathbb{p}} = \begin{cases} ie\overleftrightarrow{\partial}^0 & (\mu=0) \\ 2e(\mathbb{p}-e\mathbb{A}) & (\mu=i) \end{cases}$$

Remember

$$\phi = \int d^3\mathbb{p} \frac{e^{i\mathbb{p}\cdot\mathbb{x}}}{(2\pi)^{3/2}} \left[\phi^{in}_{\mathbb{p}} a^{in}_{\mathbb{p}} + \phi^{in*}_{\mathbb{p}} b^{in\dagger}_{-\mathbb{p}} \right]$$

$$= \int d^3\mathbb{p} \frac{e^{i\mathbb{p}\cdot\mathbb{x}}}{(2\pi)^{3/2}} \left[\phi^{ad}_{\mathbb{p}} a^{ad}_{\mathbb{p}} + \phi^{ad*}_{\mathbb{p}} b^{ad\dagger}_{-\mathbb{p}} \right]$$

$$\Rightarrow (a^{in}_{\mathbb{p}} \ b^{in\dagger}_{-\mathbb{p}}) \begin{pmatrix} \phi^{in}_{\mathbb{p}} \\ \phi^{in*}_{\mathbb{p}} \end{pmatrix} = (in \to ad).$$

and

$$\begin{pmatrix} a^{ad}_{\mathbb{p}} \\ b^{ad\dagger}_{-\mathbb{p}} \end{pmatrix} = \begin{pmatrix} \alpha_{\mathbb{p}} & \beta^*_{\mathbb{p}} \\ \beta_{\mathbb{p}} & \alpha_{\mathbb{p}} \end{pmatrix} \begin{pmatrix} a^{in}_{\mathbb{p}} \\ b^{in\dagger}_{-\mathbb{p}} \end{pmatrix}$$

Combine these two

$$(a^{in}_{\mathbb{p}} \ b^{in\dagger}_{-\mathbb{p}}) \begin{pmatrix} \phi^{in}_{\mathbb{p}} \\ \phi^{in*}_{\mathbb{p}} \end{pmatrix} = \overbrace{(a^{in}_{\mathbb{p}} \ b^{in\dagger}_{-\mathbb{p}}) \begin{pmatrix} \alpha_{\mathbb{p}} & \beta_{\mathbb{p}} \\ \beta^*_{\mathbb{p}} & \alpha_{\mathbb{p}} \end{pmatrix}}^{(a^{ad}_{\mathbb{p}} \ b^{ad\dagger}_{-\mathbb{p}})} \begin{pmatrix} \phi^{ad}_{\mathbb{p}} \\ \phi^{ad*}_{\mathbb{p}} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \phi^{in}_{\mathbf{p}} \\ \phi^{in*}_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \alpha_{\mathbf{p}} & \beta_{\mathbf{p}} \\ \beta^*_{\mathbf{p}} & \alpha_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} \phi^{ad}_{\mathbf{p}} \\ \phi^{ad*}_{\mathbf{p}} \end{pmatrix}$$

$\Leftrightarrow$ mode function $\leftrightarrow$ creation/annihilation ops.

So, mode functions are also related w/ each other by the Bogoliubov trans.

Then, from the expectation value

$$J^\mu_{mat} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Big[\phi^{in}_{\mathbf{p}} \Gamma_{\mathbf{p}} \phi^{in*}_{\mathbf{p}} - \phi^{ad}_{\mathbf{p}} \Gamma_{\mathbf{p}} \phi^{ad*}_{\mathbf{p}} \Big]$$

↪ rewrite i.t.o. $\phi^{ad}_{\mathbf{p}}$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Big[(\alpha_{\mathbf{p}} \phi^{ad}_{\mathbf{p}} + \beta_{\mathbf{p}} \phi^{ad*}_{\mathbf{p}}) \Gamma_{\mathbf{p}} (\cdots)^* - \phi^{ad}_{\mathbf{p}} \Gamma_{\mathbf{p}} \phi^{ad*}_{\mathbf{p}} \Big]$$

used $|\alpha_{\mathbf{p}}|^2 - |\beta_{\mathbf{p}}|^2 = 1$.

# of particle, 1-particle contr., # of anti-particle, 1-anti-part. contr.

$$\to = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \Big[|\beta_{\mathbf{p}}|^2 \phi^{ad*}_{\mathbf{p}} \Gamma_{\mathbf{p}} \phi^{ad}_{\mathbf{p}} + |\beta_{-\mathbf{p}}|^2 \left(\phi^{ad*}_{-\mathbf{p}} \Gamma^t_{\mathbf{p}} \phi^{ad}_{\mathbf{p}}\right)^*$$

$$+ \underbrace{2\,\mathrm{Re}\left\{ \alpha_{\mathbf{p}} \beta^*_{\mathbf{p}} \phi_{\mathbf{p}} \Gamma_{\mathbf{p}} \phi_{\mathbf{p}} \right\}}_{\parallel} \Big]$$

interference b/w positive & negative energies $\propto e^{-2i\omega_{\mathbf{p}} t}$

(cf. Zitterbewegung.)

For the current operator

$$\Gamma_{\mathbb{p}} = \begin{cases} ie \overleftrightarrow{\partial_0} & (\mu=0) \\ 2e\,\mathbb{p} & (\mu=i) \end{cases}$$

kinetic momentum $\mathbb{P}_{kin} = \mathbb{P}_{can} - e\mathbb{A}$

$$\Rightarrow J^0_{mat} = e\int \frac{d^3\mathbb{p}}{(2\pi)^3}\Big[|\beta_{\mathbb{p}}|^2 \underbrace{i\,\phi^{ad*}_{\mathbb{p}} \overleftrightarrow{\partial_t} \phi^{ad}_{\mathbb{p}}}_{=1} \quad -1.$$

$$+ |\beta_{-\mathbb{p}}|^2 \underbrace{\left(-i\,\phi^{ad*}_{\mathbb{p}} \overleftrightarrow{\partial_t} \phi^{ad}_{\mathbb{p}}\right)}_{=-1}$$

$$+ 2\mathrm{Re}\big[\underbrace{\alpha_{\mathbb{p}}\beta^*_{\mathbb{p}}\, i\,\phi_{\mathbb{p}} \overleftrightarrow{\partial_t} \phi_{\mathbb{p}}}_{=0}\big]\Big]$$

$$= 0 \quad \left(\because |\beta_{\mathbb{p}}|^2 = |\beta_{-\mathbb{p}}|^2\right)$$

($\rightarrow$ Gauge invariance
no spontaneous charge prod.)

$$\mathbb{J}_{mat} = \int \frac{d^3\mathbb{p}}{(2\pi)^3}\Big[|\beta_{\mathbb{p}}|^2\, 2e\,\mathbb{p}\, \underbrace{|\phi^{ad}_{\mathbb{p}}|^2}_{=\frac{1}{2\omega_{\mathbb{p}}}} \quad (-e) \times (-\mathbb{p})$$

$$+ |\beta_{-\mathbb{p}}|^2\, 2e\,\mathbb{p}\, |\phi^{ad}_{-\mathbb{p}}|^2$$

$$+ 2\mathrm{Re}\big[\alpha_{\mathbb{p}}\beta^*_{\mathbb{p}}\, 2e\,\mathbb{p}\, \phi^{ad\,2}_{\mathbb{p}}\big]\Big]$$

$$= \int \frac{d^3\mathbb{p}}{(2\pi)^3}\Big[|\beta_{\mathbb{p}}|^2\, e\,\frac{\mathbb{p}}{\omega_{\mathbb{p}}}$$

$$+ 2e\,\mathbb{p}\, \mathrm{Re}\big[\alpha_{\mathbb{p}}\beta^*_{\mathbb{p}} (\phi^{ad}_{\mathbb{p}})^2\big]\Big]$$

Clearly, the first term is the conduction current.
Then, what is the second term? $\Rightarrow$ polarization

To understand the meaning of the second term, to be precise (see page 42) remember

$$\begin{cases} \alpha_{\mathbf{p}} = +i\,\phi_{\mathbf{p}}^{ad*} \overleftrightarrow{\partial_t} \phi_{\mathbf{p}}^{in} \\ \beta_{\mathbf{p}} = -i\,\phi_{\mathbf{p}}^{ad} \overleftrightarrow{\partial_t} \phi_{\mathbf{p}}^{in} \end{cases}$$

$$= -i\left(\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} - \dot{\phi}_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in}\right)$$

$$= -i\left(\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} + i\omega_{\mathbf{p}}\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in}\right)$$

$$\Rightarrow \quad \dot{S} = \frac{d|\beta_{\mathbf{p}}|^2}{dt} = 2\,\mathrm{Re}\left(\beta_{\mathbf{p}}^{*}\partial_t\beta_{\mathbf{p}}\right)$$

$$\partial_t\beta_{\mathbf{p}} = -i\partial_t\left(\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} + i\omega_{\mathbf{p}}\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in}\right)$$

$$\partial_t\phi_{\mathbf{p}}^{ad} = \left(-i\omega_{\mathbf{p}} - \frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\right)\phi_{\mathbf{p}}^{ad}$$

$$\begin{aligned}
&= -i\Big[\Big(-\cancel{i\omega_{\mathbf{p}}} - \frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\Big)\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} \\
&\qquad - \cancel{\omega_{\mathbf{p}}^2}\,\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in} \\
&\qquad + i\dot{\omega}_{\mathbf{p}}\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in} \\
&\qquad + i\omega_{\mathbf{p}}\Big(-\cancel{i\omega_{\mathbf{p}}} - \frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\Big)\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in} \\
&\qquad + i\cancel{\omega_{\mathbf{p}}}\,\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in}\Big] \\
&= i\frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\Big[\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} - i\omega_{\mathbf{p}}\phi_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in}\Big] \\
&= i\frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\Big[\phi_{\mathbf{p}}^{ad}\partial_t\phi_{\mathbf{p}}^{in} + \dot{\phi}_{\mathbf{p}}^{ad}\phi_{\mathbf{p}}^{in}\Big] \\
&= \frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\Big[i\phi_{\mathbf{p}}^{ad*}\overleftrightarrow{\partial_t}\phi_{\mathbf{p}}^{in}\Big]e^{-2i\int\omega_{\mathbf{p}}dt} \\
&= \frac{\dot{\omega}_{\mathbf{p}}}{2\omega_{\mathbf{p}}}\alpha_{\mathbf{p}}\,e^{-2i\int\omega_{\mathbf{p}}dt} \\
&= \alpha_{\mathbf{p}}\,\dot{\omega}_{\mathbf{p}}\left(\phi_{\mathbf{p}}^{ad}\right)^2
\end{aligned}$$

$$= 2\,\mathrm{Re}\left[\beta_{\mathbf{p}}^{*}\alpha_{\mathbf{p}}\,\dot{\omega}_{\mathbf{p}}\left(\phi_{\mathbf{p}}^{ad}\right)^2\right].$$

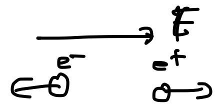
Therefore,

$$(\text{2nd term}) = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{2e\mathbf{p}}{\dot{\omega}_{\mathbf{p}}}\dot{S}.$$

$$\begin{cases} \bar{A} \propto \mathbf{e}_z. \\ \dot{\omega}_{\mathbf{p}} = \dfrac{\dot{\mathbf{p}}\cdot\mathbf{p}}{\omega_{\mathbf{p}}} = \dfrac{eE p}{\omega_{\mathbf{p}}} \end{cases}$$

$$(\quad)_z = \int\frac{d^3\mathbf{p}}{(2\pi)^3}2e\frac{\omega_{\mathbf{p}}}{eE}\dot{S}.$$

The meaning of the second term is now clear

Schwinger effect 

energy required $2\omega_p$
$\Downarrow$
time needed $\tau \sim \dfrac{2\omega_p}{eE}$
$\Downarrow$
distance when they born $d \sim \dfrac{2\omega_p}{eE}$
$\Downarrow$
dipole moment per pair $\mu = e \times d$
$= \dfrac{2e\omega_p}{eE}$

$\therefore$ polarization current $\dot{\mathbb{P}} \sim \mu\,\Gamma$

Thus, $J^{\mu}_{mat} = J^{\mu}_{cond} + J^{\mu}_{pol}$.

- The polarization current is crucial for the energy conservation because it carries energy

For simplicity, consider the homogeneous case, in which $\mathbb{B}$ field is absent

$$\mathcal{E}_{field} = \frac{1}{2} E^2 \qquad \cancel{\nabla\times\mathbb{B}}^{\,0} = \frac{\partial \mathbb{E}}{\partial t} + J.$$

$$\Rightarrow \dot{\mathcal{E}}_{field} = E\dot{E} = -EJ.$$

can be shown by calculating $\langle : \hat{T}^{00} : \rangle$.

And

$$\mathcal{E}_{\text{particles}} = \int \frac{d^3p}{(2\pi)^3} \underset{e^+ \& e^-}{2} \underbrace{\omega_p}_{\text{1 particle energy}} \underbrace{f_p}_{\text{dist.}}$$

$$\Rightarrow \dot{\mathcal{E}}_{\text{particle}} = \int \frac{d^3p}{(2\pi)^3} 2\dot{\omega}_p f_p + \int \frac{d^3p}{(2\pi)^3} 2\omega_p \dot{f}_p$$

$\dot{\omega}_p = \frac{d}{dt}\sqrt{m^2+p^2}$

$= eF_{0i} v^i$

$= e\mathbb{E}\cdot v = e\mathbb{E}\cdot\frac{\mathbb{p}}{\omega_p} = eE\frac{p_z}{\omega_p}$

Lorentz eq.

$\dot{p}_\mu = eF_{\mu\nu}v^\nu$, $v^\nu = \frac{p^\nu}{p^0}$

$\dot{f}_p = S$

$$= E\left[\underbrace{\int \frac{d^3p}{(2\pi)^3} 2e\frac{p_z}{\omega_p} f_p}_{J_{\text{cond}}} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{2\omega_p}{eE} S}_{J_{\text{pol}}}\right]$$

$$= EJ$$

$$\therefore \dot{\mathcal{E}}_{\text{tot}} = \dot{\mathcal{E}}_{\text{field}} + \dot{\mathcal{E}}_{\text{particle}}$$

$$= -EJ + EJ$$

$$= 0.$$

Note: You can show the necessity of J^{μ}_{pol} more generally (i.e., without assuming the homogeneity) and identify more general form of J^{μ}_{pol} as follows.

First, rewriting $\frac{df}{dt} = \dot{S}$ using the Boltzman kernel:

$$\dot{S} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\boldsymbol{x}}{dt} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \frac{dp^{\mu}}{dt} \frac{\partial f}{\partial p^{\mu}},$$

which can be made covariant by multiplying $p^0 = \omega_{\mathbf{p}}$ to both-hand side:

$$p^0 \dot{S} = p^{\mu} \partial_{\mu} f + m \underbrace{\frac{dp^{\mu}}{d\tau}}_{eF^{\mu\nu} p_{\nu}} \frac{\partial f}{\partial p^{\mu}}$$

$$eF^{\mu\nu} p_{\nu} \frac{\partial f}{\partial p^{\mu}}.$$

Lorentz eq

Next, calculating the expectation value of $\hat{T}^{\mu\nu}$, one can show

$$T^{\mu\nu}_{mat} = \langle 0; in | : \hat{T}^{\mu\nu} : | 0; in \rangle$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \, 2 \, \frac{p^{\mu} p^{\nu}}{p^0} f_{\mathbf{p}} \quad (p^0 = \omega_{\mathbf{p}})$$

$e^+ \& e^-$

$$\Rightarrow \partial_{\mu} T^{\mu\nu}_{mat} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \, 2 \, \frac{p^{\nu}}{p^0} p^{\mu} \partial_{\mu} f_{\mathbf{p}}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, 2\frac{p^\nu}{p^0}\left(p^0 S - eF^{\alpha\beta}\frac{p_\beta}{m}\frac{\partial f}{\partial p^\alpha}\right)$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, p^\nu S - 2eF^{\lambda}{}_{\mu}\underbrace{\int \frac{d^3p}{(2\pi)^3 p^0}\, p^\nu p^\mu \frac{\partial f}{\partial p^\lambda}}_{=}$$

$$- \int \frac{d^3p}{(2\pi)^3 p^0}\, f\, \frac{\partial (p^\mu p^\nu)}{\partial p^\lambda}$$

$$= -\delta^\mu{}_\lambda \int \frac{d^3p}{(2\pi)^3 p^0}\, p^\nu f - \delta^\nu{}_\lambda \int \frac{d^3p}{(2\pi)^3 p^0}\, p^\mu f$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, p^\nu S + 2eF^\nu{}_\mu \int \frac{d^3p}{(2\pi)^3}\, \frac{p^\mu}{p^0} f$$

$$g^{-1}\tilde F^{\mu}{}_{\lambda}$$

$$= eF^\nu{}_\mu \left[\int \frac{d^3p}{(2\pi)^3}\, 2F^{-1\,\mu}{}_{\lambda}\, p^\lambda S + \int \frac{d^3p}{(2\pi)^3}\, 2\frac{p^\mu}{p^0} f\right]$$

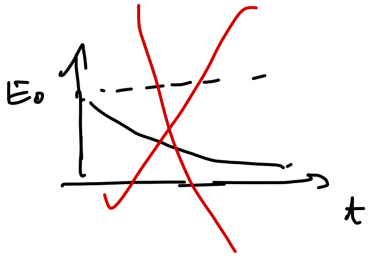
$$= F^\nu{}_\mu \left[\underbrace{\int \frac{d^3p}{(2\pi)^3}\, \frac{2}{g} e\tilde F^{\mu}{}_{\lambda}\, p^\lambda S}_{J^\mu_{pol}} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\, 2e\frac{p^\mu}{p^0} f}_{J^\mu_{cond}}\right]$$

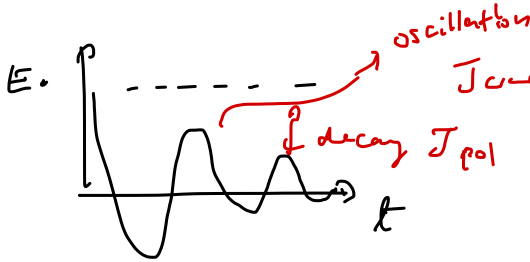
And it can be shown easily that

$$\partial_\mu T^{\mu\nu}_{field} = -F^{\mu\nu} J_\nu$$

- Typical behavior: Doesn't decay smoothly

$\longrightarrow$ oscillates.





Why: $\dot{E} = -J$

* J_{cond} contributes as: Initially $E>0 \Rightarrow \dot{v} > 0$

& v

$\downarrow$ $\dot{E} < 0$.

At some point $E=0 \Rightarrow \dot{v}=0$ but $v>0$.

$\downarrow$ $\dot{E}$ continues decreasing

$E<0 \rightarrow \dot{v}<0$

$\downarrow$

$v=0$, E takes min. $\Rightarrow \dot{v}<0$

$\downarrow$

$v>0 \rightarrow \dot{E}>0$

$\downarrow$

E increases

$\downarrow$

$E=0$

$\downarrow$

$\vdots$

Or roughly

$\dot{E} = -J_{cond} \sim -\# v$

$\Rightarrow \ddot{E} = -\# \dot{v} \sim -\# E$

$\Rightarrow E = e^{i\#t}$.

oscillation (plasma oscillation)

BUT, pair production is not important.

* J_{pol} contributes to decay because it has the information of pair production and pair production dissipates energy whenever it happens.

Mathematically,

$$\dot{E} = -J_{pol} = -\frac{1}{E}\underbrace{\int d^3p\, 2\omega_p \dot{f}}_{>0.}$$

$$\Rightarrow \underbrace{E\dot{E}}_{\frac{d}{dt}\left(\frac{1}{2}E^2\right) = \dot{\mathcal{E}}_{field}} = -\overset{>0.}{\#}$$

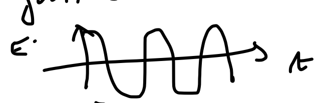
$\Rightarrow$ Field energy always decreases

or QED_{3+1} in strong B field.

Note: If $m=0$ and $p_\perp = 0$, i.e., massless QED_{1+1}, the polarization current J_{pol} vanishes because $\omega_p \to 0$, then there is no dissipation (i.e., pair production can occur w/o energy cost).

(... ~ Iwazaki (2015))

For this case, E just oscillates due to J_{cond} like



§3 Summary and discussion

What I explained

- Introduction to strong-field physics
  - * why and where
  - * relevance to HIC: early-time dynamics.
- Schwinger effect
  - * vacuum pair production by strong E field,
  - * basic theoretical framework.
    - Furry-picture perturbation theory.
    - Bogoliubov-transformation technique.
    - Heisenberg-Euler effective Lagrangian.
    - Adiabatic particle picture
    - Backreaction problem.

(mean-field theory with spatially homogeneous E field)

Old theory (1992), but still is the latest/best

$\Rightarrow$ Lots of open questions

e.g., · spatially inhomogeneous case?

· scattering beyond MFA?

· more sophisticated QFT formulation like Schwinger-Keldysh

· If ↑ are feasible, how to implement numerically?

· Is the "oscillating decay" picture still valid?

· applications to HIC and other related