Simulatie (FEB22013X)

Tutorial 6: Variance reduction

In some cases, one has to run a simulation model very often to obtain reliable results. In this tutorial, you are going to apply a method that can be used to reduce the variance in the simulation outcomes, and thereby to reduce the number of simulation runs needed. The answers of your group must be handed in as one PDF file of at most 4 pages. The file can be handed in via Blackboard (in the dropbox for Assignment 6). The deadline is Tuesday April 11, 2017 at 23:59h. Formulate your answers in a concise way. Mention your names and student numbers on the first page. The PDF file must have the following name:
`studentnr1_studentnr2_assignment6.pdf`

Opgave 1 In this exercise, we will use simulation to estimate $\theta = E(\sqrt[3]{U})$, where $U \sim U(0,1)$.

a. How can Monte Carlo simulation be used to estimate θ? Describe the corresponding simulation algorithm, and implement it in Matlab.

b. Determine the theoretical value of the variance of the Monte Carlo estimator you proposed in question **a**. Verify your answer by performing the simulation in Matlab.

c. Obviously, it is beneficial to use a control variate when estimating θ by means of Monte Carlo estimation. One candidate control variable is the variable U itself. Explain why.

d. Determine the control variate estimator of θ for which the variance reduction is maximized. Derive the theoretical value of this variance, and verify your answer by performing the simulation in Matlab.

Opgave 2 This exercise is based on the work by Kemna & Vorst (1990), and deals with pricing an Asian Call Option. An Asian Call Option is also referred to as an average value (AV) option. We consider a given stock that is traded in the period $[0, T]$, where 0 refers to the moment at which the stock is bought. During the period $[0, T]$, the stock can be traded exactly n times. The stock price at times $0 \leq t_i \leq T$ are denoted by S_{t_i}. The stock price S_0 is known, whereas for $t_i > 0$, S_{t_i} is characterized by:

$$S_{t_i} = S_{t_{i-1}} \exp(X_i) \tag{1}$$

Here, X_i is the rate of return during the time-interval between two moments the stock can be traded. The rate of return is X_i is normally distributed with mean μ_X and standard deviation σ_X, where

$$\mu_X = (r - \frac{1}{2}\sigma_A^2)\frac{T}{n}, \qquad \sigma_X = \sigma_A\sqrt{\frac{T}{n}}. \tag{2}$$

Here, r denotes the risk-free interest rate, and σ_A the volatility of the stock.

The revenue earned is given by

$$Y_T = \max\left(A_T - K, 0\right), \tag{3}$$

where K denotes the so-called stike price, and where A_T, the average value of the stock during the period $[0, T]$, is given by

$$A_T = \frac{1}{n+1}\sum_{i=0}^{n} S_{t_i}$$

Note that Y_T is a random variable. Unfortunately, it is not possible to obtain an analytical expression for $E[Y_T]$. Therefore, the value of $E[Y_T]$ needs to be estimated by means of simulation.

a. Write a Matlab-function that, given S_0, r, σ_A, K and n, simulates the stock price for the period $[0, T]$ and returns the simulated revenue of the AV-option, Y_T.

b. Use Monte Carlo simulation to estimate the expected value and the variance of Y_T. Use $s = 100.000$ runs to make these estimates. Express T in years, consider a period of four months, and divide these into 90 trading days (i.e., $T = 1/3$ and $n = 90$). Furthermore, use that $r = \ln(1.05)$, $\sigma_A = 0.2$, $S_0 = 45$, and $K = 40$.

c. Use the results from exercise **b.** to determine the number of simulation runs that are theoretically needed to obtain a 99% asymptotic confidence interval for $E[Y_T]$ that has length at most 0.01 (i.e., so that $E[Y_T]$ is estimated with a precision of 1 cent).

Pricing an option often needs to be done online (i.e., in a continuous way). It is therefore very important to minimize the number of simulations required to obtain the estimate. We will use two techniques to achieve this. First we will analyse the effect of antithetic variables. After that we will use a control variable.

To determine the revenue on the stock, you have used randomly generated normal distributed variables with mean μ_X and standard deviation σ_X. These variables can also be generated based on standard normally distributed variables. Note that for a standard normally distributed variable Z it holds that $-Z$ is also standard normally distributed.

We will use the Matlab-function of question **a.** to determine the revenue Y_T using antithetic variables.

d. Write a new Matlab-function (based on your function in question **a.**) in which you generate a vector with standard normally distributed variables Z. Use this vector to generate the returns $X_1^1, ..., X_n^1$. Also generate returns $X_1^2, ..., X_n^2$ using the antithetic variables $-Z$. Calculate the revenues Y_T^1 and Y_T^2 using the returns $X_1^1, ..., X_n^1$ and $X_1^2, ..., X_n^2$ respectively. Let your function return the average revenue $Y_T = \frac{1}{2}(Y_T^1 + Y_T^2)$.

e. Repeat the simulation from question **b.** and determine the mean and variance of Y_T based on $s = 100.000$ simulation runs.

f. Based on your results in question **e.** determine how many simulations runs you need theoretically in order to obtain a 99% asymptotic confidence interval for $E[Y_T]$ that has length at most 0.01.

g. Explain the difference between the number of simulation runs required in question **c.** and the number required in question **f.**

Another way to reduce the variance is by making use of a control variable. A suitable candidate is the following:

$$U_T = \max(G_T - K, 0)$$

where

$$G_T = \left(\prod_{i=0}^{n} S_{t_i}\right)^{1/(n+1)} = \exp\left\{\frac{1}{n+1}\sum_{i=0}^{n} \ln S_{t_i}\right\}.$$

The expected value of U_T, called μ_U, can be determined analytically:

$$\mu_U = \exp\left(\mu_G + \frac{\sigma_G^2}{2}\right)\Phi\left(\frac{\mu_G - \ln(K) + \sigma_G^2}{\sigma_G}\right) - K\Phi\left(\frac{\mu_G - \ln(K)}{\sigma_G}\right) \tag{4}$$

where $\Phi(\cdot)$ denotes the standard cumulative distribution function and where

$$\mu_G = \frac{1}{2}n\mu_X + \ln(S_0), \qquad \sigma_G = \sqrt{\frac{n(2n+1)}{6(n+1)}}\sigma_X. \tag{5}$$

Hence, $E[Y_T + c(U_T - \mu_U)]$ is an unbiased estimate of $E[Y_T]$.

h. Repeat the simulation from question **b.**, but now also calculate the value of U_T for every run. Use $s = 100.000$ simulation runs to estimate the expected value of U_T. Verify that you calculate U_T correctly by comparing this estimate with the theoretical value μ_U.
Hint: Use the function `normcdf()` to determine μ_U.

i. Write a Matlab-function that determines $E[Y_T + c(U_T - \mu_U)]$. How many simulation runs are needed theoretically to obtain a 99% asymptotic confidence interval for $E[Y_T + c(U_T - \mu_U)]$ that has length at most 0.01?
Hint: First determine the optimal value of c.

j. Explain the difference between the number of simulation runs required in question **c.** and the number required in question **i.**