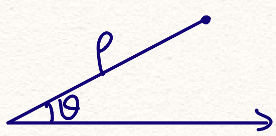
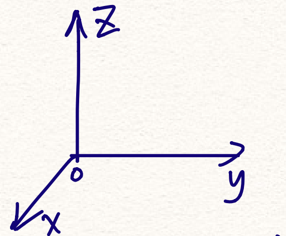
I. Generalized Coordinates / D.O.F / constraints

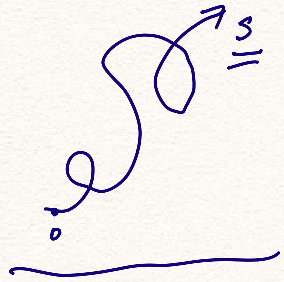
△ Generalized Coordinates : can be any coordinates describing motions

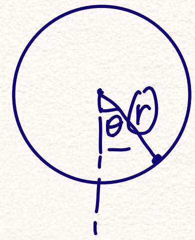


polar (ρ,θ)



Cartesian (x,y,z)





$$\begin{cases} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{cases} \rightarrow \begin{cases} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \vdots \\ \dot{q}_n(t) \end{cases}$$ (generalized velocities)

△ Degree of freedom (D.O.F).

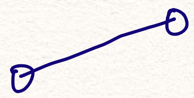
Total number of independent functions needed to describe the system

△ Free particle in 3D : $f=3$. eg. (x,y,z), (ρ,θ,φ), (ρ,φ,z)

△ Constraints can deduce D.O.F

△ Free diatomic molecule (双原子分子) : $f=5$

(x_2,y_2,z_2)



(x_1,y_1,z_1)

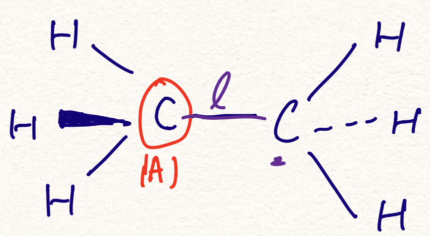
$3+3-1$

$$(x_1-y_1)^2+(x_2-y_2)^2=\ell^2$$

△ polyatomic moleule (多原子分子) :

eg. C_2H_6 :

α, $\circlearrowleft_1$

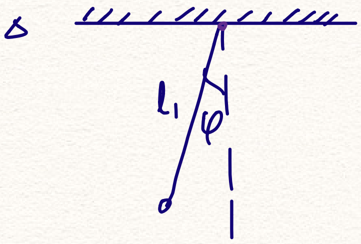


$\circlearrowleft_2$ β

$\boxed{f=7}$

x_A, y_A, z_A, θ, φ, rotating 1, rotating 2

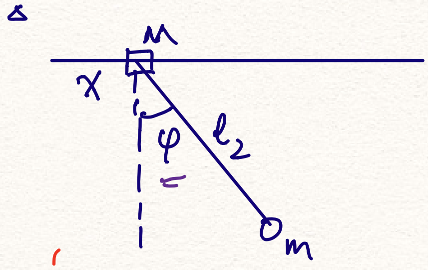
Recall in VC210, $C_{v,m(1)} = \frac{3}{2}R$, $C_{v,m(2)} = \frac{5}{2}R$, $C_{v,m(3)} = 3R$ or $\frac{7}{2}R$



simple pendulum

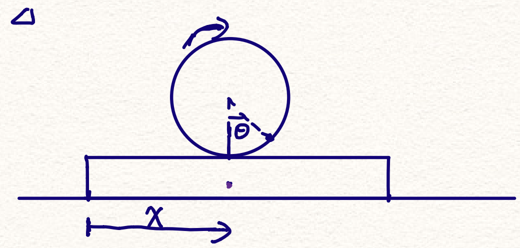
$f = 1$

$q_1 = \varphi$



$f = 2$

$$\begin{cases} q_1 = x \\ q_2 = \varphi \end{cases}$$



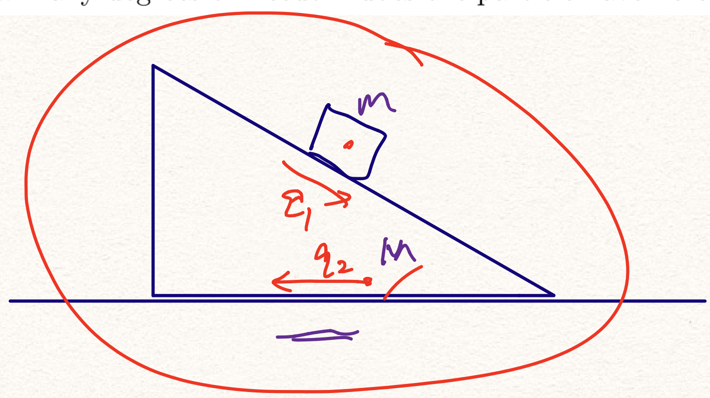
pure rotating

$f = 2$

$$\begin{cases} q_1 = x \\ q_2 = \theta \end{cases}$$

1. A point particle of mass m moves without friction down a wedge of mass M that is free to slide on a frictionless table. The wedge is inclined at the angle α to the horizontal.

   How many degrees of freedom does the particle have here? Identify the generalized coordinates here.



$f = 2$

II. Lagraugian mechanics

- Lagraugian function. $L \overset{\text{def}}{=} K - U$
- trajectory. $\vec{q} = \vec{q}(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{pmatrix}$

- Hamilton's action $S = S[\vec{q}] = \int_{t_A}^{t_B} L(\vec{q}, \dot{\vec{q}}, t)\, dt$

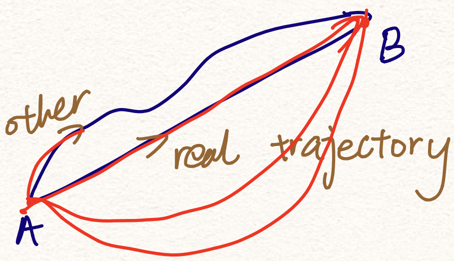
S : functional : maps functions into numbers.

△ Hamilton's principle

$$\text{Real path} \iff \boxed{\delta S = 0} \qquad (\delta : \text{variational differential})$$

变分

Meaning: $\vec{q}$ for real path must let the S attains an extremum.



$$\delta q_i(t) = \tilde{q}_i(t) - q_i(t)$$

$$\delta \vec{q}(t) = \begin{pmatrix} \tilde{q}_1(t) - q_1(t) \\ \tilde{q}_2(t) - q_2(t) \\ \vdots \end{pmatrix}$$

$$\delta \vec{q}(t_A) = \delta \vec{q}(t_B) = 0$$

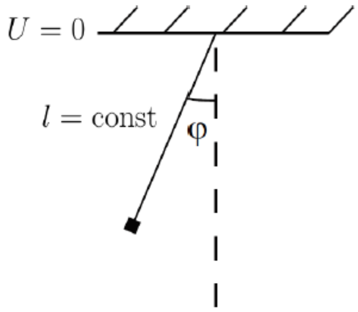
$$\begin{cases} \delta S = \delta \int L(\vec{q}, \dot{\vec{q}}, t)\, dt = 0 \\ \delta \vec{q}(t_A) = \delta \vec{q}(t_B) = 0 \end{cases} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

△ Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 , \quad \text{for } i = 1, 2, 3, \cdots, f}$$

Example (c). Simple Pendulum

System: Simple pendulum (mass m on an inextensible cord with length l, swinging in a plane, acted upon the gravitational force). Here $f = 1$ and $q_1 = \varphi$.



Speed of the bob $|\overline{v}| = |l\dot{\varphi}|$

Lagrangian

$$\mathcal{L} = K - U = \frac{1}{2}m(l\dot{\varphi})^2 - (-mgl\cos\varphi)$$

$$= \frac{1}{2}ml^2\dot{\varphi}^2 + mgl\cos\varphi$$

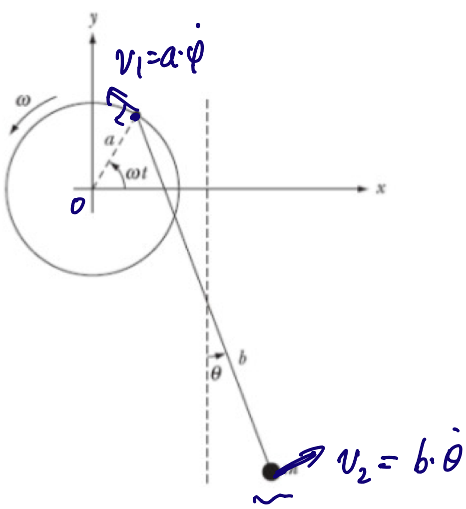
and $\frac{\partial \mathcal{L}}{\partial \varphi} = -mgl\sin\varphi, \quad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = ml^2\dot{\varphi}$

Euler–Lagrange equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \implies ml^2\ddot{\varphi} + mgl\sin\varphi = 0$$

$$\boxed{\ddot{\varphi} = -\frac{g}{l}\sin\varphi}$$

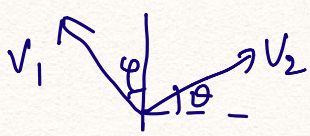
2. A simple pendulum of length b and mass m moves attached to a massless rim of radius a rotating with constant angular velocity ω. How many degrees of freedom do we have here? Find the Lagrangian.



$f = 2$, $\begin{cases} q_1 = \varphi = \omega t \\ q_2 = \theta \end{cases}$

$U(\varphi, \theta) = (a\sin\varphi - b\cos\theta) \cdot mg$

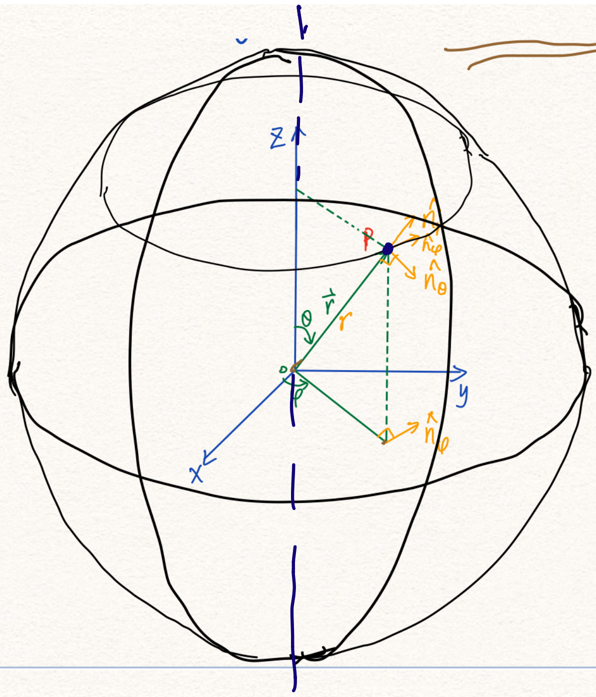
$K(\varphi, \theta) = \frac{1}{2}m\left((a\dot{\varphi}\cos\varphi + b\dot{\theta}\sin\theta)^2 + (a\dot{\varphi}\sin\varphi - b\dot{\theta}\cos\theta)^2\right)$



$L = K - U$

3. Find the equations of motion of a particle of mass m constrained to move on the surface of a sphere, acted upon a conservative force $\mathbf{F} = F_0 \hat{n}_\theta$ with F_0 a constant.

Hint. To find the potential energy find the scalar product $\mathbf{F} \cdot d\mathbf{r}$ for the infinitesimal displacement on the sphere and use the fact that it is equal to $-dU$ (the force is conservative).



$f = 3$ (r, θ, φ) $-du = \vec{F} \cdot d\vec{r}$

$\vec{r} = r \cdot \hat{n}_r$

$d\vec{r}$?

$\vec{v}(t) = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta + r\dot{\varphi}\sin\theta\,\hat{n}_\varphi$

$d\vec{r} = \vec{v} \cdot dt = dr \cdot \hat{n}_r + r d\theta \cdot \hat{n}_\theta + r d\varphi \sin\theta\,\hat{n}_\varphi$

$\int -du = \vec{F} \cdot d\vec{r} = \int F_0 \cdot r \cdot d\theta$.

Let $u = -F_0 r \theta$

$K = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2 \sin^2\theta)$

$L = K - U = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta) + F_0 r \theta$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \quad \text{for } i = 1, 2, 3, \dots, f}$$

$\frac{\partial L}{\partial r} = m\dot{\theta}^2 r + m\dot{\varphi}^2 \sin^2\theta\, r + F_0 \theta$

$\frac{\partial L}{\partial \theta} = mr^2\dot{\varphi}^2 \cdot \sin\theta\cos\theta + F_0 r$

$\frac{\partial L}{\partial \varphi} = 0$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$

$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = mr^2\sin^2\theta\,\ddot{\varphi}$

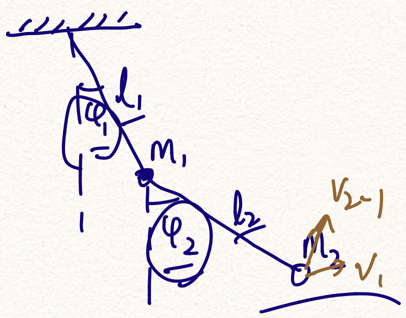
$$r \quad \left(m\ddot{r} - (m\dot{\theta}^2 r + m\dot{\varphi}^2 \sin^2\theta\, r + F_0\theta) = 0\right)$$

$$\theta \quad \left(mr^2\ddot{\theta} - (mr^2\dot{\varphi}^2 \sin\theta\cos\theta + F_0 r) = 0\right)$$

$$\varphi: \quad \boxed{mr^2\sin^2\theta\,\ddot{\varphi} - 0 = 0} \qquad mr^2\sin^2\theta\cdot\dot{\varphi}$$

Actually angular momentum conservation!

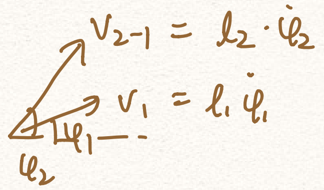
4. Double pendulum: (1) identify the generalized coordinates; (2) find the Lagrangian; (3) write down the Euler–Lagrange equations of motion (you may skip this part); (4) please look up some animations of trajectories of such a pendulum.



$$U = -l_1\cos\varphi_1\, m_1 g - (l_1\cos\varphi_1 + l_2\cos\varphi_2)\, m_2 g$$

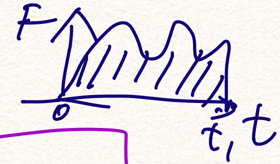
$$K = \tfrac{1}{2}m_1(l_1\dot{\varphi}_1)^2 + \tfrac{1}{2}m_2\Big((l_1\dot{\varphi}_1\cos\varphi_1 + l_2\dot{\varphi}_2\cos\varphi_2)^2$$

$$+ (l_1\dot{\varphi}_1\sin\varphi_1 + l_2\dot{\varphi}_2\sin\varphi_2)^2\Big)$$



$$L = K - U.$$

III. Momentum and Conservation of Momentum.



$$F = ma \quad \xrightarrow{t} \quad \int F\cdot dt = m\int a\,dt = \boxed{m\Delta v} \iff \boxed{I = m\Delta v} = \Delta(mv)$$

$$\xrightarrow{\vec{r}} \quad \int F\cdot dr = \int m\cdot a\,dr = \int m\left(\frac{dr}{dt}\right)\cdot\frac{dv}{dr}\,dr = \frac{1}{2}m\Delta(v^2)$$

v

$\frac{dv}{dt}$

$$F = \frac{dp}{dt}, \qquad p = mv$$

$$\vec{P_2} - \vec{P_1} = \int_{t_1}^{t_2} \vec{F_x}\,dt \qquad \boxed{\text{Impulse Theorem}}$$

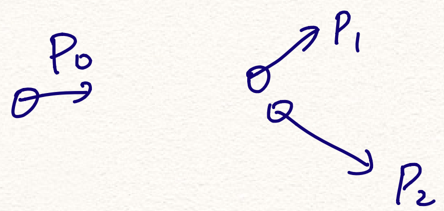
△ If $\vec{F}_{ext} = 0$, For a system, $\Delta P = 0$, $P = \text{Const}$.

Conservation of Momentum

Conservation: Deal with $\boxed{\text{states}}$, not processes.

IV. Collision

△ Non-center collision



$$\begin{cases} P_{1\perp} = P_{2\perp} \\ P_{1\parallel} + P_{2\parallel} = P_0 \end{cases}$$

△ Center collision

$$m_1 V_0 = m_1 V_1 + m_2 V_2 \quad ①$$

①③ ✓

① 完全彈：

$$\frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_1 V_0^2 \quad ②$$

$$\boxed{V_0 - 0 = V_2 - V_1} \quad ③$$

$V_{相遇} = V_{分離}$.

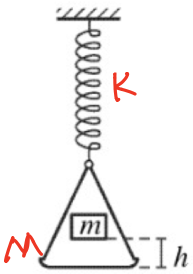
② Inelastic $\quad 0 < e = \dfrac{V_2 - V_1}{V_0 - 0} = \dfrac{V_{分離}}{V_{相遇}} < 1$

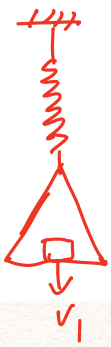
$e = 1 \to$ elastic

$e = 0 \to$ completely inelastic

③ Completely Inelastic $\quad e = 0$

6. A block with mass m falls down from height h on a horizontal plane (M) suspended on a spring with spring constant k, and remains on the plane. Find the amplitude of resulting oscillations.





① $v_0 = \sqrt{2gh}$

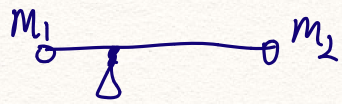
② $mV_0 = (m+M) \cdot V_1$ ✓

$$\to V_1 = \frac{mV_0}{m+M}$$

③ equibrium : $\quad x_0 = \dfrac{(m+M)g}{K}$. $\quad \Delta x = x_0 - x(0) = \dfrac{(m+M)g}{K} - \dfrac{Mg}{K} = \dfrac{mg}{K}$

④ $A^2 = \Delta x^2 + \dfrac{V_1^2}{\omega_1^2}$, $\quad \omega_1 = \sqrt{\dfrac{K}{m+M}}$

Center of Mass



$$\boxed{\vec{r}_c = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\int dm \cdot \vec{r}}{\int dm}}$$

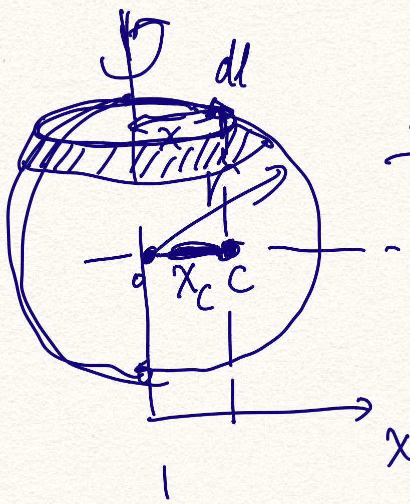
① $\vec{P}_{system} = \vec{P}_c = m \cdot v_c = m \cdot \dot{\vec{r}}_c$

② take r_c as ROE, $\vec{P}_c' = 0$

③ $\vec{F}_{ext} = 0 \Rightarrow \vec{P} = \text{Const} \Rightarrow \vec{V}_c = \text{Const}$

$$\vec{r}_c = \frac{\int r \cdot dm}{\int dm} \Rightarrow \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{pmatrix} \frac{\int x \cdot dm}{\int dm} \\ \vdots \end{pmatrix}$$

①



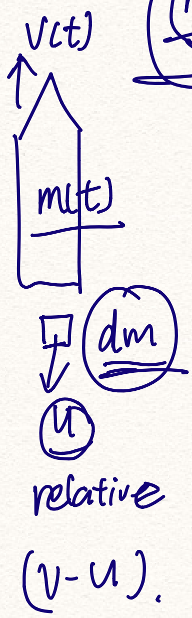
λ

$$x_c = \frac{\int \lambda \cdot dl \cdot x}{\int \lambda \cdot dl}$$

$$= \frac{\int \lambda (dl \cdot x \cdot 2\pi)}{2\pi \int \lambda \cdot dl}$$

$$= \frac{\lambda \cdot \int 球带面积}{2\pi \cdot m} = \frac{\lambda \cdot 球壳面积}{2\pi \cdot m}$$

$$= \frac{\lambda \cdot 4\pi r^2}{2\pi \cdot \pi r \lambda} = \boxed{\frac{2}{\pi} r}$$



$$mv \overset{\Delta t}{=} (m+dm)(v+dv) - dm\cdot(v-u)$$

$$m\,dv = -u\,dm$$

$$\int \frac{dm}{m} = \int \frac{1}{u}\, dv$$

$$\ln m = -\frac{1}{u} \Delta v$$

$$\boxed{v = v_0 + u \cdot \ln \frac{m_0}{m(t)}}$$