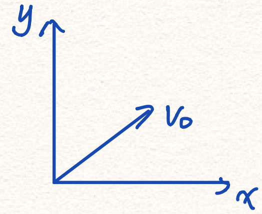
I. Linear Air Drag



$\vec{F} = m\vec{a}$ $\qquad$ $\vec{F} = m\vec{g} + (-k\vec{v})$

$$\Rightarrow \begin{cases} ma_x = -kv_x & ① \\ ma_y = -mg - kv_y & ② \end{cases} \qquad \begin{cases} v_x(0) = v_{x0} \\ v_y(0) = v_{y0} \end{cases}$$

①: $m\frac{dv_x}{dt} = -kv_x$

$$\int \frac{dv_x}{v_x} = \int -\frac{k}{m}\,dt \Longrightarrow \ln\frac{v_x}{v_{x0}} = \frac{-k}{m}t \Rightarrow v_x = v_{x0}\,e^{-\frac{k}{m}t}$$

②: $m\frac{dv_y}{dt} = -mg - kv_y$

A. $\frac{dv_y}{dt} = -g - \frac{k}{m}v_y = -\frac{k}{m}\left(v_y + \frac{mg}{k}\right)$

$$\int_{v_0+\frac{mg}{k}}^{v_y+\frac{mg}{k}} \frac{dv_y = d\left(v_y+\frac{mg}{k}\right)}{v_y+\frac{mg}{k}} = \int_0^t -\frac{k}{m}\,dt \Rightarrow \ln\frac{v_y+\frac{mg}{k}}{v_{y0}+\frac{mg}{k}} = -\frac{k}{m}t \Rightarrow \underline{v_y = \left(v_{y0}+\frac{mg}{k}\right)\cdot e^{-\frac{k}{m}t} - \frac{mg}{k}}$$

B.

Solutions to Inhomogeneous, Linear Systems $\qquad$ Slide 244 $\qquad$ JOINT INSTITUTE 交大密西根学院

Inhomogeneous Linear Systems

1.10.10. Theorem. The solution of the initial value problem

$$\frac{dx}{dt} = A(t)x + b(t), \qquad x(t_0) = x_0, \qquad (1.10.7)$$

is given by

$$x_{\text{inhom}}(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)$$

$$\frac{dv_y}{dt} = -g - \frac{k}{m}v_y$$

① homo: $\frac{dv_y}{dt} = -\frac{k}{m}v_y$ $\qquad$ $\int\frac{dv_y}{v_y} = \int -\frac{k}{m}\,dt$

$\ln\frac{v_y}{A} = -\frac{k}{m}(t-t_0)$

$v_y = Ae^{-\frac{k}{m}(t-t_0)} \Rightarrow v_y(t) = A'e^{-\frac{k}{m}t}$ $\qquad$ (A' is a Const).

② part: Let "$\frac{dv_y}{dt} = 0$" $\qquad -g = +\frac{k}{m}v_y \rightarrow v_y(t) = -\frac{mg}{k}$.

③ $v_{y\,\text{inhomo}} = v_{y\,\text{homo}} + v_{y\,\text{part}} = \underline{A'e^{-\frac{k}{m}t} - \frac{mg}{k}}$

$v_y(0) = v_{y0} \Rightarrow A' = v_{y0} + \frac{mg}{k} \Rightarrow v_y = \left(v_{y0}+\frac{mg}{k}\right)e^{-\frac{k}{m}t} - \frac{mg}{k}$

If Not Linear?

Q1. Consider fall of an object (mass m) without initial speed. Assuming quadratic air drag find the time dependence of object's velocity and position. Find the terminal speed.

$V(0)=0 \quad x(0)=0$

$f=-kv^2$

$$m\frac{dv}{dt} = mg - kv^2 \quad \rightarrow \quad \frac{dv}{dt} = g - \frac{k}{m}v^2$$

Inhomogeneous ODE.

But Not linear!

$$\frac{dv}{dt} = -\frac{k}{m}v^2 + g$$

$$\frac{dv}{-\frac{k}{m}v^2+g} = dt$$

$$\int \frac{dv}{g(1-\frac{k}{mg}v^2)} = \int dt \quad \Rightarrow \quad t = \frac{\arctan\left(\sqrt{\frac{k}{mg}}\,v\right)}{\sqrt{\frac{k}{m}g}}$$

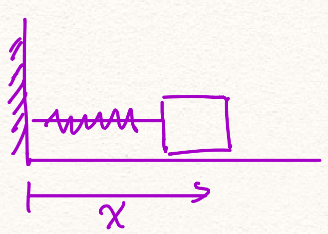
$$\sqrt{\frac{k}{mg}}\,v = \tan\left(\sqrt{\frac{kg}{m}}\,t\right)$$

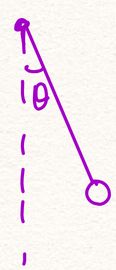
$$v(t) = \sqrt{\frac{mg}{k}}\tan\left(\sqrt{\frac{kg}{m}}\,t\right)$$

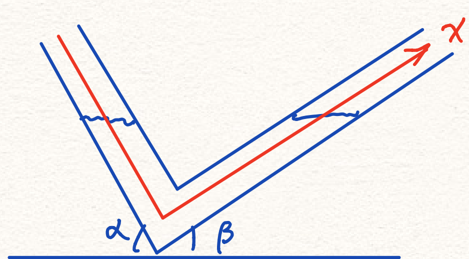
II. Simple Harmonic Oscillation

$$\boxed{\ddot{x} + \frac{k}{m}x = 0} \xrightarrow{\omega_0^2=\frac{k}{m}} \ddot{x} + \omega_0^2 x = 0$$

Note: ① x can be any "generalized coordinate vector"







② The general (homogeneous) solution of $\ddot{x} + \omega_0^2 x = 0$:

$$\boxed{x(t) = A\cos(\omega_0 t + \varphi)}$$

A : amplitude

ω_0 : angular frequency → "Initial value Problem"

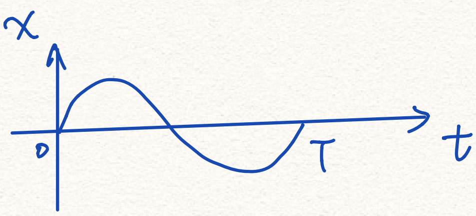
φ : phase shift

Or : $x(t) = A\sin\omega_0 t + B\cos\omega_0 t$ "Auxiliary Angle Formula"

Or : $x(t) = \mathrm{Re}\left(A e^{i(\omega_0 t + \varphi)}\right)$

③ The period T

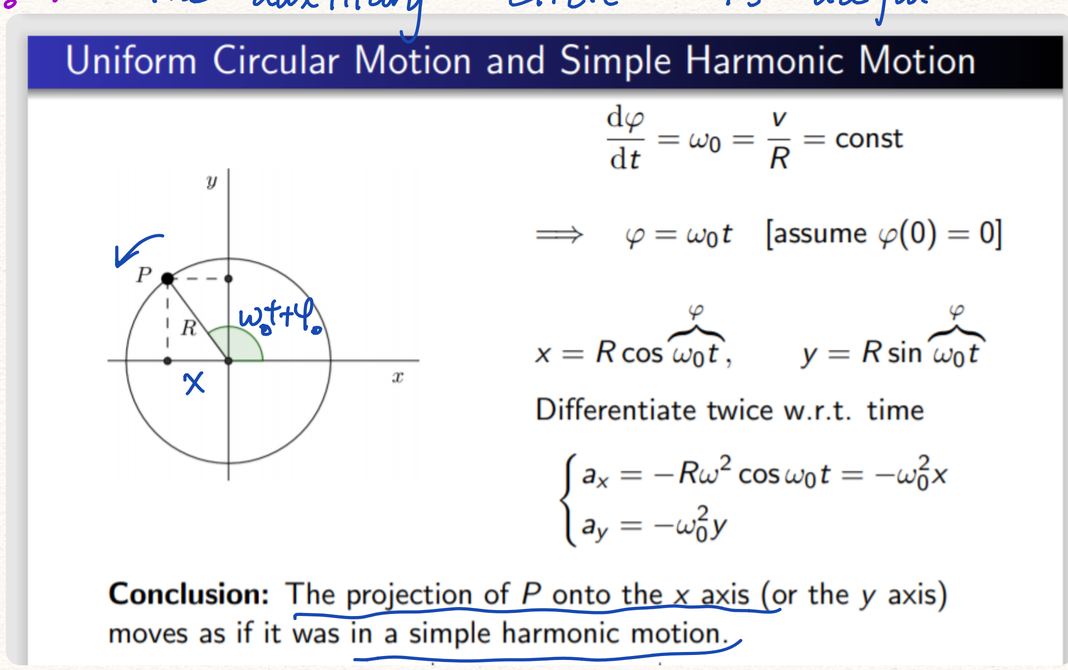
$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{K}}$$



☆ Useful tips when solving the Initial-Value Problem.

① $\omega_0 = \sqrt{\frac{K}{m}}$ get directly from the ODE

② φ_0 . The "auxiliary circle" is useful



$\cos(\omega_0 t + \varphi_0) = \frac{x}{A}$

mind the direction !

③ A. In general, if you release the system at rest, then the A is just the distance between the releasing point and the equilibrium point.

If not, you may use the energy (discuss later):

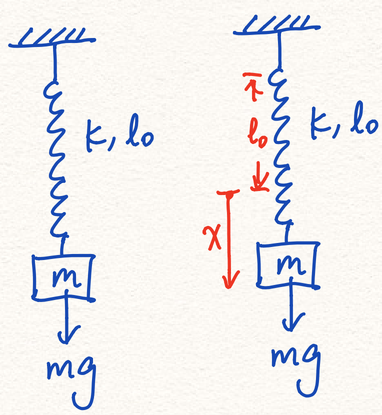
$$E_k + E_p = E = \frac{1}{2}kA^2 \iff \underline{\underline{A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2}}}$$

④ Always try to find the equilibrium point first, and then use the infinitestimal method to write down Newton's second law.

☆ Some models of simple harmonic Oscillation

① A constant force added.

eg.



$$m\ddot{x} = F = mg - kx$$

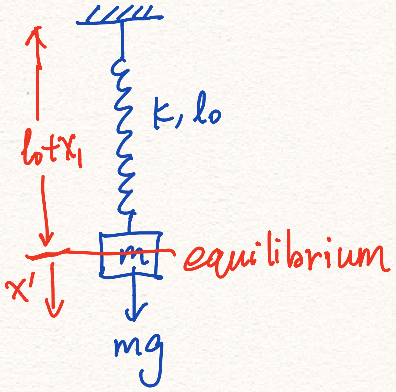
$$\ddot{x} + \frac{k}{m}x - g = 0$$

☆ let $x_1 = \frac{mg}{k}$

$$\frac{k}{m}x - g = \frac{k}{m}(x - x_1)$$

$$\ddot{(x - x_1)} + \frac{k}{m}(x - x_1) = 0$$

$$\Rightarrow \quad x - x_1 = A\cos(\omega_0 t + \varphi)$$

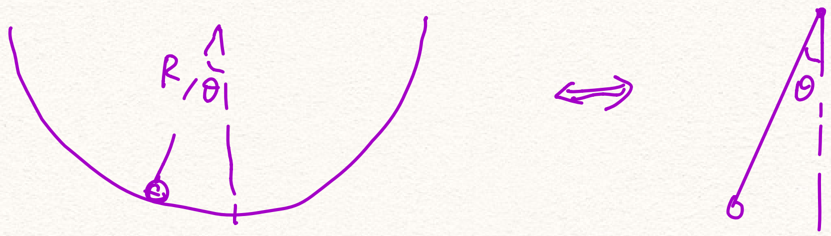


"Change the origin of the coordinate"

$x' = A\cos(\omega_0 t + \varphi)$

② "fimilar model"

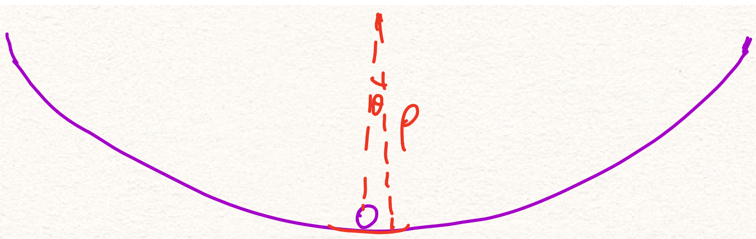
Q2. Discuss motion of a particle that is placed on the inner surface of a spherical pot, close to its bottom, and released from hold (no friction).



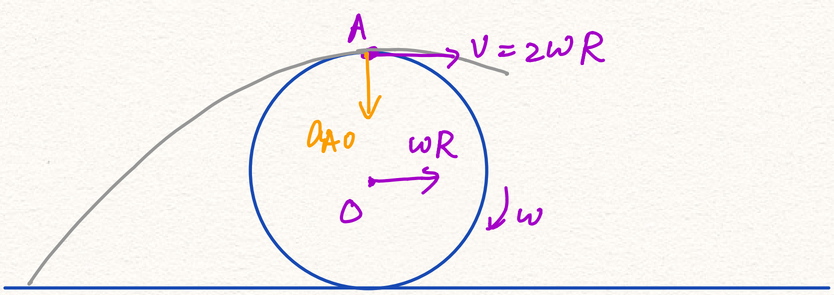
Q3. (more difficult) The same for a pot with cross-section in the shape of a cycloid placed upside-down

$$x = R(\gamma + \sin\gamma), \qquad y = R(1 - \cos\gamma),$$

where $-\pi \le \gamma \le \pi$.



TODO: Solve for the curvature ρ.



$v = \omega R + \omega R = 2\omega R$

$a_o = 0$

$a_{A-O} = \omega^2 R$

So $a_{An} = \omega^2 R$

$\rho = \frac{v^2}{a_{An}} = \frac{4\omega^2 R^2}{\omega^2 R} = 4R$

So $\omega_0{}^2 = \sqrt{\frac{g}{l}} = \sqrt{\frac{g}{4R}}$

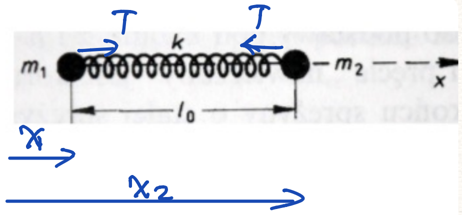
③ Multi-body problem

First choice : ~~Mess center !~~ (Now we don't know that)

Second choice : change of variables in Newton's law.

Problem 6. Show that a one-dimensional system of two point masses m_1 and m_2 connected by a massless spring with spring constant k and equilibrium length l_0 (see the figure) is a harmonic oscillator. Find its natural angular frequency.

(*5 points*)



$$T = k(x_2 - x_1 - l_0)$$

$$\begin{cases} m_1 \ddot{x}_1 = k(x_2 - x_1 - l_0) & ① \\ m_2 \ddot{x}_2 = -k(x_2 - x_1 - l_0) & ② \end{cases}$$

$m_1 \cdot ② - m_2 \cdot ①$: $m_1 m_2 (\ddot{x}_2 - \ddot{x}_1) = k(x_2 - x_1 - l_0)(-m_1 - m_2)$ ③

Set $x_{12} = x_2 - x_1 - l_0$

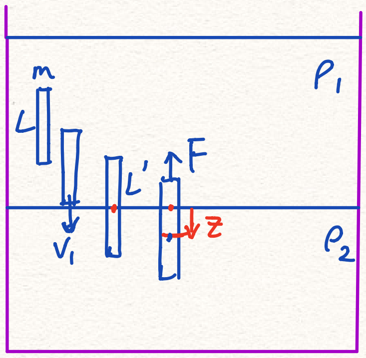
③ : $\frac{m_1 m_2}{m_1 + m_2} \ddot{x}_{12} + k x_{12} = 0$

$$\omega_0^2 = \sqrt{\frac{k}{\frac{m_1 m_2}{m_1 + m_2}}}$$

The simplist "Normal mode" (简正模)

④ Comprehensive time problem.

【练习 8-4】 一个大容器中装有互不相溶的两种液体，它们的密度分别为 ρ_1 和 ρ_2（$\rho_1 < \rho_2$）。现让一长为 L、密度为 $(\rho_1 + \rho_2)/2$ 的均匀木棍，竖直地放在上面的液体内，其下端离两液体分界面的距离为 $3L/4$，由静止开始下落。试计算木棍到达最低处所需的时间。假定由于木棍运动而产生的液体阻力可以忽略不计，且两液体都足够深，保证木棍始终都在液体内部运动，既未露出液面，也未与容器底相碰。



① Uniformly accelerated motion

$$ma_1 = mg - F_u = \frac{1}{2}(\rho_1+\rho_2)LSg - \rho_1 LSg$$

$$a_1 = \frac{ma_1}{m} = \frac{ma_1}{\frac{1}{2}(\rho_1+\rho_2)\cdot SL} = \frac{\rho_2-\rho_1}{\rho_1+\rho_2}g$$

$$\frac{3}{4}L = \frac{1}{2}a_1 t_1^2 \Rightarrow t_1 = \sqrt{\frac{3(\rho_1+\rho_2)L}{2(\rho_2-\rho_1)g}}$$

② Simple Harmonic Oscillation

It's obvious that when $L' = \frac{1}{2}L$, it's equilibrium point

$$F = \frac{1}{2}(\rho_1+\rho_2)LSg - \left[\rho_2\left(\frac{L}{2}+z\right)Sg + \rho_1\left(\frac{L}{2}-z\right)Sg\right]$$

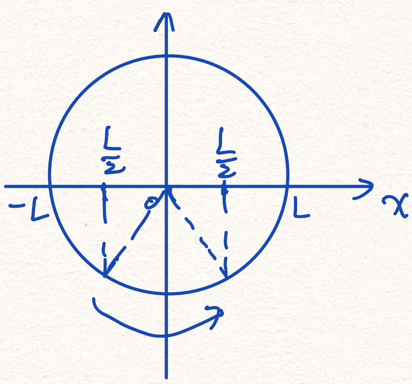
$$= (\rho_2-\rho_1)Sgz$$

$$\frac{1}{2}(\rho_1+\rho_2)LS\ddot{z} + (\rho_2-\rho_1)Sgz = 0$$

$$\omega_0 = \sqrt{\frac{2(\rho_2-\rho_1)}{\rho_2+\rho_1}}, \quad T = 2\pi\sqrt{\frac{(\rho_2+\rho_1)L}{2(\rho_2-\rho_1)g}}$$

$$v_1 = a_1 t_1,$$

$$A^2 = z_0^2 + \frac{v_0^2}{\omega_0^2} \Rightarrow A = L$$



$$t_2 = \frac{1}{6}T = \frac{\pi}{3}\sqrt{\frac{(\rho_2+\rho_1)L}{2(\rho_2-\rho_1)g}}$$

③ Uniformly accelerated motion

$$t_3 = t_1$$

$$t_{total} = t_1 + t_2 + t_3 = \frac{6\sqrt{6}+\sqrt{2}\pi}{6}\sqrt{\frac{(\rho_2+\rho_1)L}{(\rho_2-\rho_1)g}}$$

⑤ Lissajous Curve

Lissajous Curve

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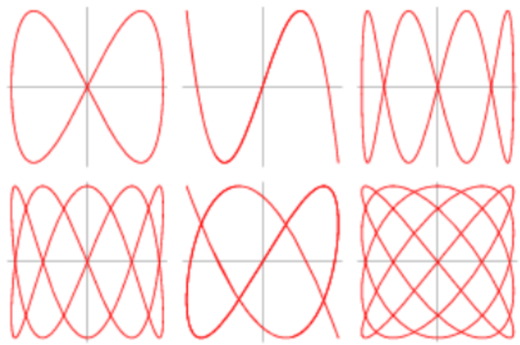

Lissajous curves are the family of curves described by the parametric equations

$$x(t) = A\cos(\omega_x t - \delta_x)$$
$$y(t) = B\cos(\omega_y t - \delta_y),$$

sometimes also written in the form

$$x(t) = a\sin(\omega t + \delta)$$
$$y(t) = b\sin t.$$

They are sometimes known as Bowditch curves after Nathaniel Bowditch, who studied them in 1815. They were studied in more detail (independently) by Jules-Antoine Lissajous in 1857 (MacTutor Archive). Lissajous curves have applications in physics, astronomy, and other sciences. The curves close iff ω_x/ω_y is rational.

IV. More Complex Oscillations.

① Damped Oscillations

$$m\ddot{x} = F = -bv - kx$$

$$\Rightarrow \boxed{\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0}$$

First, we need the theory of differential equations...

Let's look some Vv286 slides

Linear Second-Order ODEs with Constant Coefficients

We will now consider the constant-coefficient equation

$$ay'' + by' + cy = 0, \qquad a, b, c \in \mathbb{R}, \ a \neq 0, \tag{1.11.8}$$

As a system, (1.11.8) has the form

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax. \tag{1.11.9}$$

with $x_1 = y$ and $x_2 = y'$. The eigenvalues of A are determined by

$$\det A = -\lambda(-b/a - \lambda) + c/a = 0$$

or

$$\boxed{a\lambda^2 + b\lambda + c = 0.} \tag{1.11.10}$$

Linear Second-Order ODEs with Constant Coefficients

The roots of (1.11.10) are given by

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.11.11}$$

We will consider two cases:

1. $b^2 \neq 4ac$. Then there are two distinct eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ of A and two corresponding eigenvectors $v_1, v_2 \in \mathbb{R}^2$. We know that the general solution of (1.11.9) will have the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Since $x_1 = y$, $x_2 = y'$, we are actually only interested in x_1 and can write

$$\boxed{y(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},} \qquad c_1, c_2 \in \mathbb{R},$$

for the general solution of (1.11.8).

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Linear Second-Order ODEs with Constant Coefficients

2. $b^2 = 4ac$. Then there is only one eigenvalue $\lambda \in \mathbb{C}$ of

$$A = \begin{pmatrix} 0 & 1 \\ -b^2/(4a^2) & -b/a \end{pmatrix}.$$

It can be shown explicitly that in this case A is not diagonalizable (see exercises). The Jordan form of A has the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so

$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

A general solution of (1.11.8) is then given by

$$\boxed{y(t; c_1, c_2) = c_1 e^{\lambda t} + c_2 te^{\lambda t},} \qquad c_1, c_2 \in \mathbb{R}.$$

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Mechanical and Electrical Vibrations

By applying Newton's laws we then obtain the differential equation

$\gamma = \beta$

$$mu'' + \gamma u' + ku = F(t), \qquad m, \gamma, k \geq 0, \; F(t) \in \mathbb{R}. \qquad (1.11.12)$$

This type of equation can be used to model vibrations in general, not just the specific spring oscillations from which it was derived. If $\boxed{F(t) = 0}$ for all t, we say the vibrations are ***free***, otherwise they are ***forced***. If $\gamma \neq 0$ the vibrations are ***damped***, otherwise ***undamped***.

Clearly, (1.11.12) is an inhomogeneous second-order ODE with constant coefficients, so we can find specific solutions. Mathematically, there is nothing further to say. However, physically, there remains a great deal to analyze, and we will now proceed to look at (1.11.12) from a physical point of view.

Damped Free Vibrations

If $\gamma \neq 0$ the solutions of

$$mu'' + \gamma u' + ku = 0$$

$$\begin{cases} m = m \\ \gamma = \beta \\ k = k = \omega_0^2 m \end{cases}$$

are determined by the eigenvalues,

$$\boxed{\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)}$$

cf. (1.11.11).

Free Vibrations (Overdamping and Critical damping)

$\Leftrightarrow \frac{b^2}{4m^2} > \omega_0^2$

(i) If $\gamma^2 - 4km > 0$ there are two distinct real eigenvalues and the solution is given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad = c_1 e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} + c_2 e^{-\left(\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} \qquad c_1, c_2 \in \mathbb{R}.$$

Note that in this case $1 - 4km/\gamma^2 < 1$ so both eigenvalues are negative. Thus $u(t) \searrow 0$ as $t \to \infty$. This is known as the ***overdamped case***.

$b^2/4m^2 = \omega_0^2$

(ii) If $\gamma^2 - 4km = 0$ there is only the single eigenvalue $\lambda = -\gamma/(2m)$ and the solution is given by

$$u(t) = (c_1 + c_2 t)e^{-\gamma t/(2m)}, \quad = (c_1 + c_2 t)e^{-\frac{bt}{2m}} \qquad c_1, c_2 \in \mathbb{R}.$$

Here, too, $u(t) \to 0$ as $t \to \infty$ without any oscillations. This solution corresponds to the fastest non-oscillating decay to zero. This case is referred to as ***critical damping***.

critical damping: may be passed through the equilibrium position at most once.

Free Vibrations (Underdamping)

$\Leftrightarrow \omega_0^2 > \frac{b^2}{4m^2}$

(iii) If $\gamma^2 - 4km < 0$ both eigenvalues are complex. However, the real part will be negative:

$$\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = -\frac{\gamma}{2m}.$$

The general solution then has the form

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = e^{-\gamma/(2m)t}(c_1 e^{i\mu t} + c_1 e^{-i\mu t}) \\ &= e^{-\gamma/(2m)t}\big(A\cos(\mu t) + B\sin(\mu t)\big) \\ &= Re^{-\gamma/(2m)t}\cos(\mu t - \delta) \end{aligned}$$

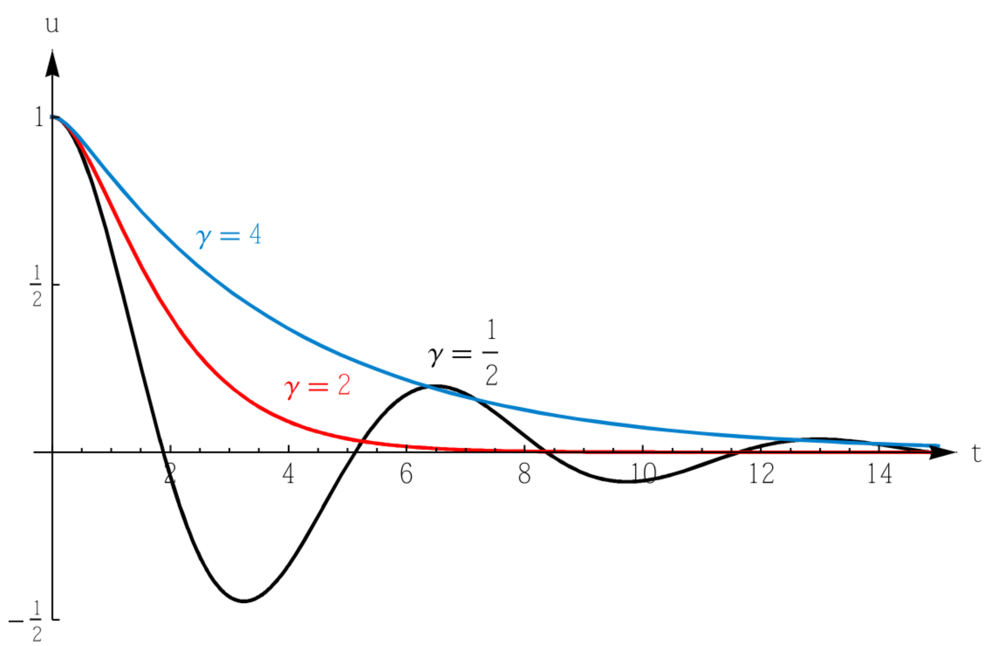
for constants $c_1, c_2, A, B \in \mathbb{R}$, R, δ as before and

$$\mu = |\operatorname{Im}\lambda| = \frac{\sqrt{4km - \gamma^2}}{2m}. = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

The motion corresponds to oscillations with decreasing amplitude. Since it is not periodic, μ is not a true frequency but rather called a ***quasifrequency***. Similarly, $T_d = 2\pi/\mu$ is called the ***quasiperiod***.

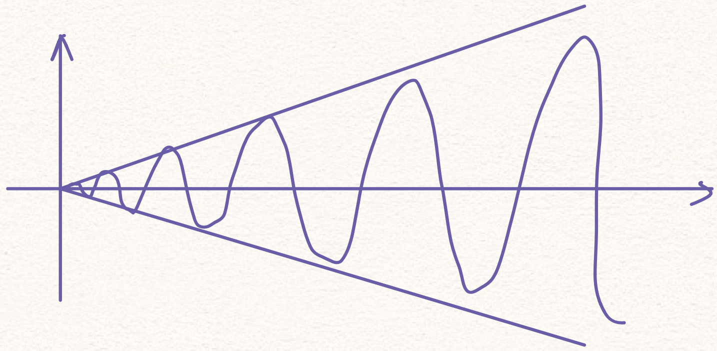
Damped Free Vibrations

For comparison, we plot solutions of the equation $u'' + \gamma u + u = 0$, $u(0) = 1$, $u'(0) = 0$, for different values of γ. Critical damping is achieved for $\gamma = 2$.



② Forced Oscillations

(i) If no damp, the amplitude will increase without bound (For a $F_w = F_0 \cos(\omega_d t)$.



(ii) If there is damp

$$\boxed{\ddot{x} + \frac{b}{m}\dot{x} + \underbrace{\frac{k}{m}}_{(\omega_0^2)} x = \frac{F_0}{m}\cos \omega_d t}$$

Linear Second-Order Equations and Vibrations Slide 277

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Damped Forced Vibrations

We finally consider the full equation $\gamma = \beta$

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

Using our known methods, we can find the general solution to be

$$u(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + R\cos(\omega t - \delta), \qquad \lambda_1 \neq \lambda_2. \tag{1.11.16}$$

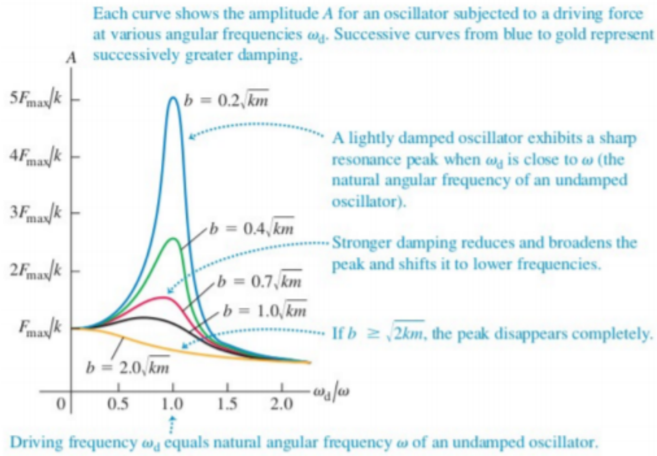
if the eigenvalues associated with the homogeneous equation are distinct, and an analogous expression if there is only one eigenvalue. In any case, the real parts of the eigenvalue(s) will be negative, so the exponential functions will approach zero as $t \to \infty$:

$$\lim_{t\to\infty} \big(u(t; c_1, c_2) - R\cos(\omega t - \delta)\big) = 0$$

Thus the exponential functions are called the ***transient solution*** while $R\cos(\omega t - \delta)$ is the ***steady-state solution*** or ***forced response***.

Discussion: Amplitude. Mechanical Resonance

$$\boxed{A = \frac{F_0}{m\sqrt{\left(\omega_0^2-\omega_{dr}^2\right)^2+\left(\frac{b}{m}\omega_{dr}\right)^2}}}$$



- A peak in the curve $A = A(\omega_{dr})$ at the **resonance frequency**, i.e. for $\omega_{dr} = \omega_{res} = \sqrt{\omega_0^2 - b^2/2m^2}$ [see Problem Set]. The sharp increase in the amplitude of oscillations when $\omega_{dr} \to \omega_{res}$ is called the **mechanical resonance**.
- Increasing damping shifts the resonance frequency downwards.
- For $\omega_{dr} \to 0$ (i.e., $T_{dr} \to \infty$; constant force), then $A \to \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$.

Discussion: Phase Shift

$$\boxed{\tan\phi = \frac{b\omega_{dr}}{m(\omega_{dr}^2 - \omega_0^2)}}$$