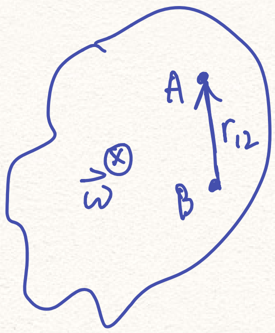
- Math Description For rigid bodies.

$$\begin{cases} \vec{r}_p = \vec{r}_{o'} + \vec{r}' \\ \vec{v}_p = \vec{v}_{o'} + \vec{\omega} \times \vec{r}' \end{cases}$$

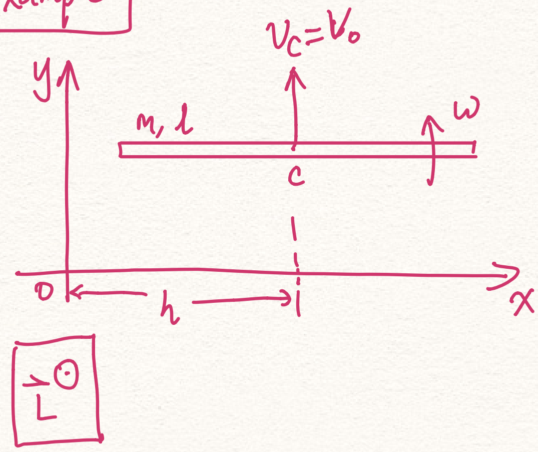


At any time, there is an instantaneous axis of rotation, which satisfy: $\vec{v}_1 - \vec{v}_2 = \vec{\omega} \times \vec{r}_{12}$

☆ Very important:

$$\begin{cases} \boxed{\vec{P} = \vec{P}_c = m\vec{v}_c} \\ \vec{L} = \vec{L}_c + \vec{L}_{in\text{-}c} \\ \quad = m \cdot \vec{r}_c \times \vec{v}_c + \vec{L}_{in\text{-}c} \\ \quad = m \cdot \vec{r}_c \times \vec{v}_c + I_c \cdot \vec{\omega} \end{cases}$$

example



$$\vec{L} = m \cdot \vec{r}_c \times \vec{v}_c + I_c \cdot \vec{\omega}$$

$$= \left(m h v_0 + \frac{1}{12} m l^2 \omega\right) \cdot \hat{n}_z$$

$$\vec{L}_{in\text{-}c} = I_c \cdot \vec{\omega}$$

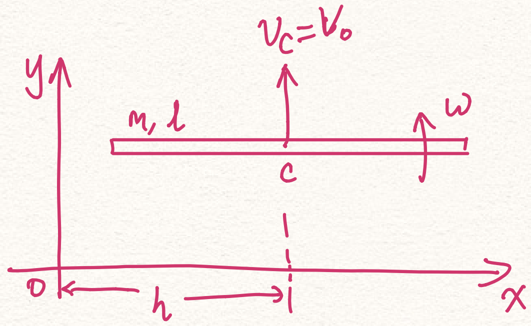
$$I_c = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \qquad \vec{L}_{in\text{-}c} = \begin{pmatrix} I_{xx}\cdot \omega_x \\ I_{yy}\cdot \omega_y \\ I_{zz}\cdot \omega_z \end{pmatrix}$$

△ Rotational Energy

$$K_{\text{rigid body}} = \boxed{K_c} + \boxed{K_{in\text{-}c}}$$

$$= \boxed{\frac{1}{2} m \cdot v_c^2} + \boxed{\frac{1}{2}\cdot I_c \cdot \omega^2}$$

example:



$$K = \frac{1}{2} m \cdot v_0^2 + \frac{1}{2}\cdot I_c \cdot \omega^2$$

$$= \frac{1}{2} m v_0^2 + \frac{1}{2}\cdot \frac{1}{12} m l^2 \cdot \omega^2$$

△ Rotation Law of Kinetic Energy

$$\Delta W = \Delta K \Rightarrow \int \tau \cdot d\theta = \Delta K$$

generalized force

generalized placement

△ Second Law of Dynamics

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d(m\vec{r}_c \times \vec{v}_c + I_c \cdot \vec{\omega})}{dt}$$

$$= \frac{d(I_c \cdot \vec{\omega})}{dt} \xlongequal[\text{axis}]{\text{principle}} I_c \cdot \frac{d\vec{\omega}}{dt}$$

$$\boxed{\vec{\tau} = I_c \cdot \frac{d\vec{\omega}}{dt}}$$

"Law of Rotation"

Understanding :	
I (Moment of inertia)	M (mass)
$\begin{cases} \theta & \text{(angular placement)} \\ \vec{\omega} \\ \vec{\varepsilon} \end{cases}$	$\begin{cases} \vec{x} & \text{(placement)} \\ \vec{v} \\ \vec{a} \end{cases}$
$\vec{\tau}$ (torque)	$\vec{F}$ (force)
$\vec{\tau} = I \cdot \frac{d\vec{\omega}}{dt} = I \cdot \vec{\varepsilon}$	$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$
$K = \frac{1}{2} I\omega^2$ (pure rotation)	$K = \frac{1}{2} m v^2$
$\int \vec{\tau} \cdot d\theta = \Delta K$	$\int \vec{F} \cdot dx = \Delta K$

△ About instantaneous rotation axis

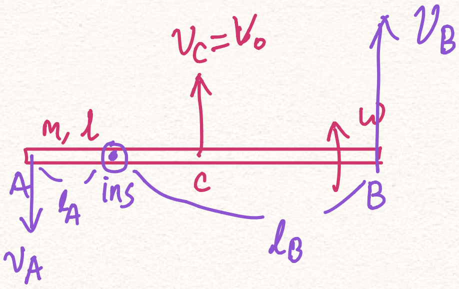
$$\vec{L} = m\vec{r}_c \times \vec{v}_c + I_c \cdot \vec{\omega} = I_{ins} \cdot \vec{\omega}$$

$$K = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \cdot \omega^2 = \frac{1}{2} \cdot I_{ins} \cdot \omega^2$$

☆ However, $\boxed{\vec{\tau} \underset{\text{maybe}}{\neq} I_{ins} \cdot \frac{d\omega}{dt}}$!!!

because I_{ins} itself will change. You must use principal axis.

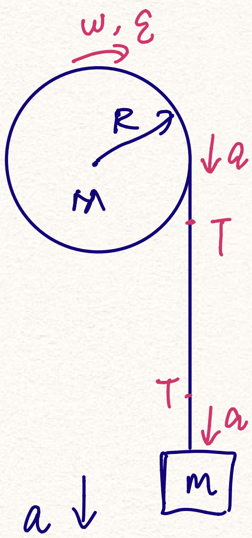
example.



$v_A < v_B$

$$\frac{v_A}{l_A} = \frac{v_B}{l_B} = \omega$$

$\boxed{Q1}$

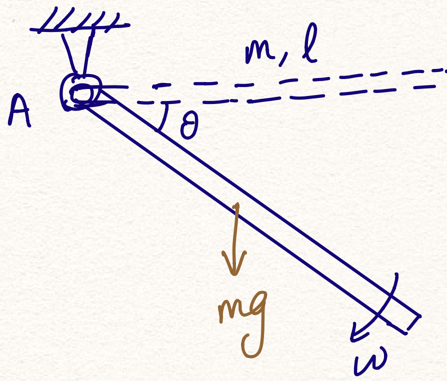


what's the value of a?

$$R \cdot \varepsilon = a \Rightarrow \varepsilon = \frac{a}{R}.$$

$$\begin{cases} T \cdot R = \frac{1}{2} M R^2 \cdot \varepsilon & ① \\ mg - T = ma & ② \\ \varepsilon = \frac{a}{R} & ③ \end{cases}$$

Q2 A is a hinge. Rest from horizontal position.



(1) $\omega(\theta)$?

(2) What the force acted on the rod acted by A?

(1).

$$\frac{1}{2}\cdot I_A\cdot\omega^2 = mg\cdot\frac{l}{2}\cdot\sin\theta$$

$$\cancel{\frac{1}{2}}\cdot\frac{1}{3}\cancel{m}l^2\cdot\omega^2 = \cancel{\frac{1}{2}}\cancel{m}gl\sin\theta \Rightarrow \omega=\sqrt{\frac{3g\sin\theta}{l}}$$

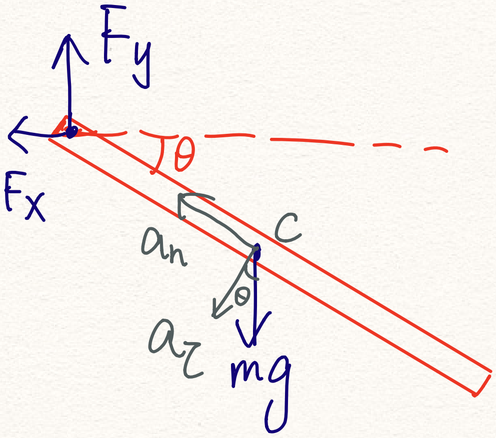
(2).

$$\tau = I_A\cdot\frac{d\omega}{dt}$$

$$mg\cdot\frac{1}{2}l\cos\theta = \frac{1}{3}ml^2\cdot\frac{d\omega}{dt}$$

$$\Rightarrow \frac{d\omega}{dt} = \boxed{\varepsilon = \frac{3g\cos\theta}{2l}}$$

(you can also get this by chain rule, i.e. $\frac{d\omega}{dt}=\frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt}=\frac{d\omega}{d\theta}\cdot\omega$)

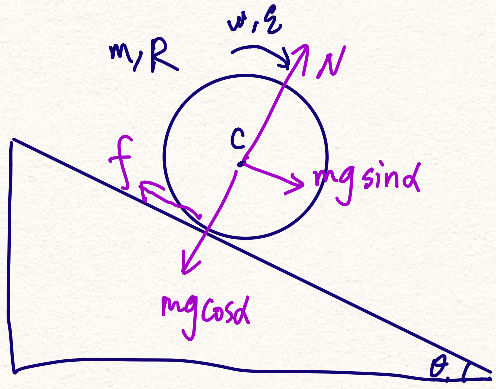


$$a_\tau = \frac{1}{2}l\cdot\varepsilon$$

$$a_n = \omega^2\cdot\frac{1}{2}l$$

$$\begin{cases} F_y - mg = m(a_n\sin\theta - a_\tau\cos\theta) \\ F_x = m(a_n\cos\theta + a_\tau\sin\theta) \end{cases}$$

Q3 Rolling Down an Incline (Ball, Cylinder, or any circle-shaped object)
(With No Slipping)



Assume $\underline{I_c = KmR^2}$

(For ball, $K = \frac{2}{5}$, for cylinder, $K = \frac{1}{2}$, for ring, $K = 1$)

No Sipping : $\underline{f \text{ unknown}}$ $(f \neq \mu N)$

But, a $\underline{\text{constraint}}$: $\begin{cases} v_C = \omega R & ① \\ a_c = \varepsilon R & ② \end{cases}$

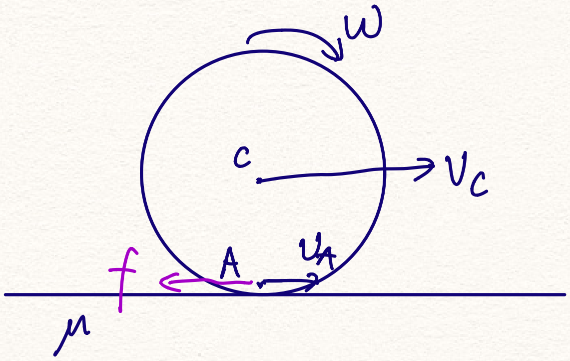
$$\begin{cases} mg\sin\alpha - f = ma_C & ③ \\ (\text{For } C):\ I_C \cdot \varepsilon = f \cdot R & ④ \end{cases}$$

$$\Rightarrow \quad \cancel{m}g\sin\alpha - \frac{K\cancel{m}R^{\cancel{2}}}{\cancel{R}}\varepsilon = \cancel{m}\cancel{a_c}\ \varepsilon R$$

$$(K+1)\varepsilon R = g\sin\alpha$$

$$\boxed{\varepsilon = \frac{g\sin\alpha}{(K+1)R}}$$

04 Rolling with sliding



If $v_C \neq \omega R$, there will be friction.

$$v_A = v_C - \omega R$$

If $v_A > 0$, f towards left.

If $v_A < 0$, f towards right.

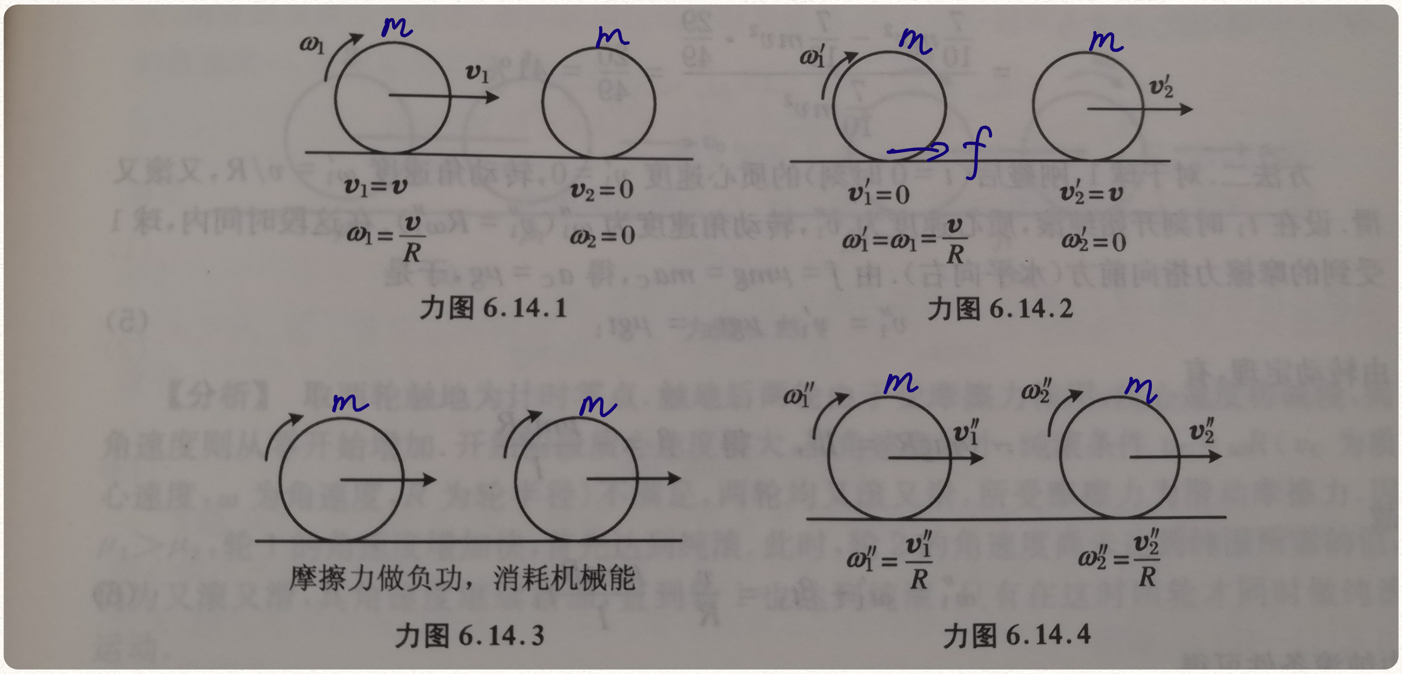
There will be:

(If $v_A > 0$)

$$\begin{cases} f = \mu N = \mu m g \\ f \cdot R = I_C \cdot \dfrac{d\omega}{dt} \\ f = -m \dfrac{dv_C}{dt} \end{cases} \Rightarrow \begin{cases} \dfrac{d\omega}{dt} = ? \\ \dfrac{dv_C}{dt} = ? \end{cases}$$

Until $\omega \cdot R = v_C$, then pure rolling.

Q5



力图 6.14.1 力图 6.14.2 力图 6.14.3 力图 6.14.4

Method 1

For ball 1 :

$$\begin{cases} f_1 = \mu m g = m a_c \Rightarrow a_c = \mu g \\ -\mu m g R = I\beta \Rightarrow \beta = \frac{-\mu m g R}{I} \end{cases}$$

$$\begin{cases} v_1'' = v_1' + \mu g t_1 = \mu g t_1 \\ \omega_1'' = \omega_1' + \beta t_1 = \frac{v}{R} - \frac{\mu m g R}{I} t_1 \end{cases}$$

$$v_1'' = \omega_1'' \cdot R$$

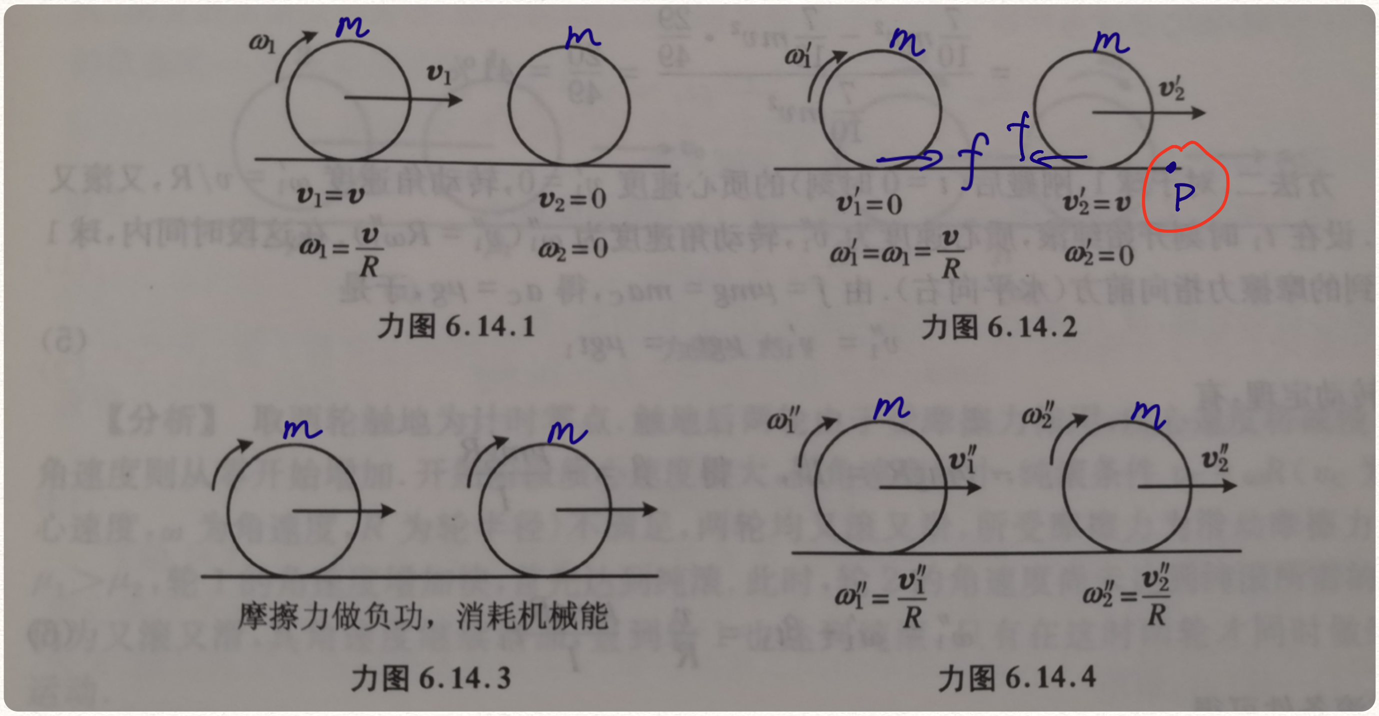
$$\Rightarrow t_1 = \frac{v}{\mu g \left(1 + \frac{mR^2}{I}\right)}, \quad v_1'' = \frac{v}{1 + \frac{mg^2}{I}} = \frac{2}{7} v$$

For ball 2:

$$\begin{cases} a_c = -\mu g \\ \beta = \dfrac{\mu m g R}{I} \end{cases}$$

$$(v - \mu g t_2) = \left(0 + \frac{\mu m g R}{I} t_2\right) \cdot R$$

$$\Rightarrow \quad v_2'' = \frac{5}{7} v.$$



力图 6.14.1

力图 6.14.2

力图 6.14.3

力图 6.14.4

$\boxed{\text{Method 2}}$

take point P as the axis

Then the angular momentum conservation!

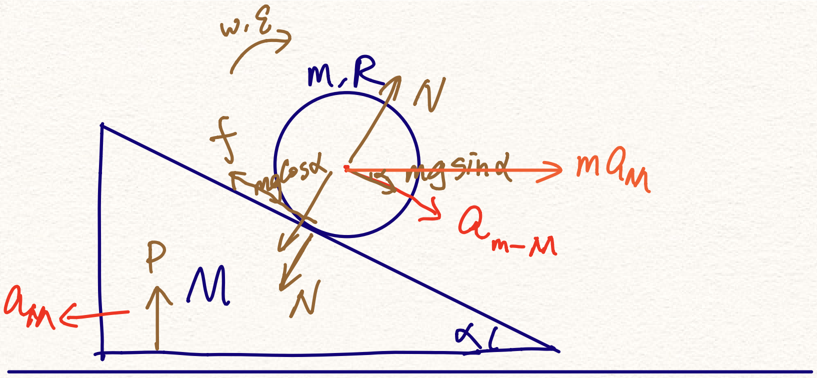
$$L_1' = I\omega_1' + mRv_1' = I\omega_1' = I\frac{v}{R}.$$

$$L_1'' = I\omega_1'' + mRv_1'' = I\frac{v_1''}{R} + mRv_1'' = v_1''\left(\frac{I}{R} + mR\right)$$

$$L_1' = L_1'' \quad \Rightarrow \quad v_1'' = \frac{I}{I + mR^2} v = \frac{2}{7} v.$$

Q6

7. A wedge with mass M and angle α rests on a frictionless horizontal surface. A cylinder with mass m rolls down the wedge without slipping. Find the acceleration of the wedge.



$$\begin{cases} a_{m-M} = R\cdot\varepsilon & (1) \\ f\cdot R = I_c\cdot\varepsilon & (2) \end{cases}$$

$$\begin{cases} mg\cos\alpha - N = a_M \sin\alpha & (3) \\ mg\sin\alpha - f = a_{m-M} - a_M\cos\alpha & (4) \end{cases}$$

$$N\sin\alpha = a_M\cdot M \qquad (5)$$

$$\boxed{a_M = \frac{g\sin 2\alpha}{\frac{3(M-m)}{m} - 2\cos^2\alpha}}$$

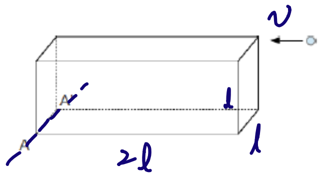
Q7

8. A ball with mass m, moving with in the horizontal direction with speed v, hits the upper edge of a rectangular box with dimensions $l \times l \times 2l$. Assuming that the box can rotate about a fixed axis containing the edge AA', and the collision of the ball with the box is elastic (and the ball moves back in the horizontal direction after the collision), find

   (a) angular velocity the box starts moving with at the moment of collision [answer: $\omega_0 = \frac{2v}{I_{AA'}/ml+l}$],

   (b) equation of motion of the box after the collision [$I_{AA'}\ddot{\alpha} + Mgl\frac{\sqrt{5}}{2}\cos\alpha = 0$],

   (c) the minimum speed of the ball needed to put the box in the upright position [$v = (\frac{I_{AA'}}{ml}+l)\frac{1}{2}\sqrt{\frac{mg}{l}(\sqrt{5}-1)}$].

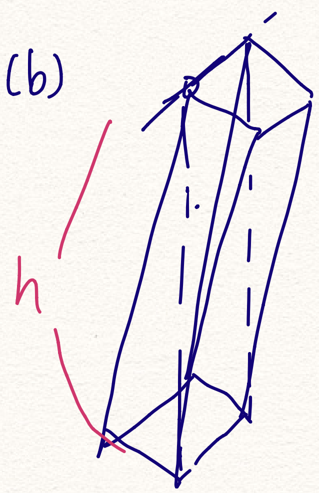


(a) $\vec{L}$ 守恒 : $mvl = mv'l + I_{AA'} \cdot \omega_0$

E 守恒 : $\frac{1}{2} m \cdot v^2 = \frac{1}{2} I_{AA'} \cdot \omega_0^2 + \frac{1}{2} m (v')^2$

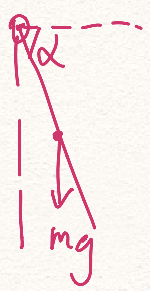
$\rightarrow \omega_0 = \dfrac{2v}{\frac{I_{AA'}}{ml} + l}$

(b)



$h = \sqrt{5}\, l$

$mg \cdot \frac{1}{2} h \cdot \cos\alpha = I_{AA'} \cdot \ddot{\alpha}$



(c) $\frac{1}{2} I_{AA'} \cdot \omega_0^2 = mg\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) l$