I. ODEs in 1D Kinematics

① Separation of Variables. (Definition from Vv286)

Separable Equations　　Slide 34　　JOINT INSTITUTE 交大密西根学院

Separation of Variables - Informally

Suppose that an equation of the form $y' = f(x) \cdot g(y)$ is given, for suitable functions f and g. Then the method of ***separation of variables*** proceeds as follows: from

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{we obtain} \qquad \frac{dy}{g(y)} = f(x)\,dx.$$

Integrating both sides, we have

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

which can be solved for y to obtain a solution.

To satisfy an initial condition $y(\xi) = \eta$ for given $\xi, \eta \in \mathbb{R}$, we insert the appropriate limits in the integrals,

$$\int_{\eta}^{y} \frac{ds}{g(s)} = \int_{\xi}^{x} f(t)\,dt.$$

eg. $a = -kv$, $v(0) = v_0$, Find the $v(t)$.

$$\frac{dv}{dt} = -kv$$

$$\int \frac{dv}{v} = \int -k \cdot dt$$

$$\ln \frac{v}{v_0} = -kt$$

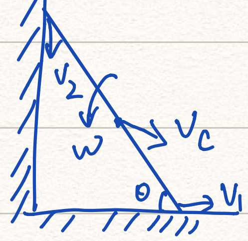
$$v = v_0 e^{-kt}$$

② Use of chain rule

※ $a = \dfrac{dv}{dt} = \dfrac{dv}{dx} \cdot \dfrac{dx}{dt} = \dfrac{dv}{dx} \cdot v$.

- How to use it? ① To avoid difficult 2-order ODEs; ② To simplify calculation.

eg.



We can express v_c using $v_c(\theta)$

$$a_c = \frac{dv_c}{dt} = \frac{dv_c(\theta)}{d\theta} \cdot \frac{d\theta}{dt} = \frac{dv_c(\theta)}{d\theta} \cdot \omega.$$

Q1. In a certain motion of a particle along a straight line, the acceleration turns out to be related to the position of the particle according to the formula $a_x = \sqrt{kx}$, where $k > 0$ is a constant and $x > 0$. How does the velocity depend on x, if we know that for $v_x(x_0) = v_0$?

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx} \cdot v_x = \sqrt{kx}$$

$$\int v_x \, dv_x = \int \sqrt{k} \cdot x^{\frac{1}{2}} \, dx$$

$$\frac{1}{2}(v_x^2 - v_0^2) = \sqrt{k} \cdot \left(x^{\frac{3}{2}} \cdot \frac{2}{3}\right)\Big|_{x_0}^{x}$$

$$= \frac{2}{3}\sqrt{k} \cdot \left(x^{\frac{3}{2}} - x_0^{\frac{3}{2}}\right).$$

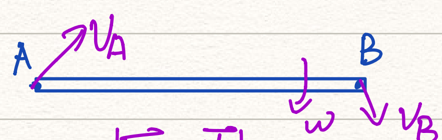
II. Relative velocity.

$$\begin{cases} v_x = v_{O'x} + v_x' \\ a_x = a_{O'x} + a_x' \end{cases}$$

$\boxed{\text{Core Essence} \Longrightarrow \text{Constraint on Kinematics}}$

△ Basic constraints

① Stick / Straight Rope.



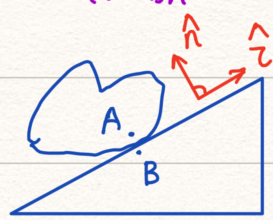
$$(\vec{v}_B - \vec{v}_A) \perp (\vec{r}_A - \vec{r}_B)$$

$$\Leftrightarrow \vec{v}_{relative} \perp \vec{l} \quad \Leftrightarrow v_{An} = v_{Bn}$$

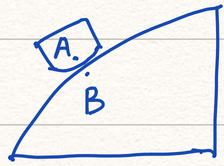
$$\omega = \frac{|\vec{v}_A - \vec{v}_B|}{l}.$$

$$a_{(B-A)n} = \omega^2 r \cdot \hat{n}$$

② Surface contact

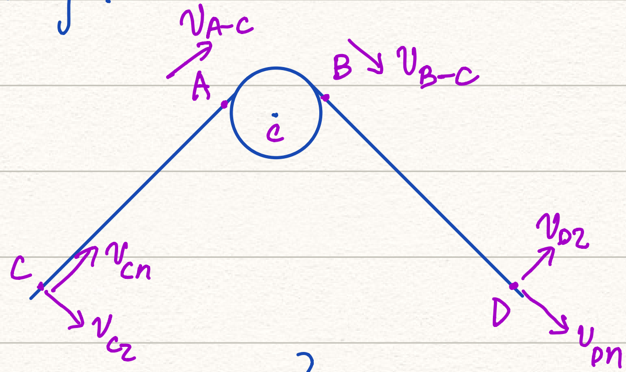


$$\begin{cases} v_{An} = v_{Bn} \\ a_{An} = a_{Bn} \end{cases}$$ (flat surface)



$$\begin{cases} v_{An} = v_{Bn} \\ a_{An} = \dfrac{v_A^2}{\rho} + a_{Bn} \end{cases}$$ (curly surface)

③ pulley.



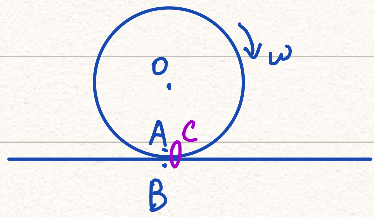
△ take the pulley as frame of reference.

△ $$\begin{cases} v_{A-C} = v_{B-C} \\ a_{A-C} = a_{B-C} \end{cases}$$

However, $v_C \overset{?}{=} v_D$. Because there may be rotation.

$v_{Cn} = v_{Dn}$, $v_{C\tau} \underset{\times}{\overset{?}{=}} v_{D\tau}$ $\quad a_{Cn} \underset{\times}{\overset{?}{=}} a_{Dn}$

④ pure rolling

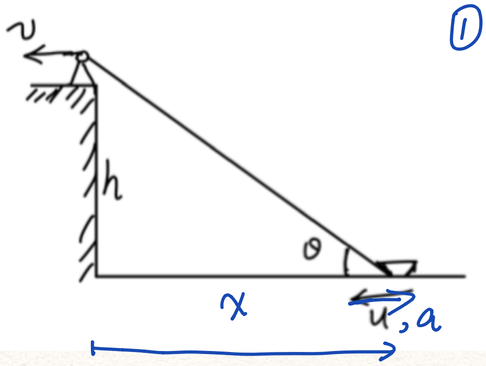


$v_A = v_B = 0$

$v_C \neq 0$

$v_C = v_o = \omega r$

Q2. A person standing on the bank is dragging rope with constant velocity v, when the angle between the rope and the water surface is θ, Find the velocity and the acceleration of the boat.



① Method 1: use x. (Chain rule)

$$x = h/\tan\theta = \frac{h}{\tan\theta}$$

$$u = \dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

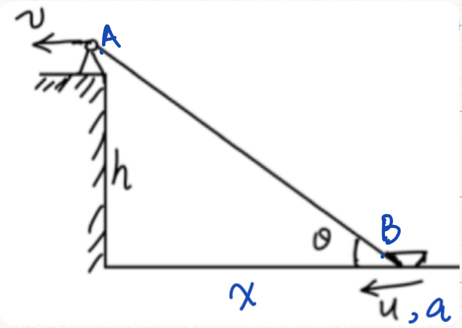
$$\frac{dx}{d\theta} = \left(\frac{h}{\tan\theta}\right)' = \frac{-\frac{h}{\cos^2\theta}}{\tan^2\theta} = \frac{-h}{\sin^2\theta}$$

$$l = \frac{h}{\sin\theta}, \quad -\frac{dl}{dt} = v \iff -\frac{dl}{d\theta} \cdot \frac{d\theta}{dt} = v \iff \frac{+\cos\theta h}{\sin^2\theta} \cdot \frac{d\theta}{dt} = v$$

$$\frac{d\theta}{dt} = \frac{+\sin^2\theta v}{\cos\theta h} = \frac{+\sin\theta\tan\theta v}{h}.$$

So finally $u = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = \frac{-h}{\sin^2\theta} \cdot \frac{+\sin\theta\tan\theta\, v}{h} = \frac{-v}{\cos\theta}$

$$a = \dot{u} = \frac{du}{dt} = \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = \frac{-\sin\theta\, v}{\cos^2\theta} \cdot \frac{+\sin\theta\tan\theta\, v}{h} = \frac{-\tan^3\theta\, v^2}{h}.$$



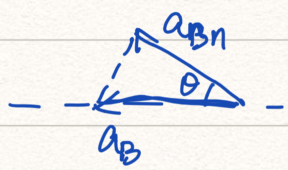
Method 2: $v_{An} = v_{Bn}$

$$v = u\cos\theta \Rightarrow u = \frac{v}{\cos\theta}$$

$$a_{An} = 0 \qquad a_{B-A)n} = \omega^2 \cdot l$$

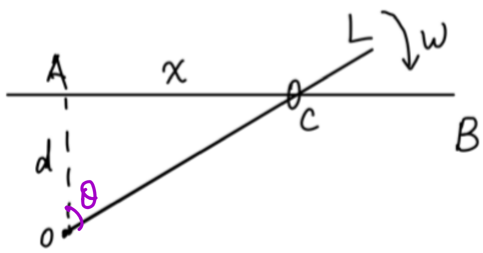
$$\omega = \frac{u\sin\theta}{l} = \frac{v\tan\theta}{h/\sin\theta} = \frac{v\tan\theta\sin\theta}{h}$$

$$a_{(B-A)n} = \omega^2 \cdot l = \frac{v^2\tan^2\theta\sin^2\theta}{h^2} \cdot \frac{h}{\sin\theta} = \frac{v^2\tan^2\theta\sin\theta}{h} = a_{Bn}$$



$$a_B = \frac{a_{Bn}}{\cos\theta} = \frac{v^2\tan^3\theta}{h}.$$

Q3. A rigid stick OL is rotating with constant angular velocity ω Around a fixed point O, and is pushing a ring C to slide on the string AB. Find the velocity and the acceleration of ring C.



$$x = d\tan\theta. \qquad \frac{d\theta}{dt} = \omega.$$

$$v = \dot{x} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= d \cdot \frac{1}{\cos^2\theta} \cdot \omega = \frac{d\omega}{\cos^2\theta} = \frac{d\omega}{\frac{d^2}{d^2+x^2}}$$

$$= \frac{(d^2+x^2)\omega}{d}$$

$$a = \dot{v} = \frac{dv}{d\theta} \cdot \frac{d\theta}{dt} = d\omega \cdot \frac{2\cos\theta\sin\theta}{\cos^4\theta} \cdot \omega = \frac{2d\omega^2\tan\theta}{\cos^2\theta}$$

$$= \frac{2d\omega^2 \cdot \frac{x}{d}}{\frac{d^2}{d^2+x^2}} = \frac{2\omega^2 x(d^2+x^2)}{d^2}$$

III. Kinematics in 2D / 3D.

① Cartesian coordinate system.

Basic : Motion Equation

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t). \end{cases}$$

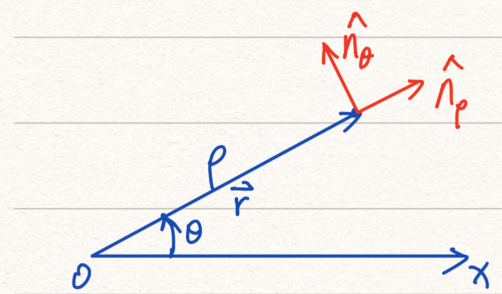
It describes all information of this motion.

$$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix}$$

△ trajectory : $F(x, y, z) = 0$ can be derived from motion equation.

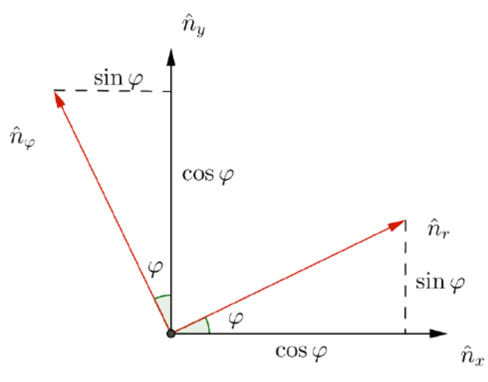
② Polar coordinate system.

$$\vec{r}(\rho, \theta) = \rho \cdot \hat{n}_\rho$$



$$\begin{cases} \hat{n}_\rho = \hat{n}_\rho(\theta) \\ \hat{n}_\theta = \hat{n}_\theta(\theta) \end{cases}$$

<1> **How to find the derivative $\hat{n}_r$ (and $\hat{n}_\varphi$) ?**



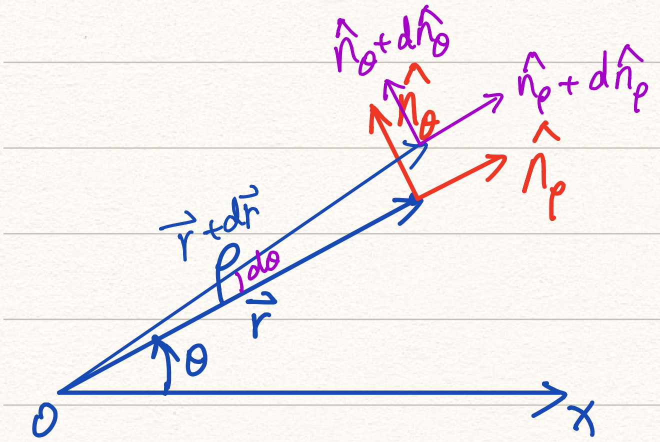
$$\begin{aligned} \hat{n}_r &= \cos\varphi\hat{n}_x + \sin\varphi\hat{n}_y \\ \hat{n}_\varphi &= -\sin\varphi\hat{n}_x + \cos\varphi\hat{n}_y \end{aligned}$$

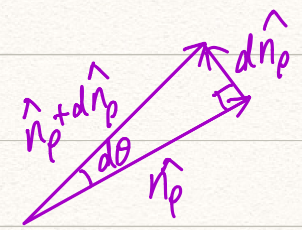
$$\begin{aligned} \boxed{\dot{\hat{n}}_r} &= -\dot{\varphi}\sin\varphi\hat{n}_x + \dot{\varphi}\cos\varphi\hat{n}_y \\ &= \dot{\varphi}(-\sin\varphi\hat{n}_x + \cos\varphi\hat{n}_y) = \\ &= \boxed{\dot{\varphi}\hat{n}_\varphi} \end{aligned}$$

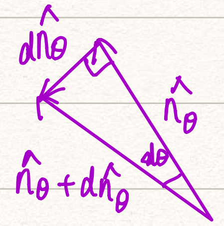
Similarly,

$$\begin{aligned} \boxed{\dot{\hat{n}}_\varphi} &= -\dot{\varphi}\cos\varphi\hat{n}_x - \dot{\varphi}\sin\varphi\hat{n}_y \\ &= -\dot{\varphi}(\cos\varphi\hat{n}_x + \sin\varphi\hat{n}_y) = \\ &= \boxed{-\dot{\varphi}\hat{n}_r} \end{aligned}$$

<2> Another Method (infinitesimal method)





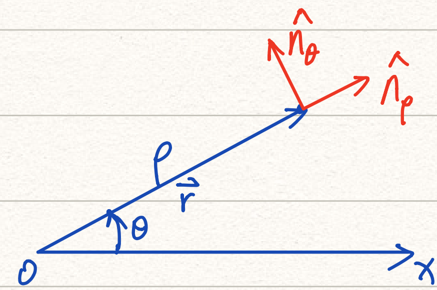


$$d\hat{n}_\rho = 1 \cdot d\theta \cdot \hat{n}_\theta$$

$$\frac{d\hat{n}_\rho}{dt} = \frac{d\theta}{dt} \hat{n}_\theta = \dot{\theta} \hat{n}_\theta$$

$$d\hat{n}_\theta = 1 \cdot d\theta \cdot - \hat{n}_\rho$$

$$\frac{d\hat{n}_\theta}{dt} = - \dot{\theta} \cdot \hat{n}_\rho$$



$$\vec{r} = \rho \cdot \hat{n}_\rho$$

$$\vec{v} = \dot{\vec{r}} = \dot{\rho} \cdot \hat{n}_\rho + \rho \cdot \dot{\hat{n}}_\rho$$

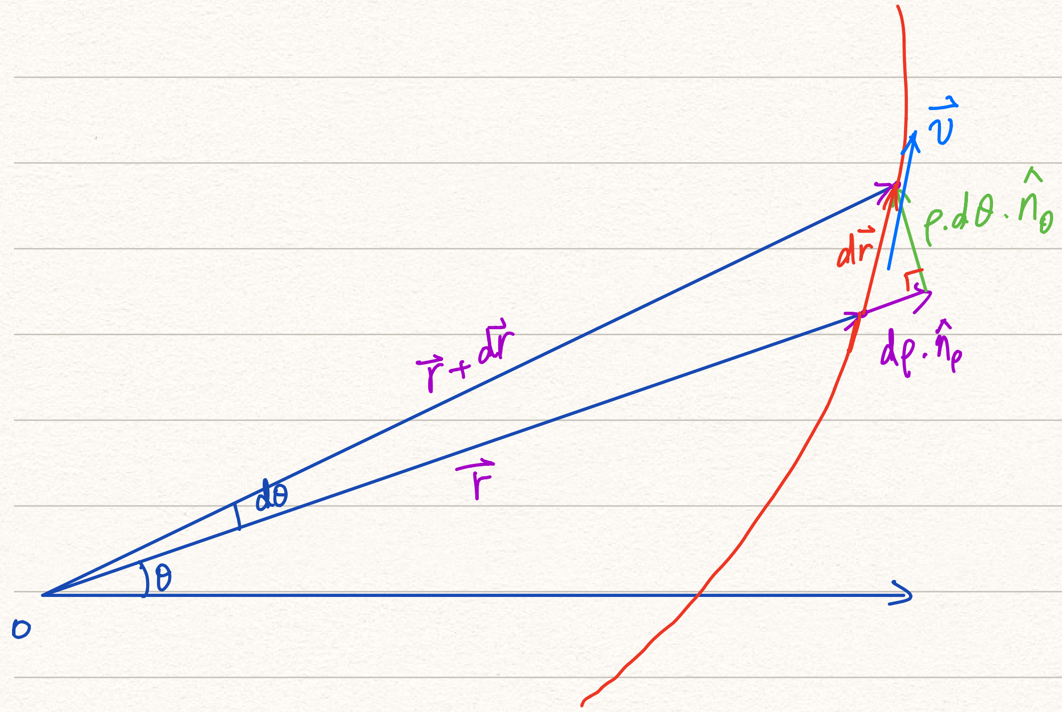
$$= \dot{\rho} \cdot \hat{n}_\rho + \rho \cdot \dot{\theta} \cdot \hat{n}_\theta$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\rho} \cdot \hat{n}_\rho + \dot{\rho} \cdot \dot{\hat{n}}_\rho + \dot{\rho}\dot{\theta}\hat{n}_\theta + \rho\ddot{\theta}\hat{n}_\theta + \rho\dot{\theta}\dot{\hat{n}}_\theta$$

$$= \ddot{\rho} \cdot \hat{n}_\rho + \dot{\rho}\dot{\theta}\hat{n}_\theta + \dot{\rho}\dot{\theta}\hat{n}_\theta + \rho\ddot{\theta}\hat{n}_\theta - \rho\dot{\theta}^2 \hat{n}_\rho$$

$$= (\ddot{\rho} - \rho\dot{\theta}^2)\hat{n}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{n}_\theta$$

$$\left\{ \begin{array}{l} \ddot{\rho} \quad \hat{n}_\rho \\ -\rho\dot{\theta}^2 \quad \hat{n}_\rho \\ \rho\ddot{\theta} \quad \hat{n}_\theta \\ 2\dot{\rho}\dot{\theta} \quad \hat{n}_\theta \end{array} \right.$$



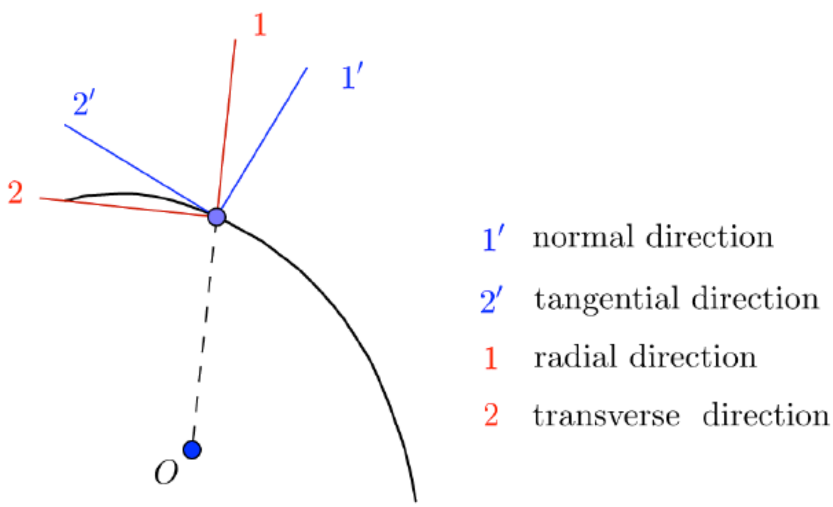
$$(ds)^2 = (d\rho)^2 + (\rho d\theta)^2$$

$$\frac{ds}{dt} = \sqrt{(\dot{\rho})^2 + (\rho\dot{\theta})^2}$$

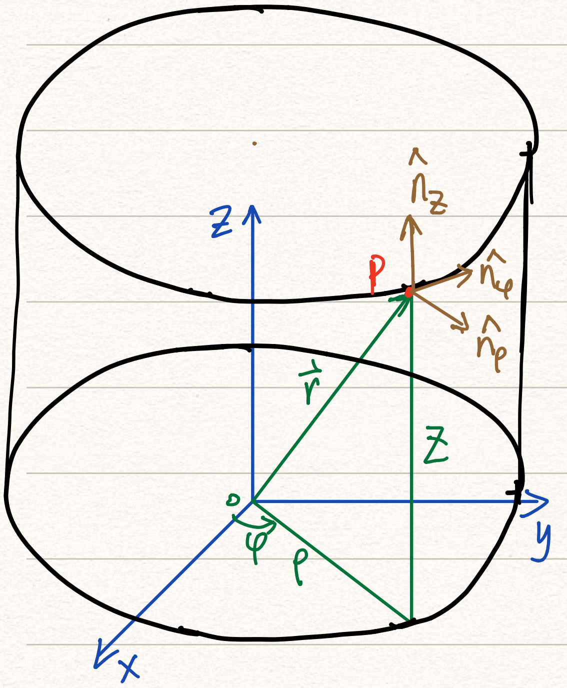
$$= |\vec{v}|^2$$

CAUTION!

In general, radial $\neq$ normal, nor transverse $\neq$ tangential!



③ Cylindrical Coordinates



Cylindrical Coordinate System (ρ, φ, z)

$$\vec{r}(\rho, \varphi, z) = \rho \cdot \hat{n}_\rho + z \cdot \hat{n}_z$$

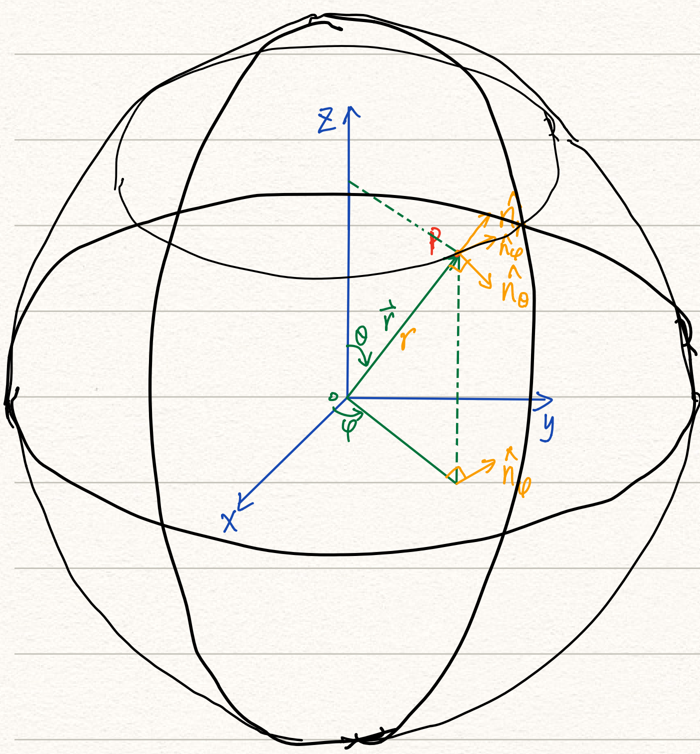
$$\begin{cases} \dot{\hat{n}}_\rho = \dot{\varphi}\, \hat{n}_\varphi \\ \dot{\hat{n}}_\varphi = -\dot{\varphi} \cdot \hat{n}_\rho \end{cases}$$

$$\vec{v}(t) = \frac{dr}{dt} = \dot{\rho}\hat{n}_\rho + \rho\, \dot{\hat{n}}_\rho + \dot{z} \cdot \hat{n}_z$$
$$= \dot{\rho}\hat{n}_\rho + \rho\dot{\varphi}\, \hat{n}_\varphi + \dot{z} \cdot \hat{n}_z$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{n}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{n}_\varphi + \ddot{z} \cdot \hat{n}_z$$

④ Spherical Coordinate System.

$$\vec{r}(r, \theta, \varphi) = r \cdot \hat{n}_r$$



with $\begin{cases} \hat{n}_r = \hat{n}_r(\theta, \varphi) \\ \hat{n}_\theta = \hat{n}_\theta(\theta, \varphi) \\ \hat{n}_\varphi = \hat{n}_\varphi(\theta, \varphi) \end{cases}$

$$\begin{cases} \dfrac{d\hat{n}_r}{dt} = \dfrac{\partial \hat{n}_r}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial t} + \dfrac{\partial \hat{n}_r}{\partial \varphi} \cdot \dfrac{\partial \varphi}{\partial t} \\ \qquad = \dot{\theta}\, \hat{n}_\theta + \dot{\varphi} \sin\theta\, \hat{n}_\varphi \\ \dfrac{d\hat{n}_\varphi}{dt} = -\dot{\varphi}\sin\theta\, \hat{n}_r - \dot{\varphi}\cos\theta\, \hat{n}_\theta \\ \dfrac{d\hat{n}_\theta}{dt} = -\dot{\theta}\, \hat{n}_r + \dot{\varphi}\cos\theta\, \hat{n}_\varphi \end{cases}$$

$$
\begin{cases}
\vec{v}(t) = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta + r\dot{\varphi}\sin\theta\,\hat{n}_\varphi \\
\vec{a}(t) = \hat{n}_r(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2) \\
\qquad + \hat{n}_\theta(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2) \\
\qquad + \hat{n}_\varphi(\sin\theta\,\dot{r}\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} + r\sin\theta\,\ddot{\varphi})
\end{cases}
$$

(B) **Non-Uniform Circular Motion**

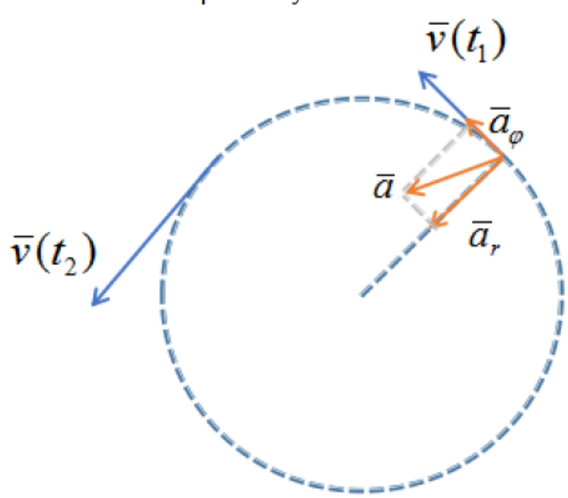
Still $r = R = \text{const}$, but now $\varphi = \varphi(t)$ is an arbitrary function of time.

$$
\begin{aligned}
\dot{\varphi} &= \dot{\varphi}(t) = \omega(t) && \text{angular velocity} \\
\ddot{\varphi} &= \ddot{\varphi}(t) = \dot{\omega}(t) = \varepsilon(t) && \text{angular acceleration}
\end{aligned}
$$

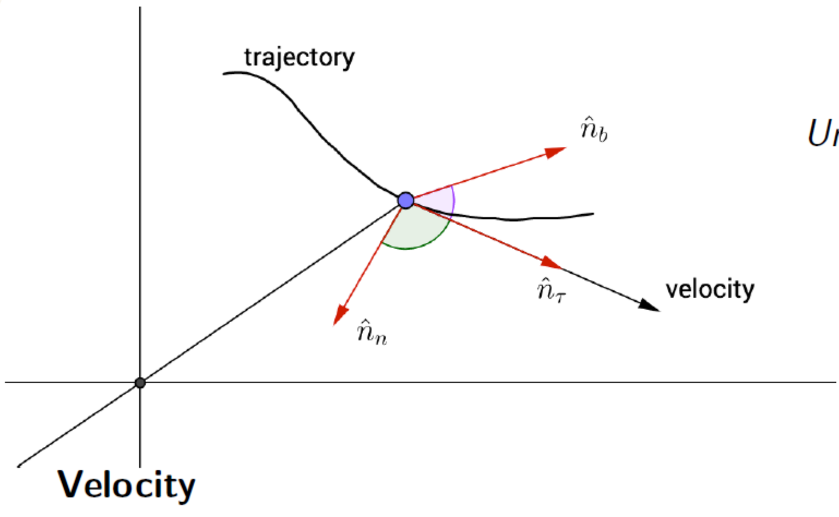
Note. Angular acceleration is in general defined as a vector quantity.

$$\overline{v} = R\omega(t)\hat{n}_\varphi$$

$$\overline{a} = \underbrace{-R\omega^2(t)\hat{n}_r}_{\text{curves the trajectory}} + \underbrace{R\varepsilon(t)\hat{n}_\varphi}_{\text{changes the speed}}$$



Natural Coordinate System.



Velocity

Unit vectors

$\hat{n}_\tau$ — tagential ($\hat{n}_\tau || \overline{v}$)
$\hat{n}_n$ — normal
$\hat{n}_b$ — binormal

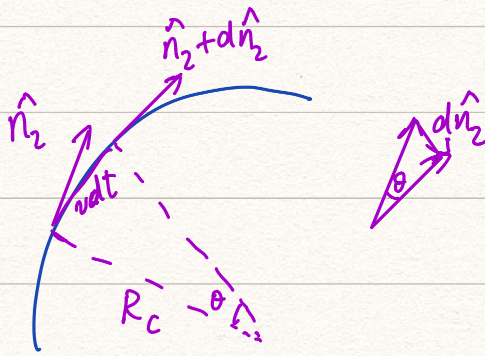
Definition : $\vec{v}(t) = v\hat{n}_\tau$

$$\hat{n}_\tau = \frac{\vec{v}}{|v|} = \frac{\vec{\dot{r}}}{|\dot{r}|}$$

$$\hat{n}_n = \frac{\frac{d\hat{n}_\tau}{dt}}{\left|\frac{d\hat{n}_\tau}{dt}\right|} \qquad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

△ Curvature

$$R_c \overset{def}{=} \frac{v}{\left|\frac{d\hat{n}_\tau}{dt}\right|}$$



$$R_c = \frac{v\,dt}{\theta} = \frac{v}{\frac{\theta}{dt}} = \frac{v}{\frac{|d\hat{n}_\tau|}{dt}}$$

$$R_c = \frac{v^2}{a_n}$$

Q4. A particle moves along a hyperbolic spiral (i.e. a curve $r = c/\varphi$, where c is a positive constant), so that $\varphi(t) = \varphi_0 + \omega t$, where φ_0 and ω are positive constants. Find its velocity and acceleration (all components and magnitudes of both vectors).

$$\varphi(t) = \varphi_0 + \omega t \qquad \dot{\varphi} = \omega. \qquad \ddot{\varphi} = 0$$

$$r(t) = \frac{c}{\varphi} = \frac{c}{\varphi_0 + \omega t} \qquad \dot{r} = \frac{-\omega c}{(\varphi_0+\omega t)^2} \qquad \ddot{r} = \frac{+2(\varphi_0+\omega t)\cdot \omega^2 c}{(\varphi_0+\omega t)^4}$$

$$= \frac{2\omega^2 c}{(\varphi_0+\omega t)^3}$$

$$\vec{v}(t) = \dot{r}\hat{n}_r + r\dot{\varphi}\cdot\hat{n}_\varphi$$

$$= \frac{-\omega c}{(\varphi_0+\omega t)^2}\cdot\hat{n}_r + \frac{c\omega}{\varphi_0+\omega t}\cdot\hat{n}_\varphi$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$$

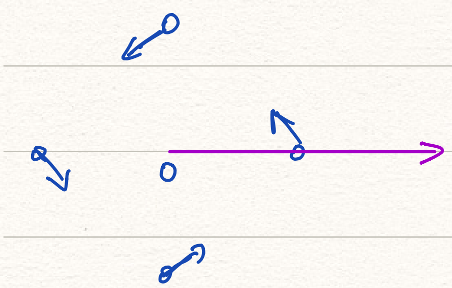
$$= \left(\frac{2c\omega^2}{(\varphi_0+\omega t)} - \frac{c\omega^2}{\varphi_0+\omega t}\right)\hat{n}_r + \left(0 + \frac{-2\omega^2 c}{(\varphi_0+\omega t)^2}\right)\hat{n}_\varphi$$

Q5. Four spiders are initially placed at the four corners of a square with side length a. The spiders crawl counter-clockwise at the same speed and each spider crawls directly toward the next spider at all times. They approach the center of the square along spiral paths. Find

(a) polar coordinates of a spider at any instant of time, assuming the origin is at the center of the square.

(b) the time after which all spiders meet,

(c) the trajectory of a spider in polar coordinates.



(a)
$$\begin{cases} \dot{r} = v_r = -\frac{\sqrt{2}}{2}v_0 & ① \\ r\dot{\varphi} = v_\theta = \frac{\sqrt{2}}{2}v_0 & ② \end{cases}$$

$$①: \quad r = \int \dot{r}\,dt = \int_0^t \left(-\frac{\sqrt{2}}{2}v_0\right)dt = r_0 - \frac{\sqrt{2}}{2}v_0 t$$

$$r(t) = \frac{\sqrt{2}}{2}(a - v_0 t) \qquad ③$$

③ → ②
$$\dot{\varphi} = \frac{\frac{\sqrt{2}}{2}v_0}{\frac{\sqrt{2}}{2}(a - v_0 t)} = \frac{v_0}{a - v_0 t}$$

$$\frac{d\varphi}{dt} = \frac{v_0}{a - v_0 t} \qquad \underline{d\varphi = \frac{v_0}{a - v_0 t}\,dt} \qquad ④$$

$$\varphi - 0 = \int_a^{a-v_0 t} \frac{v_0}{a - v_0 t}\, d(a - v_0 t) \cdot \frac{-1}{v_0}$$

$$= -\ln \frac{a - v_0 t}{a} = \ln \frac{v_0 t - a}{a}$$

$$\varphi(t) = \ln\left(\frac{v_0 t - a}{a}\right)$$

(c) $\varphi = -\ln \frac{\sqrt{2} r}{a}$ $\qquad \ln \frac{\sqrt{2} r}{a} = -\varphi$

$$\underline{r = \frac{a}{\sqrt{2}} e^{-\varphi}}$$

Q6. A particle moves in the x–y plane so that

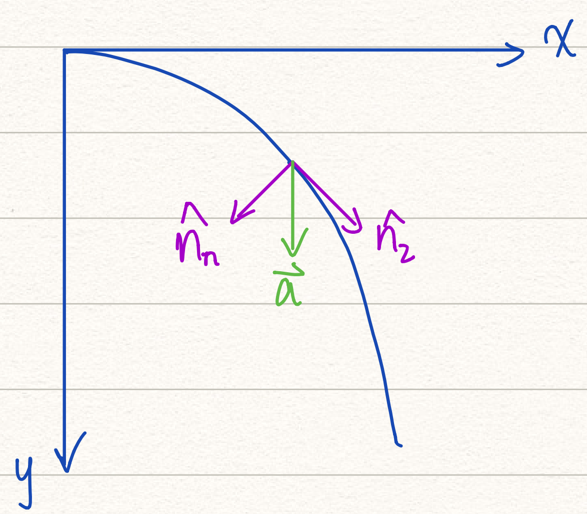
$$x(t) = at, \qquad y(t) = bt^2,$$

where a, b are positive constants. Find its trajectory, velocity, and acceleration (its tangential and normal components).

trajectory: $t = \frac{x}{a} \rightarrow y = b \cdot \frac{x^2}{a^2} = \frac{b}{a^2} x^2$

$v_x = a, \quad v_y = 2bt$

$a_x = 0 \quad a_y = 2b$



$$\vec{v} = \begin{pmatrix} a \\ 2bt \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} 0 \\ 2b \end{pmatrix}$$

$$\hat{n}_\tau = \frac{\vec{v}}{|v|} = \begin{pmatrix} a/\sqrt{a^2+4b^2t^2} \\ 2bt/\sqrt{a^2+4b^2t^2} \end{pmatrix}$$

$$\hat{n}_n = \begin{pmatrix} -2bt/\sqrt{a^2+4b^2t^2} \\ a/\sqrt{a^2+4b^2t^2} \end{pmatrix}$$

$a_\tau = \vec{a} \cdot \hat{n}_\tau \qquad a_n = \vec{a} \cdot \hat{n}_n \qquad R_c = \frac{v^2}{a_n}$

Q7. (*solution provided*) A disc of radius R rotates about its axis of symmetry (perpendicular to the disk surface) with constant angular velocity $\dot{\varphi} = \omega =$ const. At the instant of time $t = 0$ a beetle starts to walk with constant speed v_0 along a radius of the disk, from its center to the edge. Find

(a) the position of the beetle and its trajectory in the Cartesian and polar coordinate systems,

(b) its velocity both systems,

(c) its acceleration in both systems (Cartesian components, polar components, as well as tangential and normal components).

(d) What is the distance covered by the beetle (write down the integral only, do not evaluate it)?

(e) What is the curvature of the trajectory?

(a) $\begin{cases} \varphi(t) = \omega t \\ r(t) = vt \end{cases} \Rightarrow r = \frac{v_0}{\omega}\varphi$ $\qquad \begin{cases} x = v_0 t \cos\omega t \\ y = v_0 t \sin\omega t \end{cases} \rightarrow \underline{x^2 + y^2 = v_0^2 t^2}$

(b) $\vec{v} = v_0 \cdot \hat{n}_r + v_0 \omega t \cdot \hat{n}_\varphi$

$= (v_0 \cos\omega t - v_0 t\omega \sin\omega t)\hat{n}_x + (v_0 \sin\omega t + v_0 t\omega \cos\omega t)\hat{n}_y$

(c) $\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi$

$= -(v_0 t \cdot \omega^2) \cdot \hat{n}_r + 2 v_0 \omega \hat{n}_\varphi$

(d) $(ds)^2 = (dr)^2 + (r d\varphi)^2$

$= (v_0 dt)^2 + (v_0 \omega t\, dt)^2$

$= (v_0^2 + v_0^2 \omega^2 t^2)(dt)^2$

$ds = v_0 \sqrt{1 + \omega^2 t^2}\, dt$

$s = \int_0^t ds = \int_0^t v_0 \sqrt{1 + \omega^2 t^2}\, dt$

(e) In polar system: $(\hat{r}, \hat{\varphi})$

$\hat{n}_z = \begin{pmatrix} 1/\sqrt{1+\omega^2 t^2} \\ \omega t/\sqrt{1+\omega^2 t^2} \end{pmatrix}$ $\qquad \hat{n}_n = \begin{pmatrix} -\omega t/\sqrt{1+\omega^2 t^2} \\ 1/\sqrt{1+\omega^2 t^2} \end{pmatrix}$

$$a_n = \vec{a} \cdot \hat{n}_n$$

$$= \frac{v_0 t\omega^2 \cdot \omega t}{\sqrt{1+\omega^2 t^2}} + \frac{2v_0 \omega}{\sqrt{1+\omega^2 t^2}} = \frac{v_0 \omega (2+\omega^2 t^2)}{\sqrt{1+\omega^2 t^2}}$$

$$R_c = \frac{v^2}{a_n} = \frac{v_0^2 (1+\omega^2 t^2)}{a_n}.$$