I. Force and Work

$$\delta W = \vec{F} \cdot d\vec{r} \quad , \quad W_{AB} = \int_{\Gamma_{AB}} \vec{F} \cdot d\vec{r}$$

- Force Field (Const)

$$\vec{F} = \vec{F}(x,y,z) = \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix}$$

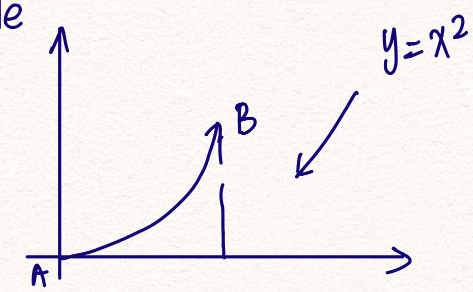
$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \cdot d\vec{r} = \int_{\Gamma_{AB}} \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Generally, we get a curve $\overset{\frown}{AB}$: $F(x,y,z)=0$

we can change it to a parametric equation : $\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}$

Then, $W_{AB} = \int_{\Gamma_{AB}} \vec{F} \cdot d\vec{r} = \int_{t_A}^{t_B} \begin{pmatrix} F_x(x(t), y(t), z(t)) \\ F_y(x(t), y(t), z(t)) \\ F_z(x(t), y(t), z(t)) \end{pmatrix} \cdot \begin{pmatrix} dx(dt) \\ dy(dt) \\ dz(dt) \end{pmatrix}$.

Example



$$\vec{F} = (x^2+y^2)\hat{n}_x + x \cdot \hat{n}_y$$

Let $\begin{cases} y = t^2 \\ x = t \end{cases}$ $t \in (0,1)$ $\begin{cases} dy = 2t\,dt \\ dx = dt \end{cases}$

$$W = \int_{\Gamma_{AB}} \begin{pmatrix} x^2+y^2 \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_0^1 \begin{pmatrix} t^2+t^4 \\ t \end{pmatrix} \cdot \begin{pmatrix} dt \\ 2t\,dt \end{pmatrix}$$

$$= \int_0^1 (t^2+t^4)+(2t^2) \cdot dt = \frac{18}{15} \; (J)$$

- Kinetic energy

$$K = \frac{1}{2} m v^2$$

- Work - Kinetic Energy Theorem

$$\underbrace{W = \Delta K}$$

- Power $P = \dfrac{\Delta W}{\Delta t}$

II. Conservative Forces and Potential Energy

- Path- Independent $\longleftrightarrow$ Conservative Force

eg. $\vec{G}$, elastic Force.

If $\vec{F}$ is conservative $\Rightarrow$ ΔW_{AB} = Constant

- Relation

$$\boxed{\vec{F} = -\nabla U = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z}\right)}$$

$\hookrightarrow$ $\mathrm{rot}\,\vec{F} = \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Example: $\vec{F}(\vec{r}) = \begin{pmatrix} 6xyz^2 + 3y^2 \\ 3x^2z^2 + 6xy - z \\ 6x^2yz - y \end{pmatrix}$

① $\nabla \times \vec{F} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} 6x^2z - 1 - (6x^2z - 1) \\ -(12yzx - 12xyz) \\ 6xz^2 + 6y - (6xz^2 + 6y) \end{pmatrix} = \vec{0}$

So it's conservative.

② Then what's the U?

$\frac{\partial u}{\partial x} = -F_x = -(6xyz^2 + 3y^2) \Rightarrow u = -3x^2yz^2 - 3xy^2 + C_1(y,z)$

$\frac{\partial u}{\partial y} = -F_y = z - 3x^2z^2 - 6xy \Rightarrow u = yz - 3x^2yz^2 - 3xy^2 + C_2(z)$

$C_1(y,z) = yz + C(z)$

$\frac{\partial u}{\partial z} = -F_z = y - 6x^2yz \Rightarrow u = yz - 3x^2yz^2 + C(x,y)$

So $U = yz - 3x^2yz^2 - 3xy^2 + C$

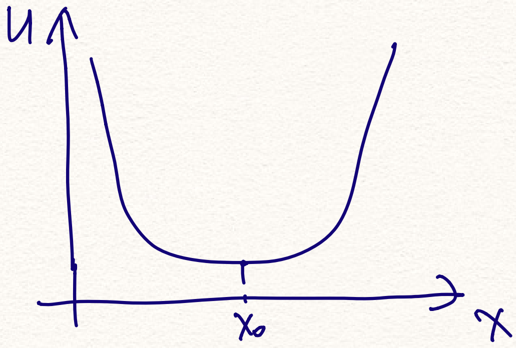
II. Work laws

① $W_{n\text{-cons}} = \Delta E$

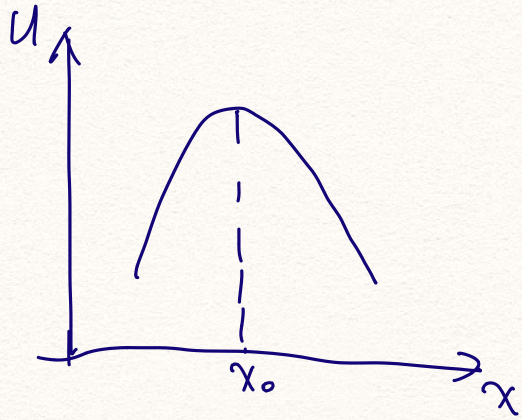
② When $F_{n\text{-cons}} = 0$, $E = K + U = \text{Const}$.

③ $W_{n\text{-cons}} = -\Delta U_{int} \Rightarrow \Delta U + \Delta K + \Delta U_{int} = 0$

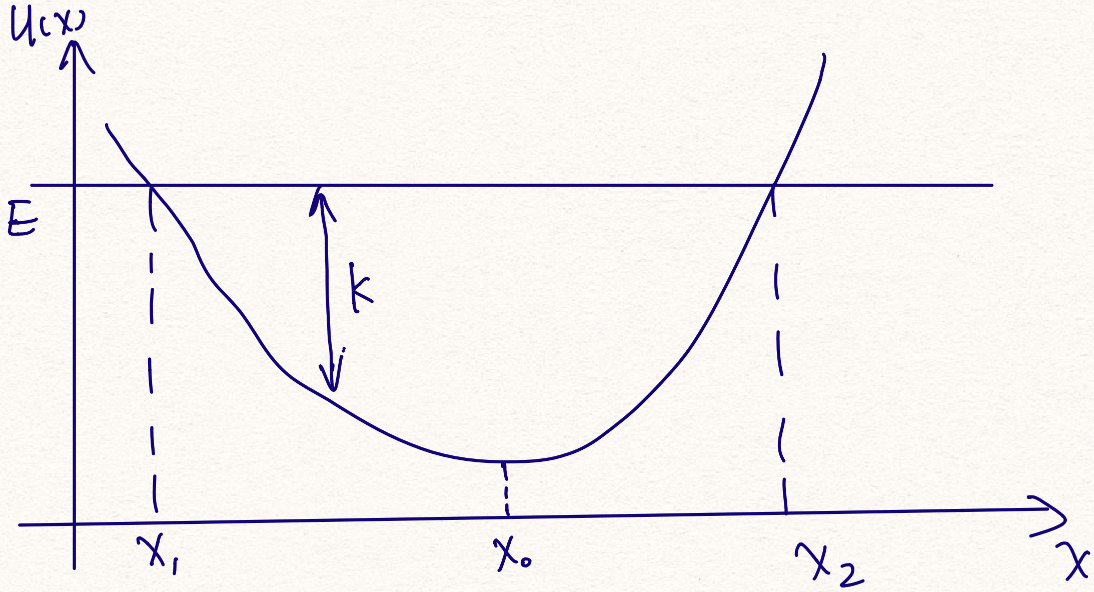
- Diagrams



stable equilibrium



unstable equilibrium



Ⅲ. With Oscillation

【练习 8-11】 单自由度系统谐振动频率的确定。

对于单自由度系统，其运动参量选为 q（通常也称广义坐标，它代表通常采用的单自由度变量 x、θ 等），其运动方程写为

$$q = q(t)$$

如果系统只受保守力作用，且动能和势能分别表示为

$$E_k = \frac{1}{2}f(q)\dot{q}^2, \quad V = V(q)$$

而且存在稳定平衡位置 $q = q_0$，则系统在此稳定平衡位置附近作微振动的角频率为

$$\omega = \sqrt{\frac{V''(q_0)}{f(q_0)}}$$

试给出证明。

$$\begin{cases} f(q) = f(q_0) + \frac{1}{1!} f'(q_0)(q-q_0) + \frac{1}{2!} f''(q_0)\cdot(q-q_0)^2 + \cdots \\ V(q) = V(q_0) + \frac{1}{1!} V'(q_0)(q-q_0) + \frac{1}{2!} V''(q_0)(q-q_0)^2 + \cdots \end{cases}$$

$\dot{q} = o(1) \quad (\dot{q})^2 = o(2)$. So $f(q) \approx f(q_0)$

$$V(q) = V(q_0) + V'(q_0)(q-q_0) + \frac{1}{2} V''(q_0)(q-q_0)^2$$

$$\Rightarrow \frac{1}{2} f(q_0)\dot{q}^2 + V(q_0) + \frac{1}{2} V''(q_0)(q-q_0)^2 = E$$

derivative $\rightarrow$ $\quad f(q_0)\ddot{q} + V''(q_0)\cdot(q-q_0) = 0$

Let $x = q - q_0$, $\quad f(q_0)\cdot\ddot{x} + V''(q_0)\cdot x = 0$

$$w = \sqrt{\frac{V''(q_0)}{f(q_0)}}$$