Unit Prefixes and Conversion

Add a prefix to the given unit to measure in a different scale.

p	n	μ	m	c	k	M
10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^{3}	10^{6}

The procedure of a unit conversion is as follows:

$$1000\mathrm{m}^3 = 1000(\frac{\mathrm{m}}{\mathrm{km}})^3 \cdot \mathrm{km}^3 = 1 \times 10^3 \cdot 1 \times 10^{-9}\mathrm{km}^3 = 1 \times 10^{-6}\mathrm{km}^3$$

Dimension Analysis: System of Units

1. We can first select some physical quantities as the "basic quantities" and specify a "basic unit of measurement" for each basic quantity, **the other physical quantities' units can be derived from the relation (definition or law) between them and the fundamental quantities.** These physical quantities are called "derived quantities" and their units It's called derived unit.
2. A set of units formed in this way, is called a certain **"system of units"**.
3. For example, the **SI system of units**, which is most commonly used, contains seven basic quantities: L, m, t, I, T, n, lv; Seven basic units: m, kg, s, A, K, mol, cd. Force (F) is an derived quantity, N is the derived unit, and the relationship with the basic unit is $N = kg \cdot m/s^2$

Dimension Analysis: Method of Undetermined Coefficients

1. We often use capital letter to represent a "dimensional quantity", and use [x] to represent the "dimensional quantity" of physical quantity x. e.g. The dimensional quantity of mass m is written as: $M = [m]$
2. In this course, we use "Method of Undetermined Coefficients" to do exercises. (Although this method is not rigorous.)

Exercise 1

A simple pendulum consists of a light inextensible string AB with length l, with the end A fixed, and a point mass m attached to B. The pendulum oscillates with a small amplitude, and the period of oscillation is T. It is suggested that T is proportional to the product of powers of m, l and g, where g is the acceleration due to gravity. Use dimensional analysis to find this relationship.

Dimension Analysis: Method of Undetermined Coefficients

$g = 9.8\ m/s^2$

$$\begin{cases} [m] = M \\ [l] = L \\ [g] = L \cdot T^{-2} \end{cases} \qquad [T] = T$$

We assume that $T = k \cdot m^{\alpha_1} \cdot l^{\alpha_2} \cdot g^{\alpha_3}$ $\qquad$ k is a constant

$$\Longleftrightarrow \quad T = M^{\alpha_1} \cdot L^{\alpha_2} \cdot (L \cdot T^{-2})^{\alpha_3}$$

$$= M^{\alpha_1} \cdot L^{\alpha_2 + \alpha_3} \cdot T^{-2\alpha_3}$$

$$\begin{cases} \alpha_1 = 0 \\ \alpha_2 + \alpha_3 = 0 \\ -2\alpha_3 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = \frac{1}{2} \\ \alpha_3 = -\frac{1}{2} \end{cases}$$

$$T \propto l^{\frac{1}{2}} \cdot g^{-\frac{1}{2}}$$

$$\boxed{T \propto \sqrt{\frac{l}{g}}}$$

Back-of-the-envelope Calculation

Definition

A quick estimation of some physical quantities.

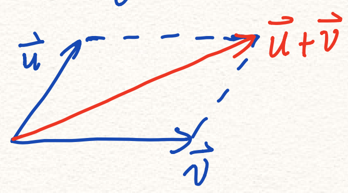
Comments

1. You should cultivate a common sense about the order of magnitude of quantities in everyday lives.
2. Tips: Try to remember the order of magnitude of some important constant.
3. This type of question will occur in your exams.

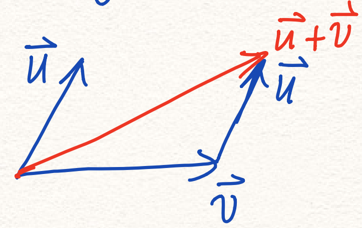
I. Vector Quantities

▵ Vector Addition

① parallelogram rule.

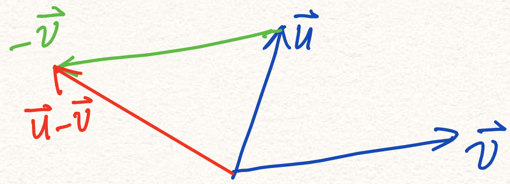


② Triangular rule



▵ Vector subtraction

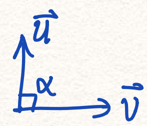
{ ① Turn $\vec{v}$ to $-\vec{v}$
② Use vector addition



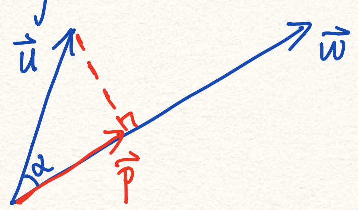
▵ Scalar product (dot product)

$$\vec{u}\cdot\vec{v} := |\vec{u}|\cdot|\vec{v}|\cdot\cos\alpha$$

$$\vec{u}\cdot\vec{v} = 0 \Rightarrow \vec{u}\perp\vec{v}$$



▵ Projection



unit vector

$$\vec{p} = \text{projection of } \vec{u} = |\vec{u}|\cdot\cos\alpha\cdot\hat{w}$$

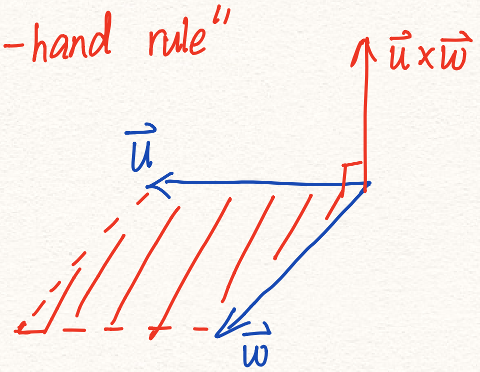
$$= \frac{\vec{u}\cdot\vec{w}}{|\vec{w}|}\cdot\frac{\vec{w}}{|\vec{w}|}$$

$$\hat{w} = \frac{\vec{w}}{|\vec{w}|}$$

▵ Vector (Cross) Product

If $\vec{b} = \vec{u}\times\vec{w}$, $|\vec{b}| = |\vec{u}|\cdot|\vec{w}|\sin\alpha$

"right-hand rule"



geometric meaning:

{
- The area of the parallelogram formed by $\vec{u}$ and $\vec{w}$.
- perpendicular to the parallelogram

∘ anticommutative : $\vec{u} \times \vec{w} = -\vec{w} \times \vec{u}$

▵ In Cartesian Coordinate System :

Let $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z \cdot v_z$ (since $\hat{n}_i \cdot \hat{n}_{j\,(i \neq j)} = 0$)

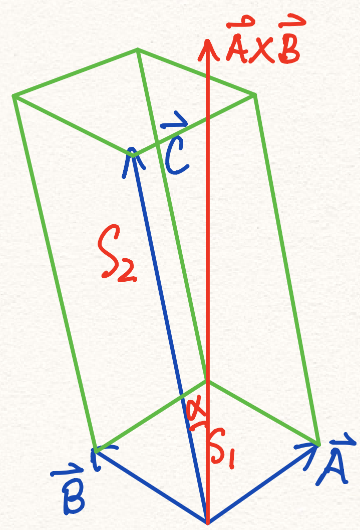
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= (u_y v_z - v_y u_z) \cdot \hat{i} - (u_x v_z - u_z v_x)\,\hat{j} + (u_x v_y - u_y v_x) \cdot \hat{k}$$

An interesting rule : $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B} = (\vec{B} \times \vec{C}) \cdot \vec{A}$

explaining :

parallelepiped



$|(\vec{A} \times \vec{B})| = S_1$

$(\vec{A} \times \vec{B}) \cdot \vec{C} = S_1 \cdot |\vec{C}| \cdot \cos\alpha$

$= S_1 \cdot h_1 = V$

$(\vec{B} \times \vec{C}) \cdot \vec{A} = S_2 \cdot \vec{A} \cdot \cos\alpha' = V$

II. 1D Kinematics

▵ Average vs. Instantaneous Quantities

$v_{xA} = \dfrac{x(t+\Delta t) - x(t)}{\Delta t}$ | $v_x = \dot{x}$

$a_{xA} = \dfrac{v_x(t+\Delta t) - v_x(t)}{\Delta t}$ | $a_x = \dot{v}_x = \ddot{x}$

III. Kinematics in 2D / 3D.

① Cartesian coordinate system.

x, y, z , (ρ, θ) (ρ, θ, z)

Basic : Motion Equation

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t). \end{cases}$$

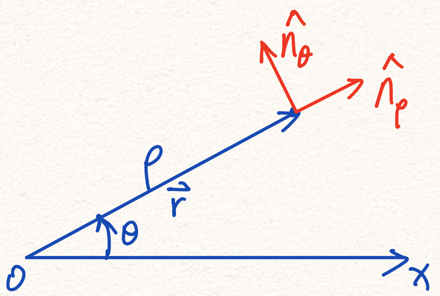
It describes all information of this motion.

$$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix}$$

△ trajectory : $F(x, y, z) = 0$ can be derived from motion equation.

② Polar coordinate system.

$\vec{r}(\rho, \theta) = \rho \cdot \hat{n}_\rho$

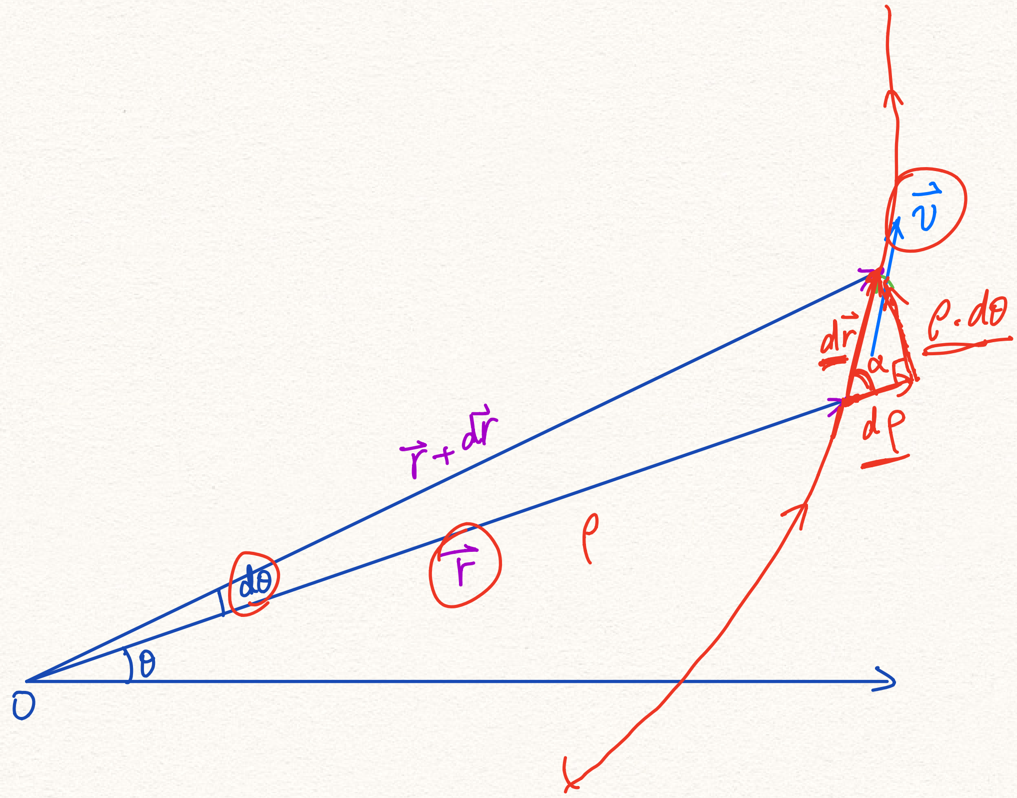


$$\begin{cases} \hat{n}_\rho = \hat{n}_\rho(\theta) \\ \hat{n}_\theta = \hat{n}_\theta(\theta) \end{cases} \qquad \begin{cases} \dot{\hat{n}}_\rho = \dot{\theta} \cdot \hat{n}_\theta \\ \dot{\hat{n}}_\theta = \dot{\theta} \cdot (-\hat{n}_\rho) \end{cases}$$

$$\vec{v}(\rho, \theta) = \dot{\rho}\hat{n}_\rho + \rho\dot{\theta} \cdot \hat{n}_\theta = v_\rho \cdot \hat{n}_\rho + v_\theta \cdot \hat{n}_\theta$$

$$\vec{a}(\rho, \theta) = (\ddot{\rho} - \rho \cdot \dot{\theta}^2) \cdot \hat{n}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\hat{n}_\theta$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases} \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$



$$\vec{v} = \dot{\rho}\cdot\hat{\rho} + \rho\dot{\theta}\cdot\hat{\theta}$$

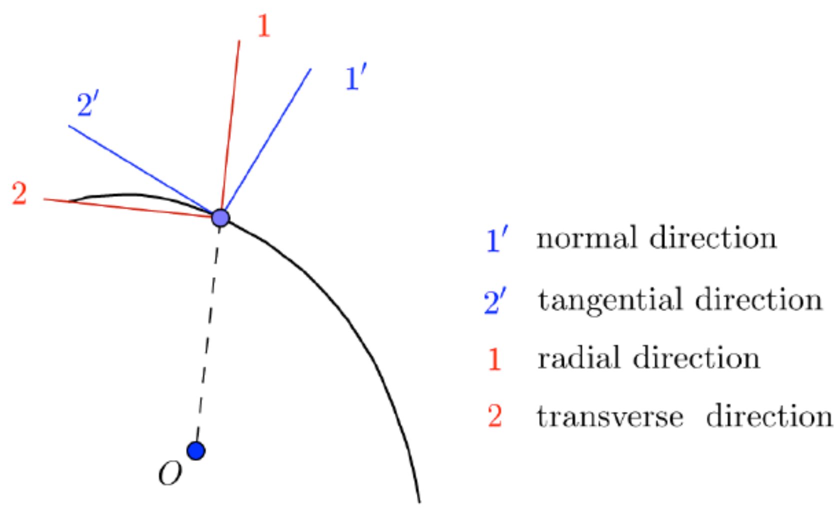
$$(ds)^2 = (d\rho)^2 + (\rho\, d\theta)^2$$

$$ds = \sqrt{ds^2}$$

$$\frac{ds}{dt} = |\vec{v}|$$

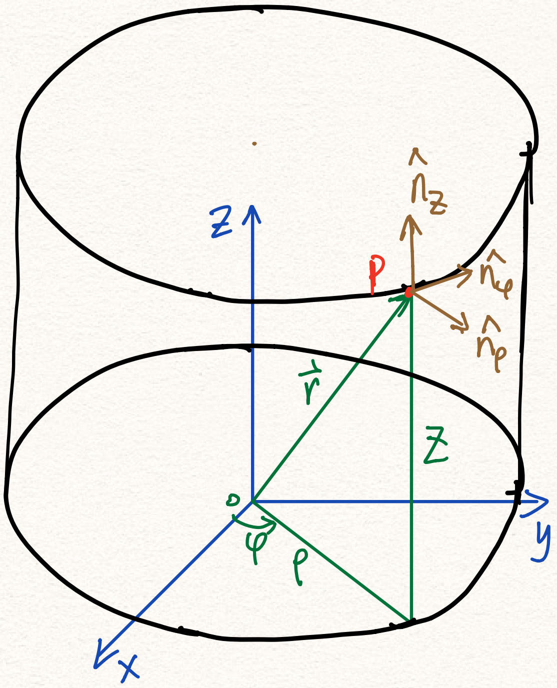
CAUTION!

In general, radial $\neq$ normal, nor transverse $\neq$ tangential!



③ Cylindrical Coordinates

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ \varphi = \arctan\left(\frac{y}{x}\right) \\ z = z \end{cases} \qquad \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$$



Cylindrical Coordinate System (ρ, φ, z)

$$\vec{r}(\rho, \varphi, z) = \rho \cdot \hat{n}_\rho + z \cdot \hat{n}_z$$

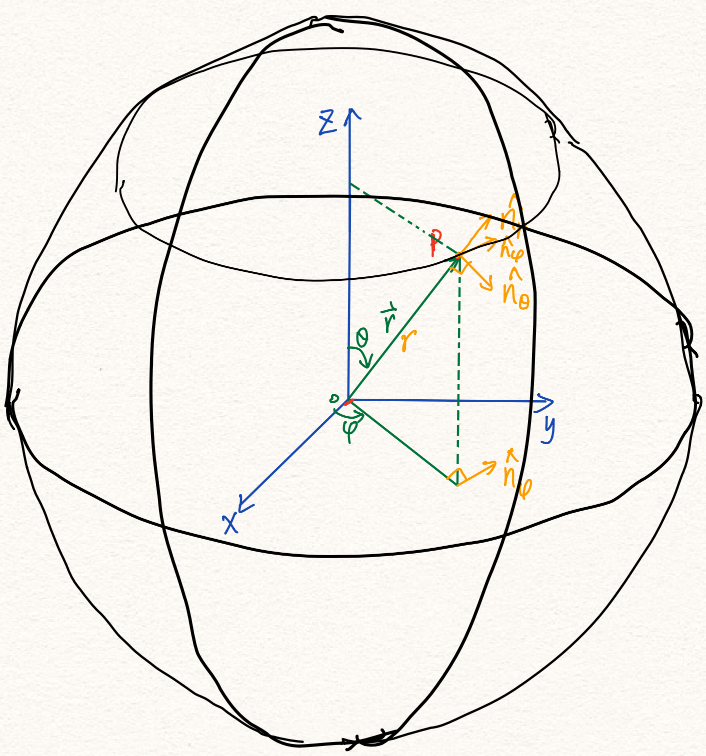
$$\begin{cases} \dot{\hat{n}}_\rho = \dot{\varphi}\, \hat{n}_\varphi \\ \dot{\hat{n}}_\varphi = -\dot{\varphi} \cdot \hat{n}_\rho \end{cases}$$

$$\vec{v}(t) = \frac{dr}{dt} = \dot{\rho}\hat{n}_\rho + \rho\,\dot{\hat{n}}_\rho + \dot{z} \cdot \hat{n}_z$$
$$= \dot{\rho}\hat{n}_\rho + \rho\dot{\varphi}\,\hat{n}_\varphi + \dot{z} \cdot \hat{n}_z$$

$$\vec{a}(t) = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{n}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{n}_\varphi + \ddot{z} \cdot \hat{n}_z$$

④ Spherical Coordinate System.

$$\vec{r}(r, \theta, \varphi) = r \cdot \hat{n}_r$$



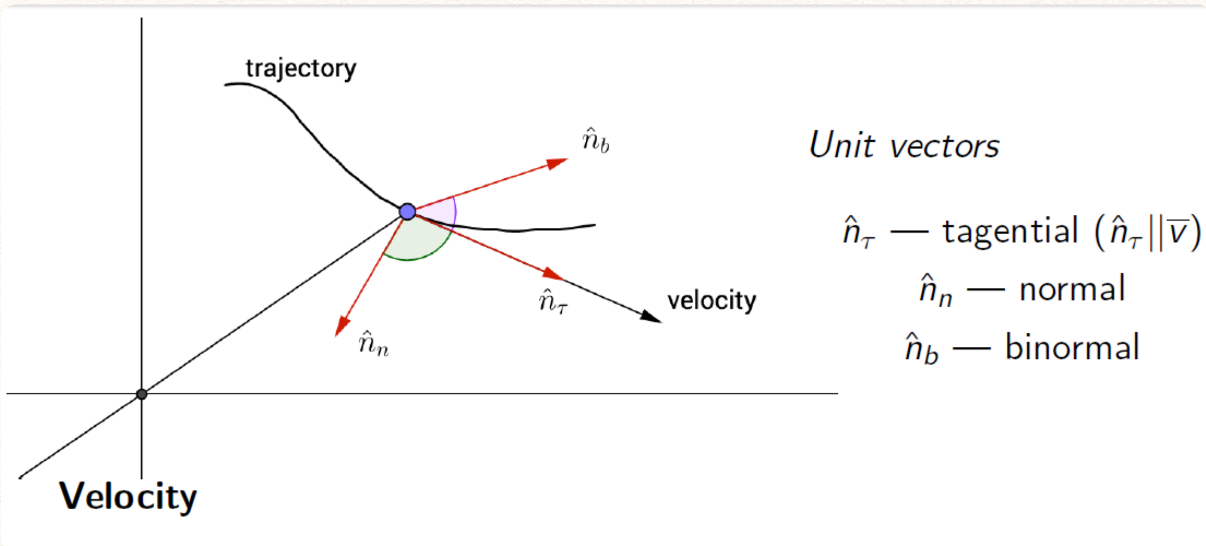
with
$$\begin{cases} \hat{n}_r = \hat{n}_r(\theta, \varphi) \\ \hat{n}_\theta = \hat{n}_\theta(\theta, \varphi) \\ \hat{n}_\varphi = \hat{n}_\varphi(\theta, \varphi) \end{cases}$$

$$\begin{cases} \dfrac{d\hat{n}_r}{dt} = \dfrac{\partial \hat{n}_r}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial t} + \dfrac{\partial \hat{n}_r}{\partial \varphi} \cdot \dfrac{\partial \varphi}{\partial t} \\ \quad = 1 \cdot \dot{\theta}\, \hat{n}_\theta + \dot{\varphi} \sin\theta\, \hat{n}_\varphi \\ \dfrac{d\hat{n}_\varphi}{dt} = \underline{-\dot{\varphi}\sin\theta\, \hat{n}_r - \dot{\varphi}\cos\theta\, \hat{n}_\theta} \\ \dfrac{d\hat{n}_\theta}{dt} = \underline{-\dot{\theta}\, \hat{n}_r + \dot{\varphi}\cos\theta\, \hat{n}_\varphi} \end{cases}$$

$$\begin{cases} \vec{v}(t) = \dot{r}\hat{n}_r + r\dot{\theta}\hat{n}_\theta + r\dot{\varphi}\sin\theta\,\hat{n}_\varphi \\ \vec{a}(t) = \hat{n}_r\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2\right) \\ \quad + \hat{n}_\theta\left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\varphi}^2\right) \\ \quad + \hat{n}_\varphi\left(\sin\theta\,\dot{r}\dot{\varphi} + 2r\cos\theta\,\dot{\theta}\dot{\varphi} + r\sin\theta\,\ddot{\varphi}\right) \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \varphi = \arctan\left(\frac{y}{x}\right) \\ \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \end{cases} \qquad \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

Natural Coordinate System.



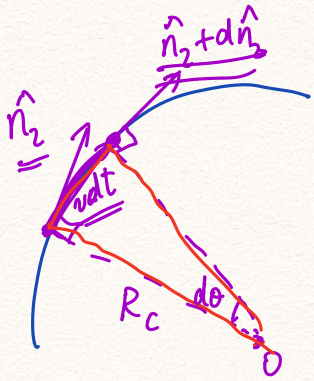
Definition : $\vec{v}(t) = v\,\hat{n}_\tau$

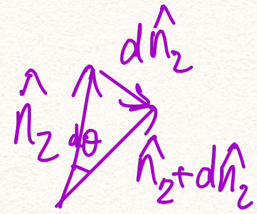
$$\hat{n}_\tau = \frac{\vec{v}}{|v|} = \frac{\dot{\vec{r}}}{|\dot{r}|}$$

$$\hat{n}_n = \frac{\frac{d\hat{n}_\tau}{dt}}{\left|\frac{d\hat{n}_\tau}{dt}\right|} \qquad \hat{n}_b = \hat{n}_\tau \times \hat{n}_n$$

△ Curvature

$$R_c \overset{def}{=} \frac{v}{\left|\frac{d\hat{n}_2}{dt}\right|}$$





$$R_c = \frac{vdt}{d\theta} = \frac{v}{\frac{d\theta}{dt}} = \frac{v}{\frac{|d\hat{n}_2|}{dt}}$$

$$|d\hat{n}_2| = 1 \cdot d\theta$$

$$R_c = \frac{v^2}{a_n}$$

$$\boxed{a_n = \frac{v^2}{R_c} = \omega^2 \cdot R_c}$$

IV. Tricks

△ Integration and Derivation

① Separation of Variables.

② Use of chain rule

※ $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$.

∘ How to use it? ① To avoid difficult 2-order ODEs
② To simplify calculation.

③ Integration by parts:

$$(uv)' = u'v + v'u$$

$$\int u'v = uv - \int v'u$$

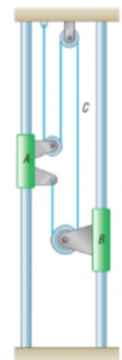
△. Relative velocity.

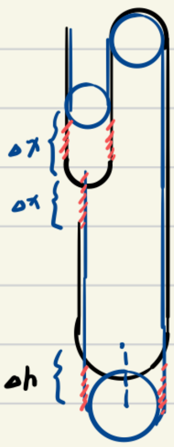
$$\begin{cases} v_x = v_{O'x} + v_x' \\ a_x = a_{O'x} + a_x' \end{cases}$$

$\boxed{\text{Core Essence} \Longrightarrow \text{Constraint on Kinematics}}$ (see RC 2)

Problem 1. Collar A starts from rest and moves upward with a constant acceleration. Knowing that after 8 s the relative velocity of collar B with respect to collar A is 24 cm/s, determine (a) the accelerations of A and B, (b) the velocity and the change in position of B after 6 s.

(*3/2 + 3/2 points*)





$2\Delta x = \Delta x + 2\Delta h$

$\Delta h = 0.5\Delta x$

$\because t_A = t_B$

$\therefore V_A = 2V_B$

$\because$ the acceleration of A is constant

$\therefore \frac{dV_A}{dt} = 2\frac{dV_A}{dt}$

$a_A = 2a_B$

a) $t_1 = 8s$ $\quad \Delta V = (a_A + a_B)t_1 = 24$ cm/s

$\because a_A = 2a_B \quad \therefore a_A = 2\,cm/s^2 \quad a_B = 1\,cm/s^2$

b) $V_B = a_B t_2 = 6$ cm/s downward

$\Delta x_B = \frac{1}{2} a_B t_2^2 = 18$ cm downward

Problem 7. A particle moves in a plane so that the angle between the position vector $\mathbf{r}$ and the velocity vector $\mathbf{v}$ is constant and equal to α.

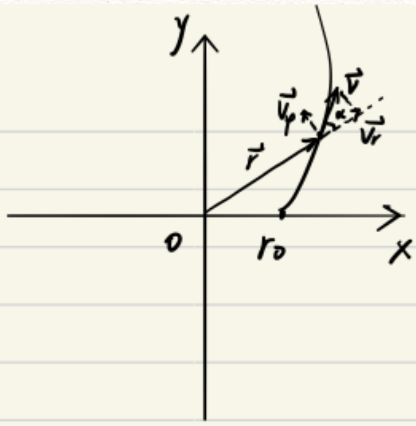
(a) Find the implicit equation of the trajectory in polar coordinates,

(b) and the total length of the trajectory.

(c) Sketch the trajectory.

(d) Discuss the solution with respect to the values of α.

Assume the initial conditions $\varphi(0) = 0$ and $r(0) = r_0$. *Hint for (a).* Separation of variables.



a) $\vec{v} = \vec{v}_r \cdot \hat{n}_r + \vec{v}_\varphi \cdot \hat{n} = \dot{r}\cdot\hat{n}_r + r\dot{\varphi}\hat{n}_\varphi$

$$\cot\alpha = \frac{v_r}{v_\varphi} = \frac{\dot{r}}{r\dot{\varphi}} = \frac{\frac{dr}{dt}}{r\cdot\frac{d\varphi}{dt}}$$

$$\therefore \frac{dr}{r} = \cot\alpha \cdot d\varphi$$

$$\int_{r_0}^{r} \frac{dr}{r} = \int_0^{\varphi} \cot\alpha \, d\varphi$$

$$\therefore \ln\frac{r}{r_0} = \cot\alpha \cdot \varphi$$

$$\therefore r = r_0 e^{\varphi\cot\alpha}$$

b) ① $a=0 \to$ a line $l=r$

② $a=\frac{\pi}{2}$ $r=r_0$ $\to$ uniform circular motion $\to$ cyclic motion

total length is $\boxed{2\pi r_0}$ for one period

③ $a \neq \frac{\pi}{2}$

$dl = \sqrt{(dr)^2+(rd\varphi)^2}$

$\frac{dr}{d\varphi} = r_0 \cdot e^{\varphi \cot\alpha} \cdot \cot\alpha$

$\therefore dr = \cot\alpha \cdot r \cdot d\varphi$

$\therefore dl = \sqrt{\cot^2\alpha+1}\, r d\varphi = \frac{r d\varphi}{|\sin\alpha|}$

$\because \alpha \in (0,\pi) \therefore \sin\alpha > 0$

$\therefore \int_0^l dl = \int_0^\varphi \frac{r_0 e^{\varphi\cot\alpha}}{\sin\alpha} d\varphi$

$\therefore l = \frac{r_0}{\sin\alpha \cdot \cot\alpha} e^{\varphi\cot\alpha} \Big|_0^\varphi$

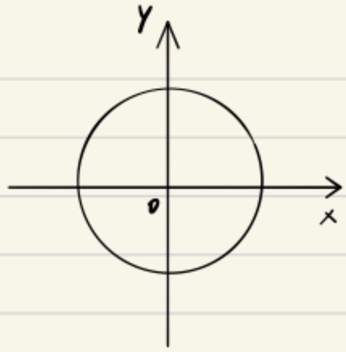
$$\boxed{= \frac{r_0}{\cos\alpha}\left(e^{\varphi\cot\alpha}-1\right)}$$

$$\begin{cases} ① \ a\in(0,\frac{\pi}{2}) \quad \cot\alpha>0, \text{ not convergent} \quad \varphi\to\infty,\ l\to\infty \\ ② \ a\in(\frac{\pi}{2},\pi) \quad \cot\alpha<0 \quad \text{convergent} \quad \varphi\to\infty,\ l\to\frac{-r_0}{\cos\alpha} \end{cases}$$

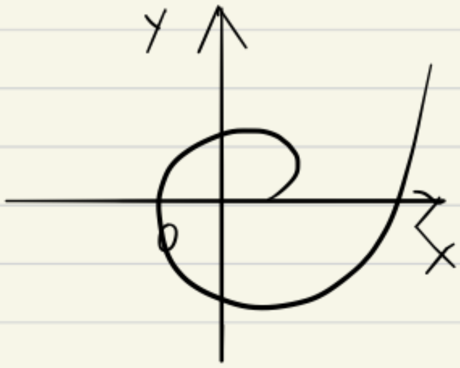
c)

$a=0$

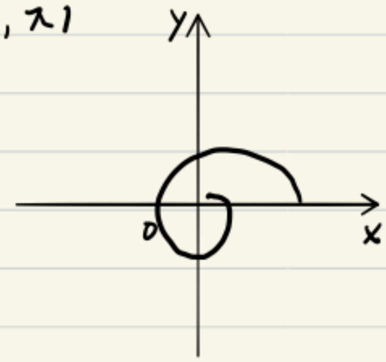
$a=\frac{\pi}{2}$



$a\in(0,\frac{\pi}{2})$



$a\in(\frac{\pi}{2},\pi)$



d) ① $\alpha=0$ or $\alpha=\pi$ → trajectory is a line

② $\alpha=\frac{\pi}{2}$ → a circle

③ $\alpha\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$ → equiangular spiral / logarithmic spiral