Le Trung Hieu . Week 1

1. To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

Answer

- Denoted those people who got Hansen's disease as H (0: Not infected; 1: Infected)

- Denoted the outcome of the test as T (0: Negative; 1: Positive)

We already known that:

$$\mathrm{P(H = 1) = 0.05 \Rightarrow P(H = 0) = 1 - 0.05 = 0.95}$$
$$\mathrm{P(T = 1|H = 1) = 0.98}$$
$$\mathrm{P(T = 1|H = 0) = 0.03}$$

$$P(H = 1|T = 1) = \frac{P(T = 1|H = 1) \cdot P(H = 1)}{P(T = 1)}$$

$$P(H = 1|T = 1) = \frac{P(T = 1|H = 1) \cdot P(H = 1)}{P(T = 1|H = 1) \cdot P(H = 1) + P(T = 1|H = 0) \cdot P(H = 0)}$$

$$P(H = 1|T = 1) = \frac{0.98 \cdot 0.05}{0.98 \cdot 0.05 + 0.03 \cdot 0.95}$$

$$P(H = 1|T = 1) = \frac{98}{155} \approx 63.22\%$$

2. Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:
   (a) Univariate normal distribution
   (b) Multivariate normal distribution (Optional)

Answer

(a) The probability density function of the Univariate Gaussian Distribution is given by:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} \qquad -\infty < x < \infty$$

To prove that the above expression is normalized, we have to show that

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2} = 1$$

or

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2} = \sqrt{2\pi\sigma^2}$$

Proof. Let

$$I = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2}\ dx$$

Squaring the above expression,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2}\ dx\ dy \tag{1}$$

To integrate this expression we make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by

$$x = r\cos\theta$$

$$y = r\sin\theta$$

We have:

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow x^2 + y^2 = r^2$$

The Jacobian of the change of variables is given by,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

Equation (1) can be rewritten as

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2\theta^2}}\ r\ dr\ d\theta$$

$$= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}}\ r\ dr$$

$$= 2\pi \int_0^{\infty} \frac{1}{2} e^{-\frac{u}{2\sigma^2}}\ du$$

$$= \pi \left[e^{-\frac{u}{2\sigma^2}}\ (-2\sigma^2) \right]_0^{\infty}$$

$$= 2\pi\sigma^2$$

$$\Rightarrow I = \sqrt{2\pi\sigma^2}$$

To prove that $\mathcal{N}(x|\mu,\theta^2)$ is normalized, we make the transformation $y = x - \mu$, so that

$$\int_{-\infty}^{\infty} \mathcal{N}(x|\mu,\theta^2)dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\ dy$$

$$= \frac{I}{\sqrt{2\pi\sigma^2}}$$

$$= \frac{\sqrt{2\pi\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$= 1$$