A LAGRANGIAN MODEL INVARIANT UNDER SUPERGAUGE TRANSFORMATIONS

J. WESS
Karlsruhe University, Germany

and

B. ZUMINO
CERN, Geneva, Switzerland

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We study, in the one-loop approximation, a Lagrangian model invariant under supergauge transformations. The model involves a scalar, a pseudoscalar and a spinor field. Supergauge invariance gives rise to relations among the masses and the coupling of these fields and implies the existence of a conserved current. The renormalization procedure is discussed and the relations among masses and couplings are shown to be preserved by renormalization.

In a recent paper [1] the authors have defined supergauge transformations in four dimensions, have studied the algebra generated by them and have given a number of representations of them as transformations on fields. Supergauge transformations transform tensor fields into spinor fields and vice versa. They depend on parameters which are themselves spinors and are treated as totally anticommuting quantities (odd elements of a Grassmann algebra). The commutator of two supergauge transformations is a four-dimensional conformal transformation accompanied by a γ_5 transformation, having parameters which are commuting quantities (even elements of the Grassmann algebra). The algebra of supergauge, conformal and γ_5 transformations closes. It is a kind of Lie algebra except for the fact that some of the parameters commute and some anticommute. For simplicity we shall use the expression Lie algebra for such a mathematical construct†[1]. In reference [1] it was also shown how to construct free Lagrangians as well as interactions invariant under supergauge transformations. The presence of the conformal and γ_5 transformations in the algebra had the consequence that only theories with massless particles could be invariant under supergauge transformations.

†[1] Lie algebras and Lie groups with some commuting and some anticommuting parameters have been studied in the mathematical literature [e.g. 2].

In order to remove the limitation to massless particles it is sufficient to restrict the requirement of invariance to a subalgebra [3], that generated by supergauges with constant parameters. The commutator of two such supergauge transformations is simply a four-dimensional translation and the algebra closes. Lorentz transformations and γ_5 transformations induce a homorphism of the algebra of supergauge transformations and translations. Conformal and scale transformations are no longer considered. The absence of conformal, scale and γ_5 transformations from the invariance algebra, has the further advantage that the Ward identities satisfied by the Lagrangian theory are now free at least of the anomalies which are known to arise in connection with scale, conformal and γ_5 invariance [e.g. 4].

In this note we present a field theory model invariant under the subalgebra mentioned above and we investigate its divergences and renormalizability in the one-loop approximation. The model provides a very simple example of a non-trivial Lagrangian theory in which there are relations among the masses and couplings of scalar, pseudoscalar and spinor fields. We have verified that these relations are preserved by renormalization.

Our Lagrangian is the sum of the free Lagrangian

$$L_0 = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2$$

and of the two invariants [1, 3]

$$L_m = m(FA + GB - \tfrac{1}{2}\mathrm{i}\bar{\psi}\psi)$$

and

$$L_g = g\,[F(A^2-B^2) + 2GAB - \mathrm{i}g\bar{\psi}(A-\gamma_5 B)\psi].$$

Here A and B are respectively a scalar and a pseudoscalar field, ψ is a Majorana spinor and F and G are auxiliary fields†[2]. A possible further invariant

$$L_\lambda = \lambda F$$

can always be eliminated by a shift in the field A or by a shift followed by a γ_5 rotation followed by a shift, depending on whether $m^2 - 4g\lambda$ is positive or negative. The auxiliary fields satisfy the equations of motion

$$F + mA + g(A^2-B^2) = 0 \quad \text{and} \quad G + mB + 2gAB = 0$$

which can be used to eliminate them from the Lagrangian. The result is

$$L = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi$$
$$-\tfrac{1}{2}m^2A^2 - \tfrac{1}{2}m^2B^2 - \tfrac{1}{2}\mathrm{i}m\bar{\psi}\psi$$
$$-gmA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2 - \mathrm{i}g\bar{\psi}(A-\gamma_5 B)\psi.$$

The equality of the masses and the relations among the various couplings are consequences of the invariance under supergauge transformations. It is remarkable that this Lagrangian appears to be renormalizable even when the masses and coupling constants are not independent and that the relations among them are preserved by renormalization. Furthermore the theory turns out to be less divergent when the above relations are satisfied†[3]. For instance, the quadratic divergence of the mass renormalization for the scalar and pseudoscalar fields cancels among the various diagrams contributing to it. Similarly the logarithmic divergence of the vertex correction to the spinor-scalar or the spinor-pseudoscalar interaction also cancels between the two diagrams where a scalar or a pseudoscalar is exchanged, leaving a finite vertex correction. All these statements have been verified first (in the one-loop approximation) using the Lagrangian in the form obtained by eliminating the fields F and G.

In order to prepare the way for a future systematic treatment of higher orders, it seems preferable to describe the renormalization procedure for the original Lagrangian containing the fields F and G, rather than after elimination of those fields†[4]. If one takes $L_0 + L_m$ as unperturbed Lagrangian, one finds, in addition to the usual propagators†[5],

$$\langle AA\rangle = \langle BB\rangle = \Delta_c,$$

propagators for the auxiliary fields

$$\langle FF\rangle = \langle GG\rangle = \Box\Delta_c$$

and mixed propagators

$$\langle AF\rangle = \langle BG\rangle = -m\Delta_c.$$

Contrary to the more complicated situation for the Lagrangian without F and G, it now turns out that the only renormalization needed in the one-loop approximation is a logarithmically divergent wave function renormalization Z, the same for all fields A, B, ψ, F and G. One finds $Z = 1 - 4g^2 I$, where I is the logarithmically divergent integral

$$I = -\mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^2} = \frac{1}{16\pi^2}\int^\infty \frac{\mathrm{d}\lambda}{\lambda}.$$

For instance, no diagonal mass for the field A and B is generated. The quadratic divergence of the self-energy cancels among the various diagrams and the remaining logarithmically divergent self-energy is proportional to p^2. Similarly, the spinor self-energy is proportional to $\gamma^\mu p_\mu$ and the corrections to the off-diagonal mass terms mFA and mGB add up to zero. Therefore, the only mass renormalization is that due to the wave-function renormalization and the renormalized mass m_r is given by $m_r = mZ$. The corrections to the couplings gFA^2, $-gFB^2$ and $2gGAB$ add up to zero, while the vertex corrections to the interactions $-\mathrm{i}g\bar{\psi}\psi A$ and $\mathrm{i}g\bar{\psi}\gamma_5\psi B$ add up to finite values and

†[2] We find it convenient to use the Majorana representation with real γ^μ's, $(\gamma^0)^2 = -1$, $(\gamma^5)^2 = -1$.

†[3] The authors are very grateful to B.W. Lee for pointing out first the occurrence of cancellations and also the fact that relations among couplings are preserved in the one-loop approximation.

†[4] In this form the theory can be regularized (for instance by the method of Pauli–Villars) without spoiling supergauge invariance. Therefore, the Ward identities following from supergauge invariance are expected to be satisfied in perturbation theory. The authors are very grateful to J. Iliopoulos for a discussion of these points.

†[5] Here Δ_c is the usual Feynman propagator, $(\Box - m^2)\Delta_c = \delta^4(x-x')$.

actually vanish for zero external momenta. All other corrections either vanish or are finite. Therefore, no renormalization of the parameter g other than that due to the wave-function renormalization is needed and the renormalized coupling constant g_r can be defined simply as

$$g_r = gZ^{3/2}.$$

Finally, no divergent trilinear or quadrilinear interactions are generated. All these cancellations among different diagrams are due to the supergauge invariance of the Lagrangian.

Invariance under supergauge transformations implies, by well known arguments, that the vector-spinor current

$$J^\mu = \gamma^\lambda \partial_\lambda (A - \gamma_5 B)\gamma^\mu \psi + m\gamma^\mu(A-\gamma_5 B)\psi + g\gamma^\mu(A-\gamma_5 B)^2 \psi$$

is conserved

$$\partial_\mu J^\mu = 0.$$

This conservation law can be checked directly by using the equations of motion and the identity†[6]

$$\psi(\bar{\psi}\psi) = \gamma_5 \psi(\bar{\psi}\gamma_5 \psi).$$

The integral $\int J^\text{o} \mathrm{d}_3 x$ is a conserved spinor charge which changes states with integral spin into states with half integral spin and vice versa†[7]. As mentioned in footnote †[4], the Ward identities, which can be derived formally from the conservation of the current, are expected to be verified in perturbation theory.

The present investigation suggests two interesting lines of research. The first is the study of higher order corrections. The second is the construction of more complex and hopefully more realistic models, invariant under a combination of supergauge and internal symmetries. Work along these lines is in progress.

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†[6] This identity is valid at the 'classical' level, where it follows from the totally anticommuting property of the Majorana field ψ. It is also formally valid when the field ψ satisfies canonical commutation relations and it is actually satisfied in the regularized version of the theory.

†[7] The model described in this note, and in general the existence of supergauge invariant field theories with interaction, seems to violate SU(6) no-go theorems like that proven by Coleman and Mandula [5]. Apparently supergauge transformations evade such no-go theorems because their algebra is not an ordinary Lie algebra, but has anticommuting as well as commuting parameters. The presence of the spinor fields in the multiplet seems therefore essential.

References

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