8.A.1 Nuclear Physics B31 (1971) 86-112. North-Holland Publishing Company

FACTORIZABLE DUAL MODEL OF PIONS*

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Received 25 March 1971

Abstract: A new dual-resonance model of mesons is constructed through the use of creation and annihilation operators having simple anticommutation properties as well as harmonic-oscillator type operators of the conventional model. This model has the following virtues not shared by the conventional one: (i) The leading trajectory (ρ-f^0) does not make a particle when it passes through zero. (ii) A π-trajectory lies one-half unit below the ρ-trajectory. (iii) Trajectories with abnormal-parity couplings to pions, such as ω-A_2 and η', also occur. (iv) The π, ρ, f^0, ω, A_2 and η' are all forced to have the proper G-parity and isospin. (v) The amplitudes for $\pi\pi \to \pi\pi$ and $\pi\pi \to \pi\omega$ are precisely the ones that have been suggested previously. (vi) The model contains neither embarrassing unobserved low-mass states nor an excessive number of high-mass states. (vii) All the physical states that we have checked have positive norms.

To be fair, we should also list some shortcomings of our model: (a) Trajectories with normal-parity couplings occur one-half unit too high. Thus, for example, $\alpha_\rho(0) = 1$ and $\alpha_\pi(0) = \frac{1}{2}$, so that the π is a tachyon and the ρ is massless. On the other hand, the abnormal-parity trajectories are nicely located. For example, $\alpha_\omega(0) = \frac{1}{2}$ and $\alpha_{\eta'}(0) = -1$. (b) A straightforward generalization to SU(3) is unsatisfactory because of the non-degeneracy of the ρ and ω. These features of the model clearly indicate that it is not accurately describing the real world. Perhaps its most important property is that it contains a gauge algebra *larger* than the Virasoro algebra of the conventional model. We believe that an understanding of this algebra may prove to be very important in the construction of more realistic models.

1. INTRODUCTION

In the nearly three years since Veneziano's classic paper [1] a great deal of effort has been expended on attempts to construct dual-resonance models that either bear a resemblance to reality or else could serve as the basis for a full-fledged unitary (albeit unrealistic) theory of hadrons. Considerable progress toward the second goal has grown out of the study of

* Research sponsored by US Atomic Energy Commission under contract AT(30-1)-4159.

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the N-point generalizations [2] of Veneziano's beta function. First, the one-loop graphs were constructed [3] and renormalized [4]. More recently, the general N-loop graphs have been constructed [5] and (in the planar case) renormalized [6]. Scherk has furthermore shown [7] that the renormalization is essentially unique when the intercept of the leading Regge trajectory $\alpha_0 = 1$, whereas in all other cases there is infinite ambiguity, i.e., the theory is non-renormalizable. The special role of $\alpha_0 = 1$ was pointed out earlier by Virasoro [8] in connection with the problem of negative-norm states (ghosts). He showed that in this case there is a large gauge algebra that probably allows for the cancellation of all ghosts.

We shall refer to the generalized Veneziano model with $\alpha_0 = 1$ as the "conventional model". Despite the manifold virtues of this model, the unitarization program encounters at least two serious problems. First, the model contains a tachyon, which causes severe difficulties, of course. Second, certain non-planar loop graphs give rise to a new singularity [9] that is not simply related to single-particle states. It has been suggested (probably correctly) that this singularity should be identified as the pomeron. Its precise character and intercept are determined by detailed characteristics of the spectrum of single-particle states. In the conventional model it turns out to be a branch point having intercept $\frac{1}{4}$ (when the linear dependences are properly accounted for). It is our strong impression that such a singularity can be reconciled with unitarity only when it is a pole. One possible attitude, closely akin to the bootstrap philosophy [10], is that of all the many dual models that might be constructed, only the "right" one is free of ghosts and tachyons and gives a pomeron *pole*. Veneziano has further suggested [11] that even the coupling constant may be determinable. However, this is an attitude we do not share.

Another shortcoming of the conventional model is its lack of resemblance to the real world. At first, attempts to incorporate more realistic quantum numbers centered on attaching superstructure describing quark spin and internal quantum numbers to the conventional model [12]. This approach encountered insuperable difficulties relating to parity doubling and ghosts. Recently, a number of more ambitious proposals have been made, but none of them has been very successful so far.

This paper presents a new model containing realistic quantum numbers without parity doubling or ghosts (as far as we can tell). The basic algebra of the model has been previously reported [13]. However, the version presented in ref. [13] contains "pions" and "rhos" of the wrong G-parity, and also has a number of other inadequacies. Shortly after submitting that work we discovered a slight but crucial modification that allows for substantial improvement.

The new model has pions and rhos with the correct quantum numbers. It also contains analogs of the Virasoro gauges of the conventional model, and it appears to be free of ghosts. More remarkably, it possesses additional gauge operators which have the almost incredible property of decoupling the ground state and the leading Regge trajectory! The algebra that is operative is not yet fully understood, but it is undoubtedly rather deep. The model contains a π, ρ, f^0, ω, A_2 and η', all with the proper quantum numbers. It does *not* contain low-mass states without an experimental identification. In fact, it does not even have a σ degenerate with the ρ or an A_1. For higher masses the degeneracy probably grows exponentially with mass as in the conventional model, although the gauge algebra is so powerful that it is not yet clear whether it grows that fast. The ρ - f^0 and ω - A_2 trajectories are not "doubled", contrary to previous suggestions [14]. The amplitude for $\pi\pi \to \pi\pi$ is the one suggested by Shapiro and Lovelace [15], while the one for $\pi\pi \to \pi\omega$ precisely coincides with Veneziano's original suggestion [1].

An empirical rule observed in the model is that those trajectories with normal-parity couplings to pions (π, ρ, f^0, etc.) occur one-half unit too high, whereas those with abnormal-parity couplings to pions (ω, A_2, η', etc.) are nicely located. This means that we have $\alpha_\pi(0) = \frac{1}{2}$, $\alpha_\rho(0) = \alpha_{f^0}(0) = 1$, $\alpha_\omega(0) = \frac{1}{2}$, and $\alpha_{\eta'}(0) = -1$. This placement of trajectories implies that the π is a tachyon and the ρ is massless. Another difficulty is that the non-degeneracy of the ρ and ω makes a straightforward generalization to SU(3) most unsatisfying. Namely, it would result in two sets of ρ - f and ω - A_2 trajectories split by one-half unit. One set would carry the normal couplings and the other the abnormal ones. In our opinion, the model should be regarded as intrinsically limited to SU(2). We suspect that a future better model might require including SU(3) as a broken symmetry.

The paper is organized as follows: sect. 2 is a review of the conventional model stressing those approaches and notations that are required in subsequent sections. Sect. 3 presents the proposed new model. In particular, it explains why the masses are constrained to be those mentioned above. Sect. 4 discusses the basic conjecture concerning the existence of extra gauge operators. The conjecture is not proved, but a strong case is made for it. In sect. 5 there is a rather detailed investigation of the spectrum. All the states at the first few mass levels and on the leading trajectories are discussed. In sect. 6 the formulas for the six-pion amplitude and the $\pi\rho \to \pi\rho$ amplitudes are presented. Sect. 7 is the conclusion.

2. REVIEW OF CONVENTIONAL DUAL-RESONANCE MODEL

The spectrum of the conventional dual-resonance model is conveniently described in the Fock space generated by four-vector creation operators $a_m^{\dagger\mu}$, $m = 1, 2, \ldots$ (ref. [16]). These operators satisfy the commutation relations

$$[a_m^\mu, a_n^\nu] = [a_m^{\dagger\mu}, a_n^{\dagger\nu}] = 0 \ , \tag{2.1a}$$

$$[a_m^\mu, a_n^{\dagger\nu}] = -g^{\mu\nu}\delta_{m,n} \ , \tag{2.1b}$$

where we use the timelike metric $g^{00} = -g^{ii} = 1$. We find it helpful to replace the a-operators and the momentum operator p by

$$\alpha_n^\mu = -i\sqrt{n}\, a_n^\mu\,, \qquad n = 1, 2, \dots\,, \tag{2.2a}$$

$$\alpha_{-n}^\mu = i\sqrt{n}\, a_n^{\dagger\mu}\,, \qquad n = 1, 2, \dots\,, \tag{2.2b}$$

$$\alpha_0^\mu = \sqrt{2}\, p^\mu\,. \tag{2.2c}$$

The algebra of eq. (2.1), re-expressed for the α-operators, takes the form

$$[\alpha_m^\mu, \alpha_n^\nu] = -m g^{\mu\nu}\, \delta_{m+n,0}\,. \tag{2.3}$$

Following Ramond [17], we define the operator

$$P^\mu(\tau) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu\, \mathrm{e}^{-in\tau}\,, \tag{2.4}$$

whose τ space average is just the momentum:

$$p^\mu = \langle P^\mu(\tau)\rangle = \frac{1}{2\pi} \int_0^{2\pi} P^\mu(\tau)\, \mathrm{d}\tau\,. \tag{2.5}$$

The Fourier moments of $P^2(\tau)$ are of particular importance:

$$L_n^{\mathrm{a}} = -\langle \mathrm{e}^{in\tau} :P^2(\tau): \rangle = -\tfrac{1}{2} : \sum_{m=-\infty}^{\infty} \alpha_{-m}\cdot\alpha_{n+m} :\,. \tag{2.6}$$

We use the superscript "a" as a reminder that an operator is defined in terms of the a oscillators since in the next section we shall introduce a "b" space as well. For $n = 0$, eq. (2.6) becomes

$$L_0^{\mathrm{a}} = -\tfrac{1}{2}\alpha_0\cdot\alpha_0 - \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m = R_{\mathrm{a}} - p^2\,, \tag{2.7}$$

while for $n > 0$,

$$L_n^{\mathrm{a}} = -\sum_{m=0}^{\infty} \alpha_{-m}\cdot\alpha_{m+n} - \tfrac{1}{2} \sum_{m=1}^{n-1} \alpha_m\cdot\alpha_{n-m}\,. \tag{2.8}$$

The operators L_n^{a} have the elegant algebra

$$[L_m^{\mathrm{a}}, L_n^{\mathrm{a}}] = (m-n)\, L_{m+n}^{\mathrm{a}}\,, \tag{2.9}$$

except when $m+n = 0$ an uninteresting constant may need to be added to the commutator. In particular, L_{-1}^{a}, L_0^{a} and L_1^{a} form an SU(1,1) algebra [18].

The construction of N-point functions can be achieved in a number of ways. The one that is most convenient for our purposes is to define a vertex operator

$$V_0(k, z) = \mathrm{e}^{ik\cdot x}\, \mathrm{e}^{\sqrt{2}k\cdot \sum_{n=1}^{\infty} \frac{\alpha_{-n} z^n}{n}}\, \mathrm{e}^{-\sqrt{2}k\cdot \sum_{n=1}^{\infty} \frac{\alpha_n z^{-n}}{n}}\, z^{-2k\cdot p}\,, \tag{2.10}$$

to describe the emission of a particle of four-momentum k^μ satisfying a mass-shell condition $k^2 = m^2$. The N-point function is then given by [19]

$$A_N(k_1, k_2, \ldots, k_N) = \int \mathrm{d}\mu_N(z) \left\{ \prod_{i=1}^{N} |z_i - z_{i+1}|^{-m^2} \right.$$

$$\left. \times \langle 0 | V_0(k_1, z_1)\, V_0(k_2, z_2) \ldots V_0(k_N, z_N) | 0 \rangle \right\} , \tag{2.11}$$

where $z_{N+1} \equiv z_1$ and

$$\mathrm{d}\mu_N(z) = (z_2 - z_N)\,\mathrm{d}z_3\,\mathrm{d}z_4 \ldots \mathrm{d}z_{N-1} \prod_{i=1}^{N-1} \frac{\theta(z_i - z_{i+1})}{z_i - z_{i+1}} . \tag{2.12}$$

The integrations in eq. (2.11) are along the real axis with limits controlled by the θ-functions of eq. (2.12); z_1, z_2 and z_N are fixed numbers satisfying $z_1 > z_2 > z_N$. The most important feature of A_N is its invariance under cyclic permutation of the momenta (duality). This fact depends critically on two properties of eq. (2.11). They are (1) the invariance of the integrand under Möbius transformations and (2) the rule for commuting $V_0(k_i, z_i)$ with $V_0(k_j, z_j)$.

Consider a Möbius transformation

$$z \to z' = Tz = \frac{az + b}{cz + d} ,$$

with a, b, c, d real numbers satisfying $ad - bc = 1$. These transformations form the group 0(2, 1), or, what is essentially the same thing, SU(1, 1). When all the variables $z_1, z_2, \ldots, z_N$ appearing in eq. (2.11) are simultaneously subjected to the *same* transformation, the resulting expression has the same form with $z_1', z_2', \ldots, z_N'$ replacing $z_1, z_2, \ldots, z_N$. To prove this one first does an easy calculation to show that the measure is form invariant, i.e., $\mathrm{d}\mu_N(z) = \mathrm{d}\mu_N(z')$. The invariance of the rest of the integrand is demonstrated by constructing a unitary operator

$$\Lambda(T) = \mathrm{e}^{i \xi_T \cdot \boldsymbol{L}^{\mathrm{a}}} \tag{2.13}$$

where $\boldsymbol{L}^{\mathrm{a}}$ is made from L_0^{a}, $L_{\pm 1}^{\mathrm{a}}$ and ξ_T is a three-vector appropriate for T. The vertex operator transforms by

$$\Lambda(T)\, V_0(k, z)\, \Lambda^{-1}(T) = (a - cz')^{-2m^2}\, V_0(k, z') . \tag{2.14}$$

The most convenient way to prove (2.14) is by first considering the infinitesimal transformations described by the formula [19, 20]

$$[L_n^{\mathrm{a}}, V_0(k, z)] = z^n \left(z \frac{\mathrm{d}}{\mathrm{d}z} - nm^2 \right) V_0(k, z) , \tag{2.15}$$

valid for all n. The proof of Möbius invariance is completed by noting that $\mathrm{e}^{i\xi_T \cdot \boldsymbol{L}^{\mathrm{a}}}$ leaves the zero-momentum ground state, $|0\rangle$, invariant, and that the factors $\prod_{i=1}^{N} (a - cz')^{-2m^2}$ arising from the transformation of the vertices are compensated for by extra factors arising in the transformation of $\prod_{i=1}^{N} |z_i - z_{i+1}|^{-m^2}$. The Möbius invariance of the integrand implies, in particular, that A_N does not depend on the values chosen for z_1, z_2 and z_N.

The proof of cyclic symmetry is completed by choosing the Möbius transformation T such that

$$z_1' = z_N \,, \qquad z_2' = z_1 \,, \qquad z_N' = z_{N-1} \,.$$

Then, in order to obtain the formula for $A_N(k_N, k_1, k_2, \dots, k_{N-1})$, it is necessary to commute $V_0(k_N, z_1')$ past $V_0(k_1, z_2') \dots V_0(k_{N-1}, z_N')$. The appropriate formula for this purpose is

$$V_0(k_i, z_i)\, V_0(k_j, z_j) = V_0(k_j, z_j)\, V_0(k_i, z_i)\, (-1)^{-2k_i \cdot k_j} \,. \tag{2.16}$$

Therefore, using momentum conservation and the mass shell condition, the complete commutation of $V_0(k_N, z_1')$ gives rise to a phase $(-1)^{2m^2}$. Fortunately, this phase is cancelled by one occurring in the transformation of $\prod_{i=1}^{N} |z_i - z_{i+1}|^{-m^2}$ suitably redefined.

Another important property of A_N is its factorizability. This is demonstrated by using

$$V_0(k, z) = z^{L_0^a} z^{m^2} V_0(k) z^{-L_0^a} \,, \tag{2.17}$$

($V_0(k)$ is a shortened notation for $V_0(k, 1)$), and carefully letting $z_1 \to \infty$, $z_1 \to 1$, $z_N \to 0$. In this way, after a suitable change of variables, one obtains

$$A_N(k_1, k_2, \dots, k_N) = \langle 0;k_1 | V_0(k_2) D_0 V_0(k_3) \dots D_0 V_0(k_{N-1}) | 0;k_N \rangle \,, \tag{2.18}$$

where the propagator D_0 is given by

$$D_0 = \int_0^1 \mathrm{d}x\, x^{L_0^a + m^2 - 1} (1-x)^{-m^2 - 1} \,, \tag{2.19}$$

and $|0;k\rangle$ represents a ground state with momentum k. (In the future we shall simply write $|0\rangle$, leaving the momentum implicit.)

A case of particular symmetry and beauty, discovered by Virasoro [8], occurs when $m^2 = -1$. In this case the leading Regge trajectory is $\alpha(s) = 1 + s$, and the ground state is a tachyon. Also, the propagator takes the simple form

$$D_0 = (L_0^a - 1)^{-1} \,, \tag{2.20}$$

so that the physical states satisfy the infinite-component wave equation

$$(L_0^a - 1)\varphi = 0 \,. \tag{2.21}$$

Furthermore, the on-shell physical states are subject to the subsidiary conditions

$$L_n^{\mathrm{a}}\varphi = 0\,, \qquad n = 1, 2, \ldots\,. \tag{2.22}$$

To prove this it is sufficient to show that $L_0^{\mathrm{a}} - L_n^{\mathrm{a}} - 1$ annihilates a physical tree

$$D_0 V_0(k_2)\, D_0 V_0(k_3) \ldots D_0 V_0(k_{N-1})\,|0\rangle\,.$$

Using eq. (2.9) and eq. (2.15) one has

$$(L_0^{\mathrm{a}} - L_n^{\mathrm{a}} - 1)\,\frac{1}{L_0^{\mathrm{a}} - 1}\,V_0(k)$$

$$= \frac{1}{L_0^{\mathrm{a}} + n - 1}\,(L_0^{\mathrm{a}} - L_n^{\mathrm{a}} + n - 1)\,V_0(k)$$

$$= \frac{1}{L_0^{\mathrm{a}} + n - 1}\,V_0(k)\,(L_0^{\mathrm{a}} - L_n^{\mathrm{a}} - 1)\,. \tag{2.23}$$

This calculation, together with the fact that $L_0^{\mathrm{a}} - L_n^{\mathrm{a}} - 1$ annihilates an on-shell ground state, proves the validity of the Virasoro subsidiary conditions. Notice that the gauge operators $L_0^{\mathrm{a}} - L_n^{\mathrm{a}} - 1$ annihilate physical states off mass shell as well as on mass shell, whereas L_n^{a} only annihilates on-mass-shell physical states.

One final construct of the conventional model that we shall need is the twist operator [21]. By definition, the twist operator, Ω_{a}, reverses the order of the external states along a tree, i.e., a tree and a twisted tree (see fig. 1) are related by

$$\Omega_{\mathrm{a}} V_0(k_1)\, D_0 V_0(k_2)\, D_0 \ldots D_0 V_0(k_{N-1})\,|0\rangle$$

$$= V_0(k_N)\, D_0 V_0(k_{N-1})\, D_0 \ldots D_0 V_0(k_2)\,|0\rangle\,. \tag{2.24}$$

A formula for Ω_{a} is

$$\Omega_{\mathrm{a}} = (-1)^{R_{\mathrm{a}}}\, \mathrm{e}^{-L_{-1}^{\mathrm{a}}} = (-1)^{L_0^{\mathrm{a}} + p^2}\, \mathrm{e}^{-L_{-1}^{\mathrm{a}}}\,. \tag{2.25}$$

A good way to prove that this formula satisfies eq. (2.24) is by letting it act on the integrand of a factorized form of eq. (2.11). Using the SU(1,1)

Fig. 1a, b. Two N-particle states related by twisting.

transformation rules, one shows in this way that

$$\Omega_a V_0(k_1, z_1)\, V_0(k_2, z_2) \ldots V_0(k_N, z_N)\, |0\rangle$$

$$= (-1)^{L_0^a + p^2}\, V_0(k_1, z_1 - 1)\, V_0(k_2, z_2 - 1) \ldots V_0(k_N, z_N - 1)\, |0\rangle$$

$$= (-1)^{p^2}\, V_0(k_1, 1 - z_1)\, V_0(k_2, 1 - z_2) \ldots V_0(k_N, 1 - z_N)\, |0\rangle$$

$$= V_0(k_N, 1 - z_N)\, V_0(k_{N-1}, 1 - z_{N-1}) \ldots V_0(k_1, 1 - z_1)\, |0\rangle \,. \tag{2.26}$$

The last step used eq. (2.16). The verification of twisting can now be completed by making the change of integration variables $z_i' = 1 - z_{N-i+1}$.

The most important fact about the twist operator for our purposes here lies in the observation that physical states are eigenstates of $\Omega^\dagger$,

$$\Omega^\dagger \varphi = (-1)^{R_a}\, e^{L_1^a}\, \varphi = (-1)^{R_a}\, \varphi \,. \tag{2.27}$$

Furthermore, when isospin is introduced by the Chan-Paton procedure [22] (which seems to be the only sensible one) and sums on all the permutations of external lines of tree graphs are included, the eigenvalue of $\Omega^\dagger$ may be identified with the charge conjugation quantum number of the neutral member of the multiplet under consideration. Therefore, if the G-parity of a physical state is known, its isospin may be readily deduced from its mass. The generalization of this procedure to SU(3) is not difficult, but will not be required in this paper.

3. NEW DUAL-RESONANCE MODEL

The Fock space described in sect. 2 can be enlarged by the addition of anticommuting creation and annihilation operators. Specifically let us consider the space generated by the oscillators of the preceding section together with $b_m^{\mu\dagger}$, $m = \frac{1}{2}, \frac{3}{2}, \ldots$. We postulate, in addition to eq. (2.3)

$$\{b_m^\mu, b_n^\nu\} = -g^{\mu\nu}\, \delta_{m+n,0} \,, \tag{3.1a}$$

$$[b_m^\mu, \alpha_n^\nu] = 0 \,, \tag{3.1b}$$

where, as a matter of convenience, we have defined $b_{-m}^\mu = b_m^{\dagger\mu}$. We next introduce the operator

$$H^\mu(\tau) = \sum_{m=-\infty}^{\infty} b_m^\mu\, e^{-im\tau} \,, \tag{3.2}$$

with the sum running over all half integers, of course. This operator has the anticommutation relations

$$\{H^\mu(\tau), H^\nu(\tau')\} = -2\pi g^{\mu\nu}\, \delta(\tau - \tau') \,. \tag{3.3}$$

Letting $\dot{H}^{\mu}(\tau) = \mathrm{d}/\mathrm{d}\tau\, H^{\mu}(\tau)$, define

$$L_n^{\mathrm{b}} = -\tfrac{1}{2}i\langle \mathrm{e}^{in\tau} :H(\tau)\cdot\dot{H}(\tau):\rangle = -\tfrac{1}{4} : \sum_{m=-\infty}^{\infty} (n+2m)\, b_{-m}\cdot b_{n+m} : . \tag{3.4}$$

For $n = 0$, eq. (3.4) becomes

$$L_0^{\mathrm{b}} = -\sum_{m=\frac{1}{2}}^{\infty} m b_{-m}\cdot b_m = R_{\mathrm{b}} , \tag{3.5}$$

while for $n = 1, 2, \ldots$ we have

$$L_n^{\mathrm{b}} = -\tfrac{1}{2}\sum_{m=\frac{1}{2}}^{\infty} (n+2m) b_{-m}\cdot b_{n+m} - \tfrac{1}{4}\sum_{m=\frac{1}{2}}^{n-\frac{1}{2}} (n-2m) b_m\cdot b_{n-m} . \tag{3.6}$$

Just as in eq. (2.9), we have the algebra

$$[L_m^{\mathrm{b}}, L_n^{\mathrm{b}}] = (m-n) L_{m+n}^{\mathrm{b}} , \tag{3.7}$$

aside from possible extra constants when $m + n = 0$. Eq. (3.1b) allows us to define

$$L_n = L_n^{\mathrm{a}} + L_n^{\mathrm{b}} , \tag{3.8}$$

with the knowledge that these operators have the same algebra. (This is analogous to addition of angular momentum.) The L_n operators will enable us to demonstrate that our new model has much the same algebraic structure as the conventional one.

The next step is to formulate N-point functions. For this purpose we replace the τ-variable by $z = \mathrm{e}^{i\tau}$ and define

$$V(k, z) = k\cdot H(z) V_0(k, z) . \tag{3.9}$$

The N-point function for N "pions" of mass m is then postulated to be

$$A_N(k_1, k_2, \ldots, k_N) = \int \mathrm{d}\mu_N(z) \prod_{i=1}^{N} \{z^{-\frac{1}{2}} |z_i - z_{i+1}|^{\frac{1}{2}-m^2}\}$$

$$\times \langle 0 | V(k_1, z_1)\, V(k_2, z_2) \ldots V(k_N, z_N) | 0 \rangle , \tag{3.10}$$

where the integration measure is given in eq. (2.12). The proof of cyclic symmetry proceeds as in the conventional model. First one proves Möbius invariance by considering the transformation of the vertex operator. Combining

$$[L_n^{\mathrm{b}}, H^{\mu}(z)] = z^n \left(z\frac{\mathrm{d}}{\mathrm{d}z} + \tfrac{1}{2}n\right) H^{\mu}(z) \tag{3.11}$$

with eq. (2.15), one finds that for $k^2 = m^2$,

$$[L_n, V(k, z)] = z^n \left[z\frac{\mathrm{d}}{\mathrm{d}z} + n(\tfrac{1}{2} - m^2) \right] V(k, z) . \tag{3.12}$$

Thus, in analogy with eq. (2.14),

$$\Lambda(T)\,\frac{V(k,z)}{\sqrt{z}}\,\Lambda^{-1}(T) = (a-cz')^{1-2m^2}\,\frac{V(k,z')}{\sqrt{z'}}\,. \tag{3.13}$$

The extra factors $\prod_{i=1}^{N} (a-cz')^{1-2m^2}$ coming from eq. (3.13) are precisely cancelled by factors that arise in the transformation of the product appearing in eq. (3.10). The proof of cyclic symmetry is completed by means of the appropriate Möbius transformation (see sect. 2) and a study of the commutation properties of the vertices. Besides the phases of eq. (2.16) one obtains minus signs from the anticommutation of the $k\cdot H$. (The δ-functions of eq. (3.3) do not contribute to the integral.) The phase $(-1)^{N-1}$ that comes about in this way is in fact required in compensating phases arising from the transformation of $\left\{\prod_{i=1}^{N} z_i^{-\frac{1}{2}}|z_i - z_{i+1}|^{\frac{1}{2}-m^2}\right\}$. Thus duality of the model is proved.

The amplitude A_N necessarily vanishes when N is odd, because the evaluation of $\langle 0|k_1\cdot H(z_1)\ldots k_N\cdot H(z_N)|0\rangle$ requires that b-operators be paired. We may therefore define the G-parity operator by

$$G = (-1)^{\sum_{m=\frac{1}{2}}^{\infty} b_{-m}\cdot b_m}\,, \tag{3.14}$$

and assert that our pions have negative G-parity.

By a construction analogous to that used to obtain eq. (2.18) we rewrite the N-point function in the form

$$A_N(k_1,k_2,\ldots,k_N) = \langle 0|k_1\cdot b_{\frac{1}{2}}\,V(k_2)DV(k_3)D\ldots DV(k_{N-1})k_N\cdot b_{-\frac{1}{2}}|0\rangle\,, \tag{3.15}$$

with

$$D = \int_0^1 \mathrm{d}x(1-x)^{-m^2-\frac{1}{2}}\,x^{m^2+L_0-\frac{3}{2}}\,, \tag{3.16}$$

$$L_0 = R_a + R_b - p^2\,. \tag{3.17}$$

For the special value of the pion mass $m^2 = -\frac{1}{2}$, one has

$$D = (L_0 - 1)^{-1}\,. \tag{3.18}$$

In this case there are also Virasoro-type subsidiary conditions, $L_n\varphi = 0$, $n = 1,2,\ldots$. In complete analogy with eq. (2.23), eq. (3.12) enables one to show that for $m^2 = -\frac{1}{2}$,

$$(L_0 - L_n - 1)\frac{1}{L_0 - 1}V(k) = \frac{1}{L_0 + n - 1}V(k)\,(L_0 - L_n - 1)\,. \tag{3.19}$$

Eq. (3.19), together with the fact that $L_0 - L_n - 1$ annihilates an on-shell pion (described by $k\cdot b_{-\frac{1}{2}}|0;k\rangle$ with $k^2 = -\frac{1}{2}$), proves the validity of the L_n subsidiary conditions. It is important to understand in this connection that we only consider states of the Fock space to be physical when they can be made from multipion systems.

Now that the model has been formulated, let us begin an exploration of its implications by calculating the four-point function depicted in fig. 2.

$$A_4 = \langle 0|k_1\cdot b_{\frac{1}{2}} V(k_2) D V(k_3) k_4\cdot b_{-\frac{1}{2}}|0\rangle$$

$$= \int_0^1 dx\, x^{m^2-s-1} (1-x)^{m^2-t-1} \Big\{ (k_1\cdot k_2)(k_3\cdot k_4)\sqrt{\frac{1-x}{x}}$$

$$+ (k_2\cdot k_3)(k_1\cdot k_4)\sqrt{\frac{x}{1-x}} - (k_1\cdot k_3)(k_2\cdot k_4)\sqrt{x(1-x)}\Big\}$$

$$= (k_1\cdot k_2)(k_3\cdot k_4) B(-\alpha(s), 1-\alpha(t)) + (k_2\cdot k_3)(k_1\cdot k_4) B(1-\alpha(s), 1-\alpha(t))$$

$$- (k_1\cdot k_3)(k_2\cdot k_4) B(1-\alpha(s), 1-\alpha(t)) , \tag{3.20}$$

where $\alpha(s) = s - m^2 + \frac{1}{2}$ is to be identified with the ρ-trajectory. If we require that there be no pole for $\alpha(s) = 0$, then it is necessary that

$$k_1\cdot k_2 = \tfrac{1}{2}(s - 2m^2) = \tfrac{1}{2}\alpha(s) , \tag{3.21}$$

and hence that $m^2 = -\frac{1}{2}$. Remarkably enough, this is the same mass condition required for the Virasoro gauges. When it is satisfied, eq. (3.20) may be simplified by some kinematics to

$$A_4 = -\tfrac{1}{4}\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} . \tag{3.22}$$

This is the formula proposed for $\pi\pi$ scattering by Lovelace and Shapiro [14] some time ago. The factor of $\frac{1}{4}$ has no deep significance since we are omitting coupling constants and our pion state is not normalized:

$$\langle\pi|\pi\rangle = \langle 0|k\cdot b_{\frac{1}{2}}\, k\cdot b_{-\frac{1}{2}}|0\rangle = -k^2 = \tfrac{1}{2} . \tag{3.23}$$

(Notice that it has positive norm only when it is a tachyon!) What we have succeeded in showing here is that for the special mass values $m_\pi^2 = -\frac{1}{2}$ and $m_\rho^2 = 0$, eq. (3.22) arises from a fully factorizable scheme.

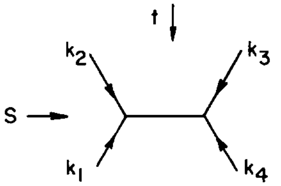

Fig. 2. Kinematics of $\pi\pi \to \pi\pi$.

4. BASIC CONJECTURE

At first sight the model we have constructed appears to have an unreasonable spectrum of states. The state described by the ground state of all the oscillators, which we shall call G, is evidently a scalar meson with $M^2 = -1$ occurring for $\alpha_\rho(s) = 0$. Furthermore, with one $b_{-\frac{1}{2}}$ and $N-1$ α_{-1} excitations, it should be possible to make a state of spin N and $M^2 = N-\frac{3}{2}$. Such states would therefore lie on a trajectory one unit higher than the π-trajectory - hence we call it the A-(for ancestor) trajectory. Our basic conjecture is that G *and the* A*-trajectory are spurious*. If true, this would imply the existence of a rather profound gauge algebra in the model, going beyond the straightforward generalization of the Virasoro gauges. We call it a conjecture because we have not yet fully proved it, but we have no doubt about its truth.

In the previous section it was shown that for special mass values the G-pole does not occur in $\pi\pi$ scattering, which implies that G does not couple to two pions. Another easy calculation is a demonstration that the pion ancestor π_A (the first state on the A-trajectory) does not couple to three pions. It is trivially decoupled from an even number of pions by G-parity, of course. The π_A state is given by $\epsilon\cdot b_{-\frac{1}{2}}|0\rangle$ with $\epsilon\cdot k = 0$ and $k^2 = -\frac{1}{2}$. Therefore, the amplitude for $\pi_A \to 3\pi$, found by almost identical arithmetic to that of eq. (3.20), is

$$\langle 0|\epsilon\cdot b_{\frac{1}{2}}\, V(k_2)\, D V(k_3)\, k_4\cdot b_{-\frac{1}{2}}|0\rangle = \epsilon\cdot k_2(k_3\cdot k_4)\, B(-\alpha(s), 1-\alpha(t))$$

$$+\, \epsilon\cdot k_4(k_2\cdot k_3)\, B(1-\alpha(s), -\alpha(t)) - \epsilon\cdot k_3(k_2\cdot k_4)\, B(1-\alpha(s), 1-\alpha(t))$$

$$= -\tfrac{1}{2}\epsilon\cdot(k_2+k_3+k_4)\,\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} = 0\ .$$

The spurious character of the ground state and A-trajectory are not independent phenomena. The following arguments show that it is sufficient to know that either the state G or the state π_A is spurious to establish our conjecture. By definition, a spurious state is one that does not couple to any multiparticle system whose constituents are all physical. Therefore, a state which couples to a two-particle system consisting of a physical state and a spurious state must itself be spurious. We use this simple truth to establish the equivalence of π_A and G being spurious by showing that there is a non-vanishing G - π - π_A vertex. Referring to fig. 3a, we have

$$\langle 0|V(k_2)\epsilon\cdot b_{-\frac{1}{2}}|0\rangle = -\epsilon\cdot k_2\ . \tag{4.1}$$

But $\epsilon\cdot k_2 \neq 0$ is kinematically possible, so π_A is spurious if and only if G is. We can further show that if G is spurious the entire A-trajectory is spurious. In the notation of fig. 3b, it is sufficient to show that $V(k_2)|0\rangle$ contains the states of the A-trajectory. In particular, for $k^2 = N-\frac{1}{2}$ we can obtain a state of spin $N+1$ from the term

$$k_2\cdot b_{-\frac{1}{2}}(k_2\cdot\alpha_{-1})^N|0\rangle\ ,$$

which arises in the expansion of $V(k_2)|0\rangle$. This term includes a spin $N+1$ part because k_2 contains a non-vanishing component orthogonal to k. Therefore, our conjecture would be established by proving G to be spurious. We have already seen that $\langle G|2\pi\rangle = 0$. In the appendix we show that $\langle G|4\pi\rangle = \langle G|6\pi\rangle = 0$. While sufficient to convince us that G is spurious, the calculations of the appendix do not constitute a complete proof of that fact.

Fig. 3. a. The $G - \pi - \pi_A$ vertex. b. The $A - \pi - G$ vertex.

The nicest way to prove our conjecture would be by finding additional gauge operators (besides the L_n operators). So far we have been unsuccessful in this quest. In our opinion, the solution of this problem is required to obtain a complete understanding of what is going on in this model. It is quite remarkable that by adding superstructure to the conventional model, one can obtain a richer gauge group. It is *not* a structure that is present and previously overlooked in the conventional model. All the physical states evidently involve b-operators in their description, since otherwise one could isolate a subspace involving states built from α-states only.

It seems likely that one of the extra gauge operators, S, will commute with L_0. The spuriousness of the ground state would then follow from the formula $\langle 0|S|0\rangle \neq 0$. Such a gauge operator is potentially very powerful in as much as there is no obvious limit to the number of spurious states that $S^\dagger$ can produce. This is in contrast to the action of L_{-n} which can only make as many spurious states of a given mass squared as the total number of states n units lower in mass squared. On the other hand, since $S^\dagger$ has the possibility of making so many spurious states it is conceivable that it makes all the spurious states, including those made by L_{-n} operators. Such a gauge operator we would choose to call a *supergauge*.

In our unsuccessful attempt to find the new gauge operators, we were led to construct some additional operators. We shall briefly mention them here since they might possibly prove useful to someone seeking the new gauges. These operators (quite similar to ones of Ramond's fermion model [23]) are

$$G_m = \sqrt{2}\langle e^{im\tau} P(\tau)\cdot H(\tau)\rangle = \sum_{n=-\infty}^{\infty} \alpha_n \cdot b_{m-n} \,, \qquad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots \,, \tag{4.2}$$

and have the commutators and anticommutators

$$[L_m, G_n] = (\tfrac{1}{2}m - n) G_{m+n} \,, \tag{4.3}$$

$$\{G_m, G_n\} = 2L_{m+n} \,. \tag{4.4}$$

The G_m are certainly *not* the new gauges, but they might play a role in their construction.

5. EXPLORATION OF THE SPECTRUM

One of the first questions that arises in the consideration of quantum numbers is how to determine isospins. For this purpose the twist operator is very useful because, as was pointed out in sect. 2, its eigenvalue gives the charge-conjugation quantum number for a physical state. Thus, as the G-parities of the states are easily determined by eq. (3.14), the isospins are immediately deduced. It is implicit in this reasoning that the only possibilities are $I = 0$ and $I = 1$ since we use the Chan-Paton procedure [21].

A natural guess for the twist operator is to replace eq. (2.25) by $(-1)^{R_a+R_b}\, e^{-L_{-1}}$. This is almost, but not quite, correct. From eq. (3.11) it follows that

$$e^{-L^b_{-1}} \frac{H_\mu(z)}{\sqrt{z}}\, e^{L^b_{-1}} = \frac{H_\mu(z-1)}{\sqrt{z-1}}\,, \tag{5.1}$$

$$(-1)^{R_b} H_\mu(z-1)\,(-1)^{-R_b} = H_\mu(1-z)\,. \tag{5.2}$$

Therefore, repeating eq. (2.26) for the b-portions of the vertices,

$$\begin{aligned}(-1)^{R_b}\, e^{-L^b_{-1}}\, k_1\cdot\frac{H(z_1)}{\sqrt{z_1}}\, k_2\cdot\frac{H(z_2)}{\sqrt{z_2}} \dots k_N\cdot\frac{H(z_N)}{\sqrt{z_N}}\,|0\rangle \\ = (-1)^{\frac{1}{2}N}\, k_1\cdot\frac{H(1-z_1)}{\sqrt{1-z_1}}\, k_2\cdot\frac{H(1-z_2)}{\sqrt{1-z_2}} \dots k_N\cdot\frac{H(1-z_N)}{\sqrt{1-z_N}}\,|0\rangle \\ = (-1)^{\frac{1}{2}N+\frac{1}{2}N(N-1)}\, k_N\cdot\frac{H(1-z_N)}{\sqrt{1-z_N}}\, k_{N-1}\cdot\frac{H(1-z_{N-1})}{\sqrt{1-z_{N-1}}} \dots k_1\cdot\frac{H(1-z_1)}{\sqrt{1-z_1}}\,|0\rangle\,.\end{aligned} \tag{5.3}$$

In the last step we have used eq. (3.3), dropping the δ-functions since they do not contribute to the integrals. We thus conclude that the twist is given by

$$\Omega = (-1)^{R_a+R_b-\frac{1}{2}N^2}\, e^{-L_{-1}}\,, \tag{5.4}$$

for an N-pion state. Eq. (5.4) is not yet in suitable operator form, however, since N has no operator representation. We therefore note that eq. (5.4) is equivalent to $(-1)^{R_a+R_b}\, e^{-L_{-1}}$ for even-G states and $(-1)^{R_a+R_b-\frac{1}{2}}\, e^{-L_{-1}}$ for odd-G states. Thus we may re-express eq. (5.4) in the form

$$\Omega = (-1)^{R_a+R_b-\frac{1}{4}(1-G)}\, e^{-L_{-1}}\,. \tag{5.5}$$

We can now correlate isospins with masses. States with $M^2 = -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \dots$ or $M^2 = 0, 2, 4, \dots$ must be isovector, whereas states with $M^2 = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \dots$ or $M^2 = 1, 3, 5, \dots$ must be isoscalar. In particular, the ρ and π are both isovector and may not be isospin doubled (i.e., degenerate with isoscalars). It also follows that there cannot be a σ degenerate with the ρ or a ρ' degenerate with f^0. Furthermore, the amplitude for $\pi\pi \to \pi\pi$ cannot contain odd daughter trajectories. The reader may wonder whether we were forced to introduce isospin or whether the model is mathematically tenable without it.

The answer is that when the operator of eq. (3.14) is identified as G-parity the isospins are determined. If we insist that all states are isoscalar, then this operator must correspond to some other unphysical "modulo 2" rule.

There is a slight ambiguity in the parity assignments of states. The reason for this is that P and GP are equally good operators, which, in the absence of electromagnetism, fermions and strange mesons, cannot be distinguished. This fact leaves the freedom to switch the parity assignments of all the odd-G states. Therefore, we may *define* the π to be a pseudoscalar. All other parities are then determined.

We now embark on a detailed investigation of the spectrum. Not having the full gauge algebra in hand, another method for identifying spurious states is required. The method employed is to accept the validity of our conjecture that the ground state is spurious and then to deduce the existence of other spurious states therefrom. For example, we showed in sect. 4 that this implies that the A-trajectory is spurious. At the lowest mass squared, $M^2 = -1$, there is one spurious state, the ground state. At $M^2 = -\frac{1}{2}$ there is the physical pion given by $k \cdot b_{-\frac{1}{2}}|0\rangle$. This state has positive norm, but, as we mentioned before, this fact depends on its being a tachyon. The three spurious states, π_A, are also at $M^2 = -\frac{1}{2}$. (We count a state of spin S as $2S+1$ states.)

At $M^2 = 0$ there are ten candidates for states given by $\alpha_{-1}^{\mu}|0\rangle$ and $b_{-\frac{1}{2}}^{\mu} b_{-\frac{1}{2}}^{\nu}|0\rangle$. One spurious state is given by

$$L_{-1}|0\rangle = \sqrt{2}\, k \cdot \alpha_{-1}|0\rangle \,. \tag{5.6}$$

This spurious scalar happens to have zero norm (because $k^2 = 0$). Additional spurious states can be found by studying the π-π_A system which can couple to spurious states only. Expanding out

$$V(k_2)\, \epsilon_1 \cdot b_{-\frac{1}{2}}|0\rangle \,, \tag{5.7}$$

with $\epsilon_1 \cdot k_1 = 0$, one finds that

$$\epsilon \cdot b_{-\frac{1}{2}}\, k \cdot b_{-\frac{1}{2}}|0\rangle \,, \tag{5.8}$$

$$\epsilon_{\mu\nu\lambda\sigma} k^{\mu} \epsilon^{\nu} b_{-\frac{1}{2}}^{\lambda} b_{-\frac{1}{2}}^{\sigma}|0\rangle \,, \tag{5.9}$$

are a spurious vector and axial vector, respectively. They also have zero norm. The only remaining candidate for a physical state is

$$|\rho\rangle = (\epsilon \cdot \alpha_{-1} + \beta\, \epsilon \cdot b_{-\frac{1}{2}}\, k \cdot b_{-\frac{1}{2}})|0\rangle \,. \tag{5.10}$$

This state is orthogonal to the spurious vector for any choice of β. To determine β one expands out the physical 2π state

$$V(k_2)\, k_1 \cdot b_{-\frac{1}{2}}|0\rangle \,.$$

This yields the result $\beta = \sqrt{2}$. The ρ is easily seen to have positive norm.

At the next mass level, $M^2 = \frac{1}{2}$, the states must be odd-G and isoscalar. There are 24 candidates, which we now enumerate. First, we find the spurious states given by the standard gauges:

$$L_{-1}\, k \cdot b_{-\frac{1}{2}}|0\rangle = (\sqrt{2}\, k \cdot \alpha_{-1}\, k \cdot b_{-\frac{1}{2}} + k \cdot b_{-\frac{3}{2}})|0\rangle \,, \tag{5.11a}$$

$$L_{-1}\epsilon\cdot b_{-\frac{1}{2}}|0\rangle = (\sqrt{2}\,k\cdot\alpha_{-1}\epsilon\cdot b_{-\frac{1}{2}}+\epsilon\cdot b_{-\frac{3}{2}})|0\rangle\ . \tag{5.11b}$$

Both of these spurious states have zero norm. Therefore, as explained by Del Guidice and Di Vecchia [24], it follows that the conjugate zero-norm states (obtained by $k\to -k$) are also spurious. Altogether this produces eight spurious zero-norm states. We obtain further spurious states by considering the two-particle system consisting of a π and a ground state. Expanding out $V(k_2)\,|0\rangle$, and doing some kinematics, one finds the following additional spurious states:

$$\epsilon_{\mu\nu}\alpha^{\mu}_{-1}b^{\nu}_{-\frac{1}{2}}|0\rangle\ , \qquad \text{(spin 2)} \tag{5.12a}$$

$$\alpha_{-1}\cdot b_{-\frac{1}{2}}|0\rangle\ , \tag{5.12b}$$

$$[\epsilon\cdot b_{-\frac{3}{2}}-\sqrt{2}(\epsilon\cdot\alpha_{-1}\,k\cdot b_{-\frac{1}{2}}+k\cdot\alpha_{-1}\,\epsilon\cdot b_{-\frac{1}{2}})]|0\rangle\ . \tag{5.12c}$$

The occurrence of the spin-2 state in (5.12a) was expected since it lies on the A-trajectory. We have now found a total of 17 spurious states, so 7 remain to be investigated.

Six of the remaining states with $M^2=\frac{1}{2}$ are

$$|\omega_1\rangle = \epsilon_{\mu\nu\lambda\sigma}\epsilon^{\mu}\alpha^{\nu}_{-1}b^{\lambda}_{-\frac{1}{2}}k^{\sigma}|0\rangle \tag{5.13a}$$

$$|\omega_2\rangle = \tfrac{1}{6}\epsilon_{\mu\nu\lambda\sigma}\epsilon^{\mu}b^{\nu}_{-\frac{1}{2}}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}|0\rangle\ . \tag{5.13b}$$

These states have the normalization $\langle\omega_1|\omega_1\rangle=1$ and $\langle\omega_2|\omega_2\rangle=-1$ and have quantum numbers appropriate for the ω. A spurious combination is generated by considering the two-particle state consisting of a pion and a spurious axial ρ, i.e.,

$$V(k_2)\,\epsilon_{\mu\nu\lambda\sigma}k_1^{\mu}\epsilon_1^{\nu}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}|0\rangle\ .$$

In this way one finds the spurious state $|\omega_1\rangle-\sqrt{2}|\omega_2\rangle$, which has negative norm. A physical ω is obtained from a π plus a physical ρ, i.e., $V(k_2)(\epsilon_1\cdot\alpha_{-1}+\sqrt{2}\epsilon_1\cdot b_{-\frac{1}{2}}k_1\cdot b_{-\frac{1}{2}})|0\rangle$. This calculation yields a state that is orthogonal to the spurious ω, namely

$$|\omega\rangle=\sqrt{2}|\omega_1\rangle-|\omega_2\rangle\ . \tag{5.14}$$

It is very pleasing to discover that this state has positive norm. The final remaining state with $M^2=\frac{1}{2}$ is the scalar $\epsilon_{\mu\nu\lambda\sigma}k^{\mu}b^{\nu}_{-\frac{1}{2}}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}|0\rangle$. This state is spurious, because it also couples to a pion and the spurious axial ρ. Altogether, we have found that only 3 of the possible 24 states with $M^2=\frac{1}{2}$ are physical. Evidently a very powerful gauge group is at work!

The amplitude for the process $\omega\to 3\pi$ can be calculated from $\langle\omega|V(k_3)DV(k_2)\,k_1\cdot b_{-\frac{1}{2}}|0\rangle$. One easily finds

$$A(\omega \to 3\pi) = \tfrac{1}{3}\epsilon_{\mu\nu\lambda\sigma}\epsilon^{\mu}k_1^{\nu}k_2^{\lambda}k_3^{\sigma}B(1-\alpha_\rho(s), 1-\alpha_\rho(t)) . \tag{5.15}$$

Eq. (5.15) is recognized to be the original Veneziano formula [1].

To extend the discussion of the spectrum given above to higher masses would be a very tedious procedure. For example, in the case $M^2 = 1$ there are 55 states to study. We therefore choose to switch our attention now to the leading Regge trajectories. We have already seen that the A-trajectory is spurious. Another trajectory we can quickly dispose of is a parity double of the ρ - f^0 trajectory. Its operator description at spin N is

$$\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma}k^{\nu}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}\alpha^{\mu_2}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle . \tag{5.16}$$

The curly brackets in (5.16) imply the extraction of maximum spin. This means symmetrizing in μ indices, and removing all traces and divergences. This trajectory is spurious because it couples to the two-particle state consisting of a physical π plus a spurious π_A.

The remaining trajectories with unit intercept are the ρ and f^0. In particular one wishes to determine their degeneracy. There are three candidates for spin N states on these trajectories:

$$\left\{\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle , \tag{5.17a}$$

$$k\cdot b_{-\frac{1}{2}}\left\{b^{\mu_1}_{-\frac{1}{2}}\alpha^{\mu_2}_{-1}\alpha^{\mu_3}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle , \tag{5.17b}$$

$$\left\{b^{\mu_1}_{-\frac{1}{2}}b^{\mu_2}_{-\frac{3}{2}}\alpha^{\mu_3}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle . \tag{5.17c}$$

By considering the physical two-pion state $V(k_2)k_1\cdot b_{-\frac{1}{2}}|0\rangle$, and doing some straightforward kinematics, one establishes that the following state is physical:

$$\left\{\left[\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1} - \sqrt{2}k\cdot b_{-\frac{1}{2}}b^{\mu_1}_{-\frac{1}{2}}\alpha^{\mu_2}_{-1} + (N-1)b^{\mu_1}_{-\frac{3}{2}}b^{\mu_2}_{-\frac{1}{2}}\right]\alpha^{\mu_3}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle . \tag{5.18}$$

For $N = 1$, (5.18) represents the physical ρ, $\left(\alpha^{\mu_1}_{-1} - \sqrt{2}k\cdot b_{-\frac{1}{2}}b^{\mu_1}_{-\frac{1}{2}}\right)|0\rangle$. The norm of each of the states on the physical ρ - f^0 trajectory of (5.18) is positive. Consideration of the two-particle state consisting of a physical π and a spurious π_A gives rise to the following spurious state

$$\left\{\left[(N-1)\,b^{\mu_1}_{-\frac{1}{2}}b^{\mu_2}_{-\frac{3}{2}} + \frac{1}{\sqrt{2}}k\cdot b_{-\frac{1}{2}}b^{\mu_1}_{-\frac{1}{2}}\alpha^{\mu_2}_{-1}\right]\alpha^{\mu_3}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle . \tag{5.19}$$

Similarly, the two-particle state consisting of a physical π and the spurious state $\epsilon\cdot\alpha_{-1}k\cdot b_{-\frac{1}{2}}|0\rangle$ gives the spurious state

$$\left\{\left[\alpha^{\mu_1}_{-1}\alpha^{\mu_2}_{-1} - \sqrt{2}\,k\cdot b_{-\frac{1}{2}}b^{\mu_1}_{-\frac{1}{2}}\alpha^{\mu_2}_{-1} + (N-2)\,b^{\mu_1}_{-\frac{3}{2}}b^{\mu_2}_{-\frac{1}{2}}\right]\alpha^{\mu_3}_{-1}\dots\alpha^{\mu_N}_{-1}\right\}|0\rangle . \tag{5.20}$$

The spurious states in (5.19) and (5.20) can be verified to be orthogonal to the physical state in (5.18) as must be the case, of course. We conclude that the physical ρ and f^0 trajectories are exchange-degenerate with one another, but are otherwise non-degenerate and contain positive norm states only.

Next let us consider the π-trajectory. It is expected to be exchange-degenerate with an odd-signature isoscalar trajectory, although we found no $J^P = 1^+$ state at $M^2 = \frac{1}{2}$. There are four candidates of spin N,

$$k\cdot b_{-\frac{1}{2}} \left\{ \alpha^{\mu 1}_{-1} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} |0\rangle\ , \tag{5.21a}$$

$$k\cdot \alpha_{-1} \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} |0\rangle\ , \tag{5.21b}$$

$$\left\{ b^{\mu 1}_{-\frac{3}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} |0\rangle\ , \tag{5.21c}$$

$$\left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-2} \alpha^{\mu 3}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} |0\rangle\ . \tag{5.21d}$$

A spurious combination can be obtained through use of the gauge operator L_{-1},

$$\begin{aligned} L_{-1} \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} |0\rangle & \\ = \Big[\sqrt{2}\, k\cdot\alpha_{-1} \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} &+ \left\{ b^{\mu 1}_{-\frac{3}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} \\ &+ (N-1) \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-2} \alpha^{\mu 3}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} \Big] |0\rangle\ . \end{aligned} \tag{5.22}$$

The states of (5.22) have zero norm, and therefore the conjugate states obtained by replacing k by $-k$ are also spurious. A third spurious combination is contained in the two-particle state consisting of a π and a G, namely

$$\begin{aligned} \Big[(N-\tfrac{1}{2}) \left\{ b^{\mu 1}_{-\frac{3}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} &+ 2(N-\tfrac{1}{2})(N-1) \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-2} \alpha^{\mu 3}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} \\ - \frac{1}{\sqrt{2}} k\cdot b_{-\frac{1}{2}} \left\{ \alpha^{\mu 1}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} &- \frac{N}{\sqrt{2}} k\cdot\alpha_{-1} \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} \Big] |0\rangle\ . \end{aligned} \tag{5.23}$$

The remaining candidate for the physical trajectory is the trajectory orthogonal to the three spurious ones,

$$\Big[\sqrt{2} k\cdot b_{-\frac{1}{2}} \left\{ \alpha^{\mu 1}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} + N \left\{ b^{\mu 1}_{-\frac{1}{2}} \alpha^{\mu 2}_{-2} \alpha^{\mu 3}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} - 2N \left\{ b^{\mu 1}_{-\frac{3}{2}} \alpha^{\mu 2}_{-1} \cdots \alpha^{\mu N}_{-1} \right\} \Big] |0\rangle\ . \tag{5.24}$$

The spin-N state in (5.24) has positive norm. To see that it is in fact physical one can consider the states coupling to a π plus a physical ρ. When this is done, one finds (5.24) multiplied by $N-1$ plus an irrelevant admixture of a zero-norm spurious state. One therefore deduces that (5.24) is physical except for $N=1$! This exception means that the Regge-residue functions have a zero at this point. The absence of this state is fortunate from a phenomenological point of view since there is no good candidate for a $J^P = 1^+$, $I^G = 0^-$ state with mass around 1 GeV. On the other hand, there is a candidate for the pion recurrence at mass 1640 MeV.

Next we consider the ω- A_2 trajectory, which happens to be degenerate with the π-trajectory in our model but is distinguished by its parity. There are three candidates for this trajectory:

$$\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} b^{\nu}_{-\frac{1}{2}} b^{\lambda}_{-\frac{1}{2}} b^{\sigma}_{-\frac{1}{2}} \alpha^{\mu_2}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}|0\rangle\ , \tag{5.25a}$$

$$\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} k^{\nu} \alpha^{\lambda}_{-1} b^{\sigma}_{-\frac{1}{2}} \alpha^{\mu_2}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}|0\rangle\ , \tag{5.25b}$$

$$\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} k^{\nu} b^{\lambda}_{-\frac{1}{2}} b^{\sigma}_{-\frac{1}{2}} b^{\mu_2}_{-\frac{3}{2}} \alpha^{\mu_3}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}|0\rangle\ . \tag{5.25c}$$

A spurious combination can be obtained from the state consisting of a π and a spurious ρ, and a different spurious combination can be found from a π and a spurious f^0. Thus there is at most one physical trajectory. Physical states are generated by a π plus the physical ρ. One finds the result

$$\Big[(1-2N)\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} b^{\nu}_{-\frac{1}{2}} b^{\lambda}_{-\frac{1}{2}} b^{\sigma}_{-\frac{1}{2}} \alpha^{\mu_2}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}$$
$$+6\sqrt{2}\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} k^{\nu} \alpha^{\lambda}_{-1} b^{\sigma}_{-\frac{1}{2}} \alpha^{\mu_2}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}$$
$$+3\sqrt{2}(N-1)\left\{\epsilon^{\mu_1}{}_{\nu\lambda\sigma} k^{\nu} b^{\lambda}_{-\frac{1}{2}} b^{\sigma}_{-\frac{1}{2}} b^{\mu_2}_{-\frac{3}{2}} \alpha^{\mu_3}_{-1} \ldots \alpha^{\mu_N}_{-1}\right\}\Big]|0\rangle\ . \tag{5.26}$$

These states also have positive norm. The ω and A_2 trajectories are exchange degenerate and quite satisfactory in all respects.

On the basis of the physical trajectories that have been found so far, we can formulate a remarkable empirical rule. Some of the trajectories of our model are approximately one-half unit higher than their experimental values, whereas others are very close to thier experimental locations. Furthermore these two classes of trajectories appear to be distinguished by their parities. The precise statement is that those states with normal-parity couplings to pions, i.e., $P = G(-1)^S$, where S is the spin of the state, occur $\frac{1}{2}$ unit too high. Equivalently, the mass is given by $M^2 = M^2_{\rm expt} - \frac{1}{2}$. On the other hand, abnormal-parity states, those for which $P = -G(-1)^S$, have $M^2 = M^2_{\rm expt}$. One may conjecture that this rule will give the model predictive power for states that may not have been seen experimentally. The relationship between the model and the real world is quite intriguing.

We conclude this section with a few comments about other states. Two states are conspicuous by their absence. They are the σ-meson, which is generally believed to be a broad resonance degenerate with the ρ, and the A_1 meson. The experimental evidence in each case is still generally considered to be inconclusive. Whether our model constitutes an argument against them is debatable.

There are two other physical states we shall comment on. One is a scalar meson degenerate with the f^0. Its existence is already evident from eq. (3.22). We have not examined its degeneracy or norm. Another state of considerable interest is a pseudoscalar at the same mass. Its quantum numbers are those of the η, but the empirical mass rule strongly suggests that an identification be made with the $\eta'(960)$. There are two candidates for this state

$$|\eta_1'\rangle = \frac{1}{\sqrt{12}}\epsilon_{\mu\nu\lambda\sigma}k^{\mu}\alpha^{\nu}_{-1}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}|0\rangle \; , \tag{5.27a}$$

$$|\eta_2'\rangle = \frac{1}{24}\epsilon_{\mu\nu\lambda\sigma}b^{\mu}_{-\frac{1}{2}}b^{\nu}_{-\frac{1}{2}}b^{\lambda}_{-\frac{1}{2}}b^{\sigma}_{-\frac{1}{2}}|0\rangle \; . \tag{5.27b}$$

These states have the normalization $\langle\eta_1'|\eta_1'\rangle = 1$ and $\langle\eta_2'|\eta_2'\rangle = -1$. By constructing the coupling to a π and a member of the A_2 trajectory it is possible to show that the physical combination is

$$|\eta'\rangle = \sqrt{3}|\eta_1'\rangle + \sqrt{2}|\eta_2'\rangle \; , \tag{5.28}$$

which has positive norm. The orthogonal combination is spurious.

6. THE SIX-PION AMPLITUDE

In order to acquire a more concrete understanding of the workings of the model, it is helpful to work out explicitly the six-pion amplitude. For the configuration shown in fig. 4 we have the formula

$$A_6 = \langle 0|k_1\cdot b_{\frac{1}{2}}\, V(k_2)\, DV(k_3)\, DV(k_4)\, DV(k_5)\, k_6\cdot b_{-\frac{1}{2}}|0\rangle \; . \tag{6.1}$$

The evaluation of this expression gives rise to 15 terms corresponding to the different ways of pairing the six sets of b-oscillators. After a straightforward, but somewhat tedious, calculation one finds

$$\begin{aligned} A_6 = \int \frac{du_{12}\,du_{13}\,du_{14}}{(1-u_{12}u_{13})(1-u_{13}u_{14})}\, & u_{12}^{-\alpha_\rho(s_{12})}\, u_{13}^{-\alpha_\pi(s_{13})-1}\, u_{23}^{-\alpha_\rho(s_{23})} \\ & \times u_{34}^{-\alpha_\rho(s_{34})}\, u_{45}^{-\alpha_\rho(s_{45})}\, u_{24}^{-\alpha_\pi(s_{24})-1}\, u_{35}^{-\alpha_\pi(s_{35})-1}\, u_{14}^{-\alpha_\rho(s_{14})} \\ & \times u_{25}^{-\alpha_\rho(s_{25})}\, Y(k,u) \; , \end{aligned} \tag{6.2}$$

where the u_{ij} variables correspond to the notation of Goebel and Sakita [2] and

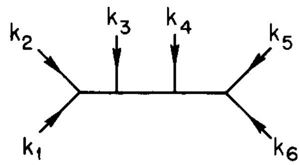

Fig. 4. Kinematics of the six-pion amplitude.

$$
\begin{aligned}
Y(k,u) = &\left[k_1\cdot k_2\, k_3\cdot k_4\, k_5\cdot k_6 \frac{1}{u_{12}u_{14}u_{34}} + k_1\cdot k_6\, k_4\cdot k_5\, k_2\cdot k_3 \frac{1}{u_{23}u_{45}u_{25}}\right] \\
&+ \left[k_1\cdot k_2\, k_3\cdot k_6\, k_4\cdot k_5 \frac{u_{24}}{u_{12}u_{45}} + k_1\cdot k_4\, k_2\cdot k_3\, k_5\cdot k_6 \frac{u_{35}}{u_{23}u_{14}}\right. \\
&\left. + k_1\cdot k_6\, k_2\cdot k_5\, k_3\cdot k_4 \frac{u_{13}}{u_{34}u_{25}}\right] + \left[k_1\cdot k_4\, k_2\cdot k_6\, k_3\cdot k_6\, u_{13}u_{24}\right. \\
&\left. + k_1\cdot k_3\, k_2\cdot k_5\, k_4\cdot k_6\, u_{24}u_{35} + k_1\cdot k_5\, k_2\cdot k_4\, k_3\cdot k_6\, u_{13}u_{35}\right] \\
&- \left[k_1\cdot k_2\, k_3\cdot k_5\, k_4\cdot k_6 \frac{u_{24}}{u_{12}} + k_1\cdot k_5\, k_2\cdot k_3\, k_4\cdot k_6 \frac{u_{35}}{u_{23}}\right. \\
&+ k_1\cdot k_5\, k_2\cdot k_6\, k_3\cdot k_4 \frac{u_{13}}{u_{34}} + k_1\cdot k_3\, k_2\cdot k_6\, k_4\cdot k_5 \frac{u_{24}}{u_{45}} \\
&\left. + k_1\cdot k_3\, k_2\cdot k_4\, k_5\cdot k_6 \frac{u_{35}}{u_{14}} + k_1\cdot k_6\, k_3\cdot k_5\, k_2\cdot k_4 \frac{u_{13}}{u_{25}}\right] \\
&- \left[k_1\cdot k_4\, k_2\cdot k_5\, k_3\cdot k_6\, u_{13}u_{24}u_{35}\right]. \qquad (6.3)
\end{aligned}
$$

In this formula we have grouped terms together which transform into one another under cyclic permutations of the momenta. Thus the cyclic symmetry of A_6 is manifest.

Fortunately there is a trick that allows a significant simplification in the formula for A_6. We know that the pion ancestor, π_A, does not couple to five pions. (This is in fact proved since we show in the appendix that G does not couple to six pions.) On the other hand, the amplitude for $\pi_A \to 5\pi$ is obtained by replacing the momentum k_1 in eq. (6.3) by a polarization vector ϵ. The only way the various terms can then add to zero is if the coefficients of $\epsilon\cdot k_2, \epsilon\cdot k_3, \dots, \epsilon\cdot k_6$ are all equal so that the sum is proportional to $\epsilon\cdot(k_2+k_3+k_4+k_5+k_6) = 0$. Although it is really not necessary to do so, we have checked that the five terms in question are, in fact, equal. Therefore, returning to the six-pion amplitude, we may replace $Y(k,u)$ by the coefficient of $k_1\cdot k_6$, say. This amounts to replacing Y by

$$\widetilde{Y}(k,u) = k_4\cdot k_5\, k_2\cdot k_3 \frac{1}{u_{23}u_{45}u_{25}} + k_2\cdot k_5\, k_3\cdot k_4 \frac{u_{13}}{u_{34}u_{25}} + k_3\cdot k_5\, k_2\cdot k_4 \frac{u_{13}}{u_{25}} . \tag{6.4}$$

The verification of cyclic symmetry when A_6 is expressed in terms of $\widetilde{Y}$ is not manifest. One method is to discover judicious integrations by parts.

The amplitude for $\rho \to 4\pi$ is obtained by taking the residue of A_6 at $\alpha_\rho(s_{14}) = 1$. One finds (see fig. 5)

$$A(\rho\to 4\pi) = \int \frac{du_{12}du_{13}}{1-u_{12}u_{13}} u_{12}^{-\alpha_\rho(s_{12})} u_{13}^{-\alpha_\pi(s_{13})-1} u_{23}^{-\alpha_\rho(s_{23})}$$
$$\times u_{34}^{-\alpha_\rho(s_{34})} u_{24}^{-\alpha_\pi(s_{24})-1} \Big[\epsilon\cdot k_4\, k_2\cdot k_3 \frac{1}{u_{23}}$$
$$+\epsilon\cdot k_2\, k_3\cdot k_4 \frac{u_{13}}{u_{34}} + \epsilon\cdot k_3\, k_2\cdot k_4 u_{13}\Big] . \tag{6.5}$$

The amplitude for $\pi\rho \to \pi\rho$ (see fig. 6a) then follows from taking the residue of eq. (6.5) at $\alpha_\rho(s_{12}) = 1$:

$$A(\pi\rho\to\pi\rho) = -\tfrac{1}{2}\epsilon_1\cdot\epsilon_2(1-\alpha_\pi(s)-\alpha_\rho(t))\, B(1-\alpha_\pi(s), 1-\alpha_\rho(t))$$
$$+ \epsilon_1\cdot k_3\, \epsilon_2\cdot k_4\, B(-\alpha_\pi(s), 1-\alpha_\rho(t))$$
$$- \epsilon_1\cdot k_4\, \epsilon_2\cdot k_3\, B(1-\alpha_\pi(s), 1-\alpha_\rho(t)) . \tag{6.6}$$

An alternative method of deriving eq. (6.6) is by directly evaluating

$$\langle 0|(\epsilon_1\cdot\alpha_1 + \sqrt{2}\epsilon_1\cdot b_{\frac{1}{2}} k_1\cdot b_{\frac{1}{2}})\, V(k_3) D V(k_4)\, (\epsilon_2\cdot\alpha_{-1} + \sqrt{2}\epsilon_2\cdot b_{-\frac{1}{2}} k_2\cdot b_{-\frac{1}{2}})|0\rangle . \tag{6.7}$$

Eq. (6.6) clearly contains a π-pole in the s-channel and a ρ-pole in the t-channel. Somewhat less obvious is the nature of the pole at $\alpha_\pi(s) = 1$. According to the analysis of sect. 5 the only possible singularity at this mass is the ω. The residue of eq. (6.6) at $\alpha_\pi(s) = 1$ is

$$\beta(t) = \tfrac{1}{2}\alpha_\rho(t)\epsilon_1\cdot\epsilon_2 + \alpha_\rho(t)\epsilon_1\cdot k_3\, \epsilon_2\cdot k_4 - \epsilon_1\cdot k_4\, \epsilon_2\cdot k_3 . \tag{6.8}$$

On the other hand, by a straightforward calculation one can show that for $s = \frac{1}{2}$,

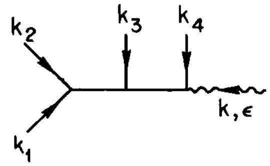

Fig. 5. Kinematics for $\rho \to 4\pi$.

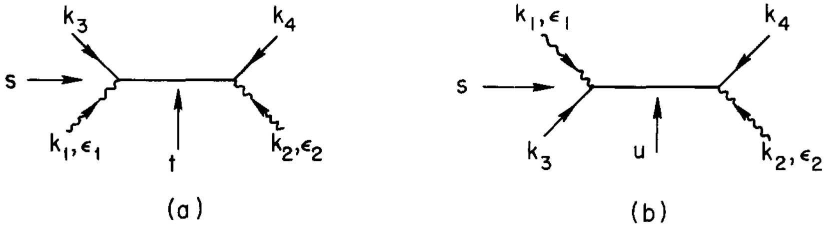

Fig. 6. a. Kinematics for $\pi\rho \to \pi\rho$. b. Kinematics for $\pi\rho \to \rho\pi$.

$$\epsilon_{\mu\nu\lambda\sigma}\epsilon_1^{\mu} k_1^{\nu} k_3^{\lambda} \epsilon_{\mu'\nu'\lambda'}{}^{\sigma} \epsilon_2^{\mu'} k_2^{\nu'} k_4^{\lambda'} = \beta(t) \; . \tag{6.9}$$

Thus the residue is completely accounted for by the ω-resonance, without any admixture of 1^+, 0^-, etc.

The amplitude shown in fig. 6b is obtained by factorizing eq. (6.5) at the pole $\alpha_\rho(s_{23}) = 1$:

$$\begin{aligned} A(\pi\rho \to \rho\pi) = & -\tfrac{1}{2}\epsilon_1\cdot\epsilon_2(1 - \alpha_\pi(s) - \alpha_\pi(u))\, B(1 - \alpha_\pi(s), 1 - \alpha_\pi(u)) \\ & + \epsilon_1\cdot k_3\, \epsilon_2\cdot k_4\, B(-\alpha_\pi(s), 1 - \alpha_\pi(u)) \\ & + \epsilon_1\cdot k_4\, \epsilon_2\cdot k_3\, B(1 - \alpha_\pi(s), -\alpha_\pi(u)) \; . \end{aligned} \tag{6.10}$$

In contrast to eq. (6.6), this expression contains odd-G poles dual to themselves. It also contains only an ω-pole at $\alpha_\pi(s) = 1$, as is easily shown by the same sort of analysis as before.

More complicated amplitudes, such as $\rho\rho \to \rho\rho$, could be obtained by methods similar to those in this section. However, a more efficient route to such results would probably begin by deriving the general operator expression for the scattering of N excited states. Such a formula has been found in the conventional model [25]. In any case, the main purpose of this section was to verify some of the assertions of the previous sections for a non-trivial example.

7. CONCLUSION

The dual model presented here bears some resemblance to the real world of mesons, while still retaining the good features of the conventional model. Its spectrum is approximately the same as predicted by a naive model in which mesons are made from $q\bar{q}$ states involving non-strange quarks only. Such a rule is required for multipion states in a dual quark model, and it accounts for finding ω and f^0 trajectories and not φ and f'. Stretching the observed mixing angle a bit, it could also account for finding an η' and not an η. In any case, just as with the empirical mass rule of sect. 5, these statements are only a description of what the model seems to be doing and are not based on any particularly deep understanding. A deeper understanding could be provided by a model for underlying quarks, for example. Such a model might be useful for seeing how to make a sensible extension to SU(3). This is an important problem, because a naive SU(3) extension of the present model gives a bad spectrum.

The most important feature of our model is its gauge algebra. It is responsible for the appearance of an enormous number of spurious states. Specifically, it was shown in sect. 5 that only 3 of the 12 leading trajectories are physical. The new gauges furthermore provide the key to obtaining a spurious state at the point where the ρ - f^0 trajectory crosses zero. An important problem remaining to be solved is the explicit construction of the new gauge operators. Such formulas would both allow for a simplification of the algebra in sect. 5 (and an elimination of the appendix!) and provide a basis for a much better understanding of some beautiful mathematics that is at present hidden from view.

Just as the Virasoro gauge algebra played an important role in motivating the construction of this model, so - we hope - will the new algebra implied by this model contribute to the construction of a still better one.

APPENDIX

In this appendix we show that the ground state G decouples from any state containing 4 or 6 pions. Using the notations of fig. 7a, we can write the amplitude for G going into four pions as:

$$A(G\to 4\pi) = 4\int dx_1\, dx_2\, x_1^{-\alpha_\rho(12)-1}\, x_2^{-\alpha_\pi(13)-1}\,(1-x_1)^{-2k_2\cdot k_3}\,(1-x_2)^{-2k_3\cdot k_4}$$

$$\times\,(1-x_1x_2)^{-2k_2\cdot k_4}\left[\frac{(k_1\cdot k_2)\,(k_3\cdot k_4)}{x_1(1-x_2)}+\frac{(k_1\cdot k_4)\,(k_2\cdot k_3)}{1-x_1}\right.$$

$$\left.-\frac{(k_1\cdot k_3)\,(k_2\cdot k_4)}{1-x_1x_2}\right]. \tag{A.1}$$

We then expand $(1-x_1x_2)^{-2k_2\cdot k_4}$ in a power series and integrate over x_1 and x_2 to get

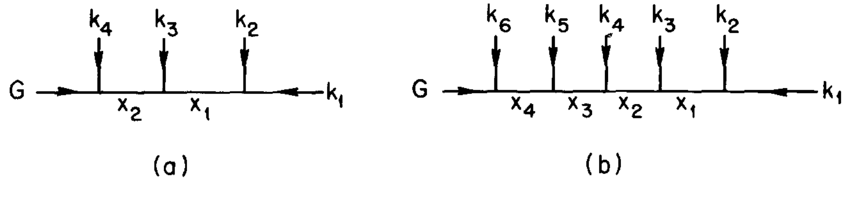

Fig. 7. a. The G → 4π amplitude. b. The G → 6π amplitude.

$$A_4 = 4(k_1\cdot k_2)(k_3\cdot k_4) \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2\cdot k_4)}{n!\,\Gamma(2k_2\cdot k_4)} B[n-\alpha_\pi(13), -2k_3\cdot k_4]$$

$$\times B[n-\alpha_\rho(12)-1, 1-2k_2\cdot k_3] + 4(k_1\cdot k_4)(k_2\cdot k_3) \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2\cdot k_4)}{n!\,\Gamma(2k_2\cdot k_4)}$$

$$\times B[n-\alpha_\pi(13), 1-2k_3\cdot k_4] B[n-\alpha_\rho(12), -2k_2\cdot k_3] - 2(k_1\cdot k_3)$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(n+2k_2\cdot k_4+1)}{n!\,\Gamma(2k_2\cdot k_4)} B[n-\alpha_\pi(13), 1-2k_3\cdot k_4] B[n-\alpha_\rho(12), 1-2k_2\cdot k_3]\,. \tag{A.2}$$

Next we express the B-functions in terms of Γ-functions and use the following kinematics:

$$2k_1\cdot k_2 = [\alpha_\rho(12)-n] + [n]\,,$$

$$2k_1\cdot k_4 = [-\alpha_\pi(13) - 2k_3\cdot k_4 + n] - [n + 2k_2\cdot k_4]\,,$$

$$2k_1\cdot k_3 = [-\alpha_\rho(12) - 2k_2\cdot k_3 + n + 1] + [\alpha_\pi(13) - n]\,.$$

It is then trivial to check that the terms in square brackets in these three equations give rise to contributions which cancel by pairs.

In the amplitude A_6 for G going into 6 pions shown in fig. 7b, there are 15 terms multiplying the ordinary Bardakçi-Ruegg integrand [2]. These fifteen terms are

$$T_1 = 8\frac{(k_1\cdot k_2)(k_3\cdot k_4)(k_5\cdot k_6)}{(1-x_2)(1-x_4)}\,,$$

$$T_2 = 8\frac{(k_1\cdot k_3)(k_2\cdot k_5)(k_4\cdot k_6)x_1x_3}{(1-x_1x_2x_3)(1-x_3x_4)}\,,$$

$$T_3 = 8\frac{(k_1\cdot k_2)(k_3\cdot k_6)(k_4\cdot k_5)x_3}{(1-x_3)(1-x_2x_3x_4)}\,,$$

$$T_4 = 8\frac{(k_1\cdot k_4)(k_2\cdot k_3)(k_5\cdot k_6)x_1}{(1-x_1)(1-x_4)}\,,$$

$$T_5 = 8\frac{(k_1\cdot k_4)(k_2\cdot k_6)(k_3\cdot k_5)x_1x_2x_3}{(1-x_2x_3)(1-x_1x_2x_3x_4)}\,,$$

$$T_6 = 8\frac{(k_1\cdot k_5)(k_2\cdot k_4)(k_3\cdot k_6)x_1x_2x_3}{(1-x_1x_2)(1-x_2x_3x_4)}\,,$$

$$T_7 = 8\frac{(k_1\cdot k_6)(k_2\cdot k_5)(k_3\cdot k_4)x_1x_2x_3}{(1-x_1x_2x_3)(1-x_2)}\,,$$

$$T_8 = 8\frac{(k_1\cdot k_6)(k_4\cdot k_5)(k_2\cdot k_3)x_1x_3}{(1-x_1)(1-x_3)}\,,$$

$$T_9 = -8\frac{(k_1\cdot k_2)(k_3\cdot k_5)(k_4\cdot k_6)x_3}{(1-x_2x_3)(1-x_3x_4)}\,,$$

$$T_{10} = -8\frac{(k_1\cdot k_3)(k_2\cdot k_6)(k_4\cdot k_5)x_1x_3}{(1-x_1x_2x_3x_4)(1-x_3)}\,,$$

$$T_{11} = -8\frac{(k_1\cdot k_4)(k_3\cdot k_6)(k_2\cdot k_5)x_1x_2x_3}{(1-x_2x_3x_4)(1-x_1x_2x_3)}\,,$$

$$T_{12} = -8\frac{(k_1\cdot k_3)(k_2\cdot k_4)(k_5\cdot k_6)x_1}{(1-x_1x_2)(1-x_4)}\,,$$

$$T_{13} = -8\frac{(k_1\cdot k_5)(k_2\cdot k_6)(k_3\cdot k_4)x_1x_2x_3}{(1-x_1x_2x_3x_4)(1-x_2)}\,,$$

$$T_{14} = -8\frac{(k_1\cdot k_5)(k_2\cdot k_3)(k_4\cdot k_6)x_1x_3}{(1-x_1)(1-x_3x_4)}\,,$$

$$T_{15} = -8 \frac{(k_1 \cdot k_6)(k_3 \cdot k_5)(k_2 \cdot k_4) x_1 x_2 x_3}{(1 - x_2 x_3)(1 - x_1 x_2)} .$$

We then expand in power series all the factors of the Bardakci-Ruegg integrand which involve more than one x. This means expanding the following product:

$$(1 - x_1 x_2)^{-2k_2 \cdot k_4} (1 - x_2 x_3)^{-2k_3 \cdot k_5} (1 - x_3 x_4)^{-2k_4 \cdot k_6} (1 - x_1 x_2 x_3)^{-2k_2 \cdot k_5}$$

$$\times (1 - x_2 x_3 x_4)^{-2k_3 \cdot k_6} (1 - x_1 x_2 x_3 x_4)^{-2k_2 \cdot k_6} .$$

Let us call n_{24}, n_{35}, n_{46}, n_{25}, n_{36}, n_{26} the integers (running from zero to infinity) involved in the expansion of each factor. We then integrate explicitly over x_1, x_2, x_3, x_4 each term of the series, and use the following kinematics:

in T_1, T_3, T_9: $\quad 2k_1 \cdot k_2 = [\alpha_\rho(12) - n_{24} - n_{25} - n_{26}] + [n_{24}] + [n_{25}] + [n_{26}]$;

in T_2, T_{10}, T_{12}: $2k_1 \cdot k_3 = [-\alpha_\rho(12) - 2k_2 \cdot k_3 + n_{24} + n_{25} + n_{26} + 1]$

$$+ [n_{35}] + [n_{36}] + [\alpha_\pi(13) - n_{24} - n_{25} - n_{26} - n_{35} - n_{36}] ;$$

in T_4: $\quad 2k_1 \cdot k_4 = [-\alpha_\pi(13) - 2k_3 \cdot k_4 + n_{24} + n_{35} + n_{25} + n_{36} + n_{26}]$

$$+ [n_{46}] - [n_{24} + 2k_2 \cdot k_4]$$

$$+ [\alpha_\rho(14) - n_{46} - n_{35} - n_{25} - n_{36} - n_{26}] ;$$

in T_5, T_{11}: $\quad 2k_1 \cdot k_4 = [-\alpha_\pi(13) - 2k_3 \cdot k_4 + n_{24} + n_{35} + n_{25} + n_{36} + n_{26} + 1]$

$$+ [n_{46}] - [n_{24} + 2k_2 \cdot k_4]$$

$$+ [\alpha_\rho(14) - n_{46} - n_{35} - n_{25} - n_{36} - n_{26} - 1] ;$$

in T_6, T_{13}, T_{14}: $2k_1 \cdot k_5 = [-\alpha_\rho(14) + n_{35} + n_{46} + n_{25} + n_{36} + n_{26} + 1 - 2k_4 \cdot k_5]$

$$- [n_{25} + 2k_2 \cdot k_5] - [n_{35} + 2k_3 \cdot k_5]$$

$$+ [\alpha_\pi(15) - n_{46} - n_{36} - n_{26}] ;$$

in T_7, T_8, T_{15}: $\quad 2k_1 \cdot k_6 = [-\alpha_\pi(15) + n_{46} + n_{36} + n_{26} - 2k_5 \cdot k_6]$

$$- [n_{26} + 2k_2 \cdot k_6] - [n_{36} + 2k_3 \cdot k_6]$$

$$- [n_{46} + 2k_4 \cdot k_6] .$$

It is left as an exercise to the reader to show that all the terms in brackets above, when taken with the appropriate series, cancel one another by pairs.

It seems that such a pedestrian proof of the decoupling of the ground state cannot be generalized in practice to more than 6 pions, owing to the enormous complexity of the formulas.

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