Honors Mathematics III
RC 8

CHEN Xiwen

UM-SJTU Joint Institute

July 24, 2018

Table of contents

The Line Integral of a Potential Function

Definition. Let $\Omega \subset \mathbb{R}^n, f : \Omega \to \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma : I \to \mathcal{C}$. We then define the ***line integral of the potential f along $\mathcal{C}^*$*** by

$$\int_{\mathcal{C}^*} f \mathrm{d}s := \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)| \mathrm{d}t,$$

which is independent of the parametrization chosen. The ***scalar line element*** is given by

$$\mathrm{d}s = |\gamma'(t)| \mathrm{d}t.$$

The Line Integral of a Potential Function

Example. Suppose a wire is in the shape of a circle, $\mathcal{C}^* : x^2 + y^2 = 1$. The density ρ at point (x, y) is $\rho(x, y) = 1 + xy$. Calculate its total mass.

The Line Integral of a Potential Function

Example. Suppose a wire is in the shape of a circle, $\mathcal{C}^* : x^2 + y^2 = 1$. The density ρ at point (x, y) is $\rho(x, y) = 1 + xy$. Calculate its total mass.

Solution. The circle can be parametrized by $\gamma(\theta) = (\cos\theta, \sin\theta)$. The total mass is calculated by

$$\begin{aligned} m &= \int_0^{2\pi} (1 + \cos\theta \cdot \sin\theta) \cdot 1 \mathrm{d}\theta \\ &= 2\pi \end{aligned}$$

Vector Fields

Definition. Let $\Omega \subset \mathbb{R}^n$. Then a function $F : \Omega \to \mathbb{R}^n$,

$$F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_n(x) \end{pmatrix}$$

is called a ***vector field*** on Ω.

Example. The ***gradient field of** f* is given by

$$F : \mathbb{R}^n \to \mathbb{R}^n, \qquad F(x) = \nabla f(x)$$

The Line Integral of a Vector Field

Definition. Let $\Omega \subset \mathbb{R}^n$, $F \to \mathbb{R}$ be a continuous vector field and $\mathcal{C}^* \subset \Omega$ an oriented open, smooth curve in $\mathbb{R}^n$. Then the ***line integral of the vector field*** F ***along*** $\mathcal{C}^*$ is given by

$$\int_{\mathcal{C}^*} F \mathrm{d}\vec{s} := \int_{\mathcal{C}^*} \langle F, T \rangle \mathrm{d}s = \int_{\mathcal{C}^*} \langle F, \mathrm{d}\vec{s} \rangle$$

Note.

- The line integral of a vector field does not depend on parametrization of $\mathcal{C}^*$.
- The ***vectorial line element*** is given by

$$\mathrm{d}\vec{s} = \gamma'(t)\mathrm{d}t$$

- To calculate line integral using parametrization $\gamma : I \to \mathcal{C}$

$$\int_{\mathcal{C}^*} F \mathrm{d}\vec{s} = \int_I \langle F \circ \gamma(t), \gamma'(t) \rangle \mathrm{d}t$$

Potential Fields

Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. A vector field $F : \Omega \to \mathbb{R}^n$ is said to be a ***potential field*** if there exists a differentiable potential function $U : \Omega \to \mathbb{R}$ such that

$$F(x) = \nabla U(x)$$

Integrals of potential fields. Since

$$\int_I \langle F \circ \gamma(t), \gamma'(t) \rangle \mathrm{d}t = \int_I (U \circ \gamma)'(t) \mathrm{d}t$$

then

$$\int_{\mathcal{C}^*} F \mathrm{d}\vec{s} = U(p_{final}) - U(p_{initial})$$

Conservative Fields

Lemma. Let $\Omega \subset \mathbb{R}^n$ be open, $F : \Omega \to \mathbb{R}^n$ a potential field and $\mathcal{C} \subset \Omega$ a closed curve. Then

$$\oint_{\mathcal{C}} F \mathrm{d}\vec{s} = 0$$

Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F : \Omega \to \mathbb{R}^n$ a vector field. If the integral along any open curve $\mathcal{C}$ depends only on the initial and final points, or equivalently,

$$\oint_{\mathcal{C}} F \mathrm{d}\vec{s} = 0$$

for any closed curve $\mathcal{C}$, then F is ***conservative***.

Note.

- Every potential field is a conservative field.
- Every continuous, conservative field on a connected open set is a potential field.

Conservative Fields

Slide 512. Proof of Theorem 3.1.17. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F : \Omega \to \mathbb{R}^n$ is a continuous, conservative field. Then F is a potential field.

Question. In the last equation, we have

$$\begin{aligned} U(x + he_i) &= U(x) + h \int_0^1 (F_i(x) + o(1))\mathrm{d}t \\ &= U(x) + F_i(x)h + o(h), \end{aligned}$$

if $F_i(x + the_i) = F_i(x) + o(1)$ for fixed t. Does the integration with respect to t still $o(1)$?

Conservative Fields

Slide 512. Proof of Theorem 3.1.17.

$$\begin{aligned} U(x+he_i) &= U(x) + h\int_0^1 (F_i(x) + o(1))\mathrm{d}t \\ &= U(x) + F_i(x)h + o(h) \end{aligned}$$

Yes. Note that we have

$$F_i(x + the_i) = F_i(x) + o(1)$$

for fixed t and any x as $h \to 0$. This is a function of x and th. Suppose we have $f(h) = o(1)$ as $h \to 0$, meaning $\lim_{h\to 0} f(h) = 0$. We then want to show

$$\int_0^1 f(th)\mathrm{d}t \leq 1 \cdot \sup_{t\in[0,1]} f(t\cdot h) \to 0 \text{ as } h \to 0$$

Conservative Fields

Then we need to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all h, if $|h| < \delta$, then

$$\left|\sup_{t\in[0,1]} f(t\cdot h)\right| < \varepsilon.$$

Since $f = o(1)$, we can choose a $\delta > 0$ such that if $|k| < \delta$, then $|f(k)| < \varepsilon/2$. Because $t \in [0,1]$, we have $|t\cdot h| < \delta$ and hence

$$|f(t\cdot h)| < \varepsilon/2 \quad \text{for all } t \in [0,1]$$

This shows that

$$\sup_{t\in[0,1]} |f(t\cdot h)| \leq \varepsilon/2 < \varepsilon.$$

Since

$$\left|\sup_{t\in[0,1]} f(t\cdot h)\right| \leq \sup_{t\in[0,1]} |f(t\cdot h)|,$$

the proof is complete.

Simply Connected Sets

Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set.

- A closed curve $\mathcal{C} \subset \Omega$ given as the image of a map $g : S^1 \to \mathcal{C}$ is said to be ***contractible to a point*** if there exist a continuous function $G : D \to \Omega$ such that $G|_{S^1} = g$.
- The set Ω is said to be ***simply connected*** if it is connected and every closed curve in Ω is contractible to a point.

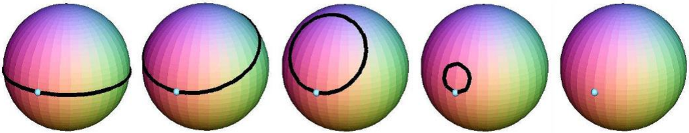
Example.

1. $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.
2. $\mathbb{R}^3 \setminus \{0\}$ is simply connected.

Simply Connected Sets

Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set.

- A closed curve $\mathcal{C} \subset \Omega$ given as the image of a map $g : S^1 \to \mathcal{C}$ is said to be ***contractible to a point*** if there exist a continuous function $G : D \to \Omega$ such that $G|_{S^1} = g$.
- The set Ω is said to be ***simply connected*** if it is connected and every closed curve in Ω is contractible to a point.



Criteria for Potential Fields

3.1.18. Lemma. Let $\Omega \subset \mathbb{R}^n$ be a connected open set and suppose that $F : \Omega \to \mathbb{R}^n$ is continuously differentiable. Then F is a potential field only if for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

3.1.21. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a ***simply connected*** open set and suppose that $F : \Omega \to \mathbb{R}^n$ is continuously differentiable. If for all $i, j = 1, \dots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

then F is a potential field.

Finding Potentials

Steps.

1. Check potential fields.
2. Integrate with respect to x_1.
3. Integrate with respect to x_2.

Finding Potentials

Example. Denote by $\mathbb{H} = \{(x, y) : y > 0\} \subset \mathbb{R}^2$ the upper half-space of $\mathbb{R}^2$ and consider the two vector fields $F, G : \mathbb{H} \to \mathbb{R}^2$ with $(x, y) \in \mathbb{H}$,

$$F(x, y) = (4x^2 + 4y^2, 8xy - \ln y), \quad G(x, y) = (x + xy, -xy)$$

1. Which of the two fields is conservative?
2. Calculate the potential function for the conservative field.

Finding Potentials

Solution 1. We calculate the partial derivatives:

$$\frac{\partial F_1}{\partial y} = 8y, \ \frac{\partial F_2}{\partial x} = 8y, \ \frac{\partial G_1}{\partial y} = x, \ \frac{\partial G_2}{\partial x} = -y$$

Since $\partial_x G_2 \neq \partial_y G_1$, G cannot be conservative. Since F is defined on a simply-connected set $\mathbb{H}$, F is conservative.

Finding Potentials

Solution 2. Integrate with respect to x and y,

$$\Phi(x,y) = \int F_1(x,y)\mathrm{d}x = \frac{4}{3}x^3 + 4y^2x + C_1(y)$$
$$\Phi(x,y) = \int F_2(x,y)\mathrm{d}y = 4xy^2 - y\ln y + y + C_2(x)$$

Then a potential function is given by

$$\Phi(x,y) = \frac{4}{3}x^3 + 4y^2x - y\ln y + y.$$

Differential Forms

Definition. Let $F_1, \ldots, F_n : \mathbb{R}^n \to \mathbb{R}$ be scalar functions. Then

$$\alpha = F_1 \mathrm{d}x_1 + \cdots + F_n \mathrm{d}x_n$$

is a ***differential one-form***.

Circulation and Flux

- ***Flux***: the normal component of a vector field that flows through the boundary of the region.

$$\int_{\mathcal{C}^*} \langle F, N \rangle \mathrm{d}s$$

- ***Circulation***: the tangential component that flows around the boundary.

$$\int_{\mathcal{C}^*} \langle F, T \rangle \mathrm{d}s$$

where

1. $\|N\| = \|T\| = 1$.
2. $\langle N, T \rangle = 0$.
3. N points ***outwards*** from the region bounded by $\mathcal{C}$.

Flux Density and the Divergence

Definition. Let $\Omega \subset \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ be a continuously differentiable vector field. Then

$$\mathrm{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

is the ***divergence*** of F. The flux density at a point x is given by the divergence of the field at x.

The Circulation Density — Rotation / Curl

Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F : \Omega \to \mathbb{R}^n$ a continuously differentiable vector field. Then the anti-symmetric, bilinear form

$$\mathscr{rot}\, F|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \mathscr{rot}\, F|_x(u, v) := \langle DF|_x u, v\rangle - \langle DF|_x v, u\rangle$$

is the ***rotation*** or ***curl*** of the vector field F at $x \in \mathbb{R}^n$. In $\Omega \subset \mathbb{R}^2$, there exists a uniquely defined continuous potential function $\mathrm{rot} F : \Omega \to \mathbb{R}$ such that

$$\mathscr{rot}\, F|_x(u, v) = \mathrm{rot} F(x) \cdot \det(u, v)$$

Rotation in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Rotation in $\mathbb{R}^2$: a scalar function $\operatorname{rot} F$:

$$\operatorname{rot} F = \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}$$

- Rotation in $\mathbb{R}^3$: for a continuously differentiable vector field on an open set Ω, there exists a uniquely defined continuous vector field $\operatorname{rot} F : \Omega \to \mathbb{R}^3$ such that

$$\mathscr{rot} F|_x(u, v) = \det(\operatorname{rot} F(x), u, v) = \langle \operatorname{rot} F(x), u \times v \rangle$$

with

$$\operatorname{rot} F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

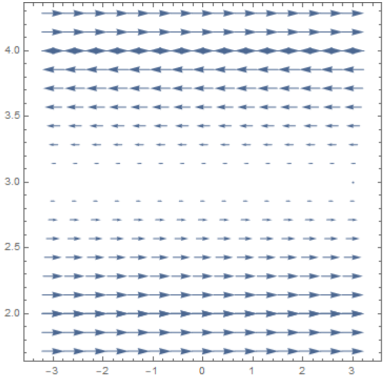
The Rotation in $\mathbb{R}^2$

Example. Is there a vector field F such that $\mathrm{rot} F = 0$ for most of the points in $\mathbb{R}^2$ but for some of the points, it is not?

The Rotation in $\mathbb{R}^2$

Example. Is there a vector field F such that $\mathrm{rot} F = 0$ for most of the points in $\mathbb{R}^2$ but for some of the points, it is not?

$$F(x,y) = \begin{cases} (2,0) & y \le 2 \\ (y^3 - 9y^2 + 24y - 18, 0) & 2 < y < 4 \\ (-2,0) & y \ge 4 \end{cases}$$



The Rotation in $\mathbb{R}^2$

Example. Given an electric field $E = c(2bxy, x^2 + ay^2)$, $a, b, c \in \mathbb{R}$, determine values for a and b such that $\mathrm{div}E = 0$ and $\mathrm{rot}E = 0$. Then find a potential function V for E with these values a and b.

The Rotation in $\mathbb{R}^2$

Example. Given an electric field $E = c(2bxy, x^2 + ay^2), a, b, c \in \mathbb{R}$, determine values for a and b such that $\mathrm{div}E = 0$ and $\mathrm{rot}E = 0$. Then find a potential function V for E with these values a and b.

Solution.

$$\begin{aligned} \mathrm{div}E = 0 \quad &\Rightarrow \quad b = -a \\ \mathrm{rot}E = 0 \quad &\Rightarrow \quad b = 1, a = -1 \end{aligned}$$

Then

$$E = c \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

with a potential

$$V = cx^2y - \frac{c}{3}y^3.$$

The Rotation in $\mathbb{R}^3$

Note. The circulation density in the plane spanned by u and v at x is given by

$$\left\langle \operatorname{rot} F|_x, \frac{u \times v}{\|u \times v\|} \right\rangle$$

The circulation density of a vector field in $\mathbb{R}^3$ is represented by a vector field $\operatorname{rot} F$ given by

$$\operatorname{rot} F(x) = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

Irrotational Fields

A continuously differentiable field $F : \Omega \to \mathbb{R}^n$ such that $\mathscr{rot} F|_x = 0$ for all $x \in \Omega$ is ***irrotational***. Then

$$(DF|_x)^T = DF|_x$$

Note. A potential field is irrotational.

Fluid Statistics

For ***potential flow***,

$$F = \nabla U, \qquad \mathrm{div} F = 0$$

Then

$$\mathrm{div}(\nabla U) = \mathrm{div} \begin{pmatrix} \frac{\partial U}{\partial x_1} \\ \vdots \\ \frac{\partial U}{\partial x_n} \end{pmatrix} = \frac{\partial^2 U}{\partial x_1^2} + \cdots + \frac{\partial^2 U}{\partial x_n^2} = \Delta U = 0$$

Triangle Calculus

Define notation

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

then

- the gradient of a potential function f: ∇f.
- the divergence of a vector field F: $\mathrm{div} F = \langle \nabla, F \rangle$.
- the rotation of a vector field F:

$$\mathrm{rot} F = \nabla \times F(x) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

- the Laplace operator: $\Delta = \langle \nabla, \nabla \rangle = \nabla^2$.

Exercises

Exercise 1. Let $g : (0, \infty) \to \mathbb{R}$ be a differentiable function and let $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Prove that the vector field

$$F : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3, \quad F(x) = g(\|x\|)x$$

is conservative.

Exercises

Exercise 1. Let $g:(0,\infty)\to\mathbb{R}$ be a differentiable function and let $\|x\|=\sqrt{x_1^2+x_2^2+x_3^2}$ for $x=(x_1,x_2,x_3)\in\mathbb{R}^3$. Prove that the vector field

$$F:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}^3,\quad F(x)=g(\|x\|)x$$

is conservative.

Solution. The set $\mathbb{R}^3\setminus\{0\}$ is simply connected. We evaluate

$$|(\mathrm{rot}F)_i|=\left|\frac{\partial F_j}{\partial x_k}-\frac{\partial F_k}{\partial x_j}\right|\quad \text{for } i\neq j\neq k\in\{1,2,3\}$$

We have

$$\frac{\partial F_j}{\partial x_k}=\frac{\partial}{\partial x_k}g(\|x\|)x_j=x_jx_k\frac{g'(\|x\|)}{\|x\|}=\frac{\partial F_k}{\partial x_j}.$$

Exercises

Exercise 2. The gravitational force in $\mathbb{R}^3$ is

$$F_3 = -\frac{GmM}{\|x\|^2}\frac{x}{\|x\|}.$$

Consider a more generalized vector space with dimension n,

$$F = \frac{cx}{\|x\|^n},$$

where $c \in \mathbb{R}$ is constant. Prove that $\mathrm{div} F = 0$, $\mathrm{rot} F = 0$.

Exercises

Exercise 2 Solution.

$$\frac{\partial F_i}{\partial x_i} = \frac{c}{\|x\|^n} - \frac{ncx_i^2}{\|x\|^{n+2}}, \qquad \frac{\partial F_i}{\partial x_j} = -\frac{ncx_ix_j}{\|x\|^{n+2}} = \frac{\partial F_j}{\partial x_i}$$

Then

$$\mathrm{div} F = 0$$

and

$$\mathscr{rot} F|_x(e_i, e_j) = \langle DF|_x e_i, e_j\rangle - \langle DF|_x e_j, e_i\rangle = 0$$

Thanks for your attention!