Honors Mathematics III
RC 7

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Table of contents

The Normal Vector of a Curve

Definition. The ***unit normal vector*** $N : \mathcal{C} \to \mathbb{R}$ of a smooth C^2-curve with parametrization $\gamma : I \to V$ is

$$N \circ \gamma(t) := \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}, \qquad t \in \operatorname{int} I$$

Note. The unit normal vector does not depend on γ on

- magnitude and
- orientation.

Curve Length

2.3.25. Theorem. $\mathcal{C} \subset V$ is a smooth and ***open*** curve with parametrization $\gamma : [a, b] \to \mathcal{C}$. Then $\mathcal{C}$ is rectifiable iff

$$\int_a^b \|\gamma'(t)\| \mathrm{d}t < \infty$$

and the ***curve length*** is

$$\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

which is independent of γ.

Curve Length

The ***length function*** is defined as

$$(\ell \circ \gamma)(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$$

The curve length gives the ***natural parametrization*** of an oriented curve $\mathcal{C}$.

$$\gamma = \ell : I \to \mathcal{C}, \qquad \mathrm{int}\, I = (0, \ell(\mathcal{C}))$$

Note. Then we also obtain

$$\|\gamma'(t)\| = \frac{\mathrm{d}\ell \circ \gamma(t)}{\mathrm{d}t}$$

Curvature

Definition. The ***curvature*** of a smooth C^2-curve $\mathcal{C} \subset V$ is

$$\kappa : \mathcal{C} \to \mathbb{R}, \qquad \kappa \circ \ell^{-1}(s) := \left\| \frac{\mathrm{d}}{\mathrm{d}s}(T \circ \ell^{-1}(s)) \right\|$$

where T is the unit tangent vector and $\ell^{-1} : I \to \mathcal{C}$ is the curve length parametrization of $\mathcal{C}$. For parametrization that is ***not*** the length parametrization,

$$\kappa \circ \gamma(t) = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}$$

Note. The curvature κ does not depend on the orientation of $\mathcal{C}$.

Curvature in $\mathbb{R}^3$

2.3.31. Lemma. Let $\mathcal{C} \subset \mathbb{R}^3$ be a smooth C^2-curve with parametrization $\gamma : I \to \mathcal{C}$, then

$$\kappa \circ \gamma(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

The Directional Derivative

Definition. Let $\Omega \in \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$ continuous and $h \in \mathbb{R}^n$, $\|h\| = 1$ be a unit vector. Then the ***directional derivative*** $D_h f$ in the direction h is defined by

$$D_h f|_x := \frac{\mathrm{d}}{\mathrm{d}t} f(x + th)\bigg|_{t=0}$$

if the right-hand side exists.

Note.

- The directional derivative is the derivative of f at x along the line segment joining x and $x + h$. It gives the slope of the tangent line of f at x in the direction of h.
- The directional derivative is a number.

The Directional Derivative

Results.

- The tangent line of $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction h:

$$t_{f,x;h}(s) = \begin{pmatrix} x + sh \\ f(x) + D_h f|_x s \end{pmatrix}$$

- If f is differentiable and the line segment is parametrized by $\gamma(t) = x + th$,

$$D_h f|_x = Df|xh = \langle \nabla f(x), h \rangle$$

The Directional Derivative

Example. Find the directional derivative of the function

$$f(x, y) = \ln(x^2 + y^2)^{1/2}$$

at $(1, 1)$ along the direction $(2, 1)$.

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$$f(x,y) = \ln(x^2 + y^2)^{1/2}$$

at $(1,1)$ along the direction $(2,1)$.
Solution. f is smooth in $(0,\infty) \times (0,\infty)$.

$$\nabla f(x,y) = \begin{pmatrix} \dfrac{x}{x^2+y^2} \\ \dfrac{y}{x^2+y^2} \end{pmatrix}$$

Then

$$Df_h|_{(1,1)} = \langle \nabla f|_{(1,1)}, h \rangle = \frac{3\sqrt{5}}{10}$$

The Normal Derivative in $\mathbb{R}^2$

Definition. Let $\Omega \subset \mathbb{R}^2$ be an open set, $f : \Omega \to \mathbb{R}$ and $\mathcal{C}$ a simple smooth C^2-curve in Ω. Let $p \in \mathcal{C}$ and $N(p)$ denote the normal vector at p. Then

$$\left.\frac{\partial f}{\partial n}\right|_p := D_{N(p)} f|_p$$

is called the ***normal derivative of f at p*** with respect to the curve $\mathcal{C}$.

The Gradient

The ***gradient*** $\nabla f(x)$ is the transpose of the Jacobian.

- $\nabla f(x)$ points in the direction of the greatest directional derivative of f at x.

$$D_h f(x) = \langle \nabla f(x), h \rangle = |\nabla f(x)| \cos \angle(\nabla f(x), h)$$

- $\nabla f(x)$ is perpendicular to the contour line of f at x.

$$\langle \nabla f(x), h_0 \rangle = 0$$

The Tangent Plane

Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ differentiable at $x_0 \in \Omega$. Then the equation

$$x_{n+1} = Tf(x; x_0), \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

defines the ***tangent plane*** to the graph $\Gamma(f) \in \mathbb{R}^n \times \mathbb{R}$ of f at the point $(x_0, f(x_0)) \in \mathbb{R}^{n+1}$.

The Tangent Plane

Results.

- The tangent plane in $\mathbb{R}^3$ is found by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The vectors

$$t_1 := \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix}, \qquad t_2 := \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

give the ***tangent vectors*** to the graph $\Gamma(f)$ at $(x_0, y_0, f(x_0, y_0))$ with normal vector given by

$$n = t_1 \times t_2 = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right)^T$$

The Second Derivative

Definition. Let X, V be finite-dimensional normed vector spaces and $\Omega \subset X$ an open set. A function $f : \Omega \to V$ is said to be ***twice differentiable*** at $x \in \Omega$ if

- f is differentiable in an open ball $B_\varepsilon(x)$ around x and
- the function $Df : B_\varepsilon(x) \to \mathcal{L}(X, V)$ is differentiable at x.

The ***second derivative*** (if it exists) is a map:

$$D(Df) =: Df^2 : \Omega \to \mathcal{L}(X, \mathcal{L}(X, V))$$

and is found by

$$Df|_{x+h} = Df|_x + D^2 f|_x h + o(h)$$

The Second Derivative

Example. (RC.5.e.g.2.) Calculate the first, second and the third derivatives of the map

$$\Phi : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathrm{Mat}(n \times n; \mathbb{R}), \qquad \Phi(A) = A^3$$

The Second Derivative

Example. (RC.5.e.g.2.) Calculate the first, second and the third derivatives of the map

$$\Phi : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathrm{Mat}(n \times n; \mathbb{R}), \qquad \Phi(A) = A^3$$

$$\begin{aligned} \Phi(A+H) &= A^3 + A^2H + AHA + HA^2 + o(H) \\ D\Phi|_A H &= A^2H + AHA + HA^2 \end{aligned}$$

then

$$\begin{aligned} D\Phi_{A+J}H = AH + AHA + HA^2 + AJH + JAH + AHJ + JHA + \\ + HAJ + HJA + o(J) \end{aligned}$$

$$D^2\Phi|_A[H,J] = AJH + AHJ + JAH + JHA + HAJ + HJA$$

The Second Derivative

The Hessian. For a differentiable potential function $f : \mathbb{R}^n \to \mathbb{R}$. The derivative is given by the Jacobian:

$$Df|_x = \begin{pmatrix} \left.\frac{\partial f}{\partial x_1}\right|_x & \cdots & \left.\frac{\partial f}{\partial x_n}\right|_x \end{pmatrix} \quad , \qquad Df|_x \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$$

and the second derivative is found by ***Hessian*** where

$$\mathrm{Hess}\, f(x) = D(\nabla f)|_x = \begin{pmatrix} \left.\frac{\partial^2 f}{\partial x_1 \partial x_1}\right|_x & \left.\frac{\partial^2 f}{\partial x_2 \partial x_1}\right|_x & \cdots & \left.\frac{\partial^2 f}{\partial x_n \partial x_1}\right|_x \\ \vdots & \vdots & & \vdots \\ \left.\frac{\partial^2 f}{\partial x_1 \partial x_n}\right|_x & \left.\frac{\partial^2 f}{\partial x_2 \partial x_n}\right|_x & \cdots & \left.\frac{\partial^2 f}{\partial x_n \partial x_n}\right|_x \end{pmatrix}$$

$$D^2 f|_x h = \mathrm{Hess}\, f(x) h$$

Bilinear Maps

- The Hessian as a bilinear map. $D^2 f|_x : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (h, \tilde{h}) \mapsto \langle \mathrm{Hess} f(x) h, \tilde{h} \rangle$.
- The second derivative as a bilinear map. $L \in \mathcal{X}, \mathcal{L}(\mathcal{X}, \mathcal{V})$, then $Lx_1 \in \mathcal{L}(X, V)$ and $(Lx_1)x_2 = \tilde{L}(x_1, x_2) \in V$.
- Multilinear Maps. Let X, V be finite-dimensional normed vector spaces. The set of multilinear maps from X to V is denoted by

  $$\mathcal{L}^{(n)}(X, V) := \{L : X \times \cdots \times X \to V : L \text{ is linear in each component}\}$$

  When $V = \mathbb{R}$, an element of $\mathcal{L}^{(n)}(X, V)$ is called a ***multilinear form***.

Bilinear forms on $\mathbb{R}^n$

- Every linear map $L \in (\mathbb{R}^n)^*$ has the form $L = \langle z, \cdot \rangle$ for some $z \in \mathbb{R}^n$.
- Interpret an element $A \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}))$ as a linear map, $A : y \mapsto L_y := \langle z_y, \cdot \rangle$, $z_y = A(y)$.
- A is actually a matrix $A : y \mapsto z_y$.
- For every $y \in \mathbb{R}^n$ we obtain a linear map $\langle Ay, \cdot \rangle \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.
- Then $L_y x = \langle Ay, x \rangle = L(x, y)$.

Schwarz's Theorem

2.5.5. Schwarz's Theorem. Let X, V be normed vector spaces and $\Omega \subset X$ an open set. Let $f \in C^2(\Omega, V)$. Then $D^2 f|_x \in \mathcal{L}^{(2)}(X \times X, V)$ is symmetric for all $x \in \Omega$.

$$D^2 f(u, v) = D^2 f(v, u) \qquad \text{for all } u, v \in X$$

This implies that if f is twice continuously differentiable, the Hessian of f at x is symmetric.

$$\langle \mathrm{Hess}\, f(x) y, z \rangle = \langle \mathrm{Hess}\, f(x) z, y \rangle, \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Note. This will be the case if all second-order partial derivatives are continuous.

Schwarz's Theorem

Sufficiency of twice-differentiablility. The Schwarz's theorem will hold if all the second-order partial derivatives are continuous.

Example. The symmetry may be broken if the function fails to have differentiable partial derivatives.

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

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The second partial derivatives are not continuous at $(0,0)$

$$\frac{\partial^2}{\partial x \partial y} f\Big|_{(0,0)} = \lim_{h\to 0} \frac{\partial_y f|_{(h,0)} - \partial_y f|_{(0,0)}}{h} = 1$$

$$\frac{\partial^2}{\partial y \partial x} f\Big|_{(0,0)} = \lim_{h\to 0} \frac{\partial_x f|_{(0,h)} - \partial_x f|_{(0,0)}}{h} = -1$$

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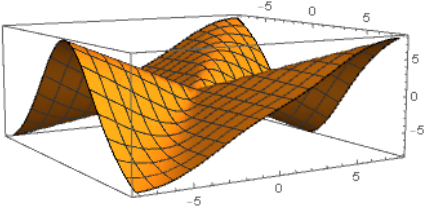

Figure: The First Partial Derivative $\partial_x f$

Exercises

Exercise 1. Calculate the length of the curve between $t = 0$ and $t = \frac{\pi}{4}$ parametrized by

$$\gamma(t) = \begin{pmatrix} t \\ \ln \cos t \end{pmatrix}$$

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Solution.

$$\begin{aligned}
\ell &= \int_0^{\pi/4} \sqrt{1 + \frac{\sin^2 t}{\cos^2 t}} \mathrm{d}t \\
&= \int_0^{\pi/4} \frac{1}{\cos t} \mathrm{d}t \\
&= \int_0^{\sqrt{2}/2} \frac{1}{1 - x^2} \mathrm{d}x \\
&= \frac{1}{2} \left(\ln(1 + x) - \ln(1 - x)\right)\Big|_0^{\sqrt{2}/2} = \frac{1}{2} \ln(3 + 2\sqrt{2})
\end{aligned}$$

Exercises.

Exercise 2. Find the curvature of the curve where the points (x, y) are parametrized with $t \in (0, \infty)$ by

$$x(t) = \int_0^t \frac{\cos u}{\sqrt{u}} \mathrm{d}u$$

$$y(t) = \int_0^t \frac{\sin u}{\sqrt{u}} \mathrm{d}u$$

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Solution.

$$x'(t) = \frac{\cos t}{\sqrt{t}}, \quad y'(t) = \frac{\sin t}{\sqrt{t}}$$

Then

$$\kappa = \frac{\|(T \circ \gamma)'\|}{\|\gamma'(t)\|} = \sqrt{t}$$

Exercises.

Exercise 3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be homogeneous of degree of k. That means that f satisfies

$$f(tx) = t^k f(x) \qquad \text{for all } x \neq 0 \text{ and } t > 0$$

Show that

$$Df|_x x = k \cdot f(x)$$

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Show that

$$Df|_x x = k \cdot f(x)$$

Solution. We differentiate with respect to t:

$$\frac{d}{dt} f(tx) = Df|_{tx} x, \quad \frac{d}{dt} t^k f(x) = kt^{k-1} f(x)$$

Setting $t = 1$, we obtain

$$Df|_x x = kf(x)$$

Frenet-Serret Formulas

Results. T is the unit tangent vector, N is the unit normal vector, $B = T \times N$ is the binormal unit vector. Then T, N, B form an orthonormal basis in $\mathbb{R}^3$.

$$\frac{dT}{ds} = \kappa N, \quad \frac{dN}{ds} = -\kappa T + \tau B, \quad \frac{dB}{ds} = -\tau N$$

and

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \qquad \kappa = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

$$N(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|} \qquad \tau = \frac{\det(\gamma'(t), \gamma''(t), \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

$$B(t) = T \times N$$

Note. $\dfrac{ds}{dt} = \|\gamma'(t)\|$.

Exercises

Exercise 4. Find the first and second derivatives of the following functions:

1. $f : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}, \qquad f(A) = \mathrm{tr}(A^2).$
2. $g : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}, \qquad g(A) = (\mathrm{tr} A)^2.$

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Solution.

- First derivatives:

$$Df|_A H = 2\mathrm{tr}(AH), \qquad Dg|_A H = 2\mathrm{tr}(A) \cdot \mathrm{tr}(H)$$

- Second derivatives:

$$D^2 f|_A[J, H] = 2\mathrm{tr}(JH), \qquad D^2 g|_A[J, H] = 2\mathrm{tr}(J) \cdot \mathrm{tr}(H)$$

Exercises

Exercise 5. Find the second derivative of the determinant of an invertible matrix A.

$$\det : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$$

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$$\det : \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$$

Solution.

1. First derivative of $\det$: $(D\det|_A H) = \det(A) \cdot \mathrm{tr}(A^{-1}H)$.
2. First derivative of tr: tr.
3. First derivative of $(\cdot)H$: $(\cdot)H$.
4. First derivative of $(\cdot)^{-1}$: $D(\cdot)^{-1}|_A H = -A^{-1}HA^{-1}$. (RC.5)

$$D^2\det|_A[J,H] = \det(A)\left(\mathrm{tr}(A^{-1}J)\cdot\mathrm{tr}(A^{-1}H) - \mathrm{tr}(A^{-1}JA^{-1}H)\right)$$

Summary

- Tangent line at a point $\gamma(t_0)$: $\{\gamma(t_0) + \gamma'(t_0)t, t \in \mathbb{R}\}$
- Unit tangent vector: $T \circ \gamma(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$.
- Unit normal vector: $N \circ \gamma(t) = \dfrac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}$.
- <u>Open</u> curve length: $\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$.
- Curve length function: $\ell \circ \gamma(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$.
- $\|\gamma'(t)\| = \dfrac{\mathrm{d}(\ell \circ \gamma)(t)}{\mathrm{d}t}$.
- Curvature: $\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)|_{s=\ell\circ\gamma(t)} = \dfrac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}$.
- Curvature in $\mathbb{R}^3$:
  $\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)|_{s=\ell\circ\gamma(t)} = \dfrac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$.

Summary

- Directional derivative $D_h f$ along h: $D_h f|_x = \frac{d}{dt} f(x + th)\Big|_{t=0}$.
- Directional derivative of smooth functions: $D_h f|_x = Df|_x h = \langle \nabla f(x), h \rangle$.
- The tangent line of $f : \mathbb{R}^n \to \mathbb{R}$ at x along h: $t_{f,x;h}(x) = \begin{pmatrix} x + sh \\ f(x) + D_h f|_x s \end{pmatrix}$.
- The normal derivative in $\mathbb{R}^2$: $\frac{\partial f}{\partial n}\Big|_p = D_{N(p)} f|_p$.
- Best linear approximation of f at x_0: $Tf(\cdot; x_0) = f(x_0) + Df|_{x_0}(\cdot - x_0)$.
- The tangent plane to the graph $\Gamma(f)$ at $(x_0, f(x_0))$: $x_{n+1} = Tf(x; x_0)$ with tangent vectors and normal vector:

$$t_1 = (1, 0, \frac{\partial f}{\partial x}\Big|_{x_0})^T,\ t_2 = (0, 1, \frac{\partial f}{\partial y}\Big|_{x_0})^T,\ n = (-\frac{\partial f}{\partial x}\Big|_{x_0}, -\frac{\partial f}{\partial y}\Big|_{x_0}, 1)^T$$

Summary

- *Frenet-Serret formulas.*

$$B = T \times N, \quad \frac{dT}{ds} = \kappa N, \quad \frac{dB}{ds} = -\tau N, \quad \frac{dN}{ds} = -\kappa T + \tau B$$

-

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \|\gamma'(t)\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

-

$$\gamma'' = \ell'' T + \kappa(\ell')^2 N, \qquad \gamma' \times \gamma'' = \kappa(\ell')^2 B$$
$$\gamma''' = (\ell''' - \kappa^2(\ell')^3) T + (3\kappa\ell'\ell'' + \kappa'(\ell)^2) N + \kappa\tau(\ell')^3 B$$

- $\tau = \dfrac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{\|\gamma' \times \gamma''\|^2} = \dfrac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2}.$

Thanks for your attention!