Honors Mathematics III
RC 3

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Matrices as Linear Maps

An $m \times n$ matrix over the complex numbers is a map

$$a : \{1, \dots, m\} \times \{1, \dots n\} \to \mathbb{C}, \quad (i, j) \mapsto a_{ij}$$

We represent the graph of a through a ***matrix***

$$A := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

We denote the set of all $m \times n$ matrices over $\mathbb{C}$ by $\mathrm{Mat}(m \times n; \mathbb{C})$.

Matrix Product

The ***matrix product*** of $A = (a_{ik}) \in \mathrm{Mat}(l \times m; \mathbb{C})$ and $B = (b_{kj}) \in \mathrm{Mat}(m \times n; \mathbb{C})$ is given by

$$C = AB \in \mathrm{Mat}(l \times n; \mathbb{C}), \qquad C := \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right)_{\substack{i=1,\ldots,l \\ j=1,\ldots,n}}$$

Interpretation: composition of linear maps.

$$\begin{aligned} j(B) \circ j(A) e_k &= j(B) \sum_{s=1}^{m} a_{sk} e_s = \sum_{s=1}^{m} a_{sk} j(B) e_s \\ &= \sum_{s=1}^{m} a_{sk} \sum_{t=1}^{l} b_{ts} e_t \\ &= \sum_{t=1}^{l} \underbrace{\left(\sum_{s=1}^{m} b_{ts} a_{sk} \right)}_{=: c_{tk}} e_t \end{aligned}$$

Transpose and Adjoint

- The ***transpose*** of $A = (a_{ij}) \in \mathrm{Mat}(m \times n; \mathbb{F})$:
  $A^T = (a_{ji}) \in \mathrm{Mat}(n \times m; \mathbb{F})$.
- The ***adjoint*** of A: $A^* = \overline{A}^T = (\overline{a_{ij}}) \in \mathrm{Mat}(n \times m; \mathbb{F})$.
  Remark: $\langle x, Ay \rangle = \langle A^* x, y \rangle$.

Elementary Matrix Manipulations

An ***elementary row manipulation*** of a matrix is one of the following:

- Swapping (interchanging) of two rows.
- Multiplication of a row with a non-zero number.
- Addition of a multiple of one row to another row.

Inverse of Matrices

A matrix $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ is called ***invertible*** if there exists some $B \in \mathrm{Mat}(n \times n; \mathbb{R})$ such that

$$AB = BA = \mathrm{id} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

We then write $B = A^{-1}$ and say that A^{-1} is the ***inverse*** of A.

Matrices as Linear Maps

Theorem 1.5.3.
Each matrix $A \in \mathrm{Mat}(m \times n; \mathbb{R})$ uniquely determines a linear map $j(A) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that the columns $a_{\cdot k}$ are the images of the standard basis vectors $e_k \in \mathbb{R}^n$; in particular,

$$j : \mathrm{Mat}(m \times n; \mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

is an isomorphism, $\mathrm{Mat}(m \times n; \mathbb{R}) \cong \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Matrices as Linear Maps

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1. $j(A)$ is a linear map uniquely determined by A.
2. j is an isomorphism that maps from a matrix to a linear map:
    - j is a linear map.
    - j is bijective.

Matrices and Linear Maps

Statement 1.
$j(A)$ is a linear map uniquely determined by A.
Proof. Given a matrix $A \in \mathrm{Mat}(m \times n; \mathbb{R})$, we define the linear map as

$$j(A) : \mathbb{R}^n \to \mathbb{R}^m, \quad e_k \mapsto a_{\cdot k}, \quad k = 1, \ldots, n$$

and given the linear map $L \in \mathcal{L}$, we have the matrix

$$j^{-1}(L) = (a_{\cdot 1}, \ldots, a_{\cdot n}), \quad a_{\cdot k} = L(e_k), \quad k = 1, \ldots, n$$

which are determined uniquely by each other. It is easy to see that $j(A)$ defined as above is a linear map.

Matrices as Linear Maps

Statement 2.

j is an isomorphism that maps from a matrix to a linear map.

Proof.

- j is a linear map:
  - Additive:
    $j(A+B)e_k = (a+b)_{\cdot k} = a_{\cdot k} + b_{\cdot k} = j(A)e_k + j(B)e_k$.
  - Homogeneity:
    $j(\lambda A)e_k = (\lambda a)_{\cdot k} = \lambda a_{\cdot k} = \lambda j(A)e_k$.
- j is bijective: j^{-1} defined previously is the inverse of j.

Properties of Matrix Product

- ***Associative:***

$$\begin{aligned} A(BC) &= j^{-1}(j(A) \circ j(BC)) = j^{-1}(j(A) \circ (j(B) \circ j(C))) \\ &= j^{-1}((j(A) \circ j(B)) \circ j(C)) = j^{-1}(j(AB) \circ j(C)) \\ &= (AB)C \end{aligned}$$

- ***Not commutative:*** $AB \neq BA$.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \quad , \quad BA = \begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix}$$

Isomorphism

Let U, V be finite-dimensional real or complex vector spaces with bases

$$\mathcal{A} = (a_1, \dots, a_n) \subset U, \qquad \mathcal{B} = (b_1, \dots, b_m) \subset U$$

and isomorphisms

$$\varphi_\mathcal{A} : U \xrightarrow{\cong} \mathbb{R}^n, \quad \varphi_\mathcal{A}(a_j) = e_j, \quad j = 1, \dots, n$$
$$\varphi_\mathcal{B} : V \xrightarrow{\cong} \mathbb{R}^n, \quad \varphi_\mathcal{B}(b_j) = e_j, \quad j = 1, \dots, m$$

Then any linear map induces a matrix $A = \Phi^\mathcal{B}_\mathcal{A}(L) \in \mathrm{Mat}(m \times n; \mathbb{R})$ through

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_\mathcal{A}"'] & V \arrow[d, "\varphi_\mathcal{B}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad \Phi^\mathcal{B}_\mathcal{A}(L) = A = \varphi_\mathcal{B} \circ L \circ \varphi_\mathcal{A}^{-1}$$

Matrix of Complex Conjugation

- $\mathcal{B} = (1, i), L : \mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$.

$$\varphi_{\mathcal{B}} : \mathbb{C} \to \mathbb{R}^2, \quad 1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \Phi_{\mathcal{B}}^{\mathcal{B}}(L) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $\mathcal{A} = (1 + i, 1 - i), L : \mathbb{C} \to \mathbb{C}, z \mapsto \bar{z}$.

$$\varphi_{\mathcal{A}} : \mathbb{C} \to \mathbb{R}^2, \quad 1 + i \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad 1 - i \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \Phi_{\mathcal{A}}^{\mathcal{A}}(L) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Matrix Manipulations

Example 1. Find a 2×2 matrix A with $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 3b \\ 3a + b \end{pmatrix}$.

Matrix Manipulations

Example 1. Find a 2×2 matrix A with $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 3a+b \end{pmatrix}$.

Solution.
Plugging in $a = 1, b = 0$ and $a = 0, b = 1$,

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Inverse Maps

- **Inverse matrices:** Applying row operations that transfer the original matrix to the unit matrix.

$$SA = \mathrm{id} \quad \Rightarrow \quad S\mathrm{id} = S$$

  Note that we also have $(AB)^{-1} = B^{-1}A^{-1}$.

- **Inverse maps:**

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad L^{-1} = \varphi_{\mathcal{A}}^{-1} \circ A^{-1} \circ \varphi_{\mathcal{B}}$$

Rotations

Rotation in $\mathbb{R}^2$. We define the rotation of vectors in $\mathbb{R}^2$ by θ through matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Active point of view.
- Passive point of view.

$$\begin{aligned} x &= \sum x_i e_i \\ &= \sum x_i R(\theta) R(\theta)^{-1} e_i \\ &= R(\theta) \left(\sum x_i e_i' \right) \end{aligned}$$

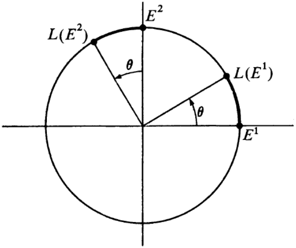

Figure: Rotation of Vectors

Changing basis as if we are applying rotation matrix on x.

Change of Basis

If $e_i' = Te_i$,

$$x = \sum_{i=1}^{n} x_i e_i, x_1, \ldots, x_n \in \mathbb{R}$$
$$x = \sum_{i=1}^{n} x_i' e_i', x_1', \ldots, x_n' \in \mathbb{R}$$

- **Goal:** Find the new coordinates $x_1', \ldots, x_n'$.
- **Method:** Apply T^{-1} to x.
- **Explanation:**

$$T^{-1} = \sum_{i=1}^{n} x_i' T^{-1} e_i' = \sum_{i=1}^{n} x_i' e_i$$

and we can identify x_i' using the original basis.

Change of Basis

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "L"] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^n \arrow[d, "\varphi_{\mathcal{B}}"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^n
\end{tikzcd}
\qquad
A \circ \varphi_{\mathcal{A}} = A = \varphi_{\mathcal{B}} \circ \mathrm{id} = \varphi_{\mathcal{B}} = T^{-1}
$$

Procedure:

1. Find matrix T so that $b_i = Te_i$.
2. Find inverse T^{-1}.
3. Operation matrix on new basis b_i.
4. Calculate TAT^{-1}.

Change Basis

Example 2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about the line through $y = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Represent R as a matrix with respect to the standard basis.

Change Basis

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Solution.

Change the standard basis to $\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\}$ via the matrix

$$S = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$$

The inverse of S is given by

$$S^{-1} = \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$$

Change Basis

Example 2. Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about the line through $y = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Represent R as a matrix with respect to the standard basis.

Solution (continued).

The map that reflects about the required line with respect to the new basis is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then

$$R = STS^{-1} = \frac{1}{13} \begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$$

System of Linear Equations — Summary

1. Concepts
   - Solution set.
   - Rank (column & row).
2. Theorems and Lemmas
   - Structure of solution set (Lemma 1.6.1 - Corollary 1.6.3).
   - Fredholm alternative (1.6.4).
   - Matrix rank and solvability (Theorem 1.6.8).

Solution Set

For a linear system of equations $Ax = b$, the ***solution set*** is given by

- $\mathrm{Sol}(A, b) = \{x \in \mathbb{R}^n : Ax = b\}$.
- $\mathrm{Sol}(A, 0) = \{x \in \mathbb{R}^n : Ax = 0\} = \ker A$

Later we will see that
$\mathrm{Sol}(A, b) = \{x_0\} + \ker A = \{y \in \mathbb{R}^n : y = x_0 + x, x \in \ker A\}$.

Rank

Let $A \in \mathrm{Mat}(m \times n; \mathbb{F})$ be a matrix with columns $a_{\cdot j} \in \mathbb{F}^m, 1 \leq j \leq n$ and rows $a_{i \cdot} \in \mathbb{F}^n, 1 \leq i \leq m$.

- The ***column rank*** of A is

$$\text{column rank } A := \dim \operatorname{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}\}.$$

- The ***row rank*** pf A is

$$\text{row rank } A := \dim \operatorname{span}\{a_{1 \cdot}, \ldots, a_{m \cdot}\}.$$

$\text{rank } A := \text{column rank } A = \text{row rank } A.$

Structure of the Solutions Set

Lemma 1.6.1. Let $x_0 \in \mathbb{R}^n$ be a particular solution to $Ax = b$, then

$$\mathrm{Sol}(A, b) = \{x_0\} + \ker\, A = \{y \in \mathbb{R}^n : y = x_0 + x, x \in \ker\, A\}$$

Fredholm Alternative 1.6.4. Let A be a $n \times n$ matrix. Then

- either $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.
  ($\ker\, A = \{0\}$ and thus A is invertible. Then the solutions is uniquely given by $x = A^{-1}b$.)
- or $Ax = 0$ has a non-trivial solution. ($\ker\, A \neq \{0\}$ and thus $Ax = b$ has no solution or infinitely many solutions.)

Matrix Rank

Theorem 1.6.8. There exists a solution x for $Ax = b$ if and only if $\mathrm{ranl}\ A = \mathrm{rank}(A|b)$, where

$$(A|b) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{pmatrix}$$

(Adding column vector b to the original matrix does not change the dimension of the column vectors.)

Proof.

$$\begin{aligned} Ax = b \text{ has solution } x \in \mathbb{R}^n &\Leftrightarrow b \in \mathrm{ran}\ A \Leftrightarrow b \in \mathrm{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}\} \\ &\Leftrightarrow \dim\ \mathrm{ran}\ A = \dim\ \mathrm{ran}(A|b) \\ &\Leftrightarrow \mathrm{rank}\ A = \mathrm{rank}(A|b) \end{aligned}$$

System of Linear Equations

Example 3. $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, prove that:

1. If $Ax = 0$ has only a trivial solution, then $A^k x = 0$ has only a trivial solution for any $k \in \mathbb{N}_+$.
2. If $Ax = 0$ has non-trivial solutions, then $A^k x = 0$ also has non-trivial solutions.

System of Linear Equations

Example 3. $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, prove that:

1. If $Ax = 0$ has only a trivial solution, then $A^k x = 0$ has only a trivial solution for any $k \in \mathbb{N}_+$.
2. If $Ax = 0$ has non-trivial solutions, then $A^k x = 0$ also has non-trivial solutions.

Solution. 1. We can prove by induction by showing that if $A^k x = 0$ has only trivial solution, then $A^{k+1} x = 0$ has only trivial solution. $A^k x = \sum_{j=1}^n x_j a_{\cdot j}^{(k)} = 0$ iff $x_j = 0$. Therefore $a_{\cdot j}^{(k)}$s are linearly independent. $\operatorname{rank} A^k = n$. ($a_{\cdot j}^{(k)}$ are the column vectors of A^k.) Then

$$a_{\cdot j}^{(k+1)} = \sum_{i=1}^{n} a_{ij} a_{\cdot i}^{(k)}, \ \sum_{j=1}^{n} \lambda_j a_{\cdot j}^{(k+1)} = 0 \Leftrightarrow \sum_{j=1}^{n} \lambda_j a_{ij} = 0, \ \forall i = 1, \dots, n$$

giving $\lambda_j = 0$ and the column vectors of A^{k+1} are independent.

System of Linear Equations

Example 3. $A \in \mathrm{Mat}(n \times n, \mathbb{R})$, prove that:

1. If $Ax = 0$ has only a trivial solution, then $A^k x = 0$ has only a trivial solution for any $k \in \mathbb{N}_+$.
2. If $Ax = 0$ has non-trivial solutions, then $A^k x = 0$ also has non-trivial solutions.

Solution. 2. If $A^k x = 0$ has a non-trivial solution x_0. Then

$$A^{k+1} x_0 = A^k \cdot Ax_0 = 0$$

has also a non-trivial solution x_0.

Rank

Example 4. Let $A \in \mathrm{Mat}(m \times n, \mathbb{F}), B \in \mathrm{Mat}(n \times l, \mathbb{F})$. Show that

$$\mathrm{rank}\ AB \leq \mathrm{rank}\ A$$

Rank

Example 4. Let $A \in \mathrm{Mat}(m \times n, \mathbb{F}), B \in \mathrm{Mat}(n \times l, \mathbb{F})$. Show that

$$\mathrm{rank}\ AB \leq \mathrm{rank}\ A$$

Solution. Let L_A and L_B be the linear maps associated with A and B, respectively. Then the multiplication of the two matrices is associated with the composition of the two linear maps.

$$\mathbb{F}^l \xrightarrow{L_B} \mathbb{F}^n \xrightarrow{L_A} \mathbb{F}^m$$

If $y \in \mathrm{ran}(L_B L_A)$, then there exists $x \in \mathbb{F}^l$ such that $L_A(L_B(x)) = y$. Therefore, $y \in \mathrm{ran}\ L_A$. Thus $\mathrm{rank}\ AB \leq \mathrm{rank}\ A$.

Inverse of a Matrix

Exercise 1. Let A be an $n \times n$ matrix, and let $a_{\cdot 1}, \ldots, a_{\cdot n}$ be its columns. Show that A is invertible if and only if $a_{\cdot 1}, \ldots, a_{\cdot n}$ are linearly independent.

Matrices as Linear Maps

Exercise 2. Define the function that maps from $\mathcal{P}_{n-1}$ to $\mathcal{P}_n$, where $\mathcal{P}_k$ is the space of polynomials in $\mathbb{R}$ of order at most k:

$$\begin{aligned} & f(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) \\ = & a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \cdots + \frac{1}{n}a_{n-1}x^n \end{aligned}$$

(1). Prove that f is linear. Find $\ker f$ and $\operatorname{ran} f$.

(2). Given bases $1, x, x^2, \ldots, x^{n-1}$ and $1, x, x^2, \ldots, x^n$ for $\mathcal{P}_{n-1}$ and $\mathcal{P}_n$, find the matrix representing f with respect to the given bases.

Basis

Exercise 3. A linear map A in $\mathbb{R}^3$ with respect to the basis

$$\eta_1 = (-1, 1, 1), \quad \eta_2 = (1, 0, -1), \quad \eta_3 = (0, 1, 1)$$

has a matrix representation

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

Find the matrix representation for A with respect to the basis

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

Basis

Exercise 4. Suppose we have a function f defined by:

$$f : \mathbb{R}^4 \to \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 + x_2 + 2x_3 + x_4 \\ -2x_2 + x_3 \\ -x_1 - x_2 + 3x_3 + x_4 \end{pmatrix}$$

We set the basis for $\mathbb{R}^4$

$$a_1 = (1, 0, 1, 1), \quad a_2 = (0, 1, 0, 1) \quad a_3 = (0, 0, 1, 0), \quad a_4 = (0, 0, 2, 1)$$

and the basis for $\mathbb{R}^3$

$$b_1 = (1, 1, 1), \quad b_2 = (1, 0, -1), \quad b_3 = (0, 1, 0)$$

Find the matrix that represents f with respect to the given bases.

Basis

Exercise 5. Find the matrix representing the following maps with respect to the given bases:

1. Use the standard basis $e_1, e_2 \in \mathbb{R}^2$. The linear map L maps a vector v to its projection onto the angle bisector for the 1st and 3rd quadrant.
2. In P_{n-1}, take basis as

$$e_0(x) = 1,\ \ e_i(x) = \frac{x(x-1)\cdots(x-i+1)}{i!},\ \ i = 1, \dots, n-1$$

Find the matrix representation A for L_A such that $L_A f(x) = f(x+1) - f(x)$.

Basis

3. In $C([a,b])$, take 6 independent vectors

$$e_1 = e^{\alpha x}\cos\beta x, \quad e_2 = e^{\alpha x}\sin\beta x, \quad e_3 = xe^{\alpha x}\cos\beta x$$
$$e_4 = xe^{\alpha x}\sin\beta x, \quad e_5 = \frac{1}{2}x^2 e^{\alpha x}\cos\beta x, \quad e_6 = \frac{1}{2}x^2 e^{\alpha x}\sin\beta x$$

Let $V = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $D : V \to V, Df \mapsto f'$. Show that D is a linear map. Find the matrix representation for D with respect to the basis given above.

Inverse of Matrix

Exercise 6. Let N be a square matrix. We say that N is nilpotent if there exists a positive integer r such that $N^r = \mathbf{0}$, where $\mathbf{0}$ is the matrix with all matrix elements being 0. Prove that if N is nilpotent then $\mathrm{id} - N$ is invertible.

Exercise 7. Let $\mathcal{P}_n$ denote the vector space of polynomials of degree $\leq n$. Then the derivative $D : \mathcal{P}_n \to \mathcal{P}_n$ is a linear map of $\mathcal{P}_n$ into itself. Let id be the identity mapping. Prove that the following linear maps are invertible:

(a). $\mathrm{id} - D^2$.

(b). $D^m - \mathrm{id}$ for any positive integer m.

(c). $D^m - c \cdot \mathrm{id}$ for any number $c \neq 0$.

P.S.

Some statements and proofs in the slides above are not mathematically rigorous. I am merely trying to give you a general idea about what is going on. Please refer to the course slides if you want to check the details!

Thanks for your attention!