Honors Mathematics III
Review — Midterm 1

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Linear maps

To prove a map from vector space $(U, \oplus, \odot)$ to another vector space $(V, \boxplus, \boxdot)$ is ***linear***, it needs to be

- Homogeneous: $L(\lambda \odot u) = \lambda \boxdot L(u)$, and
- Additive: $L(u \oplus u') = L(u) \boxplus L(u')$.

Examples.

1. For $I \in \mathbb{R}$, the map $D : C^1(I) \to C(I), f \mapsto f'$ is linear.
2. The complex conjugation map in $\mathbb{C}$ is linear if $\mathbb{C}$ is regarded as a real vector space.
3. Exercise 2.6. $\mathcal{P}_n$ is the vector space of real polynomials over $\mathbb{R}$ of degree at most n. The map

$$\alpha : \mathcal{P}_n \to \mathbb{R}, \qquad \alpha(p) = \int_{-1}^{1} p(x)\mathrm{d}x$$

is linear.

Range and Kernel

Definitions.

- ***Dual space***: $V^* = \mathcal{L}(V, \mathbb{F})$ for a finite-dimensional vector space V with basis $\{b_1, \ldots, b_n\}$.
- ***Dual basis***: $\{b_1^*, \ldots, b_n^*\}$ with

$$b_k^*(b_j) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

Exercise 2.5

1. $L = \sum_{i=1}^{n} L(b_i) b_i^*$ for a linear map $L \in \mathcal{L}(V, \mathbb{F})$.
2. $V \cong V^*$.

Range and Kernel

Definitions.

$$\operatorname{ran} L := \left\{ v \in V : \underset{u \in U}{\exists} v = Lu \right\}, \qquad \ker L := \{ u \in U : Lu = 0 \}$$

Results.

1. A linear map $L \in \mathcal{L}(U, V)$ is injective iff $\ker L = \{0\}$.
2. Dimension formula. $\dim \operatorname{ran} L + \dim \ker L = \dim U$ (for finite-dimensional vector spaces).

Isomorphisms

- For n-dimensional vector spaces U, V, an isomorphism maps from basis $(b_1, \dots, b_n)$ to basis $(Lb_1, \dots, Lb_n)$.
- For finite-dimensional vector spaces U, V, $U \cong V \quad \Leftrightarrow \quad \dim U = \dim V$.
- If $\dim U = \dim V$, then for a linear map $L \in \mathcal{L}(U, V)$, injective $\Leftrightarrow$ surjective $\Leftrightarrow$ bijective.

Operator Norm

- Equivalent definitions:

$$\|L\| := \sup_{\substack{u \in U \\ u \neq 0}} \frac{\|Lu\|_V}{\|u\|_U} = \sup_{\substack{u \in U \\ \|u\|_U = 1}} \|Lu\|_V$$

- Additional property:

$$\|L_2 L_1\| \leq \|L_2\| \cdot \|L_1\|, \qquad L_1 \in \mathcal{L}(U, V), \quad L_2 \in \mathcal{L}(V, W)$$

Linear Maps as Matrices

Every linear map between finite-dimensional vector spaces can be expressed as a matrix.

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad \Phi_{\mathcal{A}}^{\mathcal{B}}(L) = A = \varphi_{\mathcal{B}} \circ L \circ \varphi_{\mathcal{A}}^{-1}$$

- Matrix multiplication.
- Transpose A^T and adjoint A^*, $\langle x, Ay \rangle = \langle A^* x, y \rangle$.
- Inverse A^{-1} of $n \times n$ matrices. (Find inverse using Gauss-Jordan algorithm.)
- ***Change basis***.

Change Basis — Passive Point of View

1. Find basis change matrix T such that $e_i' = Te_i$.
2. Find inverse of T.
3. Find matrix A representing the operation with respect to the new basis.
4. Calculate TAT^{-1}.

Solution Set

Structure of solution set.

$$\mathrm{Sol}(A, b) = \{x_0\} + \ker A$$

Fredholm alternatives.

1. $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.
    - A is invertible.
    - $\det A \neq 0$.
    - $\ker A = \{0\}$.
2. $Ax = 0$ has a non-trivial solution. ($Ax = b$ either has no solution or infinitely many solutions.)
    - A is not invertible.
    - $\det A = 0$.
    - $\ker A \neq \{0\}$.

In general, a system $Ax = b$ is solvable if b is in the range of A, i.e., in the span of column vectors of matrix A.

Determinant

- Determinants in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Permutations.
- Group and group actions.
- Determinant in $\mathbb{R}^n$.

Properties of Determinant

- Normed. $\det \mathrm{id} = 1$.
- Multilinear.
- Alternating:
  - $\det$ is alternating.
  - $$\begin{aligned} &\det(a_{\cdot 1}, \dots, a_{\cdot (j-1)}, a_{\cdot j}, a_{\cdot (j+1)}, \dots, a_{\cdot (k-1)}, a_{\cdot k}, a_{\cdot (k+1)}, \dots, a_{\cdot p}) \\ = &-\det(a_{\cdot 1}, \dots, a_{\cdot (j-1)}, a_{\cdot k}, a_{\cdot (j+1)}, \dots, a_{\cdot (k-1)}, a_{\cdot j}, a_{\cdot (k+1)}, \dots, a_{\cdot p}) \end{aligned}$$
  - $\det(a_{\cdot 1}, \dots, a_{\cdot p}) = 0$ if $a_{\cdot 1}, \dots, a_{\cdot p}$ are linearly dependent.

Properties of Determinant

1. Elementary column operations.
   - $\det(a_2, a_1, \ldots, a_n) = -\det(a_1, a_2, \ldots, a_n)$.
   - $\det(a_1, \ldots, \lambda a_j, \ldots, a_n) = \lambda \det(a_1, \ldots, a_j, \ldots, a_n)$.
   - $\det(a_1, \ldots, a_j, \ldots, a_k + \lambda a_j, \ldots, a_n) =$
     $\det(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n)$.
2. $\det A = \det A^T$.
3. $\det(AB) = (\det A)(\det B)$.
4. $\det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \ a_{1\pi(1)} \cdots a_{n\pi(n)}$.
5. $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

Results from Determinant

- Solve linear system of equations.

$$x_i = \frac{1}{\det\ A}\det(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n), \quad i = 1, \ldots, n$$

- Find inverse.

$$A^{-1} = \frac{1}{\det A} A^\sharp$$

Calculating Determinants

Few methods to calculate determinant:

1. In $\mathbb{R}^3$: expand to the case of $\mathbb{R}^2$ and calculate directly.
2. In higher dimensions: use properties of determinant. Apply ***elementary row or column operations*** combined with Laplace expansion.
3. Leibnitz Formula.
4. Induction.

Systems of Linear Equations

Gauss-Jordan algorithm.

Finite-Dimensional Vector Spaces

Definition of Linear Independence.

$$v = \sum_{i=1}^{n} \lambda_i v_i = 0 \quad \Leftrightarrow \quad \lambda_i = 0$$

- v is a n-dimensional vector.
- v is a linear map: v acting on any vector (basis) gives 0.

Exercise 2.5. The maps $b_1^*, \dots, b_n^*$ form a basis of V^*.

1. $L = \sum_{i=1}^{n} \lambda_i b_i^* = 0 \ \Leftrightarrow \ Lb_i = 0 \ \Leftrightarrow \ \lambda_i = 0.$
2. $\forall L \in \mathcal{L}(V, \mathbb{F}), L = \sum_{i=1}^{n} L(b_i) b_i^*.$

Finite-Dimensional Vector Spaces

Basis (Any two combined.)

- $\mathrm{span}\{b_1, \ldots, b_n\} = V$.
- $\{b_1, \ldots, b_n\}$ is linearly independent.
- The length of the set $\{b_1, \ldots, b_n\}$ is $\dim V$.

Proof using basis.

- Definition: unique representation.
- Characterization of basis:
  1. $\{b_1, \ldots, b_n\}$ is independent.
  2. $V = \mathrm{span}\{b_1, \ldots, b_n\}$.
- Finite-dimensional vector spaces (subspaces) — basis extension theorem.

  e.g. $\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$.

Inner Product Spaces

Inner Product.

- Definitions of inner product. — proof of inner products.
  1. $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$.
  2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.
  3. $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$.
  4. $\langle u, v\rangle = \overline{\langle v, u\rangle}$.
- Induced norm. $\|v\|^2 = \langle v, v\rangle$.
  e.g. $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.
- Normalization and projection.
  e.g. (2.4) The vector space $C([-1, 1])$ of real continuous functions on $[-1, 1]$ endowed with the scalar product $\langle f, g\rangle := \int_{-1}^{1} fg$ and the induced norm $\|f\| = \sqrt{\langle f, f\rangle}$.
  1. $m_k(x) = x^k, k = 0, 1, \ldots$.
  2. $\{1, \sin(n\pi x), \cos(n\pi x)\}_{n=1}^{\infty}$.
- ***Note:*** $V = A \oplus B, V = A \oplus C \not\Rightarrow B = C$.

Linear Maps

- Vector spaces:
  1. Dimension.
  2. Basis.
- Linear map:
  1. Range.
  2. Kernel.

Matrices

Matrices as Linear Maps

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\varphi_{\mathcal{A}}"'] & V \arrow[d, "\varphi_{\mathcal{B}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad \Phi_{\mathcal{A}}^{\mathcal{B}}(L) = A = \varphi_{\mathcal{B}} \circ L \circ \varphi_{\mathcal{A}}^{-1}$$

- $\operatorname{ran} L = \operatorname{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}\}$.
- Matrix acting on x.
- Matrix elements. $\langle e_i, Ae_j \rangle = a_{ij}$.

Basis change. Use a basis that is convenient for the operation.

- Projection.
- Rotation.
  e.g.(3.5.) Find the matrix describing rotation about the axis through the origin and the point $(1, 2, -1)^T$.

Theory of Systems of Linear Equations

- Fredholm alternatives.
- Matrix rank.
- Presenting solution sets.

Determinants

- Properties. (Row and column operations.)
- Matrix implications.
- Relations to system of linear equations.

Final Remarks

1. Vectors and vector spaces. $(\operatorname{ran} V, \ker V, \operatorname{span} \mathcal{B}$ are all vector spaces!)
2. If the notations do not occur in the problem statement, define then clearly.
3. Do not forget to normalize the vectors using the ***specified*** inner product in Gram-Schmidt orthonormalization.
4. Consider using properties to reduce work.
5. Be careful with calculations.

Thanks for your attention!

Good Luck!