Honors Mathematics III
RC 6

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Integrals

2.2.25. Definition. A ***step function*** with respect to a partition $P = (a_0, \dots, a_n)$ with elements $y_i \in V$, $f(t) = y_i$ whenever $a_{i-1} < t < a_i, i = 1, \dots, n$.

2.2.29. Theorem. Let $f : [a, b] \to V$ be a step function with respect to some partition P. Then the ***integral*** of f is

$$I_P(f) := (a_1 - a_0)y_1 + \cdots + (a_n - a_{n-1})y_n \in V$$

and is independent of the choice of P.

$$\left\| \int_a^b f(x)\mathrm{d}x \right\|_V \leq \int_a^b \|f(x)\|_V \mathrm{d}x \leq |b - a| \cdot \sup_{x \in [a,b]} \|f(x)\|_V$$

Mean Value Theorem

2.2.30. Mean Value Theorem. X, V are finite-dimensional vector spaces, $\Omega \subset X$ is open and $f \in C(\Omega, V)$. $x, y \in \Omega$ and the line segment $x + ty, 0 \leq t \leq 1$ is wholly contained in Ω. Then

$$f(x+y) - f(x) = \int_0^1 Df|_{x+ty} y \mathrm{d}t = \left(\int_0^1 Df_{x+ty} \mathrm{d}t \right) y$$

Differentiating Under an Integral

2.2.33. Theorem.

1. X, V are finite-dimensional vector spaces.
2. $I = [a, b] \subset \mathbb{R}$, $\Omega \subset X$ an open set.
3. $f : I \times \Omega \to V$, $Df(t, \cdot)$ exists and is continuous for every $t \in I$.

Then

$$g(x) = \int_a^b f(t, x)\mathrm{d}t, \quad Dg(x) = \int_z^b Df(t, \cdot)|_x \mathrm{d}t$$

Curves

Definition. Let V be a finite-dimensional vector space and $I \subset \mathbb{R}$ an interval.

- ***Curve:*** a set $\mathcal{C} \subset V$ with a continuous, surjective and locally injective map $\gamma : I \to \mathcal{C}$.
- ***Parametrization:*** the map γ.
- ***Parametrized curve:*** $\mathcal{C}$ together with its parametrization γ.
- ***Simple curve:*** γ is globally injective.
- ***Closed:*** $\lim_{t\to a} \gamma(t) = \lim_{t\to b} \gamma(t)$.
- ***Open:*** $x := \lim_{t\to a} \gamma(t), y := \lim_{t\to b} \gamma(t), x \neq y$. x is the ***initial point*** and y is the ***final point***.

Curves

Example. The set

$$S := \{(x_1, x_2) \in \mathbb{R} : x_1^2 + x_2^2 = 1\}$$

with parametrization:

- $$\gamma : [0, 2\pi] \to S, \qquad \gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

  or
- $$\tilde{\gamma} : [0, 1] \to S, \qquad \tilde{\gamma}(t) = \begin{pmatrix} \cos(2\pi t) \\ -\sin(2\pi t) \end{pmatrix}$$

are parametrized curves.

Reparametrization of Curves

Definition. Suppose $\mathcal{C} \subset V$ is a curve with parametrization $\gamma : I \to \mathcal{C}$.

- ***Reparametrization:*** a continuous, bijective map $r : J \to I$.
- ***Orientation-preserving r:*** r is increasing.
- ***Orientation-reversing r:*** r is decreasing.

Curves in Polar Coordinates

A ***curve in polar coordinates*** is parametrized by

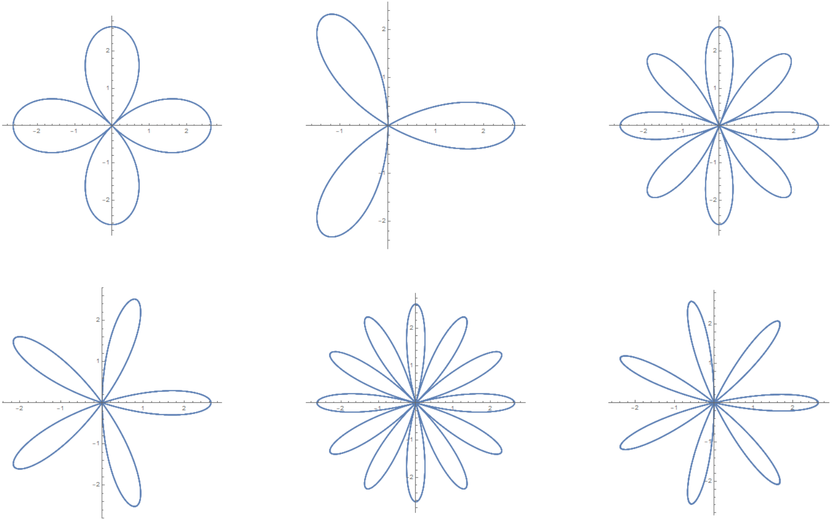
$$\gamma(t) = \begin{pmatrix} f(t)\cos(t) \\ f(t)\sin(t) \end{pmatrix}$$

where $f : \mathbb{R} \to \mathbb{R}$.
Example. The curves $r = \sqrt{7}\cos(at)$ for $a = 2, 3, \ldots, 7$ represent the curves

$$\gamma(t) = \begin{pmatrix} \sqrt{7}\cos(at)\cos(t) \\ \sqrt{7}\cos(at)\sin(t) \end{pmatrix}$$

Curves in Polar Coordinates

Example (continued). The graphs are shown below.



Smooth Curves

Definition. A curve $\mathcal{C} \subset V$ with parametrization $\gamma : I \to \mathcal{C}$ is ***smooth*** if

- γ is continuously differentiable on $\operatorname{int} I$ and
- $D\gamma|_t \neq 0$ for all $t \in \operatorname{int} I$.

Definition. A ***smooth parametrization*** is

- continuously differentiable and
- the derivative is non-vanishing in the interior of its domain.

Tangent Lines and Tangent Vectors

Definition.

- The ***tangent line*** to a curve $\mathcal{C} \subset \mathbb{R}^n$ at $p = \gamma(t_0)$ is

$$T_p\mathcal{C} = \{x \in \mathbb{R}^n : x = \gamma(t_0) + \gamma'(t_0)t : t \in \mathbb{R}\}$$

- The ***unit tangent vector*** of an oriented smooth curve $\mathcal{C}^* \subset \mathbb{R}^n$ with parametrization γ at $p = \gamma(t)$ is

$$T \circ \gamma(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \qquad t \in \operatorname{int} I$$

which defines the ***tangent vector field*** $T : \mathcal{C}^* \to \mathbb{R}^n$ on $\mathcal{C}$. For reparametrization $r : J \to I, \tilde{\gamma} = \gamma \circ r$,

$$\tilde{\gamma}'(\tau) = \gamma'(t)r'(\tau), \quad T \circ \tilde{\gamma}(\tau) = \frac{r'(\tau)}{|r'(\tau)|} T \circ \gamma(t)$$

Note. The unit tangent vector depends on parametrization only w.r.t. the orientation.

The Normal Vector of a Curve

Definition. The ***unit normal vector*** $N : \mathcal{C} \to \mathbb{R}$ of a smooth C^2-curve with parametrization $\gamma : I \to V$ is

$$N \circ \gamma(t) := \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}, \qquad t \in \operatorname{int} I$$

Note. The unit normal vector does not depend on γ on

- magnitude and
- orientation.

Curve Length

2.3.25. Theorem. $\mathcal{C} \subset V$ is a smooth and ***open*** curve with parametrization $\gamma : [a, b] \to \mathcal{C}$. Then $\mathcal{C}$ is rectifiable iff

$$\int_a^b \|\gamma'(t)\| \mathrm{d}t < \infty$$

and the ***curve length*** is

$$\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

which is independent of γ.

Curve Length

The ***length function*** is defined as

$$(\ell \circ \gamma)(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$$

The curve length gives the ***natural parametrization*** of an oriented curve $\mathcal{C}$.

$$\gamma = \ell : I \to \mathcal{C}, \qquad \mathrm{int}\, I = (0, \ell(\mathcal{C}))$$

Note. Then we also obtain

$$\|\gamma'(t)\| = \frac{\mathrm{d}\ell \circ \gamma(t)}{\mathrm{d}t}$$

Curvature

Definition. The ***curvature*** of a smooth C^2-curve $\mathcal{C} \subset V$ is

$$\kappa : \mathcal{C} \to \mathbb{R}, \qquad \kappa \circ \ell^{-1}(s) := \left\| \frac{\mathrm{d}}{\mathrm{d}s}(T \circ \ell^{-1}(s)) \right\|$$

where T is the unit tangent vector and $\ell^{-1} : I \to \mathcal{C}$ is the curve length parametrization of $\mathcal{C}$.

Note. The curvature κ does not depend on the orientation of $\mathcal{C}$.

Curvature in $\mathbb{R}^3$

2.3.31. Lemma. Let $\mathcal{C} \subset \mathbb{R}^3$ be a smooth C^2-curve with parametrization $\gamma : I \to \mathcal{C}$, then

$$\kappa \circ \gamma(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

Chain Rule

Exercise 1. Suppose a function $u(x, y)$ is differentiable in $\mathbb{R}^2$, find the representation of Laplace operator $\Delta_{(r,\theta)}$ in polar coordinates, where

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Exercises.

Exercise 2. Prove the Euler's integral formula for $n!$.

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n!$$

Exercises

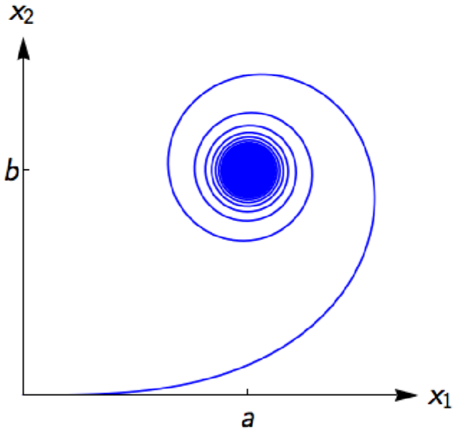
Exercise 3. A plane curve $\mathcal{C} \subset \mathbb{R}^2$ is parametrized by

$$\gamma : [0, \infty) \to \mathbb{R}^2, \qquad \gamma(t) = \begin{pmatrix} \int_0^t \cos(s^2)\mathrm{d}s \\ \int_0^t \sin(s^2)\mathrm{d}s \end{pmatrix}$$

1. Sketch the curve. How does it behave as $t \to \infty$?
2. Show that γ is the curve length parametrization of $\mathcal{C}$.
3. Find the curvature of $\mathcal{C}$.

Exercises

Exercise 2.



Summary

- Tangent line at a point $\gamma(t_0)$: $\{\gamma(t_0) + \gamma'(t_0)t, t \in \mathbb{R}\}$
- Unit tangent vector: $T \circ \gamma(t) = \dfrac{\gamma'(t)}{\|\gamma'(t)\|}$.
- Unit normal vector: $N \circ \gamma(t) = \dfrac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}$.
- <u>Open</u> curve length: $\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$.
- Curve length function: $\ell \circ \gamma(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$.
- $\|\gamma'(t)\| = \dfrac{\mathrm{d}(\ell \circ \gamma)(t)}{\mathrm{d}t}$.
- Curvature: $\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)|_{s=\ell\circ\gamma(t)} = \dfrac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}$.
- Curvature in $\mathbb{R}^3$:
  $\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)|_{s=\ell\circ\gamma(t)} = \dfrac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$.

Thanks for your attention!