VV285 Honors Mathematics III
Solution Manual for RC 2

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Exercise 1.

(1).

For $f_i, f_j \in \mathcal{B}$, we have

$$
\begin{aligned}
\langle f_n, f_m \rangle &= \int_{-\pi}^{\pi} \overline{f_n(t)} f_m(t) \mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imt} \cdot e^{-int} \mathrm{d}t \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)t} \mathrm{d}t \\
&= \begin{cases} 1 & \text{if } m = n \\ \dfrac{1}{2\pi} \cdot \dfrac{1}{i(m-n)} \cdot e^{i(m-n)t} \Bigg|_{-\pi}^{\pi} = 0 & \text{if } m \neq n \end{cases}
\end{aligned}
$$

Therefore, $\mathcal{B}$ is an orthonormal basis.

(2).

The coefficients of the linear combination can be found by taking the projection of h onto each basis vector.

Then

$$
\begin{aligned}
h(t) &= \sum_{n=-\infty}^{\infty} \langle f_n, h \rangle f_n \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} t \cdot e^{-int} \mathrm{d}t \cdot e^{int} \\
&= \sum_{n=-\infty}^{-1} -\frac{1}{in}(-1)^n \cdot e^{int} + \sum_{n=1}^{\infty} -\frac{1}{in}(-1)^n \cdot e^{int}
\end{aligned}
$$

Note that this is a real-valued function since

$$
\begin{aligned}
\sum_{n=-\infty}^{-1} -\frac{1}{in}(-1)^n \cdot e^{int} + \sum_{n=1}^{\infty} -\frac{1}{in}(-1)^n \cdot e^{int} &= \sum_{n=-\infty}^{\infty} -\frac{1}{in}(-1)^n \cdot (\cos(nt) + i \sin(nt)) \\
&= -\sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin(nt)
\end{aligned}
$$

The plots of $h(t)$ and the expression with 20 summands are shown in Figure 1.

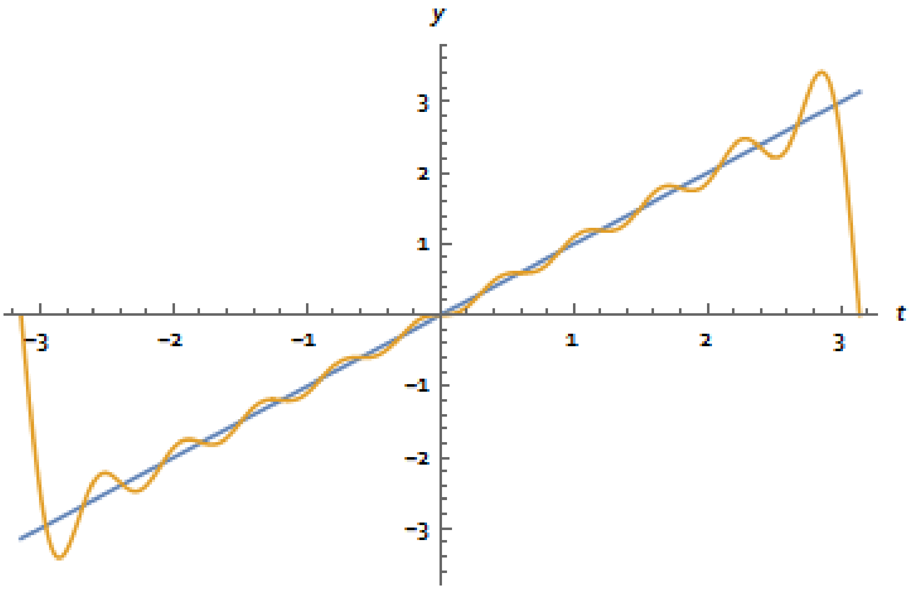

Figure 1: Approximation of $h(t) = t$.

Exercise 2.

To show that $\mathcal{B} = \{v_1, \dots, v_m\}$ is a basis of V, we need to show that span $\mathcal{B} = V$ and $\mathcal{B}$ is an independent set. Since the elements in this set are orthogonal to each other, it suffices to show that they are independent. To show $\mathcal{B}$ spans V, for any $v \in V$, let $w = \sum_{i=1}^m \langle v_i, v\rangle v_i$ and we have

$$\begin{aligned}\|v-w\|^2 &= \langle v-w, v-w\rangle \\ &= \|v\|^2 + \|w\|^2 - \langle v, w\rangle - \langle w, v\rangle \\ &= 2\sum_{i=1}^m \langle v_i, v\rangle^2 - 2\langle v, \sum_{i=1}^m \langle v_i, v\rangle v_i\rangle \\ &= 2\sum_{i=1}^m \langle v_i, v\rangle^2 - 2\sum_{i=1}^m \langle v_i, v\rangle^2 = 0\end{aligned}$$

Therefore, $v = w$ and V is spanned by $\mathcal{B}$.

Exercise 3.

The dual space of V^* is the space of linear map $V^{**} = \mathcal{L}(V^*, \mathbb{F})$. Therefore, for a fixed $v \in V$ we can define a linear map

$$\lambda_v : V^* \to \mathbb{F}, \qquad \varphi^* \mapsto \varphi^*(v)$$

where φ^* is in the dual space of V.

Then defining a map $\Phi : V \to V^{**}, \Phi(v) = \lambda_v$, we want to show that this map is an isomorphism of V and V^{**}. Namely, we want to show that it is linear and bijective.

- **Linear:** For any $v_1, v_2 \in V$,

$$\lambda_{v_1+v_2}(\varphi^*) = \varphi^*(v_1+v_2) = \varphi^*(v_1) + \varphi^*(v_2) = \lambda_{v_1}(\varphi^*) + \lambda_{v_2}(\varphi^*)$$

  So $\Phi(v_1+v_2) = \Phi(v_1) + \Phi(v_2)$. Similarly, we show the homogeneity $\Phi(cv) = c\Phi(v)$.

- **Bijective:** Set a basis in V as $\mathcal{B} = \{b_1, \dots, b_n\}$, then considering the dual basis for V^*, we can show that the maps λ_{b_i} are linearly independent:

$$\sum_{i=1}^{n} \mu_i \lambda_{b_i} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{n} \mu_i \left(\sum_{j=1}^{n} \eta_{ij} b_j^* \right) b_i = 0 \text{ for arbitrary } \eta_{ij}$$

  namely,

$$\sum_{i=1}^{n} \mu_i \eta_{ii} = 0 \qquad \text{for arbitrary } \eta_{ii}$$

  Therefore, $\mu_i = 0$ and λ_{b_i} are independent. Since $\dim V^* = n$, the dimension of the dual space $\dim V^{**} = n$. Therefore, λ_v maps from basis to another basis, it is an isomorphism for V and V^{**}.

Note:

1. $\varphi^* \in V^*$ is a map that takes a vector in V and returns $\varphi^*(v)$.
2. $\lambda_v \in V^{**}$ is a map that takes a map in V^* and returns the value of this map acting on v, i.e., $\varphi^*(v)$.
3. To prove that V is isomorphic to V^{**}, we want to find a map Φ which is a bijection. Specifically, $\Phi(v) = \lambda_v$, which is in $\mathcal{L}(V^*, \mathbb{F})$. (Namely, we have $\Phi(v)\varphi^* = \varphi^*(v)$.
4. To show this bijection, we want to prove that Φ maps from basis to basis. We define $\lambda_{b_i} \in V^{**}$, which is determined by each b_i in V.
   (a) λ_{b_i} are linearly independent. Note that λ_{b_i} is a map in $\mathcal{L}(V^*, \mathbb{F})$. So if we define

$$L := \sum_{i=1}^{n} \mu_i \lambda_{b_i}$$

   and let L equals 0, this means that for every $\varphi^* \in V^*$, $\varphi^*(b_i) = 0$, because $\lambda_{b_i}(\varphi^*) = \varphi^*(b_i)$.

   (b) Then we show that λ_{b_i} spans V^{**}. We do this by showing $\dim V^{**} = n$. This follows from $\dim V = \dim V^* = \dim V^{**}$.

Exercise 4.

We prove this by induction. When $k = 1$, we have

$$AB - BA = \mathrm{id}$$

The statement holds.

Then suppose the statement holds for $k \geq 2$. Namely,

$$A^k B - BA^k = kA^{k-1}$$

Then for $k + 1$,

$$\begin{aligned} A^{k+1}B - BA^{k+1} &= A^k \cdot AB - BA^k \cdot A \\ &= A^k(\mathrm{id} + BA) - (A^k B - kA^{k-1})A \\ &= A^k + kA^k = (k+1)A^k \end{aligned}$$

verifying the statement.

Therefore, the statement holds for all $k \in \mathbb{N}_+$.

Exercise 5.

(1).

Since V is finite, the kernel of L is also finite. Set a basis of the kernel as $\{b_1, \ldots, b_r\}$ which can be extended to a basis in V as $\mathcal{B} = \{b_1, \ldots, b_r, b_{r+1}, \ldots, b_n\}$. Set $\mathrm{span}\{b_{r+1}, \ldots, b_n\} = W$. Take an arbitrary vector $v_1 \in W$, then

$$L^2 v_1 = Lv_1 \qquad \Rightarrow \qquad L(Lv_1 - v_1) = 0$$

If there exists v_1 such that $Lv_1 \neq v_1$, then $Lv_1 - v_1 = v_1' \in \ker L$ and $v' \neq 0$. Therefore, $\ker L \cap \mathrm{ran}\, L \neq \{0\}$.

Then there exists $v'' \in W$ such that $Lv'' \in \ker L$. However, $L^2 v'' = 0 \neq Lv''$, giving a contradiction. Therefore, by setting $v_1 = \sum_{i=1}^{r} \langle b_i, v_1 \rangle b_i$ and $v_2 = v - v_1$, a representation can be obtained.

We then need to show that this representation is unique. Suppose there exist v_1, v_2, v_1', v_2' such that $Lv_1 = v_1, Lv_1' = v_1', Lv_2 = Lv_2' = 0$ and $v = v_1 + v_2 = v_1' + v_2'$, then $v_1 - v_2 = v_1' - v_2' \in \ker L$, giving a contradiction. Therefore, the original statement is verified.

(2).

Represent $v = v_1 + v_2$ with the properties described above, we then have

$$Lv = L(v_1 + v_2) = v_1 = -v$$

Set

$$v = \sum_{i=1}^{n} \lambda_i b_i, \qquad , v_2 = \sum_{i=1}^{r} \lambda_i b_i, \qquad v_1 = \sum_{i=r+1}^{n} \lambda_i b_i$$

Then we have

$$\sum_{i=1}^{r} \lambda_i b_i + \sum_{i=r+1}^{n} 2\lambda_i b_i = 0 \quad \Rightarrow \lambda_i = 0$$

giving $v = 0$.

Exercise 6.

The proof includes two steps. First we need to show that the norm exists because the definition of the operator norm is bounded above. Namely,

$$\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \leq C$$

Second we show that the supremum has the desired value as is stated in the problem.

$$C = \max_{1 \leq j \leq n} \sum_{i=1}^{n} a_{ij}$$

Let C be defined as above. For $x \neq 0$, we have

$$\|Ax\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j|$$

Using triangle inequality and interchanging the order of summation, we then obtain

$$\|Ax\|_1 = \sum_{j=1}^{n} |x_j| \sum_{i=1}^{n} |a_{ij}| \leq C \sum_{j=1}^{n} |x_j|$$

and therefore,

$$\|Ax\|_1 \leq C\|x\|_1, \qquad \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq C$$

The supremum is bounded above.

Then we need to show that this bound can be achieved by some vector x_0. This then proves that C is the maximum. Analyzing the columns of the matrix, we note that there exists an integer $1 \leq k \leq n$ such that

$$\sum_{i=1}^{n} |a_{ik}| = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

Then setting $x_0 = e_k$, where e_k is the kth standard basis vector, we have

$$\frac{\|Ax_0\|_1}{\|x_0\|_1} = \sum_{i=1}^{n} |a_{ik}| = C$$

verifying that the upper bound can be reached. Therefore, the operator norm induced by the norm of vector is

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$