VV285 Honors Mathematics III
Solution Manual for RC 3

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June 20, 2018

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Exercise 1.

Let A represents a linear map on $V = \mathbb{C}^n$.

($\Rightarrow$): If A is invertible, then ker $A = \{0\}$. Therefore, for any $x \in V$, $Ax = 0$ iff $x = 0$. Namely,

$$\forall x = (x_1, \dots, x_n)^T, \quad Ax = \sum_{i=1}^{n} x_i a_{\cdot i} = 0 \text{ iff } x_i = 0$$

Therefore, the columns of A are linearly independent.

($\Leftarrow$): If the columns of A are independent, using the similar arguments,

$$\forall x = (x_1, \dots, x_n)^T, \quad \sum_{i=1}^{n} x_i a_{\cdot i} = Ax = 0 \text{ iff } x_i = 0$$

Therefore, ker $A = \{0\}$

Exercise 2.

(1).

Choosing the basis $1, x, x^2, \dots, x^{n-1}$, p can be represented as $p = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$. To prove that f is linear, we show the followings.

First, it is additive.

$$\begin{aligned} f(p + p') &= f((a_0 + a_0') + \cdots + (a_{n-1} + a_{n-1}')x^{n-1}) \\ &= a_0 x + \cdots + \frac{1}{n} a_{n-1} x^n + a_0' x + \cdots + \frac{1}{n} a_{n-1}' x^n \\ &= f(p) + f(p') \end{aligned}$$

Second, it is homogeneous.

$$\begin{aligned} f(\lambda p) &= f(\lambda(a_0 + \cdots + a_{n-1})x^{n-1}) \\ &= \lambda(a_0 x + \cdots + \frac{1}{n} a_{n-1} x^n) \\ &= \lambda f(p) \end{aligned}$$

To find the kernel, we let $p \in \mathcal{P}_{n-1}$ and set

$$f(p) = a_0 x + \cdots + \frac{1}{n} a_{n-1} x^n = 0$$

which implies that $a_0 = a_1 = \cdots = a_{n-1} = 0$. Therefore, ker $f = \{0\}$.

f can be found by

$$f : \mathcal{P}_{n-1} \to \mathcal{P}_n, \quad f(x^k) = x^{k+1}, \quad k = 0, 1, \dots, n-1$$

where $\{1, \dots, x^{n-1}\}$ is a basis of $\mathcal{P}_{n-1}$ and $\{x, \dots, x^n\}$ is an independent set. Since $\dim \mathcal{P}_{n-1} = n = \dim \operatorname{ran} f$, we find the range of f as $\operatorname{span}\{x, \dots, x^n\}$.

(2).

Taking isomorphisms $\mathcal{P}_{n-1} \cong \mathbb{R}^n$ and $\mathcal{P}_n \cong \mathbb{R}^{n+1}$, we represent the matrix $F \in \operatorname{Mat}((n+1) \times n, \mathbb{R})$ as

$$F = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \mathbf{0} \\ & 1/2 & \ddots & \\ \mathbf{0} & & \ddots & 0 \\ & & & 1/n \end{pmatrix}$$

Exercise 3.

$$\begin{tikzcd} \mathbb{R}^3(e) \arrow[r, "A'"] \arrow[d, "T"'] & \mathbb{R}^3(e) \arrow[d, "T"] \\ \mathbb{R}^3(\eta) \arrow[r, "A"] & \mathbb{R}^3(\eta) \end{tikzcd} \qquad A' = TAT^{-1}$$

The change of basis map and its inverse (by calculations) are given by

$$T = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Therefore, with respect to the standard basis, the linear map is given by

$$A' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Exercise 4.

The relations representing the vector spaces and the linear maps are shown in the following graph, where $e^{(4)}, e^{(3)}$ are the standard bases in $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively.

$$\begin{tikzcd} \mathbb{R}^4(e^{(4)}) \arrow[r, "F"] \arrow[d, "T_1"'] & \mathbb{R}^3(e^{(3)}) \arrow[d, "T_2"] \\ \mathbb{R}^4(a) \arrow[r, "A"] & \mathbb{R}^3(b) \end{tikzcd} \qquad A = T_2^{-1} F T_1$$

The bases change matrices and their inverses (with calculations) are given by

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & -2 \\ -1 & -1 & 0 & 1 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad T_2^{-1} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}$$

Therefore, the matrix representing f with respect to $\mathcal{A}$ and $\mathcal{B}$ is given by

$$F = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 & 1 \\ 0 & -2 & 1 & 0 \\ -1 & -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5/2 & 5/2 & 15/2 \\ -2 & -1/2 & -1/2 & -5/2 \\ -3 & -9/2 & -3/2 & -11/2 \end{pmatrix}$$

Exercise 5.

1.

The map is illustrated in Figure 1.

For convenience, we change the basis to

$$b_1 = (1,1)^T, \qquad b_2 = (-1,1)^T$$

by

$$T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad b_i = Te_i, \quad i = 1, 2$$

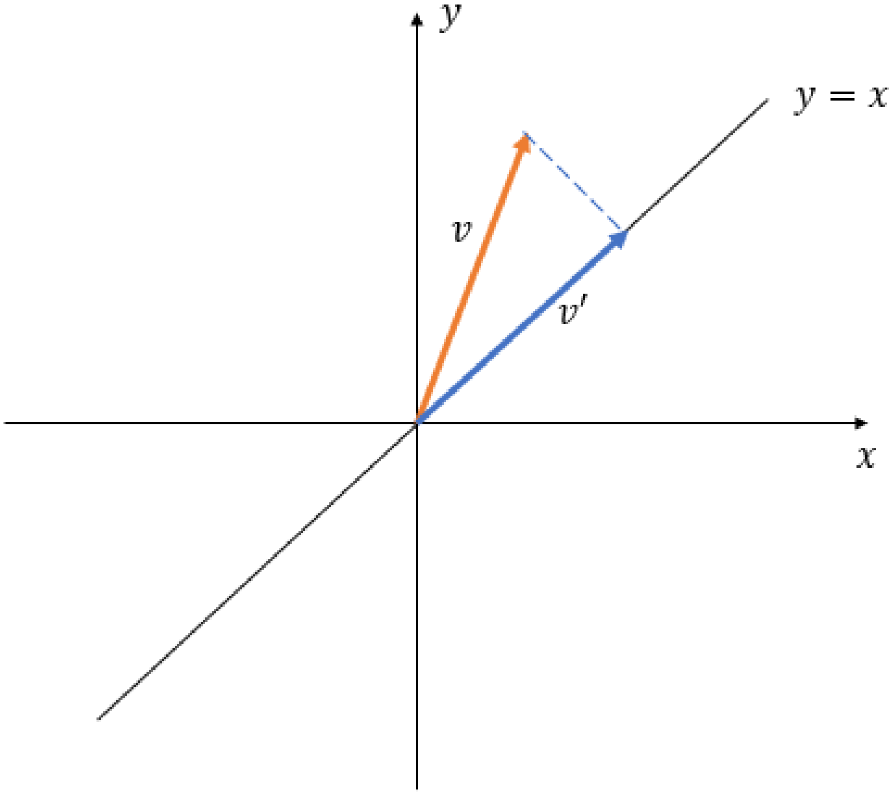

Figure 1: Projection onto Angle Bisector.

and

$$T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The projection matrix with respect to the new basis is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore,

$$A' = TAT^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2.

The linear map acts on each basis vector gives

$$L_A e_i(x) = e_i(x+1) - e_i(x) = \frac{x(x-1)\cdots(x-i+2)}{(i-1)!} = e_{i-1}(x), \qquad i \geq 2$$

and

$$L_A e_1(x) = e_1(x+1) - e_1(x) = 1 = e_0(x), \qquad L_A e_0(x) = 0$$

Therefore, the matrix for this map is given by

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

3.

By calculations,

$$\begin{aligned}
De_1 &= \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x = \alpha e_1 - \beta e_2 \\
De_2 &= \alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x = \alpha e_2 + \beta e_1 \\
De_3 &= e^{\alpha x}\cos\beta x + \alpha x e^{\alpha x}\cos\beta x - \beta x e^{\alpha x}\sin\beta x = e_1 + \alpha e_3 - \beta e_4 \\
De_4 &= e^{\alpha x}\sin\beta x + \alpha x e^{\alpha x}\sin\beta x + \beta x e^{\alpha x}\cos\beta x = e_2 + \alpha e_4 + \beta e_3 \\
De_5 &= x e^{\alpha x}\cos\beta x + \frac{1}{2}\alpha x^2 e^{\alpha}\cos\beta x - \frac{1}{2}\beta x^2 e^{\alpha x}\sin\beta x = e_3 + \alpha e_5 - \beta e_6 \\
De_6 &= x e^{\alpha x}\sin\beta x + \frac{1}{2}\alpha x^2 e^{\alpha}\sin\beta x + \frac{1}{2}\beta x^2 e^{\alpha x}\cos\beta x = e_4 + \alpha e_6 + \beta e_5
\end{aligned}$$

Therefore, the matrix representing D is given by

$$D = \begin{pmatrix} \alpha & \beta & 1 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta & 1 & 0 \\ 0 & 0 & -\beta & \alpha & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & -\beta & \alpha \end{pmatrix}$$

Exercise 6.

From the distribution property of matrix multiplication,

$$(\mathrm{id} - N)(\mathrm{id} + N + \cdots + N^{r-1}) = (\mathrm{id} + N + \cdots + N^{r-1})(\mathrm{id} - N) = \mathrm{id} - N^r = \mathrm{id}$$

Therefore, $\mathrm{id} - N$ is invertible. Its inverse is given by

$$(\mathrm{id} - N)^{-1} = \mathrm{id} + N + \cdots + N^{r-1}$$

This property remains the same with linear maps on a vector space.

Exercise 7.

For any integers $p \geq n+1$ we have $D^p = \mathbf{0}$, and if a is a constant and q is a positive integer, then aD^q is nilpotent since

$$(aD^q)^{n+1} = a^{n+1}(D^{n+1})^q = \mathbf{0}$$

Therefore, the results follow from the previous exercise that

(a). The map $\mathrm{id} - D^2$ is invertible.

(b). The map $\mathrm{id} - D^m$ is invertible for all positive integers m.

(c). The map $\frac{1}{c}D^m$ is nilpotent, so $\frac{1}{c}D^m - \mathrm{id}$ is invertible and therefore the map $D^m - c \cdot \mathrm{id}$ is invertible for any number $c \neq 0$.