Honors Mathematics III
RC 9

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Cuboids

Definition. ***n-cuiboid***: Let $a_k, b_k, k = 1, \ldots, n$ be pairs of numbers with $a_k < b_k$. Then the set $\Omega \subset \mathbb{R}^n$ given by

$$\begin{aligned} Q &= [a_1, b_1] \times \cdot \times [a_n, b_n] \\ &= \{x \in \mathbb{R}^n : x_k \in [a_k, b_k], k = 1, \ldots, n\} \end{aligned}$$

with volume

$$|Q| := \prod_{k=1}^{n} (b_k - a_k)$$

Upper/Lower Volumes of Sets

Definition. ***Outer/inner volume***:

$$\overline{V}(\Omega) := \inf\left\{\sum_{k=0}^{r}|Q_k| : r\in\mathbb{N}, Q_0,\dots,Q_r\in\mathcal{Q}_n, \Omega\subset\bigcup_{k=0}^{r}Q_k\right\},$$

$$\underline{V}(\Omega) := \sup\left\{\sum_{k=0}^{r}|Q_k| : r\in\mathbb{N}, Q_0,\dots,Q_r\in\mathcal{Q}_n, \Omega\supset\bigcup_{k=0}^{r}Q_k, \bigcap_{k=0}^{r}Q_k=\emptyset\right\}.$$

Measurable Sets

Definition. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is said to be ***(Jordan) measurable*** if either

1. $\overline{V}(\Omega) = 0$ (***measure zero***) or
2. $\overline{V}(\Omega) = \underline{V}(\Omega)$.

Set of measure zero.

1. A bounded set in lower dimensions in high-dimensional spaces.
2. The set of rational numbers in the interval $[0, 1]$.

Step Functions on Cuboids

Definition. A ***partition*** P of an n-cuboid $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a tuple $P = (P_1, \ldots, P_n)$ such that $P_k = (a_{k0}, \ldots, a_{km_k})$ is a partition of the interval $[a_k, b_k]$. The partition P of Q induces cuboids of the form

$$Q_{j_1 j_2 \ldots j_n} := [a_{1(j_1-1)}, a_{1(j_1)}] \times \cdots \times [a_{n(j_n-1)}, a_{n(j_n)}]$$

Definition. A ***step function with respect to a partition*** P:

$$f(x) = y_{j_1 j_2 \ldots j_n}, \quad x \in \operatorname{int} Q_{j_1 j_2 \ldots j_n}, \quad j_k = 1, \ldots, m_k$$

3.3.11.Theorem. Let $Q \subset \mathbb{R}^n$ be a cuboid and $f : Q \to \mathbb{R}$ a step function with respect to some partition P of Q, then the ***integral*** is

$$\int_Q f := I_P(f) = \sum_{\substack{j_1 = 1, \ldots, m_1 \\ \vdots \\ j_n = 1, \ldots, m_n}} |Q_{j_1 \ldots j_n}| \cdot y_{j_1 \ldots j_n}$$

Integration over Cuboids

Definition. Let $Q \subset \mathbb{R}^n$ be an n-cuboid and f a bounded real function on Q. Let $\mathcal{U}_f$ denote the set of all step functions u on Q such that $u \geq f$ and $\mathcal{L}_f$ the set of all step functions v on Q such that $v \leq f$. The function f is ***Darboux-integrable*** if

$$\sup_{v \in \mathcal{L}_f} \int_Q v = \inf_{u \in \mathcal{U}_f} \int_Q u.$$

3.3.13.Theorem. A bounded function $f : Q \to \mathbb{R}$ is ***Rieman-integrable*** if an only if for every $\varepsilon > 0$ there exist step functions u_ε and v_ε such that $u_\varepsilon \leq f \leq u_\varepsilon$ and

$$\int_Q u_\varepsilon - \int_Q v_\varepsilon \leq \varepsilon.$$

3.3.14.Proposition. f bounded and continuous almost everywhere $\Rightarrow$ f is (Riemann) integrable.

Integration over Jordan-Measurable Sets

3.3.16.Lemma. Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is Jordan-measurable if and only if its boundary $\partial\Omega$ has Jordan measure zero.

3.3.17.Corollary. Let $\Omega \subset \mathbb{R}^n$ be a bounded Jordan-measurable set and let $f : \Omega \to \mathbb{R}$ be continuous a.e. Then f is integrable on Ω.

Practical Integration over Cuboids

3.4.1.Fubini's Theorem. Let Q_1 be an n_1-cuboid and Q_2 an n_2-cuboid so that $Q := Q_1 \times Q_2 \subset \mathbb{R}^{n_1+n_2}$ is an $(n_1 + n_2)$-cuboid. Assume that $f : Q \to \mathbb{R}$ is integrable on Q and that for every $x \in Q_1$ the integral

$$g(x) = \int_Q f(x, \cdot)$$

exists. Then

$$\int_Q f = \int_{Q_1 \times Q_2} f = \int_{Q_1} g = \int_{Q_1} \left(\int_{Q_2} f \right).$$

Ordinate and Simple Regions in $\mathbb{R}^2$

Definition. A set $D \subset \mathbb{R}^2$ is called an ***ordinate region with respect to*** x_2, if there exists an interval $I \subset \mathbb{R}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi_2 : I \to \mathbb{R}$ such that

$$D = \{(x_1, x_2) \subset \mathbb{R}^2 : x_1 \in I, \varphi_1(x_1) \leq x_2 \leq \varphi_2(x_1)\}.$$

If D is an ordinate region both with respect to x_1 and x_2, then D is a ***simple region***.

Ordinate Regions in $\mathbb{R}^n$

Definition. A subset $U \subset \mathbb{R}^n$ is said to be an ***ordinate region (with respect to x_k)*** if there exists a measurable set $\Omega \subset \mathbb{R}^{n-1}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi 2 : \Omega \to \mathbb{R}$, such that

$$U = \{x \in \mathbb{R}^n : x \in \Omega, \varphi_1(\hat{x}^{(k)}) \leq x_k \leq \varphi_2(\hat{x}^{(k)})\}$$

If U is an ordinate region with respect to each $x_k, k = 1, \dots, n$, it is said to be a ***simple region***.

The Substitution Rule

3.4.12.Substitution Rule. Let $\Omega \subset \mathbb{R}^n$ be open and $g : \Omega \to \mathbb{R}^n$ injective and continuously differentiable. Suppose that $\det J_g(y) \neq 0$ for all $y \in \Omega$. Let K be a compact measurable subset of Ω. Then $g(K)$ is compact and measurable and if $f : g(K) \to \mathbb{R}$ is integrable, then

$$\int_{g(K)} f(x)\mathrm{d}x = \int_K f(g(y)) \cdot |\det J_g(y)| \mathrm{d}y.$$

Coordinate Systems

- Polar coordinates:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad |\det J(r,\phi)| = r$$

- Cylindrical coordinates:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = \zeta, \quad |\det J(r,\phi,\zeta)| = r$$

- Spherical coordinates:

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta$$

$$|\det J(r,\phi,\theta)| = r^2\sin\theta.$$

Coordinate Systems

- Spherical coordinates in $\mathbb{R}^n$:

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\
x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

$$|\det J(r,\theta_1,\ldots,\theta_{n-1})| = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$$

Note. $r > 0, 0 < \theta_k < \pi, k = 1,\ldots,n-2,\ 0 < \theta_{n-1} < 2\pi.$

The Gauss Integral

To integrate

$$\lim_{a\to\infty} I(a) := \lim_{a\to\infty} \int_{-a}^{a} e^{-x^2/2}\mathrm{d}x$$

we have

$$\lim_{a\to\infty} I(a)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\mathrm{d}y\right)$$

Green's Theorem

3.4.18. Green's Theorem. Let $R \subset \mathbb{R}^2$ be a bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\int_{\partial R^*} F \mathrm{d}\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathrm{d}x$$

Exercises

Exercise 1. Evaluate the integral

$$I = \int_{\mathbb{R}^2} e^{-(x^2+(y-x)^2+y^2)} \mathrm{d}x \mathrm{d}y$$

Exercises

Solution. We introduce the substitution

$$\begin{pmatrix} u \\ v \end{pmatrix} = \Phi(x,y) = \begin{pmatrix} x+y \\ x-y \end{pmatrix}, \quad \Phi^{-1}(u,v) = \frac{1}{2}\begin{pmatrix} u+v \\ u-v \end{pmatrix}.$$

Then

$$x^2 + (y-x)^2 + y^2 = \frac{1}{2}(u^2 + 3v^2)$$

and

$$|\det J_{\Phi^{-1}}| = \frac{1}{4}\left|\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = \frac{1}{2}.$$

Exercises

Solution (continued). Then

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-(x^2+(y-x)^2+y^2)}\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_{\mathbb{R}^2} e^{\frac{1}{2}(u^2+3v^2)}\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-u^2/2}\mathrm{d}u \cdot \int_{-\infty}^{\infty} e^{-3v^2/2}\mathrm{d}v \\ &= \frac{\pi}{\sqrt{3}}\end{aligned}$$

Exercises

Exercise 2. Consider the potential U:

$$U(x,y) = -x - y.$$

Let

$$\Omega = \{(x,y) \in \mathbb{R}^2 : \pi \leq y \leq 2\pi, |x| \leq |\sin y|\}$$

1. Sketch Ω.
2. Calculate $\int_\Omega U(x,y)$.

Exercises

Exercise 3. Assume an object is distributed at the region $\Omega : x^2 + y^2 + 2z^2 \leq x + y + 2z$. Its density function is $\rho(x, y, z) = x^2 + y^2 + z^2$. Calculate its mass

$$M = \int_\Omega \rho = \iiint_\Omega (x^2 + y^2 + z^2) \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Exercises

Exercise 4. Consider the vector field G:

$$G(x, y) = (x + xy, -xy).$$

Let

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : \pi \leq y \leq 2\pi, |x| = |\sin y|\}.$$

Calculate $\displaystyle\int_\Gamma G$ (in positive orientation).

Exercises

Exercise 5. Calculate the volume $V(B)$ of the n-dimensional ball:

$$B = \{x = (x_1, \dots, x_n) : \|x\| \leq R\}$$

Thanks for your attention!