Honors Mathematics III
Review — Final

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Integration Over Cuboids

By Fubini's Theorem, we have

$$\int_Q f = \int_{a_n}^{b_n} \cdots \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_n) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

or equivalently,

$$\int_Q f = \int_Q f(x) \mathrm{d}x$$

Integration Over Ordinate Regions

Definitions.

- Ordinate region (with respect to x_k): there exists a measurable set $\Omega \subset \mathbb{R}^{n-1}$ and continuous, almost everywhere differentiable functions $\varphi_1, \varphi_2 : \Omega \to \mathbb{R}$ such that

$$U = \{x \in \mathbb{R}^n : x \in \Omega, \varphi_1(\hat{x}^{(k)}) \leq x_k \leq \varphi_2(\hat{x}^{(k)})\}.$$

- Simple region: U is an ordinate region with respect to each $x_k, k = 1, \ldots, n$.

Integration Over Ordinate Regions

For an ordinate region $U \subset \mathbb{R}^n$ with respect to x_k over a measurable set Ω, the indicator function $\mathbb{1}_U$ takes the form

$$\mathbb{1}_U(x) = \mathbb{1}_\Omega \cdot \mathbb{1}_{[\varphi_1(\hat{x}^{(k)}), \varphi_2(\hat{x}^{(k)})]}(x_k).$$

It then follows that

$$\int_U f(x) \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_\Omega \left(\int_{\varphi_1(\hat{x}^{(k)})}^{\varphi_2(\hat{x}^{(k)})} f(x) \mathrm{d}x_k \right) \mathrm{d}\hat{x}^{(k)}$$

if

$$\int_{\varphi_1(\hat{x}^{(k)})}^{\varphi_2(\hat{x}^{(k)})} f(x) \mathrm{d}x_k$$

exists for every $\hat{x}^{(k)} \in \Omega$.

Substitution Rule

3.4.12. Substitution Rule. Let $\Omega \subset \mathbb{R}^n$ be open and $g : \Omega \to \mathbb{R}^n$ injective and continuously differentiable. Suppose that $\det J_g(y) \neq 0$ for all $y \in \Omega$. Let K be a compact measurable subset of Ω. Then $g(K)$ is compact and measurable and if $f : g(K) \to \mathbb{R}$ is integrable, then

$$\int_{g(K)} f(x)\mathrm{d}x = \int_K f(g(y)) \cdot |\det J_g(y)| \mathrm{d}y.$$

Coordinate Systems

- Polar coordinates:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad |\det J(r,\phi)| = r$$

- Cylindrical coordinates:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = \zeta, \quad |\det J(r,\phi,\zeta)| = r$$

- Spherical coordinates:

$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta$$

$$|\det J(r,\phi,\theta)| = r^2\sin\theta.$$

Coordinate Systems

- Spherical coordinates in $\mathbb{R}^n$:

$$
\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\
x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}
$$

$$|\det J(r,\theta_1,\ldots,\theta_{n-1})| = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}$$

Note. $r>0, 0<\theta_k<\pi, k=1,\ldots,n-2,\ 0<\theta_{n-1}<2\pi.$

The Gauss Integral

The Gauss Integral.

$$\lim_{a\to\infty} I(a) := \int_{-\infty}^{\infty} e^{-x^2/2}\mathrm{d}x = \sqrt{2\pi}.$$

Variants. For $k > 0$,

$$\int_{-\infty}^{\infty} e^{-kx^2}\mathrm{d}x = \sqrt{\frac{\pi}{k}}.$$

Green's Theorem

Green's Theorem. Let $R \subset \mathbb{R}^2$ be bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F : \Omega \to \mathbb{R}^2$ be continuously differentiable vector field. Then

$$\int_{\partial R^*} F \mathrm{d}\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathrm{d}x$$

Physical Interpretation of Green's Theorem

- For $F = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$,

$$\begin{aligned} \text{circulation along } \partial R &= \int_{\partial R^*} F \mathrm{d}\vec{s} \\ &= \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathrm{d}x \\ &= \int_R \mathrm{rot} F \mathrm{d}x \\ &= \text{integral of circulation density over } R. \end{aligned}$$

Physical Interpretation of Green's Theorem

- For $\tilde{F} = \begin{pmatrix} -F_2(x) \\ F_1(x) \end{pmatrix}$,

$$\begin{aligned} \text{flux through } \partial R &= \int_{\partial R^*} \langle F, N \rangle \mathrm{d}s = \int_{\partial R^*} \tilde{F} \mathrm{d}\vec{s} \\ &= \int_R \left(\frac{\partial \tilde{F}_2}{\partial x_1} - \frac{\partial \tilde{F}_1}{\partial x_2} \right) \mathrm{d}x \\ &= \int_R \mathrm{div} F \mathrm{d}x \\ &= \text{integral of flux density over } R. \end{aligned}$$

Tangent Spaces of Surfaces

Definition. Let $\mathcal{S} \subset \mathbb{R}^n$ be a parametrized m-surface with parametrization $\varphi : \Omega \to \mathcal{S}$. Then

$$t_k(p) = \frac{\partial}{\partial x_k} \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_2(x) \end{pmatrix} \Bigg|_{x=\varphi^{-1}(p)}, \qquad k = 1, \dots, m$$

is called the ***k-th tangent vector of $\mathcal{S}$ at $p \in \mathcal{S}$*** and

$$T_p\mathcal{S} := \operatorname{ran} D\varphi|_x = \operatorname{span}\{t_1(p), \dots, t_m(p)\}$$

is called the ***tangent space*** to $\mathcal{S}$ at p. The vector field

$$t_k : \mathcal{S} \to \mathbb{R}^n, \qquad p \mapsto t_k(p)$$

is called the ***k-th tangent vector field*** on $\mathcal{S}$.

The Normal Vector to Hypersurfaces

Definition. Let $\mathcal{S} \subset \mathbb{R}^n$ be a hypersurface. Then a unit vector that is orthogonal to all tangent vectors to $\mathcal{S}$ at p is called a ***unit normal vector to $\mathcal{S}$ at p*** and denoted by $N(p)$. The vector field

$$N : \mathcal{S} \to \mathbb{R}^n, \qquad p \mapsto N(p)$$

is called the ***normal vector field*** on $\mathcal{S}$.

Orientation of Hypersurfaces

Definitions.

- A hypersurface $\mathcal{S} \subset \mathbb{R}^n$ such that it admits a continuous normal vector field is said to be ***orientable***.
- A choice of direction for the normal vector field is called an ***orientation of*** $\mathcal{S}$.
- A hypersurface that is the boundary of a measurable set $\Omega \subset \mathbb{R}^n$ with non-zero measure is said to be a ***closed surface***.
- A closed hypersurface is said to have ***positive orientation*** if the normal vector field is chosen so that the normal vectors point ***outwards*** from Ω.

The Metric Tensor

Definition. Let $\mathcal{S} \subset \mathbb{R}^n$ be an m-surface with parametrization φ and tangent vector fields $t_1, \ldots, t_m$. Then $G \in \mathrm{Mat}(m \times m; \mathbb{R})$ given by

$$G := \begin{pmatrix} \langle t_1, t_1 \rangle & \cdots & \langle t_1, t_m \rangle \\ \vdots & \ddots & \vdots \\ \langle t_m, t_1 \rangle & \cdots & \langle t_m, t_m \rangle \end{pmatrix}$$

is said to be the ***metric tensor*** on $\mathcal{S}$ with respect to φ. The coefficients

$$g_{ij} := \langle t_i, t_j \rangle, \qquad i, j = 1, \ldots, m,$$

are called the ***metric coefficients*** of G.

Scalar Surface Integrals

Definition. Let $f : \mathcal{S} \to \mathbb{R}$ be a potential function. $\mathcal{S}$ is a parametrized m-surface with parametrization $\varphi : \Omega \to \mathcal{S}, \Omega \subset \mathbb{R}^m$. Then the ***(scalar) surface integral of f over $\mathcal{S}$*** is defined as

$$\int_{\mathcal{S}} f \,\mathrm{d}A := \int_{\Omega} f \circ \varphi \sqrt{g(x)} \,\mathrm{d}x$$

Flux Through Hypersurfaces

Definition. Ley $F : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a vector field defined in a neighborhood of a hypersurface $\mathcal{S}$ with parametrization $\varphi : \Omega \to \mathbb{R}^{n+1}$, $\Omega \subset \mathbb{R}^n$. Then we define the ***flux of*** F ***through*** $\mathcal{S}$ by

$$\begin{aligned}\int_{\mathcal{S}} F \mathrm{d}\vec{A} &:= \int_{\mathcal{S}} \langle F, N \rangle \mathrm{d}A \\ &= \int_{\Omega} \langle F \circ \varphi(x), N \circ \varphi(x) \rangle \sqrt{g(x)} \mathrm{d}x_1 \ldots \mathrm{d}x_n\end{aligned}$$

Admissible Regions

Definitions.

- A subset $R \subset \mathbb{R}^n$ is called a ***region*** if it is open and (pathwise) connected.
- A region $R \subset \mathbb{R}^n$ is said to be ***admissible*** if it is bounded and its boundary is the union of a finite number of parametrized hypersurfaces whose normal vectors point outwards from R.
- A hypersurface $\mathcal{S} \subset \mathbb{R}^3$ with parametrization $\varphi : R \to \mathcal{S}$ is said to be ***admissible*** if
  1. the interior $\operatorname{int} R$ is an admissible region in $\mathbb{R}^2$ with an oriented boundary curve ∂R^* and
  2. R is closed, i.e., $R = \overline{R}$.

Closed Hypersurfaces in $\mathbb{R}^3$

Definition. Let $\mathcal{S} \subset \mathbb{R}^3$ be an admissible hypersurface with parametrization $\varphi : R \to \mathcal{S}$. Let $\partial R^* = \mathcal{C}_1^* \cup \mathcal{C}_2^* \cup \cdots \cup \partial_k^*$, where each $\mathcal{C}_i^*$ is an oriented smooth curve in $\mathbb{R}^2$ and all $\mathcal{C}_i^*$ are pairwise disjoint.

- We say that φ ***annihilates*** a chain of curves $\mathcal{C}_{i_1} \cup \cdots \cup \mathcal{C}_{i_j}$ if

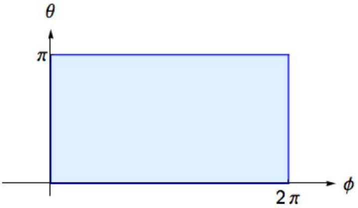
$$\int_{\varphi(\mathcal{C}_{i_1} \cup \cdots \cup \mathcal{C}_{i_j})} 1 \mathrm{d}s = 0.$$

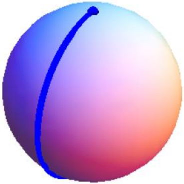
- If φ annihilates ∂R, $\mathcal{S}$ is said to be a ***closed surface***.
- Denote by $\mathcal{C}' \subset \partial R$ the largest chain of curves that is annihilated by φ. If $\mathcal{C}' \neq \partial R$ we say that $\mathcal{S}$ is a ***surface with boundary*** and define

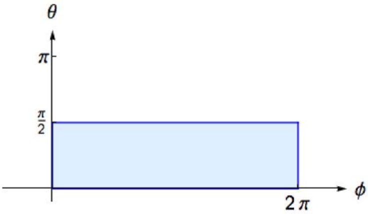
$$\partial \mathcal{C} := \varphi(\partial R \setminus \mathcal{C}').$$

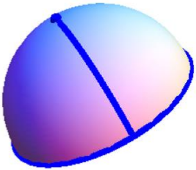
Closed Hypersurfaces in $\mathbb{R}^3$

Examples.









Stokes's Theorem in $\mathbb{R}^3$

3.6.7. Stokes's Theorem. Let $\Omega \subset \mathbb{R}^3$ be an open set, $\mathcal{S} \subset \Omega$ a parametrized, admissible surface in $\mathbb{R}^3$ with boundary $\partial\mathcal{S}$ and let $F : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Then

$$\int_{\partial\mathcal{S}^*} F \mathrm{d}\vec{s} = \int_{\mathcal{S}^*} \mathrm{rot}\, F \mathrm{d}\vec{A}$$

with positive orientation and normal vectors pointing in the direction of the thumb of the right hand if the four fingers point in the direction of the tangent vector to $\partial\mathcal{S}^*$.

Gauss's Theorem

3.6.9. Gauss's Theorem. Let $R \subset \mathbb{R}^n$ be an admissible region and $F : \overline{R} \to \mathbb{R}^n$ a continuously differentiable vector field. Then

$$\int_R \operatorname{div} F \mathrm{d}x = \int_{\partial \mathcal{R}^*} F \mathrm{d}\vec{A}$$

Green's Identities

3.6.13. Green's Identities. Let $R \subset \mathbb{R}^n$ be an admissible region and $u, v : \overline{R} \to \mathbb{R}$ be twice continuously differentiable potential functions. Then we have:

- ***Green's first identity:***

$$\int_R \langle \nabla u, \nabla v \rangle \mathrm{d}x = -\int_R u \cdot \Delta v \mathrm{d}x + \int_{\partial R^*} u \frac{\partial v}{\partial n} \mathrm{d}A.$$

- ***Green's second identity:***

$$\int_R (u \cdot v - v \cdot u) \mathrm{d}x = \int_{\partial R^*} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \mathrm{d}A.$$

Various Kinds of Elements

- Scalar line element:

$$\mathrm{d}s = |\gamma'(t)|\mathrm{d}t.$$

Line integral of potential f along $\mathcal{C}^*$:

$$\int_{\mathcal{C}^*} f\mathrm{d}s = \int_I (f \circ \gamma)(t) \cdot |\gamma'(t)|\mathrm{d}t.$$

- Vectorial line element:

$$\mathrm{d}\vec{s} = \gamma'(t)\mathrm{d}t.$$

The line integral of vector field F along $\mathcal{C}^*$:

$$\int_{\mathcal{C}^*} F\mathrm{d}\vec{s} = \int_{\mathcal{C}^*} \langle F, T\rangle \mathrm{d}s.$$

Various Kinds of Elements

- Volume element: (take spherical coordinates as example.)

$$\mathrm{d}x = |\det J_\Phi(r, \theta, \varphi)| \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi.$$

Integration of potentials in a $\mathbb{R}^3$ region:

$$\int_\Omega f = \int_{\Phi^{-1}(\Omega)} f \circ \Phi(r, \theta, \phi) \cdot |\det J_\Phi(r, \theta, \varphi)| \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi.$$

- Scalar surface element of a hypersurface in $\mathbb{R}^n$:

$$\mathrm{d}A = |\det(t_1, t_2, \ldots, t_{n-1}, N) \circ \varphi| \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1}.$$

Volume or area of $\mathcal{S}$:

$$|\mathcal{S}| = \int_\Omega |\det(t_1, \ldots, t_{n-1}, N) \circ \varphi(x)| \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1}.$$

Various Kinds of Elements

- Infinitesimal surface element of arbitrary surfaces in $\mathbb{R}^n$:

$$\mathrm{d}A = \sqrt{g(x)}\mathrm{d}x,$$

where

$$G = \begin{pmatrix} \langle t_1, t_1 \rangle & \cdots & \langle t_1, t_m \rangle \\ \vdots & \ddots & \vdots \\ \langle t_m, t_1 \rangle & \cdots & \langle t_m, t_m \rangle \end{pmatrix}, \qquad g(x) = \det G(\varphi(x))$$

The scalar (surface) integral of f over $\mathcal{S}$:

$$\int_{\mathcal{S}} f \,\mathrm{d}A = \int_{\Omega} f \circ \varphi(x) \sqrt{g(x)}\mathrm{d}x.$$

Various Kinds of Elements

- Vectorial surface element:

$$\mathrm{d}\vec{A} = N(\varphi(x)) \cdot \sqrt{g(x)}\mathrm{d}x.$$

The flux of F through $\mathcal{S}$ integral:

$$\int_{\mathcal{S}} F\mathrm{d}\vec{A} = \int_{\Omega} \langle F \circ \varphi(x), N \circ \varphi(x) \rangle \sqrt{g(x)}\mathrm{d}x_1 \ldots \mathrm{d}x_n.$$

Integration in Practice

- Ordinate region:

$$\int_U f(x)\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_\Omega \left(\int_{\varphi_1(\hat{x}^{(k)})}^{\varphi_2(\hat{x}^{(k)})} f(x)\mathrm{d}x_k \right) \mathrm{d}\hat{x}^{(k)}$$

- Substitution rule:

$$\int_{g(K)} f(x)\mathrm{d}x = \int_K f(g(y)) \cdot |\det J_g(y)| \mathrm{d}y.$$

- Green's theorem: $(\mathbb{R}^2)$

$$\int_{\partial R^*} F \mathrm{d}\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \mathrm{d}x.$$

Integration in Practice

- Stokes's theorem: $(\mathbb{R}^3)$

$$\int_{\partial \mathcal{S}^*} F \mathrm{d}\vec{s} = \int_{\mathcal{S}^*} \mathrm{rot}\, F \mathrm{d}\vec{A}.$$

- Gauss's theorem: $(\mathbb{R}^3)$

$$\int_R \mathrm{div}\, F \mathrm{d}x = \int_{\partial R^*} F \mathrm{d}\vec{A}.$$

- Green's identities:

$$\int_R \langle \nabla u, \nabla v \rangle \mathrm{d}x = -\int_R u \cdot \Delta v \mathrm{d}x + \int_{\partial R^*} u \frac{\partial v}{\partial n} \mathrm{d}A,$$
$$\int_R (u \cdot v - v \cdot u) \mathrm{d}x = \int_{\partial R^*} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \mathrm{d}A.$$

Thanks for your attention!

Good Luck!