Honors Mathematics III
RC 1

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Linear System

A ***linear system*** of m (algebraic) equations in n unknowns $x_1, \ldots, x_n \in V$ is a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $b_1, \ldots, b_m \in V$ and $a_{ij} \in \mathbb{F}, i = 1, \ldots, m, j = 1, \ldots, n$.

- $b_1 = b_2 = \cdots = b_m = 0$ (***homogeneous***). Otherwise (***inhomogeneous***).
- $m < n$ (***underdetermined***). $m > n$ (***overdetermined***).
- $x_1 = x_2 = \cdots x_n = 0$ (***trivial solution***).

Notations

Diagonal Form:

$$\left.\begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array}\right.$$

Echelon Form:

$$\left.\begin{array}{ccccc|c} 1 & * & * & * & * & \diamond \\ 0 & 1 & * & * & * & \diamond \\ 0 & 0 & 0 & 1 & * & \diamond \\ 0 & 0 & 0 & 0 & 1 & \diamond \end{array}\right.$$

Upper Triangular Form:

$$\left.\begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array}\right.$$

$$\left.\begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right.$$

Solution Set

The ***solution set*** S is the set of all n-tuples of numbers $x_1, \cdots, x_n$ that satisfy the system of equations. Presented as a ***set***:

$$S = \left\{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n : x \text{ satisfies some constraints.} \right\}$$

Existence of Solutions

1.1.8 Lemma.

The Fundamental Lemma for Homogeneous Equations:

The homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

of m equations in n real or complex unknowns $x_1, \ldots, x_n$ has a non-trivial solution if $n > m$.

Existence of Solutions

Proof.

Induction in m (with $n > m$):

1. $m = 1$, induction in n:
   - $n = 2$. $a_{11}x_1 + a_{12}x_2 = 0$ has a non-trivial solution.
   - $n > 2$. The linear equation

     $$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

     has a non-trivial solution implies that the equation

     $$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + a_{1(n+1)}x_{n+1} = 0$$

     has a non-trivial solution.

Existence of Solutions

Proof (continued).

2. $m > 1$, extend a system with $n-1$ unknowns and $m-1$ equations (with $n > m$) to a system with n unknowns and m equations.

$$\begin{array}{cccc|c l}
\color{red}{a_{11}} & a_{12} & \dots & a_{1n} & 0 & \cdot\left(\frac{a_{21}}{a_{11}}\right) \quad \cdot\left(\frac{a_{31}}{a_{11}}\right) \quad \cdot\left(\frac{a_{m1}}{a_{11}}\right) \\
a_{21} & a_{22} & \dots & a_{2n} & 0 & + \\
a_{31} & a_{32} & \dots & a_{3n} & 0 & \quad + \\
\vdots & \vdots & & \vdots & \vdots & \\
a_{m1} & a_{m2} & \dots & a_{mn} & 0 & \qquad +
\end{array}$$

Existence of Solutions

Proof (continued).

2. $m > 1$, extend a system with $n-1$ unknowns and $m-1$ equations (with $n > m$) to a system with n unknowns and m equations.

$$
\sim \quad \left(\begin{array}{c|ccc|c}
a_{11} & a_{12} & \dots & a_{1n} & 0 \\
\hline
0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} & \dots & a_{22} - \frac{a_{21}a_{1n}}{a_{11}} & 0 \\
0 & a_{32} - \frac{a_{31}a_{12}}{a_{11}} & \dots & a_{32} - \frac{a_{31}a_{3n}}{a_{11}} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
0 & a_{m2} - \frac{a_{m1}a_{12}}{a_{11}} & \dots & a_{m2} - \frac{a_{m1}a_{mn}}{a_{11}} & 0 \\
\hline
\end{array}\right)
$$

Uniqueness of Solutions

- A system of m equations with n unknowns will have a unique solution iff it is ***diagonizable***.
- Or equivalently, iff the system can be transformed into an ***upper triangular form***. (Backward substitution always works.)

Solutions of Homogeneous Systems

A homogeneous system of equations has

- a unique solution:

$$2x_1 + x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \qquad x_1 - x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- no solution:

$$x_1 + x_2 = 1, \qquad x_1 + x_2 = 2$$

- infinite number of solutions:

$$x_1 + x_2 = 1, \qquad x_1 + x_2 = 1$$

Gauss-Jordan Algorithm

Example.

Find the solution set of the linear system of equations

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= -2 \\ x_1 - x_2 \qquad &= 3 \end{aligned}$$

for x_1, x_2, x_3 using Gauss-Jordan algorithm.

Gauss-Jordan Algorithm

Solution.

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & -2 \\ 1 & -1 & 0 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 3 \end{array}\right) \begin{array}{l} \cdot(-2) \quad \cdot(-1) \\ + \\ \quad\quad + \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & -3 & -1 & 5 \\ 0 & -3 & -1 & 5 \end{array}\right) \begin{array}{l} \\ \mid : (-3) \quad \cdot 3 \\ + \end{array}$$

Gauss-Jordan Algorithm

Solution (continued).

$$
\sim \quad \begin{array}{ccc|c} 1 & 2 & 1 & -2 \\ 0 & 1 & 1/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{l} + \\ \cdot(-2) \\ \end{array}
$$

$$
\sim \quad \begin{array}{ccc|c} 1 & 0 & 1/3 & 4/3 \\ 0 & 1 & 1/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{array}
$$

Gauss-Jordan Algorithm

Solution (concluded).

This gives

$$x_1 = -\frac{1}{3}\lambda + \frac{4}{3}, \qquad x_2 = -\frac{1}{3}\lambda - \frac{5}{3}, \qquad x_3 = \lambda \in \mathbb{R}$$

Therefore, the solution set is given by

$$\left\{ x \in \mathbb{R}^3 : x = \lambda \begin{pmatrix} -1/3 \\ -1/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4/3 \\ -5/3 \\ 0 \end{pmatrix} \right\}$$

Vector Spaces — Summary

1. Concepts
   - Linear independence.
   - Linear combination and span.
   - Basis (ordered and unordered).
     - Coordinates.
     - Standard (canonical) basis.
   - Dimension.
   - Maximal subsets.
   - Sums of vector spaces.
     - Direct sums.

Vector Spaces — Summary

2. Theorems and Lemmas
   - Linear independence.
     - Linear independence and spans (Lemma 1.2.6).
     - Dependence and linear combination (Lemma 1.2.17).
   - Basis.
     - Characterization of bases (Theorem 1.2.10).
     - Length of a basis (Theorem 1.2.13).
     - Maximal subsets (Theorem 1.2.20).
     - ***Basis extension theorem*** (Theorem 1.2.21).
     - Independent set with proper length $\Rightarrow$ basis (Corollary 1.2.22).
     - Length of an independent set (Corollary 1.2.23).
   - Direct sums.
     - Unique representation (Lemma 1.2.27).
     - Dimension (Theorem 1.2.28).

Definitions

- ***Linear independence:***

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

- ***Independent set:*** a set with independent elements.
- ***Span (linear hull):*** — is a vector space

$$\mathrm{span}\{v_1, \ldots, v_n\} = \left\{ y \in V : y = \sum_{k=1}^{n} \lambda_k v_k, \lambda_1, \ldots, \lambda_n \right\}$$

- ***Basis:*** unique representation $v = \sum_{i=1}^{n} \lambda_i b_i \in V$.
- ***Dimension:***
  - $V = \{0\} : \dim\ V = 0.$
  - $V \neq \{0\} : \dim\ V = n$ (basis length).

Definitions

- ***Maximal subset F of A:***
  - $F \subset A$ independent.
  - $\forall x \in A$, x is a linear combination of elements in F. (A finite: $\operatorname{span} F = \operatorname{span} A$.)
- ***Sum of U and W:***
  - ***Sum:***

$$U + W = \left\{ v \in V : \underset{u \in U}{\exists} \underset{w \in W}{\exists} v = u + w \right\}$$

  - ***Direct sum $U \oplus W$:*** $U \cap W = \{0\}$.

Linear Independence

Lemma 1.2.6.

Vectors $v_1, \ldots, v_n \in V$ are independent iff none of them is contained in the span of all the others.

Proof.

$$
\begin{aligned}
& v_k \in \operatorname{span}\{v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\} \\
\Leftrightarrow & \lambda_1 v_1 + \cdots + \lambda_{k-1} v_{k-1} - \lambda_k v_k + \lambda_{k+1} v_{k+1} + \cdots + \lambda_n v_n = 0
\end{aligned}
$$

Linear Independence

Lemma 1.2.17.

If $a_1, \ldots, a_n$ are independent and $a_1, \ldots, a_{n+1}$ are dependent then a_{n+1} is a linear combination of $a_1, \ldots, a_n$.

Proof.

$$\sum_{i=1}^{n} \lambda_i a_i + \lambda_{n+1} a_{n+1} = 0$$

- $\lambda_{n+1} \neq 0$.
- $\lambda_1, \ldots, \lambda_n$ not all 0.

Basis

Theorem 1.2.10.

$\mathcal{B} = (b_1, \ldots, b_n) \in V^n$ is a basis iff

- $b_1, \ldots, b_n$ are linearly independent.
- $V = \operatorname{span} \mathcal{B}$.

Theorem 1.2.13.

Bases have the same length.

Proof. $(m > n)$.

$$b_j = \sum_{i=1}^{n} c_{ij} a_i \quad \Rightarrow \quad \sum_{j=1}^{m} \lambda_j b_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} c_{ij} \lambda_j \right) a_i$$

Basis

Theorem 1.2.20.

Independent sets $A' \subset A$ are contained in some maximal subset $F \subset A$. ***(Finite.)***

Proof. Repeatedly adding elements from A that is not in the span of F (initially set to A').

Basis Extension Theorem

There exists a basis of a finite-dimensional vector space containing an independent set A' in it.

Proof. Choose a basis $\mathcal{A}$ and set $A = \mathcal{A} \cup A'$. Find a maximal subset F of A containing A'.

Direct Sum

Lemma 1.2.27

$U+W$ is direct iff the representation $x=u+w, x\neq 0, u\in U, w\in W$ is unique.

Proof.

- ($\Rightarrow$) contraposition: $x=u+w=u'+w'$ implies $u-u'=w-w'\neq 0$ $(u\neq u', w\neq w')$, then $U\cap W\neq\{0\}$.
- ($\Leftarrow$) contraposition: $x=x+0=\dfrac{1}{2}x+\dfrac{1}{2}x$.

Direct Sum

Theorem 1.2.28

Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U + W) + \dim(U \cap W) = \dim\ U + \dim\ W$$

Proof. Suppose $\{t_1, \ldots, t_r\}$ is a basis of $U \cap W$, then by basis extension theorem, we can extend this basis to find a basis of U as $\{t_1, \ldots, t_r, a_1, \ldots, a_n\}$ and W as $\{t_1, \ldots, t_r, b_1, \ldots, b_m\}$. Therefore, we have

$$\dim(U \cap W) = r, \quad \dim\ U = r + n, \quad \dim\ W = r + m$$

Direct Sum

Theorem 1.2.28

Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U+W)+\dim(U\cap W)=\dim\ U+\dim\ W$$

Proof (continued). Then we can show that

$$\mathcal{B} := \{t_1,\dots,t_r,a_1,\dots,a_n,b_1,\dots,b_m\}$$

is a basis of $U+W$. To do this, we show the following:

- $\operatorname{span}\mathcal{B}=U+W$. For $\forall v\in U+W, v=u+w, u\in U, w\in W$.
- $\mathcal{B}$ is independent. Namely,

$$\sum_{i=1}^{r}\lambda_i t_i+\sum_{j=1}^{n}\mu_j a_j+\sum_{k=1}^{m}\eta_k b_k=0\Rightarrow\lambda_i=\mu_j=\eta_k=0$$

Direct Sum

Theorem 1.2.28

Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U+W) + \dim(U \cap W) = \dim\ U + \dim\ W$$

Proof (continued). Transforming the equation,

$$\sum_{i=1}^{r} \lambda_i t_i + \sum_{j=1}^{n} \mu_j a_j = -\sum_{k=1}^{m} \eta_k b_k \in U \cap W$$

with the observation that the lhs of the equation is a linear combination of a basis in U and similarly for the rhs. Hence,

$$\exists \lambda_i' \in \mathbb{R} \text{ s.t.} \quad -\sum_{k=1}^{m} \eta_k b_k = \sum_{i=1}^{r} \lambda_i' t_i \Rightarrow \eta_k = \lambda_i' = 0$$

and similarly for η_j.

Direct Sum

Theorem 1.2.28

Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U+W)+\dim(U\cap W)=\dim\ U+\dim\ W$$

Proof (concluded). Then the remaining term becomes

$$\sum_{i=1}^{r}\lambda_i t_i = 0$$

which immediately implies that $\lambda_i = 0$ because $\{t_1,\dots,t_r\}$ is a basis of $U\cap W$. Therefore, it can be shown that $\lambda_i = \mu_j = \eta_k = 0$ for $1\le i\le r, 1\le j\le n, 1\le k\le m$, namely, $\{t_1,\dots,t_r,a_1,\dots,a_n,b_1,\dots,b_m\}$ is independent. Thus we have $\dim(U+W) = r+m+n = \dim\ U+\dim\ W$.

Comments

- An n-dimensional vector space cannot have independent sets with more than n elements.
- An ***independent*** set with n elements in an n-dimensional vector space is a basis.
- A set of n vectors ***spanning*** the whole n-dimensional vector space is a basis.

Comments

Example.

Suppose $\{a_1, \ldots, a_n\}$ is a basis of V. $\{b_1, \ldots, b_n\}$ is a set of vectors in V. If each a_i can be expressed as a linear combination of $b_1, \ldots, b_n$, then $\{b_1, \ldots, b_n\}$ is also a basis of V.

Comments

Example.

Suppose $\{a_1, \ldots, a_n\}$ is a basis of V. $\{b_1, \ldots, b_n\}$ is a set of vectors in V. If each a_i can be expressed as a linear combination of $b_1, \ldots, b_n$, then $\{b_1, \ldots, b_n\}$ is also a basis of V.

Proof. (Independence) Suppose for contradiction that $\{b_1, \ldots, b_n\}$ are not independent, then there exists k such that

$$b_k = \lambda_1 b_1 + \cdots + \lambda_{k-1} b_{k-1} + \lambda_{k+1} b_{k+1} + \cdots + \lambda_n b_n$$

Then the dimension of V $\dim\ V \leq n-1$, resulting in contradiction $(\dim\ V = n)$. Therefore, $\{b_1, \ldots, b_n\}$ are independent.

Inner Product Spaces — Summary

1. Concepts
   - Inner product spaces.
   - The Induced norm and angle.
   - Orthogonality and orthogonal complement.
   - Orthonormal systems.
     - Orthonormal basis.
     - Projection

Inner Product Spaces — Summary

2. Theorems and Lemmas
   - Cauchy-Schwarz inequality (1.3.6).
   - Pythagoras's theorem (1.3.13).
   - Orthonormal systems are independent (Lemma 1.3.16).
   - Basis representation (Theorem 1.3.18).
   - Parseval's theorem (1.3.20).
   - Projection theorem (1.3.21).
   - Dimensions of orthogonal complement (Corollary 1.3.24).
   - Bessel inequality (1.3.25).
3. Practical Methods
   - Gram-Schmidt Orthonormalization.

Inner Product Spaces

1.3.1. Definition.

Let V be a real or complex vector space. Then a map $\langle\cdot,\cdot\rangle : V \times V \mapsto \mathbb{F}$ is called a ***scalar product*** or ***inner product*** if for all $u, v, w \in V$ and all $\lambda \in \mathbb{F}$,

- $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$,
- $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$,
- $\langle u, \lambda v\rangle = \lambda\langle u, v\rangle$,
- $\langle u, \lambda v\rangle = \overline{\langle v, \lambda u\rangle}$.

The pair $(V, \langle\cdot,\cdot\rangle)$ is called an inner product space.

The Induced Norm and Angle

- ***Induced norm:*** $\|\cdot\| : V \mapsto \mathbb{R}, \quad \|v\| = \sqrt{\langle v, v \rangle}.$
  - In $\mathbb{R}^n$ and $\mathbb{C}^n$:

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \|x\|_2$$

  - On $C([a, b])$:

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b |f(x)|^2 \mathrm{d}x} = \|f\|_2$$

- ***Angle $\alpha(u, v) \in [0, \pi]$:*** $\cos \alpha(u, v) = \dfrac{\langle u, v \rangle}{\|u\| \|v\|}$

Orthogonality

- ***Orthogonal (perpendicular):*** $u \perp v \Leftrightarrow \langle u, v\rangle = 0$.
- ***Orthogonal complement of $M \subset V$:***

$$M^{\perp} := \left\{ v \in V : \underset{m\in M}{\forall} \langle m, v\rangle = 0 \right\}$$

- ***Orthonormal system:*** a tuple of vectors $(v_1, \dots, v_r)$ such that

$$\langle v_j, v_k\rangle = \delta_{jk} := \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

for $j, k = 1, \dots, r$.
- ***Orthonormal projection:*** $\pi_U v := \sum_{i=1}^{r} \langle e_i, v\rangle e_i$.

The Induced Norm

Cauchy-Schwarz Inequality 1.3.6.

$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|, \quad \forall u, v \in V.$

Proof.

$$|\langle u, v\rangle|^2 = \|v\|^2 \cdot \left|\left\langle u, \frac{v}{\|v\|}\right\rangle\right|^2 \leq \|u\|^2 \cdot \|v\|^2$$

Lemma 1.3.12.

The orthogonal complement $M^\perp$ is a subspace of V.

Pythagoras's Theorem 1.3.13.

$z = x + y, x \in M, y \in M^\perp, M \subset V \Rightarrow \|z\|^2 = \|x\|^2 + \|y\|^2.$

Orthonormal System

Lemma 1.3.16.

The elements in an orthonormal system are linearly independent.

Proof.

$$\sum_{i=0}^{r} \lambda_i v_i = 0 \Rightarrow 0 = \langle v_i, 0 \rangle$$

$$= \langle v_i, \sum_{i=0}^{r} \lambda_i v_i \rangle$$

$$= \lambda_i$$

Orthonormal System

Theorem 1.3.18.
Basis representation $v = \sum_{j=1}^{n} \langle e_j, v \rangle e_j$.

Parseval's Theorem 1.3.21.
$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$.

Orthonormal System

Projection Theorem 1.3.22.

Let $(V, \langle\cdot,\cdot\rangle)$ be a ***possibly infinite-dimensional*** inner product vector space and $(e_1, \ldots, e_r), r \in \mathbb{N}$ be an orthonormal system in V. $U := \operatorname{span}\{e_1, \ldots, e_r\}$. Then there exists a unique representation

$$v = u + w \qquad \text{where } u \in U \text{ and } w \in U^\perp$$

and $u = \sum_{i=1}^{r} \langle e_i, v\rangle e_i$, $w := v - u$.

Proof.

Uniqueness: $0 = \|u - u' + (w - w')\|^2 = \|u - u'\|^2 + \|w - w'\|^2$.
Existence: $u = \sum_{i=1}^{r} \langle e_i, v \rangle e_i$, $w = v - u$ and

$$\|u\|^2 = \sum_{i=1}^{r} \|\langle e_i, v \rangle\|^2 \quad \Rightarrow \quad \langle v - u, u \rangle = 0$$

verifying that $w \perp u$.

Corollary 1.3.24

$V = U \oplus U^\perp$ (V possibly infinite-dimensional). If V is finite-dimensional, then $\dim V = \dim U + \dim U^\perp$.

Orthonormal System

Bessel Inequality 1.3.25.

For a ***possibly infinite-dimensional*** inner product space V and an orthonormal system $(e_1, \ldots, e_n)$,

$$\sum_{k=1}^{r} |\langle e_k, v\rangle|^2 \leq \|v\|^2$$

for all $v \in V$.

Proof. It follows from:

- $\|v - u\|^2 = \|v\|^2 - \|u\|^2 \geq 0.$
- $\|u\|^2 = \sum_{i=1}^{r} |\langle e_i, v\rangle|^2.$

Gram-Schmidt Orthonormalization

To orthonormalize a system of vectors $(v_1, \dots, v_n)$ to obtain an orthonormal system $(w_1, \dots, w_m)$, we have

$$
\begin{aligned}
w_1 &:= \frac{v_1}{\|v_1\|} \\
w_2 &:= \frac{v_2 - \langle w_1, v_2 \rangle w_1}{\|v_2 - \langle w_1, v_2 \rangle w_1\|} \\
w_3 &:= \frac{v_3 - \langle w_2, v_3 \rangle w_2 - \langle w_1, v_3 \rangle w_1}{\|v_3 - \langle w_2, v_3 \rangle w_2 - \langle w_1, v_3 \rangle w_1\|} \\
&\dots \\
w_k &:= \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\|}
\end{aligned}
$$

(Repeatedly searching for the orthogonal complement of the vector space spanned by the previous orthonormal system that has been constructed.)

Gram-Schmidt Orthonormalization

Example 1.

Suppose $v_1 = (2,1,1)^T, v_2 = (1,2,-1)^T, v_3 = (1,1,0)^T$. Find an orthonormal basis for the vector space $\mathrm{span}\{v_1, v_2, v_3\}$.

Gram-Schmidt Orthonormalization

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Solution.

Define

$$w_1 = \frac{1}{\sqrt{2^2+1^2+1^2}} u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

then

$$u_2 - \langle w_1, u_2 \rangle w_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \frac{3}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/2 \\ -3/2 \end{pmatrix}$$

Gram-Schmidt Orthonormalization

Example 1.

Suppose $v_1 = (2,1,1)^T, v_2 = (1,2,-1)^T, v_3 = (1,1,0)^T$. Find an orthonormal basis for the vector space $\mathrm{span}\{v_1, v_2, v_3\}$.

Solution (continued).

Then we calculate

$$w_2 = \frac{\sqrt{2}}{3}\begin{pmatrix} 0 \\ 3/2 \\ -3/2 \end{pmatrix}$$

We then attempt to calculate w_3 and notice that

$$u_3 - \langle w_1, u_3\rangle w_1 - \langle w_2, u_3\rangle w_2 = 0$$

$\mathrm{span}\{v_1, v_2, v_3\}$ is actually 2-dimensional and its orthonormal basis can be given by (w_1, w_2).

P.S.

Some statements and proofs in the slides above are not mathematically rigorous at all. I am merely trying to give you a general idea about what is going on. Please refer to the course slides if you want to check the details!

Thanks for your attention!