Honors Mathematics III
RC 5

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Convergence and Continuity — Summary

1. Definitions
   - Opens sets,closed sets, compact sets.
   - Equivalence of norms.
   - Interior, exterior and boundary points.
   - Image and pre-image of Sets.
   - Continuity and uniform continuity.
2. Theorems and Lemmas
   - Theorem of Bolzano-Weierstrass in $\mathbb{R}^n$ (2.1.13).
   - All norms are equivalent in a finite-dimensional vector space (Theorem 2.1.11).
   - Open(closed) sets and interior, exterior, and boundary points (Lemma 2.1.21).
   - Continuity of functions (Theorem 2.1.26).
   - Pre-image of open sets (Theorem 2.1.28).
   - Compact sets vs. closed and bounded sets (Theorem 2.1.32 and 2.1.33).
   - Compact sets and continuity (Proposition 2.1.35).
   - Compact sets contains maximum for a continuous function (Theorem 2.1.36).

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Open Sets

- ***Open ball:***

$$B_\varepsilon(a) := \{x \in V : \|x - a\| < \varepsilon\} \qquad a \in V, \varepsilon > 0$$

- ***Open sets:*** A set $U \subset V$ is ***open***, where $(V, \|\cdot\|)$ is a normed vector space, if $\forall a \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.
    1. $B_\varepsilon(a), \varepsilon > 0, a \in V$.
    2. $\emptyset$.
    3. The entire space V.

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Equivalence of Norms

- ***Equivalent norms:***

$$\exists C_1, C_2 > 0 \quad \text{such that} \quad C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_2$$

- Theorem 2.1.11. In a ***finite-dimensional*** vector space, all norms are equivalent.

Equivalence of Norms

Theorem 2.1.11. In a ***finite-dimensional*** vector space, all norms are equivalent.

Proof. We want to show that for any norm in V, $\|\cdot\|$,

$$\exists C_1, C_2 \quad \textit{such that} \quad C_1\|v\|_0 \leq \|v\| \leq C_2\|v\|_0$$

where

$$v = \sum_{i=1}^{n} \lambda_i v_i, \qquad \|v\|_0 = \sum_{i=1}^{n} |\lambda_i|$$

This is realized by

1. $\exists C_1, \|v\| \geq C_1 \sum_{i=1}^{n} |\lambda_i|$.
   1.1 Theorem of Bolzano-Weierstrass in $\mathbb{R}^n$.
   1.2 Norm inequality.
2. $\exists C_2, \|v\| \leq C_2 \sum_{i=1}^{n} |\lambda_i|$: Triangle inequality.

Equivalence of Norms

2.1.13. Theorem of Bolzano-Weierstrass in $\mathbb{R}^n$. Let $(x^{(m)})_{m\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^n$, i.e., $x^{(m)} = (x_1^{(m)}, \dots, x_n^{(m)})$. Suppose that there exists a constant $C > 0$ such that $|x_k^{(m)}| < C$ for all $m \in \mathbb{N}$ and each $k = 1, \dots, n$. Then there exists a subsequence $(x^{(m_j)})_{j\in\mathbb{N}}$ that converges to a vector $y \in \mathbb{R}^n$ in the sense that

$$x_k^{(m_j)} \xrightarrow{j\to\infty} y_k$$

Proof. Finite steps of selecting subsequences.

- Convergent subsequence $(x_1^{m(j_1)})$ of $(x_1^{(m)})_{m\in\mathbb{N}}$, $x_1^{m(j_1)} \to y_1$.
- Convergent subsequence $(x_2^{m(j_2)})$ of $(x_2^{m(j_1)})$, $x_2^{m(j_2)} \to y_2$.
- Convergent subsequence $(x_3^{m(j_3)})$ of $(x_3^{m(j_2)})$, $x_3^{m(j_3)} \to y_3$.
- $\dots$

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Closed Sets

Definitions. For a normed vector space $(V, \|\cdot\|)$, $M \subset V$, a point x is a(an)

- ***Interior point:*** $\exists \varepsilon > 0, B_\varepsilon(x) \subset M$. ($int\, M$).
- ***Boundary point:*** $\forall \varepsilon > 0, B_\varepsilon(x) \cap M \neq \emptyset, B_\varepsilon(x) \cap (V \setminus M) \neq \emptyset$. ($\partial M$).
- ***Exterior point:*** x is neither a boundary nor an interior point.

Closed and open sets. A set is ***closed*** if its complement is open.

1. The empty set $\emptyset$.
2. The entire space V.
3. $V = \bigcup_{i=1}^{n} V_i$, where V_is are disjoint open sets. Then V_i is a closed and open set.

Closed Sets

Question. Sets that are neither closed nor open?

Closed Sets

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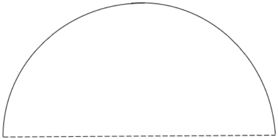

The set can be described as the set of all points (x, y) that are less than or equal to 1 unit away from the origin and have positive y coordinates.

1. This set is not open: points on the arc.
2. Its complement is not open: points on the base line.

Closed Sets

2.1.21. Lemma. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$, then

1. The set M is open iff $M = int\, M$.
2. The set M is closed iff $\partial M \subset M$.

Closure of M:

$$\overline{M} := M \cup \partial M = \left\{ x \in V : \underset{(x_n)_{n\in\mathbb{N}}}{\exists} x_n \in M \text{ and } x_n \to x \right\}$$

which is the smallest set that contains both M and is closed.

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Compact Sets

Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M, K \subset V$.

- M is ***bounded:*** $\exists R > 0, M \subset B_R(0)$.
- K is ***compact:*** Every sequence in K has a convergent subsequence with limit contained in K.

Results.

1. ***Finite-dimensional space:*** Compact $\Leftrightarrow$ closed and bounded.
2. ***Infinite-dimensional space:***
   2.1 Compact $\Rightarrow$ closed and bounded.
   2.2 Closed and bounded $\not\Rightarrow$ compact.

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Continuous Functions

Definitions.

- ***Continuous at*** $a \in U$***:***

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall} \|x-a\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(a)\|_2 < \varepsilon$$

  iff

$$\underset{\substack{(x_n)_{n\in\mathbb{N}} \\ x_n \in U}}{\forall} x_n \to a \quad \Rightarrow \quad f(x_n) \to a$$

- ***Uniform Continuous on*** $\Omega \subset U$***:***

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in \Omega}{\forall} \|x-y\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(y)\|_2 < \varepsilon$$

- ***Image of*** A***:*** $f(A) := \{y \in N : y = f(x) \text{ for some } x \in A\}$.
- ***Pre-image of B:*** $f^{-1}(B) := \{x \in M : f(x) = y \text{ for some } y \in B\}$.

Continuous Functions

2.1.28. Theorem. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f : U \to V$ a function. Then

$$f \text{ continuous} \quad \Leftrightarrow \quad f^{-1}(\Omega) \text{ is } \underline{\text{open}} \text{ for any } \underline{\text{open}} \text{ set } \Omega \subset V.$$

Note: The following is ***not true***.

$$f \text{ continuous} \quad \Leftrightarrow \quad f^{-1}(\Omega) \text{ is } \underline{\text{closed}} \text{ for any } \underline{\text{closed}} \text{ set } \Omega \subset V.$$

Continuous Functions on Compact Sets

Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f : U \to V$ a function.

- 2.1.35. Proposition. f continuous $\to$ $ran\, f = f(K)$ where $K \subset U$ is compact on V.
- 2.1.36. Theorem. $V = \mathbb{R}$, f has a maximum on a compact set K.
- 2.1.38. Theorem. $f : K \to V$ is continuous on $K \Rightarrow f$ is uniformly continuous on K.

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Functions and Derivatives — Summary

1. Definitions.
   - Differentiability and derivative.
   - Jacobian.
   - Generalized product.
2. Theorems and results.
   - The existence of derivative.
   - Product rule and chain rule.
   - Regulated functions.
   - Mean value theorem.

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The Derivative of a Function

- The ***derivative*** of $f : \Omega \to V$ at x:

$$L_x = Df|_x \quad \textit{such that} \quad f(x+h) = f(x) + L_x h + o(h) \quad \textit{as } h \to 0$$

- The ***derivative*** as a map D

$$D : C^1(\Omega, V) \to C(\Omega, \mathcal{L}(X, V)), \qquad f \mapsto Df$$

The Derivative of a Function

Example 1. Calculate the derivative of inverse of matrices A^{-1} of A in $GL(n;\mathbb{R})$. Namely, the derivative of $F : A \mapsto A^{-1}$.

The Derivative of a Function

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Solution.

1. $F(A+H) = (A+H)^{-1} = (A(id+A^{-1}H))^{-1} = (id+A^{-1}H)^{-1}A^{-1}$.
2. $(id + A^{-1}H)^{-1}$ exists because $\|A^{-1}H\| \le \|A^{-1}\|\|H\| < 1$ for arbitrarily small $\|H\|$.
3. $(id + A^{-1}H)(id - A^{-1}H) = id - A^{-1}HA^{-1}H = id + o(\|H\|)$.
4. $(id + A^{-1}H)^{-1} = (id - A^{-1}H) + o(\|H\|)$.
5. $F(A+H) = ((id - A^{-1}H) + o(\|H\|))A^{-1} = A^{-1} - A^{-1}HA^{-1} + o(\|H\|)$.
6. Therefore, $DF|_A H = -A^{-1}HA^{-1}$.

The Derivative of a Function

Example 2. Calculate the first and second derivative of the map

$$\Psi : Mat(n \times n, \mathbb{R}) \to Mat(n \times n, \mathbb{R}), \qquad \Psi(A) = A^3$$

The Derivative of a Function

Example 2. Calculate the first and second derivative of the map

$$\Psi : Mat(n \times n, \mathbb{R}) \to Mat(n \times n, \mathbb{R}), \qquad \Psi(A) = A^3$$

Solution. We calculate

$$\begin{aligned}\Psi(A+H) &= (A+H)^3 = (A+H)(A^2 + AH + HA + H^2) \\ &= A^3 + A^2H + AHA + AH^2 + HA^2 + HAH + H^2A + H^3 \\ &= A^3 + A^2H + AHA + HA^2 + o(H)\end{aligned}$$

Therefore

$$D\Psi|_A H = A^2H + AHA + HA^2$$

The Derivative of a Function

Example 2. Calculate the first and second derivative of the map

$$\Psi : Mat(n \times n, \mathbb{R}) \to Mat(n \times n, \mathbb{R}), \qquad \Psi(A) = A^3$$

Solution (continued). To find the second derivative, we need to find the derivative of $D\Psi|_A H$ with respect to A. Note that now $D\Psi|_A H$ is viewed as a function of A.

$$\begin{aligned} D\Psi|_{A+J}H &= (A+J)^2H + (A+J)H(A+J) + H(A+J)^2 \\ &= A^2H + AHA + HA^2 + AJH + JAH + AHJ + JHA + \\ &\qquad + HAJ + HJA + o(J) \end{aligned}$$

Therefore,

$$D^2\Psi|_A[H,J] = AJH + AHJ + JAH + JHA + HAJ + HJA$$

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The Jacobian

Definition. Let $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$. Assume that all partial derivatives $\partial_{x_j} f_i$ of f exist at $x \in \Omega$. The matrix

$$J_f(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \Bigg|_x$$

is called the ***Jacobian*** of f.

Note. The existences of all partial derivatives (and thus the Jacobian) do not guarantee the existence of derivative of the original function.

e.g. $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

The Jacobian

Note. The existence of Jacobian does not guarantee the existence of derivative.

e.g. $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

We calculate partial derivatives

$$\left.\frac{\partial g}{\partial x_1}\right|_{x=0} = \lim_{h \to 0} \frac{g(0+h, 0) - g(0)}{h} = 0$$

$$\left.\frac{\partial g}{\partial x_2}\right|_{x=0} = \lim_{h \to 0} \frac{g(0, 0+h) - g(0)}{h} = 0$$

However, g is not continuous at 0 since

$$\lim_{h \to 0} g(h, h) = \frac{h^2}{h^2 + h^2} = \frac{1}{2}, \quad \lim_{h \to 0} g(-h, h) = \frac{-h^2}{(-h)^2 + h^2} = -\frac{1}{2}$$

The Jacobian

Example 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & x_1^2 + x_2^2 \neq 0 \\ 0 & x_1^2 + x_2^2 = 0 \end{cases}$$

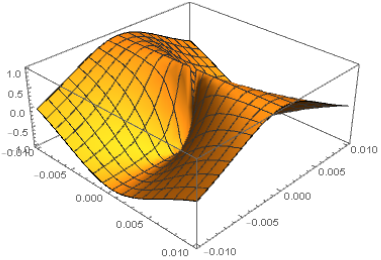
The Jacobian

Example 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & x_1^2 + x_2^2 \neq 0 \\ 0 & x_1^2 + x_2^2 = 0 \end{cases}$$

Then

$$\lim_{x_2 \to 0} \left(\lim_{x_1 \to 0} f(x_1, x_2) \right) = -1, \qquad \lim_{x_1 \to 0} \left(\lim_{x_2 \to 0} f(x_1, x_2) \right) = 1$$



The Jacobian

2.2.18. Theorem. Let $\Omega \subset \mathbb{R}^n$ be an open set and $f :\to \mathbb{R}^m$ such that all partial derivatives $\partial_{x_j} f_i$ exist on Ω.

1. $\partial_{x_j} f_i$ are bounded on $\Omega \Rightarrow f \in C(\Omega, \mathbb{R}^m)$.
2. $\partial_{x_j} f_i$ are continuous on $\Omega \Rightarrow f \in C^1(\Omega, \mathbb{R}^m)$ with derivative given by the Jacobian.

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow f$ is not differentiable.

The Jacobian

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow$ f is not differentiable.

Example 4. Does the derivative exist for the following function?

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

The Jacobian

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow$ f is not differentiable.

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Solution. To find the derivative, we notice that

$$\frac{\|f(h_1,h_2)\|}{\|(h_1,h_2)\|} = \sqrt{h_1^2+h_2^2}\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right) \to 0$$

as $(h_1,h_2) \to (0,0)$. Furthermore,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{h(h,0)-f(0,0)}{h} = \lim_{h\to 0} h\sin(\frac{1}{|h|}) = 0$$

and similarly for $\dfrac{\partial f}{\partial y}(0,0) = 0$.

The Jacobian

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow$ f is not differentiable.

Example 4. Does the derivative exist for the following function?

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Solution (continued). Away from the origin, we calculate the partial derivatives as

$$\frac{\partial f}{\partial x}(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\frac{1}{\sqrt{x^2+y^2}}}$$

$$\frac{\partial f}{\partial x}f(x,y) = 2y\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{y\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\frac{1}{\sqrt{x^2+y^2}}}$$

The Jacobian

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow$ f is not differentiable.

Example 4. Does the derivative exist for the following function?

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Solution (continued). Both partial derivatives oscillate wildly near the origin.

$$\frac{\partial f}{\partial x}(x,0) = 2x\sin\left(\frac{1}{|x|}\right) - \textit{sign}(x)\cos\left(\frac{1}{|x|}\right)$$

$$\frac{\partial f}{\partial y}(0,y) = 2y\sin\left(\frac{1}{|y|}\right) - \textit{sign}(y)\cos\left(\frac{1}{|y|}\right)$$

The sin terms vanish but the cos terms oscillate rapidly between -1 and 1. The partial derivatives are not continuous at 0.

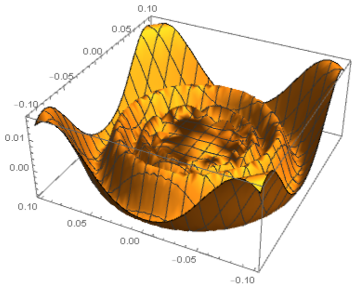
The Jacobian

Note. For 2., $\partial_{x_j} f_i$ are not continuous $\not\Rightarrow$ f is not differentiable.

Example 4. Does the derivative exist for the following function?

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

Solution (concluded.) The graph is shown below.



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Product Rule

Definition. Let X_1, X_2, V be normed vector spaces. A map $\odot : X_1 \times X_2 \to V$ is called a ***(generalized) product*** if

1. $\odot$ is bilinear and
2. $\|u \odot v\|_V \leq \|u\|_{X_1} \|v\|_{X_2}$ for all $u \in X_1, v \in X_2$.

2.2.22. Product Rule.

- U, X_1, X_2, V are finite-dimensional vector spaces.
- $f : \Omega \to X_1, g : \Omega \to X_2$ are differentiable.
- $\odot : X_1 \times X_2 \to V$ is a generalized product.

Then

- $f \odot g : \Omega \to V$ is differentiable.
- $D(f \odot g) = (Df) \odot g + f \odot (Dg)$.

Chain Rule

2.2.23. Chain Rule. U, X, V are finite-dimensional vector spaces, $\Omega \subset U, \Sigma \subset X$ are open sets.

$$g : \Omega \to \Sigma, \qquad f : \Sigma \to V$$

are differentiable maps. Then

1. $f \circ g : \Omega \to V$ is differentiable
2. $D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x$.

e.g. Polar coordinates $(r, \phi) \in (0, \infty) \times [0, 2\pi)$ with

- $\Phi : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2, \Phi(r, \phi) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$.
- $U : \mathbb{R}^2 \to \mathbb{R}, U(x_1, x_2) = x_1^2 + x_2^2$.

Then $D(U \circ \Phi)|_{(r,\phi)} = DU|_{(r\cos\phi, r\sin\phi)} D\Phi_{(r,\phi)} = (2r, 0)$.

Chain Rule

Example 4. Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinates: $(r, \varphi, \theta) \in (0, \infty) \times [0, 2\pi) \times [0, \pi]$ and

$$\Phi(r, \varphi, \theta) = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

Chain Rule

Example 4. Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinates: $(r, \varphi, \theta) \in (0, \infty) \times [0, 2\pi) \times [0, \pi]$ and

$$\Phi(r, \varphi, \theta) = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

Solution. The derivative of Φ is given by

$$D\Phi\Big|_{(r,\varphi,\theta)} = \begin{pmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}$$

and

$$Df\Big|_{(x,y,z)} = (2x, 2y, 2z)$$

Chain Rule

Example 4. Calculate the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in spherical coordinates: $(r, \varphi, \theta) \in (0, \infty) \times [0, 2\pi) \times [0, \pi]$ and

$$\Phi(r, \varphi, \theta) = \begin{pmatrix} r \sin\theta \cos\varphi \\ r \sin\theta \sin\varphi \\ r \cos\theta \end{pmatrix}$$

Solution (continued). Therefore,

$$\begin{aligned} D(f \circ \Phi)\Big|_{(r,\varphi,\theta)} &= Df\Big|_{(r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta)} D\Phi\Big|_{(r,\varphi,\theta)} \\ &= (2r \sin\theta \cos\varphi, 2r \sin\theta \sin\varphi, 2r \cos\theta) \\ &\qquad \cdot \begin{pmatrix} \sin\theta \cos\varphi & -r \sin\theta \sin\varphi & r \cos\theta \cos\varphi \\ \sin\theta \sin\varphi & r \sin\theta \cos\varphi & r \cos\theta \sin\varphi \\ \cos\theta & 0 & -r \sin\theta \end{pmatrix} \\ &= (2r, 0, 0) \end{aligned}$$

Chain Rule

Example 5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$f(u, v) = \begin{pmatrix} u^2 + v^2 \\ u^2 - v^2 \\ uv \end{pmatrix}$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$g(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Calculate the derivative of $f \circ g$.

Chain Rule

Solution.

$$Df\Big|_{(u,v)} = \begin{pmatrix} 2u & 2v \\ 2u & -2v \\ v & u \end{pmatrix} \quad , \qquad Dg\Big|_{(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and

$$D(f \circ g)\Big|_{(r,\theta)} = \begin{pmatrix} 2r & 0 \\ 2r\cos 2\theta & -2r^2\sin 2\theta \\ r\sin 2\theta & r^2\cos 2\theta \end{pmatrix}$$

Convergence and Continuity

- Summary
- Open Sets
- Equivalence of Norms
- Closed Sets
- Compact Sets
- Continuous Functions

Functions and Derivatives

- Summary
- The Derivative of a Function
- The Jacobian
- Product Rule and Chain Rule
- Integrals

Integrals

2.2.25. Definition. A ***step function*** with respect to a partition $P = (a_0, \dots, a_n)$ with elements $y_i \in V$, $f(t) = y_i$ whenever $a_{i-1} < t < a_i, i = 1, \dots, n$.

2.2.29. Theorem. Let $f : [a, b] \to V$ be a step function with respect to some partition P. Then the ***integral*** of f is

$$I_P(f) := (a_1 - a_0)y_1 + \cdots + (a_n - a_{n-1})y_n \in V$$

and is independent of the choice of P.

$$\left\| \int_a^b f(x)dx \right\|_V \leq \int_a^b \|f(x)\|_V dx \leq |b-a| \cdot \sup_{x \in [a,b]} \|f(x)\|_V$$

Mean Value Theorem

2.2.30. Mean Value Theorem. X, V are finite-dimensional vector spaces, $\Omega \subset X$ is open and $f \in C(\Omega, V)$. $x, y \in \Omega$ and the line segment $x + ty, 0 \leq t \leq 1$ is wholly contained in Ω. Then

$$f(x+y) - f(x) = \int_0^1 Df|_{x+ty} y dt = \left(\int_0^1 Df_{x+ty} dt \right) y$$

Differentiating Under an Integral

2.2.33. Theorem.

1. X, V are finite-dimensional vector spaces.
2. $I = [a, b] \subset \mathbb{R}$, $\Omega \subset X$ an open set.
3. $f : I \times \Omega \to V$, $Df(t, \cdot)$ exists and is continuous for every $t \in I$.

Then

$$g(x) = \int_a^b f(t, x)dt, \quad Dg(x) = \int_z^b Df(t, \cdot)|_x dt$$

Thanks for your attention!