VV285 Honors Mathematics III
Final Review

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August 9, 2018

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1 Substitution Rule.

Calculate the integral

$$I = \iint_D \frac{3x}{y^2 + xy^3} \mathrm{d}x\mathrm{d}y$$

where D is the area bounded by $xy = 1, xy = 3, y^2 = x, y^2 = 3x$.
Solution. Let

$$u = xy, \qquad v = \frac{y^2}{x}.$$

Then D becomes $\{(u, x) : 1 \le u, v \le 3\}$. Taking the inverse of the composition:

$$x = u^{\frac{2}{3}} v^{-\frac{1}{3}}, \qquad y = (uv)^{\frac{1}{3}}.$$

the determinant of the Jacobian is then

$$\begin{aligned} |\det J| &= \left|\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right| \\ &= \left|\det\begin{pmatrix} \frac{2}{3}u^{-\frac{1}{3}}v^{-\frac{1}{3}} & -\frac{1}{3}u^{\frac{2}{3}}v^{-\frac{4}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{1}{3}} & \frac{1}{3}u^{\frac{1}{3}}v^{-\frac{2}{3}} \end{pmatrix}\right| \\ &= \frac{1}{3v}. \end{aligned}$$

Therefore, by substitution rule,

$$\begin{aligned} \iint_D \frac{3x}{y^2 + xy^3} \mathrm{d}x\mathrm{d}y &= \iint_D \frac{1}{\frac{y^2}{x}(1+xy)} \mathrm{d}x\mathrm{d}y \\ &= \iint_D \frac{1}{v(1+u)} \cdot \frac{1}{3v} \mathrm{d}u\mathrm{d}v \\ &= \int_1^3 \frac{\mathrm{d}u}{1+u} \int_1^3 \frac{\mathrm{d}v}{v^2} \\ &= \frac{2}{3}\ln 2 \end{aligned}$$

Note. The determinant of the Jacobian of the inverse can be found by taking the reciprocal of the determinant calculated from the original substitution function.

2 Polar Coordinates.

Calculate the integral

$$I = \iint_D \frac{1}{xy} \mathrm{d}x\mathrm{d}y,$$

where

$$D = \left\{(x,y) : \frac{x}{x^2+y^2} \in [2,4], \frac{y}{x^2+y^2} \in [2,4]\right\}$$

Solution. Let

$$x = r\cos t, \qquad y = r\sin t$$

Then D is

$$D = \left\{(r,t) : \frac{\cos t}{r} \in [2,4], \frac{\sin t}{r} \in [2,4]\right\}$$

and

$$D = D_1 \cup D_2,$$

where

$$D_1 = \left\{(r,t) : r \in [\frac{1}{4}\cos t, \frac{1}{2}\sin t], t \in [\arctan\frac{1}{2}, \frac{\pi}{4}]\right\},$$
$$D_2 = \left\{(r,t) : r \in [\frac{1}{4}\sin t, \frac{1}{2}\cos t], t \in [\frac{\pi}{4}, \arctan 2]\right\}.$$

Then the integral is given by

$$\begin{aligned}
\iint_D \frac{1}{xy}\mathrm{d}x\mathrm{d}y &= 2\iint_{D_1} \frac{1}{xy}\mathrm{d}x\mathrm{d}y \\
&= 2\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} \int_{\frac{1}{4}\cos t}^{\frac{1}{2}\sin t} \frac{1}{r^2\cos t\sin t} \cdot r\mathrm{d}r\mathrm{d}t \\
&= 2\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} \frac{\mathrm{d}t}{\cos t\sin t}\ln(2\tan t) \\
&= 2\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} \frac{\ln(2\tan t)}{2\tan t}\mathrm{d}(2\tan t) \\
&= 2\int_{\arctan\frac{1}{2}}^{\frac{\pi}{4}} \ln(2\tan t)\mathrm{d}(\ln 2\tan t) \\
&= (\ln 2)^2
\end{aligned}$$

3 Improper Integrals.

$$\begin{aligned}
\int_0^\infty \int_0^\infty \cos(x^2+y^2)\mathrm{d}x\mathrm{d}y &= \int_0^\infty \int_0^\infty \cos x^2 \cos y^2 \mathrm{d}x\mathrm{d}y - \int_0^\infty \int_0^\infty \sin x^2 \sin y^2 \mathrm{d}x\mathrm{d}y \\
&= \left(\int_0^\infty \cos x^2 \mathrm{d}x\right)^2 - \left(\int_0^\infty \sin x^2 \mathrm{d}x\right)^2 \\
&= \frac{\pi}{8} - \frac{\pi}{8} \\
&= 0 \\
\int_0^\infty \int_0^\infty \cos(x^2+y^2)\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}} \int_0^\infty \cos r^2 \cdot r\mathrm{d}r\mathrm{d}\theta \\
&= \frac{\pi}{2}\int_0^\infty r\cos r^2 \mathrm{d}r \\
&= \frac{\pi}{4}\int_0^\infty \cos r^2 \mathrm{d}r^2 \to \text{does not exist.}
\end{aligned}$$

Note. When we use the substitution rule, the region should be compact. To evaluate the improper integral formally, the following steps are necessary:

1. Find $I(a)$ such that when $a \to \infty$, $I(a)$ represents the improper integral.
2. Prove that the improper integral exists.
3. Evaluate the value of $I(a)$, the value you get should be a function (number) with respect to a.
4. Let $a \to \infty$ and evaluate the result.

Formally speaking, step 2 is necessary and the form of $I(a)$ is needed.

4 Stokes's Theorem.

Let

$$F : \mathbb{R}^3 \to \mathbb{R}^3, \qquad F(x,y,z) = \begin{pmatrix} x^2+y-4 \\ 3xy \\ 2xz+z^2 \end{pmatrix}$$

1. Find $\operatorname{rot} F$.
2. Calculate

$$\int_S \operatorname{rot} F \mathrm{d}\vec{A}$$

for

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 16, z \geq 0\}.$$

Solution. The rotation is given by

$$\operatorname{rot} F = \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2z \\ 3y-1 \end{pmatrix}$$

By Stokes's theorem,

$$\int_S \operatorname{rot} F \mathrm{d}\vec{A} = \int_{\partial S} \langle F, \mathrm{d}\vec{s} \rangle = \int_D \operatorname{rot} F \mathrm{d}\vec{A}$$

where

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 16, z = 0\}$$

has the same boundary as S. Then

$$\begin{aligned} \int_D \operatorname{rot} F \mathrm{d}\vec{A} &= \int_D \langle \operatorname{rot} F, e_3 \rangle \mathrm{d}A \\ &= \int_D (3y-1) \mathrm{d}x\mathrm{d}y \\ &= \int_0^4 \int_0^{2\pi} (3r\sin\varphi - 1) r \mathrm{d}r \mathrm{d}\varphi \\ &= -\int_0^4 \int_0^{2\pi} r \mathrm{d}\varphi \mathrm{d}r \\ &= -16\pi \end{aligned}$$

5 Gauss's Theorem.

The vector field F satisfies

$$F = \begin{pmatrix} x-y+z \\ y-z+x \\ z-x+y \end{pmatrix}$$

and

$$\partial S := \{(x,y,z) : |x-y+z| + |y-z+x| + |z-x+y| = 1\}.$$

Calculate

$$\int_{\partial S} F \mathrm{d}\vec{A}.$$

Solution. Using Gauss's theorem,

$$\int_{\partial S} F\mathrm{d}\vec{A} = \int_S \mathrm{div}F\mathrm{d}x\mathrm{d}y\mathrm{d}z = 3\int_S 1\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Substituting

$$u = x - y + z, \quad v = y - z + x, \quad w = z - x + y,$$

we have $\partial S = |u| + |v| + |w| = 1$ and

$$\begin{aligned}\int_{\partial S} F\mathrm{d}\vec{A} &= \int_S \mathrm{div}F\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= 3\int_S 1\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= 3\int_S |\det J|\mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= \frac{3}{4}\int_S \mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &= \frac{3}{4}\cdot\left(\frac{1}{6}\times 8\right) = 1.\end{aligned}$$

6 Green's Identities.

Let $\Omega \subset \mathbb{R}^n$ be an admissible region and define

$$\langle u, v\rangle := \int_\Omega \overline{u(x)}\cdot v(x)\mathrm{d}x$$

for $u, v \in C(\Omega, \mathbb{R})$. Let

$$M = \{u \in C^2(\Omega, \mathbb{R}) : u|_{\partial\Omega} = 0\}$$

be the set of all twice continuously differentiable functions on Ω that vanish on the boundary of Ω. Show that the operator

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is negative definite on M, i.e.,

$$\langle u, \Delta u\rangle < 0$$

if $u \in M$ is not the constant zero function.
Solution. By Green's first identity,

$$\int_\Omega \langle\nabla u, \nabla v\rangle\mathrm{d}x = -\int_\Omega u\cdot\Delta v\mathrm{d}x + \int_{\partial\Omega^*} u\frac{\partial u}{\partial n}\mathrm{d}A.$$

Then we have

$$\int_\Omega u \cdot \Delta u \mathrm{d}x = -\int_\Omega \langle \nabla u, \nabla u \rangle \mathrm{d}x + \int_{\partial\Omega^*} u \frac{\partial u}{\partial n} \mathrm{d}A.$$

Since $u = 0$ on the boundary, the second integral vanishes and

$$\int_\Omega u \cdot \Delta u \mathrm{d}x = -\int_\Omega |\nabla u|^2 \mathrm{d}x \le 0.$$

Furthermore,

$$\int_\Omega u \cdot \Delta u \mathrm{d}x = 0 \quad \Rightarrow \quad \int_\Omega |\nabla u|^2 \mathrm{d}x = 0.$$

Since u is twice continuously differentiable, ∇u is continuous and therefore this implies that $\nabla u = 0$ on Ω. But then u must be constant on Ω and, moreover, constantly equal to zero, due to the boundary condition $u|_{\partial\Omega} = 0$ and the continuity of u. Hence, if $u \neq 0$, we have

$$\int_\Omega u \cdot \Delta u \mathrm{d}x < 0.$$