VV285 Honors Mathematics III
Some Questions for Mid 2

Sample Exercise 1 (i).

Question. Let X, V be finite-dimensional vector spaces and $L : X \to V$ a linear map. Then the second derivative of L always exists and equals 0.

Solution. Suppose $L \in \mathcal{L}(X, V)$, then its first derivative is found by

$$L(x + h) = Lx + Lh,$$

implying $DL|_x = L, DL|_x h = Lh$.

The first derivative is independent of x, meaning the second derivative is always 0. To see this, we have

$$DL|_{x+p}h = Lh = DL|_x h,$$

implying

$$D^2L|_x = 0.$$

Note that the second derivative is a linear approximation of the first derivative. Namely,

$$DL|_{x+p}h = DL|_x h + (D^2L|_x p)h + o(h)$$

where $D^2L|_x p$ is a linear map (that acts on h).

Sample Exercise 1 (ii).

Question. The set $\Omega = \{A \in \text{Mat}(2 \times 2) : \det A = 1\}$ is closed.

Solution. The closeness of the set Ω is found by showing its complement $\Phi = \{A \in \text{Mat}(2 \times 2) : \det A \neq 1\}$ is open.

Recall that to prove a set is open, we need to show that for any point in this set, there is a small open ball contained in this set. In this case, we need to show that for any matrix $A \in \Phi$, i.e., $\det A \neq 1$, we can find a small ball $B_\delta(A)$ such that $B_\delta(A) \subset \Phi$. For a 2-by-2 matrix, the determinant is given by

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \det A = a_{11}a_{22} - a_{12}a_{21}.$$

We take the norm

$$\|A\|_1 = \max_{i,j} |a_{ij}|.$$

Then for any $A \in \Phi$, without the loss of generality, suppose $\det A = 1 + k$ with $k > 0$. Then since the set $\mathbb{R} \setminus \{1\}$ is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(1+k) \subset \mathbb{R}$. To find an open ball that is wholly contained in Φ, suppose we have any $A' \in \Phi$ such that

$$\|A - A'\|_1 < \delta, \qquad \delta > 0 \quad \Rightarrow \quad |a_{ij} - a'_{ij}| < \delta$$

then

$$\begin{aligned} |\det A - \det A'| &= |a_{11}a_{22} - a_{12}a_{21} - a'_{11}a'_{22} + a'_{12}a'_{21}| \\ &\le (|a_{11}| + |a_{12}| + |a_{21}| + |a_{22}|)\delta + 2\delta^2, \end{aligned}$$

where each a'_{ij} satisfies $a_{ij} - \delta < a'_{ij} < a_{ij} + \delta$. Denote $C = |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}|$, then for any fixed point A, this value C is also fixed. We can thus choose δ as small as possible so that $\det A' \in B_\varepsilon(1+k)$ and $\det A' \neq 1$. Then $A' \in \Phi$.

Therefore, for any A such that $\det A \neq 1$, we can find such δ that for any $A' \in B_\delta(A)$, $A' \in \Phi$, i.e., $B_\delta(A) \subset \Phi$. Thus Φ is open and its complement is closed.

Alternatively, since det is a continuous function and the set $\mathbb{R} \setminus \{1\}$ is open, then the set Φ is open can also be verified.

Sample Exercise 4 (ii).

Question. Suppose a plane is defined by

$$x = x_0 + sx_1 + tx_2$$

where $x_0, x_1, x_2 \in \mathbb{R}^3$ are fixed and x is a point on this plane. Then this plane will include the origin if and only if

$$\det(x_0, x_1, x_2) = 0$$

Solution. If the origin is included in the plane, we can find $s, t \in \mathbb{R}$ such that

$$x_0 + sx_1 + tx_2 = 0 \quad \Rightarrow \quad sx_1 + tx_2 = -x_0.$$

Namely, x_0 is in the range of x_1 and x_2. Then the vectors x_0, x_1, x_2 are dependent. Therefore, $\det(x_0, x_1, x_2) = 0$.

In a geometric sense, if the plane includes the origin, then the volume of the cube spanned by the three vectors is 0 (which is the geometric interpretation of determinant in $\mathbb{R}^3$).

Euler's Formula (RC 6 (Ex.2) and Review).

Question. Use differentiation under integral to prove the Euler's integral formula for $n!$

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = n!.$$

Solution. Define function

$$g(t) = \int_0^\infty e^{-xt} \mathrm{d}x, \qquad t > 0$$

which evaluates to

$$g(t) = \frac{1}{t}$$

Taking derivatives with respect to t for n times in both representations, we can obtain

$$g^{(n)}(t) = \int_0^\infty (-x)^n e^{-xt} \mathrm{d}x = (-1)^n \frac{n!}{t^n},$$

Taking $t = 1$, the Euler's integral formula is derived.

Assignment 5 Ex.5.3(ii)

Question. Show that the map $\mathbb{R}^n \to \mathbb{R}, x \mapsto d(x, M)$ is continuous.

Solution. The distance function is defined as

$$d(x, M) = \inf_{y \in M} \|x - y\|.$$

Then for any $x \in \mathbb{R}^n$,

$$\begin{aligned} d(x, M) &= \inf_{y \in M} \|x - y\| \\ &= \inf_{y \in M} \|x - x' + x' - y\| \\ &\leq \inf_{y \in M} \|x - x'\| + \inf_{y \in M} \|x' - y\| \\ &= \|x - x'\| + \inf_{y \in M} \|x' - y\| \\ &= \|x - x'\| + d(x', M) \end{aligned}$$

Making x' as close as to x, the difference in the distances will also converge to 0. This shows that the distance function is continuous.