Honors Mathematics III
RC 4

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Determinant in $\mathbb{R}^2$

The ***determinant*** in $\mathbb{R}^2$ is defined as a map

$$\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad \det\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = a_1 b_2 - a_2 b_1$$

It is

- ***normed***: $\det(e_1, e_2) = 1$.
- ***bilinear***:

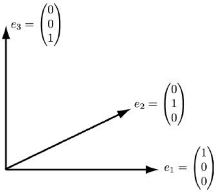
$$\begin{aligned} \det(\lambda a, b) &= \det(a, \lambda b) = \lambda \det(a, b) \\ \det(a + b, c) &= \det(a, c) + \det(b, c) \\ \det(a, b + c) &= \det(a, b) + \det(a, c) \end{aligned}$$

- ***alternating***: $\det(a, b) = -\det(b, a)$.

Vector product in $\mathbb{R}^3$

The ***vector product*** $a \times b \in \mathbb{R}^3$ of two vectors $a, b \in \mathbb{R}^3$ is determined by

1. length: $|a \times b| = A(a, b)$.
2. direction: $a \times b \perp \operatorname{span}\{a, b\}$.
3. orientation: $(a, b, a \times b)$ form a "right-hand system".



$$e_1 \times e_2 = e_3, \;\; e_2 \times e_3 = e_1, \;\; e_3 \times e_1 = e_2$$

$$a \times b = \begin{pmatrix} +\det\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \\ -\det\begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix} \\ +\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \end{pmatrix}$$

Determinant in $\mathbb{R}^3$

The determinant in $\mathbb{R}^3$ is defined as an ***oriented volume***.

$$\det : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3, \qquad \det(a, b, c) = \langle a \times b, c \rangle$$

- $\det(a, b, c) > 0$ if (a, b, c) form a right-hand system.
- $\det(a, b, c) < 0$ if (a, b, c) form a left-hand system.
- $\det(a, b, c) = 0$ if $a = \lambda b$ or $a = \lambda c$ or $b = \lambda c$ for any $\lambda \in \mathbb{R}$.

Determinant in $\mathbb{R}^3$

The determinant in $\mathbb{R}^3$ is calculated as

$$\begin{aligned}\det(a, b, c) &= \langle b \times c, a\rangle \\ &= a_1 \det\begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} - a_2 \det\begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} + a_3 \det\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \\ &= \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\end{aligned}$$

The determinant for $\mathbb{R}^3$ is also

- multilinear,
- alternating and
- normed.

Permutations

An ordered list of elements is endowed with a relation $(x_1, \ldots, x_n)$ with $x_1 \prec x_2 \cdots \prec x_n \prec x_1$.

- A ***permutation*** is a bijective map defined by

$$\pi : \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n\}, \quad x_k \to \pi(x_k) = x_r \text{ for some } r$$

- A ***cyclic permutation*** is a permutation if

$$\pi(x_1) \prec \pi(x_2) \prec \cdots \prec \pi(x_n) \prec \pi(x_1)$$

- A permutation of n elements is represented by

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$$

Transpositions

A ***transposition*** switches the positions of two elements while leaving the other $n-2$ elements invariant. For a transposition τ,

$$\tau(k) = \begin{cases} i & \text{if } k = j \\ j & \text{if } k = i \\ k & \text{otherwise} \end{cases}$$

for some $i, j \in \{1, \ldots, n\}$.

Permutations as Transpositions

Lemma 1.7.7. Every permutation $\pi \in S_n, n \geq 2$, is a composition of transpositions, $\pi = \tau_1 \circ \cdots \circ \tau_k$.

Definition and Theorem 1.7.8 Let $\pi \in S_n$ be represented as a composition of k transpositions, $\pi = \tau_1 \circ \cdots \circ \tau_k$. Then the ***sign*** of π,

$$\operatorname{sgn} \pi := (-1)^k$$

does not depend on the representation chosen.

Determinants

Summary

Determinant in $\mathbb{R}^2$ and $\mathbb{R}^3$

Permutations

Group

Determinant in $\mathbb{R}^n$

Exercises

Calculate Determinant

Determinant and Linear System

Group

A ***group*** is a pair $(G, \circ)$ with a set G and a ***group operation*** $\circ : G \times G \to G$ s.t.

1. $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in G$ (associativity),
2. $\exists e \in G$ such that $a \circ e = e \circ a = a$ for all $a \in G$ (unit element),
3. for every $a \in G$ there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$ (inverse).

Additionally for commutative group:

4. $a \circ b = b \circ a$ for all $a, b \in G$ (commutativity).

Group Actions

Let $(G, \circ)$ be a group and X a set. Then an ***action (or operation) of G on X from the left*** is a map

$$\Phi : G \times X \to X \qquad (g, x) \mapsto \Phi(g, x) = \Phi_g x = gx$$

with the properties

1. $ex = x$ ($e \in G$ is the unit element),
2. $(a \circ b)x = a(bx)$ for $a, b \in G, x \in X$.

We say that G acts (operates) on X.

Proposition 1.7.10. Let X be the set of all maps $f : \mathbb{R}^n \to \mathbb{R}$. Then S_n acts on X via

$$(\pi f)(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)}), \qquad \pi \in S_n$$

Lemma 1.7.11. Denote by $\Delta : \mathbb{R}^n \to \mathbb{R}$ the function

$$\Delta(x_1, \dots, x_n) = \prod_{i<j}(x_j - x_i)$$

Then $\tau\Delta = -\Delta$ for any transposition $\tau \in S_n$.
Corollary 1.7.12. For every permutation $\pi = \tau_1 \circ \cdots \circ \tau_k \in S_n$,

$$\pi\Delta = (\tau_1 \circ \cdots \circ \tau_k)\Delta = (-1)^k\Delta$$

and in particular, $\operatorname{sgn} \pi = (-1)^k$.

Characterization of Alternating Forms

Lemma 1.7.14. Let $f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be a p-multilinear map, then the followings are equivalent:

1. f is alternating.
2. $$\begin{aligned} &f(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_p) \\ = -\, &f(a_1, \ldots, a_{j-1}, a_k, a_{j+1}, \ldots, a_{k-1}, a_j, a_{k+1}, \ldots, a_p) \end{aligned}$$
3. $f(a_1, \ldots, a_p) = 0$ if $a_1, \ldots, a_p$ are linearly dependent.

Determinants in $\mathbb{R}^n$

Theorem 1.7.15. For every $n \in \mathbb{N}, n > 1$, there exists a unique, normed, alternating n-multilinear form $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \cong \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$ given by

$$\det(a_1, \ldots, a_n) = \det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \; a_{\pi(1)1} \cdots a_{\pi(n)n}$$

Proof.

1. The $\det$ function above is multilinear, normed and alternating.
2. The linear map with the above properties is unique.

$$\begin{aligned}\det(a_1, \ldots, a_n) &= \det\left(\sum_{j_1=1}^{n} a_{j_1 1} e_{j_1}, \ldots, \sum_{j_n=1}^{n} a_{j_n n} e_{j_n}\right) \\ &= \sum_{\pi \in S_n} a_{\pi(1)1} \cdots a_{\pi(n)n} \det(e_{\pi(1)}, \ldots, e_{\pi(n)})\end{aligned}$$

Determinants in $\mathbb{R}^n$

1. Elementary column operations.
   - $\det(a_2, a_1, \ldots, a_n) = -\det(a_1, a_2, \ldots, a_n)$.
   - $\det(a_1, \ldots, \lambda a_j, \ldots, a_n) = \lambda \det(a_1, \ldots, a_j, \ldots, a_n)$.
   - $\det(a_1, \ldots, a_j, \ldots, a_k + \lambda a_j, \ldots, a_n) =$
     $\det(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n)$.
2. $\det\ A = \det\ A^T$.
3. Leibnitz Formula. $\det\ A = \sum_{\pi \in S_n} \operatorname{sgn}\ \pi\ a_{1\pi(1)} \cdots a_{n\pi(n)}$.
4. Proposition 1.7.19. For $A \in \mathrm{Mat}(n \times n)$ in upper triangular form with diagonal elements $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\det A = \lambda_1 \cdots \lambda_n$.
5. $\det(AB) = (\det\ A)(\det\ B)$.
6. Laplace Expansion. $\det\ A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$.

Results from Determinant

The properties of determinant can be applied in matrices and system of linear equations.

- Proposition 1.7.20. A matrix $A \in \mathrm{Mat}(n \times n)$ is invertible iff $\det A \neq 0$.
- Fredholm Alternative 1.7.21.
  - $\det A = 0$. $(\ker A \neq \{0\})$.
  - $\det A \neq 0 \Rightarrow A$ is invertible $\Rightarrow x = A^{-1}b$ is a unique solution for any $b \in \mathbb{R}^n$.
- Cramer's Rule 1.7.22. Find solution for the system $Ax = b$ with invertible A:

$$x_i = \frac{1}{\det A}\det(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n), \quad i = 1, \ldots, n$$

Results from Determinant

We can use determinants to find inverses.

1. The (i, j)th ***minor*** of A: $m_{ij} := \det A_{ij}$.
2. The (i, j)th ***cofactor*** of A: $c_{ij} := (-1)^{i+j} \det A_{ij}$.
3. The ***cofactor matrix*** of A: $\mathrm{Cof}\, A := (c_{ij})_{1 \le i,j \le n}$.
4. The ***adjugate*** of A: $A^\sharp := (\mathrm{Cof}\, A)^T$.

Then we have

- Theorem 1.7.25. For an invertible matrix,

$$A^{-1} = \frac{1}{\det A} A^\sharp$$

- Lemma 1.7.26. Denoting e_i as the ith standard basis vector in $\mathbb{R}^n$,

$$\det(a_1, \ldots, a_{j-1}, e_i, a_{j+1}, \ldots, a_n) = (-1)^{i+j} \det A_{ij} = c_{ij}$$

Calculate Determinant

Few methods to calculate determinant:

1. In $\mathbb{R}^3$: expand to the case of $\mathbb{R}^2$ and calculate directly.
2. In higher dimensions: use properties of determinant. Apply ***elementary row or column operations*** combined with Laplace expansion.
3. Leibnitz Formula.

Calculate Determinant

Exercise 1. Calculate the determinant for the following matrix:

$$A = \begin{pmatrix} -1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Calculate Determinant

Exercise 2. Calculate the determinant for the matrix $A \in \mathrm{Mat}(2n \times 2n, \mathbb{R})$.

$$A^{(2n)} = \begin{pmatrix} a & 0 & \cdots & 0 & b \\ 0 & a & \cdots & b & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b & \cdots & a & 0 \\ b & 0 & \cdots & 0 & a \end{pmatrix}$$

Calculate Determinant

Exercise 3. Calculate the determinant for the following $n \times n$ matrix, where $a, b, c \in \mathbb{R}$.

$$D_n = \begin{pmatrix} a & b & b & \cdots & b \\ c & a & b & \cdots & b \\ c & c & a & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b \\ c & \cdots & \cdots & c & a \end{pmatrix}$$

Cramer's Rule

Exercise 4. Use Cramer's rule to solve the following systems of equations.

1.

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 2x_4 & = & 6 \\ 3x_1 - 3x_2 + 3x_3 + 2x_4 & = & 5 \\ 3x_1 - x_2 - x_3 + 2x_4 & = & 3 \\ 3x_1 - x_2 + 3x_3 - x_4 & = & 4 \end{cases}$$

2.

$$\begin{cases} x_1 + 2x_2 + 3x_3 - 2x_4 & = & 6 \\ 2x_2 - x_2 - 2x_3 - 3x_4 & = & 8 \\ 3x_1 + 2x_2 - x_3 + 2x_4 & = & 4 \\ 2x_1 - 3x_2 + 3x_3 + x_4 & = & -8 \end{cases}$$

Thanks for your attention!