VV285 Honors Mathematics III
Solution Manual for RC 4

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Exercise 1.

Expanding the determinant with respect to the 3rd column, we get

$$\det A = (-1)^{1+3}\det A_{13} = \det A_{13}$$

with

$$A' = A_{13} = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

Expanding the determinant with respect to the second row,

$$\det A = (-1)\cdot(-1)^{2+2}\det A'_{22} = -\det A'_{22}$$

where

$$A'_{22} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad , \qquad \det A'_{22} = -5$$

Therefore, $\det A = 5$.

Exercise 2.

The determinant can be found by

$$\det A^{(2n)} = a\cdot(-1)^{1+1}\det A^{(2n)}_{11} + b\cdot(-1)^{2n+1}\det A^{(2n)}_{(2n)1} = a\det A^{(2n)}_{11} - b\det A^{(2n)}_{(2n)1}$$

where

$$A^{(2n)}_{11} = \begin{pmatrix} a & \cdots & b & 0 \\ \vdots & & \vdots & \vdots \\ b & \cdots & a & 0 \\ 0 & \cdots & 0 & a \end{pmatrix} \in \mathrm{Mat}((2n-1)\times(2n-1),\mathbb{R})$$

$$A^{(2n)}_{(2n)1} = \begin{pmatrix} 0 & \cdots & 0 & b \\ a & & b & 0 \\ \vdots & & \vdots & \vdots \\ b & \cdots & a & 0 \end{pmatrix} \in \mathrm{Mat}((2n-1)\times(2n-1),\mathbb{R})$$

Then expanding the $(2n-1)$th columns for the two matrices, we obtain

$$\det A = a^2 \det A^{(2n-2)} - b^2 \det A^{(2n-2)}$$

Then we attempt to prove $\det A = (a^2 - b^2)^n$ by induction on $n \geq 1$.

- When $n = 1$, it is clear that

$$\det A^{(2)} = a^2 - b^2$$

- Then based on the hypothesis on $n = k$, from the relation deduced above, we have verified that $\det A^{(2k+2)} = (a^2 - b^2)\det A^{(2k)} = (a^2 - b^2)^n$.

Therefore, the the determinant is given by

$$\det A^{(2n)} = (a^2 - b^2)^n$$

Exercise 3.

The determinant can be transferred to

$$\det D_n = \det \begin{pmatrix} a-b & 0 & 0 & \cdots & 0 \\ c & a & b & \cdots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b \\ c & \cdots & \cdots & c & a \end{pmatrix} + \det \begin{pmatrix} b & b & b & \cdots & b \\ c & a & b & \cdots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b \\ c & \cdots & \cdots & c & a \end{pmatrix}$$

$$= (a-b)\det D_{n-1} + b \det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ c & a & b & \cdots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b \\ c & \cdots & \cdots & c & a \end{pmatrix}$$

$$= (a-b)\det D_{n-1} + b \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ c & a-c & b-c & \cdots & b-c \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & b-c \\ c & \cdots & \cdots & 0 & a-c \end{pmatrix}$$

$$= (a-b)\det D_{n-1} + b(a-c)^{n-1}$$

Similarly, if we decompose the column vector in the first step, then we will get the result

$$\det D_n = (a-c)\det D_{n-1} + c(a-b)^{n-1}$$

Then if $b \neq c$, eliminating D_{n-1} we then obtain

$$\det D_n = \frac{b(a-c)^n - c(a-b)^n}{b-c}$$

When $b = c$, we obtain

$$\begin{aligned}\det D_n &= (a-b)\det D_{n-1} + b(a-b)^{n-1} \\ &= (a-b)^2\det D_{n-2} + 2b(a-b)^{n-1} \\ &= \cdots \\ &= (a-b)^{n-1}\det D_1 + (n-1)b(a-b)^{n-1} \\ &= (a-b)^n + nb(a-n)^{n-1}\end{aligned}$$

Exercise 4.

1.

The system can be described as

$$Ax = b, \qquad \text{where} \quad A = \begin{pmatrix} 2 & -1 & 3 & 2 \\ 3 & -3 & 3 & 2 \\ 3 & -1 & -1 & 2 \\ 3 & -1 & 3 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 5 \\ 3 \\ 4 \end{pmatrix}$$

The determinant of A is (with calculations)

$$\det A = -70$$

Therefore,

$$x_1 = \frac{1}{\det A}\det\begin{pmatrix} 6 & -1 & 3 & 2 \\ 5 & -3 & 3 & 2 \\ 3 & -1 & -1 & 2 \\ 4 & -1 & 3 & -1 \end{pmatrix} = 1$$

$$x_2 = \frac{1}{\det A}\det\begin{pmatrix} 2 & 6 & 3 & 2 \\ 3 & 5 & 3 & 2 \\ 3 & 3 & -1 & 2 \\ 3 & 4 & 3 & -1 \end{pmatrix} = 1$$

$$x_3 = \frac{1}{\det A} \det \begin{pmatrix} 2 & -1 & 6 & 2 \\ 3 & -3 & 5 & 2 \\ 3 & -1 & 3 & 2 \\ 3 & -1 & 4 & -1 \end{pmatrix} = 1$$

$$x_4 = \frac{1}{\det A} \det \begin{pmatrix} 2 & -1 & 3 & 6 \\ 3 & -3 & 3 & 5 \\ 3 & -1 & -1 & 3 \\ 3 & -1 & 3 & 4 \end{pmatrix} = 1$$

2.

The system can be described as

$$Ax = b, \qquad \text{where} \quad A = \begin{pmatrix} 1 & 2 & 3 & -2 \\ 2 & -1 & -2 & -3 \\ 3 & 2 & -1 & 2 \\ 2 & -3 & 3 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 6 \\ 8 \\ 4 \\ -8 \end{pmatrix}$$

The determinant of A is (with calculations)

$$\det A = 360$$

Therefore,

$$x_1 = \frac{1}{\det A} \det \begin{pmatrix} 6 & 2 & 3 & -2 \\ 8 & -1 & -2 & -3 \\ 4 & 2 & -1 & 2 \\ -8 & -3 & 3 & 1 \end{pmatrix} = \frac{11}{10}$$

$$x_2 = \frac{1}{\det A} \det \begin{pmatrix} 1 & 6 & 3 & -2 \\ 2 & 8 & -2 & -3 \\ 3 & 4 & -1 & 2 \\ 2 & -8 & 3 & 1 \end{pmatrix} = \frac{37}{20}$$

$$x_3 = \frac{1}{\det A} \det \begin{pmatrix} 1 & 2 & 6 & -2 \\ 2 & -1 & 8 & -3 \\ 3 & 2 & 4 & 2 \\ 2 & -3 & -8 & 1 \end{pmatrix} = -\frac{9}{10}$$

$$x_4 = \frac{1}{\det A} \det \begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & -1 & -2 & 8 \\ 3 & 2 & -1 & 4 \\ 2 & -3 & 3 & -8 \end{pmatrix} = -\frac{39}{20}$$