Honors Mathematics III
RC 2

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Linear and Structure-Preserving Maps

- ***Linear map*** L from $(U, \oplus, \odot)$ to $(V, \boxplus, \boxdot)$ ***(both real or both complex)*** is:
  - ***Homogeneous:*** $L(\lambda \odot u) = \lambda \boxdot L(u), \lambda \in \mathbb{F}$.
  - ***Additive:*** $L(u \oplus u') = L(u) \boxplus L(u')$.

  Remark: For the conjugate map in $\mathbb{C}$, if $\mathbb{C}$ is regarded as a complex vector space, is not linear:

  $$a + bi \mapsto a - bi, \quad i(a + bi) = -b + ai \mapsto -b - ai \neq i(a - bi)$$

- ***Structure-preserving map (homomorphism):***

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, "\lambda\odot"'] & V \arrow[d, "\lambda\boxdot"] \\ U & V \arrow[l, "L^{-1}"'] \end{tikzcd}$$

Homomorphism

A homomorphism $L \in \mathcal{L}(U, V)$ is said to be

- ***isomorphism***: L is bijective;
- ***endomorphism:*** $U = V$;
- ***automorphism:*** $U = V$ and L is bijective;
- ***epimorph:*** L is surjective;
- ***monomorph:*** L is injective.

Coordinate Map and Dual Space

- The ***coordinate map*** is linear and bijective:

$$\varphi : V \to \mathbb{F}^n, \quad v = \sum_{k=1}^{n} \lambda_k b_k \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

- ***Dual space:*** $V^* = \mathcal{L}(V, \mathbb{F})$.
- ***Dual basis:***

$$b_k^* : V \to \mathbb{F}, \quad b_k^*(b_j) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Range and Kernel

- ***Range:***

$$\operatorname{ran} L := \left\{ v \in V : \underset{u \in U}{\exists} v = Lu \right\}$$

- ***Kernel:***

$$\ker L := \{ u \in U : Lu = 0 \}$$

Remark: $L \in \mathcal{L}(U, V)$ is injective iff $\ker L = \{0\}$.

$$Lx = L(x + y - y) = L(x + y) - Ly \neq 0 \text{ if } x \neq 0$$

This can be used to prove injective property.

Normed Vector Spaces and Bounded Linear Maps

- A linear map $L : U \to V$ between two normed vector spaces is ***bounded*** if $\exists c > 0 \text{ s.t. } \|Lu\|_V \leq c \cdot \|u\|_U$.
  Remark. Every linear map in a finite-dimensional vector space is bounded.

  $$Lb_k = v_k \Rightarrow \|Lu\| = \|\sum \lambda_k v_k\| \leq \sum |\lambda_k| \cdot \|v_k\|$$

- ***Operator norm:***

  $$\|L\| := \sup_{u \in U,\ u \neq 0} \frac{\|Lu\|_V}{\|u\|_U} = \sup_{u \in U,\ \|u\|_U = 1} \|Lu\|_V$$

Homomorphisms

Theorem 1.4.4.
For a basis $(b_1, \ldots, b_n)$ of U and every $(v_1, \ldots, v_n) \in V^n$, $\exists!\ L : U \to V$ s.t. $Lb_k = v_k$.

Proof.

- Uniqueness:

$$\begin{aligned} Lu &= \sum_{k=1}^{n} \lambda_k L(b_k) = \sum_{k=1}^{n} \lambda_k v_k \\ &= \sum_{k=1}^{n} \lambda_k M(b_k) = Mu \end{aligned}$$

- Existence: Define a map L by $Lu = \sum_{k=1}^{n} \lambda_k v_k$ and show that L is linear.

Isomorphisms

Theorem 1.4.11.

$L \in \mathcal{L}(U, V)$, where U, V are finite-dimensional, is an isomorphism iff it maps from basis to basis.

Proof.

$(\Rightarrow)$

- A representation for each y exists:

$$y = L\left(\sum_{k=1}^{n} \lambda_k b_k\right) = \sum_{k=1}^{n} \lambda_k \cdot Lb_k$$

- This representation is unique:

$$\begin{aligned} y = \sum_{k=1}^{n} \lambda_k \cdot Lb_k = \sum_{k=1}^{n} \mu_k \cdot Lb_k \;\Rightarrow\; L^{-1}y &= \sum_{k=1}^{n} \lambda_k b_k \\ &= \sum_{k=1}^{n} \mu_k b_k \end{aligned}$$

Theorem 1.4.11.
$L \in \mathcal{L}(U, V)$, where U, V are finite-dimensional, is an isomorphism iff it maps from basis to basis.

Proof (continued).

$(\Leftarrow)$

- Surjective: $\forall y \in V, y = \sum \lambda_k \cdot Lb_k$ is the image of $x = \sum \lambda_k b_k$.
- Injective: $\ker L = \{0\}$

$$Lx = \sum \lambda_k \cdot Lb_k = 0 \Leftrightarrow \lambda_k = 0$$

Dimension

Lemma 1.4.13.

Isomorphic is equivalent to having the same dimension: $U \cong V \Leftrightarrow \dim\ U = \dim\ V$.

Dimension Formula 1.4.14.

$\dim\ \mathrm{ran}\ L + \dim\ \ker\ L = \dim\ U, \dim\ U < \infty$.

Proof.

- Basis of $\ker\ L : (a_1, \ldots, a_r)$
  Basis of $U : (a_1, \ldots, a_r, a_{r+1}, \ldots, a_n)$.
- $(La_{r+1}, \ldots, La_n)$ is independent and forms a basis of $\mathrm{ran}\ L$.

Corollary 1.4.15.

If $\dim\ U = \dim\ V$, then for a linear map $L \in \mathcal{L}(U, V)$: injective $\Leftrightarrow$ surjective.

The Operator Norm

Theorem 1.4.19.

The ***operator norm*** $\|L\| = \sup_{u \in U, u \neq 0} \frac{\|Lu\|_V}{\|u\|_U}$ defines a norm:

- $\|L\| \geq 0$.
- $\|\lambda \cdot L\| = |\lambda| \cdot \|L\|$.
- $\|L_1 + L_2\| \leq \|L_1\| + \|L_2\|$.

Additionally,

- $\|L_1 L_2\| \leq \|L_1\| \cdot \|L_2\|, \quad L_1 \in \mathcal{L}(U, V), \quad L_2 \in \mathcal{L}(V, W)$.

Linear Maps

Example 1.

Suppose that a vector space $V = V_1 \oplus V_2$. L is a linear map on V and

$$\forall v_1 \in V_1, \quad \forall v_2 \in V_2 \Rightarrow L(v_1 + v_2) = v_1$$

Find $\mathrm{ran}\ L$ and $\ker\ L$.

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Find $\mathrm{ran}\ L$ and $\ker\ L$.

Solution.

- $v_1 = 0 \Rightarrow \forall v_2 \in V_2, Lv_2 = 0.$
- $v_2 = 0 \Rightarrow \forall v_1 \in V_1, Lv_1 = v_1 = 0 \text{ iff } v_1 = 0.$
- $\forall v \in V, v = v_1 + v_2, Lv = v_1.$

Therefore $\ker\ L = V_2, \mathrm{ran}\ L = V_1.$

Linear Maps

Example 2.

Suppose L is a linear map from U to V. V' is a subspace of V and $V' \subset \mathrm{ran}\ L$. Show that

1. $L^{-1}(V') := \{u \in U | L(u) \in V'\}$ is a subspace of U.
2. $\dim\ V' + \dim\ \ker\ L = \dim\ L^{-1}(V')$.

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Solution.

1.
   - $0 \in L^{-1}(V')$.
   - $u, v \in L^{-1}(V') \Rightarrow L(u+v) = Lu + Lv \in V' \Rightarrow (u+v) \in L^{-1}(V')$.
   - $v \in L^{-1}(V') \Rightarrow L(\lambda v) \in V' \Rightarrow \lambda v \in L^{-1}(V')$.

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Solution (continued).

2. Define

$$L' : L^{-1}(V) \to V', \quad L'(v) \mapsto L(v)$$

then $\dim\ L^{-1}(V') - \dim\ V' = \dim\ \ker\ L' = \ker\ L$.
Note: $0 \in V'$ implies $\ker\ L = \ker\ L'$.

Exercise 1.4

(iii).

$$\begin{cases} \sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3 + \sin\alpha_4 = 0 \\ \cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 + \cos\alpha_4 = 0 \\ \sin 2\alpha_1 + \sin 2\alpha_2 + \sin 2\alpha_3 + \sin 2\alpha_4 = 0 \\ \cos 2\alpha_1 + \cos 2\alpha_2 + \cos 2\alpha_3 + \cos 2\alpha_4 = 0 \\ 3\sin\alpha_1 + \sin\alpha_2 - \sin\alpha_3 - 3\sin\alpha_4 = 0 \\ 3\cos\alpha_1 + \cos\alpha_2 - \cos\alpha_3 - 3\cos\alpha_4 = 0 \\ 3\sin 2\alpha_1 + \sin 2\alpha_2 - \sin 2\alpha_3 - 3\sin 2\alpha_4 = 0 \\ 3\cos 2\alpha_1 + \cos 2\alpha_2 - \cos 2\alpha_3 - 3\cos 2\alpha_4 = 0 \end{cases}$$

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Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we then obtain

$$\begin{cases} e^{i\alpha_1} + e^{i\alpha_2} + e^{i\alpha_3} + e^{i\alpha_4} = 0 \\ 3e^{i\alpha_1} + e^{i\alpha_2} - e^{i\alpha_3} - 3e^{i\alpha_4} = 0 \\ e^{2i\alpha_1} + e^{2i\alpha_2} + e^{2i\alpha_3} + e^{2i\alpha_4} = 0 \\ 3e^{2i\alpha_1} + e^{2i\alpha_2} - e^{2i\alpha_3} - 3e^{2i\alpha_4} = 0 \end{cases} \Rightarrow \begin{cases} 2e^{i\alpha_1} + e^{i\alpha_2} = e^{i\alpha_4} \\ 2e^{2i\alpha_1} + e^{2i\alpha_2} = e^{2i\alpha_4} \end{cases}$$

Exercise 1.7

(i). To show that U is a subspace of the real vector space $\mathbb{R}^4$,

- $0 \in U$.
- $\forall u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$.
- $\forall u \in U, \lambda \in \mathbb{R} \Rightarrow \lambda u \in U$.

(ii).

$$\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}, \qquad x = \alpha \begin{pmatrix} 7 \\ -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}$$

Inner Product Spaces

Exercise 1. Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$. If $f, g \in V$, we define

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(t)} g(t) \mathrm{d}t$$

and choose a basis

$$\mathcal{B} = \left\{ f_n(t) = \frac{1}{\sqrt{2\pi}} e^{int} \right\}_{n=-\infty}^{\infty}$$

(1). Show that the basis is an orthonormal basis.

(2). Represent the function $h : \mathbb{R} \to \mathbb{R}, h(t) = t$ as a linear combination of this basis.

Inner Product Spaces

Exercise 2. Let V be a finite dimensional space over $\mathbb{R}$, with an inner product $\langle \cdot, \cdot \rangle$. Let $\{v_1, \ldots, v_m\}$ be a set of elements of V, of norm 1, and mutually perpendicular (i.e., $\langle v_i, v_j \rangle = 0$ if $i \neq j$). Assume that for every $v \in V$ we have

$$\|v\|^2 = \sum_{i=1}^{m} \langle v_i, v \rangle^2$$

Show that $\{v_1, \ldots, v_m\}$ is a basis of V.

Dual Spaces

Exercise 3. Let V be a vector space of dimension n. Let V^{**} be the dual space of V^*, where V^* is the dual space for V. Show that each element $v \in V$ gives rise to an element λ_v in V^{**} and that the map $v \mapsto \lambda_v$ gives an isomorphism of V and V^{**}.

Linear Maps

Exercise 4. Suppose there are two linear maps A and B satisfying $AB - BA = \mathrm{id}$. Show that for any $k \in \mathbb{N}_+$, we have

$$A^k B - BA^k = kA^{k-1}$$

Exercise 5. Suppose L is a linear map in a finite dimensional vector space V satisfying $L^2 = L$. Show that

(1). Any $v \in V$ has a unique representation $v = v_1 + v_2$, where $Lv_1 = v_1$ and $Lv_2 = 0$.

(2). If $Lv = -v$ for some $v \in V$, then $v = 0$.

Operator Norm

Exercise 6. (*Column-sum Norm*) Show that the operator norm induced by

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

is

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

where $x = (x_1, \ldots, x_n)^T$ is a vector in $\mathbb{R}^n$ and A is a matrix representing a linear map.

P.S.

Some statements and proofs in the slides above are not mathematically rigorous. I am merely trying to give you a general idea about what is going on. Please refer to the course slides if you want to check the details!

Thanks for your attention!