Honors Mathematics III
Review — Midterm 2

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July 16, 2018

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Equivalence of Norms

Theorem. In a ***finite-dimensional*** vector space, all norms are equivalent.

- Prove continuity. Exercise 5.6. Use a suitable norm to prove $\det$ is continuous. ($\|A\| = \max_{i,j} |a_{ij}|$)
- Exercise 5.4. Does not hold for infinite-dimensional vector spaces: in the vector space of continuous functions on $[0, 1]$, the norms

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|, \quad \|f\|_1 = \int_0^1 |f(x)| \mathrm{d}x$$

are not equivalent.

Open, Closed and Compact Sets

For a set $U \subset V$ where $(V, \|\cdot\|)$ is a normed vector space, it is

- Open: if $\forall a \in U$, there exists $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.
- Closed: if its complement is open.
- Compact: if every sequence in U has a convergent subsequence with limit contained in U.

Open, Closed and Compact Sets

Exercise 5.1.

1. If a set $A \subset \mathbb{R}$ is closed and $f \in C(A, \mathbb{R})$, the set $f(A) = \operatorname{ran} f$ does not have to be closed.
   - The set $\mathbb{R}$ is closed.
   - A function defined on $\mathbb{R}$ (e.g., $f(x) = x$) can have an open range.

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2. If a set $B \subset \mathbb{R}$ is open and $g \in C(B, \mathbb{R})$, the set $g(B)$ does not have to be open.
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   - The set $\mathbb{R}$ is open.
   - A function defined on $\mathbb{R}$ (e.g., cosine and sine) have a closed range.
3. If $f \in C(\mathbb{R}^n, \mathbb{R}^m)$ and $K \subset \mathbb{R}^m$ is compact, then $f^{-1}(K)$ does not have to be compact.
   - $f : (0, 2\pi) \to [-1, 1], \quad f(x) = \sin x.$
   - ***Note.*** f is continuous, if $C \subset \mathbb{R}$ is compact, then $f(C)$ is also compact. But the inverse is not true.

Interior, Exterior and Boundary Points

- Interior point: $\exists \varepsilon > 0, B_\varepsilon(x) \subset M$. $(\mathrm{int}\, M)$.
- Boundary point: $\forall \varepsilon > 0, B_\varepsilon(x) \cap M \neq \emptyset, B_\varepsilon(x) \cap (V \setminus M) \neq \emptyset$. (∂M).
- Exterior point: x is neither a boundary nor an interior point.
- Closure: $\overline{M} = M \cup \partial M$.

Continuous and Uniformly Continuous

Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces, $\Omega \subset U$ and $f : \Omega \to V$ a function. Then f is a

- Continuous function on Ω:
  $$\underset{\varepsilon>0}{\forall}\ \underset{x\in U}{\forall}\ \underset{\delta>0}{\exists}\ \underset{y\in U}{\forall} \|x-y\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(y)\|_2 < \varepsilon.$$
  Note. A function is continuous at $a \in U$ if and only if
  $$\underset{\substack{(x_n),n\in\mathbb{N}\\ x_n\in U}}{\forall} x_n \to a \quad \Rightarrow \quad f(x_n) \to f(a)$$
  (Often used to prove or disprove continuity.)
- Uniformly continuous function on Ω:
  $$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall}\ \underset{y\in U}{\forall} \|x-y\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(y)\|_2 < \varepsilon.$$

Continuous Functions on Compact Sets

1. On a normed vector space $(V, \|\cdot\|)$,
   - $K \subset V$ is compact $\Rightarrow$ K is closed and bounded.
   - $K \subset V$ is closed and bounded, ***and V is finite-dimensional***, $\Rightarrow$ K is compact.
2. For a continuous function f defined on a compact set $K \subset V$,
   - $f(K)$ is compact.
   - f has a maximum on K.
   - f is uniformly continuous on K.

Differentiability and Derivative

- The first derivative.

$$f(x+h) = f(x) + Df|_x h + o(h), \qquad h \to 0$$

- The second derivative.

$$Df|_{x+h} = Df|_x + D^2 f_x h + o(h), \qquad h \to 0$$

The Derivatives of Functions

1. Determinant. Exercise 5.6 & RC 7. (Second derivative for invertible matrices.)

$$(D\det)|_A H = \det A \cdot \mathrm{tr}(A^{-1}H) = \mathrm{tr}(A^{\sharp}H)$$
$$D^2\det|_A[J,H] = \det(A)\left(\mathrm{tr}(A^{-1}J)\cdot\mathrm{tr}(A^{-1}H) - \mathrm{tr}(A^{-1}JA^{-1}H)\right)$$

2. $\Phi(A) = A^3$.

$$D\Phi|_A H = A^2H + AHA + HA^2$$
$$D^2\Phi|_A[H,J] = AJH + AHJ + JAH + JHA + HAJ + HJA$$

3. Inverse. $D(\cdot)^{-1}|_A H = -A^{-1}HA^{-1}$.
4. Linear maps. $DL|_x = L$.
   - Complex conjugation when $\mathbb{C}$ is regarded as a real vector space. $D\overline{(\cdot)}|_z h = \overline{h}$.
   - Trace of matrix. $D\mathrm{tr}|_A = \mathrm{tr}$.

Partial Derivatives and Jacobian

Evaluate partial derivatives

1. on an open interval. (Usually for continuous partial derivatives.) Calculate based on formula.
2. at a specific point. (Not continuous partial derivatives.) Use definition

$$\left.\frac{\partial f}{\partial x_j}\right|_x = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

Note. The partial derivative at a point should be evaluated up to the specific value of this point.

Partial Derivatives and Jacobian

e.g.

1. Exercise 5.7.

$$g(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2)\sin((x_1^2 + x_2^2)^{-1/2}) & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

2. Exercise 6.7.

$$f(x, y) = \begin{cases} \dfrac{x^3}{y^2} e^{-x^2/y} & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 f(x, y)\mathrm{d}y \bigg|_{x=0} \neq \int_0^1 \frac{\partial}{\partial x} f(x, y)\mathrm{d}y \bigg|_{x=0}$$

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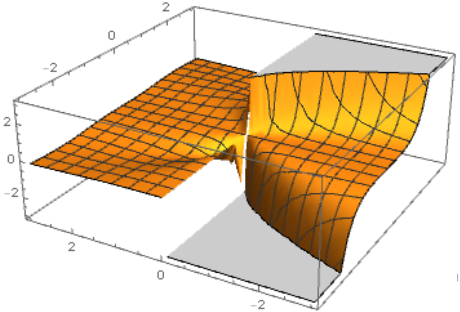


Jacobian

If all partial derivatives of f at x,

$$J_f(x) := \left. \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \right|_x$$

- All partial derivatives are continuous on $\Omega \Rightarrow f$ is continuously differentiable on Ω with derivative given by Jacobian.
- Not all partial derivatives are continuous $\not\Rightarrow f$ is not differentiable.

  e.g. Exercise 5.7.

$$g(x_1, x_2) = \begin{cases} (x_1^2 + x_2^2) \sin((x_1^2 + x_2^2)^{-1/2}) & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

Finding Derivatives

- Definition.

$$f(x+h) = f(x) + Df|_x h + o(h), \qquad h \to 0$$
$$Df|_{x+h} = Df|_x + D^2 f_x h + o(h), \qquad h \to 0$$

- Product Rule.

$$D(f \odot g) = (Df) \odot g + f \odot (Dg)$$

- Chain Rule.

$$D(f \circ g)|_x = Df|_{g(x)} \circ Dg|_x$$

Note. Composition of two functions.

Chain Rule

Change of coordinates.

$$\Phi(x,y)=\begin{pmatrix}u\\v\end{pmatrix},\quad f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=g(u,v)$$

then

$$Df|_{(x,y)}=Dg|_{\Phi(x,y)}\circ D\Phi|_{(x,y)}=\begin{pmatrix}\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}$$

Namely,

$$\frac{\partial}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v},\quad \frac{\partial}{\partial y}=\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}$$

e.g. Exercise 6.4.

$$\Delta_{(r,\theta)}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Integrals of Functions

- Mean value theorem.

$$f(x+y)-f(x)=\int_0^1 Df|_{x+ty}y\mathrm{d}t=\left(\int_0^1 Df_{x+ty}\mathrm{d}t\right)y$$

- Differentiating under an integral.

$$g(x)=\int_a^b f(t,x)\mathrm{d}t,\quad Dg(x)=\int_a^b Df(t,\cdot)|_x\mathrm{d}t$$

e.g.

1. Evaluate integral
$$\int_0^\infty \frac{\sin t}{t}\mathrm{d}t$$
2. Prove Euler's integral formula
$$\int_0^\infty x^n e^{-x}\mathrm{d}x=n!$$

Length

- Curve length:

$$\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

- Curve length function:

$$\ell \circ \gamma(t) = \int_a^t \|\gamma'(\tau)\| \mathrm{d}\tau$$

- Length parametrization:

$$\|\gamma'(t)\| = \frac{\mathrm{d}(\ell \circ \gamma)(t)}{\mathrm{d}t}$$

Note. This means that $\dfrac{\mathrm{d}s}{\mathrm{d}t} = \|\gamma'\|$.

Vectors

- Unit tangent vector, unit normal vector, binormal vector.

$$T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}, \quad N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}, \quad B = T \times N$$

- Relations of derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \|\gamma'(t)\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Curvature and Torsion

- Curvature in $\mathbb{R}^3$:

$$\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)|_{s=\ell\circ\gamma(t)} = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$$

Note. The curvature is calculated differently in length parametrization.

- Torsion in $\mathbb{R}^3$.

$$\tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{\|\gamma' \times \gamma''\|^2} = \frac{\det(\gamma', \gamma'', \gamma''')}{\|\gamma' \times \gamma''\|^2}$$

Tangent Line, Tangent Plane, Tangent Vectors

- Tangent line at t_0:

$$\gamma(t_0) : \{\gamma(t_0) + \gamma'(t_0)t, t \in \mathbb{R}\}$$

- Tangent line of $f : \mathbb{R}^n \to \mathbb{R}$ along h:

$$t_{f,x;h}(x) = \begin{pmatrix} x + sh \\ f(x) + D_h f|_x s \end{pmatrix}$$

- Tangent plane at $(x_0, f(x_0))$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

Tangent Line, Tangent Plane, Tangent Vectors

- Tangent vectors at $(x_0, f(x_0))$:

$$t_1 := \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x_0, y_0) \end{pmatrix}, \qquad t_2 := \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{pmatrix}$$

- Normal vectors at $(x_0, f(x_0))$:

$$n = t_1 \times t_2 = \begin{pmatrix} -\frac{\partial f}{\partial x}(x_0, y_0) \\ -\frac{\partial f}{\partial y}(x_0, y_0) \\ 1 \end{pmatrix}$$

- Best linear approximation of f at x_0:

$$Tf(\cdot; x_0) = f(x_0) + Df|_{x_0}(\cdot - x_0)$$

Directional Derivatives

- Directional derivative:

$$D_h f|_x = \frac{\mathrm{d}}{\mathrm{d}t} f(x + th)\Big|_{t=0}$$

For smooth functions,

$$D_h f|_x = Df|_x h = \langle \nabla f(x), h \rangle$$

Note. $\|h\| = 1$.

- Normal derivative:

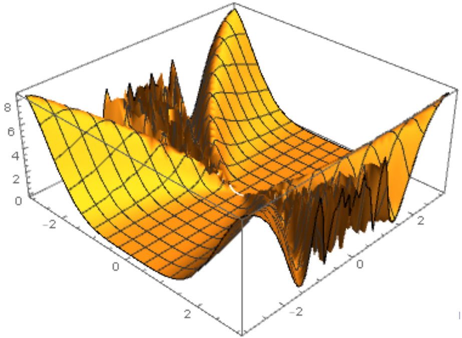
$$\frac{\partial f}{\partial n}\Big|_p = D_{N(p)} f|_p$$

Directional Derivatives

Note. The existence of directional derivatives does not guarantee differentiability.

e.g. Sample 2 Exercise 8. $f : \mathbb{R}^2 \to \mathbb{R}$, all directional derivatives exist, but f is not differentiable.

$$f(x,y) = \begin{cases} \left(1 - \cos \dfrac{x^2}{y}\right) \sqrt{x^2+y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$



Thanks for your attention!

Good Luck!