VV286 Honors Mathematics IV
Solution Manual for RC 5

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Example 1.

Show that if $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx.$$

This means that $e^{-\pi x^2}$ is its own Fourier transform (later). If $\xi = 0$, the formula is the know integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

Solution. Suppose that $\xi > 0$, consider the function $f(z) = e^{-\pi z^2}$, which is holomorphic in the interior of the toy contour γ_R shown in Figure 1.

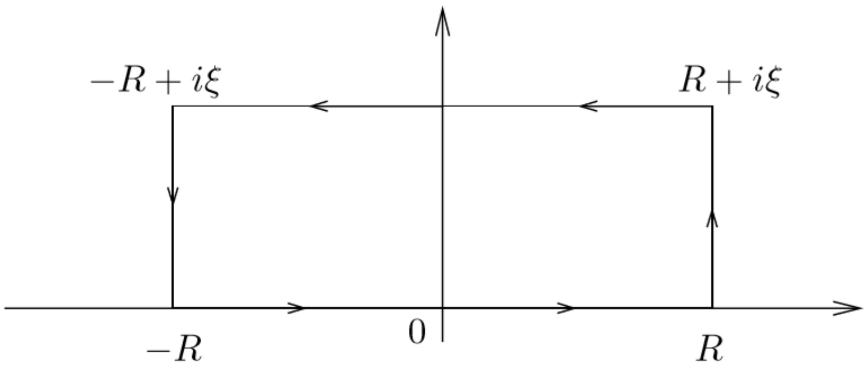

Figure 1: The contour γ_R in Example 1.

The contour γ_R consists of a rectangle with vertices $R, R+i\xi, -R+i\xi, -R$ and the positive counterclockwise orientation. By Cauchy's theorem,

$$\begin{aligned}\int_{\gamma_R} f(z)dz &= \int_{-R}^{R} e^{-\pi x^2} dx + I_r(R) - I_l(R) - \int_{-R}^{R} e^{-\pi(x+i\xi)^2} dx \\ &= \int_{-R}^{R} e^{-\pi x^2} dx + I_r(R) - I_l(R) - e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx \overset{!}{=} 0,\end{aligned}$$

where

$$\begin{aligned}I_r(R) &= \int_0^{\xi} f(R+iy)idy = \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)} idy \xrightarrow{R\to\infty} 0, \\ I_l(R) &= \int_0^{\xi} f(-R+iy)idy = \int_0^{\xi} e^{-\pi(R^2-2iRy-y^2)} idy \xrightarrow{R\to\infty} 0.\end{aligned}$$

Therefore,

$$\int_{\gamma_R} f(z)dz = 1 - e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx = 0,$$

giving

$$e^{-\pi\xi^2} = \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx.$$

Exercise 1.

Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}.$$

This is the complex version of the chain rule.

Solution. Suppose $f(x,y) = s(x,y) + it(x,y), g(s,t) = u(s,t) + iv(s,t)$, therefore,

$$\begin{aligned}
\frac{\partial g}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{1}{i}\frac{\partial}{\partial t}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial s} + i\frac{\partial v}{\partial s} + \frac{1}{i}\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}\right),\\
\frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)(s+it) = \frac{1}{2}\left(\frac{\partial s}{\partial x} + i\frac{\partial t}{\partial x} + \frac{1}{i}\frac{\partial s}{\partial y} + \frac{\partial t}{\partial y}\right),\\
\frac{\partial g}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial s} - \frac{1}{i}\frac{\partial}{\partial t}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial s} + i\frac{\partial v}{\partial s} - \frac{1}{i}\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right),\\
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)(s+it) = \frac{1}{2}\left(\frac{\partial s}{\partial x} + i\frac{\partial t}{\partial x} - \frac{1}{i}\frac{\partial s}{\partial y} - \frac{\partial t}{\partial y}\right).
\end{aligned}$$

and similar for their conjugates. Therefore,

$$\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{1}{i}\frac{\partial v}{\partial y}\right), \qquad \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{1}{i}\frac{\partial v}{\partial y}\right).$$

Exercise 2.

Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\ln z = \ln r + i\theta \quad \text{where } z = e^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.
Solution. Using chain rule and $x = r\cos\theta, y = r\sin\theta$, we have

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}, \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y},
\end{aligned}$$

giving

$$\frac{\partial u}{\partial x} = \frac{1}{r}\left(r\cos\theta\frac{\partial u}{\partial r} - \sin\theta\frac{\partial u}{\partial \theta}\right), \qquad \frac{\partial u}{\partial y} = \frac{1}{r}\left(r\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}\right),$$

and similarly for v. Then plugging in the Cauchy-Riemann equations, we obtain

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Exercise 3.

Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin(bx)dx$$

by integrating e^{-Az}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.
Solution. We integrate $f(z) = e^{-Az}$ around a circular sector of radius R with $0 \le \theta \le \omega$, where $\omega = \cos^{-1}(a/A)$ is strictly between 0 and $\dfrac{\pi}{2}$.

If $b = 0$, then the two integrals are trivially equal to $\dfrac{1}{a}$ and 0, respectively. If $b \neq 0$, then the integral along the x axis is

$$\int_0^R e^{-Ax}dx \xrightarrow{R\to\infty} \int_0^\infty e^{-Ax}dx = \frac{1}{A}.$$

Then for the sector part, using the inequality

$$\cos\theta \ge 1 - \frac{2\theta}{\pi}, \qquad \text{when } 0 \le \theta \le \frac{\pi}{2},$$

then we have

$$\begin{aligned}\left|\int_0^{\omega} e^{-ARe^{i\theta}} Re^{i\theta} d\theta\right| &\le \int_0^{\omega} \left|e^{-ARe^{i\theta}} Re^{i\theta}\right| d\theta \\ &= R\int_0^{\omega} e^{-AR\cos\theta} d\theta \\ &\le R\int_0^{\omega} e^{-AR} e^{2AR\theta/\pi} d\theta \\ &= Re^{-AR}\frac{\pi}{2AR} e^{2AR\theta/\pi}\Big|_0^{\omega} \\ &= \frac{\pi}{2A}\left(e^{-AR(1-2\omega/\pi)} - e^{-AR}\right) \xrightarrow{R\to\infty} 0.\end{aligned}$$

Finally, on the segment with $\theta = \omega$, $z = re^{i\omega} = r\dfrac{a+bi}{A}$, the integral is

$$\int_R^0 e^{-Ar(a+bi)/A}\frac{a+bi}{A}dr = \frac{a+bi}{A}\int_R^0 e^{-ar}e^{-ibr}dr.$$

Then using Cauchy's theorem and letting $R \to \infty$, we have

$$\frac{a+bi}{A}\int_{\infty}^0 e^{-ax}e^{-ibx}dx + \frac{1}{A} = 0 \quad \Rightarrow \quad \int_0^{\infty} e^{-ax}e^{ibx} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}.$$

Comparing real and imaginary parts, we have

$$\int_0^{\infty} e^{-ax}\cos(bx)dx = \frac{a}{a^2+b^2}, \qquad \int_0^{\infty} e^{-ax}\sin(bx)dx = \frac{b}{a^2+b^2}.$$

Exercise 4.

Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z)dz = 0.$$

Solution. Let γ_ε be a circle of radius ε centered at w, where ε is sufficiently small that γ_ε lies within the interior of T. Then use a toy contour formed by this circle and the triangle T. Since f is holomorphic in the region R between T and γ_ε,

$$\int_{\partial R} f(z)dz = \int_T f(z)dz - \int_{\gamma_\varepsilon} f(z)dz = 0 \quad \Rightarrow \quad \int_T f(z)dz = \int_{\gamma_\varepsilon} f(z)dz.$$

Since f is bounded near w and the length of γ_ε goes to 0 as $\varepsilon \to 0$. Therefore,

$$\int_{\gamma_\varepsilon} f(z)dz \to 0 \quad \Rightarrow \quad \int_T f(z)dz = 0.$$

Exercise 5.

If f is a holomorphic function on the strip $-1 < y < 1, x \in \mathbb{R}$ with

$$|f(z)| \leq A(1+|z|)^\eta$$

for all z in that strip, where η is a fixed real number. Show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta$$

for all $x \in \mathbb{R}$. [Hint: Cauchy inequality from Assignment 5.6.]

Solution. For any x, consider a circle C centered at x of radius $\frac{1}{2}$. Then applying Cauchy's inequalities to the circle, we have

$$|f^{(n)}(x)| \leq \frac{n!M}{(1/2)^n},$$

where $M = \sup_{z\in C}|f(z)|$. Then for $z \in C$,

$$1+|z| \leq 1+|x|+|z-x| = \frac{3}{2}+|x| < 2(1+|x),$$

giving

$$|f(z)| \leq A(1+|z|)^\eta \leq A\cdot 2^\eta(1+|x|)^\eta.$$

Therefore,

$$M \leq A\cdot 2^\eta(1+|x|)^\eta \quad \Rightarrow \quad |f^{(n)}(x)| \leq n!2^n A 2^\eta(1+|x|)^\eta = A_n(1+|x|)^\eta,\ A_n = n!2^{n+\eta}A.$$