Honors Mathematics IV
RC 6

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November 3, 2018

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The Complex Logarithm

Theorem. Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \ln_\Omega(z)$ so that

1. F is holomorphic in Ω.
2. $e^{F(z)} = z$ for all $z \in \Omega$.
3. $F(x) = \ln x$ whenever $x \in \mathbb{R}$ and near 1.

The Complex Logarithm

Proof. Define

$$\ln_\Omega(z) = F(z) = \int_{\mathcal{C}} f(w)dw, \qquad f(z) = \frac{1}{z},$$

where $\mathcal{C}$ is any curve in Ω connecting 1 to z. We show that

1. F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$.
2. $e^{F(z)} = z$ for all $z \in \Omega$:

$$\begin{aligned}\frac{d}{dz}\left(ze^{-F(z)}\right) &= e^{-F(z)} - zF'(z)e^{-F(z)} \\ &= (1 - zF'(z))e^{-F(z)} = 0.\end{aligned}$$

   Therefore, $ze^{-F(z)} = \mathrm{constant}$. Evaluating at 1 gives the constant 1, meaning $e^{F(z)} = z$.

The Complex Logarithm

Proof. Define

$$\ln_{\Omega}(z) = F(z) = \int_{\mathcal{C}} f(w)dw, \qquad f(z) = \frac{1}{z},$$

where $\mathcal{C}$ is any curve in Ω connecting 1 to z. We show that

3. Finally, if $x \in \mathbb{R}$ and close to 1,

$$F(x) = \int_1^x \frac{ds}{s} = \ln x.$$

The Complex Logarithm

Definition. On any simply connected set Ω and any simple curve joining 1 and z,

$$\ln z := \int_{\mathcal{C}} \frac{dz}{z}.$$

Let

$$\mathbb{R}_{-}^{0} := \{x \in \mathbb{R} : x \leq 0\}, \qquad \mathbb{R}_{+}^{0} := \{x \in \mathbb{R} : x \geq 0\}.$$

- Principle branch: $\ln : \mathbb{C} \setminus \mathbb{R}_{-}^{0} \to \mathbb{C}$.

$$\ln(re^{i\varphi}) = \ln r + \varphi i, \qquad r > 0, -\pi < \varphi < \pi.$$

- $\ln : C \setminus \mathbb{R}_{+}^{0} \to \mathbb{C}$.

$$\ln(re^{i\varphi}) = \ln r + \varphi i, \qquad r > 0, 0 < \varphi < 2\pi.$$

Note. This branch is not the analytic expansion of the logarithm in $\mathbb{R}$.

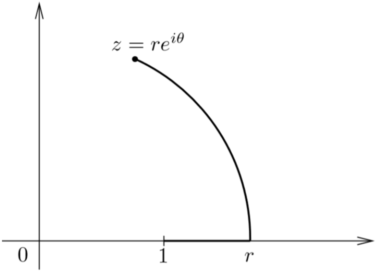
The Complex Logarithm

The principle branch. $\ln : \mathbb{C} \setminus \mathbb{R}_-^0 \to \mathbb{C}$.

$$\ln(re^{i\theta}) = \ln r + \theta i, \qquad r > 0, -\pi < \theta < \pi.$$

Proof. Using the path below, if $|\theta| < \pi$, then

$$\begin{aligned}\ln z &= \int_1^r \frac{dx}{x} + \int_\eta \frac{dw}{w} \\ &= \ln r + \int_0^\theta \frac{ire^{it}}{re^{it}} dt \\ &= \ln r + i\theta.\end{aligned}$$



Complex Power and Roots

- Complex power.

$$z^\alpha := e^{\alpha \ln z}, \qquad \alpha \in \mathbb{C}.$$

- Complex root.

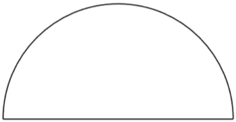
$$\sqrt[n]{z} := z^{1/n}.$$

Note. For $n \in \mathbb{N}$,

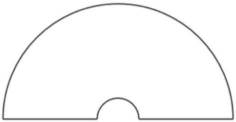
$$(z^{1/n})^n = \prod_{k=1}^{n} e^{\frac{1}{n}\ln z} = e^{\sum_{k=1}^{n} \frac{1}{n \ln z}} = e^{\frac{n}{n}\ln z} = e^{\ln z} = z.$$

Contours — Semi-circle

Semi-circle.



Semicircle



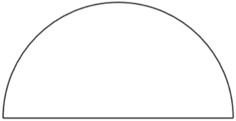
Indented semicircle

Integrals. We have used this contour to find

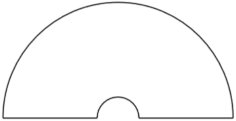
1. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$,
2. $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\,dx$,
3. $\displaystyle\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}\,dx$,

Contours — Semi-circle

Semi-circle.



Semicircle



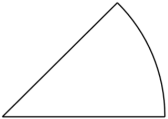
Indented semicircle

Integrals. We have used this contour to find

4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$,
5. $\displaystyle\int_{0}^{\infty} \frac{x \sin x}{(x^2+4)^2}\,dx$,
6. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$,
7. $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$.

Contours — Sector

Sector.



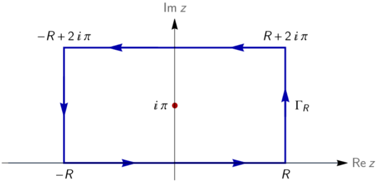
Sector

Integrals.

- Integral containing $\sin(x^n), \cos(x^n)$. (choose central angle $\pi/(2n)$.)
- We have used this contour to find
  1. $\displaystyle\int_0^\infty \sin x^2\,dx$,
  2. $\displaystyle\int_0^\infty \cos x^2\,dx$.

Contours — Rectangle

Rectangle.



Integrals.

1. $\displaystyle\int_0^\infty \frac{e^{ax}}{1+e^x}\,dx, 0 < a < 1.$

Contours — Rectangle

Example 1. Verify

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dz = \frac{1}{\cosh \pi\xi},$$

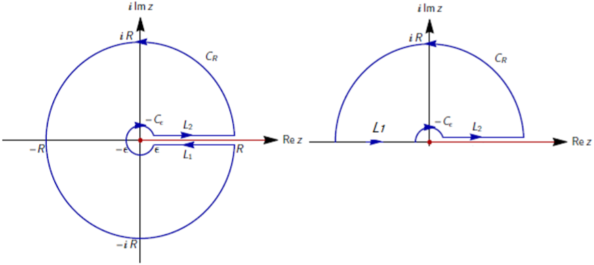
where

$$\cosh z = \frac{e^{z} + e^{-z}}{2}.$$

(This implies that $1/\cosh \pi x$ is its own Fourier transform.)

Contours — (Semi-)Circle without a Half-axis

Contours — (Semi-)Circle without a Half-axis.



Integrals. Integrals containing $x^{1/n}, \ln x$(with branch $\mathbb{C} \setminus \mathbb{R}^0_+$).

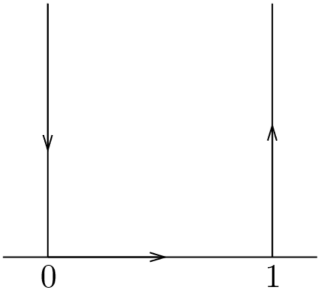
1. $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+a^2}\,dx,$
2. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx.$

Contours — (Semi-)Circle without a Half-axis

Example 2. Show that

$$\int_0^1 \ln(\sin \pi x) dx = -\ln 2$$

using the following contour.



The Heaviside Operator Method

Heaviside Operator Method. Treating the operator D as a number so that $D\{f\} = \{f'\}$.

$$
\begin{tikzcd}
f \arrow[r] \arrow[d, "\frac{d}{dt}"'] & \{f\} \arrow[d, "D"] \\
f' \arrow[r] & \{f'\} = D\{f\}
\end{tikzcd}
$$

The Heaviside Function and Delta Function

The Heaviside function.

$$H : \mathbb{R} \to \mathbb{R}, \quad H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

The delta function (not a function in mathematical sense).

- $t \neq 0$,

$$\delta(t) = 0.$$

- $0 \in I \subset \mathbb{R}$,

$$\int_I \delta(t) f(t) dt = f(0).$$

Definition

Definition. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$\sup_{t \in [0,\infty)} e^{-\beta t}|f(t)| < \infty \qquad \text{for some } \beta \geq 0.$$

Then the function $F : (\beta, \infty) \to \mathbb{R}$,

$$F(p) := (\mathcal{L}f)(p) := \int_0^\infty e^{-pt} f(t) dt$$

is called the ***(unilateral) Laplace transform*** of f.

The bilateral Laplace Transform.

$$(\tilde{L}f)(p) := \int_{-\infty}^\infty f(t) e^{-pt} dt, \quad \mathcal{L}f = \tilde{L}(Hf).$$

Derivatives

- First derivative.

$$(\mathcal{L}f')(p) = p(\mathcal{L}f)(p) - f(0).$$

- Second derivative.

$$(\mathcal{L}f'')(p) = p^2(\mathcal{L}f)(p) - pf(0) - f'(0).$$

- Higher-order derivatives.

$$(\mathcal{L}(f^{(n)}))(p) = p^n(\mathcal{L}f)(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0).$$

Table of Laplace Transform

$f(t)$	$(\mathscr{L}f)(p)$	Comment / Domain of $\mathscr{L}f$
1	$\dfrac{1}{p}$	$p > 0$
t^n	$\dfrac{n!}{p^{n+1}}$	$n \in \mathbb{N},\ p > 0$
e^{at}	$\dfrac{1}{p-a}$	$p > a$
$\sin(bt)$	$\dfrac{b}{p^2+b^2}$	$b \in \mathbb{R},\ p > 0$
$\cos(bt)$	$\dfrac{p}{p^2+b^2}$	$b \in \mathbb{R},\ p > 0$

Table of Laplace Transform

$f(t)$	$(\mathscr{L}f)(p)$	Comment
$H(t-a)$	e^{-ap}/p	$a, p > 0$
$g(t-a)H(t-a)$	$e^{-ap}(\mathscr{L}g)(p)$	$a > 0$
$e^{at}g(t)$	$(\mathscr{L}g)(p-a)$	$a \in \mathbb{R}$
$g(at)$	$\frac{1}{a}(\mathscr{L}g)\left(\frac{p}{a}\right)$	$a > 0$
$g^{(n)}(t)$	$p^n(\mathscr{L}g)(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$n \in \mathbb{N}$
$(-t)^n g(t)$	$(\mathscr{L}g)^{(n)}(p)$	$n \in \mathbb{N}$

Laplace Transform

Example 3. Find the inverse Laplace transform of the function

$$F(p) = \frac{2p^2 + 3}{(p^2 + 4)(p^2 + 1)}.$$

Laplace Transform

Example 3. Find the inverse Laplace transform of the function

$$F(p) = \frac{2p^2 + 3}{(p^2 + 4)(p^2 + 1)}.$$

Solution. The function can be converted to

$$F(p) = \frac{5}{3(p^2 + 4)} + \frac{1}{3(p^2 + 1)}.$$

Looking up the transform table, the inverse Laplace function is

$$f(t) = \frac{5}{6}\sin(2t) + \frac{1}{3}\sin(t).$$

The Bromwich Integral

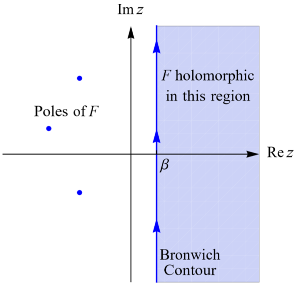
Definition. $\Omega \subset \mathbb{C}$ is an open set, $\beta \in \mathbb{R}$, $F : \Omega \to \mathbb{C}$ is analytic for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z \geq \beta$. Then the ***Bromwich integral*** of F is

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i} \int_{\mathcal{C}^*} e^{pt} F(p) dp,$$

where $\mathcal{C} = \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$ is the ***Bromwich contour***.

Often, the integral is written as

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{pt} F(p) dp.$$



Mellin Inversion Formula

2.6.8. Theorem. The Bromwich integral is the inverse of the Laplace transform. In particular, if f is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$ and has Laplace transform $\mathcal{L}f$, then

$$f(s) = [\mathcal{M}(\mathcal{L}f)](s) \qquad \text{for all } s \in [0, \infty).$$

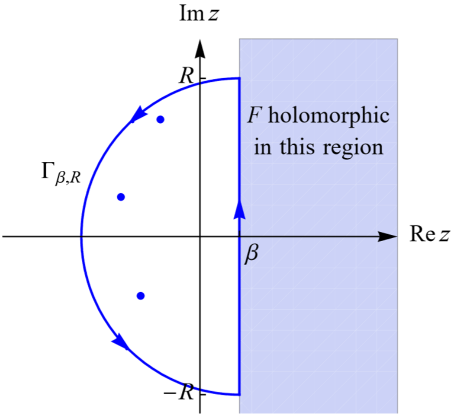
Evaluating the Bromwich Integral

1. Choose contour for $t > 0$ and $t < 0$.
2. Find poles and residue contained in the contour. Usually, the contour is chosen as a semi-circle oriented to the left or right.
3. Write out residue theorem.
4. Save the part for Bromwich integral and evaluate other parts (which usually goes to zero).

Evaluating the Bromwich Integral

- $t > 0$. Use contour

$$\gamma_{\beta,R}(s) = \beta + Re^{is}, \qquad \frac{\pi}{2} \leq s \leq \frac{3\pi}{2}.$$



Evaluating the Bromwich Integral

- $t > 0$. Then

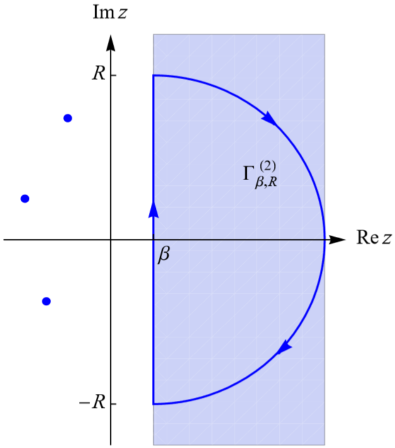
$$\begin{aligned}
&\int_{\Gamma_{\beta,R}} e^{pt} F(p)dp \\
=&\int_{\pi/2}^{3\pi/2} e^{t(\beta+R\exp(is))} F(\beta + Re^{is}) iRe^{is} ds \\
=&e^{\beta t} \int_0^{\pi} e^{tR\exp(is+i\pi/2)} F(\beta + Re^{i(s+\pi/2)}) iRe^{is+i\pi/2} ds \\
=&ie^{\beta t} \int_0^{\pi} e^{itR\exp(is)} F(\beta + iRe^{is}) iRe^{is} ds \\
=&ie^{\beta t} \int_{C_R} e^{itp} F(\beta + ip)dp \quad \xrightarrow{R\to\infty} 0,
\end{aligned}$$

where C_R is a semi-circle of radius R in the upper half-plane.

Evaluating the Bromwich Integral

- $t < 0$. Use contour

$$\gamma_{\beta,R}(s) = \beta + Re^{is}, \qquad -\frac{\pi}{2} \le s \le \frac{\pi}{2}.$$



Evaluating the Bromwich Integral

- $t < 0$. Then

$$
\begin{aligned}
&\int_{\Gamma^{(2)}_{\beta,R}} F(p)dp \\
&= -\int_{-\pi/2}^{\pi/2} e^{t\gamma_{\beta,R}(s)} F(\beta + Re^{is}) iRe^{is} ds \\
&= -\int_{0}^{\pi} e^{t\gamma_{\beta,R}(s-\pi/2)} F(\beta + Re^{i(s-\pi/2)}) iRe^{is} ds \\
&= -e^{\beta t} \int_{0}^{\pi} e^{-itR\exp(is)} F(\beta - iR^{is}) iRe^{is} ds \\
&= -e^{\beta s} \int_{C_R} e^{i|t|p} F(\beta - ip) dp \quad \xrightarrow{R\to\infty} 0,
\end{aligned}
$$

where C_R is a semi-circle of radius R in the upper half-plane.

Evaluating the Bromwich Integral

In sum, the Bromwich integral gives

- When $t < 0$,

$$f(t) = 0.$$

- When $t > 0$,

$$f(t) = \sum_{k=1}^{N} \mathrm{res}_{p_k} \left(e^{pt} F(p) \right),$$

where $F(p) \to 0$ as $|p| \to \infty$.

Evaluating the Bromwich Integral

Example 4. Find the inverse Laplace transform of

$$F(p) = p^{-1/2}$$

using Bromwich integral.

Convolution

Definition. The ***convolution*** of f and g is given by

$$(f * g)(t) := \int_0^t f(t-s)g(s)ds.$$

2.6.10. Theorem.

$$\mathcal{L}(f * g) = (\mathcal{L}f) \cdot (\mathcal{L}g).$$

A Green's Function for a 2nd Order Linear ODE

Initial Value Problem. The linear, second order, inhomogeneous ODE with constant coefficients is given by

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'.$$

1. Take the Laplace transform of both sides.

$$(ap^2 + bp + c)Y - (ap + b)y_0 - ay_1 = F(p).$$

2. Solve for Y.

$$Y(p) = \underbrace{\frac{(ap + b)y_0 + ay_1}{ap^2 + bp + c}}_{\Phi(p)} + \underbrace{\frac{F(p)}{ap^2 + bp + c}}_{\Psi(p)}.$$

A Green's Function for a 2^{nd} Order Linear ODE

Initial Value Problem. The linear, second order, inhomogeneous ODE with constant coefficients is given by

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'.$$

3. Find the inverse Laplace transform for Y.

$$\begin{aligned} y_{\text{part}}(t) &= \mathcal{L}^{-1}(\Psi)(t) \\ &= \mathcal{L}^{-1}\left(\frac{F(p)}{ap^2 + bp + c}\right)(t) = f * g(t). \end{aligned}$$

where

$$\mathcal{L}g(p) = \frac{1}{ap^2 + bp + c}.$$

Then $y(t) = y_{\text{hom}}(t) + y_{\text{part}}(t)$.

Applying the Laplace Transform

1. Apply the Laplace transform to both sides of the ODE/IVP.
2. The transformed equation is algebraic; solve for the Laplace transform of the unknown function.
3. Find the inverse Laplace transform of the unknown function by looking up the transform table.

Applying the Laplace Transform

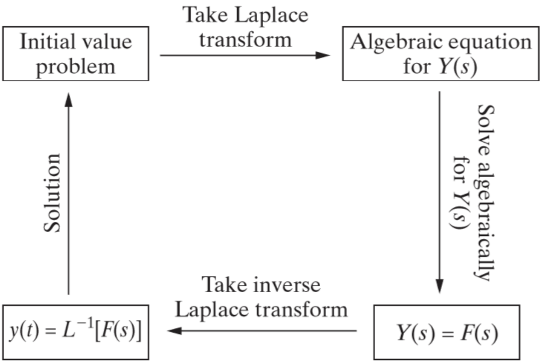
Example 5. Solve the initial value problem

$$y'' + \omega^2 y = f(t), \qquad y(0) = \alpha, \qquad y'(0) = \beta,$$

where α, β and ω are constants with $\omega \neq 0$ and f is an arbitrary function in $(0, \infty)$.

Applying the Laplace Transform

Laplace Transform for IVP. The steps for applying Laplace Transform to solve initial value problems can be illustrated in the following graph.



Thanks for your attention!