VV286 Honors Mathematics IV
Solution Manual for RC 1

Chen Xiwen

September 25, 2018

University of Michigan–Shanghai Jiao Tong University Joint Institute

Exercise 1. (Omitted in RC.)

Find the equation of the orthogonal trajectories to the family

$$x^2 + y^2 - 2cx = 0. \tag{1}$$

(Completing the square in x, we obtain $(x - c)^2 + y^2 = c^2$, which represents the family of circles centered at $(c, 0)$ with radius c.)

Solution. First we need an expression for the slope of the given family at the point (x, y). Differentiating Equation (1) implicitly with respect to x yields

$$2x + 2y\frac{dy}{dx} - 2c = 0,$$

which simplifies to

$$\frac{dy}{dx} = \frac{c - x}{y}. \tag{2}$$

This is not the differential equation of the given family, since it still contains the constant c, and hence is dependent on the individual curves in the family. Therefore, we must eliminate c to obtain an expression for the slope of the family that is independent of any particular curve in the family. From Equation (1) we have

$$c = \frac{x^2 + y^2}{2x}.$$

Substituting this expression for c into Equation (2) and simplifying gives

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

Therefore, the differential equation for the family of orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{2xy}{y^2 - x^2}. \tag{3}$$

This differential equation is first-order and homogeneous. Substituting $y = xu(x)$ into Equation (3) yields

$$\frac{d}{dx}(xu) = \frac{2u}{1 - u^2}.$$

so that

$$x\frac{du}{dx} + u = \frac{2u}{1 - u^2}.$$

Hence

$$x\frac{du}{dx} = \frac{u+u^3}{1-u^2},$$

or in separated form,

$$\frac{1-u^2}{u(1+u^2)}du = \frac{1}{x}dx.$$

Integrating both sides, we have

$$\ln|u| - \ln(1+u^2) = \ln|x| + c_1,$$

or equivalently,

$$\ln\left(\frac{|u|}{1+u^2}\right) + \ln|x| + c_1,$$

where c_1 is a constant.

Substituting back and exponentiating both sides, we obtain

$$\frac{xy}{x^2+y^2} = c_2 x \quad \Leftrightarrow \quad x^2+y^2 = c_3 y.$$

Namely,

$$x^2 + (y-k)^2 = k^2, \tag{4}$$

where $k = c_3/2$. Equation (4) is the equation of the family of orthogonal trajectories. This is the family of circles centered at $(0, k)$ with radius k (circles along the y-axis). The trajectories are shown in Figure 1.

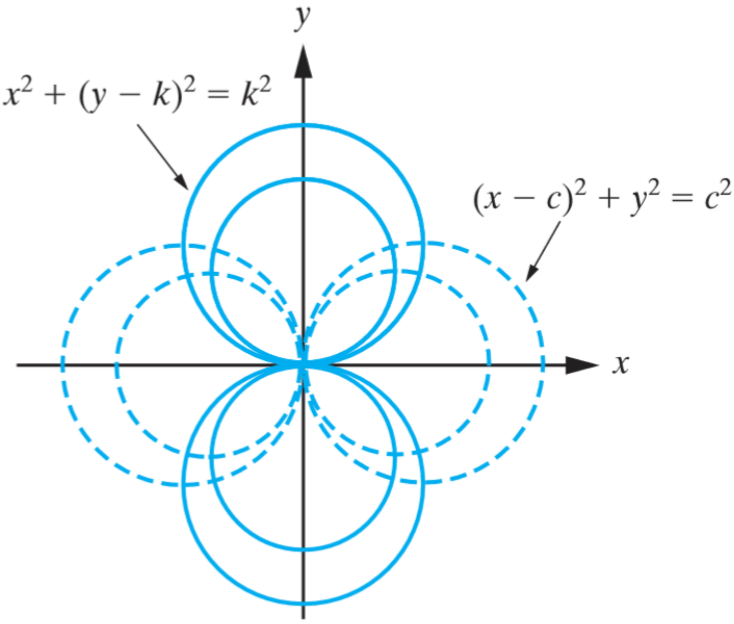

Figure 1: The family $(x-c)^2 + y^2 = c^2$ and its orthogonal trajectories $x^2 + (y-k)^2 = k^2$.

Exercise 2.

Solve the IVP

$$y' - y = f(x), \qquad y(0) = 0,$$

where

$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ 2 - x & \text{if } x \geq 1. \end{cases}$$

Solution 1 — Duhamel's Principle. We identify that

$$a_1(x) = 1, \qquad a_0(x) = -1$$

and $f(x)$ is given above. Then the solution to the IVP is given by

$$\begin{aligned} y(x) &= e^{\int_0^x 1dt} \cdot \int_0^x e^{\int_0^s 1dt} \cdot f(s)ds \\ &= e^x \cdot \int_0^x e^{-s} f(s)ds \end{aligned}$$

We discuss the case for $x < 1$ and $x \geq 1$.

- $\underline{x < 1}$.

$$\begin{aligned} y(x) &= e^x \cdot \int_0^x e^s f(s) ds \\ &= e^x \cdot \int_0^x e^{-s} ds \\ &= e^x - 1. \end{aligned}$$

- $\underline{x \geq 1}$.

$$\begin{aligned} y(x) &= e^x \cdot \int_0^x e^{-s} f(s) ds \\ &= e^x \cdot \left(\int_0^1 e^{-s} ds + \int_1^x e^{-s}(2 - s) ds \right) \\ &= e^x \cdot (-e^{-1} + 1 + (x - 1)e^{-x}) \\ &= (1 - e^{-1})e^x + x - 1. \end{aligned}$$

Therefore, the solution to the IVP is given by

$$y(x) = \begin{cases} e^x - 1 & \text{if } x < 1, \\ (1 - e^{-1})e^x + x - 1 & \text{if } x \geq 1. \end{cases}$$

Solution 2 — Integrating Factor. We identify that in this IVP,

$$h(x, y) = 1, \qquad g(x, y) = -y - f(x).$$

Noticing

$$\frac{g_y - h_x}{h} = -1$$

does not depend on y, an integrating factor can be found by

$$M = e^{-x}.$$

Then

1. Find the potential $U(x, y)$ of the vector field

$$F(x, y) = \begin{pmatrix} -e^{-x}(y + f(x)) \\ e^{-x} \end{pmatrix}.$$

   We discuss the following cases:

   - $\underline{x < 1}$.

$$U_x = -e^{-x}(y + 1), \qquad U_y = e^{-x},$$

giving

$$U(x,y) = (y+1)e^{-x} = c$$

is the solution to the equation. Plugging in the initial condition, we obtain

$$y(x) = e^x - 1.$$

- $\underline{x \geq 1}$.

$$U_x = -e^{-x}(y - x + 2), \qquad U_y = e^{-x},$$

giving

$$U(x,y) = ye^{-x} + (1-x)e^{-x} = c'$$

is the solution to the equation. Then

$$y(x) = c'e^x + x - 1.$$

We aim at deriving a continuous solution to the original IVP. Therefore,

$$c' = 1 - e^{-1}$$

2. Then the overall solution to the IVP is

$$y(x) = \begin{cases} e^x - 1 & \text{if } x < 1, \\ (1-e^{-1})e^x + x - 1 & \text{if } x \geq 1. \end{cases}$$

Note. Differentiating both branches of the solution, we have

$$y'(x) = \begin{cases} e^x & \text{if } x < 1, \\ (1-e^{-1})e^x + 1 & \text{if } x \geq 1. \end{cases} \quad \text{and} \quad y''(x) = \begin{cases} e^x & \text{if } x < 1, \\ (1-e^{-1})e^x & \text{if } x \geq 1. \end{cases}$$

Then

1. Even though the function f is not differentiable at $x = 1$, the solution to the initial-value problem has a continuous derivative at that point.
2. The discontinuity of f does has an effect on the second derivative of the solution.

Exercise 3.

Solve the IVP

$$y' + \frac{y}{x} - \sqrt{y} = 0, \qquad y(1) = 0.$$

Solution. This is a Bernoulli's equation with

$$g(x) = \frac{1}{x}, \quad h(x) = -1, \quad \alpha = \frac{1}{2}$$

and initial condition $y(1) = 0$. Here $\alpha > 0$ and we have the trivial solution $y = 0$.

1. Multiply with $\frac{1}{2}y^{-1/2}$ (when $y \neq 0$ in a neighborhood of 1):

$$\frac{1}{2}y^{-1/2}y' + \frac{1}{2}\frac{y^{1/2}}{x} - \frac{1}{2} = 0.$$

2. Define

$$u = y^{1/2} \quad \Rightarrow \quad u' = \frac{1}{2}y^{-1/2}y'.$$

3. Then the equation is transformed to

$$u' + \frac{1}{2x}u = \frac{1}{2},$$

which is a linear equation of u with initial condition $u(1) = 0$. We can solve the equation by

$$\begin{aligned} u(x) &= e^{-\int_1^x \frac{1}{2t}dt} \cdot \int_1^x e^{\int_1^s \frac{1}{2t}dt} \cdot \frac{1}{2}ds \\ &= \frac{1}{3}x - \frac{1}{3}x^{-1/2}. \end{aligned}$$

4. Find $y(x)$ by

$$y(x) = \left(\frac{1}{3}x - \frac{1}{3}x^{-1/2}\right)^2.$$