Honors Mathematics IV
RC 7

CHEN Xiwen

UM-SJTU Joint Institute

November 18, 2018

Table of contents

Transforms

The Fourier Transform

Exercises

The Fourier Transform

Definition. The ***Fourier transform*** of a function $f : \mathbb{R} \to \mathbb{C}$ is given by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx.$$

$\widehat{f}$ exists for all $\xi \in \mathbb{R}$ if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Derivatives

1. $\widehat{f'}(\xi) = i\xi \cdot \widehat{f}(\xi)$.
2. $\dfrac{d}{d\xi}\widehat{f}(\xi) = \widehat{(-ix)f}(\xi)$.

Properties of Fourier Transform

Properties of Fourier transform for continuous functions .

$f(t)$	$\widehat{f}(\xi) = F(\xi)$
$f(at+b), a \neq 0$	$\frac{1}{\lvert a \rvert} e^{i\xi b/a} F\left(\frac{\xi}{a}\right)$
$f(t - t_0)$	$F(\xi)e^{-i\xi t_0}$
$f(-t)$	$F(-\xi)$
$f(at), a \neq 0$	$\frac{1}{\lvert a \rvert} F\left(\frac{\xi}{a}\right)$
$f_1(t) * f_2(t)$	$\sqrt{2\pi} F_1(\xi) \cdot F_2(\xi)$

Properties of Fourier Transform

Properties of Fourier transform for continuous functions (continued).

$f(t)$	$\widehat{f}(\xi) = F(\xi)$
$f_1(t) \cdot f_2(t)$	$\frac{1}{\sqrt{2\pi}} F_1(\xi) * F_2(\xi)$
$f(t)e^{i\xi_0 t}$	$F(\xi - \xi_0)$
$f(t)\cos(\xi_0 t)$	$\frac{F(\xi - \xi_0) + F(\xi + \xi_0)}{2}$
$\frac{d^n}{dt^n} f(t)$	$(i\xi)^n F(\xi)$
$(-it)^n f(t)$	$\frac{d^n}{d\xi^n} F(\xi)$

Common Fourier Transform Pairs

Table of Fourier transform pairs.

$f(t)$	$\sqrt{2\pi} \cdot \widehat{f}(\xi)$
$\frac{1}{b^2+t^2}$	$\frac{\pi}{b} e^{-b\lvert\xi\rvert}$
$e^{-b\lvert t\rvert}$	$\frac{2b}{b^2+\xi^2}$
$e^{i\xi_0 t}$	$2\pi\delta(\xi-\xi_0)$
e^{-bt^2}	$\sqrt{\pi/b}e^{-\xi^2/(4b)}$

Common Fourier Transform Pairs

Table of Fourier transform pairs (continued).

$f(t)$	$\sqrt{2\pi} \cdot \widehat{f}(\xi)$
$\delta(t)$	1
1	$\sqrt{2\pi}\delta(\xi)$
$\cos(\xi_0 t)$	$\pi\delta(\xi - \xi_0) + \pi\delta(\xi + \xi_0)$
$\sin(\xi_0 t)$	$\frac{\pi}{i}\delta(\xi - \xi_0) - \frac{\pi}{i}\delta(\xi + \xi_0)$

Fourier Transform

Example 1. Calculate the Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 1/(1+x^2)^2$.

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{(1+x^2)^2} dx.$$

Decay Behavior at Infinity

Definition. Let $\Omega \subset \mathbb{R}$ be bounded and $f : \mathbb{R} \setminus \Omega \to \mathbb{C}$.

- ***Polynomial decay.***

$$f(x) = O(x^{-n}) \quad \text{as } |x| \to \infty \text{ for some } n > 0.$$

- ***Faster-than-polynomial decay.***

$$f(x) = O(x^{-n}) \quad \text{as } |x| \to \infty \text{ for all } n > 0.$$

- ***Exponential decay.***

$$f(x) = O(e^{-b|x|}) \quad \text{as } |x| \to \infty \text{ for some } b > 0.$$

Analytic Theory of the Fourier Transform

2.7.4. Definition. Let $a > 0$ be some constant. The set $\mathcal{F}_a$ of functions $f : S_a \to \mathbb{C}$ where $S_a = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < a\}$ such that

1. f is analytic on S_A and
2. there exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < a.$$

2.7.6. Theorem. Let $f \in \mathcal{F}_a$ for some $a > 0$. Then for any $0 \leq b < a$ there exists a constant $B > 0$ such that

$$|\widehat{f}(\xi)| \leq Be^{-b|\xi|} \qquad \text{for all } \xi \in \mathbb{R}.$$

Thus, <u>complex analyticity</u> of f implies <u>exponential decay</u> of $\widehat{f}$.

Fourier Inversion Theorem

Definition.

$$\mathcal{F} = \{f : \mathbb{C} \to \mathbb{C}, \exists a > 0, f \in \mathcal{F}_a\}.$$

2.7.8. Fourier Inversion Theorem. If $f \in \mathcal{F}$, then $\widehat{f}$ exists and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi \qquad \text{for all } x \in \mathbb{R}.$$

The Complex Fourier Transform

Definition. The Fourier transform of $f : \mathbb{C} \to \mathbb{C}$ at $\xi + i\eta \in \mathbb{C}$ if given by

$$\widehat{f}(\xi + i\eta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(\xi+i\eta)x} dx,$$

where in the integral f is evaluated on the real axis only.

The existence of the complex Fourier transform. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) = O(e^{-b|x|})$ as $|x| \to \infty$ for some $b > 0$. Then $\widehat{f}$ exists and is analytic in the strip

$$S_b = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < b\}.$$

The Fourier Inversion Formula

2.7.10. Fourier inversion theorem. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is absolutely integrable and satisfies the following condition: there exists a set of numbers $\{a_1, \ldots, a_n\}, n \in \mathbb{N}$, such that f is continuously differentiable on the intervals (a_k, a_{k+1}), where we set $a_0 = \infty, a_{n+1} = \infty$, and such that the one-sided limits of f and f' at $a_1, \ldots, a_n$ exists. Then $\widehat{f}$ exists and for all $x \in \mathbb{R}$,

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi} d\xi,$$

where the limit on the right exists for all $x \in \mathbb{R}$ and

$$f(x^+) := \lim_{y \searrow x} f(y), \qquad f(x^-) := \lim_{y \nearrow x} f(y).$$

Exercises

Exercise 1. Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}.$$

Thanks for your attention!