VV286 Honors Mathematics IV
Solution Manual for RC 3

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Example 1.

Find the generalized eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} .$$

Solution. The characteristic polynomial is given by

$$p(\lambda) = (3 - \lambda)(1 - \lambda)^2,$$

giving eigenvalues $\lambda_1 = 3, \lambda_2 = 1$ (with multiplicity 2). The we calculate the eigenvectors for each eigenvalue.

1. $\underline{\lambda_1 = 3}$. We solve the linear system $(A - \lambda_1 \mathbb{1})v = 0$, namely

$$\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

   and find $v_1 = (1, 2, 2)$.

2. $\underline{\lambda_2 = 1}$. We solve the linear system $(A - \lambda_2 \mathbb{1})v = 0$, namely

$$\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}$$

   and find $v_2 = (1, 0, 0)$.

For $\lambda_2 = 1$, the geometric multiplicity is less than its algebraic multiplicity. Therefore, we need to find the generalized eigenvectors for λ_2.

- Bottom-up method. We solve the system $(A - \lambda_2 \mathbb{1})v_3 = v_2$, namely

$$\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}$$

  and find $v_3 = (0, 1, 0)$.

- Top-down method. The highest rank for the generalized eigenvector is

$$m = a_{\lambda_2} - \dim V_{\lambda_2} + 1 = 2.$$

So we solve the system $(A - \lambda_2 \mathbb{1})^2 v_3 = 0$ with the condition $(A - \lambda_2 \mathbb{1})v_3 \neq 0$, namely

$$\left.\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right. , \qquad (A - \lambda_2 \mathbb{1})v_3 \neq 0$$

and we can set $v_3 = (0, 1, 0)$ so that v_2 is given by

$$v_2 = (A - \lambda_2 \mathbb{1})v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

Example 2.

Write out a Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 0 & 0 & 4 & 0 & 0 \\ 0 & 7 & 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Solution. The eigenvalue of A is 7 with algebraic multiplicity 6. From calculations, the dimension of the eigenspace is 3. Thus, we will have three possibilities.

1. One 4×4 block and two 1×1 blocks.
2. One 3×3 block, one 2×2 block and one 1×1 block.
3. Three 2×2 blocks.

Observing $(A - 7\mathbb{1})^2 = 0$, it is impossible to build a cycle of generalized eigenvectors of length greater than 2. Therefore, we can only have the Jordan matrix as

$$J = \begin{pmatrix} 7 & 1 & & & & \\ 0 & 7 & & & & \\ & & 7 & 1 & & \\ & & 0 & 7 & & \\ & & & & 7 & 1 \\ & & & & 0 & 7 \end{pmatrix}$$

Example 3.

Solve the system

$$\begin{aligned} x_1' &= 9x_1 + 6x_2 \\ x_2' &= -10x_1 - 7x_2 \end{aligned}$$

for $x(t)$.

Solution. The linear system is given by

$$x' = Ax, \qquad A = \begin{pmatrix} 9 & 6 \\ -10 & -7 \end{pmatrix} .$$

The eigenvalues are given by calculating

$$\det \begin{pmatrix} 9-\lambda & 6 \\ -10 & -7-\lambda \end{pmatrix} = (\lambda+1)(\lambda-3) = 0 \quad \Rightarrow \quad \lambda_1 = 3, \lambda_2 = -1$$

with eigenvectors $v_1 = (-1, 1), v_2 = (-3, 5)$. Thus

$$U = \begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix} , \qquad U^{-1} = -\frac{1}{2} \begin{pmatrix} 5 & 3 \\ -1 & -1 \end{pmatrix} , \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then the solution is given by

$$\begin{aligned} x = e^{At}x_0 &= Ue^D U^{-1} \\ &= -\frac{1}{2} \begin{pmatrix} -1 & -3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -1 & -1 \end{pmatrix} x_0 \\ &= -\frac{1}{2} \begin{pmatrix} -5e^{3t} + 3e^{-t} & -3e^{3t} + 3e^{-t} \\ 5e^{3t} - 5e^{-t} & 3e^{3t} - 5e^{-t} \end{pmatrix} x_0 \end{aligned}$$

Therefore, the general solution is given by

$$x_1(t) = -c_1e^{3t} + 3c_2e^{-t}, \quad x_2(t) = c_1e^{3t} - 5c_2e^{-t}.$$

Example 4.

Solve the linear system

$$\begin{aligned} x_1' &= -9x_1 + 9x_2 \\ x_2' &= -16x_1 + 15x_2 \end{aligned}$$

for $x(t)$.
Solution. The linear system is given by

$$x' = Ax, \qquad A = \begin{pmatrix} -9 & 9 \\ -16 & 15 \end{pmatrix} .$$

The eigenvalues are given by calculating

$$\det \begin{pmatrix} -9-\lambda & 9 \\ -16 & 15-\lambda \end{pmatrix} = (\lambda-3)^2 = 0 \quad \Rightarrow \quad \lambda = 3$$

with eigenvector $v_1 = (9, 12)$. To find a generalized eigenvector, we can solve

$$(A - \lambda \mathbb{1})^2 v_2 = 0, \qquad (A - \lambda \mathbb{1}) v_2 \neq 0.$$

Since $(A - \lambda \mathbb{1})^2 = 0$, we can set $v_2 = (0, 1)$. Thus

$$U = \begin{pmatrix} 9 & 0 \\ 12 & 1 \end{pmatrix} , \qquad U^{-1} = \frac{1}{9} \begin{pmatrix} 1 & 0 \\ -12 & 9 \end{pmatrix} , \qquad J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} .$$

Then the solution is given by

$$\begin{aligned} x &= e^{At} x_0 = U e^J \cdot e^N U^{-1} \\ &= \frac{1}{9} \begin{pmatrix} 9 & 0 \\ 12 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -12 & 9 \end{pmatrix} x_0 \\ &= \begin{pmatrix} e^{3t} - 12te^{3t} & 9te^{3t} \\ -16te^{3t} & 12te^{3t} + e^{3t} \end{pmatrix} x_0 \end{aligned}$$

Therefore, the general solution is given by

$$x_1(t) = c_1(e^{3t} - 12te^{3t}) + 9c_1te^{3t}, \quad x_2(t) = -16c_1te^{3t} + c_2(12te^{3t} + e^{3t}).$$