VV286 Honors Mathematics IV
Solution Manual for RC 6

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Example 1.

Verify

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dz = \frac{1}{\cosh \pi\xi},$$

where

$$\cosh z = \frac{e^z + e^{-z}}{2}.$$

(This implies that $1/\cosh \pi x$ is its own Fourier transform.)
Solution. For a fixed $\xi \in \mathbb{R}$, let

$$f(z) = \frac{e^{-2\pi i z\xi}}{\cosh \pi z}.$$

We use the contour shown in Figure 1. Letting $e^{\pi z} = -e^{-\pi z}$, the poles are given by

$$\alpha = \frac{i}{2}, \qquad \beta = \frac{3i}{2}.$$

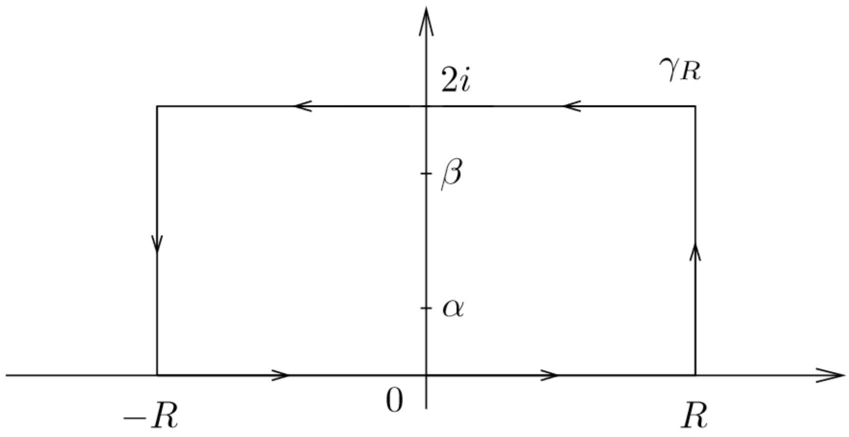

Figure 1: The contour γ_R in Example 1.

Then we find the residue of f at α and β by

$$\begin{aligned}
\mathrm{res}_\alpha f &= \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha} e^{-2\pi i z\xi}\frac{2(z-\alpha)}{e^{\pi z}+e^{-\pi z}} \\
&= 2e^{-2\pi i\alpha\xi}e^{\pi\alpha}\frac{1}{2\pi e^{2\pi\alpha}} = \frac{e^{\pi\xi}}{\pi i}, \\
\mathrm{res}_\beta f &= \lim_{z\to\beta}(z-\beta)f(z) = -\frac{e^{3\pi\xi}}{\pi i}.
\end{aligned}$$

Then for the vertical side on the right, we have

$$I_r = \int_0^{2i} \frac{e^{-2\pi i\xi(R+ix)}}{\cosh(\pi(R+xi))} \cdot i dx,$$

where

$$|e^{-2\pi i\xi(R+ix)}| = e^{2\pi|\xi|x} \leq e^{4\pi|\xi|},$$

and

$$\begin{aligned}|\cos(\pi(R+xi))| &= \left|\frac{e^{\pi z}+e^{-\pi z}}{2}\right| \\ &\geq \frac{1}{2}\left||e^{\pi z}-|e^{-\pi z}|\right| \\ &\geq \frac{1}{2}\left(e^{\pi R}-e^{-\pi R}\right) \xrightarrow{R\to\infty} \infty.\end{aligned}$$

Thus the integral vanishes as $R \to \infty$ and similarly for the left integral. Using residue theorem. we obtain

$$I - e^{4\pi\xi}I = 2\pi i\left(\frac{e^{\pi\xi}}{\pi i} - \frac{e^{3\pi i}}{\pi i}\right) = -2e^{2\pi\xi}\left(e^{\pi\xi}-e^{-\pi\xi}\right),$$

and

$$I = 2\cdot\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{2\pi\xi}-e^{-2\pi\xi}} = \frac{1}{\cosh(\pi\xi)}.$$

Example 2.

Show that

$$\int_0^1 \ln(\sin\pi x)dx = -\ln 2.$$

Solution. Choosing the principle branch $0 < \theta < 2\pi$, we have

$$\begin{aligned}\ln(\sin\pi x) &= \ln\left(\frac{e^{i\pi x}-e^{-\imath\pi x}}{2i}\right) \\ &= \ln(e^{2i\pi x}-1) - \ln\left(2e^{i(\pi x+\pi/2)}\right) \\ &= \ln(e^{2i\pi x}-1) - \ln 2 - i\pi x - i\frac{\pi}{2}.\end{aligned}$$

We use the contour shown in Figure 2.

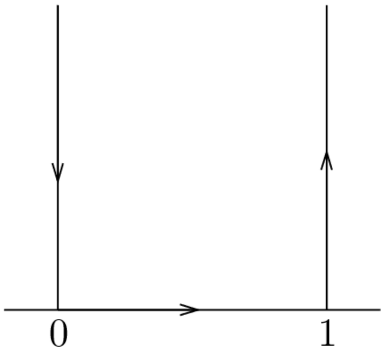

Figure 2: The contour in Example 2.

Let

$$f(z) = \ln\left(e^{2i\pi z} - 1\right),$$

and set the upper bound for the integral as iR. Then from Cauchy's theorem, the integral is given by

$$0 = \int_0^1 \ln(e^{2i\pi x} - 1)dx + \int_0^R i\ln\left(e^{2\pi i - 2\pi r} - 1\right) dr$$
$$- \int_0^1 \ln(e^{2i\pi(x+iR)} - 1)dx - \int_0^R i\ln(e^{-2\pi r} - 1)dr.$$

Since as $R \to \infty$,

$$\int_0^1 \ln(e^{2i\pi(x+iR)} - 1)dx \to \pi i.$$

Therefore,

$$\int_0^1 \ln(e^{2i\pi x} - 1)dx = \pi i,$$

and

$$\int_0^1 \ln\left(\sin(\pi x)\right) dx = \pi i - \int_0^1 \ln 2 + i\pi x + i\frac{\pi}{2}dx = -\ln 2.$$

Example 4.

Find the inverse Laplace transform of

$$F(p) = p^{-1/2}$$

using Bromwich integral.

Solution. Using Bromwich integral, we need to evaluate

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} p^{-1/2} e^{pt} dp.$$

When $t < 0$, we close the contour to the right and obtain $f(t) = 0$. When $t > 0$, we choose branch $-\pi < \theta < \pi$ and the contour shown in Figure 3, the contour of the larger circle is γ_R and the contour of the smaller circle is γ_ε.

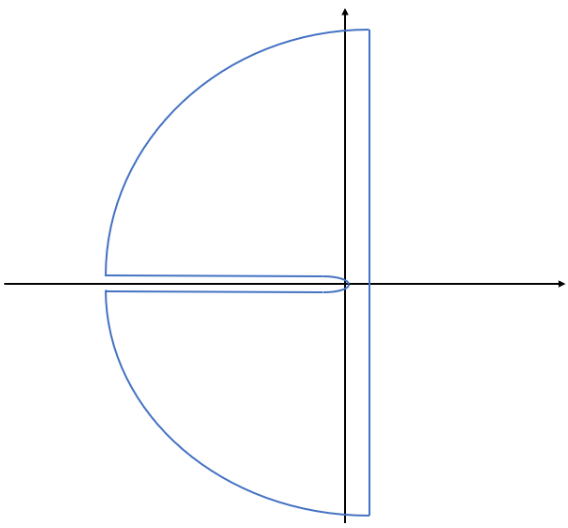

Figure 3: The contour in Example 3.

On the smaller circle,

$$I_{\gamma_\varepsilon} = \int_{-\pi/2}^{\pi/2} \varepsilon^{-1/2} e^{-i\theta/2} e^{\varepsilon \exp(i\theta) t} i\varepsilon e^{i\theta} d\theta = \mathcal{O}(\varepsilon^{1/2}) \xrightarrow{\varepsilon \to 0} 0.$$

Similarly, on the large circle,

$$\begin{aligned} I_{\gamma_R} &= \int_{\pi/2}^{\pi} (\beta + Re^{i\theta})^{-1/2} e^{(\beta + Re^{i\theta})t} iRe^{i\theta} d\theta + \int_{-\pi}^{-\pi/2} (\beta + Re^{i\theta})^{-1/2} e^{(\beta + Re^{i\theta})t} iRe^{i\theta} d\theta \\ &\leq \int_{\pi/2}^{\pi} \frac{R}{\sqrt{R - \beta}} e^{\beta t + Rt\cos\theta} d\theta + \int_{-\pi}^{-\pi/2} \frac{R}{\sqrt{R - \beta}} e^{\beta t + Rt\cos\theta} d\theta \xrightarrow{R \to \infty} 0, \end{aligned}$$

since $\cos\theta \leq -\frac{2}{\pi}\left(\theta - \frac{\pi}{2}\right)$ and $t > 0$ (for the first integral). Then the required integral is equal to the sum of integrals on both sides of the branch cut. Namely, for $p = re^{i\pi}$ and $p = re^{-i\pi}$,

$$\begin{aligned} f(t) &= -\frac{1}{2\pi i}\left[\int_{\infty}^{0} r^{-1/2}e^{-i\pi/2}e^{-rt}\cdot(-1)dr - \int_{\infty}^{0} r^{-1/2}e^{i\pi/2}e^{-rt}\cdot(-1)dr\right] \\ &= \frac{2}{\pi}\int_0^{\infty} e^{-s^2 t}ds \\ &= \frac{1}{\sqrt{\pi t}}. \end{aligned}$$

where we substitute $r = s^2$. Therefore, the inverse Laplace transform is given by

$$(\mathcal{L}^{-1}F)(t) = \frac{1}{\sqrt{\pi t}}.$$

<u>Note</u>. This is a generalization of the result that

$$(\mathcal{L}(\cdot)^n)(p) = \frac{n!}{p^{n+1}}$$

to

$$(\mathcal{L}(\cdot)^{\alpha})(p) = \frac{\Gamma(\alpha+1)}{p^{\alpha+1}},$$

where the *Gamma function* is defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx.$$

Example 5.

Solve the initial value problem

$$y'' + \omega^2 y = f(t), \qquad y(0) = \alpha, \qquad y'(0) = \beta,$$

where α, β and ω are constants with $\omega \neq 0$ and f is an arbitrary function in $(0, \infty)$.
Solution. Taking the Laplace transform of the differential equation and imposing the initial conditions yields

$$s^2 Y(s) - \alpha s - \beta + \omega^2 Y(s) = F(s),$$

where $F(s)$ is the Laplace transform of f. Simplifying, we obtain

$$Y(s) = \frac{F(s)}{s^2+\omega^2} + \frac{\alpha s}{s^2+\omega^2} + \frac{\beta}{s^2+\omega^2}.$$

Taking the inverse Laplace transform of both sides and using convolution theorem, we obtain

$$y(t) = \frac{1}{\omega} \int_0^t \sin(\omega(t-\tau)) f(\tau) d\tau + \alpha \cos(\omega t) + \frac{\beta}{\omega} \sin(\omega t).$$