Honors Mathematics IV
RC 4

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Homogeneous System

1.10.1. Lemma. For the homogeneous system $\dot{x} = Ax$ and any basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$, the ***fundamental system*** is given by

$$\mathcal{F} = \{e^{At}v_1, \ldots, e^{At}v_n\}.$$

To find the fundamental matrix,

- $v_i = e_i$: The fundamental matrix is $X(t) = e^{At}$.
- $v_i = u_i$: Let $U = (u_1, \ldots, u_n)$.
  1. A is diagonalizable: $\{u_1, \ldots, u_n\} \in \mathbb{R}^n$ is a basis of ***eigenvectors***. $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = U^{-1}AU$ is a ***diagonal matrix***. The fundamental matrix is

     $$X(t) = Ue^{\mathrm{diag}(\lambda_1, \ldots, \lambda_n)t} = (e^{\lambda_1 t}u_1, \ldots, e^{\lambda_n t}u_n).$$

  2. A is non-diagonalizable: $\{u_1, \ldots, u_n\} \in \mathbb{R}^n$ is a basis of ***generalized eigenvectors***. $J = U^{-1}AU$ is a ***Jordan matrix***. The fundamental matrix is

     $$X(t) = Ue^{Jt}.$$

Inhomogeneous System

1.10.4. Theorem. The solution to the IVP

$$\frac{dx}{dt} = Ax + b(t), \qquad x(t_0) = x_0$$

is given by

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-As}b(s)ds.$$

Inhomogeneous System

1.10.5. Theorem. The general solution of the system

$$\frac{dx}{dt} = Ax + b(t)$$

is given by

$$x(t; c_1, \dots, c_n) = \sum_{k=1}^{n} c_k x^{(k)}(t) + e^{At} \int e^{-As} b(s) ds,$$

where $\mathcal{F} = \{x^{(1)}, \dots, x^{(n)}\}$ is a fundamental system of the associated homogeneous system

$$\frac{dx}{dt} = Ax$$

and $c_1, \dots, c_n \in \mathbb{R}$ are arbitrary.

Solving Higher-Order ODEs (Constant A)

1. Transform the ODE into a linear system

$$\frac{dx}{dt} = Ax + b(t).$$

2. Find eigenvalues and (generalized) eigenvectors of A.
3. Construct a fundamental system by $X(t) = Ue^{Jt}$. ($x_{\mathrm{hom}}(t)$.)
4. Find x_{part}.
   - Calculate e^{At} and e^{-As},

$$x_{\mathrm{part}}(t) = e^{At} \int e^{-As} b(s) ds.$$

   - Use variation of parameters: make ansatz

$$x_{\mathrm{part}}(t) = \sum_{k=1}^{n} c_k(t) x^{(k)}(t)$$

   and use Cramer's rule to solve $X(t)c'(t) = b(t)$.

The Wronskian

- The Wronskian of n solutions of a system. $x^{(1)}, \dots, x^{(n)}$ are n arbitrary solutions of the homogeneous system

$$\frac{dx}{dt} = A(t)x.$$

Then the ***Wronskian*** is given by

$$W_{x_1,\dots,x_n}(t) := \det(x^{(1)}(t), \dots, x^{(n)}(t)).$$

- 1.10.8. Lemma and Abel's equation.

$$\frac{dW}{dt} = a(t)W, \quad a(t) = \operatorname{tr} A(t), \quad W(t) = W(t_0)e^{-\int_{t_0}^{t} a(s)ds}.$$

- 1.10.9. Corollary. Either $W(t) = 0$ for all t or $W(t) \neq 0$ for all t.

Variation of Parameters for Linear Systems

Problem. Given the fundamental system $x^{(1)}, \ldots, x^{(n)}$ of the homogeneous equation

$$\frac{dx}{dt} = A(t)x, \qquad A : \mathbb{R} \to \mathrm{Mat}(n \times n, \mathbb{R}),$$

we wish to find the general solution to the inhomogeneous equation

$$\frac{dx}{dt} = A(t)x + b(t), \qquad b : \mathbb{R} \to \mathbb{R}^n.$$

Variation of Parameters for Linear Systems

Method.

1. Make the ansatz

$$x_{\mathrm{part}}(t) = c_1(t)x^{(1)}(t) + \cdots + c_n(t)x^{(n)}(t).$$

2. Find $c_k(t)$ by

$$c_k(t) = \int \frac{W^{(k)}(\tau)}{W(\tau)}\, d\tau,$$

   where

   - $W(t) = \det X(t)$ is the Wronskian,
   - $W^{(k)}(t) = \det X^{(k)}(t)$, where $X^{(k)}(t)$ is the fundamental matrix where the kth column has been replaced with b.

IVP for Inhomogeneous Linear Systems

To solve the IVP for an inhomogeneous linear system with variable coefficients

$$\frac{dx}{dt} = A(t)x + b(t), \qquad x(t_0) = x_0,$$

Find

1. $x_{\text{hom}}(t)$:

$$\frac{dx_{\text{hom}}}{dt} = A(t)x_{\text{hom}}, \qquad x_{\text{hom}}(t_0) = x_0,$$

2. $x_{\text{part}}(t)$: For some fundamental system $(x^{(1)}, \dots, x^{(n)})$,

$$x_{\text{part}}(t) = \sum_{k=1}^{n} x^{(k)}(t) \int_{t_0}^{t} \frac{W^{(k)}(\tau)}{W(\tau)} d\tau.$$

3. $x_{\text{inhom}}(t)$:

$$x_{\text{inhom}}(t) = x_{\text{hom}}(t) + x_{\text{part}}(t).$$

Linear Second-Order ODEs

Definition. A ***linear differential equation of order 2*** is of the form

$$r(t)y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

Let $r(t) = 1$, the equation is equivalent to the linear system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} x + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

with $x_1 = y$ and $x_2 = y'$. The IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, y'(t_0) = y_0'$$

has a unique solution y that exists throughout I.

Linear Second-Order ODEs with Constant Coefficients

Homogeneous.

$$ay'' + by' + cy = 0, \qquad a, b, c \in \mathbb{R}, a \neq 0.$$

with characteristic polynomial

$$a\lambda^2 + b\lambda + c = 0.$$

Solution.

- $b^2 \neq 4ac$. There are two distinct eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ and two corresponding eigenvectors $v_1, v_2 \in \mathbb{C}^2$.

$$y(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{C}.$$

- $b^2 = 4ac$. There is only one eigenvalue $\lambda \in \mathbb{R}$

$$y(t; c_1, c_2) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Linear Second-Order ODEs with Constant Coefficients

Example 1. Find the general solution of the equation

$$y'' - 2y' + y = \frac{e^x}{2x}.$$

Vibrations

Undamped Free Vibrations.

$$mu'' + ku = 0.$$

Solution.

$$\begin{aligned} u(t) &= A\cos(\omega_0 t) + B\sin(\omega_0 t), \quad A, B \in \mathbb{R} \\ &= R\cos(\omega_0 t - \delta), \quad R = \sqrt{A^2 + B^2}, \delta = \arctan(B/A), \end{aligned}$$

with ***natural frequency***

$$\omega_0 := \sqrt{\frac{k}{m}}.$$

Vibrations

Damped Free Vibrations.

$$mu'' + \gamma u' + ku = 0.$$

Solution.

- Overdamping. $\gamma^2 - 4km > 0.$

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R}.$$

- Critical damping. $\gamma^2 - 4km = 0.$

$$u(t) = (c_1 + c_2 t) e^{-\gamma t/(2m)}, \qquad c_1, c_2 \in \mathbb{R}.$$

Vibrations

Damped Free Vibrations.

$$mu'' + \gamma u' + ku = 0.$$

Solution.

- **Underdamping.** $\gamma^2 - 4km < 0.$

$$\begin{aligned} u(t) &= e^{-\gamma/(2m)t}(A\cos(\mu t) + B\sin(\mu t)) \\ &= Re^{-\gamma/(2m)t}\cos(\mu t - \delta), \end{aligned}$$

where

$$R = \sqrt{A^2 + B^2}, \quad \delta = \arctan(B/A), \quad c_1, c_2, A, B \in \mathbb{R}.$$

and

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m}.$$

Vibrations

Undamped Forced Vibrations.

$$mu'' + ku = F_0 \cos(\omega t), \qquad F_0, \omega \in \mathbb{R}.$$

Solutions.

- General solution.

$$u(t; c_1, c_2) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

- Initial condition $u(0) = u'(0) = 0$.

$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} \big(\cos(\omega t) - \cos(\omega_0 t)\big) \\ &= \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}}_{A(t)} \sin \frac{(\omega_0 + \omega)t}{2}. \end{aligned}$$

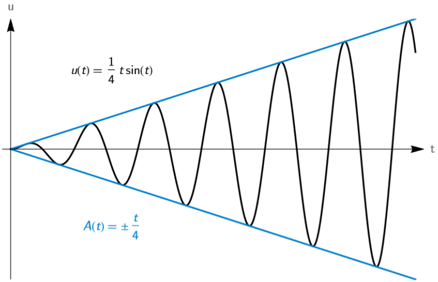
Vibrations

Undamped Resonance.

$$mu'' + ku = F_0 \cos(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Solutions.

$$u(t; c_1, c_2) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$



Vibrations

Damped Forced Vibrations.

$$mu'' + \gamma u' + ku = F_0 \cos(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Solutions.

- General solution is given by

$$u(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + R\cos(\omega t - \delta), \quad \lambda_1 \neq \lambda_2,$$

where

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2},$$

and

$$R = \frac{F_0}{\Delta}, \quad \cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin\delta = \frac{\gamma\omega}{\Delta}.$$

Vibrations

Damped Forced Vibrations.

$$mu'' + \gamma u' + ku = F_0 \cos(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Solutions.

- The amplitude R of the forced response

$$R(\omega) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma\omega^2}}.$$

Maximum occurs when

$$\omega_{\max}^2 = \omega_0^2 \left(1 - \frac{\gamma^2}{2km}\right), \quad R(\omega_{\max}) = \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/(4mk)}}.$$

Vibrations

Damped Forced Vibrations.

$$mu'' + \gamma u' + ku = F_0 \cos(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

Solutions.

- ***Phase angle*** δ:

$$\delta(\omega) = \arctan \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}.$$

1. ω is close to zero, $\delta(\omega) \approx 0$. The response $R\cos(\omega t - \delta)$ is nearly in phase with $F_0 \cos(\omega t)$.
2. $\omega \approx \omega_0, \delta(\omega) \approx \pi/2$. The phase difference is $1/4$ period.
3. ω increases further. The phase difference increases until it reaches π.

Linear Second-Order ODEs with Variable Coefficients

For a linear second-order differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

We have

- $y_{\text{hom}}(t)$:

$$y_{\text{hom}}(t; c_1, c_2) = c_1 y^{(1)}(t) + c_2 y^{(2)}(t).$$

- $y_{\text{part}}(t)$:

$$y_{\text{part}}(t) = -y^{(1)}(t) \int \frac{g(t)y^{(2)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt + y^{(2)}(t) \int \frac{g(t)y^{(1)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt.$$

- $y_{\text{inhom}}(t)$: $y_{\text{inhom}}(t; c_1, c_2) = y_{\text{hom}}(t; c_1, c_2) + y_{\text{part}}(t)$.

Reduction of Order

Finding homogeneous solutions. (Variable coefficients.)

1. Given solution y_1 of
$$y'' + p(t)y' + q(t)y = 0,$$
let
$$y_2(t) = v(t)y_1(t).$$
2. Plug y_2 into the original equation, simplify it to and solve
$$y_1 v'' + (2y_1' + py_1)v' = 0$$
to obtain v.
3. Find $y_2(t)$ using
$$y_2(t) = v(t)y_1(t).$$

Linear Second-Order ODEs with Variable Coefficients

Example 2 (*Legendre's Equation*). Verify that $y(x) = x$ solves

$$(1 - x^2)y'' - 2xy' + 2y = 0, \qquad -1 < x < 1,$$

and find another independent solution of the equation.

Linear Second-Order ODEs with Variable Coefficients

Example 3 (*Bessel's Equation*). Use the reduction of order to find the general solution of the following differential equation. A solution y_1 is given.

$$t^2 y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0, \qquad y_1(t) = \frac{\sin t}{\sqrt{t}}.$$

Solving Linear Second-Order ODEs

Constant coefficients and homogeneous.

$$ay'' + by' + cy = 0, \qquad q, b, c \in \mathbb{R}, a \neq 0.$$

1. Solve $a\lambda^2 + b\lambda + c = 0$.
2. $\lambda_1 \neq \lambda_2 \in \mathbb{C}$:
   - $\lambda_1, \lambda_2 \in \mathbb{R}$:

     $$y^{(1)}(x) = e^{\lambda_1 x}, \qquad y^{(2)}(x) = e^{\lambda_2 x}.$$

   - $\lambda_1, \lambda_2 \in \mathbb{C}$:

     $$y^{(1)}(x) = \cos(\mathrm{Im}\lambda_i x)e^{\mathrm{Re}\lambda_i x}, \quad y^{(2)}(x) = \sin(\mathrm{Im}\lambda_i x)e^{\mathrm{Re}\lambda_i x}.$$

3. $\lambda_1 = \lambda_2 \in \mathbb{R}$:

   $$y^{(1)}(x) = e^{\lambda x}, \qquad y^{(2)}(x) = xe^{\lambda x}.$$

Solving Linear Second-Order ODEs

Constant coefficients and inhomogeneous.

$$ay'' + by' + cy = g(x), \qquad q, b, c \in \mathbb{R}, a \neq 0.$$

1. Solve the homogeneous equation

$$ay'' + by' + cy = 0,$$

   and obtain $y^{(1)}, y^{(2)}$. Then $y_{\mathrm{hom}} = c_1 y^{(1)} + c_2 y^{(2)}$.

2. Find particular solution by letting $y_{\mathrm{part}} = c_3(x) y^{(1)} + c_4(x) y^{(2)}$. Find c_3, c_4 through Cramer's rule and

$$y_{\mathrm{part}} = -y^{(1)} \int \frac{gy^{(2)}}{W} + y^{(2)} \int \frac{gy^{(1)}}{W}.$$

3. $y_{\mathrm{inhom}} = c_1 y^{(1)} + c_2 y^{(2)} + y_{\mathrm{part}}$.

Solving Linear Second-Order ODEs

Example 4. Find the general solution to

$$y''' + 3y'' + 3y' - 7y = 0.$$

Solving Linear Second-Order ODEs

Note.

- Initial conditions are satisfied by the homogeneous solutions.
- The inhomogeneity $g(x)$ is satisfied by the particular solution.
- To incorporate the initial conditions $y(x_0) = y_0, y'(x_0) = y_0'$,
  - Fit the initial condition in the homogeneous solution in Step 1.
  - Use the definite integral $\int_{x_0}^{x}$ in Step 2.

Then plugging in $x = x_0$,

$$y_{\mathrm{inhom}}(x_0) = \underbrace{y_{\mathrm{hom}}(x_0)}_{y_0} - \underbrace{y^{(1)} \int_{x_0}^{x_0} \frac{gy^{(2)}}{W} ds}_{0} + \underbrace{y^{(2)} \int_{x_0}^{x_0} \frac{gy^{(1)}}{W} ds}_{0},$$

$$y'_{\mathrm{inhom}}(x_0) = \underbrace{y'_{\mathrm{hom}}(x_0)}_{y_0'} - \underbrace{y^{(1)} \cdot \frac{gy^{(2)}}{W}\bigg|_{x=x_0} + y^{(2)} \cdot \frac{gy^{(1)}}{W}}_{0}\bigg|_{x=x_0}.$$

Solving Linear Second-Order ODEs

Example 5. (Assignment 4.3.) Find the solution to the initial value problem

$$y''' + y' = \sec t \tan t, \qquad y''(0) = y'(0) = y(0) = 0.$$

Thanks for your attention!