VV286 Honors Mathematics IV
Solution Manual for RC 2

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Example 1.

Solve the differential equation

$$y = (yy' + 2x)y'.$$

Solution.

1. The straight line solution is given by $y = 0$.

2. Using slope parametrization, the original equation is transformed to

$$y(p) = p^2 y(p) + 2x(p)p, \quad y'(p) = px'(p).$$

This gives

$$y(p) = \frac{2px}{1-p^2}.$$

Differentiating both sides with respect to p,

$$\begin{aligned} y'(p) &= \frac{(2x + 2px')(1-p^2) - 2px(-2p)}{(1-p^2)^2} \\ &= \frac{2p(1-p^2)x' + 2x + 2p^2 x}{(1-p^2)^2} = px' \end{aligned}$$

Then

$$\begin{aligned} x' &= -\frac{2}{p(1-p^2)}x \\ &= -\frac{1}{p}\left(\frac{1}{1-p} + \frac{1}{1+p}\right)x \\ &= -\left(\frac{2}{p} + \frac{1}{1-p} - \frac{1}{1+p}\right)x \end{aligned}$$

This is a separable equation for x and we have

$$\ln|x| = -2\ln|p| + \ln|p-1| + \ln|p+1| + c_1,$$

namely,

$$x(p) = c_2 \cdot \left(1 - \frac{1}{p^2}\right).$$

3. Plugging this in the equation for y, we have

$$y(p) = \frac{2px}{1-p^2} = -\frac{c_2}{p}.$$

Therefore, the solution is given by

$$x = c_2 - \frac{y^2}{4c_2}.$$

In sum, the solution is given by

$$x = c_2 - \frac{y^2}{4c_2}, \qquad c_2 \in \mathbb{R}$$

with straight line solution $y = 0$.

Example 2.

Solve the differential equation

$$2y = 2x^2 + 4xy' + (y')^2.$$

Solution. Using slope parametrization, the original equation is transformed to

$$2y = 2x^2 + 4xp + p^2.$$

Differentiating both sides and plugging in $y' = px'$, we have

$$\begin{aligned} 2y' &= 4xx' + 4x + 4x'p + 2p \\ &= 2px' \quad \Rightarrow \quad (2x+p)(x'+1) = 0. \end{aligned}$$

Then we discuss two cases

- $\underline{2x + p = 0}$. Then

$$2x + y' = 0 \quad \Rightarrow \quad y(x) = -x^2 + c_1.$$

  To determine c_1, we plug this in the original equation and obtain

$$2(-x^2 + c_1) = 2x^2 + 4x(-2x) + 4x^2 \quad \Rightarrow \quad c_1 = 0.$$

  Therefore, the first solution is $y_1(x) = -x^2$.

- $\underline{x' + 1 = 0}$. Then

$$x = -p + c_2 \quad \Rightarrow \quad p = c_2 - x,$$

and

$$
\begin{aligned}
y &= x^2 + 2px + \frac{p^2}{2} \\
&= x^2 + 2(c_2 - x)x + \frac{(c_2 - x)^2}{2} \\
&= c_2^2 - \frac{(x - c_2)^2}{2}.
\end{aligned}
$$

Therefore, the final answer is

$$
y = c_2^2 - \frac{(x - c_2)^2}{2} \quad \text{or} \quad y = -x^2,
$$

where $c_2 \in \mathbb{R}$.

Example 3.

Determine all the solutions for the following Clairaut's differential equations in explicit forms.

1. $y = xy' - \sqrt{y' - 1}$.
2. $y = xy' + y'^2$.

Solution.

1. Using slope parametrization, the original equation is transformed to

$$
y = xp - \sqrt{p - 1}, \qquad p \geq 1.
$$

Differentiating both sides,

$$
y' = x'p + x - \frac{1}{2\sqrt{p - 1}} = x'p \quad \Rightarrow \quad x = \frac{1}{2\sqrt{p - 1}}.
$$

Then $y(x)$ is found by

$$
y(x) = x - \frac{1}{4x}
$$

with straight line solutions $y = cx - \sqrt{c - 1}, c \geq 1 \in \mathbb{R}$.

Alternatively, we can find the solution by finding the envelope of straight line solutions

$$
\gamma(c, x) = \begin{pmatrix} x \\ cx - \sqrt{c - 1} \end{pmatrix}.
$$

The envelope is then found by

$$
\frac{\partial \gamma_1}{\partial c} \frac{\partial \gamma_2}{\partial x} = 0, \qquad \frac{\partial \gamma_1}{\partial x} \frac{\partial \gamma_2}{\partial c} = x - \frac{1}{2\sqrt{c - 1}},
$$

yielding the same result.

2. Using slope parametrization, the original equation is transformed to

$$y = xp + p^2.$$

Differentiating both sides,

$$y' = x'p + x + 2p = x'p \quad \Rightarrow \quad x = -2p$$

Then $y(x)$ is found by

$$y(x) = -\frac{x^2}{4}$$

with straight line solutions $y = cx + c^2, c \in \mathbb{R}$.

Example 4.

Determine the solutions of the following differential equation.

$$y = xy'^2 + \ln y'^2.$$

Solution.

1. Using slope parametrization, the equation is given by

$$y(p) = x(p)p^2 + 2\ln|p|.$$

2. Using the relation $\dot{y}(p) = p\dot{x}(p)$ and differentiating both sides, we have

$$\dot{y} = p\dot{x} = p^2\dot{x} + 2xp + \frac{2}{p} \quad \Leftrightarrow \quad (p^2 - p)\dot{x} + 2xp + \frac{2}{p} = 0, \quad p \neq 0,$$

which is a linear ODE for x. Using the method of finding integral curves, we identify that if $p \neq 1$,

$$g(p,x) = 2xp + \frac{2}{p}, \qquad h(p,x) = p^2 - p.$$

Furthermore, we notice that

$$g_x = 2p, \qquad h_p = 2p - 1$$

and an integrating factor is given by

$$M = \frac{p-1}{p}.$$

Then we find the integral curve of

$$F(p,x)=\begin{pmatrix} 2(p-1)x+\dfrac{2(p-1)}{p^2} \\ (p-1)^2 \end{pmatrix},$$

which is given by

$$U(p,x)=(p-1)^2x+\frac{2}{p}+2\ln|p|=C$$

Therefore, representing x and y both by p, we have the solutions given by

$$\begin{cases} x=\dfrac{C}{(p-1)^2}-\dfrac{2\ln|p|}{(p-1)^2}-\dfrac{2}{p(p-1)^2}, \\ y=\dfrac{Cp^2}{(p-1)^2}-\dfrac{2p^2\ln|p|}{(p-1)^2}-\dfrac{2p^2}{p(p-1)^2}+2\ln|p|. \end{cases}$$

When $p=1$, then the solution is given by $y=x$.

Example 5.

Find the approximated sequence $x^{(k)}$ for the IVP

$$x'=2t(1+x), \qquad x(0)=0.$$

Solution.

1. Start from $x^{(0)}(t)=0$.

2. The recurrent relation is given by

$$x^{(k+1)}(t)=\int_0^t 2s(1+x^{(k)}(s))ds.$$

Then

$$x^{(1)}(t)=\int_0^t 2s(1+0)ds=t^2,$$

and in fact, it can be shown that

$$x^{(k)}(t)=t^2+\frac{t^4}{2}+\frac{t^6}{3!}+\cdots+\frac{t^{2k}}{n!}.$$

This sequence converges to the solution of the original differential equation with a limit

$$x(t)=e^{t^2}-1.$$