Honors Mathematics IV
Final Review

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December 8, 2018

Table of contents

Uniqueness of Solution

Example 1. Prove the uniqueness of solution of the three dimensional wave on $\Omega \subset \mathbb{R}^3$

$$c^2 u_{tt} = \Delta u$$

which satisfy the boundary conditions

$$u(x, y, z, t) = F(x, y, z, t), \qquad (x, y, z) \in \partial\Omega$$

and initial conditions

$$u(x, y, z, 0) = G(x, y, z), \qquad u_t(x, y, z, 0) = H(x, y, z).$$

Hint. Green's identity

$$\int_{\Omega} \langle \nabla u, \nabla v \rangle d\tau = -\int_{\Omega} u \cdot \Delta v d\tau + \int_{\partial\Omega^*} u \frac{\partial v}{\partial n} dA.$$

Separation of Variables for PDEs

1. Make ansatz

$$u(x_1, \ldots, x_n) = u_1(x_1) \cdot u_2(x_2) \cdots u_n(x_n).$$

2. Write out the boundary conditions with the ansatz. (Usually this will reduce to initial conditions for the ODE with only one variable.)
3. Solve the ODEs one-by-one. (Begin with the one that has boundary conditions.)
   - Obtain ODE in the form $Lu = \lambda u$ with boundary condition.
   - Solve this ODE and obtain eigenvalues & eigenfunctions.
   - Plug in these eigenvalues to solve other equations.
4. Gather all the eigenfunctions for each variable to obtain the general solution of the PDE.
5. Fit the general solution into the initial conditions.
   - The eigenfunctions for one ODE normally form an orthonormal system. (Usually Fourier series or Bessel functions.)
   - Expand the initial condition with this orthonormal system.

Orthonormal Systems

Fitting initial conditions. There are (most commonly) two series that we can expand the initial conditions to. On the interval $[0, L]$, we have

- Fourier series (usually cosine or sine series).

$$\left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}, \quad \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}.$$

- Bessel functions. Expand the initial condition into

$$\left\{ \frac{1}{\sqrt{L}|J_n'(\alpha_{n,m})|} J_n(\alpha_{n,m}\sqrt{x/L}) \right\}_{m=1}^{\infty}$$

so that

$$f(x) = \sum_{m=1}^{\infty} \frac{1}{L \cdot J_n'(\alpha_{n,m})^2} \langle J_n(\alpha_{n,m}\sqrt{(\cdot)/L}), f \rangle J_n(\alpha_{n,m}\sqrt{x/L}).$$

Inhomogeneous PDEs

Step 2. ***Solve the ODEs one-by-one. (Begin with the one that has boundary conditions.)***

Example 2. Solve the inhomogeneous heat equation

$$u_{xx} - u_t = -2x, \qquad (x, t) \in (0, 1) \times \mathbb{R}_+$$

with Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(1, t) = 0, \qquad t > 0$$

and initial temperature distribution

$$u(x, 0) = x - x^2, \qquad x \in [0, 1].$$

Orthogonality of Eigenfunctions

Step 3. ***Solve this ODE and obtain eigenvalues & eigenfunctions.***

The choice of basis largely depends on the eigenfunctions obtained from the Sturm-Liouville problem.

Mixed Boundary Conditions

Step 3. ***Solve this ODE and obtain eigenvalues & eigenfunctions.***

Example 3. Show how a solution to the heat equation

$$u_{xx} - u_t = 0, \qquad (x,t) \in (0,1) \times \mathbb{R}_+$$

with mixed boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 0, \qquad t > 0$$

and initial temperature distribution

$$u(x,0) = f(x), \qquad x \in [0,1]$$

can be obtained.

Orthogonality of Eigenfunctions

Step 3. ***Solve this ODE and obtain eigenvalues & eigenfunctions.***

Example 4 (RC 9). Solve the wave equation problem

$$
\begin{gathered}
4u_{tt} = u_{xx}, \\
u_x(-\pi, t) = u_x(\pi, t) = 0, \quad u(x, 0) = x^2, \quad u_t(x, 0) = 0.
\end{gathered}
$$

The basis is chosen as

$$
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin\left(\left(n - \frac{1}{2} \right) x \right) \right\}_{n=1}^{\infty}.
$$

Orthogonality of Eigenfunctions

Step 3. ***Solve this ODE and obtain eigenvalues & eigenfunctions.***

Example 5. The Sturm-Liouville problem can have various forms deciding the eigenfunctions.

- Wave equation $c^2 u_{xx} = u_{tt}, 0 < x < l, t > 0$ gives two ODEs

$$X'' + \lambda X = 0, \qquad T'' + \lambda c^2 T = 0.$$

- The equation for the suspended chain $u_{tt} = g \cdot x u_x$ yields

$$(xX')' + \lambda X = 0 \quad \Rightarrow \quad \begin{aligned} y^2 Y'' + yY' + \lambda y^2 Y &= 0, \\ T'' + \lambda g T &= 0. \end{aligned}$$

Fit into Initial Conditions

Step 5. ***Fit the general solution into the initial conditions.***

Expand the initial condition using the orthonormal system formed by the eigenfunctions.

Fit into Initial Conditions

The Fourier series for $L^2([0, L])$:

- The Fourier-Euler Basis.

$$\mathcal{B}_1 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

- The Fourier-Cosine Basis.

$$\mathcal{B}_2 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

- The Fourier-Sine Basis.

$$\mathcal{B}_3 := \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

Fit into Initial Conditions

- The complex Fourier-Euler Basis.

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$$

- The Bessel Functions.

$$\left\{ \frac{1}{\sqrt{L}|J_n'(\alpha_{n,m})|} J_n(\alpha_{n,m}\sqrt{x/L}) \right\}_{m=1}^{\infty}$$

Final Remarks

- The choice of eigenvalues need to be discussed in order to satisfy the boundary conditions.
- Scale the basis functions according to the length of the interval.
- The expression of eigenfunctions can be either as sine/cosine functions or exponential functions. CHOOSE SMARTLY :)

Good luck for your Final!