Honors Mathematics IV
RC 8

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The Power Series Ansatz

Equation.

$$x'' + p(t)x' + q(t)x = 0$$

Solution.

1. Make power series ansatz of the solution:

$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$

2. Plug the ansatz into the equation using

$$x'(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}, \quad x''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2}.$$

The Power Series Ansatz

Equation.

$$x'' + p(t)x' + q(t)x = 0$$

Solution.

3. Find the recurrence formula of a_ks.
4. Decide the first few terms (by initial conditions) to obtain two independent solutions $x_1(t), x_2(t)$.
5. Find $x(t)$ by

$$x(t) = x_1(t) + x_2(t).$$

The Power Series Ansatz

Example 1. Determine the terms up to x^5 in each of the two linearly independent power series solutions to

$$y'' + (2 - 4x^2)y' - 8xy = 0$$

centered at $x = 0$. Also find the radius of convergence of these solutions.

Euler's Equation

Euler's equation.

$$t^2x'' + \alpha t x' + \beta x = 0, \qquad \alpha, \beta \in \mathbb{R}.$$

Solution $(t > 0)$. We make ansatz $x(t) = x^r$ and obtain

$$(r^2 + (\alpha - 1)r + \beta)t^r = 0, \quad r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 - 4\beta}.$$

Then

- $(\alpha - 1)^2 - 4\beta > 0$. $r_1 \neq r_2 \in \mathbb{R}$.

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_2}, \qquad c_1, c_2 \in \mathbb{R}.$$

Euler's Equation

Euler's equation.

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$$(r^2 + (\alpha - 1)r + \beta)t^r = 0, \quad r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 - 4\beta}.$$

Then

- $(\alpha - 1)^2 - 4\beta = 0$. $r_1 = r_2 = \dfrac{1 - \alpha}{2}$.

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_1} \ln t, \qquad c_1, c_1 \in \mathbb{R}.$$

Euler's Equation

Euler's equation.

$$t^2x'' + \alpha t x' + \beta x = 0, \qquad \alpha, \beta \in \mathbb{R}.$$

Solution $(t > 0)$. We make ansatz $x(t) = x^r$ and obtain

$$(r^2 + (\alpha - 1)r + \beta)t^r = 0, \quad r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 - 4\beta}.$$

Then

- $(\alpha - 1)^2 - 4\beta < 0$. $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu \in \mathbb{C}$, where

$$\lambda = \frac{1 - \alpha}{2}, \quad \mu = \frac{1}{2}\sqrt{4\beta - (\alpha - 1)^2}.$$

  Then the two solutions are $x_1(t) = t^{r_1}, x_2(t) = t^{r_2}$ and

$$x(t; c_1, c_2) = c_1 t^\lambda \cos(\mu \ln t) + c^2 t^\lambda \sin(\mu \ln t), \quad c_1, c_2 \in \mathbb{R}.$$

Euler's Equation

Euler's equation.

$$t^2 x'' + \alpha t x' + \beta x = 0, \qquad \alpha, \beta \in \mathbb{R}.$$

General solution. For $t > 0$ or $t < 0$,

$$x(t; c_1, c_2) = \xi(-t; c_1, c_2) = \begin{cases} c_1 |t|^{r_1} + c_2 |t|^{r_2}, \\ c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|, \\ c_1 |t|^{\lambda} \cos(\mu \ln |t|) + c_2 |t|^{\lambda} \sin(\mu \ln |t|). \end{cases}$$

The solutions may not defined at $t = 0$.

Regular Singular Points

Definition. The equation

$$x'' + p(t)x' + q(t)x = 0$$

has a ***regular singular point*** at t_0 if

- $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic in a neighborhood of t_0 or, equivalently,
- we have convergent power series such that

$$p(t) = \frac{p_{-1}}{t - t_0} + \sum_{j=0}^{\infty} p_j (t - t_0)^j,$$

$$q(t) = \frac{q_{-2}}{(t - t_0)^2} + \frac{q_{-1}}{t - t_0} + \sum_{j=0}^{\infty} q_j (t - t_0)^j$$

in a neighborhood of t_0.

The Method of Frobenius

Equation.

$$t^2x'' + t(tp(t))x' + t^2q(t)x = 0, \qquad t > 0$$

Solution.

1. Make ***Frobenius ansatz*** and identify $tp(t), t^2q(t)$ and their coefficients,

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k, \qquad a_0 \neq 0.$$

2. Plug in the ansatz (or memorize) to obtain the ***indicial equation***

$$F(r) = 0, \quad F(x) := x(x-1) + p_0 x + q_0,$$

$$a_m F(r+m) = -\sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k})a_k, \quad m \geq 1.$$

The Method of Frobenius

3. Find two independent solutions to the ODE. The first solution is (find by recurrent equation)

$$x_t(t) = t^{r_1} \sum_{k=0}^{\infty} a_k(r_1) t^k,$$

and

- $r_1 - r_2 \notin \mathbb{N}$. Find x_1, x_2 by two different recurrent equations given by r_1 and r_2.
- $r_1 - r_2 \in \mathbb{N}$.
  - Both sides vanish in the indicial equation for $N \in \mathbb{N}$: choose an arbitrary a_N and find the second independent solution.
  - One side does not vanish for $N \in \mathbb{N}$:

$$x_2(t) = \frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k \right)\Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a_k'(r_2) t^k,$$

where $c \in \mathbb{R}$ may vanish. If $r_1 = r_2$, then $c = 1$.

The Method of Frobenius

Example 2. Find a series solution about $x = 0$ of

$$x^2 y'' - xy' + (1 - x)y = 0.$$

The Method of Frobenius

Example 3. Find the first few terms of the second series solution about $x = 0$ of

$$x^2y'' - xy' + (1 - x)y = 0.$$

The Bessel Functions

Bessel equations. The ***Bessel equation*** of order ν is given by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x \in \mathbb{R}.$$

The Bessel Equation of Order 1/2

Equation.

$$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0.$$

Solution (Frobenius method).

1. Find $tp(t) = 1, t^2q(t) = t^2 - 1/4$, which are analytic at $t = 0$ and

$$p_0 = 1, \qquad\qquad\qquad\qquad p_k = 0 \ (k \geq 1),$$
$$q_0 = -\frac{1}{4}, \qquad q_1 = 0, \qquad q_2 = 1, \qquad q_k = 0 \ (k \geq 3).$$

2. Write and solve the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = (r + \frac{1}{2})(r - \frac{1}{2}) = 0,$$

giving $r_1 = 1/2$ and $r_2 = -1/2$.

The Bessel Equation of Order 1/2

Equation.

$$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0.$$

Solution (Frobenius method).

3. Write out the recurrent relation:

$$a_mF(r+m) = -\sum_{k=0}^{m-1}(q_{m-k} + (r+k)p_{m-k})a_k, \quad m \geq 1,$$

which gives

$$-a_1F(r+1) = 0, \quad -a_mF(r+m) = a_{m-2}$$

for $m \geq 2$. Set $a_0 = 1$.

The Bessel Equation of Order 1/2

Equation.

$$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0.$$

Solution (Frobenius method).

4. Find the two independent solutions. $(r_1 - r_2 = 1.)$
   - $r_1 = \frac{1}{2}$. For $m \geq 2$, the recurrent equation is

     $$a_m = -\frac{a_{m-2}}{m(m+1)},$$

     giving

     $$a_{2k+1} = 0, \quad a_{2k} = \frac{1}{2k(2k+1)}\frac{1}{(2k-2)(2k-1)}a_{2(k-2)}.$$

     Then $x_1(t)$ is given by

     $$x_1(t) = t^{r_1}\sum_{m=0}^{\infty} a_m t^m = \frac{\sin t}{\sqrt{t}}.$$

The Bessel Equation of Order 1/2

Equation.

$$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0.$$

Solution (Frobenius method).

4. Find the two independent solutions. $(r_1 - r_2 = 1.)$
   - $r_2 = -\frac{1}{2}$. Both sides of the recurrent relation vanish for $N = 1$ in $F(r_2 + N)$. So a_1 is arbitrary (set to 0). For $m \geq 2$, the recurrent relation is

     $$a_m = -\frac{a_{m-2}}{m(m-1)},$$

     giving

     $$a_{2k+1} = 0, \qquad a_{2k} = \frac{(-1)^k}{(2k)!}.$$

     Then $x_2(t)$ is given by

     $$x_2 = t^{r_2}\sum_{m=0}^{\infty} a_m t^m = \frac{\cos t}{\sqrt{t}}.$$

The Bessel Equation of Order 0

Equation.

$$t^2x'' + tx' + t^2x = 0.$$

Solution (Frobenius method).

1. Find $tp(t) = 1, t^2q(t) = t^2$, which are analytic at $t = 0$ and

$$\begin{aligned} &p_0 = 1, \quad p_1 = 0, && && p_k = 0 \ (k \geq 2), \\ &q_0 = 0, \quad q_1 = 0, && q_2 = 1, && q_k = 0 \ (k \geq 3). \end{aligned}$$

2. Write and solve the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = r^2 = 0,$$

giving $r_1 = r_2 = 0$.

The Bessel Equation of Order 0

Equation.

$$t^2x'' + tx' + t^2x = 0.$$

Solution (Frobenius method).

3. Write out the recurrent relation:

$$a_m F(r+m) = -\sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k})a_k, \quad m \geq 1,$$

which gives

$$(r+1)^2 a_1 = 0, \quad a_m(r+m)^2 = -a_{m-2}(r)$$

for $m \geq 2$. Then we know $a_1 = 0$ and set $a_0 = 1$.

The Bessel Equation of Order 0

Equation.

$$t^2 x'' + t x' + t^2 x = 0.$$

Solution (Frobenius method).

4. Find the two independent solutions. $r_1 = r_2 = 0$.
   - x_1. From the recurrent relations we know that

$$a_{2k+1} = 0, \quad a_{2k} = \frac{1}{(r+2k)^2} \frac{1}{(r+2k-2)^2} a_{2k-4}(r).$$

Then $x_1(t)$ is given by

$$x_1(t) = t^{r_1} \sum_{m=0}^{\infty} a_m t^m = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} t^{2k}.$$

The Bessel Equation of Order 0

Equation.

$$t^2 x'' + t x' + t^2 x = 0.$$

Solution (Frobenius method).

4. Find the two independent solutions. $r_1 = r_2 = 0$.
   - x_2. The second solution is given by

$$x_2(t) = \frac{d}{dt}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k \right)\Bigg|_{r=r_2=0}$$

$$= x_1(t) \ln t + \sum_{k=0}^{\infty} a_k'(0) t^k,$$

with

$$a_{2k}(r) = \frac{(-1)^k}{(2+r)^2(4+r)^2 \cdots (2k+r)^2}.$$

The Bessel Equation of Order 0

Equation.

$$t^2x'' + tx' + t^2x = 0.$$

Solution (Frobenius method).

4. Find the two independent solutions. $r_1 = r_2 = 0$.
   - x_2. Then we have

$$\frac{a_{2k}'(r)}{a_{2k}(r)} = -2\sum_{j=1}^{k} \frac{1}{2j+r}.$$

evaluated at $r_2 = 0$,

$$a_{2k}'(0) = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) 2_{2n}(0).$$

Therefore,

$$x_2(t) = x_1(t)\ln t + \sum_{k=0}^{\infty}\left(\sum_{j=1}^{k}\frac{1}{j}\right)\frac{(-1)^{k+1}}{2^{2k}(k!)^2}t^{2k}.$$

The Bessel Functions

Generating function. The function

$$\Psi(x,t) = e^{\frac{x}{2}(t-1/t)}.$$

is a ***generating function*** for the Bessel functions of the first kind and integer order. More precisely,

$$e^{\frac{x}{2}(t-1/t)} = \sum_{-\infty}^{\infty} J_n(x) t^n.$$

The Bessel Functions

Bessel functions of integer order. For $n \in \mathbb{N}$ the function $J_n : \mathbb{R} \to \mathbb{R}$ is given by

$$J_n(x) := \sum_{r=0}^{\infty} \frac{(-1)^r}{(r+n)!r!} \left(\frac{x}{2}\right)^{2r+n}, \quad J_{-n} = (-1)^n J_n.$$

General Bessel functions. The Bessel functions $J_\nu(x)$ of order $\nu \in \mathbb{R}$ is given by

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(\nu+r+1)r!} \left(\frac{x}{2}\right)^{2r+\nu},$$

where Γ is the Gamma function.

The Bessel Functions

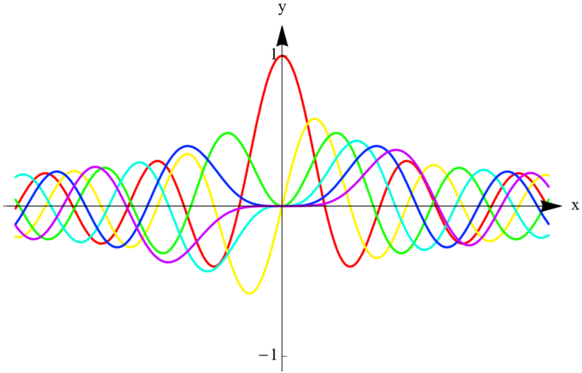
Trigonometric approximation.

$$J_n(x) \sim \frac{1}{\sqrt{x}} \cos\left(x - \frac{2n+1}{4}\pi\right), \quad \text{as } x \to \infty.$$

Graph of some Bessel functions.

The graph shows

- J_0 (red),
- J_1 (yellow),
- J_2 (green),
- J_3 (light blue),
- J_4 (dark blue),
- J_5 (violet).



The Bessel Functions

Recurrence relations.

$$2nJ_n(x) = xJ_{n+1}(x) + xJ_{n-1}(x),$$
$$J_n'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)).$$

For $\nu \in \mathbb{R}$.

$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x), \quad \frac{d}{dx}(x^{-\nu} J_\nu(x)) = -x^{-\nu} J_{\nu+1}(x).$$

Applications of Bessel Functions

1. Use substitution or separation of variables ansatz to obtain a Bessel function

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x \in \mathbb{R}.$$

2. Once the Bessel function is obtained, we can directly write out the solution depending on the practical conditions:
   - Solution is finite (most common):

$$y(x) = c \cdot J_\nu(x).$$

   - Solution is infinite:

$$y(x) = c \cdot Y_\nu(x).$$

3. Substitute back for the original variables and insert boundary/initial conditions.

The Suspended Chain

Model equation. PDE with boundary condition at $x = l$:

$$\frac{1}{g} u_{tt}(x,t) = x \cdot u_{xx}(x,t) + u_x(x,t), \quad y(l) = 0.$$

Solution.

1. Suppose u is periodic with respect to time:

$$u(x,t) = y(x) \cdot e^{i\omega t}$$

   for some frequency ω, then we obtain the ODE

$$\frac{\partial}{\partial x}(x \cdot y') + \frac{\omega^2}{g} y = 0.$$

2. Use substitution

$$x = \frac{gz^2}{4\omega^2}, \qquad z = 2\omega\sqrt{\frac{x}{g}}.$$

The Suspended Chain

Model equation. PDE with boundary condition at $x = l$:

$$\frac{1}{g} u_{tt}(x,t) = x \cdot u_{xx}(x,t) + u_x(x,t), \quad y(l) = 0.$$

Solution.

3. Obtain the Bessel equation of order zero:

$$zw'' + w' + zw = 0.$$

4. Write out the finite solution for the Bessel equation

$$w(z) = c \cdot J_0(z).$$

5. Substitute back for the original variables and insert the boundary conditions:

$$y(x) = c \cdot J_0\left(2\omega\sqrt{\frac{x}{g}}\right), \quad \omega = \frac{1}{2}\sqrt{\frac{g}{l}} \cdot \alpha_{0,n}.$$

Airy's Equation

Airy's equation.

$$y'' + xy = 0.$$

Solution.

1. Use substitution

$$u(t) = x^{-1/2}y(x), \qquad t = \frac{2}{3}x^{3/2}$$

to obtain the Bessel equation of order $\nu = 1/3$:

$$\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} = -\frac{1}{2}x^{-2}y + x^{-1}\frac{dy}{dx},$$
$$\frac{d^2u}{dt^2} = \frac{d}{dt}\left(\frac{du}{dt}\right) = x^{-7/2}y - \frac{3}{2}x^{-5/2}\frac{dy}{dx} + x^{-3/2}\frac{d^2y}{dx^2}.$$

Airy's Equation

Airy's equation.

$$y'' + xy = 0.$$

Solution.

1. Use substitution

$$u(t) = x^{-1/2} y(x), \qquad t = \frac{2}{3} x^{3/2}$$

to obtain the Bessel equation of order $\nu = 1/3$: Then

$$\frac{dy}{dx^2} = x^{3/2} \frac{d^2 u}{dt^2} - \frac{1}{4} x^{-3/2} u + \frac{3}{2} \frac{du}{dt},$$

substituting $t = \frac{2}{3} x^{3/2}$, we then have

$$t^2 u'' + tu' + (t^2 - \frac{1}{9}) u = 0.$$

Airy's Equation

Airy's equation.

$$y'' + xy = 0.$$

Solution.

2. Write out the general solution using Bessel functions:

$$u(t) = c_1 J_{1/3}(t) + c_2 J_{-1/3}(t).$$

3. Substitute back for the original variables.

$$y(x) = c_1 \cdot x^{1/2} J_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2 \cdot x^{1/2} J_{-1/3}\left(\frac{2}{3}x^{3/2}\right).$$

Transforming an Equation into a Bessel's Equation

A Bessel function is of the form

$$t^2 u'' + tu' + (t^2 - \nu^2)u = 0.$$

Using the substitution $t = ax^b, u = yx^c$, we have

$$y = x^{-c}u(t(x)), \qquad \frac{dt}{dx} = abx^{b-1}$$

and thus

$$\begin{aligned} y' &= -cx^{-c-1}u + abx^{b-c-1}u' \\ y'' &= c(c+1)x^{-2}y + ab(b-2c-1)x^{b-c-2}u' + a^2b^2x^{2b-c-2}u''. \end{aligned}$$

Transforming an Equation into a Bessel's Equation

Solving for u, u' and u'', we have

$$u = yx^c, \qquad u' = \frac{y' + cx^{-1}y}{abx^{b-c-1}},$$

$$u'' = \frac{y'' - c(c+1)yx^{-2} - ab(b-2c-1)x^{b-c-2} \cdot \frac{y'+cx^{-c-1}u}{abx^{b-c-1}}}{a^2b^2x^{2b-c-2}}$$

Therefore, we can transform the original equation into

$$x^2y'' + x(2c+1)y' + (a^2b^2x^{2b} - b^2\nu^2 + c^2)y = 0.$$

Differential equations of this form can be transformed into a Bessel's equation using substitution

$$t = ax^b, \qquad u = yx^c.$$

Transforming an Equation into a Bessel's Equation

Example. For Airy's Equation, we have

$$x^2 y'' + x^3 y = 0,$$

where $c = -\frac{1}{2}$, $b = \frac{3}{2}$ and $a = \frac{2}{3}$, therefore, the transformation is

$$t = ax^b = \frac{2}{3}x^{3/2}, \qquad u(t) = yx^c = x^{-1/2}y.$$

and the order is given by

$$c^2 = b^2\nu^2 \quad \Rightarrow \quad \nu = \frac{1}{3}.$$

Exercises

Exercise 1. The *Legendre differential equation of order* $\lambda \in \mathbb{R}$ is given by

$$(1 - x^2)y'' - 2xy' + \lambda(\lambda + 1)y = 0.$$

1. Find the general solution of the equation in power series form.
2. Verify that if $\lambda \in \mathbb{N}$ there exists a non-zero polynomial solution.

Exercises

Exercise 2. Use the method of Frobenius to solve

$$5x^2y'' + x(1+x)y' - y = 0.$$

Thanks for your attention!