Honors Mathematics IV
RC 3

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The Eigenvalue Problem

Let V be a real or complex vector space. For $L \in \mathcal{L}(V, V)$,

- ***eigenvalue of*** L: $\lambda \in \mathbb{F}$ such that $\exists x, Lx = \lambda x$.
- ***eigenvector for the eigenvalue*** λ: x such that $Lx = \lambda x$.
- ***eigenspace for the eigenvalue*** λ:

$$V_\lambda = \{x \in V : Lx = \lambda x\}.$$

- ***geometric multiplicity of*** λ: $\dim V_\lambda$.

The Eigenvalue Problem

Note. From above we highlight the followings:

1. An eigenvalue can be <u>real</u> or <u>complex</u>.
2. If λ is a complex eigenvalue of A with corresponding eigenvector v, then $\overline{\lambda}$ is an eigenvalue of A with corresponding eigenvector $\overline{v}$.
3. The concept of eigenspace and geometric multiplicity is associated with a specific <u>eigenvalue</u>. (Same with algebraic multiplicity.)

The Eigenvalue Problem

Let V be a real or complex vector space with dimension n. For $L \in \mathcal{L}(V, V)$,

- L has at most n distinct eigenvalues.
- If L has n eigenvalues $\lambda_1, \ldots, \lambda_n$, then it has precisely n independent eigenvectors $v_1, \ldots, v_n$. Thus $\mathcal{B} = (v_1, \ldots, v_n)$ constitutes as basis of V and

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}.$$

- If the sum of geometric multiplicities equals n, there exists a basis of eigenvectors of $\mathbb{R}^n$.

The Eigenvalue Problem for Matrices

Finding eigenvalues and eigenvectors for matrices.

- For $V = \mathbb{R}^n, A \in \mathrm{Mat}(n \times n, \mathbb{R})$,

$$Ax = \lambda x \quad \Leftrightarrow \quad (A - \lambda \mathbb{1})x = 0$$

  and $p(\lambda) = \det(A - \lambda \mathbb{1})$ is the ***characteristic polynomial***.
- Solve $p(\lambda) = 0$ to obtain eigenvalues $\lambda_1, \dots, \lambda_k$. (Or else the column vectors of $A - \lambda \mathbb{1}$ should be independent, and thus $x = 0$.)
- Plug in each eigenvalue

$$Ax = \lambda_i x, \qquad i = 1, \dots, k$$

  and solve for the eigenvectors.

Matrix Power of Diagonalizable Matrices

- ***Diagonalizable matrix***: $A \in \mathrm{Mat}(n \times n, \mathbb{R})$ whose eigenvectors form a basis.
- ***Diagonal form*** of A:

$$D := U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Matrix Power of Diagonalizable Matrices

- Matrix power of diagonalizable matrices:

$$A^k = UD^kU^{-1}, \quad D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix},$$

and in particular,

$$e^A = U \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} U^{-1}.$$

Self-Adjoint Matrices

If a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{F})$ is ***self adjoint***, then

- $\langle x, Ay \rangle = \langle Ax, y \rangle, A = \overline{A}^T$.
- All eigenvalues of A are real.
- If $\mathbb{F} = \mathbb{R}$, A is a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, then it has at least one eigenvalue.
- If $U \subset \mathbb{R}^n$ is ***invariant*** under A (if $x \in U$, then $Ax \in U$), then $U^\perp$ is invariant under A.

Self-Adjoint Matrices

If a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{F})$ is ***self adjoint***, then

- 1.8.6. Spectral Theorem. There exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
- A is diagonalizable.
- Let $U = (v_1, \ldots, v_n)$ is an orthonormal basis consisting of eigenvectors. Then $U^{-1} = U^*$.
- The diagonal form of A is given by

$$D = U^* A U.$$

Positive Definite Linear Maps

For a matrix $A \in \mathrm{Mat}(n \times n, \mathbb{R})$,

- it is ***positive definite*** if

$$\langle x, Ax \rangle > 0 \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

- it is ***negative definite*** if $-A$ is positive definite

$$\langle x, Ax \rangle < 0 \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

- if A is self-adjoint, then A is positive definite iff all eigenvalues of A are strictly greater than zero.

Generalized Eigenvectors

1.9.1. Definition. Let λ be an eigenvalue of a matrix A. Then v is a ***generalized eigenvector*** of rank r, $r \in \mathbb{N} \setminus \{0\}$, if

$$(A - \lambda\mathbb{1})^r v = 0 \quad \text{and} \quad \underline{(A - \lambda\mathbb{1})^{r-1} v \neq 0}.$$

Denote

$$E_k = \{v \in V : (A - \lambda\mathbb{1})^k v = 0\}.$$

Then a generalized eigenvector of rank r is an element in $E_r \setminus E_{r-1}$.

Finding Generalized Eigenvectors

Bottom-up Method. (For specific λ.)

1. Find the ordinary eigenspace E_1 using $(A - \lambda\mathbb{1})v^{(1)} = 0$. Set $E = E_1$.
2. If $\dim E < a_\lambda$, where a_λ is the algebraic multiplicity, use a suitable $v^{(1)} \in E_1$ to find $v^{(2)}$ using
$$(A - \lambda\mathbb{1})v^{(2)} = v^{(1)}.$$
3. $E = E_1 \oplus \mathrm{span}\{v^{(2)}\}$.
4. Repeat step 2 and 3 for one higher dimension until there is no solution.

Finding Generalized Eigenvectors

Top-down Method. (For specific λ.)

1. Find the highest rank necessary: $m := a_\lambda - \dim V_\lambda + 1$.
2. Solve
$$(A - \lambda\mathbb{1})^m v = 0, \qquad (A - \lambda\mathbb{1})^{m-1} \neq 0$$
to obtain $v^{(m)}$.
3. Set
$$v^{(m-1)} := (A - \lambda\mathbb{1})v^{(m)}$$
and similarly for lower ranks to find a set of generalized eigenvectors $\{v^{(m)}, v^{(m-1)}, \ldots, v^{(1)}\}$.

Chain of Generalized Eigenvectors

Note. The vectors found by a chain of multiplication in the top-down method are linearly independent. Suppose we have r and v such that

$$(A - \lambda \mathbb{1})^r v = 0, \qquad (A - \lambda \mathbb{1})^{r-1} v \neq 0.$$

Then for the chain of generalized eigenvectors $\{(A - \lambda \mathbb{1})^{r-1} v, (A - \lambda \mathbb{1})^{r-2} v, \ldots, v\}$, assume

$$a_0 v + a_1 (A - \lambda \mathbb{1})^1 v + a_2 (A - \lambda \mathbb{1})^3 v + \cdots + a_{r-1} (A - \lambda \mathbb{1})^{r-1} v = 0.$$

Then multiplying by $(A - \lambda \mathbb{1})^{r-1}$, we obtain

$$a_0 (A - \lambda \mathbb{1})^{r-1} v = 0 \quad \Rightarrow \quad a_0 = 0.$$

Continuing this process, we can verify that $a_0 = \cdots = a_{r-1} = 0$.

Finding Generalized Eigenvectors

Example 1. Find the generalized eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} .$$

Matrix Power of Non-diagonalizable Matrices

- The generalized eigenvectors give a “nearly diagonalized” matrix.
- We will then calculate the matrix power using this “nearly diagonalized matrix”.

Jordan Matrices

Definition.

- ***Jordan block of size*** $k \in \mathbb{N} \setminus \{0\}$, $\lambda \in \mathbb{C}$:

$$J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in \mathrm{Mat}(k \times k, \mathbb{C}).$$

- ***Jordan matrix*** with not necessarily distinct $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ and $k_1, \dots, k_m \in \mathbb{N}$:

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_m}(\lambda_m) \end{pmatrix}.$$

Jordan Normal Form and Nilpotent Matrix

- ***Jordan normal form*** of $A \in \mathrm{Mat}(n \times n, \mathbb{C})$: there exists a basis of $\mathbb{C}^n$ consisting of generalized eigenvectors such that

$$J := U^{-1}AU$$

is a Jordan matrix, where U is the transformation into this basis.
- ***Nilpotent matrix***: there exists $k \in \mathbb{N}$ such that $N^k = 0$.

Jordan Normal Form

Note. Given a matrix A, we can directly write out a Jordan normal form without calculating $U^{-1}AU$. This follows from:

1. The number of Jordan blocks is the number of linearly independent eigenvectors of A.
2. The size of a Jordan block for an eigenvector v is number of vectors in the corresponding cycle of generalized eigenvectors of A.

Jordan Normal Form

Example 2. Write out a Jordan normal form of the matrix

$$A = \begin{pmatrix} 7 & 0 & 0 & 4 & 0 & 0 \\ 0 & 7 & 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 0 & 0 & 6 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

Jordan Normal Form and Nilpotent Matrix

Note. For a Jordan matrix $J = D + N$, where N is a nilpotent matrix $N \in \mathrm{Mat}(n \times n, \mathbb{R})$ and D is a diagonal matrix $D \in \mathrm{Mat}(n \times n, \mathbb{R})$,

$$e^J = e^D \cdot e^N = e^N \cdot e^D,$$

which is in general not the case for D, N.

Matrix Power of Non-diagonalizable Matrices

To find e^A,

1. Find generalized eigenvectors $\{v_1, \ldots, v_n\}$.
2. Construct a basis consisting of these generalized eigenvectors and find the transformation U to this basis. Then

$$J := U^{-1}AU = D + N,$$

   where D is a diagonal matrix and N is a nilpotent matrix.
3. Then

$$e^A = Ue^J U^{-1} = U(e^{J_{k_1}(\lambda_1)}, \ldots, e^{J_{k_m}(\lambda_m)})U^{-1},$$

   where e^N is found by expanding the series.

Solving Linear System of Equations Using Matrix Power

Example 3. Solve the system

$$
\begin{aligned}
x_1' &= 9x_1 + 6x_2 \\
x_2' &= -10x_1 - 7x_2
\end{aligned}
$$

for $x(t)$.

Solving Linear System of Equations Using Matrix Power

Example 4. Solve the linear system

$$\begin{aligned} x_1' &= -9x_1 + 9x_2 \\ x_2' &= -16x_1 + 15x_2 \end{aligned}$$

for $x(t)$.

Thanks for your attention!