Honors Mathematics IV
Midterm 2 Review

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Singularities and Poles

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Classification of Singularities

Definition. $\Omega \subset \mathbb{C}$ is open, $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. (f has a ***point singularity at z_0*.**) The singularity is

- ***removable***: there exists an analytic continuation $\tilde{f} : \Omega \to \mathbb{C}$. (i.e., $\lim_{z \to z_0} f(z)$ exists.)
- a ***pole***:
  1. $g = 1/f$ is holomorphic on $\Omega \setminus \{z_0\}$.
  2. g has a removable singularity at z_0.
  3. $\tilde{g}(z_0) = 0$.
- ***essential***: it is neither removable nor a pole.

Zeros

2.3.5. Theorem. f is holomorphic in a connected open set Ω with a zero at $z_0 \in \Omega$ and does not vanish identically in Ω. In a neighborhood $U \subset \Omega$ of z_0,

$$f(z) = (z - z_0)^n g(z) \qquad \text{for all } z \in U,$$

where g is non-vanishing and holomorphic.

- n, g are both unique.
- n is the ***multiplicity*** or ***order*** of the zero.
- The zero is ***simple*** if $n = 1$.

Poles

2.3.8. Theorem. $f : \Omega \to \mathbb{C}$ has a pole at $z_0 \in \Omega$, then in a neighborhood U of z_0,

$$f(z) = (z - z_0)^{-n} h(z) \qquad \text{for all } z \in U,$$

where h is non-vanishing and holomorphic.

- n, h are both unique.
- n is the ***multiplicity*** or ***order*** of the pole.
- The pole is ***simple*** if $n = 1$.

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Representation Near Poles

2.3.11. Theorem. If $f : \Omega \to \mathbb{C}$ has a pole of order n at $z_0 \in \Omega$, then there exists a neighborhood $U \subset \Omega$ of z_0, numbers $a_{-n}, \dots, a_{-1} \in \mathbb{C}$ and a holomorphic function $G : U \to \mathbb{C}$ such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

for all $z \in U$.

- ***Principal part***:

$$P(z) := \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}.$$

- ***Residue***:

$$\operatorname{res}_{z_0} f := a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z)).$$

Representation Near Poles

Example 1. Does the complex logarithm have an essential singularity at $z = 0$?

A. No, it has a pole because $\lim_{z \to 0} |\ln(z)| = \infty$.

B. Yes.

C. No, because it is not an isolated singularity.

Example 2. Let $a, b \in \mathbb{C}$ and $f(z) = \dfrac{z-a}{z-b}$. The residue of f at $z = b$ is

A. $b - a$.

B. $-(a + b)$.

C. a/b.

Representation Near Poles

Example 3. Find the principal part for the Laurent series

$$f(z) = \frac{\pi^2}{(\sin \pi z)^2}$$

centered at $k \in \mathbb{Z}$.

Representation Near Poles

Solution 3. We know from the power series for sine function and $\sin \pi z = (-1)^k \sin[\pi(z-k)]$,

$$
\begin{aligned}
\sin^2 \pi z &= \sin^2[\pi(z-k)] \\
&= \left(\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}(z-k)^{2n+1}}{(2n+1)!} \right)^2 \\
&= \left(\pi(z-k) - \frac{\pi^3 (z-k)^3}{3!} + \cdots \right)^2 \\
&= \pi^2 (z-k)^2 - \frac{\pi^4}{3}(z-k)^4 + \mathcal{O}((z-k)^6).
\end{aligned}
$$

Representation Near Poles

Solution 1 (continued). Then

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{\pi^2(z-k)^2\left(1 - \frac{\pi^2}{3}(z-k)^2 + \mathcal{O}((z-k)^3)\right)}$$

$$= \frac{\pi^2}{\pi^2(z-k)^2}\left(1 + \frac{\pi^2(z-k^2)}{3} + \cdots\right),$$

and thus the principal part is $\dfrac{1}{(z-k)^2}$.

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Summary

- Singularities.
  1. Removable singularity.
  2. Pole.
  3. Essential singularity.
- Zeros and Poles.
  1. Multiplicity or order.
  2. Simple zero/pole.
- Representation near poles.
  1. Principle part.
  2. Residue.

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The Complex Logarithm

Definition. Let

$$\mathbb{R}_-^0 := \{x \in \mathbb{R} : x \leq 0\}, \qquad \mathbb{R}_+^0 := \{x \in \mathbb{R} : x \geq 0\}.$$

- Principle branch: $\ln : \mathbb{C} \setminus \mathbb{R}_-^0 \to \mathbb{C}$.

$$\ln(re^{i\varphi}) = \ln r + \varphi i, \qquad r > 0, -\pi < \varphi < \pi.$$

- $\ln : C \setminus \mathbb{R}_+^0 \to \mathbb{C}$.

$$\ln(re^{i\varphi}) = \ln r + \varphi i, \qquad r > 0, 0 < \varphi < 2\pi.$$

Note. This branch is not the analytic expansion of the logarithm in $\mathbb{R}$.

Complex Power and Roots

- Complex power.

$$z^\alpha := e^{\alpha \ln z}, \qquad \alpha \in \mathbb{C}.$$

- Complex root.

$$\sqrt[n]{z} := z^{1/n}.$$

Note. For $n \in \mathbb{N}$,

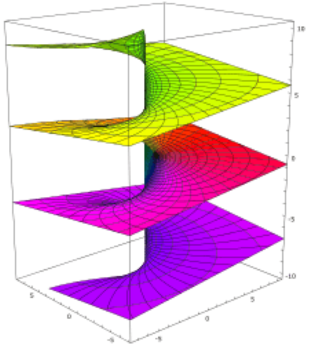
$$(z^{1/n})^n = \prod_{k=1}^{n} e^{\frac{1}{n} \ln z} = e^{\sum_{k=1}^{n} \frac{1}{n \ln z}} = e^{\frac{n}{n} \ln z} = e^{\ln z} = z.$$

The Complex Logarithm

Branches. We have many options regarding the choice of branch.

- The evaluated integral should be continuous.
- The branch should exhibits a measurable integral.

The choice of branch can be visualized as below.



The Complex Logarithm

Evaluation using a branch.

- Whatever the branch chosen, it should cover the whole complex plane without overlapping and excluding half of an axis.
- The decision of ϕ should rely on geometric considerations.
- When evaluating complex logarithms, complex numbers represented in $x + yi$ can be transformed to

$$x + yi = Re^{i\theta}, \qquad R = \sqrt{x^2 + y^2},$$

where θ is in the branch.

The Complex Logarithm

Example. Using different branches and $t \in (0,1)$,

1. $(0, 2\pi)$. $\lim\limits_{\varepsilon \to 0} \dfrac{1}{\sqrt{(\varepsilon - it)^2 + 1}} = -\dfrac{1}{\sqrt{1 - t^2}}$,
2. $(-2\pi, 0)$. $\lim\limits_{\varepsilon \to 0} \dfrac{1}{\sqrt{(\varepsilon - it)^2 + 1}} = \dfrac{1}{\sqrt{1 - t^2}}$.

Note.

- The sign of the imaginary part and the real part determines the position of the complex number in the complex plane, and should be considered in complex logarithm.
- Branch appears in integrals involving square root, logarithm, etc.

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Evaluating Real Integrals Using Residue Calculus

1. Extend the real domain to complex domain.
   - Change $x \in \mathbb{R}$ to $z \in \mathbb{C}$.
   - Consider e^{iz} for $\sin x, \cos x$.
2. Find a suitable contour and the branch (if needed).
3. Find poles for $f(z)$.
4. Calculate residues for poles. (If the contour cannot be decided yet, find residue for all poles.)
   - Write out expression near poles.
   - Use
   $$\mathrm{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}((z - z_0)^n f(z)).$$
5. Write out residue theorem.
6. Save the desired integral and solve other parts.

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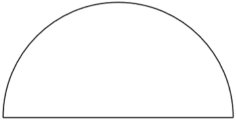
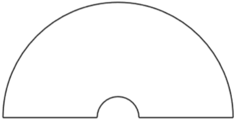
Contours — Semi-circle

Semi-circle.



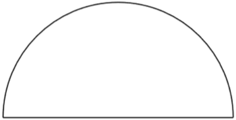


Integrals. We have used this contour to find

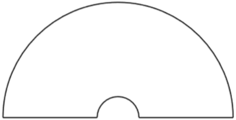
1. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx,$
2. $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2}\,dx,$
3. $\displaystyle\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}\,dx,$

Contours — Semi-circle

Semi-circle.



Semicircle



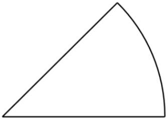
Indented semicircle

Integrals. We have used this contour to find

4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$,
5. $\displaystyle\int_{0}^{\infty} \frac{x \sin x}{(x^2+4)^2}\,dx$,
6. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$,
7. $\displaystyle\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx$.

Contours — Sector

Sector.



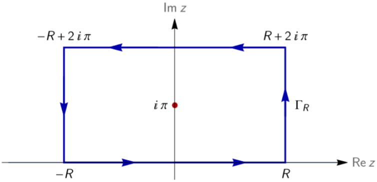
Sector

Integrals.

- Integral containing $\sin(x^n), \cos(x^n)$. (choose central angle $\pi/(2n)$.)
- We have used this contour to find
  1. $\int_0^\infty \sin x^2 dx$,
  2. $\int_0^\infty \cos x^2 dx$.

Contours — Rectangle

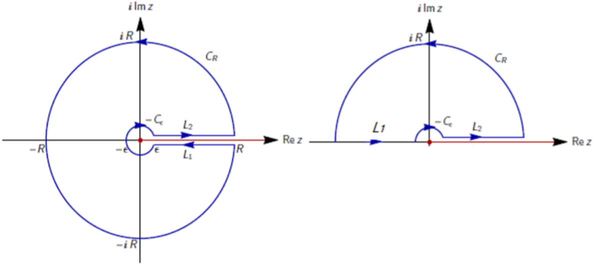
Rectangle.



Integrals.

1. $\displaystyle\int_0^\infty \frac{e^{ax}}{1+e^x}\,dx, 0 < a < 1.$

Contours — (Semi-)Circle without a Half-axis.

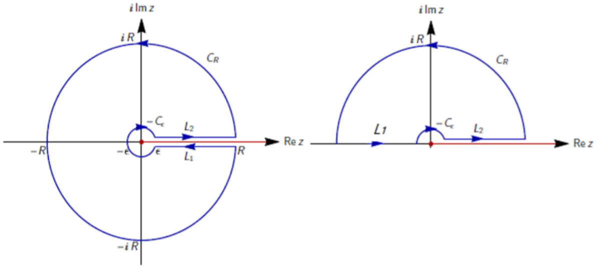


Integrals. Integrals containing $x^{1/n}, \ln x$(with branch $\mathbb{C} \setminus \mathbb{R}_+^0$).

1. $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2 + a^2}\,dx,$
2. $\displaystyle\int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx.$

Contours — (Semi-)Circle without a Half-axis

Contours — (Semi-)Circle without a Half-axis.



Integrals. When approaching from downside to positive real axis (with branch $\mathbb{C} \setminus \mathbb{R}_+^0$),

1. $\sqrt{z} = \sqrt{Re^{i\theta}} = R^{1/2}e^{i\theta/2} \to -R^{1/2}, \theta \in (0, 2\pi)$.
2. $\ln z = \ln Re^{i\theta} \to \ln R + i\theta, \theta \in (0, 2\pi)$.

Final Remarks

1. Describe the contour along which you are integrating.
2. Clearly choose the branch if complex logarithm is required.
3. A brief proof is required when you want to conclude that a part of the integral vanishes. (Sometimes you can apply Jordan's lemma.)
4. Describe necessary details.

Good luck for your Midterm 2!