Honors Mathematics IV
RC 5

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Holomorphic Functions

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is ***complex differentiable***, or ***holomorphic***, as $z \in \mathbb{C}$ if

$$f'(z) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

exists.

- A holomorphic function is infinitely often differentiable.
- A holomorphic function is analytic.
- Any closed integral of a holomorphic function vanished.

Complex Differentiability

Cauchy-Riemann differential equations. The complex function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x + yi) = u(x, u) + iv(x, y)$$

is holomorphic only if its components u, v satisfy

$$u_x = v_y, \qquad u_y = -v_x.$$

Complex Differentiability

Differential operators.

- Define
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$
- If $f = u + iv : \mathbb{C} \to \mathbb{C}$ is holomorphic at $z \in \mathbb{C}$, then
$$f'(z) = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}, \qquad \frac{\partial f}{\partial \overline{z}} = 0.$$

Sets in the Complex Plane

Let $\Omega \subset \mathbb{C}$, then it is

- ***Open***:

$$\forall z \in \Omega, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\} \subset \Omega.$$

- ***Bounded***: $\Omega \subset B_R(0)$ for some $R > 0$.
- ***Compact*** (denoted as K):
  - Definition. Every sequence in K has a converging subsequence with limit in K.
  - K is compact $\Leftrightarrow$ K is closed and bounded.
- ***Disconnected***: For an open(closed) set Ω, there exists two open(closed) sets $\Omega_1, \Omega_2 \in \mathbb{C}$ such that

$$\Omega = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \emptyset.$$

- ***Region(domain)***: An open and connected set in $\mathbb{C}$.

Power Series

- Convergence. The series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence ρ: $(1/\rho = \lim_{n\to\infty} \sup |a_n|^{1/n}.)$
  - converges if $|z| < \rho$.
  - diverges if $|z| > \rho$.
- Differentiation. Within the disc of convergence,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

is the derivative of f and has the same radius of convergence as f.

Trigonometric Functions

- Series definition.

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

- Euler formulas.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Analytic Functions

Definition. A function f defined on an open set $\Omega \subset \mathbb{C}$ is said to be ***analytic*** (or have a power series expansion) at z_0 if there exists a power series centered at z_0, with positive radius of convergence such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all z in a neighborhood of z_0. Moreover, with the same radius of convergence,

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}.$$

Note. A holomorphic function is automatically analytic.

Integrals along Complex Curves

Definition. Let $\Omega \subset \mathbb{C}$ be an open set, f is holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ and oriented smooth curve, the integral of f along $\mathcal{C}^*$ is defined as

$$\int_{\mathcal{C}^*} f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt.$$

In particular, we have

- ***Curve length***:

$$\ell(\mathcal{C}) := \left| \int_{\mathcal{C}} dz \right|.$$

- Orientation:

$$\int_{-\mathcal{C}^*} f(z)dz = -\int_{\mathcal{C}^*} f(z)dz.$$

- Inequality:

$$\left| \int_{\mathcal{C}^*} f(z)dz \right| \leq \ell(\mathcal{C}) \cdot \sup_{z \in \mathcal{C}} |f(z)|.$$

Primitives

Consider a continuous function f in an open set $\Omega \subset \mathbb{C}$ and an oriented curve $\mathcal{C}^*$.

- 2.2.5. Theorem. If $\mathcal{C}^*$ begins at w_1 and ends at w_2 and has primitive F,

$$\int_{\mathcal{C}^*} f(z)dz = F(w_2) - F(w_1), \quad \oint_{\mathcal{C}} f(z)dz = 0.$$

- 2.2.9 and 10. Theorem. f is holomorphic. $D \subset \Omega$ is a triangle or a rectangle whose interior is contained in Ω

$$\oint_D f(z)dz = 0.$$

- 2.2.11. Theorem. f is holomorphic in an open disc $\Rightarrow$ f has a primitive in that disc.

Cauchy's Theorem

- 2.2.12. Cauchy's Theorem. f is holomorphic in a disc, then for any closed curve $\mathcal{C}$ in that disc,

$$\oint_{\mathcal{C}} f(z)dz = 0.$$

- Morera's Theorem (converse of Cauchy's theorem). f is a continuous function in the open disc D, such that for any triangle T contained in D,

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Evaluating Real Integrals using Cauchy's Theorem

1. Transform the integrand into a complex function $f(z)$.
2. Select appropriate contour and use Cauchy's theorem to obtain

$$\int_{-R}^{-\varepsilon} f(z)dz + \underbrace{\int_{-C_\varepsilon} f(z)dz}_{I_\varepsilon} + \int_{\varepsilon}^{R} f(z)dz + \underbrace{\int_{C_R} f(z)dz}_{I_R} = 0.$$

3. Let $\varepsilon \to 0, R \to \infty$. In many cases,

$$I_R \to 0, \qquad I_\varepsilon \to I_0.$$

4. Then

$$\int_{\mathbb{R}} f(x)dx = -I_0.$$

Note. $-I_0$ is usually not the desired integral. We need to take some transformations.

Evaluating Real Integrals using Cauchy's Theorem

Example 1. Show that if $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx.$$

This means that $e^{-\pi x^2}$ is its own Fourier transform (later). If $\xi = 0$, the formula is the know integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx.$$

Evaluating Real Integrals using Cauchy's Theorem

Remarks. Evaluate the following integrals.

1. Slide Example 2.2.14. Verify that

$$\int_0^\infty \frac{1-\cos x}{x^2}dx = \frac{\pi}{2},$$

by integrating

$$f(z) = \frac{1-e^{iz}}{z^2}.$$

2. Assignment 5.4 Show that

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

by integrating

$$f(z) = \frac{e^{iz}-1}{2iz}.$$

Jordan's Lemma

2.2.15. Jordan's Lemma. Assume that for some $R_0 > 0$ the function $g : \mathbb{R} \setminus B_{R_0}(0) \to \mathbb{C}$ is holomorphic, Let

$$f(z) = e^{iaz} g(z), \qquad \text{for some } a > 0.$$

Let

$$C_R = \{z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leq \theta \leq \pi\}$$

be a semi-circle segment in the upper half-plane and assume that

$$\sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \xrightarrow{R \to \infty} 0.$$

Then

$$\lim_{R \to \infty} \int_{C_R} f(z) dz = 0.$$

Cauchy's Integral Formula

2.2.16. Cauchy's Integral Formula. Suppose f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$. If D is an open disc whose closure is contained in Ω, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \text{for all } z \in D,$$

where $C = \partial D$ is the (positively oriented) boundary circle of D. Furthermore,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \qquad \text{for all } z \in D.$$

Analytic Functions.

2.2.20. Theorem. Suppose f is a holomorphic function in an open set Ω. If D is an open disc centered at z_0 whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

for all $z \in D$ and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \qquad n \in \mathbb{N}.$$

Analytic Functions

2.2.23. Theorem. Let $\Omega \subset \mathbb{C}$ be a region and $f, g : \Omega \to \mathbb{C}$ two holomorphic functions. Suppose $S \subset \Omega$ has an accumulation point that is contained in Ω and that

$$f(z) = g(z) \qquad \text{for all } z \in S.$$

Then $f(z) = g(z)$ for all $z \in \Omega$.

Analytic Continuation. $g : \Omega \to \mathbb{C}, \Omega \subset \mathbb{C}$ is an analytic continuation of $f : M \to \mathbb{C}, M \subset \Omega$ if

- g is holomorphic.
- $g(z) = f(z)$ for $z \in M$.

Classification of Singularities

Definition. $\Omega \subset \mathbb{C}$ is open, $z_0 \in \Omega$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. (f has a ***point singularity at z_0.***) The singularity is

- ***removable***: there exists an analytic continuation $\tilde{f} : \Omega \to \mathbb{C}$. (i.e., $\lim_{z \to z_0} f(z)$ exists.)
- a ***pole***:
  1. $g = 1/f$ is holomorphic on $\Omega \setminus \{z_0\}$.
  2. g has a removable singularity at z_0.
  3. $\tilde{g}(z_0) = 0$.
- ***essential***: it is neither removable nor a pole.

Zeros

2.3.5. Theorem. f is holomorphic in a connected open set Ω with a zero at $z_0 \in \Omega$ and does not vanish identically in Ω. In a neighborhood $U \subset \Omega$ of z_0,

$$f(z) = (z - z_0)^n g(z) \qquad \text{for all } z \in U,$$

where g is non-vanishing and holomorphic.

- n, g are both unique.
- n is the ***multiplicity*** or ***order*** of the zero.
- The zero is ***simple*** if $n = 1$.

Poles

2.3.8. Theorem. $f : \Omega \to \mathbb{C}$ has a pole at $z_0 \in \Omega$, then in a neighborhood U of z_0,

$$f(z) = (z - z_0)^{-n} h(z) \qquad \text{for all } z \in U,$$

where h is non-vanishing and holomorphic.

- n, h are both unique.
- n is the ***multiplicity*** or ***order*** of the pole.
- The pole is ***simple*** if $n = 1$.

Representation Near Poles

2.3.11. Theorem. If $f : \Omega \to \mathbb{C}$ has a pole of order n at $z_0 \in \Omega$, then there exists a neighborhood $U \subset \Omega$ of z_0, numbers $a_{-n}, \ldots, a_{-1} \in \mathbb{C}$ and a holomorphic function $G : U \to \mathbb{C}$ such that

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$$

for all $z \in U$.

- ***Principal part***:

$$P(z) := \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}.$$

- ***Residue***:

$$\operatorname{res}_{z_0} f := a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}((z-z_0)^n f(z)).$$

The Residue Theorem

2.4.2. Theorem. Suppose that f is holomorphic in an open set containing a (positively oriented) toy contour $\mathcal{C}$ and its interior, except for poles at the points $z_1, \ldots, z_N$ inside $\mathcal{C}$. Then

$$\int_{\mathcal{C}} f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_k} f.$$

Residue Calculus

2.4.5. Theorem. Let P and Q be polynomials of degree m and n, respectively, where $n \geq m+2$. If $Q(x) \neq 0$ for $x > 0$, if Q has a zero of order at most 1 at the origin and if

$$f(z) = \frac{z^\alpha P(z)}{Q(z)}, \qquad 0 < \alpha < 1,$$

then

$$\int_0^\infty \frac{x^\alpha P(x)}{Q(x)} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{j=1}^{k} \operatorname{res}_{z_j} f,$$

where $z_1, \ldots, z_k$ are the nonzero poles of P/Q.

Evaluating Real Integrals Using Residue Calculus

1. Extend the real domain to complex domain.
   - Change $x \in \mathbb{R}$ to $z \in \mathbb{C}$.
   - Consider e^{iz} for $\sin x, \cos x$.
2. Find a suitable contour and the branch (if needed).
3. Find poles for $f(z)$.
4. Calculate residues for poles. (If the contour cannot be decided yet, find residue for all poles.)
   - Write out expression near poles.
   - Use
   $$\operatorname{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)).$$
5. Write out residue theorem.
6. Save the desired integral and solve other parts.

Exercises.

Exercise 1. Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial f}{\partial \overline{z}}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial f}{\partial z}.$$

This is the complex version of the chain rule.

Exercises.

Exercise 2. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\ln z = \ln r + i\theta \quad \text{where } z = e^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Exercises.

Exercise 3. Evaluate the integrals

$$\int_0^\infty e^{-ax}\cos(bx)dx \quad \text{and} \quad \int_0^\infty e^{-ax}\sin(bx)dx$$

by integrating e^{-Az}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

Exercises.

Exercise 4. Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(z)dz = 0.$$

Exercises.

Exercise 5. If f is a holomorphic function on the strip $-1 < y < 1, x \in \mathbb{R}$ with

$$|f(z)| \leq A(1 + |z|)^{\eta}$$

for all z in that strip, where η is a fixed real number. Show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$|f^{(n)}(x)| \leq A_n(1 + |x|)^{\eta}$$

for all $x \in \mathbb{R}$.

Thanks for your attention!