Honors Mathematics IV
Midterm 1 Review

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Generalized Eigenvectors

1.9.1. Definition. Let λ be an eigenvalue of a matrix A. Then v is a ***generalized eigenvector*** of rank r, $r \in \mathbb{N} \setminus \{0\}$, if

$$(A - \lambda \mathbb{1})^r v = 0 \quad \text{and} \quad \underline{(A - \lambda \mathbb{1})^{r-1} v \neq 0}.$$

Denote

$$E_k = \{v \in V : (A - \lambda \mathbb{1})^k v = 0\}.$$

Then a generalized eigenvector of rank r is an element in $E_r \setminus E_{r-1}$.

Finding Generalized Eigenvectors

Bottom-up Method. (For specific λ.)

1. Find the ordinary eigenspace E_1 using $(A - \lambda \mathbb{1})v^{(1)} = 0$. Set $E = E_1$.
2. If $\dim E < a_\lambda$, where a_λ is the algebraic multiplicity, use a suitable $v^{(1)} \in E_1$ to find $v^{(2)}$ using
$$(A - \lambda \mathbb{1})v^{(2)} = v^{(1)}.$$
3. $E = E_1 \oplus \mathrm{span}\{v^{(2)}\}$.
4. Repeat step 2 and 3 for one higher dimension until there is no solution.

Finding Generalized Eigenvectors

Top-down Method. (For specific λ.)

1. Find the highest rank necessary: $m := a_\lambda - \dim V_\lambda + 1$.
2. Solve
$$(A - \lambda\mathbb{1})^m v = 0, \qquad (A - \lambda\mathbb{1})^{m-1} \neq 0$$
   to obtain $v^{(m)}$.
3. Set
$$v^{(m-1)} := (A - \lambda\mathbb{1})v^{(m)}$$
   and similarly for lower ranks to find a set of generalized eigenvectors $\{v^{(m)}, v^{(m-1)}, \dots, v^{(1)}\}$.

Jordan Normal Form

Note. Given a matrix A, we can directly write out a Jordan normal form without calculating $U^{-1}AU$. This follows from:

1. The number of Jordan blocks is the number of linearly independent eigenvectors of A.
2. The size of a Jordan block for an eigenvector v is number of vectors in the corresponding cycle of generalized eigenvectors of A.

Matrix Power of Non-diagonalizable Matrices

To find e^A,

1. Find generalized eigenvectors $\{v_1, \ldots, v_n\}$.
2. Construct a basis consisting of these generalized eigenvectors and find the transformation U to this basis. Then

   $$J := U^{-1}AU = D + N,$$

   where D is a diagonal matrix and N is a nilpotent matrix.
3. Then

   $$e^A = Ue^J U^{-1} = U(e^{J_{k_1}(\lambda_1)}, \ldots, e^{J_{k_m}(\lambda_m)})U^{-1}.$$

The Wronskian

- The Wronskian of n solutions of a system. $x^{(1)}, \ldots, x^{(n)}$ are n arbitrary solutions of the homogeneous system

$$\frac{dx}{dt} = A(t)x.$$

Then the ***Wronskian*** is given by

$$W_{x_1,\ldots,x_n}(t) := \det(x^{(1)}(t), \ldots, x^{(n)}(t)).$$

- 1.10.8. Lemma and Abel's equation.

$$\frac{dW}{dt} = a(t)W, \quad a(t) = \operatorname{tr} A(t), \quad W(t) = W(t_0)e^{-\int_{t_0}^{t} a(s)ds}.$$

Linear Systems

General linear systems.

$$\frac{dx}{dt} = A(t)x + b(t), \qquad A \in \mathrm{Mat}(n \times n), t \in I, x \in \mathbb{R}^n.$$

- Constant A.
  - Homogeneous: $b(t) = 0$ — Fundamental system constructed by (generalized) eigenvectors and matrix power.
  - Inhomogeneous: $b(t) \neq 0$ — Particular solution found from the fundamental system and Wronskian.
- Variable A.
  - Homogeneous: $b(t) = 0$ — Given fundamental systems.
  - Inhomogeneous: $b(t) \neq 0$ — Particular solution found from the fundamental system and Wronskian.

Linear Systems

General linear systems.

$$\frac{dx}{dt} = A(t)x + b(t), \qquad A \in \mathrm{Mat}(n \times n), t \in I, x \in \mathbb{R}^n.$$

- Constant A:
  - Homogeneous: Find the fundamental matrix:
    1. A is diagonalizable: $\{u_1, \ldots, u_n\} \in \mathbb{R}^n$ is a basis of ***eigenvectors***. $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = U^{-1}AU$ is a ***diagonal matrix***. The fundamental matrix is

       $$X(t) = Ue^{\mathrm{diag}(\lambda_1, \ldots, \lambda_n)t} = (e^{\lambda_1 t}u_1, \ldots, e^{\lambda_n t}u_n).$$

    2. A is non-diagonalizable: $\{u_1, \ldots, u_n\} \in \mathbb{R}^n$ is a basis of ***generalized eigenvectors***. $J = U^{-1}AU$ is a ***Jordan matrix***. The fundamental matrix is

       $$X(t) = Ue^{Jt}.$$

Linear Systems

General linear systems.

$$\frac{dx}{dt} = A(t)x + b(t), \qquad A \in \mathrm{Mat}(n \times n), t \in I, x \in \mathbb{R}^n.$$

- Constant A:
  - Inhomogeneous:
    1. Make the ansatz
    $$x_{\mathrm{part}}(t) = c_1(t)x^{(1)}(t) + \cdots + c_n(t)x^{(n)}(t).$$
    2. Find $c_k(t)$ by
    $$c_k(t) = \int \frac{W^{(k)}(\tau)}{W(\tau)}\, d\tau.$$

Linear Systems

General linear systems.

$$\frac{dx}{dt} = A(t)x + b(t), \qquad A \in \mathrm{Mat}(n \times n), t \in I, x \in \mathbb{R}^n.$$

- Constant A: To incorporate the initial condition

$$x(t_0) = x_0, \qquad x_0 \in \mathbb{R}^n,$$

we need

1. Fit the homogeneous solutions into the initial condition.
2. Adjust the integral of Wronskian to $\displaystyle\int_{t_0}^{t}$.

For an inhomogeneous linear system with initial conditions,

1. The homogeneous solution is used to fit in the initial conditions.
2. The particular solution is to fit in $b(t)$.

Linear Second-Order ODEs

General linear second-order ODEs.

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- Constant coefficients: $ay'' + by' + cy = g(t)$.
  - Homogeneous: $g(t) = 0$ — Find homogeneous solutions through the characteristic polynomial.
  - Inhomogeneous: $g(t) \neq 0$ — Find particular solution from homogeneous solutions and Wronskian.
- Variable coefficients: $y'' + p(t)y' + q(t)y = g(t)$.
  - Homogeneous: Given one homogeneous solution y_1, use reduction of order to find another independent solution y_2.
  - Inhomogeneous: Find particular solution using homogeneous solutions and Wronskian.

Linear Second-Order ODEs

General linear second-order ODEs.

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- Constant coefficients: $ay'' + by' + cy = g(t)$.
  - Homogeneous: $ay'' + by' + cy = 0$.
    1. $b^2 \neq 4ac, \lambda_1, \lambda_2 \in \mathbb{R}$.

       $$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R}.$$

    2. $b^2 \neq 4ac, \lambda_1, \lambda_2 \in \mathbb{C}$.

       $$y(t) = c_1 e^{\mathrm{Re}\lambda_i t} \sin\left(\mathrm{Im}\lambda_i t\right) + c_2 e^{\mathrm{Re}\lambda_i t} \cos\left(\mathrm{Im}\lambda_i t\right), \quad c_1, c_2 \in \mathbb{R}.$$

    3. $b^2 = 4ac, \lambda_1 = \lambda_2 \in \mathbb{R}$.

       $$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Linear Second-Order ODEs

General linear second-order ODEs.

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- <u>Constant coefficients</u>: $ay'' + by' + cy = g(t)$.
  - Inhomogeneous: $ay'' + by' + cy = g(t)$.
    1. Find two independent solutions y_1, y_2 of the homogeneous e-quation.
    2. Find particular solution $ay'' + by' + cy = 0$.

$$y_{\text{part}}(t) = -y^{(1)}(t) \int \frac{g(t)y^{(2)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt + y^{(2)}(t) \int \frac{g(t)y^{(1)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt.$$

    3. $y_{\text{inhom}}(t; c_1, c_2) = y_{\text{hom}}(t; c_1, c_2) + y_{\text{part}}(t)$.

Linear Second-Order ODEs

General linear second-order ODEs.

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- Constant coefficients: To incorporate the initial condition

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \qquad y_0, y_0' \in \mathbb{R},$$

we need

1. Fit the homogeneous solutions into the initial condition.
2. Adjust the integral of Wronskian to $\displaystyle\int_{t_0}^{t}$.

For an inhomogeneous second-order ODE with initial conditions,

1. The homogeneous solution is used to fit in the initial conditions.
2. The particular solution is to fit in $g(t)$.

Linear Second-Order ODEs

General linear second-order ODEs.

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- Variable coefficients: Reduction of order to find another independent solution of the homogeneous equation $y'' + p(t)y' + q(t)y = 0$.
  1. Given solution y_1, let
  $$y_2(t) = v(t)y_1(t).$$
  2. Plug y_2 into the original equation, simplify it to and solve
  $$y_1 v'' + (2y_1' + py_1)v' = 0$$
  to obtain v.
  3. Find $y_2(t)$ using
  $$y_2(t) = v(t)y_1(t).$$

Definitions

1. Trajectory for vector field and path for direction field.
2. Initial value problem.
3. Homogeneous and inhomogeneous linear equations.
4. Equilibrium, steady-state and transient solutions.
5. Integral curves.
6. Envelope.
7. Picard iteration.
8. Superposition principle of solutions.
9. Positive definite and negative definite matrices.
10. Geometric multiplicity and algebraic multiplicity.
11. Eigenvalue, eigenvector, and eigenspace.
12. Jordan block, Jordan matrix and Jordan normal form.
13. Fundamental system.

Definitions

Example 1. Which of the following ordinary differential equations are linear?

A. $\sin x(y' + x^2 y)'' - y = x^3$

B. $y'' = -y^2$

C. $y \cdot y'' - x^2 y = \cos(x)$

Definitions

Example 2. Let A be an $n \times n$ matrix. Then A will be diagonalizable if

A. all eigenvalues are distinct.

B. A is self-adjoint.

C. A is invertible.

Definitions

Example 3. A fundamental system of solutions to the system $\dot{x} = Ax$ is given by the column vectors of the matrix

A. $e^{Jt}U^{-1}$.

B. Ue^{Jt}.

C. $Ue^{Jt}U^{-1}$.

Separation of Variables

Equation.

$$y' = f(x) \cdot g(y), \qquad y(\xi) = \eta.$$

Solution.

$$\int_{\eta}^{y} \frac{ds}{g(s)} = \int_{\xi}^{x} f(t)dt.$$

Examples.

1. Ex 1.5.
   a. $y' = (1 + x)(1 + y)$.
   b. $y' = e^{x+y+3}$.
2. RC 1.

$$\frac{dy}{dx} + 2xy = x, \qquad y(1) = 2.$$

First-Order Linear Equations

Equation.

$$a_1(x)y' + a_0(x)y = f(x), \quad y(\xi) = \eta, \quad x \in I.$$

Solution.

$$y_{\text{inhom}}(x) = \eta \cdot e^{-G(x)} + e^{-G(x)} \int_\xi^x \frac{f(s)}{a_1(s)} e^{G(s)} ds,$$

$$G(x) := \int_\xi^x \frac{a_0(t)}{a_1(t)} dt.$$

Examples.

1. Slide 61. $y' + y \sin x = \sin^3 x$.

Transformable Equations

Equation.

$$y' = f(ax + by + c), a, b, c \in \mathbb{R}.$$

Solution.

$$u(x) := ax + by(x) + c \quad \Rightarrow \quad \begin{cases} u' = a + bf(u), \\ y(x) = \dfrac{u(x) - ax - c}{b}. \end{cases}$$

Examples.

1. Slide 68. $y' = (x + y)^2$.

Transformable Equations

Equation.

$$y' = f\left(\frac{y}{x}\right).$$

Solution.

$$u(x) = \frac{y(x)}{x}, \qquad x \neq 0 \quad \Rightarrow \quad \begin{cases} u' = \dfrac{f(u) - u}{x}, \\ y(x) = x \cdot u(x). \end{cases}$$

Examples.

1. Slide 70. $y' = \dfrac{y}{x} - \dfrac{x^2}{y^2}$.
2. RC 1. $\dfrac{dy}{dx} = \dfrac{4x + y}{x - 4y}$.

Transformable Equations

Equation.

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$$

Solution.

1. Define

$$u = a_1x + b_1y + c_1, \quad v = a_2x + b_2y + c_2.$$

2. Calculate

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = \left(a_1 + b_1 f\left(\frac{u}{v}\right)\right) \frac{b_2 \cdot \frac{du}{dv} - b_1}{a_1 b_2 - a_2 b_1}.$$

3. Transform into and solve

$$\frac{du}{dv} = g\left(\frac{u}{v}\right).$$

Bernoulli's Equation

Equation.

$$y' + gy + hy^{\alpha} = 0, \qquad \alpha \neq 1.$$

Solution. Multiplying with $(1-\alpha)y^{-\alpha}$ and,

$$u(x) := y^{1-\alpha}(x) \quad \Rightarrow \quad u' + (1-\alpha)g(x)u + (1-\alpha)h(x) = 0.$$

Examples.

1. Slide 74. $y' + \dfrac{y}{1+x} + (1+x)y^4 = 0.$
2. RC 1. $y' + \dfrac{4}{x}y = x^3y^2.$
3. Sample 1. $\dot{x} = t^4x + t^4x^4, y' = 5y - 5xy^3.$

Ricatti's Equation

Equation.

$$y' + gy + hy^2 = k.$$

Solution. Given a solution ϕ,

$$u := y - \phi \quad \Rightarrow \quad u' + (g + 2\phi h)u + hu^2 = 0.$$

Examples.

1. Transformation into a second-order linear differential equation.

$$u(x) = e^{\int h(x)y(x)dx} \quad \Rightarrow \quad u'' + \left(g - \frac{h'}{h}\right)u' - khu = 0.$$

Integral Curves

Equation.

$$h(x,y)y' + g(x,y) = 0, \quad x \in I \subset \mathbb{R}, \quad h(x,y) \neq 0.$$

Solution. Find potential $U(x,y) = \text{constant}$ of the vector field

$$F(x,y) = \begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix} \text{ or } F(x,y) = \begin{pmatrix} M(x,y)g(x,y) \\ M(x,y)h(x,y) \end{pmatrix},$$

$$M_y g + M g_y = M_x h + M h_x, \quad \left((\ln M)' = \frac{g_y - h_x}{h} \right).$$

Examples.

1. Slide 92. $y' = -\dfrac{(2x^2 + 2xy^2 + 1)y}{3y^2 + x}$.
2. Sample 1. $\left(\dfrac{x^2}{2} + 2xe^t \right) dt + (x + e^t) dx = 0.$

Implicit Equations

Equation.

$$F(y, y'; x) = 0, \qquad \gamma(p) = (x(p), y(p)).$$

Solution.

$$F(y(p), p; x(p)) = 0, \qquad \dot{y}(p) = p\dot{x}(p)$$

Examples.

1. RC 2. $y = (yy' + 2x)y', 2y = 2x^2 + 4xy' + (y')^2.$

Clairaut's Equation

Equation.

$$y = xy' + g(y').$$

Solution 1. Straight line solutions $y = cx + g(c)$ and,

$$x(p) = -\dot{g}(p), \qquad y(p) = -p\dot{g}(p) + g(p).$$

Solution 2. Straight line solutions $y = cx + g(c)$ and, by envelope equation,

$$\frac{\partial \gamma_1}{\partial c}\frac{\partial \gamma_2}{\partial x} = \frac{\partial \gamma_1}{\partial x}\frac{\partial \gamma_2}{\partial c}, \qquad \gamma(c,x) = \begin{pmatrix} x \\ cx + g(c) \end{pmatrix}.$$

Examples.

1. Slide 113. $y = x\left(y' + \dfrac{1}{y'}\right) + (y')^4$.

d'Alembert's Equation

Equation.

$$y = xf(y') + g(y').$$

Solution. Straight line solution $y = cx + d$ (if $f(c) = c$ and $d = g(c)$), and,

$$\dot{x} = \frac{x\dot{f}(p) + \dot{g}(p)}{p - f(p)}, \qquad \dot{y} = \dot{x}f + x\dot{f} + \dot{g}.$$

Examples.

1. Slide 103. $y = xy' + e^{y'}$.
2. RC 2. $y = xy' - \sqrt{y' - 1}, y = xy' + y'^2$.
3. RC 2. $y = x(y')^2 + \ln(y')^2$.

Final Remarks

1. Review assignments and sample exam.
2. Be prepared for integration.
3. Pay attention to details (minus sign, order of independent solutions in Wronskian, etc.).
4. Make sure you are able to recognize the equations and apply corresponding methods.
5. Fit in the initial conditions.

Good luck for your Midterm 1!