Honors Mathematics IV
RC 9

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Table of contents

Measure Zero

Definition. Let $\varepsilon > 0$. A set $\Omega \subset \mathbb{R}$ is said to have ***measure less than*** ε if there exists a (possibly countably infinite) family of intervals $\{I_k\}$ such that $\Omega \subset \bigcup I_k$ and the total length of the intervals is less than ε.

Measure Zero. A set $\Omega \subset \mathbb{R}$ is said to have ***measure zero*** if it has measure less than ε for any $\varepsilon > 0$. A property is said to hold ***almost everywhere*** (abbreviated by ***a.e.***) on a subset $D \subset \mathbb{R}$ if the set of all points of D where it does not hold has measure zero.

Orthogonality and Orthonormal Systems

Definition. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product vector space.

- ***Normed (normalized).*** $\langle v, v \rangle = 1$.
- ***Orthogonal (perpendicular).*** If $\langle u, v \rangle = 0$, then $u \perp v$.
- ***Orthonormal system.*** A family of vectors $\{v_k\}_{k \in I} \subset V$, $I \subset \mathbb{N}$, with

$$\langle v_j, v_k \rangle = \delta_{jk} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \quad j, k \in I,$$

i.e.m if $\|v_k\| = 1$ and $v_j \perp v_k$ for $j \neq k$.

Fourier Series

The Fourier series for $L^2([0,L])$:

- The Fourier-Euler Basis.

$$\mathcal{B}_1 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

- The Fourier-Cosine Basis.

$$\mathcal{B}_2 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

- The Fourier-Sine Basis.

$$\mathcal{B}_3 := \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

Exponential Fourier Series

Definition. The sequence

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{2L}} e^{inx\pi/L} \right\}_{n=-\infty}^{\infty}$$

is called a ***complex Fourier-Euler basis*** of the space $L^2([-L, L])$.

The Bessel functions.

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} e^{ix \sin t} dt$$

The Cosine and Sine Fourier Transforms

Definition.

- Cosine Fourier transform.

$$\mathcal{F}_c f(\xi) := \int_0^\infty f(y)\cos(\xi y)dy$$

- Sine Fourier transform.

$$\mathcal{F}_s f(\xi) := \int_0^\infty f(y)\sin(\xi y)dy$$

Convergence of Fourier Series

Dirichlet's rule. Let $f \in L^2([a,b])$ be piecewise continuously differentiable. Then

1. On any subinterval $[a', b'] \subset [a,b]$ with $a' > a$, $b' < b$ on which f is continuous, the Fourier series converges uniformly towards f.
2. At any point $c \in [a,b]$, we have the pointwise limit

$$S_N(x) \xrightarrow{N\to\infty} \frac{\lim_{y \nearrow x} f(y) + \lim_{y \searrow x} f(y)}{2}.$$

Fourier Series

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + 2\pi) = f(x)$ and

$$f(x) = e^x, \qquad -\pi < x < \pi.$$

Find the Fourier series of f and use it to evaluate

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + 1}.$$

Boundary Value Problems

Definition.

- ***(Second-order) initial value problem (IVP).***

$$y'' = f(y', y, x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1.$$

- ***(Second-order) boundary value problem (BVP)*** is in the form of

1. $$y'' = f(y', y, x), \quad y(x_0) = y_0, \quad y(x_1) = y_1$$
2. $$y'' = f(y', y, x), \quad y'(x_0) = y_0, \quad y'(x_1) = y_1$$

or mixture.

The Sturm-Liouville Equation

- ***Sturm-Liouville operator***:

$$L := -\frac{1}{r(x)}\left(\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right)$$

- ***Sturm-Liouville equation***: (Defined on $I \in \mathbb{R}$)

$$Lu = \lambda u, \qquad x \in I$$

or

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + (q(x) + \lambda r(x))u = 0, \quad x \in I.$$

- ***Regular Sturm-Liouville operator***:
  1. $I = (a, b)$ is a finite interval.
  2. $p, p', q, r \in C([a, b])$.
  3. $p(x) > 0$ and $r(x) > 0$ for all $x \in [a, b]$.

Generality of the Sturm-Liouville Equation

A 2nd order linear ODE is given by

$$Lu = \lambda u, \qquad x \in I,$$

where

$$p(x) = e^{\int \frac{a_1}{a_2}}, \quad r(x) = -\frac{p(x)}{a_2(x)}, \quad q(x) = -a_0(x)r(x).$$

Then the equation in the Sturm-Liouville form can be found with p, r and q.

Sturm-Liouville Problems

Definition. A ***regular Sturm-Liouville boundary value problem*** on an interval $[a, b]$ is given by

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + (q(x) + \lambda r(x))u = 0, \quad x \in (a, b)$$

together with boundary conditions

$$B_a u := \alpha_1 u(a) + \beta_1 u'(a) = 0,$$
$$B_b u := \alpha_2 u(b) + \beta_2 u'(b) = 0,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $|\alpha_1| + |\beta_1| \neq 0, |\alpha_2| + |\beta_2| \neq 0$. B_a, B_b are ***boundary operators***.

Solving a Sturm-Liouville Problem

To solve a Sturm-Liouville problem with boundary conditions

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0,$$

1. Plug in the ansatz
$$y_\lambda(x) = e^{\rho(\lambda)}x$$
to obtain the ***characteristic polynomial*** with
$$(\rho(\lambda)^2 + \lambda)y_\lambda(x) = 0, \qquad x \in [0, \pi].$$
2. Discuss the value of λ to obtain the general equation and plug in the boundary conditions.

Solving a Sturm-Liouville Problem

2. Discuss the value of λ to obtain the general equation and plug in the boundary conditions.
   - $\lambda > 0$. In this case

     $$y_\lambda(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x).$$

     $$y(0) = y(\pi) = 0 \quad \Rightarrow \quad C_1 = C_2 \sin(\sqrt{\lambda}\pi) = 0.$$

     Then

     $$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots \quad \Rightarrow \quad \underline{y_n(x) = C \cdot \sin(nx)}.$$

   - $\lambda < 0$. $y_\lambda(x) = C_1 \cosh(\sqrt{|\lambda|}x) + C_2 \sinh(\sqrt{|\lambda|}x)$. No solution satisfying $y(0) = y(\pi) = 0$.
   - $\lambda = 0$. $y_0(x) = c_1 x + c_2$. No eigenfunction satisfying $y(0) = y(\pi) = 0$.
3. Conclusion — eigenvalues: $\lambda_n = n^2, n = 1, 2, 3, \ldots$, eigenfunctions: $y_n(x) = C \cdot \sin(nx)$.

Separation of Variables for PDEs

1. Make ansatz

$$u(x_1, \ldots, x_n) = u_1(x_1) \cdot u_2(x_2) \cdots u_n(x_n).$$

2. Write out the boundary conditions with the ansatz. (Usually this will reduce to initial conditions for the ODE with only one variable.)
3. Solve the ODEs one-by-one. (Begin with the one that has boundary conditions.)
   - Obtain ODE in the form $Lu = \lambda u$ with boundary condition.
   - Solve this ODE and obtain eigenvalues & eigenfunctions.
   - Plug in these eigenvalues to solve other equations.
4. Gather all the eigenfunctions for each variable to obtain the general solution of the PDE.
5. Fit the general solution into the initial conditions.
   - The eigenfunctions for one ODE normally form an orthonormal system. (Usually Fourier series or Bessel functions.)
   - Expand the initial condition with this orthonormal system.

Orthonormal Systems

Fitting initial conditions. There are (most commonly) two series that we can expand the initial conditions to. On the interval $[0, L]$, we have

- Fourier series (usually cosine or sine series).

$$\left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}, \quad \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}.$$

- Bessel functions. Expand the initial condition into

$$\left\{ \frac{1}{\sqrt{L}|J_n'(\alpha_{n,m})|} J_n(\alpha_{n,m}\sqrt{x/L}) \right\}_{m=1}^{\infty}$$

so that

$$f(x) = \sum_{m=1}^{\infty} \frac{1}{L \cdot J_n'(\alpha_{n,m})^2} \langle J_n(\alpha_{n,m}\sqrt{(\cdot)/L}), f \rangle J_n(\alpha_{n,m}\sqrt{x/L}).$$

Orthogonality of Bessel Functions

Let α, β be two zeros of Bessel function J_ν of order ν. Then in $L^2([0,1])$,

- $\alpha \neq \beta$.

$$\langle J_\nu(\alpha\sqrt{\cdot}), J_\nu(\beta\sqrt{\cdot})\rangle_{L^2([0,1])} = \int_0^1 J_\nu(\alpha\sqrt{t})J_\nu(\beta\sqrt{t})dt = 0.$$

- $\alpha = \beta$.

$$\|J_\nu(\alpha\sqrt{\cdot})\|^2_{L^2([0,1])} = J_\nu'(\alpha)^2.$$

The Wave Equation for a Finite String

Wave equation.

$$c^2 u_{xx} = u_{tt}, \qquad 0 < x < l, t > 0.$$

With the following two conditions

1. ***Initial conditions***:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l,$$

   where $f, g \in L^2([0,l])$.

2. ***Boundary conditions***:
   - ***Dirichlet boundary conditions*** (Fixed ends.)

$$u(0,t) = u(l,t) = 0, \qquad t > 0.$$

   - ***Neumann boundary conditions*** (Free ends allowing vertical movements.)

$$u_x(0,t) = u_x(l,t) = 0, \qquad t > 0.$$

Solutions of Wave Equation

1. Make ansatz

$$u(x,t) = X(x)T(t),$$

which gives

$$\frac{1}{c^2 T} T_{tt} = \frac{1}{X} X_{xx} =: \lambda \in \mathbb{R}.$$

2. Dirichlet boundary conditions. Write out the boundary conditions with the ansatz.

$$X(0)T(t) = X(l)T(t) = 0 \quad \Rightarrow \quad X(0) = X(l) = 0$$

with trivial solution $u(x,t) = 0$.

Solutions of Wave Equation

3. Dirichlet boundary conditions. Solve the ODEs one-by-one.
   - Start with
     $$X'' = \lambda X, \qquad X(0) = X(l) = 0.$$
     Then we obtain eigenvalues
     $$\lambda_n = -\left(\frac{n\pi}{l}\right)^2, \qquad n = 1, 2, 3, \ldots$$
     and eigenfunctions
     $$X_n(x) = C_n \sin\left(\frac{n\pi x}{l}\right), \qquad C_n \in \mathbb{R}.$$
   - Plugging in the eigenvalues, we then need to solve
     $$T'' = -\left(\frac{n\pi}{l}\right)^2 c^2 T, \qquad n = 1, 2, 3, \ldots$$
     The general solution is
     $$T_n(t) = D_n \cos\left(\frac{cn\pi t}{l}\right) + E_n \sin\left(\frac{cn\pi t}{l}\right), \quad D_n, E_n \in \mathbb{R}.$$

Solutions of Wave Equation

4. Dirichlet boundary conditions. Gather all the eigenfunctions for each variable to obtain the general solution of the PDE:

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) \\
&= \sum_{n=1}^{\infty} \left(F_n \cos\left(\frac{cn\pi t}{l}\right) + G_n \sin\left(\frac{cn\pi t}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right),
\end{aligned}
$$

where $F_n, G_n \in \mathbb{R}$.

Solutions of Wave Equation

5. Dirichlet boundary conditions. Fit the general solution into the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l.$$

Then

$$u(x,0) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{l}\right), \quad u_t(x,0) = \sum_{n=1}^{\infty} G_n \frac{cn\pi}{l} \sin\left(\frac{n\pi x}{l}\right).$$

Expanding f and g into Fourier-Sine series, we obtain

$$F_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx,$$
$$G_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right) dx.$$

Solutions of Wave Equation

1. Make ansatz

$$u(x,t) = X(x)T(t),$$

   which gives

$$\frac{1}{c^2 T} T_{tt} = \frac{1}{X} X_{xx} =: \lambda \in \mathbb{R}.$$

2. Neumann boundary conditions. Write out the boundary conditions with the ansatz.

$$X'(0)T(t) = X'(l)T(t) = 0 \quad \Rightarrow \quad X'(0) = X'(l) = 0$$

   with trivial solution $u(x,t) = 0$.

Solutions of Wave Equation

3. Neumann boundary conditions. Solve the ODEs one-by-one.
   - Start with
   $$X'' = \lambda X, \qquad X'(0) = X'(l) = 0.$$
   Then we obtain eigenvalues
   $$\lambda_n = -\left(\frac{n\pi}{l}\right)^2, \qquad n = 0, 1, 2, 3, \ldots$$
   and eigenfunctions
   $$X_n(x) = B_n \cos\left(\frac{n\pi x}{l}\right), \qquad B_n \in \mathbb{R}, n = 0, 1, 2, \ldots$$
   - Plugging in the eigenvalues, we then need to solve
   $$T'' = -\left(\frac{n\pi}{l}\right)^2 c^2 T, \qquad n = 0, 1, 2, 3, \ldots$$
   The general solution is (with solution when $n = 0$ given by $T_0(t) = At + B, A, B \in \mathbb{R}$)
   $$T_n(t) = D_n \cos\left(\frac{cn\pi t}{l}\right) + E_n \sin\left(\frac{cn\pi t}{l}\right), \quad D_n, E_n \in \mathbb{R}, n \geq 1.$$

Solutions of Wave Equation

4. Neumann boundary conditions. Gather all the eigenfunctions for each variable to obtain the general solution of the PDE:

$$u(x,t) = At + B + \\ + \sum_{n=1}^{\infty} \left(F_n \cos\left(\frac{cn\pi t}{l}\right) + G_n \sin\left(\frac{cn\pi t}{l}\right) \right) \cos\left(\frac{n\pi x}{l}\right),$$

where $A, B, F_n, G_n \in \mathbb{R}$.

Solutions of Wave Equation

5. Neumann boundary conditions. Fit the general solution into the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l.$$

Then

$$u(x,0) = B + \sum_{n=1}^{\infty} F_n \cos\left(\frac{n\pi x}{l}\right)$$
$$u_t(x,0) = A + \sum_{n=1}^{\infty} G_n \frac{cn\pi}{l} \cos\left(\frac{n\pi x}{l}\right).$$

Expanding f and g into Fourier-Sine series, we obtain

$$B = \frac{1}{l}\int_0^l f(x)dx, \quad F_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$$
$$A = \frac{1}{l}\int_0^l g(x)dx, \quad G_n = \frac{2}{n\pi c}\int_0^l g(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

The Suspended Chain

Equation for the suspended chain.

$$u_{tt} = g \cdot x u_x, \qquad 0 < x < l, t > 0.$$

With the following two conditions:

1. ***Initial conditions***:

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq l,$$

where $f, g \in L^2([0, l])$.

2. ***Boundary conditions***:

$$u(0, t) < \infty, \quad u(l, t) = 0, \quad t > 0.$$

Solution for the Suspended Chain

1. Make ansatz
$$u(x,t) = X(x)T(t),$$
which gives
$$\frac{T''}{gT} = \frac{(xX')'}{X} =: -\lambda \in \mathbb{R}.$$
2. Write out the boundary conditions with the ansatz.
$$X(0)T(t) < \infty, \qquad X(l)T(t) = 0 \quad \Rightarrow \quad X(0) < \infty, X(l) = 0$$
with trivial solution $u(x,t) = 0$.

Solution for the Suspended Chain

3. Solve the ODEs one-by-one.
   - Start with

     $$(xX')' = -\lambda X \;\Leftrightarrow\; xX' + X' + \lambda X = 0, \quad X(0) < \infty, X(l) = 0.$$

     Substitute $x = y^2/4, y = 2\sqrt{x}$ and define $Y(y(x)) = X(x)$, then the equation becomes

     $$y^2 Y'' + yY' + \lambda y^2 Y = 0.$$

     Substitute $z = \sqrt{\lambda} y$ and define $Z(z(y)) = Y(y)$, then the equation becomes

     $$z^2 Z'' + zZ' + z^2 Z = 0.$$

Solution for the Suspended Chain

3. Solve the ODEs one-by-one.
   - (Continued.)
     - $\lambda \geq 0$. Then

       $$Z(z) = c_1 J_0(z) + c_2 Y_0(z), \qquad c_1, c_2 \in \mathbb{R}$$

       and considering the boundary conditions $X(0) < \infty$,

       $$X(x) = c_1 J_0(2\sqrt{\lambda x}).$$

     - $\lambda < 0$. Then

       $$X(x) = c_1 J_0(2i\sqrt{|\lambda| x}).$$

   For both cases, the other boundary condition gives $X(l) = 0$. Therefore, we only allow non-negative λ and the eigenvalues are given by

   $$\lambda_n = \frac{\alpha_{0,n}^2}{4l}, \qquad n = 1, 2, 3, \ldots$$

   where $\alpha_{0,n}$ is the nth zero of the Bessel function J_0.

Solution for the Suspended Chain

3. Then we need to solve the other ODE with the obtained eigenvalues
$$T'' = -\frac{\alpha_{0,n}^2 g}{4l} T,$$
which leads to
$$T_n(t) = F_n \cos\left(\frac{\alpha_{0,n}}{2}\sqrt{\frac{g}{l}}\,t\right) + G_n \sin\left(\frac{\alpha_{0,n}}{2}\sqrt{\frac{g}{l}}\,t\right).$$

Solution for the Suspended Chain

4. Gather all the eigenfunctions for each variable to obtain the general solution of the PDE.

$$u(x,t) = \sum_{n=1}^{\infty} \left(F_n \cos\left(\frac{\alpha_{0,n}}{2} \sqrt{\frac{g}{l}}\, t \right) + G_n \sin\left(\frac{\alpha_{0,n}}{2} \sqrt{\frac{g}{l}}\, t \right) \right) \times$$
$$\times J_0\left(\alpha_{0,n} \sqrt{\frac{x}{l}} \right).$$

Solution for the Suspended Chain

5. Fit the general solution into the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l.$$

Then

$$u(x,0) = \sum_{n=1}^{\infty} F_n J_0\left(\alpha_{0,n}\sqrt{\frac{x}{l}}\right),$$
$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{G_n \alpha_{0,n}}{2}\sqrt{\frac{g}{l}} J_0\left(\alpha_{0,n}\sqrt{\frac{x}{l}}\right).$$

Expanding f and g into a series of Bessel functions of order 0, we obtain

$$F_n = \frac{1}{l \cdot J_1(\alpha_{0,n})^2}\langle J_0\left(\alpha_{0,n}\sqrt{(\cdot)/l}\right), f\rangle,$$
$$G_n = \frac{2}{\sqrt{gl}\alpha_{0,n} \cdot J_1(\alpha_{0,n})^2}\langle J_0\left(\alpha_{0,n}\sqrt{(\cdot)/l}\right), g\rangle.$$

Separation of Variables for PDEs

Example 2. Solve the wave equation problem

$$
\begin{gathered}
4u_{tt} = u_{xx}, \\
u_x(-\pi, t) = u_x(\pi, t) = 0, \quad u(x, 0) = x^2, \quad u_t(x, 0) = 0.
\end{gathered}
$$

Exercises

Exercise 1. Find the Fourier series for a periodic function f with period $2L$ and

$$f(x) = \begin{cases} L & -L \leq x \leq 0, \\ 2x & 0 < x \leq L. \end{cases}$$

Thanks for your attention!