VV286 Honors Mathematics IV
Solution Manual for Final Review

Chen Xiwen

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University of Michigan–Shanghai Jiao Tong University Joint Institute

Example 1.

Prove the uniqueness of solution of the three dimensional wave on $\Omega \subset \mathbb{R}^3$

$$c^2 u_{tt} = \Delta u$$

which satisfy the boundary conditions

$$u(x,y,z,t) = F(x,y,z,t), \qquad (x,y,z) \in \partial\Omega$$

and initial conditions

$$u(x,y,z,0) = G(x,y,z), \qquad u_t(x,y,z,0) = H(x,y,z).$$

Solution. Suppose we have two solutions to the equation u and v, then by setting $w = v - u$, we have

$$c^2 w_{tt} = \Delta w$$

with boundary condition $w = 0$ on $\partial\Omega$ and initial conditions

$$w(x,y,z,0) = 0, \qquad w_t(x,y,z,0) = 0.$$

Then consider the volume integral

$$E(t) = \frac{1}{2}\int_\Omega c^2 w_t^2 + (\nabla w)^2 d\tau.$$

Then taking the derivative with respect to t, we obtain

$$E'(t) = \int_\Omega \left(c^2 w_{tt} w_t + \nabla w \cdot \nabla w_t\right) d\tau.$$

Using Green's first identity

$$\int_\Omega \langle \nabla u, \nabla v\rangle d\tau = -\int_\Omega u \cdot \Delta v d\tau + \int_{\partial\Omega^*} u\frac{\partial v}{\partial n} dA$$

we then obtain

$$\begin{aligned} E'(t) &= \int_\Omega c^2 w_{tt} w_t d\tau + \int_{\partial\Omega^*} w_t \nabla w \cdot d\vec{A} - \int_\Omega w_t \Delta w d\tau \\ &= \int_\Omega w_t (c^2 w_{tt} - \Delta w) d\tau + \int_{\partial\Omega^*} w_t \nabla w \cdot d\vec{A} \\ &= \int_{\partial\Omega^*} w_t \nabla w \cdot d\vec{A}. \end{aligned}$$

Since $w = 0$ on $\partial\Omega$ and thus $w_t = 0$. Therefore, $E(t)$ is a constant. Using the initial conditions, we further have $E(t) = 0$. Therefore, the solution to the boundary value problem is unique.

Example 2.

Solve the inhomogeneous heat equation

$$u_{xx} - u_t = -2x, \qquad (x,t) \in (0,1) \times \mathbb{R}_+$$

with Dirichlet boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t > 0$$

and initial temperature distribution

$$u(x,0) = x - x^2, \qquad x \in [0,1].$$

Solution. We first find the homogeneous solution for the PDE. First we transform the associated homogeneous PDE using separation of variables ansatz as

$$X''T - XT' = 0 \quad \Rightarrow \quad \frac{X''}{X} = \frac{T'}{T} = -\lambda,$$

with boundary conditions

$$X(0) = X(1) = 0.$$

Then we obtain boundary value problem for X with homogeneous boundary conditions

$$X'' + \lambda X = 0, \qquad X(0) = X(1) = 0.$$

Then assuming $X(x) = e^{\rho(\lambda)x}$, we obtain $\rho^2 + \lambda = 0$ and discuss the following cases.

- $\underline{\lambda = 0}$. Then we have $X(x) = ax + b$. Inserting the boundary conditions, we then have $a = b = 0$.
- $\underline{\lambda < 0}$. Set $\alpha = \sqrt{|\lambda|}$. Then the solutions are given by

  $$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x},$$

  and inserting the boundary conditions gives

  $$\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{\alpha} + c_2 e^{-\alpha} = 0, \end{cases} \quad \Rightarrow \quad c_1 = c_2 = 0.$$

  Thus this case does not yield a solution for the boundary value problem.
- $\underline{\lambda > 0}$. The general solution is given by

  $$X(x) = c_3 \cos(\sqrt{\lambda} x) + c_4 \sin(\sqrt{\lambda} x),$$

and inserting the boundary conditions gives

$$\begin{cases} c_3 = 0, \\ c_3 \cos(\sqrt{\lambda}) + c_4 \sin(\sqrt{\lambda}) = 0, \end{cases} \quad \Rightarrow \quad c_3 = 0, \lambda = (n\pi)^2, n = 1, 2, \ldots$$

Therefore the ODE for X gives the eigenvalues $\lambda_n = (n\pi)^2$ and eigenfunctions $X_n(x) = \sin(n\pi x)$.

The to incorporate the inhomogeneity of the PDE, we plug in the solution for X into the original equation rather than replacing it into the equation for T. Specifically,

$$X(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x), \qquad u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x) T_n(t)$$

and

$$-\sum_{n=1}^{\infty} (n\pi)^2 \sin(n\pi x) T_n - \sum_{n=1}^{\infty} \sin(n\pi x) T_n' = -2x.$$

Using the basis

$$\left\{\sqrt{2}\sin(n\pi x)\right\}_{n=1}^{\infty}$$

to expand the function $-2x$ as $F(x,t) = \sum_{n=1}^{\infty} F_n(t)\sin(n\pi x)$, we obtain

$$\begin{aligned} -2x &= \sum_{n=1}^{\infty} 2\int_0^1 -2x\sin(n\pi x)dx \cdot \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n\pi}\sin(n\pi x), \qquad F(t) = 4\frac{(-1)^n}{n\pi}, \end{aligned}$$

and the equation becomes

$$-\sum_{n=1}^{\infty} (n\pi)^2 \sin(n\pi x) T_n - \sum_{n=1}^{\infty} \sin(n\pi x) T_n' = \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n\pi}\sin(n\pi x).$$

Then the remaining problem is to solve an ODE for $T_n, n = 1, 2, \ldots$, we have

$$T_n' + (n\pi)^2 T_n = -4\frac{(-1)^n}{n\pi},$$

with initial condition

$$\begin{aligned}
u(x,0) &= x - x^2 \\
&= \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}(1-(-1)^n)\sin(n\pi x) \\
&= \sum_{n=1}^{\infty} T_n(0)X_n(x) \quad \Rightarrow \quad T_n(0) = \frac{4}{(n\pi)^3}(1-(-1)^n).
\end{aligned}$$

Solving the initial value problems, we obtain

$$T_n(t) = \frac{4}{(n\pi)^3}\left(e^{-(n\pi)^2 t} - (-1)^n\right).$$

Therefore, the solution to the original PDE is given by

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}\left(e^{-(n\pi)^2 t} - (-1)^n\right)\sin(n\pi x).$$

Example 3.

Show how a solution to the heat equation

$$u_{xx} - u_t = 0, \qquad (x,t) \in (0,1) \times \mathbb{R}_+$$

with mixed boundary conditions

$$u(0,t) = 0, \qquad u_x(1,t) = 0, \qquad t > 0$$

and initial temperature distribution

$$u(x,0) = f(x), \qquad x \in [0,1]$$

can be obtained.

Solution. We make the separation of variable ansatz and obtain

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda,$$

and thus

$$X'' + \lambda X = 0, \qquad X(0) = X'(1) = 0.$$

Then we discuss the values of λ.

- <u>If $\lambda = 0$</u>. Then $X(x) = ax+b$ and inserting the boundary conditions for X gives $a = b = 0$. Therefore, there is no nontrivial solution.

- <u>If $\lambda < 0$</u>. Setting $\alpha = \sqrt{|\lambda|}$, the general solution can be expressed as

$$X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}.$$

Inserting the boundary conditions yields

$$\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{\alpha} - c_2 e^{-\alpha} = 0, \end{cases} \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Therefore, no nontrivial solution exists that satisfy the boundary conditions.

- <u>If $\lambda > 0$</u>. Then the general solution is obtained as

$$X(x) = c_3 \cos(\sqrt{\lambda} x) + c_4 \sin(\sqrt{\lambda} x).$$

Inserting the boundary conditions gives

$$\begin{cases} c_3 = 0, \\ -c_3 \sin(\sqrt{\lambda}) + c_4 \cos(\sqrt{\lambda}) = 0, \end{cases} \quad \Rightarrow \quad c_3 = 0, \lambda = \left(n + \frac{1}{2}\right)^2 \pi^2, n = 0, 1, 2, \ldots$$

Then we obtain the eigenvalues $\lambda_n = \left(n + \frac{1}{2}\right)^2 \pi^2$ with eigenfunctions

$$X_n(x) = A_n \sin\left(\left(n + \frac{1}{2}\right) \pi x\right), \qquad n = 0, 1, 2, \ldots$$

Using these eigenvalues to solve the ODE for T, we obtain

$$T_n(t) = B_n e^{-(n+1/2)^2 \pi^2 t}, \qquad n = 0, 1, 2, \ldots$$

and the general solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} C_n \sin\left(\left(n + \frac{1}{2}\right) \pi x\right) e^{-(n+1/2)^2 \pi^2 t}.$$

It can be shown that

$$\left\{ \sqrt{2} \sin\left(\left(n + \frac{1}{2}\right) \pi x\right) \right\}_{n=0}^{\infty}$$

forms an orthonormal system in $L^2[0,1]$, and the initial condition can be expanded by

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} 2\int_0^1 f(x)\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)dx \cdot \sin\left(\left(n+\frac{1}{2}\right)\pi x\right) \\ &= u(x,0) = \sum_{n=0}^{\infty} C_n \sin\left(\left(n+\frac{1}{2}\right)\pi x\right), \end{aligned}$$

which enables us to find C_n as

$$C_n = 2\int_0^1 f(x)\sin\left(\left(n+\frac{1}{2}\right)\pi x\right)dx,$$

and thus determining the solution to the heat equation $u(x,t)$.

Example 4.

Solve the wave equation problem

$$4u_{tt} = u_{xx}, \qquad u_x(-\pi,t) = u_x(\pi,t) = 0, \qquad u(x,0) = x^2, \qquad u_t(x,0) = 0.$$

Solution. We make the separation of variables ansatz $u(x,t) = X(x)T(t)$, then

$$4XT'' = X''T \quad \Rightarrow \quad \frac{X''}{X} = 4\frac{T''}{T} = -\lambda.$$

Together with the boundary conditions, we have

$$\begin{aligned} &X'' + \lambda X = 0, \qquad X'(-\pi) = X'(\pi) = 0, \\ &T'' + 4\lambda T = 0, \qquad T'(0) = 0, \qquad X(x)T(0) = x^2. \end{aligned}$$

Then we can discuss the values of λ in three cases.

- $\underline{\lambda = 0.}$ Then

  $$X(x) = a_1 + a_2 x.$$

  Plugging in the initial conditions, we have $a_2 = 0$ and similarly for T,

  $$T(t) = a_3 + a_4 x,$$

  therefore we conclude that $u(x,t) = a$, where $a \in \mathbb{R}$ is a constant.

- $\underline{\lambda < 0.}$ Then

  $$X(x) = b_1 e^{\sqrt{-\lambda}x} + b_2 e^{-\sqrt{-\lambda}x}.$$

Plugging in the initial conditions,

$$\begin{cases} b_1\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - b_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} = 0, \\ b_1\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} - b_2\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} = 0, \end{cases}$$

giving $b_1 = b_2 = 0$.

- $\underline{\lambda > 0.}$ Then the eigenfunctions are given by

$$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x).$$

Plugging in the initial conditions, we have

$$\begin{cases} -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0, \\ c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0, \end{cases}$$

giving

$$c_1\sin(\sqrt{\lambda}\pi) = c_2\cos(\sqrt{\lambda}\pi) = 0.$$

1. If $c_2 = 0$, then $\lambda = n^2, n = 1, 2, \ldots$ and

$$X_n(x) = c_n\cos(nx).$$

Using the same eigenvalues, we have

$$T(t) = c_3\cos\left(\frac{n}{2}t\right) + c_4\sin\left(\frac{n}{2}t\right), \qquad T'(0) = 0 \quad \Rightarrow \quad T(t) = c_3\cos\left(\frac{n}{2}t\right).$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} c_n\cos(nx)\cos\left(\frac{n}{2}t\right).$$

2. If $c_1 = 0$, then $\lambda = \left(n - \frac{1}{2}\right)^2$. Similarly, we have

$$u(x,t) = \sum_{n=1}^{\infty} d_n\sin\left(\left(n-\frac{1}{2}\right)x\right)\cos\left(\frac{1}{2}\left(n-\frac{1}{2}\right)t\right).$$

Then the general solution is given by

$$u(x,t) = a + \sum_{n=1}^{\infty} c_n\cos(nx)\cos\left(\frac{n}{2}t\right) + \sum_{n=1}^{\infty} d_n\sin\left(\left(n-\frac{1}{2}\right)x\right)\cos\left(\frac{1}{2}\left(n-\frac{1}{2}\right)t\right).$$

Fitting into the initial condition, we have

$$u(x,0) = a + \sum_{n=1}^{\infty} c_n \cos(nx) + \sum_{n=1}^{\infty} d_n \sin\left(\left(n - \frac{1}{2}\right)x\right) = x^2.$$

We can prove that the functions

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin\left(\left(n-\frac{1}{2}\right)x\right)\right\}_{n=1}^{\infty}$$

forms an orthonormal basis on $[-\pi, \pi]$. Therefore, we expand the function $f(x) = x^2$, to this basis with

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{1}{n\pi}\left(\left(x^2\sin(nx)\right)\Big|_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} x\sin(nx)dx\right) = \frac{4(-1)^n}{n^2},
\end{aligned}$$

with the knowledge that it is an even function. Therefore, the solution to the wave equation is

$$u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)\cos\left(\frac{n}{2}t\right).$$