VV286 Honors Mathematics IV
Solution Manual for RC 8

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November 26, 2018

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Example 1.

Determine the terms up to x^5 in each of the two linearly independent power series solutions to

$$y'' + (2 - 4x^2)y' - 8xy = 0$$

centered at $x = 0$. Also find the radius of convergence of these solutions.
Solution. We make the power series ansatz

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and substitute it into the given differential equation,

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^{n-1} - 4\sum_{n=1}^{\infty} n a_n x^{n+1} - 8\sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Replacing n by $k+2$ in the first summation, $k+1$ in the second summation, and $k-1$ in the third and fourth summations, we obtain

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + 2\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - 4\sum_{k=2}^{\infty}(k-1)a_{k-1}x^k - 8\sum_{k=1}^{\infty} a_{k-1}x^k = 0.$$

Separating out the terms corresponding to $k = 0$ and $k = 1$, it follows that this can be written as

$$(2a_2 + 2a_1) + (6a_3 + 4a_2 - 8a_0)x + \sum_{k=2}^{\infty}\left[(k+2)(k+1)a_{k+2} + 2(k+1)a_{k+1} - (4(k-1)+8)\,a_{k-1}\right]x^k = 0.$$

Setting the coefficients of all powers x^k to zero yields the followings.

- For $k = 0$ and $k = 1$, we have

$$2a_2 + 2a_1 = 0, \qquad 6a_3 + 4a_2 - 8a_0 = 0.$$

- For $k \geq 2$, we have

$$(k+2)(k+1)a_{k+2} + 2(k+1)a_{k+1} - (4(k-1)+8)a_{k-1} = 0.$$

Then we obtain

$$a_2 = -a_1, \qquad a_3 = \frac{2}{3}(2a_0 + a_1),$$

and

$$a_{k+2} = \frac{4a_{k-1} - 2a_{k+1}}{k+2}, \qquad k = 2, 3, 4, \ldots$$

Then we are able to find the first terms for powers up to x^5. We proceed to calculate when $k = 2$,

$$a_4 = \frac{1}{4}(4a_1 - 2a_3) = \frac{1}{4}\left(4a_1 - \frac{4}{3}(2a_0 + a_1)\right),$$

which simplifies to

$$a_4 = \frac{2}{3}(a_1 - a_0).$$

Then when $k = 3$,

$$a_5 = \frac{1}{5}(4a_2 - 2a_4) = \frac{1}{5}\left(-4a_1 - \frac{4}{3}(a_1 - a_0)\right) = \frac{4}{15}(a_0 - 4a_1).$$

Therefore,

$$\begin{aligned} y(x) &= a_0 + a_1 x - a_1 x^2 + \frac{2}{3}(2a_0 + a_1)x^3 + \frac{2}{3}(a_1 - a_0)x^4 + \frac{4}{15}(a_0 - 4a_1)x^5 + \cdots \\ &= a_0\left(1 + \frac{4}{3}x^3 - \frac{2}{3}x^4 + \frac{4}{15}x^5 + \cdots\right) + a_1\left(x - x^2 + \frac{2}{3}x^3 + \frac{2}{3}x^4 - \frac{16}{15}x^5 + \cdots\right). \end{aligned}$$

Thus, the two linearly independent solutions are

$$\begin{aligned} y_1(x) &= 1 + \frac{4}{3}x^3 - \frac{2}{3}x^4 + \frac{4}{15}x^5 + \cdots, \\ y_2(x) &= x - x^2 + \frac{2}{3}x^3 + \frac{2}{3}x^4 - \frac{16}{15}x^5 + \cdots. \end{aligned}$$

Example 2.

Find a series solution about $x = 0$ of

$$x^2 y'' - xy' + (1 - x)y = 0.$$

Solution. The singular point at $x = 0$ is regular. We rewrite the equation as

$$y'' - \frac{1}{x}y' + \frac{1-x}{x^2}y = 0.$$

Then

$$\begin{aligned} p_0 &= \lim_{x \to 0} xp(x) = -1, \\ q_0 &= \lim_{x \to 0} x^2 q(x) = 1. \end{aligned}$$

The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - 2r + 1 = (r-1)^2 = 0,$$

giving $r_1 = r_2 = 1$ and

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

The differential equation implies

$$\sum_{n=0}^{\infty} a_n n(n+1)x^{n+1} - \sum_{n=0}^{\infty} a_n (n+1)x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Simplifying the result, we obtain

$$\sum_{m=1}^{\infty} [m^2 a_m - a_{m-1}]x^{m+1} = 0 \quad \Rightarrow \quad a_m = \frac{1}{m^2} a_{m-1}, \qquad m \geq 1.$$

For $m = 1, 2, 3$ we have

$$\begin{aligned} a_1 &= a_0, \\ a_2 &= \frac{1}{2^2} a_1, \\ a_3 &= \frac{1}{3^2} a_2 = \frac{1}{(3 \times 2)^2} a_0. \end{aligned}$$

It can be verified that

$$a_m = \frac{1}{(m!)^2} a_0,$$

and a series solution can be found as

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 \sum_{m=0}^{\infty} \frac{1}{(m!)^2} x^{m+1}.$$

Example 3.

Find the first few terms of the second series solution about $x = 0$ of

$$x^2 y'' - xy' + (1-x)y = 0.$$

Solution. We use the formerly calculated solution y_1 to obtain

$$y_2(x) = y_1 \ln x + \sum_{n=0}^{\infty} a_n'(1) x^{n+1}.$$

Then the recurrent relation gives

$$a_n = \frac{1}{(n+r-1)^2} a_{n-1} = \cdots = \frac{1}{[(n+r-1)(n+r-2)\cdots r]^2} a_0.$$

From

$$\frac{a_n'}{a_n} = (\ln a_n)' = \left(-2\sum_{k=0}^{n-1} \ln(k+r)\right)' = -\sum_{k=0}^{n-1} \frac{2}{k+r},$$

we then obtain

$$a_n'(1) = \left(-\sum_{k=0}^{n-1} \frac{2}{k+1}\right) a_n(1).$$

Therefore, the second independent solution is found as

$$y_2(x) = y_1 \ln(x) - a_0 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{2}{k+1}\right) \frac{1}{(n!)^2} x^{n+1}.$$

Exercise 1.

The *Legendre differential equation of order* $\lambda \in \mathbb{R}$ is given by

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0.$$

1. Find the general solution of the equation in power series form.
2. Verify that if $\lambda \in \mathbb{N}$ there exists a non-zero polynomial solution.

Solution. The point $x = 0$ is a regular point of the ODE, so we make the ansatz

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Let $\alpha = \lambda(\lambda+1)$. Then plugging the ansatz into the ODE, we have

$$\begin{aligned} 0 &= (1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \alpha \sum_{n=0}^{\infty} a_n x^n, \end{aligned}$$

then

$$\begin{aligned} n &= 0, & \alpha a_0 + 2a_2 &= 0, \\ n &= 1, & (\alpha - 2)a_1 + 6a_3 &= 0, \\ n &\geq 2, & (n+2)(n+1)a_{n+2} - (n^2 + n + \alpha)a_n &= 0. \end{aligned}$$

The recurrence relation is given by

$$a_{n+2} = \frac{n(n+1) - \alpha}{(n+2)(n+1)} a_n, \qquad n \geq 2.$$

One independent solution is given for $a_0 = c_0, a_1 = 0$ and another is given for $a_0 = 0, a_1 = c_1$. Therefore,

$$\begin{aligned} a_2 &= -\frac{\alpha}{2}, \\ a_4 &= \frac{2 \times 3 - \alpha}{3 \times 4} a_2 = -\alpha \frac{2 \times 3 - \alpha}{2 \times 3 \times 4} a_0 = -\frac{\alpha}{4} \left(1 - \frac{\alpha}{6}\right) a_0, \\ a_6 &= \frac{4 \times 5 - \alpha}{5 \times 6} a_4 = -\alpha \frac{4 \times 5 - \alpha}{4 \times 5 \times 6} \left(1 - \frac{\alpha}{6}\right) \left(1 - \frac{\alpha}{20}\right) a_0. \end{aligned}$$

Then it can be verified that

$$a_{2n} = -\frac{a_0 \alpha}{2n} \prod_{k=2}^{n} \left(1 - \frac{\alpha}{(2k-2)(2k-1)}\right), \qquad n = 2, 3, 4, \ldots$$

In a similar pattern we can verify that

$$a_{2n+1} = \frac{a_1}{2n+1} \prod_{k=2}^{n} \left(1 - \frac{\alpha}{(2k-1) \cdot 2k}\right), \qquad n = 1, 2, 3, 4, \ldots$$

Therefore, the general solution is given by

$$\begin{aligned} y(x; c_0, c_1) = c_0 &\left(1 - \frac{\lambda(\lambda+1)}{2} x^2 - \sum_{k=2}^{\infty} \frac{\lambda(\lambda+1)}{2n} \prod_{k=2}^{n} \left(1 - \frac{\lambda(\lambda+1)}{(2k-2)(2k-1)}\right) x^{2n}\right) \\ &+ c_1 \left(x + \sum_{n=1}^{\infty} \frac{1}{2n+1} \prod_{k=2}^{n} \left(1 - \frac{\lambda(\lambda+1)}{(2k-1)2k}\right) x^{2n+1}\right). \end{aligned}$$

For the second question, we discuss three cases.

- $\lambda = 0$, then the solution is given by $y(x) = c_0$, which is a polynomial.
- $\lambda = 2m - 2, m = 2, 3, 4, \ldots$, then the terms in the first column will vanish for $k \geq m$. Therefore, y is a onzero polynomial.
- $\lambda = 2m - 1, m = 1, 2, 3, \ldots$, then the terms in the second column will vanish for $k \geq m$, y is a nonzero polynomial.

Exercise 2.

Use the method of Frobenius to solve

$$5x^2y'' + x(1+x)y' - y = 0.$$

Solution. The original equation is transformed to

$$x^2y'' + \frac{1}{5}x(1+x)y' - \frac{1}{5}y = 0,$$

with

$$xp = \frac{1}{5}(1+x), \qquad x^2q = -\frac{1}{5}.$$

Therefore,

$$\begin{aligned} p_0 &= \frac{1}{5}, \qquad & p_1 &= \frac{1}{5}, \\ q_0 &= -\frac{1}{5}, \qquad & q_1 &= 0. \end{aligned}$$

The indicial equation gives

$$\begin{aligned} F(r) &= r(r-1) + \frac{1}{5}r - \frac{1}{5} = \frac{1}{5}(5r+1)(r-1) = 0 \quad \Rightarrow \quad r_1 = 1, r_2 = -\frac{1}{5}, \\ a_mF(r+m) &= -\sum_{k=0}^{m-1}\left(q_{m-k} + (r+k)p_{m-k}\right)a_k \\ &= -(q_1 + (r+m-1)p_1)a_{m-1} \\ &= -\frac{1}{5}(r+m-1)a_{m-1}. \end{aligned}$$

Then we can obtain two independent solutions from the two roots.

- **For** $r_1 = 1$**.** The recurrence relation is given by

  $$\frac{1}{5}m(5m+6)a_m = -\frac{1}{5}ma_{m-1} \quad \Rightarrow \quad a_m = -\frac{1}{5m+6}a_{m-1}.$$

  Therefore,

  $$a_m = (-1)^m a_0 \prod_{k=1}^{m}\frac{1}{5k+6}, \qquad m = 1, 2, \ldots$$

  and the series solution is given by

  $$y_1(x) = x\sum_{n=0}^{\infty}\left((-1)^n a_0 \prod_{k=1}^{n}\frac{1}{5k+6}\right)x^n.$$

- **For $r_2 = -\dfrac{1}{5}$.** The relation is given by

$$a_m = -\frac{a_{m-1}}{5m} = \frac{(-1)^n}{5^m m!}a_0, \qquad m = 1, 2, \ldots$$

In this case, the solution has a closed form expression as

$$y_2(x) = x^{-1/5}\sum_{n=0}^{\infty}\frac{(-1)^n}{5^n n!}a_0 x^n = a_0 x^{-1/5}e^{-x/5}.$$