Honors Mathematics IV
RC 2

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Slope Parametrization

Using slope parametrization γ, we have the followings for the curve.

- A point:
$$\gamma(p) = (x(p), y(p)).$$
- Slope at this point: p.
- Relation:
$$\dot{y}(p) = p\dot{x}(p).$$

Note. y'' should exists and $y'' \neq 0$ (i.e., y' is monotonic) for the validity of slope parametrization.

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The Envelope Equation

Consider a family of smooth curves in $\mathbb{R}^2$

$$F = \{\mathcal{C}_s, s \in I\}$$

with each curve $\mathcal{C}_s$ parametrized by

$$\gamma(s, \cdot) : J \to \mathcal{C}_s, \qquad t \mapsto \gamma(s, t).$$

Then we have

- ***envelope***: a curve $\mathcal{E}$ such that every point of $\mathcal{E}$ is tangent to a curve in F,
- the tangent point on the envelope $p = \gamma(s, \psi(s))$, and
- the ***envelope equation***:

$$\frac{\partial \gamma_1}{\partial s}\frac{\partial \gamma_2}{\partial t} = \frac{\partial \gamma_1}{\partial t}\frac{\partial \gamma_2}{\partial s}, \qquad t = \psi(s).$$

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Implicit Differential Equations

Equation.

$$F(y, y'; x) = 0, \qquad \gamma(p) = (x(p), y(p)).$$

Solution.

1. Substitute parametrization to obtain

$$F(y(p), p; x(p)) = 0.$$

2. Solve the equation using

$$\dot{y}(p) = p\dot{x}(p).$$

3. Find $y(x)$ from $x(p)$ and $y(p)$. ***(Straight line solutions.)***

Implicit Differential Equations

Example 1. Solve the differential equation

$$y = (yy' + 2x)y'.$$

Implicit Differential Equations

Example 2. Solve the differential equation

$$2y = 2x^2 + 4xy' + (y')^2.$$

$y = xy' + g(y')$ (Clairaut's Equation)

Equation.

$$y = xy' + g(y').$$

Solution 1.

1. Straight line solution:

$$y = cx + g(c), \qquad c \in I.$$

2. Use slope parametrization and differentiate to obtain

$$x(p) = -\dot{g}(p), \qquad y(p) = -p\dot{g}(p) + g(p).$$

3. Find $y(x)$ from $x(p)$ and $y(p)$.

$y = xy' + g(y')$ (Clairaut's Equation)

Equation.

$$y = xy' + g(y').$$

Solution 2.

1. Straight line solution:

$$y = cx + g(c), \qquad c \in I.$$

2. Find the envelope of straight line solutions using envelope equation $\frac{\partial \gamma_1}{\partial c}\frac{\partial \gamma_2}{\partial x} = \frac{\partial \gamma_1}{\partial x}\frac{\partial \gamma_2}{\partial c}$.

$$\gamma(c, x) = \begin{pmatrix} x \\ cx + g(c) \end{pmatrix} \quad \Rightarrow \quad 0 = x + g'(c).$$

3. The parametrization of the envelope is $\gamma(c, -\dot{g}(c))$ and

$$y(c) = -c\dot{g}(c) + g(c).$$

$y = xy' + g(y')$ (Clairaut's Equation)

Example 3. Determine all the solutions for the following Clairaut's differential equations in explicit forms.

1. $y = xy' - \sqrt{y' - 1}$.
2. $y = xy' + y'^2$.

$y = xy' + g(y')$ (Clairaut's Equation)

Example 3.

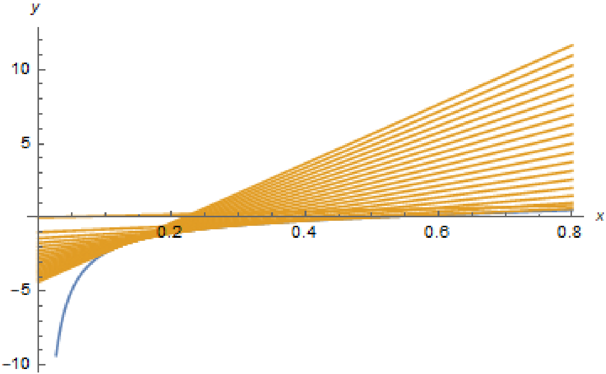

Figure: Solution Curves (1).

$y = xy' + g(y')$ (Clairaut's Equation)

Example 3.

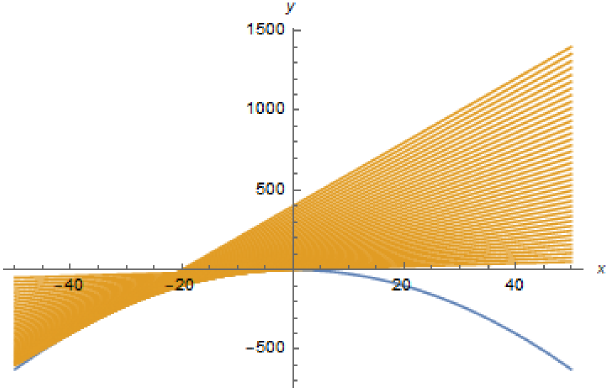

Figure: Solution Curves (2).

$y = xf(y') + g(y')$ (d'Alembert's Equation)

Method.

- Form:

$$y = xf(y') + g(y').$$

- Solution:
  1. Straight line solution $y = cx + d$ (if $f(c) = c$ and $d = g(c)$).
  2. Use slope parametrization and differentiate to obtain

$$\dot{x} = \frac{x\dot{f}(p) + \dot{g}(p)}{p - f(p)}, \qquad \dot{y} = \dot{x}f + x\dot{f} + \dot{g}.$$

  3. Solve the first ODE to obtain $x(p)$ and then obtain $y(p)$.
  4. Find $y(x)$ from $x(p)$ and $y(p)$.

Implicit Differential Equations

Example 4. Determine the solutions of the following differential equation.

$$y = xy'^2 + \ln y'^2.$$

General Implicit Equations

Method.

- Form:

$$F(y, y'; x) = 0.$$

- General rule: a coupled system

$$\dot{x} = -\frac{F_p}{F_x + pF_y}, \qquad \dot{y} = -\frac{pF_p}{F_x + pF_y}.$$

- Special case: the system above decouples such as $F(y, y'; x) = G(x, y') - y$ or $F(y, y'; x) = H(y, y') - x$. Then respectively,

$$\dot{x} = \frac{G_p(x, p)}{p - G_x(x, p)}, \qquad y(p) = G(x(p), p),$$

$$\dot{y} = \frac{pH_p(y, p)}{1 - H_y(y, p)}, \qquad x(p) = H(y(p), p).$$

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Systems of First-Order ODEs

- ***Explicit systems of n first-order differential equations***:

$$\dot{x}(t) = F(x, t)$$

where

$$x : \mathbb{R} \to \mathbb{R}^n, \qquad F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$$

Systems of First-Order ODEs

- ***Higher Order equations***:

$$x^{(n)}(t) = f(x, x', x'', \ldots, x^{(n-1)}, t).$$

Introducing

$$x_1 := x, \quad x_2 := x', \quad x_3 := x'', \quad \ldots, \quad x_n := x^{(n-1)},$$

we have

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ f(x_1, x_2, \ldots, x_n, t) \end{pmatrix}.$$

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IVP and Integral Equations

For the IVP

$$\frac{dx}{dt} = F(x,t), \qquad x(t_0) = x_0 \in \mathbb{R}^n,$$

we have

- ***integral equation***:

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)ds.$$

- ***Picard iteration***: guess $x^{(0)}(t)$, then

$$x^{(k+1)}(t) := x_0 + \int_{t_0}^{t} F(x^{(k)}(s), s)ds, \qquad k \in \mathbb{N}$$

converges to a unique function $x(t)$ under suitable conditions.

Picard Iteration

The IVP is given by

$$\frac{dx}{dt} = F(x,t), \qquad x(t_0) = x_0.$$

Picard Iteration.

1. Start from

$$x^{(0)}(t) = x_0.$$

2. Find an approximating sequence of x_n given by the recurrent relation

$$x^{(k+1)}(t) = x_0 + \int_{t_0}^{t} F(x^{(k)}(s), s)ds.$$

Picard Iteration

Example 5. Find the approximating sequence $x^{(k)}$ for the IVP

$$x' = 2t(1+x), \qquad x(0) = 0.$$

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The Fundamental Existence and Uniqueness Theorem

1.6.5. Theorem of Picard-Lindelöf. Let $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is open and let $t_0 \in I$, where $I \subset \mathbb{R}$ is an interval. Suppose $F : \Omega \times I \to \mathbb{R}^n$ is a continuous function satisfying a ***Lipschitz estimate*** in x: there exists an $L > 0$ such that for all $x, y \in \Omega$ and all $t \in I$,

$$\|F(x,t) - F(y,t)\| \leq L\|x - y\|.$$

Then the initial value problem

$$\frac{dx}{dt} = F(x,t), \qquad x(t_0) = x_0$$

has a unique solution in some t-interval containing t_0.

The Stability of Solutions

1.6.7. Gronwall's Inequality. Suppose that all the conditions of Theorem 1.6.5 are satisfied and that x and y satisfy the differential equation with initial values $x_0, y_0 \in \mathbb{R}^n$, i.e.,

$$\begin{aligned} \frac{dx}{dt} &= F(x,t), \qquad & x(t_0) &= x_0, \\ \frac{dy}{dt} &= F(y,t), \qquad & y(t_0) &= y_0. \end{aligned}$$

Then

$$\|x(t) - y(t)\| \leq e^{L|t-t_0|}\|x_0 - y_0\|.$$

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Linear System of ODEs

Definition. A ***linear system of equations*** have the matrix form

$$\frac{dx}{dt} = A(t)x + b(t), \qquad t \in I \subset \mathbb{R},$$

where $A : I \to \mathrm{Mat}(n \times n, \mathbb{R})$ is a matrix-valued function of t and $b : I \to \mathbb{R}^n$.

Example. The second-order ODE

$$\ddot{x}(t) = a(t)\dot{x}(t) + bx(t) + c$$

with variable coefficients and $x_1 = x, x_2 = \dot{x}$ can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ ax_2 + bx_1 + c \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} .$$

Fundamental System of Solutions

1.6.13. Proposition. Let $\{b_1, \ldots, b_n\}$ be a basis of $\mathbb{R}^n$, $I \subset \mathbb{R}$ an interval and let $x^{(k)} : I \to \mathbb{R}^n$, $k = 1, \ldots, n$ satisfy the system

$$\frac{dx^{(k)}}{dt} = A(t)x^{(k)}, \qquad x^{(k)}(t_0) = b_k$$

with initial point $t_0 \in I$. Then $\{x^{(1)}, \ldots, x^{(n)}\}$ is a ***fundamental system*** for the equation $\dot{x} = A(t)x$, $t \in I$. The matrix $X : I \to \mathrm{Mat}(n \times n, \mathbb{R})$ given by

$$X(t) = (x^{(1)}, \ldots, x^{(n)})$$

is a ***fundamental matrix*** for the IVP.

Construction of Solutions

For the linear system of differential equation with initial condition

$$\frac{dx}{dt} = A(t)x + b(t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad t \in I \subset \mathbb{R},$$

we have solutions

Construction of Solutions

For the linear system of differential equation with initial condition

$$\frac{dx}{dt} = A(t)x + b(t), \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad t \in I \subset \mathbb{R},$$

we have solutions

1. $x_{\mathrm{hom}}(t)$: $x_{\mathrm{hom}}(t) = \sum_{k=1}^{n} \lambda_k x^{(k)}(t)$, where $x^{(k)}(t), k = 1, \dots, n$ satisfies

$$\frac{dx^{(k)}}{dt} = A(t)x^{(k)}, \quad \left(x^{(k)}(t_0) = b_k, x_0 = \sum_{k=1}^{n} \lambda_k b_k \right)$$

   for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and

$$\forall t \in I, \sum_{k=1}^{n} \lambda'_k x^{(k)}(t) = 0 \quad \Rightarrow \quad \lambda'_1 = \cdots = \lambda'_n = 0.$$

2. $x_{\mathrm{part}}(t)$: Discussed later.
3. $x_{\mathrm{inhom}}(t)$: $x_{\mathrm{inhom}}(t) = x_{\mathrm{hom}}(t) + x_{\mathrm{part}}(t)$.

The Most Basic Case of Linear ODE Systems

Consider the linear, homogeneous system

$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0,$$

where the matrix A is constant.

- Attempt: the unique solution is given by

$$x(t) = e^{At}x_0.$$

- Well-defined: using operator norm, we have

$$\sum_{k=1}^{\infty} \left\| \frac{A^k t^k}{k!} \right\| \leq \sum_{k=1}^{\infty} \frac{\|A\|^k \cdot |t|^k}{k!} = e^{|t|\|A\|} - 1 < \infty.$$

The Most Basic Case of Linear ODE Systems

Consider the linear, homogeneous system

$$\frac{dx}{dt} = Ax, \qquad x(0) = x_0,$$

where the matrix A is constant.

- Justification of the solution: A formal calculation gives

$$\begin{aligned}\frac{d}{dt}e^{At} &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k t^k}{k!} = \sum_{k=0}^{\infty}\frac{d}{dt}\frac{A^k t^k}{k!} \\ &= \sum_{k=1}^{\infty}\frac{A^k t^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty}\frac{A^k}{k!} \\ &= Ae^{At} = Ax\end{aligned}$$

and

$$e^{At}|_{t=0} = \mathbb{1}.$$

Thanks for your attention!