Honors Mathematics IV
RC 1

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September 17, 2018

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Vector Fields and Trajectories

- A ***vector field*** on $\mathbb{R}^2$ is a map:

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}$$

  with ***trajectory***:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

  so that

$$\frac{dx}{dt} = F_1(x, y), \qquad \frac{dy}{dt} = F_2(x, y).$$

Direction Fields and Paths

- A ***direction field*** consists of line elements (x, y, p). Without vertical vectors in the vector field, this can be written as

$$G : \mathbb{R}^2 \to \mathbb{R}^2, \qquad G(x, y) = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix},$$

where $p = f(x, y)$ is the slope of the line element at $(x, y) \in \mathbb{R}^2$. The ***path*** of this direction field is given by

$$\gamma : I \to \Omega, \qquad \gamma(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix}$$

for a suitable interval $I \subset \mathbb{R}$. This gives

$$\gamma'(x) = \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} = G(x, y), \qquad y'(x) = f(x, y)$$

Initial Value Problems (I.V.P.)

The problem of finding a continuously differentiable function $y : I \to \mathbb{R}^n$ such that

$$y'(x) = f(x, y), \qquad x \in I$$

together with ***initial condition***

$$y(\xi) = \eta$$

for $\xi \in \bar{I}$ and $\eta \in \mathbb{R}^n$ is an ***initial value problem***.

Separation of Variables

Conditions. $y' = f(x) \cdot g(y)$.

1. f is continuous in an interval $I_x \subset \mathbb{R}$,
2. g is continuous in an interval $I_y \subset \mathbb{R}$,
3. $\xi \in I_x, \eta \in I_y$,
4. $g(\eta) \neq 0$.

Conclusion. In a neighborhood of ξ in I_x, the IVP

$$y' = f(x)g(y), \quad y(\xi) = \eta$$

has a unique solution found by solving for y in

$$\int_\eta^y \frac{ds}{g(s)} = \int_\xi^x f(t)dt.$$

Separation of Variables

Conclusion — $g(\eta) = 0$.

In

$$\int_\eta^y \frac{ds}{g(s)} = \int_\xi^x f(t)dt,$$

- the first solution

$$y(x) = \eta,$$

- the existence of the second solution depends on whether the integral on the left-hand side exists for y in a small neighborhood of η.

Separation of Variables

Example: $y' = \beta y, y(0) = y_0$.

$$\frac{dy}{dx} = \beta y \quad \Rightarrow \quad \frac{1}{y} dy = \beta dx$$

integrating both sides (with initial condition $y(0) = y_0$), the unique solution to the IVP is given by

$$y(x) = y_0 e^{\beta x}.$$

Equilibrium, Steady-State, Transient Solutions

Given a solution $x(t)$, we define

- the ***equilibrium solution*** x_{equi} by
$$x_{\mathrm{equi}} = \text{constant},$$
- the ***steady-state*** solution x_{ss} by
$$x_{\mathrm{ss}} = \lim_{t\to\infty} x(t),$$
- the ***transient*** component by
$$x_{\mathrm{trans}}(t) = x(t) - x_{\mathrm{ss}}.$$

Note. The steady-state solution may or may not equal the equilibrium solution.

Linear Equations

- Form:

$$a_1(x)y' + a_0(x)y = f(x), \qquad x \in I.$$

with $f(x) = 0$(***homogeneous***) or $f(x) \neq 0$(***inhomogeneous***).

- Differential operator:

$$L = a_1 \frac{d}{dx} + a_0.$$

- Solution:

$$y_{\mathrm{inhom}} := y_{\mathrm{part}} + y_{\mathrm{hom}}.$$

Solution to Linear Equations

For the IVP $a_1(x)y' + a_0(x)y = f(x), y(\xi) = \eta$ ($a_1 \neq 0$ for now),

- Homogeneous: data $\{\eta, 0\}$.

$$y_{\mathrm{hom}} = \eta \cdot e^{-G(x)}, \qquad G(x) := \int_{\xi}^{x} \frac{a_0(t)}{a_1(t)} dt.$$

- Particular: data $\{0, f\}$.

$$y_{\mathrm{part}}(x) = e^{-G(x)} \int_{\xi}^{x} \frac{f(s)}{a_1(s)} e^{G(s)} ds.$$

- Inhomogeneous: data $\{\eta, f\}$.

$$y(x) = \eta \cdot e^{-G(x)} + e^{-G(x)} \int_{\xi}^{x} \frac{f(s)}{a_1(s)} e^{G(s)} ds.$$

Solution to Linear Equations

Integrating factor for Duhamel's principle. We want to find $\mu(x)$ such that

$$\mu(x)y' + \mu(x)\frac{a_0(x)}{a_1(x)}y = \frac{d}{dx}g(x) = \mu(x)\frac{f(x)}{a_1(x)}. \tag{1}$$

Namely, the left-hand side can be written as the derivative of some function of x. Observing

$$\frac{d}{dx}\mu(x)y = \mu y' + \mu' y,$$

and letting $\mu(\xi) = 1$, we can set

$$\mu'(x) = \mu(x)\frac{a_0(x)}{a_1(x)},$$

which is a separable ODE.

Solution to Linear Equations

Integrating factor for Duhamel's principle. Then $\mu(x)$ can be found as

$$\mu(x) = e^{\int_{\xi}^{x} \frac{a_0(t)}{a_1(t)}dt}.$$

By integrating both sides of Equation (1) from $x = \xi$ to x, we have

$$e^{\int_{\xi}^{x} \frac{a_0(t)}{a_1(t)}dt} y(x) - \eta = \int_{\xi}^{x} e^{\int_{\xi}^{s} \frac{a_0(t)}{a_1(t)}dt} \cdot \frac{f(s)}{a_1(s)} ds.$$

Therefore,

$$y(x) = \eta \cdot e^{-\int_{\xi}^{x} \frac{a_0(t)}{a_1(t)}dt} + e^{-\int_{\xi}^{x} \frac{a_0(t)}{a_1(t)}dt} \cdot \int_{\xi}^{x} e^{\int_{\xi}^{s} \frac{a_0(t)}{a_1(t)}dt} \cdot \frac{f(s)}{a_1(s)} ds,$$

as is the same with the previous discussion. (This is the "integrating factor" method to solve linear ODEs.)

Linear Equations

Example. Find the solution of the IVP

$$\frac{dy}{dx} + 2xy = x, \qquad y(1) = 2.$$

Linear Equations

Solution. Here we have

$$a_1(x) = 1, \quad a_0(x) = 2x, \quad f(x) = x.$$

Therefore, the solution is given by

$$\begin{aligned} y(x) &= 2 \cdot e^{-\int_1^x 2tdt} + e^{-\int_1^x 2tdt} \cdot \int_1^x \left(e^{\int_1^s 2tdt} \right) \cdot sds \\ &= \frac{3}{2} e^{-x^2+1} + \frac{1}{2}. \end{aligned}$$

$y' = f(ax + by + c); b \neq 0$

Equation.

$$y' = f(ax + by + c), a, b, c \in \mathbb{R}.$$

Solution.

1. Define
$$u(x) := ax + by(x) + c.$$
2. Solve
$$u' = a + bf(u).$$
3. Find $y(x)$ by
$$y(x) = \frac{u(x) - ax - c}{b}.$$

$y' = f(y/x)$

Equation.

$$y' = f\left(\frac{y}{x}\right).$$

Solution.

1. Define

$$u(x) = \frac{y(x)}{x}, \qquad x \neq 0.$$

2. Solve

$$u' = \frac{f(u) - u}{x}.$$

3. Find $y(x)$ by

$$y(x) = x \cdot u(x).$$

$y' = f(y/x)$

Example. Find the general solution to the ODE

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}.$$

$y' = f(y/x)$

Solution. Let

$$u(x) = \frac{y(x)}{x}, \qquad x \neq 0.$$

Then we solve

$$\frac{du}{dx} = \frac{\frac{4+u}{1-4u} - u}{x} = \frac{4(1+u^2)}{(1-4u)x}$$

to obtain

$$\ln|x| = \frac{1}{4}\arctan u - \frac{1}{2}\ln(u^2+1) + c_1.$$

Plugging in $u = y/x$, we have

$$2\ln(x^2+y^2) = \arctan\left(\frac{y}{x}\right) + c_2$$

decided up to a constant.

$y' = f(y/x)$

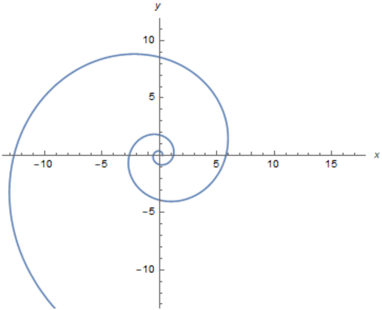
Solution (continued). We can describe the solution in polar coordinates.

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then

$$\ln r = \frac{1}{4}\theta + c_3 \quad \text{or} \quad r = c_4 e^{\theta/4}$$

is the general solution to the ODE. When $c_4 = 1$, the curve is plotted below:



$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$

Equation

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$$

Solution.

1. Define

$$u = a_1x + b_1y + c_1, \quad v = a_2x + b_2y + c_2.$$

2. Calculate

$$\frac{du}{dv} = \frac{du}{dx} \cdot \frac{dx}{dv} = \left(a_1 + b_1 f\left(\frac{u}{v}\right)\right) \frac{b_2 \cdot \frac{du}{dv} - b_1}{a_1 b_2 - a_2 b_1}.$$

3. Transform into and solve

$$\frac{du}{dv} = g\left(\frac{u}{v}\right).$$

$y' + gy + hy^\alpha = 0, \alpha \neq 1$ (Bernoulli's Equation)

Equation.

$$y' + gy + hy^\alpha = 0, \alpha \neq 1.$$

Solution.

1. Multiply with $(1-\alpha)y^{-\alpha}$:

$$(y^{1-\alpha})' + (1-\alpha)g(x)y^{1-\alpha} + (1-\alpha)h(x) = 0.$$

2. Define

$$u(x) = y^{1-\alpha}(x).$$

3. Solve linear equation for u

$$u' + (1-\alpha)g(x)u + (1-\alpha)h(x) = 0.$$

$y' + gy + hy^{\alpha} = 0, \alpha \neq 1$ (Bernoulli's Equation)

Solution.

4. Find $y(x)$: initial condition $y(\xi) = \eta, u(\xi) = \eta^{1-\alpha}$.

$$y_+(x) = |u(x)|^{1/(1-\alpha)}, \alpha \in \mathbb{R}.$$

- If $\alpha > 0$, $y(x) = 0$ is a trivial solution.
- If $\alpha \in \mathbb{Z}$ and is odd,

$$y_-(x) = -y_+(x)$$

  is a negative solution with initial condition $y_-(\xi) = -\eta < 0$.
- If $\alpha \in \mathbb{Z}$ and is even,

$$y_-(x) = -|u(x)|^{1/(1-\alpha)}$$

  is a negative solution with initial condition $y(\xi) = \eta < 0$.

$y' + gy + hy^\alpha = 0, \alpha \neq 1$ (Bernoulli's Equation)

Example. Solve the IVP

$$y' + \frac{4}{x}y = x^3y^2, \quad y(2) = -1, \quad x > 0$$

$y' + gy + hy^\alpha = 0, \alpha \neq 1$ (Bernoulli's Equation)

Solution. Here $g(x) = 4/x, h(x) = -x^3, \alpha = 2$. (The trivial solution does not satisfy the initial condition.)

1. Multiply with y^{-2}

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3.$$

2. Define

$$u(x) = y^{-1}(x), \quad \Rightarrow \quad u'(x) = -y^{-2}(x)y'(x).$$

3. Then the equation is transformed to

$$-u' + \frac{4}{x}u = x^3,$$

which is a linear differential equation with initial condition $u(2) = -1$.

$y' + gy + hy^{\alpha} = 0, \alpha \neq 1$ (Bernoulli's Equation)

Solution.

4. The solution of the linear ODE is given by

$$u(x) = -1 \cdot e^{\int_2^x \frac{4}{t} dt} - e^{\int_2^x \frac{4}{t} dt} \cdot \int_2^x e^{-\int_2^s \frac{4}{t} dt} \cdot s^3 ds$$
$$= \left(\ln 2 - \frac{1}{16}\right) x^4 - x^4 \ln x$$

5. Substitute back to obtain $y(x)$,

$$y(x) = -\frac{16}{x^4(1 + 16\ln(x/2))}.$$

Therefore, we have the solution in the intervals

$$0 < x < 2e^{-\frac{1}{16}}, \qquad 2e^{-\frac{1}{16}} < x < \infty.$$

$y' + gy + hy^2 = k$ (Ricatti's Equation)

Equation.

$$y' + gy + hy^2 = k.$$

Solution.

1. Given a solution ϕ.
2. Define
$$u = y - \phi.$$
3. Solve the Bernoulli's equation with $\alpha = 2$
$$u' + (g + 2\phi h)u + hu^2 = 0.$$
which can be transformed by $z = u^{-1}$ into
$$z' - (g + 2\phi h)z = h.$$
4. Find $y(x)$ by
$$y = \phi + \frac{1}{z}.$$

$y' + gy + hy^2 = k$ (Ricatti's Equation)

Example. Show that the Ricatti's equation on an open interval $I \subset \mathbb{R}$,

$$y' + g(x)y + h(x)y^2 = k(x)$$

with $g, h \in C(I), h \in C^1(I), h \neq 0$ on I, can be transformed into the linear differential equation of second order,

$$u'' + \left(g - \frac{h'}{h}\right) u' - khu = 0,$$

using the transformation

$$u(x) = e^{\int h(x)y(x)dx}.$$

Integral Curves of Vector Fields

Definition.

- ***Integral curve:***

$$\gamma : I \to \mathcal{C}, \quad \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad I \subset \mathbb{R}.$$

- Corresponding ***vector field:***

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F \circ \gamma(t) = \gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \quad t \in I.$$

Integral Curves and ODEs

Equation.

$$h(x,y)y' + g(x,y) = 0, \quad x \in I \subset \mathbb{R}, \quad h(x,y) \neq 0.$$

Solution.

1. Find potential $U(x,y) = \text{constant}$ of the vector field

$$F(x,y) = \begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix}.$$

2. The integral curves are general solutions to the ODE.

Integral Curves and ODEs

Example. Solve the following differential equation:

$$(2y + x^2 + 1)y' + 2xy - 9x^2 = 0.$$

Integral Curves and ODEs

Solution. Here we have

$$h(x,y) = 2y + x^2 + 1, \qquad g(x,y) = 2xy - 9x^2.$$

1. Find potential $U(x,y) = \text{constant}$ of the vector field

$$F(x,y) = \begin{pmatrix} 2xy - 9x^2 \\ 2y + x^2 + 1 \end{pmatrix}.$$

2. The integral curves

$$y^2 + (x^2+1)y - 3x^3 = c$$

is the (implicit) solution of the differential equation.

Integral Curves and ODEs — Integrating Factors

Equation.

$$h(x,y)y' + g(x,y) = 0, \quad x \in I \subset \mathbb{R}, \quad h(x,y) \neq 0.$$

Solution.

1. Find potential $U(x,y) = \text{constant}$ of the vector field

$$F(x,y) = \begin{pmatrix} M(x,y)g(x,y) \\ M(x,y)h(x,y) \end{pmatrix}.$$

2. The integral curves are general solutions to the ODE.

Finding Integrating Factors

- General rule:

$$M_y g + M g_y = M_x h + M h_x.$$

- Suppose M depends only on x or only on y (or xy): find through

$$(\ln M)' = \frac{g_y - h_x}{h}.$$

Integral Curves and ODEs — Integrating Factors

Example. Find all integral curves of the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Integral Curves and ODEs — Integrating Factors

Solution. We identify that with $g(x,y) = 3xy + y^2$ and $h(x,y) = x^2 + xy$,

$$\frac{g_y - h_x}{h} = \frac{3x + 2y - 2x - y}{x(x+y)} = \frac{1}{x}$$

depends only on x. The integrating factor is given by

$$M = x.$$

Integral Curves and ODEs — Integrating Factors

Solution (continued). Then the equation becomes

$$(3x^2y + y^2x)dx + (x^3 + x^2y)dy = 0,$$

and the potential function is found by

$$U(x,y) = \int (3x^2y + y^2x)dx = x^3y + \frac{1}{2}y^2x^2 + c_1(y),$$
$$U(x,y) = \int (x^3 + x^2y)dy = x^3y + \frac{1}{2}y^2x^2 + c_2(x).$$

Then the integral curves are given by

$$x^3y + \frac{1}{2}y^2x^2 = c, \qquad c \in \mathbb{R}.$$

Example. Solve the IVP

$$y' + xy = xe^{x^2/2}, \qquad y(0) = 1.$$

Integral Curves and ODEs — Integrating Factors

Solution. We notice that with $g(x, y) = xy - xe^{x^2/2}, h(x) = 1$,

$$\frac{g_y - h_x}{h} = x$$

depends only on x. Then the integrating factor is given by

$$M = e^{x^2/2}.$$

Integral Curves and ODEs — Integrating Factors

Solution.

1. Find the potential of the vector field

$$F(x,y) = \begin{pmatrix} e^{x^2/2}(xy - xe^{x^2/2}) \\ e^{x^2/2} \end{pmatrix}.$$

2. The solution is then given by

$$ye^{x^2/2} - \frac{1}{2}e^{x^2} = c.$$

Plugging in the initial condition, we have $c = \dfrac{1}{2}$ and

$$y(x) = \frac{1}{2}(e^{x^2/2} + e^{-x^2/2}) = \cosh\left(\frac{x^2}{2}\right)$$

is the solution to the IVP.

Exercises

Exercise 1. Solve the IVP

$$y' - y = f(x), \qquad y(0) = 0,$$

where

$$f(x) = \begin{cases} 1 & \text{if } x < 1, \\ 2 - x & \text{if } x \geq 1. \end{cases}$$

Exercises

Exercise 2. Solve the IVP

$$y' + \frac{y}{x} - \sqrt{y} = 0, \qquad y(1) = 0.$$

Thanks for your attention!