VV286 Honors Mathematics IV
Solution Manual for RC 7

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Example 1.

Calculate the Fourier transform of the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 1/(1+x^2)^2$.

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{(1+x^2)^2} dx.$$

Solution. Suppose $\xi < 0$, then set the function

$$f_\xi(z) = \frac{e^{-iz\xi}}{(1+z^2)^2},$$

which is holomorphic in the upper half-plane and we integrate along the semicircle shown in Figure 1.

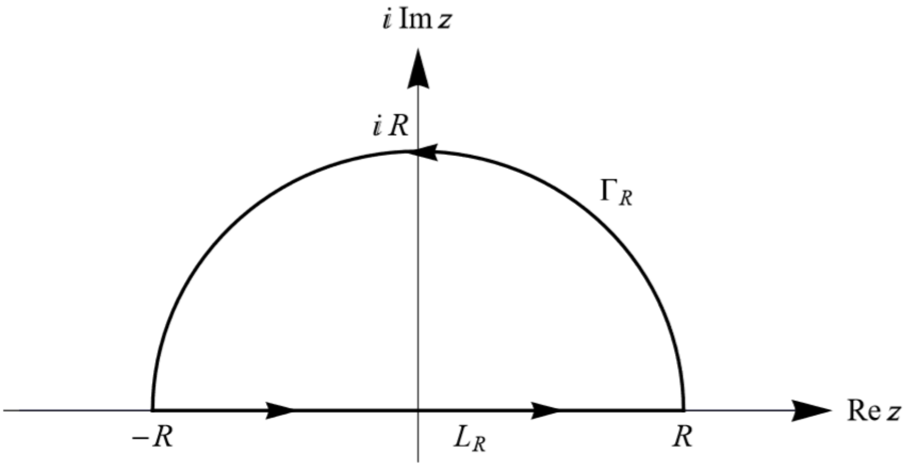

Figure 1: Contour for Example 1.

Then we evaluate the integral by parts. The integral over the semi-circle converges to zero as R goes to infinity by Jordan's lemma. For $R > 1$,

$$\sup_{\theta \in [0,\pi]} \frac{1}{|(1+R^2 e^{2i\theta})^2|} \le \frac{1}{(R^2-1)^2} \xrightarrow{R\to\infty} 0,$$

verifying the requirement by Jordan's lemma. Therefore,

$$\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{(1+x^2)^2} dx = 2\pi i \operatorname{res}_i f_\xi(z),$$

since the contour includes only the pole $z = i$ in the upper half plane. Then for a pole of order 2, which is seen from

$$\frac{1}{(1+z^2)^2} = \frac{1}{(z-i)^2(z+i)^2},$$

we have

$$\begin{aligned}
\operatorname{res}_i f_\xi(z) &= \lim_{z\to i} \frac{d}{dz} \frac{(z-i)^2 e^{-iz\xi}}{(z-i)^2(z+i)^2} \\
&= \lim_{z\to i} \frac{d}{dz} \frac{e^{-iz\xi}}{(z+i)^2} \\
&= \lim_{z\to i} e^{-iz\xi} \frac{(-i\xi)(z+i)^2 - 2(z+i)}{(z+i)^4} \\
&= e^{\xi} \frac{2(\xi-1)}{-8i} \\
&= e^{\xi} \frac{(\xi-1)}{4} i.
\end{aligned}$$

Then

$$\int_{-\infty}^{\infty} \frac{e^{-ix\xi}}{(1+x^2)^2} dx = \pi e^{\xi} \frac{1-\xi}{2} = \pi e^{-|\xi|} \frac{1+|\xi|}{2}$$

for $\xi < 0$. Since f is even, so is $\hat{f}$ and

$$\widehat{f}(\xi) = \sqrt{\frac{\pi}{2}} \frac{1+|\xi|}{2} e^{-|\xi|}$$

for all $\xi \neq 0$. Since f decays sufficiently fast at infinity, $\hat{f}$ is differentiable and hence continuous, so the formula also holds for $\xi = 0$.

Exercise 1.

Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}.$$

Solution. When $\xi > 0$, using a semicircle contour in the upper half plane and define the complex function

$$f(z) = \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = \frac{a}{(z+ai)(z-ai)} e^{-2\pi i z\xi},$$

we can verify from Jordan's lemma that

$$\sup_{\theta\in[0,\pi]} \frac{a}{|a^2+R^2e^{2i\theta}|} \leq \frac{a}{R^2-a^2} \xrightarrow{R\to\infty} 0.$$

Therefore, the integral along the semicircle vanishes as R goes to infinity. By residue theorem, the integral is found as

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx &= 2\pi i \operatorname{res}_{ai} f \\ &= 2\pi i \lim_{z\to ai} \frac{a}{z+ai} e^{-2\pi i z\xi} \\ &= \pi e^{2\pi a\xi}.\end{aligned}$$

Since the integral is an even function for ξ, and is continuous at $\xi = 0$, then we can conclude that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}.$$