VV286 Honors Mathematics IV
Solution Manual for RC 9

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Example 1.

Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + 2\pi) = f(x)$ and

$$f(x) = e^x, \qquad -\pi < x < \pi.$$

Find the Fourier series of f and use it to evaluate

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}.$$

Solution. To find the Fourier series, we calculate

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-inx} f(x)dx &= \int_{-\pi}^{\pi} e^{(1-in)x} dx \\ &= \frac{1}{1-in}\left(e^{(1-in)\pi} - e^{-(1-in)\pi}\right) \\ &= \frac{1+in}{1+n^2}(e^{\pi} - e^{-\pi})(-1)^n.\end{aligned}$$

Therefore, the Fourier series is given by

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1+in}{1+n^2}(e^{\pi} - e^{-\pi})(-1)^n e^{inx}.$$

Setting $x = \pi$, the series then becomes

$$\frac{1}{2\pi}(e^{\pi} - e^{-\pi}) \sum_{-\infty}^{\infty} \frac{1+in}{1+n^2} = \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\right).$$

The series converges pointwise at $x = \pi$ to the average of the left- and right-hand values of f, therefore

$$\lim_{\varepsilon \to 0} \frac{f(\pi+\varepsilon) + f(\pi-\varepsilon)}{2} = \frac{e^{\pi} + e^{-\pi}}{2} = \cosh(\pi) = \frac{\sinh(\pi)}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\right),$$

giving

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{2\tanh(\pi)} + \frac{1}{2}.$$

Example 2.

Solve the wave equation problem

$$4u_{tt} = u_{xx}, \qquad u_x(-\pi, t) = u_x(\pi, t) = 0, \qquad u(x, 0) = x^2, \qquad u_t(x, 0) = 0.$$

Solution. We make the separation of variables ansatz $u(x,t) = X(x)T(t)$, then

$$4XT'' = X''T \quad \Rightarrow \quad \frac{X''}{X} = 4\frac{T''}{T} = -\lambda.$$

Together with the boundary conditions, we have

$$\begin{aligned} &X'' + \lambda X = 0, && X'(-\pi) = X'(\pi) = 0, \\ &T'' + 4\lambda T = 0, && T'(0) = 0, \qquad X(x)T(0) = x^2. \end{aligned}$$

Then we can discuss the values of λ in three cases.

- $\underline{\lambda = 0.}$ Then

  $$X(x) = a_1 + a_2 x.$$

  Plugging in the initial conditions, we have $a_2 = 0$ and similarly for T,

  $$T(t) = a_3 + a_4 x,$$

  therefore we conclude that $u(x,t) = a$, where $a \in \mathbb{R}$ is a constant.

- $\underline{\lambda < 0.}$ Then

  $$X(x) = b_1 e^{\sqrt{-\lambda}x} + b_2 e^{-\sqrt{-\lambda}x}.$$

  Plugging in the initial conditions,

  $$\begin{cases} b_1\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} - b_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} = 0, \\ b_1\sqrt{-\lambda}e^{-\sqrt{-\lambda}\pi} - b_2\sqrt{-\lambda}e^{\sqrt{-\lambda}\pi} = 0, \end{cases}$$

  giving $b_1 = b_2 = 0$.

- $\underline{\lambda > 0.}$ Then the eigenfunctions are given by

  $$X(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x).$$

  Plugging in the initial conditions, we have

  $$\begin{cases} -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0, \\ c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = 0, \end{cases}$$

  giving

  $$c_1\sin(\sqrt{\lambda}\pi) = c_2\cos(\sqrt{\lambda}\pi) = 0.$$

1. If $c_2 = 0$, then $\lambda = n^2, n = 1, 2, \ldots$ and

$$X_n(x) = c_n \cos(nx).$$

Using the same eigenvalues, we have

$$T(t) = c_3 \cos\left(\frac{n}{2}t\right) + c_4 \sin\left(\frac{n}{2}t\right), \qquad T'(0) = 0 \quad \Rightarrow \quad T(t) = c_3 \cos\left(\frac{n}{2}t\right).$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos(nx) \cos\left(\frac{n}{2}t\right).$$

2. If $c_1 = 0$, then $\lambda = \left(n - \frac{1}{2}\right)^2$. Similarly, we have

$$u(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\left(n - \frac{1}{2}\right)x\right) \cos\left(\frac{1}{2}\left(n - \frac{1}{2}\right)t\right).$$

Then the general solution is given by

$$u(x,t) = a + \sum_{n=1}^{\infty} c_n \cos(nx) \cos\left(\frac{n}{2}t\right) + \sum_{n=1}^{\infty} d_n \sin\left(\left(n - \frac{1}{2}\right)x\right) \cos\left(\frac{1}{2}\left(n - \frac{1}{2}\right)t\right).$$

Fitting into the initial condition, we have

$$u(x,0) = a + \sum_{n=1}^{\infty} c_n \cos(nx) + \sum_{n=1}^{\infty} d_n \sin\left(\left(n - \frac{1}{2}\right)x\right) = x^2.$$

We can prove that the functions

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin\left(\left(n - \frac{1}{2}\right)x\right)\right\}_{n=1}^{\infty}$$

forms an orthonormal basis on $[-\pi, \pi]$. Therefore, we expand the function $f(x) = x^2$, to this basis with

$$\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}, \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) dx = \frac{1}{n\pi} \left(\left(x^2 \sin(nx)\right) \Big|_{-\pi}^{\pi} - 2 \int_{-\pi}^{\pi} x \sin(nx) dx \right) = \frac{4(-1)^n}{n^2},
\end{aligned}$$

with the knowledge that it is an even function. Therefore, the solution to the wave equation is

$$u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \cos\left(\frac{n}{2}t\right).$$

Exercise 1.

Find the Fourier series for a periodic function f with period $2L$ and

$$f(x) = \begin{cases} L & -L \le x \le 0, \\ 2x & 0 < x \le L. \end{cases}$$

Solution. We calculate

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\left[\int_{-L}^{0} f(x)dx + \int_{0}^{L} f(x)dx\right] \\ &= \frac{1}{2L}\left[\int_{-L}^{0} Ldx + \int_{0}^{L} 2xdx\right] \\ &= \frac{1}{2L}[L^2 + L^2] = L, \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{1}{L}\left[\int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx\right]. \end{aligned}$$

Evaluating the two terms separately, we obtain

$$\begin{aligned} \int_{-L}^{0} L\cos\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{0} = \frac{L^2}{n\pi}\sin(n\pi) = 0, \\ \int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\cos\left(\frac{n\pi x}{L}\right) + n\pi x\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{0}^{L} \\ &= \left(\frac{2L}{n^2\pi^2}\right)(L\cos(n\pi) + n\pi L\sin(n\pi) - L\cos(0)) \\ &= \left(\frac{2L^2}{n^2\pi^2}\right)((-1)^n - 1). \end{aligned}$$

Putting the expressions together, we have

$$\begin{aligned} a_n &= \frac{1}{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[0 + \left(\frac{2L^2}{n^2\pi^2}((-1)^n - 1)\right)\right] \\ &= \frac{2L}{n^2\pi^2}((-1)^n - 1), \qquad n = 1, 2, 3, \ldots \end{aligned}$$

Similarly for b_n, we calculate

$$\begin{aligned}b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\int_{-L}^{0} f(x)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]\\ &= \frac{1}{L}\left[\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx\right].\end{aligned}$$

Evaluating them individually, we have

$$\begin{aligned}\int_{-L}^{0} L\sin\left(\frac{n\pi x}{L}\right)dx &= \left(-\frac{L^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\bigg|_{-L}^{0} = \frac{L^2}{n\pi}((-1)^n - 1)\\ \int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)dx &= \left(\frac{2L}{n^2\pi^2}\right)\left(L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right)\bigg|_{0}^{L}\\ &= \left(\frac{2L}{n^2\pi^2}\right)(L\sin(n\pi) - n\pi L\cos(n\pi))\\ &= \left(\frac{2L}{n^2\pi^2}\right)(-n\pi(-1)^n) = -\frac{2L^2}{n\pi}(-1)^n.\end{aligned}$$

Putting them together, we obtain

$$\begin{aligned}b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[\frac{L^2}{n\pi}((-1)^n - 1) - \frac{2L^2}{n\pi}(-1)^n\right]\\ &= \frac{L}{n\pi}(-1 - (-1)^n) = -\frac{L}{n\pi}(1 + (-1)^n), \qquad n = 1, 2, 3, \ldots\end{aligned}$$

Therefore, the Fourier series of f is

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\\ &= L + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty}\frac{L}{n\pi}(1 + (-1)^n)\sin\left(\frac{n\pi x}{L}\right).\end{aligned}$$