VV286 Honors Mathematics IV
Solution Manual for RC 4

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Example 1.

Find the general solution of the equation

$$y'' - 2y' + y = \frac{e^x}{2x}.$$

Solution. The associated homogeneous equation is given by

$$y'' - 2y' + y = 0.$$

The equation given by the characteristic polynomial is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

Therefore, the only eigenvalue is $\lambda = 1$. The general solution of the homogeneous equation is then

$$y_{\text{hom}}(x) = c_1 e^x + c_2 x e^x, \qquad c_1, c_2 \in \mathbb{R}.$$

To find the particular solution, we use the Wronskian

$$W(x) = \det \begin{pmatrix} e^x & xe^x \\ e^x & e^x(1+x) \end{pmatrix} = e^{2x}.$$

Then the particular solution can be found by

$$\begin{aligned} y_{\text{part}}(x) &= -y^{(1)} \int \frac{g(x) y^{(2)}(x)}{W(y^{(1)}(x), y^{(2)}(x))} dx + y^{(2)} \int \frac{g(x) y^{(1)}(x)}{W(y^{(1)}(x), y^{(2)}(x))} dx \\ &= -e^x \int \frac{\frac{e^x}{2x} x e^x}{e^{2x}} dx + x e^x \int \frac{\frac{e^x}{2x} e^x}{e^{2x}} dx \\ &= -\frac{x}{2} e^x + \ln(\sqrt{x}) x e^x. \end{aligned}$$

Therefore, the general solution of the inhomogeneous equation is

$$y_{\text{inhom}}(x) = c_1 e^x + (c_2 + \ln(\sqrt{x})) x e^x, \qquad c_1, c_2 \in \mathbb{R}.$$

Example 2.

Verify that $y(x) = x$ solves

$$(1 - x^2) y'' - 2xy' + 2y = 0, \qquad -1 < x < 1,$$

and find another independent solution of the equation.
Solution. From $y(x) = x$ we know that

$$x'' = 0, \qquad x' = 1.$$

Therefore, $y(x) = x$ is a solution. Using reduction of order, we let

$$y_2(x) = c(x) \cdot x,$$

which gives

$$\begin{aligned}(1-x^2)y'' - 2xy' + 2y &= (1-x^2)(xc'' + 2c') - 2x(xc' + c) + 2xc \\ &= (1-x^2)(xc'' + 2c') - 2x^2c' \\ &= (1-x^2)xc'' + 2(1-2x^2)c' = 0.\end{aligned}$$

Therefore, substituting $u(x) = c'(x)$, we need to solve the equation

$$u' = -\frac{2(1-2x^2)}{(1-x^2)x}u.$$

Then

$$\begin{aligned}\int -\frac{2(1-2x^2)}{(1-x^2)x}dx &= \int -\frac{2}{x}\left(1 - \frac{x^2}{1-x^2}\right)dx \\ &= \int -2\left(\frac{1}{x} - \frac{x}{1-x^2}\right)dx \\ &= -\ln(x^2(1-x^2)).\end{aligned}$$

Therefore, the general solution of u is given by

$$u(x) = c_1 e^{-\ln(x^2(1-x^2))} = \frac{c_1}{x^2(1-x^2)}.$$

Then

$$c(x) = \int \frac{c_1}{x^2(1+x)(1-x)}dx = -\frac{c_1}{x} + \frac{c_1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

Therefore, the second independent solution is

$$y_2(x) = c_1\left(-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right).$$

Example 3.

Use the reduction of order to find the general solution of the following differential equation. A solution y_1 is given.

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right) y = 0, \qquad y_1(t) = \frac{\sin t}{\sqrt{t}}.$$

Solution. Setting $y_2 = u(x)y_1$, we have

$$t^2(u''y_1 + 2u'y_1' + uy_1'') + tu'y_1 + tuy_1' + \left(t^2 - \frac{1}{4}\right) uy_1 = 0,$$

which simplifies to

$$t^2(u''y_1 + 2u'y_1') + tu'y_1 = 0$$

using that y_1 is a solution. Then plugging in y_1, we have

$$t^{3/2}\sin(t)u'' + 2t^2u'\frac{\cos(t)}{\sqrt{t}} - t^{1/2}\sin(t)u' + t^{1/2}\sin(t)u' = 0 \quad\Rightarrow\quad \sin(t)u'' + 2\cos(t)u' = 0,$$

namely,

$$(\sin(t)u)'' + \sin(t)u = 0.$$

Set $\omega(t) = \sin(t)u(t)$. Then the solution to $\omega'' + \omega = 0$ is

$$\omega(t) = c_1\cos(t) + c_2\sin(t),$$

and we have

$$u(t) = c_1\frac{\cos(t)}{\sin(t)} + c_2$$

so that

$$y_2(t) = c_1\frac{\cos(t)}{\sqrt{t}} + c_2\frac{\sin t}{\sqrt{t}}, \qquad c_1, c_2 \in \mathbb{R}.$$

Example 4.

Find the general solution to

$$y''' + 3y'' + 3y' - 7y = 0.$$

Solution. The characteristic polynomial gives the equation

$$\lambda^3 + 3\lambda^2 + 3\lambda - 7 = 0 \quad\Rightarrow\quad (\lambda - 1)(\lambda^2 + 4\lambda + 7),$$

giving

$$\lambda_1 = 1, \quad \lambda_2 = -2 + \sqrt{3}i, \quad \lambda_3 = -2 - \sqrt{3}i.$$

Then the three independent solutions are given by

$$y^{(1)}(x) = e^x, \quad y^{(2)}(x) = e^{-2x}\cos\sqrt{3}x, \quad y^{(3)}(x) = e^{-2x}\sin\sqrt{3}x.$$

Then the general solution is

$$y(x) = c_1 e^x + c_2 e^{-2x}\cos\sqrt{3}x + c_3 e^{-2x}\sin\sqrt{3}x.$$

Example 5 (Assignment 4.3.)

After substituting $x = y'$, The associated homogeneous equation is

$$x'' + x = 0, \qquad x(0) = x'(0) = 0.$$

The eigenvalues are $\lambda_1 = i, \lambda_2 = -i$. Then the two independent solutions are

$$x^{(1)}(t) = \sin t, \qquad x^{(2)}(t) = \cos t,$$

and the general solution to the homogeneous equation is

$$x_{\text{hom}}(t) = c_1\sin t + c_2\cos t, \qquad x(0) = x'(0) = 0.$$

Therefore, $x_{\text{hom}}(t) = 0$.

To obtain a particular solution, we have

$$\begin{aligned}
W &= \det\begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -1, \\
W^{(1)} &= \det\begin{pmatrix} 0 & \cos t \\ \sec t\tan t & -\sin t \end{pmatrix} = -\tan t, \quad W^{(2)} = \det\begin{pmatrix} \sin t & 0 \\ \cos t & \sec t\tan t \end{pmatrix} = \tan^2 t.
\end{aligned}$$

Then the particular solution is given by

$$\begin{aligned}
x_{\text{part}}(t) &= -x^{(1)}(t)\int_0^t \frac{g(s)x^{(2)}(s)}{W(s)}ds + x^{(2)}(t)\int_0^t \frac{g(s)x^{(1)}(s)}{W(s)}ds \\
&= -\sin(t)\int_0^t \frac{\sec(s)\tan(s)\cos(s)}{-1}ds + \cos(t)\int_0^t \frac{\sec(s)\tan(s)\sin(s)}{-1}ds \\
&= \sin(t)\int_0^t \tan(s)ds - \cos(t)\int_0^t \tan^2(s)ds \\
&= -\sin(t)\ln|\cos(t)| - \cos(t)\cdot\left(\frac{\sin(t)}{\cos(t)} - t\right) \\
&= -\sin(t)\ln|\cos(t)| - \sin(t) + t\cos(t)
\end{aligned}$$

Therefore, the inhomogeneous solution of the IVP for x is

$$x_{\text{inhom}}(t) = x_{\text{hom}}(t) + x_{\text{part}}(t) = -\sin(t)\ln|\cos(t)| - \sin(t) + t\cos(t),$$

and integrating from 0, we obtain

$$\begin{aligned} y(t) &= \int_0^t -\sin(s)\ln|\cos(s)| - \sin(s) + s\cos(s)ds \\ &= \cos(t)\ln|\cos(t)| + t\sin(t) + \cos(t) - 1. \end{aligned}$$

Assignment 3.1.

(i).

If $\psi \in V$, then

$$\begin{aligned} H\psi &= \left(-\frac{d^2}{dx^2} + x^2\right)\left(e^{-x^2/2}p(x)\right) \\ &= -\frac{d}{dx}\left(-xe^{-x^2/2}p + e^{-x^2/2}p'\right) + x^2e^{-x^2/2}p \\ &= e^{-x^2/2}p - x^2e^{-x^2/2}p + xe^{-x^2/2}p' + xe^{-x^2/2}p' - e^{-x^2/2}p'' + x^2e^{-x^2/2}p \\ &= e^{-x^2/2}p + 2xe^{-x^2/2}p' - e^{-x^2/2}p'' \end{aligned}$$

which is also a polynomial in V.

(ii).

Suppose

$$\psi = p_1, \qquad \varphi = p_2.$$

From definition we can see that

$$\begin{aligned} \langle H\psi, \varphi\rangle &= \langle e^{-x^2/2}p_1 + 2xe^{-x^2/2}p_1' - e^{-x^2/2}p_1'', e^{-x^2/2}p_2\rangle \\ &= \langle e^{-x^2/2}p_1, e^{-x^2/2}p_2\rangle + 2\langle xe^{-x^2/2}p_1', e^{-x^2/2}p_2\rangle - \langle e^{-x^2/2}p_1'', e^{-x^2/2}p_2\rangle. \\ &= \int_{-\infty}^{\infty} e^{-x^2}p_1p_2dx + 2\int_{-\infty}^{\infty} xe^{-x^2}p_1'p_2dx - \int_{-\infty}^{\infty} e^{-x^2}p_1''p_2dx, \\ \langle \psi, H\varphi\rangle &= \int_{-\infty}^{\infty} e^{-x^2}p_1p_2dx + 2\int_{-\infty}^{\infty} xe^{-x^2}p_1p_2'dx - \int_{-\infty}^{\infty} e^{-x^2}p_1p_2''dx. \end{aligned}$$

Therefore,

$$\begin{aligned}\langle H\psi,\varphi\rangle-\langle\psi,H\varphi\rangle&=\int_{-\infty}^{\infty}2xe^{-x^2}(-p_1p_2'+p_1'p_2)-e^{-x^2}(p_1''p_2-p_1p_2'')dx\\&=\int_{-\infty}^{\infty}-p_1(p_2'e^{-x^2})'+p_2(p_1'e^{-x^2})'dx.\end{aligned}$$

Since

$$\begin{aligned}\int_{-\infty}^{\infty}-p_1(p_2'e^{-x^2})dx&=-p_1p_2'e^{-x^2}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}p_1'p_2'dx,\\ \int_{-\infty}^{\infty}p_2(p_1'e^{-x^2})'dx&=p_1'p_2e^{-x^2}\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}p_1'p_2'dx,\end{aligned}$$

and

$$-p_1p_2'e^{-x^2}\Big|_{-\infty}^{\infty}\to 0,\qquad p_1'p_2e^{-x^2}\Big|_{-\infty}^{\infty}\to 0,$$

we have

$$\langle H\psi,\varphi\rangle=\langle\psi,H\varphi\rangle.$$

(iii).

According to the definition,

$$\begin{aligned}HA&=\left(-\frac{d^2}{dx^2}+x^2\right)\left(-\frac{dx}{x}+x\right)=\frac{d^3}{dx^3}-2\frac{d}{dx}-x\frac{d^2}{dx^2}-x^2\frac{d}{dx}+x^3\\AH&=\left(-\frac{dx}{x}+x\right)\left(-\frac{d^2}{dx^2}+x^2\right)=\frac{d^3}{dx^3}-2x-x^2\frac{d}{dx}-x\frac{d^x}{dx^2}+x^3.\end{aligned}$$

Then

$$HA-AH=-2\frac{d}{dx}+2x=2A.$$

(iv).

Since ψ is an eigenfunction of H with eigenvalue λ, then $H\psi=\lambda\psi$. To show that $A\psi$ is an eigenfunction of H, we have

$$HA\psi=(2A+AH)\psi=(\lambda+2)A\psi,$$

showing that $A\psi$ is an eigenfunction of H or eigenvalue $\lambda+2$.

(v).

Omitted.

(vi).

We first prove the two equations.

$$\begin{aligned}H(e^{-x^2/2}) &= \left(-\frac{d^2}{dx^2}+x^2\right)e^{-x^2/2}\\ &= -\frac{d}{dx}\left(-xe^{-x^2/2}\right)+x^2e^{-x^2/2}\\ &= e^{-x^2/2},\end{aligned}$$

and

$$\begin{aligned}e^{x^2/2}\left(-\frac{d}{dx}\right)\left(e^{-x^2/2}f(x)\right) &= e^{x^2/2}\left(xe^{-x^2/2}f(x)-e^{-x^2/2}f'(x)\right)\\ &= xf(x)-f'(x)\\ &= Af(x).\end{aligned}$$

To prove that the eigenfunctions of H to eigenvalues $\lambda_n = 2n+1$ can be written as

$$\psi_n(x) = e^{-x^2/2}H_n(x),$$

we use induction.

- Basic step. For $n = 0$, it is trivial that the relation is true.
- Induction step. Suppose for $n = k$, the relation is true, then this means that

$$\begin{aligned}\psi_k &= (-1)^k e^{x^2/2}\frac{d^k}{dx^k}\left(e^{-x^2/2}\right),\\ A\psi_k &= e^{x^2/2}\left(-\frac{d}{dx}\right)\left((-1)^k\frac{d^k}{dx^k}\left(e^{-x^2/2}\right)\right) = \psi_{k+1}.\end{aligned}$$

  Therefore, if for $n = k$, the relation is satisfied, then

$$HA\psi_k - AH\psi_k = 2A\psi_k \quad\Rightarrow\quad H\psi_{k+1}-(2k+1)\psi_{k+1} = 2\psi_{k+1},$$

  and thus

$$H\psi_{k+1} = (2k+3)\psi_{k+1}.$$

Therefore, the eigenfunctions of H to eigenvalues $\lambda_n = 2n+1$ can be written as the form of ψ_n.

(vii).

First we have

$$H_n' = (-1)^n 2xe^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)+(-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right).$$

Therefore,

$$H_{n+1} = 2xH_n - H_n'.$$

Then for $n = 1$,

$$H_1 = -e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = 2x.$$

The statement is true for the basic step. Then suppose for $n = k$, the statement holds, then for $n = k+1$, we want to show that

$$H_{k+1}' = 2H_k + 2xH_k' - H_k'' = 2(k+1)H_k,$$

which, by subtracting one term of H_k and multiplying with $e^{-x^2/2}$ on both sides, is equivalent to

$$\underbrace{e^{-x^2/2}H_k + 2xe^{-x^2/2}H_k' - e^{-x^2/2}H_k''}_{H\psi_k} = (2k+1)\underbrace{e^{-x^2/2}H_k}_{\psi_k}.$$

This follows from the fact that ψ_n is the eigenfunction of H for eigenvalue $\lambda_n = 2n+1$.

(viii).

We use induction to prove this statement. For the basic step, $n = 0$, we have

$$\langle\psi_0,\psi_0\rangle = \int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\pi}.$$

Then suppose the statement holds for $n = k$, which means that

$$\int_{\mathbb{R}} e^{-x^2}H_k dx = \int_{\mathbb{R}} e^{-x^2}H_k\cdot\frac{H_{k+1}'}{2(k+1)}dx = \sqrt{\pi}2^k k!.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} e^{-x^2}H_k\cdot H_{k+1}'dx &= \underbrace{e^{-x^2}H_kH_{k+1}\Big|_{-\infty}^{\infty}}_{0} - \int_{\mathbb{R}} H_{k+1}\underbrace{\left(-2xe^{-x^2}H_k + e^{-x^2}H_k'\right)}_{-e^{-x^2}H_{k+1}}dx \\ &= \langle\psi_{k+1},\psi_{k+1}\rangle,\end{aligned}$$

verifying the result for $n = k+1$.