VE401 Probabilistic Methods in Eng.
Solution Manual for RC 8

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Assignment 8.4

Recall that

$$P = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \qquad H = X(X^TX)^{-1}X^T$$

where X is the model specification matrix for multiple linear regression.

1. Show that $PH = HP = P$. Conclude that $H - P$ is an orthogonal projection and that

$$\mathrm{SS_R} = \langle (H-P)Y, (H-P)Y\rangle.$$

2. Show that $\operatorname{tr} P = 1$ and conclude $\operatorname{tr}(H - P) = p$.

3. Follow the steps in the lecture slides to show that if $\beta = (\beta_0, 0, \ldots, 0)$ (i.e., if $\beta_1 = \cdots = \beta_p = 0$), then $\mathrm{SS_R}/\sigma^2$ follows a chi-squared distribution with p degrees of freedom.

4. Show that $(\mathbb{1} - H)(P - H) = (P - H)(\mathbb{1} - H) = 0$. Deduce that

$$\operatorname{ran}(P - H) \subset \ker(\mathbb{1} - H) \qquad \text{and} \qquad \operatorname{ran}(\mathbb{1} - H) \subset \ker(P - H).$$

Explain why this means that the eigenvectors of $H - P$ for the eigenvalue 1 are also eigenvectors of $\mathbb{1} - H$ for the eigenvalue 0 and vice-versa. Construct a matrix U which diagonalizes both $P - H$ and $\mathbb{1} - H$. Use U to show that $\mathrm{SS_R}$ and $\mathrm{SS_E}$ are the sums of squares of independent standard normal variables. Deduce that $\mathrm{SS_R}$ and $\mathrm{SS_E}$ are independent.

Solution.

1. Since H is an orthogonal projection, $H^2 = H$, and

$$\det\big(X(X^TX)^{-1}X^T\big) = \det\big((X^TX)^{-1}X^TX\big) = \det \mathbb{1}_{p+1} = 1,$$

which means H is invertible. Then

$$P = H^{-1}HP = H^{-1}H^2P = HP, \qquad P = PH^{-1}H = PH^{-1}H^2 = PH.$$

Therefore,

$$\begin{aligned}(H-P)^2 &= H^2 + P^2 - HP - PH = H - P, \\ (H-P)^T &= H^T - P^T = H - P,\end{aligned}$$

and $H-P$ is an orthogonal projection. Furthermore,

$$\begin{aligned}
\mathrm{SS_R} &= \mathrm{SS_T} - \mathrm{SS_E} \\
&= \langle Y, (\mathbb{1}-P)Y\rangle - \langle Y, (\mathbb{1}-H)Y\rangle \\
&= \langle Y, (H-P)Y\rangle \\
&= \langle Y, (H-P)^2 Y\rangle = \langle (H-P)Y, (H-P)Y\rangle.
\end{aligned}$$

2. We know that

$$\operatorname{tr} P = \frac{1}{n}\sum_{i=1}^{n} 1 = 1,$$

and thus

$$\operatorname{tr}(H-P) = \operatorname{tr} H - \operatorname{tr} P = \operatorname{tr}(X(X^TX)^{-1}X^T) - 1 = p+1-1 = p.$$

3. Since $H-P$ is an orthogonal projection, the sum of its eigenvalues is equal to the number of eigenvalues that equal 1. Since $H-P$ is symmetric, there exists U consisting of columns of eigenvectors of $H-P$ such that

$$U^{-1}(H-P)U = U^T(H-P)U = D_p = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & 0 \end{pmatrix}$$

and thus $H-P = UD_pU^T$. Since given that $\beta = (\beta_0, 0, \dots, 0)$,

$$\begin{aligned}
(H-P)(X\beta + E) &= HX\beta - PX\beta + (H-P)E \\
&= (\mathbb{1}-P)X\beta + (H-P)E \\
&= (\mathbb{1}-P)\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_0 \end{pmatrix} + (H-P)E = (H-P)E.
\end{aligned}$$

we then have

$$\begin{aligned}
\frac{\mathrm{SS_R}}{\sigma^2} &= \frac{1}{\sigma^2}\langle (H-P)(X\beta+E), (H-P)(X\beta+E)\rangle \\
&= \left\langle (H-P)\frac{E}{\sigma}, (H-P)\frac{E}{\sigma}\right\rangle \\
&= \langle Z, (H-P)Z\rangle \\
&= \langle Z, UD_pU^TZ\rangle \\
&= \sum_{i=1}^{p}(U^TZ)_i^2,
\end{aligned}$$

where Z is standard normally distributed. Therefore, $\mathrm{SS_R}/\sigma^2$ follows a chi-squared distribution with p degrees of freedom.

4. Since $HP = PH = P$, we have

$$(\mathbb{1} - H)(P - H) = P - H - HP + H^2 = P - H - P + H = 0,$$
$$(P - H)(\mathbb{1} - H) = P - PH - H + H^2 = P - P - H + H = 0.$$

Then for any $v \in \operatorname{ran}(P - H)$, there exists a $u \in \mathbb{R}^n$ such that $v = (P - H)u$, and thus

$$(\mathbb{1} - H)(P - H)u = (\mathbb{1} - H)v = 0 \quad \Rightarrow \quad v \in \ker(\mathbb{1} - H).$$

Similarly, for any $v \in \operatorname{ran}(\mathbb{1} - H)$, there exists a $u \in \mathbb{R}^n$ such that $v = (\mathbb{1} - H)u$, and thus

$$(P - H)(\mathbb{1} - H)u = (P - H)v = 0 \quad \Rightarrow \quad v \in \ker(P - H).$$

Therefore, $\operatorname{ran}(P - H) \subset \ker(\mathbb{1} - H)$ and $\operatorname{ran}(\mathbb{1} - H) \subset \ker(P - H)$. Then if v is an eigenvector of $H - P$ for 1, then $(H - P)v = v$, which means $v \in \operatorname{ran}(P - H)$ and thus $v \in \ker(\mathbb{1} - H)$, indicating that v is also an eigenvector of $\mathbb{1} - H$ for 0. It is similar with the eigenvectors of $\mathbb{1} - H$ for 1. We can construct a matrix U

$$U = (b_1, \ldots, b_p, b_{p+1}, \ldots, b_n),$$

where $(b_1, \ldots, b_p)$ is an orthonormal basis of eigenvectors of $H - P$ for eigenvalue 1, and $(b_{p+1}, \ldots, b_n)$ is an orthonormal basis of eigenvectors of $H - P$ for eigenvalue 0, among which $(b_{p+2}, \ldots, b_n)$ consists of eigenvectors of $\mathbb{1} - H$ for 1. Then it satisfies that

$$U^T(H - P)U = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & 0 \end{pmatrix} =: D_p, \qquad U^T(\mathbb{1} - H)U = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{n-p-1} \end{pmatrix} =: D_{n-p},$$

and U diagonalizes both $P - H$ and $\mathbb{1} - H$. Since

$$\mathrm{SS_E} = \langle Y, (\mathbb{1} - H)Y \rangle = \langle Y, UD_{n-p-1}U^T \rangle = \sum_{i=p+2}^{n} (U^T Y)_i^2,$$
$$\mathrm{SS_R} = \langle Y, (H - P)Y \rangle = \langle Y, UD_pU^T Y \rangle = \sum_{i=1}^{p} (U^T Y)_i^2,$$

which are independent of each other.