VE401 Probabilistic Methods in Eng.
RC 1

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March 12, 2020

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Random Variables and Probability Density Function

Definition. Let S be a sample space and Ω a countable subset of $\mathbb{R}$. A ***discrete random variable*** is a map

$$X : S \to \Omega$$

together with a function

$$f_X : \Omega \to \mathbb{R}$$

having the properties that

(i) $f_X(x) \geq 0$ for all $x \in \Omega$ and

(ii) $\displaystyle\sum_{x \in \Omega} f_X(x) = 1.$

The function f_X is called the ***probability density function*** or ***probability distribution*** of X. A random variable is given by the pair (X, f_X).

Cumulative Distribution Function

Definition. The ***cumulative distribution function*** of a random variable is defined as

$$F_X : \mathbb{R} \to \mathbb{R}, \qquad F_X(x) := P[X \leq x].$$

For a discrete random variable,

$$F_X(x) = \sum_{y \leq x} f_X(y).$$

Expectation and Variance

Definition. Let (X, f_X) be a discrete random variable.

- The ***expected value*** or ***expectation*** of X is

$$\mu_X = \mathrm{E}[X] := \sum_{x \in \Omega} x \cdot f_X(x),$$

provided that the sum (possibly series, if Ω is infinite) on the right converges absolutely.

- The ***variance*** is defined by

$$\sigma_X^2 = \mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$$

which is defined as long as the right-hand side exists.

- The ***standard deviation*** is $\sigma_X = \sqrt{\mathrm{Var}[X]}$.

Properties

- Expectation.

  (a). Suppose $\varphi : \Omega \to \mathbb{R}$ is some function, then

$$\mathsf{E}[\varphi \circ X] = \sum_{x \in \Omega} \varphi(x) \cdot f_X(x).$$

  (b). $\mathsf{E}[aX + bY + c] = a\mathsf{E}[X] + b\mathsf{E}[Y] + c$, where $a, b, c \in \mathbb{R}$ and X, Y are random variables.

  (c). $\mathsf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathsf{E}[X_i]$.

  (d). If $X_1, \ldots, X_n$ are independent random variables, and $g_i, i = 1, \ldots, n$ are functions, then

$$\mathsf{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathsf{E}[X_i], \quad \mathsf{E}\left[\prod_{i=1}^{n} g_i(X_i)\right] = \prod_{i=1}^{n} \mathsf{E}[g_i(X_i)].$$

Properties

- Variance.
  (a). $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$.
  (b). $\mathrm{Var}[aX + b] = a^2\mathrm{Var}[X]$, where $a, b \in \mathbb{R}$.
  (c). If $X_1, \ldots, X_n$ are independent random variables, then

$$\mathrm{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \mathrm{Var}[X_i].$$

  Note. If X and Y are not independent, then according to definitions,

$$\begin{aligned}\mathrm{Var}[X + Y] &= \mathrm{E}\left[(X + Y - (\mu_X + \mu_Y))^2\right] \\ &= \mathrm{E}\left[(X - \mu_X)^2\right] + \mathrm{E}\left[(Y - \mu_Y)^2\right] + \\ &\qquad + 2\mathrm{E}\left[(X - \mu_X)(Y - \mu_Y)\right] \\ &\neq \mathrm{Var}[X] + \mathrm{Var}[Y].\end{aligned}$$

Ordinary and Central Moments

Definition. The n^{th} ***(ordinary) moments*** of a random variable X is given by

$$\mathrm{E}[X^n], \qquad n \in \mathbb{N}.$$

The n^{th} ***central moments*** of X is given by

$$\mathrm{E}\left[\left(\frac{X-\mu}{\sigma}\right)^n\right], \qquad \text{where } n = 3, 4, 5, \ldots$$

Moment-Generating Function

Definition. Let (X, f_X) be a random variable and such that the sequence of moments $\mathrm{E}[X^n]$, $n \in \mathbb{N}$, exists. If the power series

$$m_X(t) := \sum_{k=0}^{\infty} \frac{\mathrm{E}[X^k]}{k!} t^k$$

has radius of convergence $\varepsilon > 0$, the thereby defined function

$$m_X(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$$

is called the ***moment-generating function*** for X.

Moment-Generating Function

Theorem. Let $\varepsilon > 0$ be given such that $\mathrm{E}[e^{tX}]$ exists and has a power series expansion in t that converges for $|t| < \varepsilon$. Then the moment-generating function exists and

$$m_X(t) = \mathrm{E}[e^{tX}] \qquad \text{for } |t| < \varepsilon.$$

Furthermore,

$$E[X^k] = \left.\frac{\mathrm{d}^k m_X(t)}{\mathrm{d}t^k}\right|_{t=0}.$$

We can hence calculate the moments of X by differentiating the moment-generating function.

Bernoulli Random Variable

Definition. Let S be a sample space and

$$X : S \to \{0, 1\} \subset \mathbb{R}.$$

Let $0 < p < 1$ and define the density function

$$f_X : \{0, 1\} \to \mathbb{R}, \qquad f_X(x) = \begin{cases} 1 - p & \text{for } x = 0, \\ p & \text{for } x = 1. \end{cases}$$

Then X is said to be a ***Bernoulli random variable*** or follow a ***Bernoulli distribution*** with parameter p, denoted by

$$X \sim \text{Bernoulli}(p).$$

Bernoulli Random Variable

Mean, variance, and M.G.F.

- Mean.

$$\mathrm{E}[X] = 0 \cdot (1-p) + 1 \cdot p = p.$$

- Variance.

$$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2 = p - p^2 = p(1-p).$$

- M.G.F.

$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = (1-p) + e^t p.$$

Binomial Random Variable

Definition. Let S be a sample space, $n \in \mathbb{N} \setminus \{0\}$, and

$$X : S \to \Omega = \{0, \ldots, n\} \subset \mathbb{R}.$$

Let $0 < p < 1$ and define the density function

$$f_X : \Omega \to \mathbb{R}, \qquad f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Then X is said to be a ***binomial random variable*** with parameters n and p, denoted by

$$X \sim \mathrm{B}(n, p),$$

and particularly, $\mathrm{B}(1, p) = \mathrm{Bernoulli}(p)$.

Binomial Distribution

Mean, variance and M.G.F.

- Mean.

$$E[X] = \sum_{i=1}^{n} E[X_i] = np.$$

- Variance.

$$\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = np(1-p).$$

- M.G.F.

$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = E[e^{tX}] = \prod_{i=1}^{n} E[e^{tX_i}] = (1 - p + pe^t)^n.$$

Geometric Random Variable

Definition. Let S be a sample space and

$$X : S \to \Omega = \mathbb{N} \setminus \{0\}.$$

Let $0 < p < 1$ and define the density function $f_X : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ given by

$$f_X(x) = (1 - p)^{x-1} p.$$

Then X is a ***geometric random variable*** with parameter p, denoted by

$$X \sim \text{Geom}(p).$$

Geometric Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{1}{p}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{1-p}{p^2}.$$

- M.G.F.

$$m_X : (-\infty, -\ln(1-p)) \to \mathbb{R}, \qquad m_X(t) = \frac{pe^t}{1-(1-p)e^t}.$$

Poisson Distribution

Definition. Let $k \in \mathbb{R}$. A random variable (X, f_X) with

$$X : S \to \mathbb{N}$$

and density function

$$f_X : \mathbb{N} \to \mathbb{R}, \qquad f_X(x) = \frac{k^x e^{-k}}{x!}$$

is said to follow a ***Poisson distribution*** with parameter k, which describes the occurrence of events that occur at a constant rate and continuous environment.

Poisson Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = k.$$

- Variance.

$$\mathrm{Var}[X] = k.$$

- M.G.F.

$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = e^{k(e^t - 1)}.$$

Pascal Distribution

Definition. Let $r \in \mathbb{N} \setminus \{0\}$. A random variable (X, f_X) with

$$X : S \to \Omega = \mathbb{N} \setminus \{0, 1, \ldots, r-1\} = \{r, r+1, \ldots\}$$

and distribution function $f_X : \Omega \to \mathbb{R}$ given by

$$f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \qquad 0 < p < 1,$$

is said to follow a ***Pascal distribution*** with parameters p and r.

Pascal Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{r}{p}.$$

- Variance. Let $q = 1 - p$,

$$\mathrm{Var}[X] = \frac{rq}{p^2}.$$

- M.G.F.

$$m_X : (-\infty, -\ln q) \to \mathbb{R}, \qquad m_X(t) = \frac{(pe^t)^r}{(1 - qe^t)^r}.$$

Negative Binomial Distribution

Definition. Let $r \in \mathbb{N} \setminus \{0\}$. A random variable (X, f_X) with

$$X : S \to \Omega = \mathbb{N}$$

and distribution function $f_X : \Omega \to \mathbb{R}$ given by

$$f_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x, \qquad 0 < p < 1,$$

is said to follow a ***negative binomial distribution*** with parameters p and r.

Negative Binomial Distribution

Mean, variance and M.G.F.

- Mean. Let $q = 1 - p$,

$$\mathrm{E}[X] = \frac{rp}{q}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{rp}{q^2}.$$

- M.G.F.

$$m_X : (-\infty, -\ln q) \to \mathbb{R}, \qquad m_X(t) = \frac{p^r}{(1 - qe^t)^r}.$$

Gamma Distribution

Definition. Let $r \in \mathbb{N} \setminus \{0\}$. A random variable (X, f_X) with

$$X : S \to \Omega = \mathbb{N}$$

and distribution function $f_X : \Omega \to \mathbb{R}$ given by

$$f_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x, \qquad 0 < p < 1,$$

is said to follow a ***negative binomial distribution*** with parameters p and r.

Poisson, Gamma, and Negative Binomial Distribution

Suppose a conditional random variable $X|\Lambda$ follows a Poisson distribution, and Λ follows a

Thanks for your attention!