VE401 Probabilistic Methods in Eng.
RC 5

CHEN Xiwen

UM-SJTU Joint Institute

April 7, 2020

Table of contents

Fisher's Null Hypothesis Test

Overview.

1. Set up a ***null hypothesis*** H_0 that compares a population parameter θ to a given null value θ_0.
   - $H_0 : \theta = \theta_0$,
   - $H_0 : \theta \leq \theta_0$,
   - $H_0 : \theta \geq \theta_0$.
2. Try to reject the null hypothesis by finding an upper bound (***significance*** or ***P-value***) of probability of obtaining the data or more extreme data (based on the null hypothesis), given that the null hypothesis is true.

$$P[D|H_0] \leq P - \text{valur}.$$

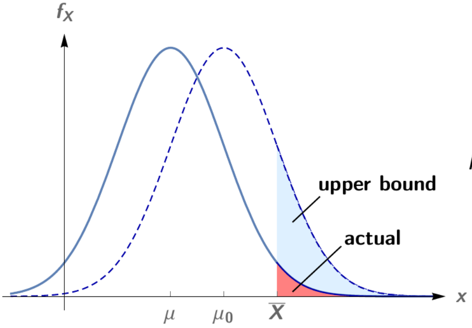
3. We either
   - fail to reject H_0 or
   - reject H_0 at the [p-value] level of significance.

One-tailed Test

Null hypothesis.

$$H_0 : \theta \leq \theta_0 \qquad \text{or} \qquad H_0 : \theta \geq \theta_0.$$

Test for mean. Suppose the sample mean $\overline{X}$ follows a normal distribution with mean μ.



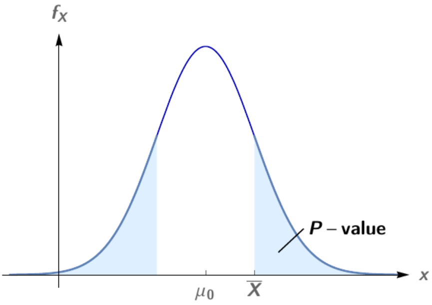
$H_0 : \mu \leq \mu_0$

Two-tailed Test

Null hypothesis.

$$H_0 : \theta = \theta_0.$$

Test for mean. Suppose the sample mean $\overline{X}$ follows a normal distribution with mean μ.



$$H_0 : \mu = \mu_0$$

Fisher's Null Hypothesis Test

Example.

Neyman-Pearson Decision Theory

Overview.

1. Set up a ***null hypothesis*** H_0 and an ***alternative hypothesis*** H_1.
2. Determine a desirable α and β, where
   - $\alpha := P[\text{accept } H_1 | H_0 \text{ true}]$, and
   - $\beta := P[\text{accept } H_0 | H_1 \text{ true}]$.
3. Use α and β to determine the appropriate sample size n. $\triangle$
4. Use α and n to determine the critical region. $\triangle$
5. Obtain sample statistics, and reject H_0 at significance level α and accept H_1 if the test statistic falls into critical region. Otherwise, accept H_0.

Choosing the Sample Size

Special case. Suppose the sample mean $\overline{X}$ follows a normal distribution with unknown mean μ and known variance σ^2, and we have hypothesis

$$H_0 : \mu = \mu_0, \qquad H_1 : |\mu - \mu_0| \geq \delta_0.$$

Relation between α, β δ, σ and n. With true mean $\mu = \mu_0 + \delta$, the test statistic $Z = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathrm{N}(\delta\sqrt{n}/\sigma, 1)$.

$$\begin{aligned} P[\text{fail to reject } H_0 | \mu = \mu_0 + \delta] &= \frac{1}{\sqrt{2\pi}} \int_{-z_{\alpha/2}}^{z_{\alpha/2}} e^{-(t-\delta\sqrt{n}/\sigma)^2/2} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{-z_{\alpha/2}-\delta\sqrt{n}/\sigma}^{z_{\alpha/2}-\delta\sqrt{n}/\sigma} e^{-t^2/2} \mathrm{d}t \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_{\alpha/2}-\delta\sqrt{n}/\sigma} e^{-t^2/2} \mathrm{d}t \stackrel{!}{=} \beta, \end{aligned}$$

giving $-z_\beta = z_{\alpha/2} - \delta\sqrt{n}/\sigma$.

Choosing the Sample Size

Special case. Suppose the sample mean $\overline{X}$ follows a normal distribution with unknown mean μ and known variance σ^2, and we have hypothesis

$$H_0 : \mu = \mu_0, \qquad H_1 : |\mu - \mu_0| \geq \delta_0.$$

Choosing the sample size n.

$$n \approx \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2},$$

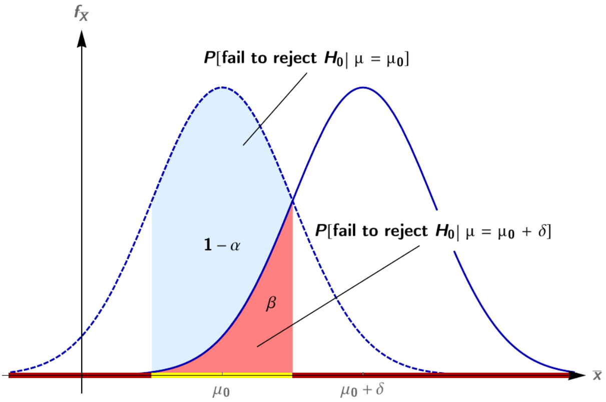
where $z_{\alpha/2}$ and z_β satisfies that

$$\Phi(z_{\alpha/2}) = 1 - \alpha/2, \qquad \Phi(z_\beta) = 1 - \beta,$$

given cumulative distribution function Φ of standard normal distribution.

Choosing the Sample Size

Special case.



Choosing the Sample Size

More general case: OC curve.

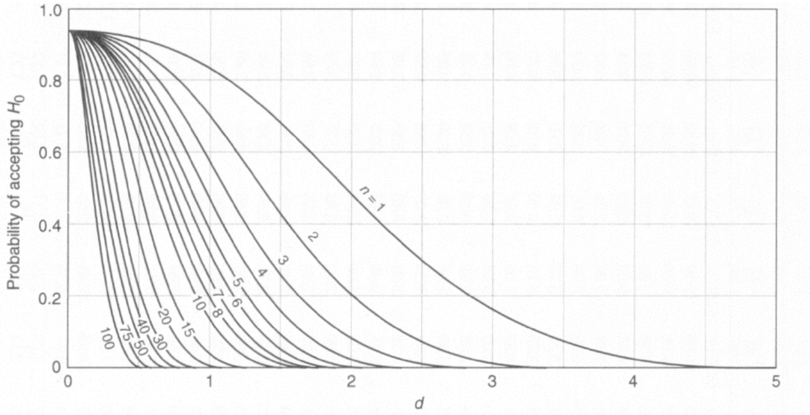
1. Calculate

$$d := \frac{|\mu - \mu_0|}{\sigma}.$$

2. Look up in OC curve for sample size n.

Choosing the Sample Size

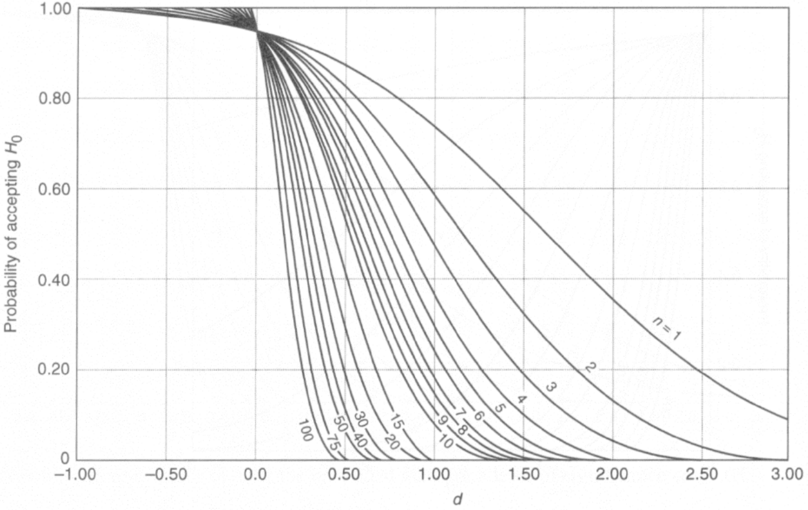
More general case: OC curve.



(a) OC curves for different values of n for the two-sided normal test for a level of significance $\alpha = 0.05$.

Choosing the Sample Size

More general case: OC curve.



(c) OC curves for different values of n for the one-sided normal test for a level of significance $\alpha = 0.05$.

Choosing the Critical Region

Determine the critical region using α and n. The ***critical region*** is chosen so that if H_0 is true, then the probability of test statistic's value falling into the critical region is no more than α.

Critical region for mean. Suppose the sample mean $\overline{X}$ follows a normal distribution with unknown mean μ and known variance σ^2, with $H_0 : \mu = \mu_0$. Then the test statistic

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1),$$

and thus the critical region is obtained from

$$\frac{|\overline{X} - \mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}.$$

We reject H_0 at significance level α and accept H_1 if $\overline{X}$ falls in this critical region.

Neyman-Pearson Decision Theory

Example.

Hypothesis Tests

- Fisher's Null Hypothesis Test
- Neyman-Pearson Decision Theory

Test for Statistics

- Tests for Mean, Median and Variance
- Inferences on Proportions
- Comparing Two Means and Two Variances

Hypothesis Tests

Fisher's Null Hypothesis Test

Neyman-Pearson Decision Theory

Test for Statistics

Tests for Mean, Median and Variance

Inferences on Proportions

Comparing Two Means and Two Variances

Thanks for your attention!