VE401 Probabilistic Methods in Eng.
RC 3

CHEN Xiwen

UM-SJTU Joint Institute

March 19, 2020

Table of contents

Continuous Random Variables

Definition. Let S be a sample space. A ***continuous random variable*** is a map $X : S \to \mathbb{R}$ together with a function $f_X : \mathbb{R} \to \mathbb{R}$ with the properties that

(i). $f_X(x) \geq 0$ for all $x \in \mathbb{R}$ and

(ii). $\displaystyle\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1.$

The integral of f_X is interpreted as the probability that X assumes values x in a given range, i.e.,

$$P[a \leq X \leq b] = \int_a^b f_X(x)\mathrm{d}x.$$

The function f_X is called the ***probability density function*** of random variable X.

Cumulative Distribution

Definition. Let (X, f_X) be a continuous random variable. The cumulative distribution function for X is defined by $F_X : \mathbb{R} \to \mathbb{R}$,

$$F_X(x) := P[X \leq x] = \int_{-\infty}^{x} f_X(y)\mathrm{d}y.$$

By the fundamental theorem of calculus, we can obtain the density function from F_X by

$$f_X(x) = F_X'(x).$$

Expectation, Variance, and M.G.F.

- Expectation.

$$\mathrm{E}[X] := \int_{\mathbb{R}} x \cdot f_X(x)\mathrm{d}x.$$

- Variance.

$$\mathrm{Var}[X] := \mathrm{E}[(X - \mathrm{E}[X])^2] = \mathrm{E}[X^2] - \mathrm{E}[X]^2.$$

- Moment-generating function.

$$m_X(t) = \mathrm{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\mathrm{d}x.$$

Note. All previous properties about expectation, variance and M.G.F. hold for continuous random variables.

Location of Continuous Distributions

Definitions.

- The ***median*** M_X is defined by $P[X \leq M_X] = 0.5$.
- The ***mean*** is given by $\mathrm{E}[X]$.
- The ***mode*** x_0, is the location of the maximum of f_X.

Transformation of Random Variables

Theorem. Let X be a continuous random variable with density f_X. Let $Y = \varphi \circ X$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly monotonic and differentiable. The density for Y is then given by

$$f_Y(y) = f_X(\varphi^{-1}(y)) \cdot \left| \frac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y} \right|, \qquad \text{for } y \in \operatorname{ran} \varphi$$

and

$$f_Y(y) = 0, \qquad \text{for } y \notin \operatorname{ran} \varphi.$$

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution.

(i). Identify and calculate φ^{-1} and $\dfrac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y}$.

(ii). Substitute x with $\varphi^{-1}(y)$ in the density function of X.

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution. We have $\varphi(x) = ve^{xt}$, and thus

$$\varphi^{-1}(y) = \frac{1}{t}\log\left(\frac{y}{v}\right), \qquad \frac{\mathrm{d}\varphi^{-1}(y)}{\mathrm{d}y} = \frac{1}{ty}.$$

Transformation of Random Variables

Example 1. A model for populations of microscopic organisms is exponential growth. Initially, v organisms are introduced into a large tank of water, and let X be the rate of growth. After time t, the population becomes $Y = ve^{Xt}$. Suppose X is unknown and has a continuous distribution

$$f_X(x) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is the distribution of Y?

Solution (continued). Therefore,

$$f_Y(y) = \begin{cases} 3\left(1 - \frac{1}{t}\log\left(\frac{y}{v}\right)\right)^2 \cdot \frac{1}{ty}, & v < y < ve^t, \\ 0 & \text{otherwise.} \end{cases}$$

Exponential Distribution

Definition. A continuous random variable (X, f_β) a ***exponential distribution*** with parameter β if the probability density function is defined by

$$f_\beta(X) = \begin{cases} \beta e^{-\beta x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Interpretation. The time between successive arrivals of a Poisson process with rate λ follows exponential distribution with parameter $\beta = \lambda$. (Recall $P[T > t] = e^{-\beta t}$.)
Note. Memoryless property:

$$P[X > x + s | X > x] = P[X > s].$$

Exponential Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{1}{\beta}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{1}{\beta^2}.$$

- M.G.F.

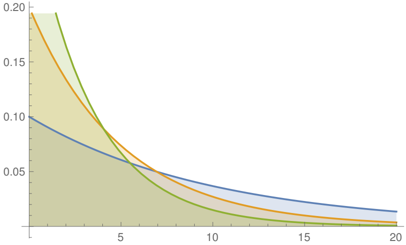
$$m_X : (-\infty, \beta) \to \mathbb{R}, \qquad m_X(t) = \frac{1}{1 - t/\beta}.$$

Exponential Distribution

Plots.

In[∘]:=
```
Plot[Table[PDF[ExponentialDistribution[β], x], {β, {0.1, 0.2, 0.3}}] // Evaluate, {x, 0, 20},
  Filling → Axis]
```

Out[∘]=



Exponential Distribution

Example 2. n light bulbs are burning simultaneously and independently, and the lifetime for each bulb follows the Exp(β).

(i). What is the distribution of the length of time Y_1 until the first failure in one of the n bulbs?

(ii). What is the distribution of the length of time Y_2 after the first failure until a second bulb fails?

Exponential Distribution

Example 2. n light bulbs are burning simultaneously and independently, and the lifetime for each bulb follows the Exp(β).

(i). What is the distribution of the length of time Y_1 until the first failure in one of the n bulbs?

(ii). What is the distribution of the length of time Y_2 after the first failure until a second bulb fails?

Solution (i). Suppose random variables $X_1, \ldots, X_n$ satisfies that $X_i \sim \text{Exp}(\beta)$, and $Y_1 = \min\{X_1, \ldots, X_n\}$. Then for any $t > 0$,

$$\begin{aligned} P[Y_1 > t] &= P[X_1 > t, \ldots, X_n > t] \\ &= P[X_1 > t] \times \cdots \times P[X_n > t] \\ &= e^{-n\beta t}, \end{aligned}$$

indicating an exponential distribution with parameter $n\beta$. What about (ii)?

Gamma Distribution

Definition. Let $\alpha, \beta \in \mathbb{R}, \alpha, \beta > 0$. A continuous random variable $(X, f_{\alpha,\beta})$ follows a ***gamma distribution*** with parameters α and β if the probability density function is given by

$$f_{\alpha,\beta}(x) = \begin{cases} \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

where $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} \mathrm{d}z, \alpha > 0$ is the Euler gamma function.

Interpretation. The time needed for the next r arrivals in a Poisson process with rate λ follows a Gamma distribution with parameters $\alpha = r, \beta = \lambda$.

Gamma Distribution

Mean, variance and M.G.F.

- Mean.

$$\mathrm{E}[X] = \frac{\alpha}{\beta}.$$

- Variance.

$$\mathrm{Var}[X] = \frac{\alpha}{\beta^2}.$$

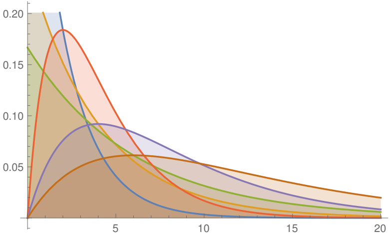
- M.G.F.

$$m_X : (-\infty, \beta) \to \mathbb{R}, \qquad m_X(t) = \frac{1}{(1 - t/\beta)^\alpha}.$$

Gamma Distribution

Plots.

In[ ]:=
```
Plot[Table[PDF[GammaDistribution[α, β], x], {α, {1, 2}}, {β, {2, 4, 6}}] // Evaluate, {x, 0, 20},
  Filling → Axis]
```

Out[ ]=


Chi-Squared Distribution

Definition. Let $\gamma \in \mathbb{N}$. A continuous random variable (X_γ^2, f_X) follows a ***chi-squared distribution*** with γ degrees of freedom if the probability density function is given by

$$f_\gamma(x) = \begin{cases} \dfrac{1}{2^\alpha \Gamma(\gamma/2)} x^{\gamma/2-1} e^{-x/2}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

which is a gamma distribution with $\alpha = \gamma/2, \beta = 1/2$. Therefore,

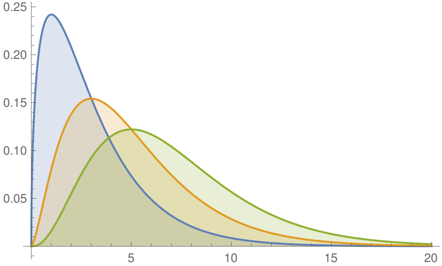
$$\mathrm{E}[X_\gamma^2] = \gamma, \qquad \mathrm{Var}[X_\gamma^2] = 2\gamma.$$

Chi-Squared Distribution

Plots.

In[ ]:=
```
Plot[Table[PDF[ChiSquareDistribution[γ], x], {γ, {3, 5, 7}}] // Evaluate, {x, 0, 20},
 Filling → Axis]
```

Out[ ]=



Normal Distribution

Definition. A continuous random variable (X, f_{μ,σ^2}) has the ***normal distribution*** with mean $\mu \in \mathbb{R}$ and variance $\sigma^2, \sigma > 0$ if the probability density function is given by

$$f_{\mu,\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \qquad x \in \mathbb{R}.$$

Normal Distribution

Mean, variance and M.G.F.

- Mean.

$$E[X] = \mu.$$

- Variance.

$$\mathrm{Var}[X] = \sigma^2.$$

- M.G.F.

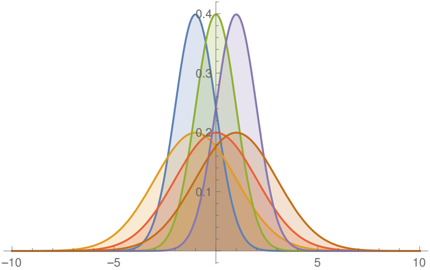
$$m_X : \mathbb{R} \to \mathbb{R}, \qquad m_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

Normal Distribution

Plots.

In[ ]:=
```
Plot[Table[PDF[NormalDistribution[μ, σ], x], {μ, {-1, 0, 1}}, {σ, {1, 2}}] // Evaluate,
 {x, -10, 10}, Filling → Axis]
```

Out[ ]=



Standardizing Normal Distribution

Suppose $X \sim \text{Normal}(\mu, \sigma^2)$. Then

$$Z = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1),$$

where the normal distribution with mean μ and variance σ^2 is the ***standard normal distribution***. Furthermore, the cumulative distribution function of X is given by

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad F^{-1}(p) = \mu + \sigma\Phi^{-1}(p),$$

where Φ is the cumulative distribution function for the standard normal distribution function.

Distributions based on of Poisson Process

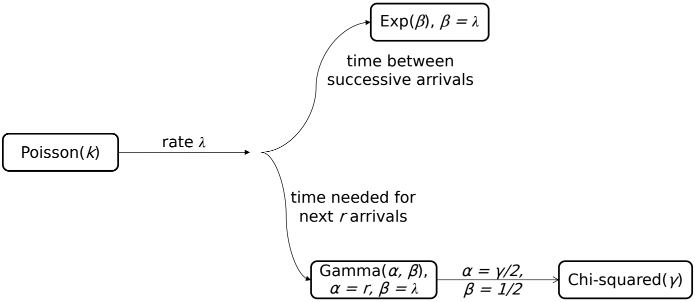


Figure: Connections of distributions based on Poisson process.

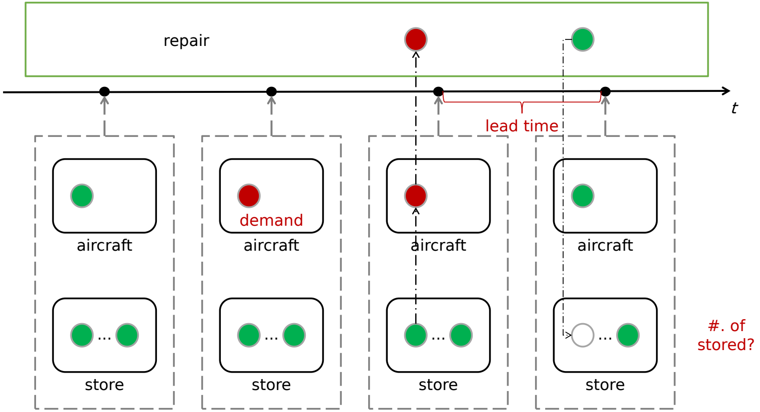
Poisson, Gamma and Negative Binomial Distribution

Application.

- To ensure stability of an aircraft, the broken components should be removed, replaced, and repaired immediately
- When the components are repaired, the replacing component is removed from the aircraft and restored to storage.
- Suppose the components are returned in the same order as they are removed from the aircraft.
- Q: How many components should be stored in case of such usage?

Poisson, Gamma and Negative Binomial Distribution

Application (continued).



Poisson, Gamma and Negative Binomial Distribution

Application (continued). We have the following models.

- The demand is a random variable R that follows a Poisson distribution

$$f_R(r|t) = \frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)},$$

  where λ is the average number of demands per unit time, and t denotes the lead time.

- The lead time t follows a gamma distribution

$$f_T(t) = \frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)}, \qquad k > 0,$$

  which can be seen from $\alpha = k, \beta = \mu$.

Poisson, Gamma and Negative Binomial Distribution

Application (continued). Then the probability of r demands during the lead time with parameter k is

$$
\begin{aligned}
p_{rk} &= \int_0^\infty f_R(r|t) f_T(t) \mathrm{d}t \qquad \text{(recall } P[A] = \sum P[A|B]P[B]\text{)} \\
&= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{\Gamma(r+1)} \times \frac{\mu e^{-\mu t}(\mu t)^{k-1}}{\Gamma(k)} \mathrm{d}t \\
&= \frac{\lambda^t \mu^k}{\Gamma(r+1)\Gamma(k)} \int_0^\infty t^{r+k-1} e^{-(\lambda+\mu)t} \mathrm{d}t \qquad \text{(let } z = (\lambda+\mu)t\text{)} \\
&= \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}\Gamma(r+1)\Gamma(k)} \int_0^\infty z^{r+k-1} e^{-z} \mathrm{d}z \\
&= \frac{\lambda^r \mu^k}{(\lambda+\mu)^{r+k}} \times \frac{\Gamma(r+k)}{\Gamma(r+1)\Gamma(k)} = \binom{r+k-1}{k-1} \frac{(\lambda/\mu)^r}{(1+\lambda/\mu)^{r+k}},
\end{aligned}
$$

implying r follows a negative binomial distribution with mean $\lambda k/\mu$ and variance $\lambda k/\mu(1+\lambda/\mu)$.

Poisson, Gamma and Negative Binomial Distribution

Application (continued). Next steps are basically the following.

1. Calculate *risk level* P_{nk}, defined by

$$P_{nk} = \sum_{r=n+1}^{\infty} p_{rk}.$$

2. Predetermine the risk level, calculate the corresponding float size n.

For more information, please see the file uploaded on canvas.

Exercises

Exercise 1. Suppose that a certain system contains three components C_1, C_2, C_3 that function independently of each other and are connected as series, so that the system fails as soon as one of the components fails. Suppose that the length of life of the C_1, C_2, C_3 has the exponential distribution with parameters

$$\beta_1 = 0.001, \qquad \beta_2 = 0.003, \qquad \beta_3 = 0.006,$$

respectively, all measured in hours. Determine the probability that the system will not fail before 100 hours.

Exercises

Exercise 2. Let X_1, X_2, X_3 be independent lifetimes of memory chips. Suppose each X_i follows the normal distribution with mean 300 hours and standard deviation 10 hours. Compute the probability that at least one of the three chips lasts at least 290 hours.

Thanks for your attention!