VE401 Probabilistic Methods in Eng.
Solution Manual for RC 4

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Assignment 3.4

A mathematics textbook has 200 pages on which typographical errors in the equations could occur. Suppose there are in fact five errors randomly dispersed among these 200 pages.

1. What is the probability that a random sample of 50 pages will contain at least one error?
2. How large must the random sample be to assure that at least three errors will be found with 90% probability? (You may use a normal approximation to the binomial distribution.)

Solution.

1. The problem is to randomly place the five errors in 200 pages, and each error has the same probability of being placed among the sampled pages.

$$\begin{aligned} P[\text{at least 1 error in 50 pages}] &= 1 - P[\text{0 error in 50 pages}] \\ &= 1 - \left(\frac{200-50}{200}\right)^5 \\ &= 76.27\%. \end{aligned}$$

2. Let the sample size be k. The number of selected errors follows a binomial distribution with

$$p = \frac{k}{200}, \qquad n = 5,$$

and thus the mean and standard deviation are given by

$$\mu = 5p = \frac{k}{40}, \qquad \sigma = \sqrt{5p(1-p)} = \sqrt{\frac{k}{40}\left(1 - \frac{k}{200}\right)}.$$

Let X be the number of errors in the sample. Then

$$P[X \geq 3] \geq 90\% \quad \Rightarrow \quad P[Y + 0.5 \geq 3] = P[Y \geq 2.5] \geq 90\%,$$

where Y follows normal distribution. Transforming to standard normal variable Z, we have

$$P\left[Z \geq \frac{2.5-\mu}{\sigma}\right] \geq 0.9 \quad \Rightarrow \quad F\left[\frac{2.5-\mu}{\sigma}\right] \leq 0.1 \quad \Rightarrow \quad \frac{2.5-\mu}{\sigma} \leq -1.28,$$

which gives $k \geq 150$.
Note. Some of you may have noticed that the requirements for "good approximation" specified in lecture slides are not satisfied. However, if we calculate using $p = 0.75$ and $n = 5$ for binomial distribution,

$$P[X \geq 3] = 1 - \texttt{CDF}[\texttt{BinomialDistribution}[5, 0.75], 2] = 0.896484,$$

which is quite close to 90%. This posterior validation shows the approximation is reasonable.

Assignment 3.10

Let $X = (X_1, X_2)$ be a random vector. Then we define the expectation vector and the variance-covariance matrix as follows.

$$\mathrm{E}[X] := \begin{pmatrix} \mathrm{E}[X_1] \\ \mathrm{E}[X_2] \end{pmatrix}, \qquad \mathrm{Var}\ X := \begin{pmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}\ X_2 \end{pmatrix}.$$

Let A be a constant 2×2 matrix and $Y = (Y_1, Y_2) = AX$.

1. Show that $\mathrm{E}[AX] = A\mathrm{E}[X]$.
2. Show that $\mathrm{Var}(AX) = A(\mathrm{Var}\ X)A^T$.
3. Suppose X_1 and X_2 follow independent normal distributions with mean μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the joint density is given by

   $$f_X(x) = f_X(x_1, x_2) = \frac{1}{2\pi\sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\langle x - \mu_X, \Sigma_X^{-1}(x - \mu_X)\rangle}$$

   where $\mu_X = (\mu_1, \mu_2)$ and $\Sigma_X = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$ is the 2×2 matrix with the variances on the diagonal and all other entries vanishing.
4. Suppose that X_1 and X_2 follow independent normal distributions with means $\mu_1, \mu_2 \in \mathbb{R}$ and variances $\sigma_1^2, \sigma_2^2 > 0$, respectively. Let $Y = AX$ where A is an invertible $n \times n$ matrix. Show that

   $$f_Y(y) = \frac{1}{2\pi\sqrt{|\det \Sigma_Y|}} e^{-\frac{1}{2}\langle y - \mu_Y, \Sigma_Y^{-1}(y - \mu_Y)\rangle} \tag{1}$$

   where $\mu_Y = \mathrm{E}[Y]$, $\Sigma_Y = \mathrm{Var}\ Y$ and $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product in $\mathbb{R}^2$.
5. Show that Eq. (1) can be written as

   $$f_Y(y_1, y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1 - \rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1 - \mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1 - \mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2 - \mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2 - \mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]}$$

   where μ_{Y_i} is the mean and $\sigma_{Y_i}^2$ the variance of $Y_i, i = 1, 2$, and ρ is the correlation of Y_1 and Y_2.

__**Solution**__.

1. Following properties for expectation, we have

$$\mathrm{E}[AX] = \mathrm{E}\left[\begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix}\right] = \begin{pmatrix} a_{11}\mathrm{E}[X_1] + a_{12}\mathrm{E}[X_2] \\ a_{21}\mathrm{E}[X_1] + a_{22}\mathrm{E}[X_2] \end{pmatrix} = A\mathrm{E}[X].$$

2. By definition, we have

$$\begin{aligned}\mathrm{Var}(AX) &= \mathrm{Var}\begin{pmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Var}(a_{11}X_1 + a_{12}) & \mathrm{Cov}(a_{11}X_1 + a_{12}X_2, a_{21}X_1 + a_{22}X_2) \\ \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) & \mathrm{Var}(a_{21}X_1 + a_{22}X_2) \end{pmatrix}.\end{aligned}$$

From the properties of variance and covariance, we have

$$\begin{aligned}\mathrm{Var}(a_{11}X_1 + a_{12}X_2) &= a_{11}^2\mathrm{Var}X_1 + a_{12}^2\mathrm{Var}X_2 + 2a_{11}a_{12}\mathrm{Cov}(X_1, X_2), \\ \mathrm{Var}(a_{21}X_1 + a_{22}X_2) &= a_{21}^2\mathrm{Var}X_1 + a_{22}^2\mathrm{Var}X_2 + 2a_{21}a_{22}\mathrm{Cov}(X_1, X_2),\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(a_{11}X_1 + a_{12}X_2, a_{21}X_1 + a_{22}X_2) &= \mathrm{Cov}(a_{21}X_1 + a_{22}X_2, a_{11}X_1 + a_{12}X_2) \\ &= a_{11}a_{21}\mathrm{Var}(X_1) + a_{12}a_{22}\mathrm{Var}(X_2) + \\ &\qquad + (a_{11}a_{22} + a_{12}a_{21})\mathrm{Cov}(X_1, X_2).\end{aligned}$$

Therefore,

$$\begin{aligned}A(\mathrm{Var}\ X)A^T &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11}\mathrm{Var}X_1 + a_{12}\mathrm{Cov}(X_1, X_2) & a_{21}\mathrm{Var}X_1 + a_{22}\mathrm{Cov}(X_1, X_2) \\ a_{11}\mathrm{Cov}(X_2, X_1) + a_{12}\mathrm{Var}X_2 & a_{21}\mathrm{Cov}(X_1, X_2) + a_{22}\mathrm{Var}X_2 \end{pmatrix} \\ &= \mathrm{Var}(AX).\end{aligned}$$

3. We have

$$\sqrt{\det \Sigma_X} = \sigma_1\sigma_2, \qquad \Sigma_X^{-1} = \begin{pmatrix} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{pmatrix}.$$

and

$$\langle x - \mu_X, \Sigma_X^{-1}(x - \mu_X)\rangle = \frac{(x_1 - \mu_1)^2}{\sigma_1^2} + \frac{(x_2 - \mu_2)^2}{\sigma_2^2}.$$

Since X_1 and X_2 are independent,

$$\begin{aligned} f_X(x) &= f_X(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} \\ &= \frac{1}{2\pi\sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\langle x-\mu_X, \Sigma_X^{-1}(x-\mu_X)\rangle}. \end{aligned}$$

4. Since $Y = AX$, from (1) and (2) we know that

$$\begin{aligned} \mu_Y = \mathrm{E}[AX] = A\mu_X, \qquad \Sigma_Y = A\Sigma_X A^T \quad &\Rightarrow \quad \Sigma_Y^{-1} = (A^T)^{-1}\Sigma_X^{-1}A^{-1}, \\ &\Rightarrow \quad \det \Sigma_Y = (\det A)^2 \det \Sigma_X. \end{aligned}$$

Using transformation of variables,

$$\begin{aligned} f_Y(y) &= f_Y \circ (A^{-1}y) \cdot |\det A^{-1}| \\ &= \frac{1}{2\pi\sqrt{\det \Sigma_X}} e^{-\frac{1}{2}\langle A^{-1}y - A^{-1}\mu_Y, \Sigma_X^{-1}(A^{-1}y - A^{-1}\mu_Y)\rangle} \cdot \frac{1}{|\det A|} \\ &= \frac{1}{2\pi\sqrt{\det \Sigma_X \cdot (\det A)^2}} e^{-\frac{1}{2}\langle y-\mu_Y, \Sigma_Y^{-1}(y-\mu_Y)\rangle} \\ &= \frac{1}{2\pi\sqrt{|\det \Sigma_Y|}} e^{-\frac{1}{2}\langle y-\mu_Y, \Sigma_Y^{-1}(y-\mu_Y)\rangle}. \end{aligned}$$

5. Rewriting $\sqrt{|\det \Sigma_Y|}$ as

$$\begin{aligned} \sqrt{|\det \Sigma_Y|} &= \sqrt{\sigma_{Y_1}^2\sigma_{Y_2}^2 - \mathrm{Cov}^2(Y_1, Y_2)} \\ &= \sigma_{Y_1}\sigma_{Y_2}\sqrt{1 - \left(\frac{\mathrm{Cov}(Y_1, Y_2)}{\sigma_{Y_1}\sigma_{Y_2}}\right)^2} \\ &= \sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}, \end{aligned}$$

and

$$\Sigma_Y = \begin{pmatrix} \sigma_{Y_1}^2 & \rho\sigma_{Y_1}\sigma_{Y_2} \\ \rho\sigma_{Y_1}\sigma_{Y_2} & \sigma_{Y_2}^2 \end{pmatrix} \quad \Rightarrow \quad \Sigma_Y^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_{Y_1}^2} & -\frac{\rho}{\sigma_{Y_1}\sigma_{Y_2}} \\ -\frac{\rho}{\sigma_{Y_1}\sigma_{Y_2}} & \frac{1}{\sigma_{Y_2}^2} \end{pmatrix},$$

we have

$$\begin{aligned} \langle y-\mu_Y, \Sigma_Y(y-\mu_Y)\rangle &= \frac{1}{1-\rho^2}\langle \begin{pmatrix} y_1 - \mu_{Y_1} \\ y_2 - \mu_{Y_2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sigma_{Y_1}^2} & -\frac{\rho}{\sigma_{Y_1}\sigma_{Y_2}} \\ -\frac{\rho}{\sigma_{Y_1}\sigma_{Y_2}} & \frac{1}{\sigma_{Y_2}^2} \end{pmatrix}\rangle \\ &= \frac{1}{1-\rho^2}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]. \end{aligned}$$

Therefore, Eq. (1) can be written as

$$f_Y(y_1, y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right) + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]}.$$

Assignment 3.11

A system consists of two independent components connected in series. The life span (in hours) of the component follows a Weibull distribution with $\alpha = 0.006$ and $\beta = 0.5$; the second has a lifespan in hours follows the exponential distribution with $\beta = 1/25000$.

1. Find the reliability of the system at 2500 hours.
2. Find the probability that the system will fail before 2000 hours.
3. If the two components are connected in parallel, what is the system reliability at 2500 hours?

Solution.

1. The reliability function for the two components are given by

$$R_1(t) = e^{-\alpha_1 t^{\beta_1}}, \qquad R_2(t) = 1 - \int_0^t f_{T_2}(x)\mathrm{d}x = 1 + e^{-\beta_2 x}\Big|_0^t = e^{-\beta_2 t}.$$

Therefore, the reliability of the system at $t = 2500$ is given by

$$R(t) = R_1(t) \cdot R_2(t) \quad \Rightarrow \quad R(2500) = 0.7408 \times 0.9048 = 0.6703.$$

2. The probability that the system fail before 2000h is given by

$$P[T < 2000] = 1 - R(2000) = 1 - 0.7059 = 0.2941.$$

3. The reliability at $t = 2500$ for the parallel system is

$$R(t) = 1 - (1 - R_1(t))(1 - R_2(t)) \quad \Rightarrow \quad R(2500) = 0.9753.$$

Assignment 4.2

Let $X_1, \ldots, X_n$ be a random sample of size n from a random variable with variance σ^2. We have seen that the sample variance

$$S_{n-1}^2 := \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$$

is an unbiased estimator for σ^2. It can be shown that

$$\mathrm{Var}(S_{n-1}^2) = \mathrm{MSE}(S_{n-1}^2) = \frac{1}{n}\left(\mathrm{E}[(X-\overline{X})^4] - \frac{n-3}{n-1}\sigma^4\right) = \frac{1}{n}\left(\gamma_2 + \frac{2n}{n-1}\right)\sigma^4 \tag{2}$$

where $\gamma_2 := \mathrm{E}[(X-\mu)^4]/\sigma^4 - 3$ is called the *excess kurtosis* of a distribution.

1. Show that if X follows a normal distribution with mean μ and variance σ^2,

$$\mathrm{MSE}(S_{n-1}^2) = \frac{2}{n-1}\sigma^4.$$

2. For $a > 0$ set

$$S_a^2 := \frac{n-1}{a} S_{n-1}^2.$$

   Find $\mathrm{MSE}(S_a^2)$ and show that the mean square error is minimized for

$$a = n + 1 + \frac{n-1}{n}\gamma_2.$$

   In the case of a normal distribution with mean μ and variance σ^2, show that this reduces to $a = n + 1$.

<u>Solution</u>.

1. We know that,

$$\mathrm{MSE}[S_{n-1}^2] = \mathrm{Var}[S_{n-1}^2] + \mathrm{bias}^2 = \mathrm{Var}[S_{n-1}^2].$$

   Since X follows a normal distribution,

$$\chi_{n-1}^2 = \frac{(n-1)S_{n-1}}{\sigma^2}$$

   follows Chi-squared distribution with $n-1$ degrees of freedom. The variance is

$$\mathrm{Var}[\chi_{n-1}^2] = 2(n-1) \quad \Rightarrow \quad \mathrm{Var}[S_{n-1}^2] = \frac{\sigma^4}{(n-1)^2} \cdot 2(n-1) = \frac{2\sigma^4}{n-1}.$$

2. By definition, the MSE for S_a^2 is given by

$$\begin{aligned}\mathrm{MSE}[S_a^2] &= \mathrm{E}[S_a^4 - 2\sigma^2 S_a^2 + \sigma^4] \\ &= \mathrm{E}[S_a^4] - 2\sigma^2 \cdot \frac{n-1}{a}\sigma^2 + \sigma^4 \\ &= \mathrm{E}[S_a^4] + \left(1 - \frac{2(n-1)}{a}\right)\sigma^4.\end{aligned}$$

Using Eq. (2) and property for variance, we have

$$\begin{aligned}
\mathrm{E}[S_a^4] &= \mathrm{Var}[S_a^2] + \mathrm{E}[S_a^2]^2 \\
&= \frac{(n-1)^2}{a^2}\mathrm{Var}[S_{n-1}^2] + \frac{(n-1)^2}{a^2}\sigma^4, \\
\mathrm{MSE}[S_a^4] &= \frac{(n-1)^2}{a^2} \cdot \frac{1}{n}\left(\gamma_2 + \frac{2n}{n-1}\right)\sigma^4 + \frac{(n-1)^2}{a^2}\sigma^4 + \left(1 - \frac{2(n-1)}{a}\right)\sigma^4 \\
&= \left[x^2 \cdot \frac{1}{n}\left(\gamma_2 + \frac{2n}{n-1}\right) + x^2 - 2x + 1\right]\sigma^4 \qquad \left(\text{let } x = \frac{n-1}{a}\right) \\
&= \left[\left(\frac{\gamma_2}{n} + \frac{2}{n-1} + 1\right)x^2 - 2x + 1\right],
\end{aligned}$$

which is maximized when

$$x = \frac{1}{\dfrac{\gamma_2}{n} + \dfrac{2}{n-1} + 1} = \frac{n-1}{a} \quad \Rightarrow \quad a = n + 1 + \frac{n-1}{n}\gamma_2.$$

In case of normal distribution with mean μ and variance σ^2, since

$$\gamma_2 = \mathrm{E}\left[\frac{(X-\mu)^4}{\sigma^4}\right] - 3 = \mathrm{E}[Z^4] - 3,$$

where Z follows a standard normal distribution and $\mathrm{E}[Z^4]$ is the 4th moment of it, given by

$$\mathrm{E}[Z^4] = \left.\frac{\mathrm{d}^4 m_Z(t)}{\mathrm{d}t^4}\right|_{t=0} = \left.(t^4 + 6t^2 + 3)e^{\frac{1}{2}t^2}\right|_{t=0} = 3,$$

and thus

$$a = n + 1.$$