VE401 Probabilistic Methods in Eng.
Solution Manual for RC 6

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Assignment 5.2

Let $X_1, \ldots, X_n$ be i.i.d. exponential random variables with parameter β. Recall that $Y = X_1 + X_2 + \cdots + X_n$ follows a Gamma distribution with parameters $\alpha = n$ and β. Transform this expression further to yield a chi-squared random variable.

Let X be an exponential random variable with parameter β. Devise a test statistic for testing $H_0 : \beta = \beta_0$ and $H_0 : \beta \le \beta_0$ in a Fisher test.

Solution. Since Y follows a Gamma distribution with parameters $\alpha = n$ and β, we have the density function

$$f_Y(y) = \frac{\beta^n}{\Gamma(n)} y^{n-1} e^{-\beta y}, \qquad y > 0$$

and $f_Y(y) = 0$ when $y \le 0$. Let $u = \varphi(y) = 2\beta y$, then

$$y = \varphi^{-1}(u) = \frac{u}{2\beta} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}u}\varphi^{-1}(u) = \frac{1}{2\beta}.$$

Then using transformation of variable, we have

$$\begin{aligned} f_U(u) &= f_Y \circ \varphi^{-1}(u) \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u}\varphi^{-1}(u) \right| \\ &= \frac{\beta^n}{\Gamma(n)} \frac{u^{n-1}}{(2\beta)^{n-1}} e^{-u/2} \cdot \frac{1}{2\beta} \\ &= \frac{1}{2^n \Gamma(n)} u^{n-1} e^{-u/2} \qquad u > 0, \end{aligned}$$

and $f_U(u) = 0$ when $u \le 0$, which is a chi-squared distribution with $2n$ degrees of freedom. Therefore, we have the distribution

$$Y \sim \frac{1}{2\beta}\chi^2_{2n}.$$

Given samples $X_1, \ldots, X_n$ with size n, we can use test statistic

$$\chi^2_{2n} = 2\beta_0 \sum_{i=1}^{n} X_i = 2n\beta_0 \overline{X}.$$

With larger true parameter β, we would expect a smaller test statistic

- For one-tailed test $\beta \le \beta_0$, "more extreme data" means smaller $\overline{X}$. Therefore, the p-value is given by

  $$P\text{−value} = F_{\chi^2_{2n}}\left(2n\beta_0\overline{X}\right),$$

  where $F_{\chi^2_{2n}}$ is the cumulative distribution function of chi-squared distribution with $2n$ degrees of freedom.

- For two-tailed test $\beta = \beta_0$, the p-value is given by

$$P\text{−value} = 2\min\left(F_{\chi^2_{2n}}\left(2n\beta_0\overline{X}\right), 1 - F_{\chi^2_{2n}}\left(2n\beta_0\overline{X}\right)\right).$$

If we want to have a critical region for the tests, with the test statistic χ^2_{2n} defined above, we reject H_0 at significance level α

- $H_0 : \beta = \beta_0$ if $\chi^2_{2n} < \chi^2_{2n,1-\alpha/2}$ or $\chi^2_{2n} > \chi^2_{2n,\alpha/2}$,
- $H_0 : \beta \le \beta_0$ if $\chi^2_{2n} < \chi^2_{2n,1-\alpha}$,
- $H_0 : \beta \ge \beta_0$ if $\chi^2_{2n} > \chi^2_{2n,\alpha}$.