



University  
of Glasgow

# HIGH FREQUENCY COMMUNICATION SYSTEMS

## Lecture 2

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- Plane waves and the wave equation
- Dielectric Properties and Materials
- Nanoscale Electromagnetics

- Maxwell's Equations are first-order partial differential equations
  - They are coupled equations (i.e. the unknown  $\vec{\mathbf{E}}$  and  $\vec{\mathbf{H}}$ ) appear in each equation
- To find the solution of the equations we treat it as a boundary value problem
- We also uncouple the equations by raising the order (*here two*).
- The result is the *wave equation*.

Recall,

$$\nabla \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} - \vec{\mathbf{M}} \quad (\text{Faraday's Law})$$

$$\nabla \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t} + \vec{\mathbf{J}} \quad (\text{Ampere's Law})$$

where,

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_i + \sigma \vec{\mathbf{E}}$$

We take the curl of the above two equations and use the vector identity,  $\nabla \times \nabla \times \vec{\mathbf{A}} \equiv \nabla(\nabla \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}}$ ,

$$\nabla(\nabla \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\nabla \times \vec{\mathbf{M}} - \mu \frac{\partial}{\partial t} (\nabla \times \vec{\mathbf{H}})$$

$$\nabla(\rho_s/\varepsilon) - \nabla^2 \vec{\mathbf{E}} = -\nabla \times \vec{\mathbf{M}} - \mu \frac{\partial \vec{\mathbf{J}}_i}{\partial t} - \mu \sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

After rearranging we get the *uncoupled* second-order differential equation for  $\vec{\mathbf{E}}$ ,

$$\nabla^2 \vec{\mathbf{E}} = \nabla \times \vec{\mathbf{M}}_i + \mu \frac{\partial \vec{\mathbf{J}}_i}{\partial t} + \mu \sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} + \frac{1}{\varepsilon} \nabla \rho_s$$

### Homework

- Derive the vector wave equation for the magnetic field  $\vec{\mathbf{H}}$
- Due on MS Teams on March 23.
- You can either scan your work or better typeset in  $\text{\LaTeX}$ .

- Simplest electromagnetic wave
- Generally propagate in a fixed direction (e.g.  $z$ )
- The EM fields are only functions of time and space coordinate  $z$ .
- No variation in transverse coordinates ( $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = 0$ )
  - $E_z = H_z = 0$

$$\vec{\mathbf{E}}(x, y, z, t) = \vec{\mathbf{E}}(z, t)$$

For a uniform plane wave, the source-free Maxwell's equations are:

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \implies \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t} \implies \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} = \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \implies \frac{\partial E_z}{\partial z} = 0$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \implies \frac{\partial H_z}{\partial z} = 0$$

- Starting with a uniform plane wave in a source-free region.
- Considering one-dimensional case
- Since  $E_z, H_z = 0$ , we start with and use the identity ( $\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}} \times \vec{\mathbf{A}}) \equiv 0$ ):

$$\hat{\mathbf{z}} \cdot \left( \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} \right) = \epsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} = 0 \implies \frac{\partial E_z}{\partial t} = 0$$

The solutions (transverse fields) must be of the form:

$$\begin{aligned}\vec{\mathbf{E}}(z, t) &= \hat{\mathbf{x}}E_x(z, t) + \hat{\mathbf{y}}E_y(z, t) \\ \vec{\mathbf{H}}(z, t) &= \hat{\mathbf{x}}H_x(z, t) + \hat{\mathbf{y}}H_y(z, t)\end{aligned}$$

The electric and magnetic fields only exist in the  $x - y$  plane which is perpendicular to the direction of propagation.



- We can also simplify 1D Maxwell's equations

$$\begin{aligned}\hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z} &= -\frac{1}{c} \eta \frac{\partial \vec{\mathbf{H}}}{\partial t} \\ \eta \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} &= -\frac{1}{c} \frac{\partial \vec{\mathbf{E}}}{\partial t}\end{aligned}$$

where,

$$c = \frac{1}{\sqrt{\mu\epsilon}}, \text{ and } \eta = \sqrt{\frac{\mu}{\epsilon}}$$

Using the BAC-CAB ( $\vec{\mathbf{A}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{B}}(\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}) - (\vec{\mathbf{B}} \cdot \vec{\mathbf{A}})\vec{\mathbf{C}}$ ) rule of vector algebra:

$$\left( \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z} \right) \times \hat{\mathbf{z}} = \frac{\partial \vec{\mathbf{E}}}{\partial z} - \hat{\mathbf{z}} \left( \hat{\mathbf{z}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial z} \right) = \frac{\partial \vec{\mathbf{E}}}{\partial z}$$

We can now write the Maxwell's equations as:

$$\begin{aligned}\frac{\partial \vec{\mathbf{E}}}{\partial z} &= -\frac{1}{c} \frac{\partial}{\partial t} \left( \eta \vec{\mathbf{H}} \times \hat{\mathbf{z}} \right) \\ \frac{\partial}{\partial z} \left( \eta \vec{\mathbf{H}} \times \hat{\mathbf{z}} \right) &= -\frac{1}{c} \frac{\partial \vec{\mathbf{E}}}{\partial t}\end{aligned}$$

We differentiate the first equation w.r.t  $z$  and use the second:

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial z^2} = -\frac{1}{c} \frac{\partial^2}{\partial t \partial z} \left( \eta \vec{\mathbf{H}} \times \hat{\mathbf{z}} \right) = \frac{1}{c^2} \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

which is the 1D wave equation. We can also write in a convenient form as:

$$\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{\mathbf{E}}(z, t) = 0$$

- Time-harmonic representation  $\exp(j\omega t)$  is convenient in finding the solutions
- We replace the derivatives  $\frac{\partial}{\partial t}$  and  $\frac{\partial^2}{\partial t^2}$  by  $j\omega$  and  $-\omega^2$  respectively
- We also call the result as the **Helmholtz equation**.
- For source-free ( $\vec{\mathbf{J}} = \vec{\mathbf{M}} = 0$ ) case, we get

$$\nabla^2 \vec{\mathbf{E}} + \omega^2 \mu \epsilon \vec{\mathbf{E}} = 0$$

$$\nabla^2 \vec{\mathbf{E}} + \beta^2 \vec{\mathbf{E}} = 0$$

- A second order differential equation leads to 2 solutions
  - We can split the fields into *forward* and *backward* components.
- We use the *Separation of variables* method to obtain the solutions of vector wave equation
  - By solving the scalar equations for each components

$$\vec{\mathbf{E}} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y + \hat{\mathbf{z}}E_z$$

As an example, for the x-component, we get:

$$\nabla^2 E_x(x, y, z) + \beta^2 E_x(x, y, z) = 0$$

The solution is of the form:

$$E_x(x, y, z) = f(x)g(y)h(z)$$

- There are different forms of solutions we can use
  - Depends on the nature of the problem
- For free-space problems, we use the travelling wave form

$$h(z) = A_1 \exp(-j\beta_z z) + B_1 \exp(+j\beta_z z)$$

For confined problems (such as a waveguide), we use the standing wave form:

$$g(x) = A_2 \sin(\beta_y y) + B_2 \cos(\beta_y y)$$

- Uniform travelling wave in the  $+z$  direction
- Equiphasic plane (increase in  $t$  also increase  $z$ )

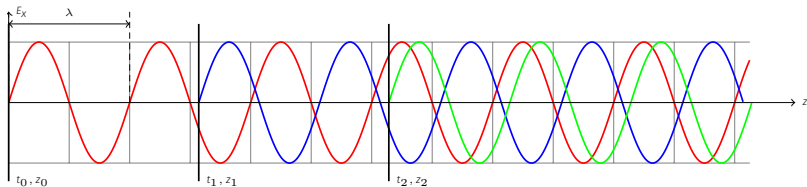


Figure 1: X-polarized Plane Wave propagation along  $z$  direction

For the above, the plane wave can be described as:

$$E_x(z, t) = \cos(\omega t - \beta z)$$

## MATERIAL PROPERTIES

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- Materials play a huge role in electromagnetic radiation and guiding
- The electrons inside the atom of a material behave differently when an external electric field is applied
  - The electric field distorts the electron distribution
  - An electric dipole moment is created
- We tend to observe it macroscopically (not at the atom level but over the volume of the material)
- We need to describe the behaviour of  $\varepsilon$  with frequency (using Classical Harmonic model)

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{e}{m} E$$

where  $\gamma$  is a measure of rate of collisions per unit time,  $\omega_0$  refers to the resonant frequency,  $e$  and  $m$  are the electron charge and mass respectively.



- Using the phaser form of the Harmonic model for a plane wave,  
 $E(t) = E_0 \exp(j\omega t)$

$$\varepsilon(\omega) = \varepsilon_0 + \frac{\varepsilon_0 + \omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} \quad (\text{Lorentz Model})$$

where  $\varepsilon_0$  is the free-space permittivity,  $\omega_p$  is the plasma frequency given by:

$$\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}$$

$N$  being the charge density.

- The real part of  $\varepsilon$  refers to the refractive properties
- The imaginary part determines the absorption or loss.

- Formation of electric dipoles in the presence of external electric fields.
- There are magnetic materials as well but we are not interested in them in this course.
  - We assume  $\mu_r = 1$  for all materials.

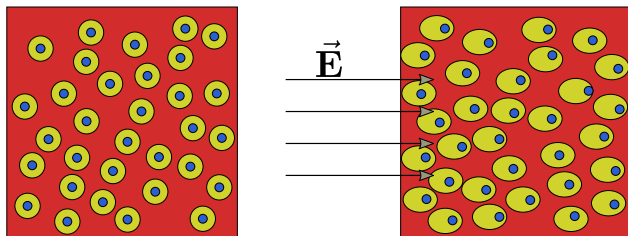


Figure 2: Effect of electric field on dipole formation.

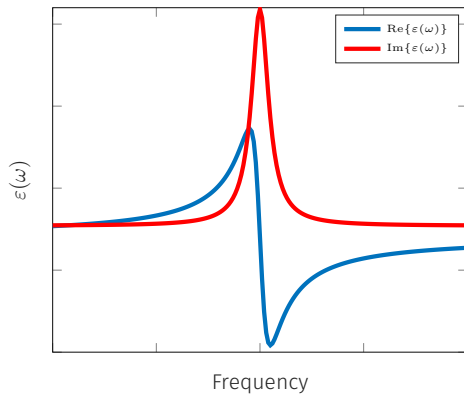


Figure 3: The dielectric function using the Lorentz Model

- Main difference from dielectrics is that the motion of electric charges and the generation of current flow.
- Conductors have *loosely held* electrons in the valence band of atoms [free electrons]
- Conductors have very high values of electric conductivity ( $\sigma \rightarrow \infty$ ).
- For perfect electric conductors, we use  $\sigma = \infty$ .

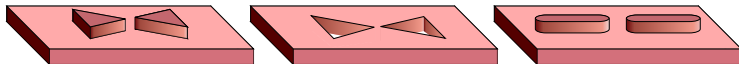
$$\varepsilon(\omega) = \varepsilon_0 + \frac{\sigma(\omega)}{j\omega} \quad (\text{Drude Model})$$

- Plasma like solid, liquid and gas is the fourth form of matter
- We consider the resonant frequency  $\omega_0 = 0$ .
- Plasma effectively acts as a switch
  - Before plasma frequency, wave is completely attenuated.
  - After  $\omega_p$ , there is zero attenuation

$$\varepsilon(\omega) = \varepsilon_0 + \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$



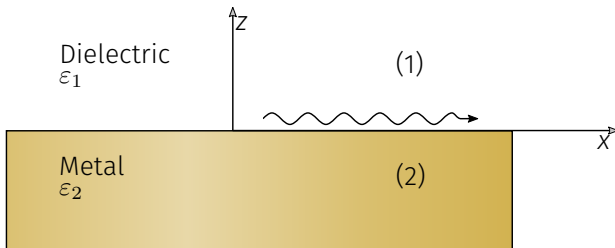
- Electronic device sizes are fast approaching the nanoscale
  - Modern transistors are typically 5 nm in size
- Maxwell's equations have remained valid at macro scale
  - However, discrepancies have lately surfaced at the nanoscale level between the theory and experiment
- Highlight of nanoscale electromagnetics is the *complex-valued* nature of the relative permittivity
- Interestingly, EM surface waves exist at metal/dielectric interfaces



- EM fields can be split into *transverse-magnetic* (TM) and *transverse-electric* (TE) components
- Observing a planar dielectric-metal interface, TM-mode means H-field only has a transverse ( $H_y$ ) component
  - The E-field has  $E_x$  and  $E_z$  components



- At optical and mid infrared frequencies ( $> 10$  THz), some materials such as gold exhibit negative dielectric constant ( $\text{Re}(\epsilon_r) < 0$ )
- At a metal-dielectric interface such as the one below:



- The TM mode fields in region 1 are expressed as:

$$\begin{aligned}\vec{\mathbf{E}}_1 &= (\hat{\mathbf{x}}E_{x1} + \hat{\mathbf{z}}E_{z1}) \exp(-j(k_x x + k_{z1} z)), \\ \vec{\mathbf{H}}_1 &= \hat{\mathbf{y}}H_{y1} \exp(-j(k_x x + k_{z1} z))\end{aligned}$$

and likewise for region 2. Using the Ampere's Law ( $\nabla \times \vec{\mathbf{H}} = -j\omega\vec{\mathbf{E}}$ ), we get the boundary conditions:

$$\begin{aligned}k_{z1}H_{y1} &= \omega\epsilon_1 E_{x1} \\ k_{z2}H_{y2} &= -\omega\epsilon_2 E_{x2}\end{aligned}$$

- Ensuring the continuity of tangential fields,  $E_{x1} = E_{x2}$  and  $H_{y1} = H_{y2}$ , we get:

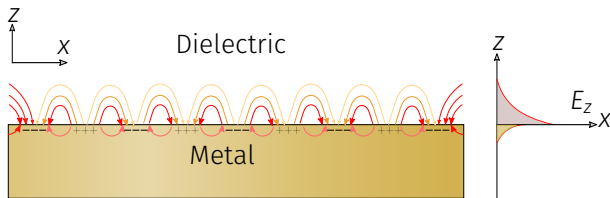
$$\frac{k_{z1}}{\varepsilon_1} + \frac{k_{z2}}{\varepsilon_2} = 0.$$

- Using the Helmholtz equation,  $(\nabla^2 \vec{\mathbf{E}} + k_i^2 \vec{\mathbf{E}} = 0)$ , where  $i = 1, 2$ , and assuming the permeabilities of all regions are that of air, we obtain,

$$k_x = k_0 \sqrt{\frac{\varepsilon_{r1} \varepsilon_{r2}}{\varepsilon_{r1} + \varepsilon_{r2}}}$$

- The dispersion relation has a solution when

$$\epsilon_1 > 0, \quad \epsilon'_2 < 0, \quad \text{and} \quad |\epsilon'_2| > \epsilon_1,$$



Surface plasmon propagation along a dielectric-metal boundary and exponential decay perpendicular to the boundary.