ELECTROMAGNETIC FIELD AND MICROWAVE TECHNOLOGY

Lecture 2: The Wave Equation and Material Properties

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- Plane waves and the wave equation
- Dielectric Properties and Materials

- Real-World Connection
  - How does a 5G signal propagate from a base station to your phone?
  - Why do signals cut out in tunnels (High-Speed Rail context)?
- The **Wave Equation** — the engine of modern communications.
  - From macro-scale communications to 5 nm chip manufacturing (nanoscale).
- **The Physics Challenge**
  - Maxwell's Equations are "coupled" (tangled together).
  - To understand wave propagation, we must "uncouple" them.

Figure 1: Mobile Connectivty in High-Speed Rail in China.

- Maxwell's Equations are first-order partial differential equations
  - They are coupled equations (i.e. the unknown $\vec{\mathbf{E}}$ and $\vec{\mathbf{H}}$) appear in each equation
- To find the solution of the equations we treat it as a boundary value problem
- We also uncouple the equations by raising the order (*here two*).
- The result is the *wave equation*.

Recall,

$$\boldsymbol{\nabla} \times \vec{\mathbf{E}} = -\mu \frac{\partial \vec{\mathbf{H}}}{\partial t} - \vec{\mathbf{M}} \qquad \text{(Faraday's Law)}$$

$$\boldsymbol{\nabla} \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t} + \vec{\mathbf{J}} \qquad \text{(Ampere's Law)}$$

where,

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_i + \sigma \vec{\mathbf{E}}$$

We take the curl of the above two equations and use the vector identity, $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{\mathbf{A}} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{\mathbf{A}}) - \nabla^2 \vec{\mathbf{A}}$,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{\mathbf{E}}) - \nabla^2 \vec{\mathbf{E}} = -\boldsymbol{\nabla} \times \vec{\mathbf{M}} - \mu \frac{\partial}{\partial t} \left(\boldsymbol{\nabla} \times \vec{\mathbf{H}} \right)$$

$$\boldsymbol{\nabla}(\rho_v / \varepsilon) - \nabla^2 \vec{\mathbf{E}} = -\boldsymbol{\nabla} \times \vec{\mathbf{M}} - \mu \frac{\partial \vec{\mathbf{J}}_i}{\partial t} - \mu\sigma \frac{\partial \vec{\mathbf{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

After rearranging we get the *uncoupled* second-order differential equation for $\vec{\mathbf{E}}$,

$$\nabla^2\vec{\mathbf{E}} = \boldsymbol{\nabla} \times \vec{\mathbf{M}}_i + \mu\frac{\partial \vec{\mathbf{J}}_i}{\partial t} + \mu\sigma\frac{\partial \vec{\mathbf{E}}}{\partial t} + \mu\varepsilon\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} + \frac{1}{\varepsilon}\boldsymbol{\nabla}\rho_v$$

- Simplest electromagnetic wave
- Generally propagate in a fixed direction (e.g. z)
- The EM fields are only functions of time and space coordinate z.
- No variation in transverse coordinates ($\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = 0$)
    - $E_z = H_z = 0$

$$\vec{\mathbf{E}}(x, y, z, t) = \vec{\mathbf{E}}(z, t)$$

For a uniform plane wave, the source-free Maxwell's equations are:

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \implies \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

$$\nabla \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t} \implies \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} = \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

$$\nabla \cdot \vec{\mathbf{E}} = 0 \implies \frac{\partial E_z}{\partial z} = 0$$

$$\nabla \cdot \vec{\mathbf{H}} = 0 \implies \frac{\partial H_z}{\partial z} = 0$$

- Starting with a uniform plane wave in a source-free region.
- Considering one-dimensional case
- Since $E_z, H_z = 0$, we start with and use the identity $(\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}} \times \vec{\mathbf{A}}) \equiv 0)$:

$$\hat{\mathbf{z}} \cdot \left(\hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} \right) = \varepsilon \frac{\partial \vec{\mathbf{E}}}{\partial t} = 0 \implies \frac{\partial E_z}{\partial t} = 0$$

The solutions (transverse fields) must be of the form:

$$\vec{\mathbf{E}}(z,t) = \hat{\mathbf{x}} E_x(z,t) + \hat{\mathbf{y}} E_y(z,t)$$
$$\vec{\mathbf{H}}(z,t) = \hat{\mathbf{x}} H_x(z,t) + \hat{\mathbf{y}} H_y(z,t)$$

The electric and magnetic fields only exist in the $x - y$ plane which is perpendicular to the direction of propagation.

- We can also simplify 1D Maxwell's equations

$$\widehat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z} = -\frac{1}{c}\eta\frac{\partial \vec{\mathbf{H}}}{\partial t}$$
$$\eta\widehat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{H}}}{\partial z} = -\frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t}$$

where,

$$c = \frac{1}{\sqrt{\mu\varepsilon}}, and \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Using the BAC-CAB ($\vec{\mathbf{A}} \times (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{B}}(\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}) - (\vec{\mathbf{B}} \cdot \vec{\mathbf{A}})\vec{\mathbf{C}}$) rule of vector algebra:

$$\left(\widehat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{E}}}{\partial z}\right) \times \widehat{\mathbf{z}} = \frac{\partial \vec{\mathbf{E}}}{\partial z} - \widehat{\mathbf{z}}\left(\widehat{\mathbf{z}} \cdot \frac{\partial \vec{\mathbf{E}}}{\partial z}\right) = \frac{\partial \vec{\mathbf{E}}}{\partial z}$$

We can now write the Maxwell's equations as:

$$\frac{\partial \vec{\mathbf{E}}}{\partial z} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\eta \vec{\mathbf{H}} \times \hat{\mathbf{z}}\right)$$

$$\frac{\partial}{\partial z}\left(\eta \vec{\mathbf{H}} \times \hat{\mathbf{z}}\right) = -\frac{1}{c}\frac{\partial \vec{\mathbf{E}}}{\partial t}$$

We differentiate the first equation w.r.t z and use the second:

$$\frac{\partial^2 \vec{\mathbf{E}}}{\partial z^2} = -\frac{1}{c}\frac{\partial^2}{\partial t \partial z}\left(\eta \vec{\mathbf{H}} \times \hat{\mathbf{z}}\right) = \frac{1}{c^2}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$$

which is the 1D wave equation. We can also write in a convenient form as:

$$\boxed{\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\mathbf{E}}(z,t) = 0}$$

- Time-harmonic representation $\exp(j\omega t)$ is convenient in finding the solutions
- We replace the derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial^2}{\partial t^2}$ by $j\omega$ and $-\omega^2$ respectively
- We also call the result as the Helmholtz equation.
- For source-free ($\vec{\mathbf{J}} = \vec{\mathbf{M}} = 0$) case, we get

$$\nabla^2 \vec{\mathbf{E}} + \omega^2 \mu \varepsilon \vec{\mathbf{E}} = 0$$
$$\nabla^2 \vec{\mathbf{E}} + \beta^2 \vec{\mathbf{E}} = 0$$

- A second order differential equation leads to 2 solutions
  - We can split the fields into *forward* and *backward* components.
- We use the *Separation of variables* method to obtain the solutions of vector wave equation
  - By solving the scalar equations for each components

$$\vec{\mathbf{E}} = \widehat{\mathbf{x}}E_x + \widehat{\mathbf{y}}E_y + \widehat{\mathbf{z}}E_z$$

As an example, for the x-component, we get:

$$\nabla^2 E_x\left(x, y, z\right) + \beta^2 E_x\left(x, y, z\right) = 0$$

The solution is of the form:

$$E_x\left(x, y, z\right) = f(x)g(y)h(z)$$

- There are different forms of solutions we can use
  - Depends on the nature of the problem
- For free-space problems, we use the travelling wave form

$$h(z) = A_1 \exp\left(-j\beta_z z\right) + B_1 \exp\left(+j\beta_z z\right)$$

For confined problems (such as a waveguide), we use the standing wave form:

$$g(x) = A_2 \sin(\beta_y y) + B_2 \cos(\beta_y y)$$

- Uniform travelling wave in the $+z$ direction
- Equiphase plane (increase in t also increase z)

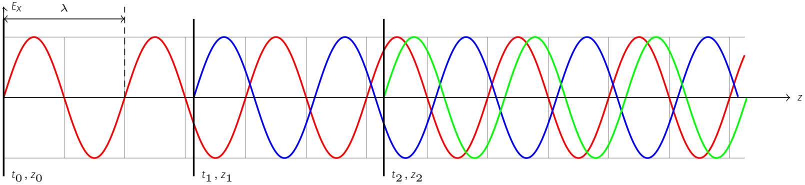

Figure 2: X-polarized Plane Wave propagation along z direction

For the above, the plane wave can be described as:

$$E_x(z,t) = \cos(\omega t - \beta z)$$

MATERIAL PROPERTIES

- Materials play a huge role in electromagnetic radiation and guiding
- The electrons inside the atom of a material behave differently when an external electric field is applied
  - The electric field distorts the electron distribution
  - An electric dipole moment is created
- We tend to observe it macroscopically (not at the atom level but over the volume of the material)
- We need to describe the behaviour of ε with frequency (using Classical Harmonic model)

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{e}{m} E$$

where γ is a measure of rate of collisions per unit time, ω_0 refers to the resonant frequency, e and m are the electron charge and mass respectively.

- Using the phasor form of the Harmonic model for a plane wave, $E(t) = E_0 \exp(j\omega t)$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}\right) \qquad \text{(Lorentz Model)}$$

where ε_0 is the free-space permittivity, ω_p is the plasma frequency given by:

$$\omega_p = \sqrt{\frac{Ne^2}{\varepsilon_0 m}}$$

N being the charge density.

- The real part of ε refers to the refractive properties
- The imaginary part determines the absorption or loss.

- Formation of electric dipoles in the presence of external electric fields.
- There are magnetic materials as well but we are not interested in them in this course.
  - We assume $\mu_r = 1$ for all materials.

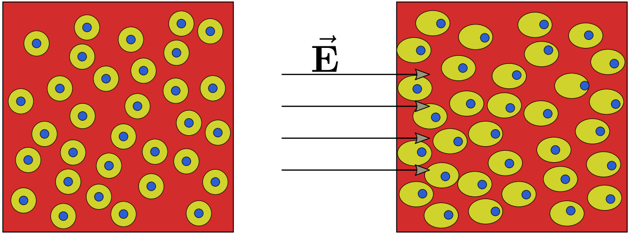

Figure 3: Effect of electric field on dipole formation.

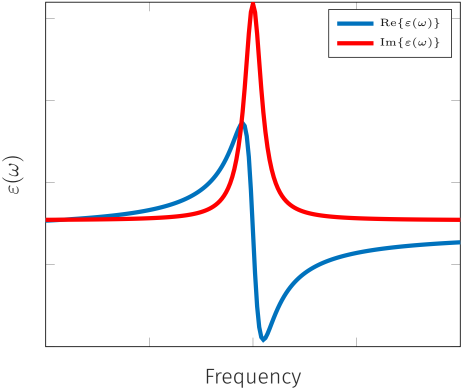

Figure 4: The dielectric function using the Lorentz Model

- Main difference from dielectrics is that the motion of electric charges and the generation of current flow.
- Conductors have *loosely held* electrons in the valence band of atoms [free electrons]
- Conductors have very high values of electric conductivity ($\sigma \to \infty$).
- For perfect electric conductors, we use $\sigma = \infty$.

$$\varepsilon(\omega) = \varepsilon_0 + \frac{\sigma(\omega)}{j\omega} \qquad \text{(Drude Model)}$$

- Plasma like solid, liquid and gas is the fourth form of matter
- We consider the resonant frequency $\omega_0 = 0$.
- Plasma effectively acts as a switch
  - Before plasma frequency, wave is completely attenuated.
  - After ω_p, there is zero attenuation

$$\varepsilon(\omega) = \varepsilon_0 + \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

CONCLUSION

- We uncoupled Maxwell's equations to find the **Wave Equation**.
- This equation predicts how EM waves propagate in space and time.
- The medium matters! $\epsilon(\omega)$ is not a constant.
- **Dielectrics:** Allow propagation (Insulators, Air).
- **Conductors:** Cause loss/attenuation (Skin effect).
- **Plasmas:** Act as high-pass filters (Spacecraft re-entry blackout).

- We have studied waves in infinite, unbounded media.
- But what happens when a wave encounters a boundary?
- **Next Time:**
  - Boundary Conditions.
  - Reflection and Transmission.
  - *Why your signal reflects off the tunnel walls instead of passing through them.*

Questions?