



University  
of Glasgow

## HIGH FREQUENCY COMMUNICATION SYSTEMS

### Lecture 6

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- Numerical Methods in Electromagnetics
- Discretisation of Maxwell's Equations
- Finite Difference Time Domain method

NUMERICAL METHODS IN ELECTROMAGNETICS

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- There are many numerical methods developed to solve Maxwell's or wave equations

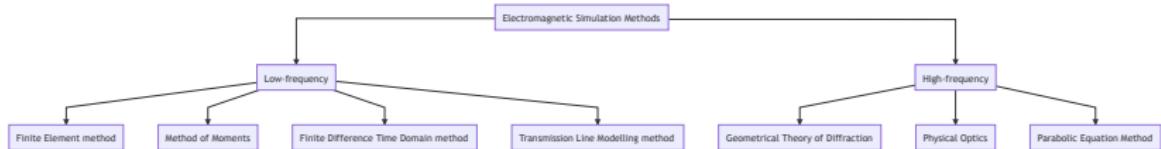


Figure 1: Various simulation methods in EM

- We solve either a differential or integral equation *numerically*
  - Only at a single frequency
- These methods are relatively more accurate than time domain methods
  - However, mathematically they are more complex
- Finite element method (FEM) is the most computational tool used in all areas of engineering including electromagnetic simulation (e.g. HFSS, CST and COMSOL)
  - It solves partial differential equations by discretisation into *finite elements*

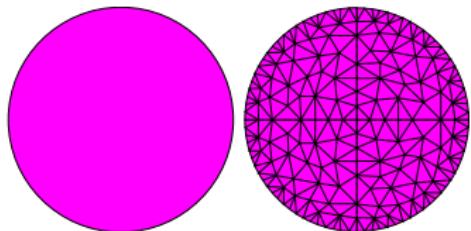


Figure 2: Typical mesh in FEM

- We observe the numerical time evolution of electromagnetic fields or currents through a region of space
- They are more intuitive as we can *visualise* wave phenomena
- We can obtain all the frequency information in one run through the Fourier transformation
- Simplicity comes at the cost of problems such as numerical stability and dispersion
- However, we will look into the finite difference time domain (FDTD) method deeply

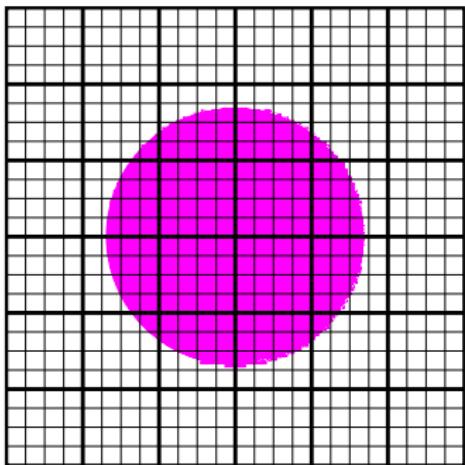


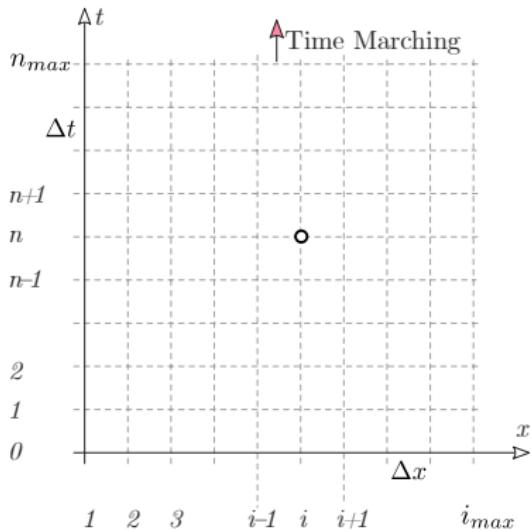
Figure 3: Typical mesh in FDTD

## FINITE DIFFERENCE METHOD

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- The goal is to solve differential equations *numerically* given some initial and boundary conditions
- Maxwell's equations has operators such as curl  $\nabla \times$ , divergence  $\nabla \cdot$ , and Laplacian  $\nabla^2$
- Lets first express the derivative term as a difference formula.

- We aim to **transform** a Calculus problem into a linear algebra one
- In other words, through discretisation, we move from the continuous domain to the discrete domain
- A partial differential equation is thus converted into a difference equation



Consider the Taylor series expansions of the function  $f(x)$  expanded about the point  $x_0$  with an offset of  $\pm\Delta/2$ :

$$f(x_0 + \frac{\Delta}{2}) = f(x_0) + \frac{\Delta}{2}f'(x_0) + \frac{1}{2!} \left(\frac{\Delta}{2}\right)^2 f''(x_0) + \frac{1}{3!} \left(\frac{\Delta}{2}\right)^3 f'''(x_0) + \dots,$$

$$f(x_0 - \frac{\Delta}{2}) = f(x_0) - \frac{\Delta}{2}f'(x_0) + \frac{1}{2!} \left(\frac{\Delta}{2}\right)^2 f''(x_0) - \frac{1}{3!} \left(\frac{\Delta}{2}\right)^3 f'''(x_0) + \dots$$

here the primes indicate differentiation. Subtracting the second equation from the first yields,

$$f\left(x_0 + \frac{\Delta}{2}\right) - f\left(x_0 - \frac{\Delta}{2}\right) = \Delta f'(x_0) + \frac{2}{3!} \left(\frac{\Delta}{2}\right)^3 f'''(x_0) + \dots \quad (1)$$

Moving on and dividing Eq. 1 by  $\Delta$  we get,

$$\frac{f(x_0 + \frac{\Delta}{2}) - f(x_0 - \frac{\Delta}{2})}{\Delta} = f'(x_0) + \frac{1}{3!} \frac{\Delta^2}{2^2} f''(x_0) + \dots$$

The term on the left is the derivative term of the  $f(x)$  at point  $x_0$  plus an approximation term,  $O(\Delta^2)$ :

$$\frac{df}{dx} \Big|_{x=x_0} = \frac{f(x_0 + \frac{\Delta}{2}) - f(x_0 - \frac{\Delta}{2})}{\Delta} + O(\Delta^2).$$

The central difference formula is thus:

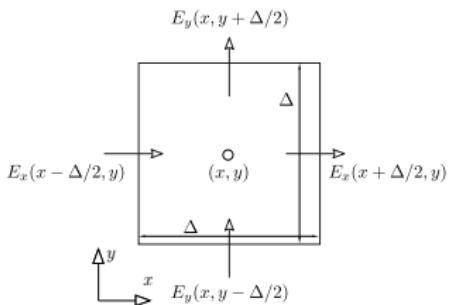
$$\frac{df}{dx} \Big|_{x=x_0} \approx \frac{f(x_0 + \frac{\Delta}{2}) - f(x_0 - \frac{\Delta}{2})}{\Delta}.$$

The divergence of a vector gives us a scalar field, and according to the Maxwell's equation:

$$\nabla \cdot E = \frac{\rho}{\varepsilon} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

For a 2D case, the above expression can be written as:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} \approx \frac{(E_x(x + \Delta/2, y) - E_x(x - \Delta/2, y) + E_y(x, y + \Delta/2) - E_y(x, y - \Delta/2))}{\Delta}$$

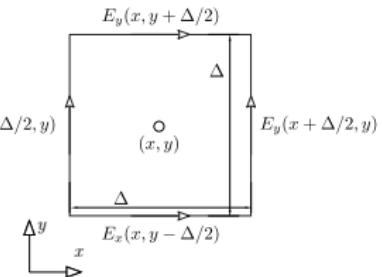


Likewise, we use the curl  $\nabla \times$  operator in Maxwell's equations:

$$\nabla \times E = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{bmatrix}$$

For a 2D case, the above expression can be written as:

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \approx \frac{(E_y(x + \Delta/2, y) - E_y(x - \Delta/2, y) - E_x(x, y + \Delta/2) + E_x(x, y - \Delta/2))}{\Delta}$$



The wave equation possesses the  $\nabla^2$  operator:

$$\nabla^2 \phi = \frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial y^2} + \frac{\partial^2 \phi_z}{\partial z^2}$$

Using the Taylor Series expansion, we can write the second derivate as:

$$\frac{d^2 f}{dx^2} \Big|_{x=x_0} \approx \frac{f(x_0 + \Delta) - 2f(x_0) + f(x_0 - \Delta)}{\Delta^2}$$

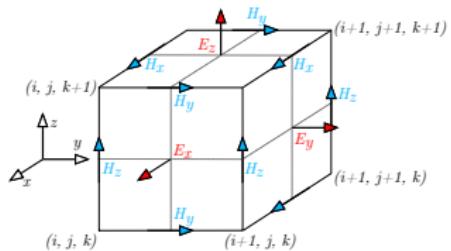
## FINITE DIFFERENCE TIME DOMAIN METHOD

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A point in space in a uniform rectangular lattice can be denoted as  $(i\Delta x, j\Delta y, k\Delta z)$ . Any function can then be expressed in terms of space and time

$$u(i\Delta x, j\Delta y, k\Delta z, n\Delta t) = u_{i,j,k}^n$$

- We are going to compute *only one* component at a given point
- There are no field components at the lattice edges
- Each component is similarly spaced.



- Developed in 1966 by Kane Yee
  - The algorithm employs second-order ( $O(\Delta^2)$ ) central differences.
1. Replace all the derivatives in Ampere's and Faraday's laws with finite differences.
    - 1.1 Discretize space and time so that the electric and magnetic fields are staggered in both space and time.
  2. Evaluate the magnetic fields one time-step into the future ( $n + 1$ )
  3. Evaluate the electric fields into the future ( $n + 1$ )
  4. Repeat Steps 3 and 4 until all the fields have been computed.

- Google Colab notebook running a Python example
- We focus on 1D FDTD first.
- Alternatively, download the Python repository from:



Let's write the time-dependent Maxwell's equations that have the curl operations:

$$\frac{\partial \vec{\mathbf{E}}}{\partial t} = \frac{1}{\epsilon_0} \nabla \times \vec{\mathbf{H}}$$
$$\frac{\partial \vec{\mathbf{H}}}{\partial t} = -\frac{1}{\mu_0} \nabla \times \vec{\mathbf{E}}$$

For one-dimensional case, we have:

$$\frac{\partial E_z}{\partial t} = \frac{1}{\epsilon_0} \frac{\partial H_y}{\partial x}$$
$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_z}{\partial x}$$

We will be using the finite difference notation to express the fields and derivatives (differences)

$$E_z(x, t) = E_z(i\Delta_x, n\Delta_t) = E_z^n[i]$$

$$H_y(x, t) = H_y(i\Delta_x, n\Delta_t) = H_y^n[i]$$

Here  $\Delta_x$  and  $\Delta_t$  are offsets in space and time. Indices  $i$  and  $n$  represent steps in space and time.

The 1D Maxwell's (Faraday's) equations then become:

$$\mu \frac{\partial H_y}{\partial t} \Big|_{(i+1/2)\Delta_x, n\Delta_t} = \frac{\partial E_z}{\partial x} \Big|_{(i+1/2)\Delta_x, n\Delta_t}$$

$$\mu \frac{H_y^{n+\frac{1}{2}}[i + \frac{1}{2}] - H_y^{n-\frac{1}{2}}[i + \frac{1}{2}]}{\Delta_t} = \frac{E_z^n[i+1] - E_z^n[i]}{\Delta_x}$$

# THE YEE ALGORITHM IN 1D - FARADAY'S LAW

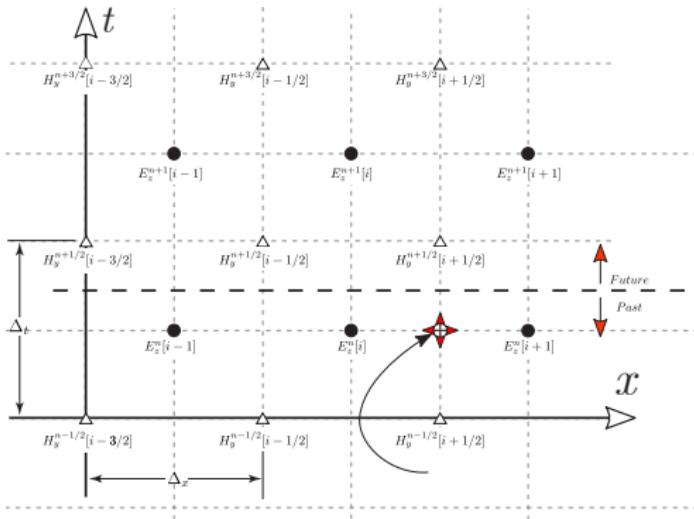


Figure 4: The interleaved  $E_z$  and  $H_y$  points in space and time.

$$H_y^{n+\frac{1}{2}} \left[ i + \frac{1}{2} \right] = H_y^{n-\frac{1}{2}} \left[ i + \frac{1}{2} \right] + \frac{\Delta_t}{\mu \Delta_x} (E_z^n [i+1] - E_z^n [i]). \quad (2)$$

Using similar techniques, we get the update equation for the electric field using the Ampere's law:

$$E_z^{n+1}[i] = E_z^n[i] + \frac{\Delta_t}{\varepsilon \Delta_x} \left( H_y^{n+\frac{1}{2}} \left[ i + \frac{1}{2} \right] - H_y^{n+\frac{1}{2}} \left[ i - \frac{1}{2} \right] \right). \quad (3)$$

- We have the update coefficients  $\frac{\Delta_t}{\varepsilon \Delta_x}$  and  $\frac{\Delta_t}{\mu \Delta_x}$  in the electric and magnetic fields expressions.
- These coefficients determine how far the energy propagates in a single step in time and space.
- Knowing  $c = 1/\sqrt{\varepsilon_0 \mu_0}$ , the maximum distance energy can travel in one time step is  $c\Delta_t$
- Courant number  $S_c$  is the ratio  $c\Delta_t/\Delta_x$ 
  - It determines the stability of the simulation.

$$\frac{1}{\varepsilon} \frac{\Delta_t}{\Delta_x} = \frac{1}{\varepsilon_r \varepsilon_0} \frac{\sqrt{\varepsilon_0 \mu_0}}{\sqrt{\varepsilon_0 \mu_0}} \frac{\Delta_t}{\Delta_x} = \frac{\sqrt{\varepsilon_0 \mu_0}}{\varepsilon_r \varepsilon_0} \frac{c \Delta_t}{\Delta_x} = \frac{1}{\varepsilon_r} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{c \Delta_t}{\Delta_x} = \frac{\eta_0}{\varepsilon_r} \frac{c \Delta_t}{\Delta_x} = \frac{\eta_0}{\varepsilon_r} S_c$$

$$\frac{1}{\mu} \frac{\Delta_t}{\Delta_x} = \frac{1}{\mu_r \mu_0} \frac{\sqrt{\varepsilon_0 \mu_0}}{\sqrt{\varepsilon_0 \mu_0}} \frac{\Delta_t}{\Delta_x} = \frac{\sqrt{\varepsilon_0 \mu_0}}{\mu_r \mu_0} \frac{c \Delta_t}{\Delta_x} = \frac{1}{\mu_r} \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{c \Delta_t}{\Delta_x} = \frac{1}{\mu_r \eta_0} \frac{c \Delta_t}{\Delta_x} = \frac{1}{\mu_r \eta_0} S_c$$

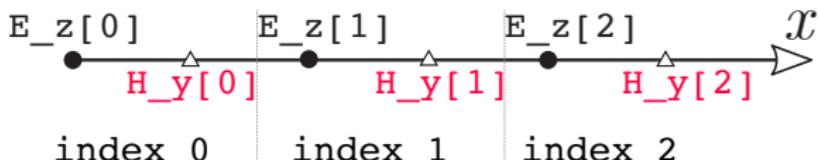


Figure 5: Array variable definition for 1D FDTD.

The update equations for a computer program therefore are:

$$\text{hy}[i] = \text{hy}[i] + (\text{ez}[i + 1] - \text{ez}[i]) / \text{imp0}$$

$$\text{ez}[i] = \text{ez}[i] + (\text{hy}[i] - \text{hy}[i - 1]) * \text{imp0}$$

Here **imp0** is the characteristic impedance, and  $S_c = 1$  is assumed 1. Note the magnetic field is calculated first and then the electric field. We place the expressions above in a loop.

- The points at the boundary need to be dealt with differently. As an example, for  $\text{ez}[0]$ , we don't have  $\text{hy}[-1]$ .
  - Similarly, for  $\text{hy}[\text{MAX}-1]$ , there is no  $\text{ez}[\text{MAX}]$
  - We can apply absorbing boundary conditions
- Only having one impedance results in a homogeneous medium
  - We can assign each node a  $\epsilon$  and  $\mu$
- Right now we haven't talked about introducing energy into the system
  - We can create a hard-wired source

## A SIMPLE PYTHON 1D FDTD SIMULATION

```
# do time stepping
for n in range(maxTime):

    # update magnetic field
    for i in range(SIZE-1):
        hy[i] = hy[i] + (ez[i + 1] - ez[i]) / imp0

    # update electric field
    for i in range(SIZE):
        ez[i] = ez[i] + (hy[i] - hy[i - 1]) * imp0

    # Additive Gaussian source node */
    if n < sourceSigma:
        ez[0] = math.exp(-((n - 1*sourcePeakTime)/)**2 / sourceSigma)
    else:
        ## PEC Boundaries
        ez[0] = 0.0
        ez[(SIZE - 1)] = 0.0

print(ez[sensorLocation])
#done with time stepping loop
```

To simulate infinite space, we use special types of boundary conditions called absorbing boundary conditions (ABCs).

ABCs are approximate and introduce dispersion especially in higher dimensions

Simple ABC's are:

$$\text{ez}[0] = \text{ez}[1]$$

$$\text{hy}[\text{MAX}-1] = \text{hy}[\text{MAX}-2]$$



Figure 6: F-16 inside an RF anechoic test chamber.

- In FDTD, we can either use *hard-wired* sources or *additive* sources.
- The former is easier to implement but suffers from severe discontinuity on the point of generation
- The latter just introduces a current term in the electric field update equation.

```
if n < sourceSigma:  
    ez[0] = math.exp(-(n - sourcePeakTime)**2 / sourceSigma)  
else:  
    ez[0] = 0.0
```

Hardwired source

```
ez[50] += math.exp(-(n - sourcePeakTime)**2 / sourceSigma)
```

Additive Source

Recall we have the material coefficients in the electric and magnetic field update equations:

$$H_y^{n+\frac{1}{2}} \left[ i + \frac{1}{2} \right] = H_y^{n-\frac{1}{2}} \left[ i + \frac{1}{2} \right] + \frac{\Delta_t}{\mu \Delta_x} (E_z^n [i+1] - E_z^n [i])$$
$$E_z^{n+1} [i] = E_z^n [i] + \frac{\Delta_t}{\varepsilon \Delta_x} \left( H_y^{n+\frac{1}{2}} \left[ i + \frac{1}{2} \right] - H_y^{n+\frac{1}{2}} \left[ i - \frac{1}{2} \right] \right)$$

We can therefore define  $\varepsilon$  and  $\mu$  at any point in the FDTD simulation:

```
hy[i] = hy[i] + (ez[i + 1] - ez[i]) / imp0 / muR[i]
ez[i] = ez[i] + (hy[i] - hy[i - 1]) * imp0 / epsR[i]
```

Chapters 2 & 3 – Understanding the FDTD Method<sup>1</sup>

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<sup>1</sup>John Schneider. *Understanding the FDTD Method*. URL:  
<https://eecs.wsu.edu/~schneidj/ufdtd/>.