Differentiation

Concepts

Topics covered are:

1. Differentiation
2. Chain Rule
3. Differentiability
4. Increasing and Decreasing Functions
5. Increment Theorem

Differentiation

The rate of change of a function is said to be the derivative of a function.

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here are different ways to write a derivative, considering $y = f(x)$:

$$\frac{dy}{dx} = f'(x) = y'$$

For differentiating it n times:

$$\frac{d^n y}{dx^n} = f^{(n)}(x) = y^{(n)}$$

Chain rule

If $y = f(u)$, $u = g(x)$, or $y = f(g(x))$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Differentiability

A function $f(x)$ is differentiable at a point if RHD = LHD, where

$$\text{RHD} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \text{LHD} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Graphically, a function which is continuous and has no sharp points throughout its domain is said to be differentiable.

Here is a list of some functions and their continuity and differentiability throughout their domain:

Function	Continuous?	Differentiable?
Polynomial $\left(\sum_{i=0}^{n} a_i x^{n-i},\ n \in \mathbb{N}\right)$	Yes	Yes
Modulus $\lvert x \rvert$	Yes	No (at 0)
Constant (a)	Yes	Yes
Signum $\left(\dfrac{\lvert x \rvert}{x},\ x \neq 0;\ 0,\ x = 0\right)$	No (at 0)	No (at 0)
Greatest Integer / Floor $(\lfloor x \rfloor)$	No (at integers)	No (at integers)
Smallest Integer / Ceil $(\lceil x \rceil)$	No (at integers)	No (at integers)
Fractional $\{x\} = x - \lfloor x \rfloor$	No (at integers)	No (at integers)
Reciprocal $\left(\dfrac{1}{f(x)},\ f(x) \neq 0\right)$	No (depends on $f(x)$)	No (depends on $f(x)$)

Increasing and Decreasing Functions

A function $f(x)$ is said to be increasing throughout (a, b) if $f'(x) \geq 0$, $x \in (a, b)$.

A function $f(x)$ is said to be decreasing throughout (a, b) if $f'(x) \leq 0$, $x \in (a, b)$.

A function $f(x)$ is said to be strictly increasing throughout (a, b) if $f'(x) > 0$, $x \in (a, b)$.

A function $f(x)$ is said to be strictly decreasing throughout (a, b) if $f'(x) < 0$, $x \in (a, b)$.

A function $f(x)$ is said to be constant throughout (a, b) if $f'(x) = 0$, $x \in (a, b)$.

Increment Theorem

If a function $f(x)$ is differentiable at a point x, then for a small increment Δx, the corresponding increment in the function, $\Delta y = f(x + \Delta x) - f(x)$, can be expressed as:

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

Here, ε is a small number that cancels out the error achieved by approximating the change in the function. It becomes negligible as the change in x is reduced, because the value of Δy gets more accurate.

Uses

Differentiation: To find the rate of change in something at a given instant, and to obtain the tangent of a curve at any given input value.

Chain Rule: Finding derivatives of nested functions.

Differentiability: Determine whether a function can be differentiated at a given input value without having to look at the curve.

Increasing and Decreasing Functions: Determine whether a function is rising or declining over a range without looking at the curve.

Increment Theorem: Highly useful for error approximation and rigorous ways to prove derivatives.