Scalars and Vectors

Concepts

Topics covered are:

1. Scalar vs Vector
2. Components of a Vector
3. Position Vector
4. Vector Addition
5. Scalar Multiplication
6. Properties of Addition and Multiplication
7. Scalar or Dot Product
8. Vector or Cross Product
9. Scalar and Vector Functions

Scalar vs Vector

Scalar	Vector
Represented by magnitude only	Represented by both magnitude as well as direction
Example: **5 units**. It could mean 5 units front, back, left, right or any direction.	Example: **5 units front**. It means 5 units front only.

Components of a Vector

Vectors which are used in graphs are usually represented by $\vec{r}$, where r is a vector quantity.

Further, A vector can represent the number of units it has moved in each direction. So

$$\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

where, the initial point of $\vec{r}$ is $P(x_1, y_1, z_1)$ and final point is $Q(x_2, y_2, z_2)$.

$\hat{i}$ is a unit vector along x-axis. Meaning, it represents 1 unit along the x direction.

$\hat{j}$ is a unit vector along y-axis. Meaning, it represents 1 unit along the y direction.

$\hat{k}$ is a unit vector along z-axis. Meaning, it represents 1 unit along the z direction.

This can also be written as

$$\vec{r} = [x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1]$$

The magnitude, or the length of a vector is

$$|\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Which is also the distance formula for two points in 3D.

Position Vector

A vector with it's initial point as the origin and terminal point $P(x, y, z)$. Hence, a vector quantity r can be written as

$$\vec{r} = [x,\, y,\, z] = x\hat{i} + y\hat{j} + z\hat{k}$$

This is obtained from the same representation used for vector, where $x_1 = y_1 = z_1 = 0$ and $x_2 = x$, $y_2 = y$ and $z_2 = z$.

Hence, magnitude of position vector is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Vector Addition

Consider two vectors: $\vec{r_1} = [a_1,\, b_1,\, c_1]$ and $\vec{r_2} = [a_2, b_2, c_2]$

Now, when two vectors are added, the resultant vector has each of components being the sum of the individual components of both the vectors. Hence

$$\vec{r_1} \pm \vec{r_2} = [a_1 \pm a_2,\ b_1 \pm b_2,\ c_1 \pm c_2]$$

Scalar Multiplication

When a vector is multiplied with a scalar quantity (a constant), the entire vector becomes the constant times the original vector. So, if $\vec{r} = [a,\ b,\ c]$

$$k\vec{r} = [ka,\ kb,\ kc]$$

Properties of Addition and Multiplication

Addition	Multiplication
$\vec{a} + \vec{b} = \vec{b} + \vec{a}$	$k \cdot (\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$	$(k + l) \cdot \vec{a} = k\vec{a} + l\vec{a}$
$\vec{a} + 0 = 0 + \vec{a}$	$(kl) \cdot \vec{a} = k \cdot (l\vec{a})$
$\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = 0$	$1 \cdot \vec{a} = \vec{a} \cdot 1$

Scalar or Dot Product

Consider two vectors $\vec{r_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{r_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Then the dot product of these two will be

$$\vec{r_1} \cdot \vec{r_2} = |\vec{r}_1||\vec{r}_2|cos\theta$$

Here θ is the angle between the two vectors. This can also be written as

$$\vec{r_1} \cdot \vec{r_2} = a_1 \cdot a_2 + b_1 \cdot b_2 + c_1 \cdot c_2$$

The dot product of two vectors will result in a directionless quantity. Hence, it gives **only the magnitude**.

Properties of Dot Product

- $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$
- $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot \vec{a} = 0 \iff a = 0$

- $(\vec{a}+\vec{b})\cdot\vec{c}=\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$
- $|\vec{a}\cdot\vec{b}|\leq|\vec{a}||\vec{b}|$
- $|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$

Vector or Cross Product

Consider two vectors $\vec{r_1}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{r_2}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$.

Then the cross product of these two will be

$$\vec{r_1}\times\vec{r_2}=|\vec{r_1}||\vec{r_2}|sin\theta\,\hat{n}$$

Here θ is the angle between the two vectors, and $\hat{n}$ is the direction of the resultant vector. This can also be written as

$$\vec{r_1}\times\vec{r_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$$

The cross product of two vectors will result in a directional quantity. Hence, it gives **both the magnitude and direction**.

Properties of Cross Product

- $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$
- $\hat{i}\times\hat{j}=\hat{k};\ \hat{j}\times\hat{k}=\hat{i};\ \hat{k}\times\hat{i}=\hat{j}$
- $\hat{j}\times\hat{i}=-\hat{k};\ \hat{k}\times\hat{j}=-\hat{i};\ \hat{i}\times\hat{k}=-\hat{j}$
- $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ but $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$
- $\vec{a}\times\vec{a}=0$
- $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$
- $l\vec{a}\times\vec{b}=l(\vec{a}\times\vec{b})=\vec{a}\times(l\vec{b})$

Scalar and Vector Functions

Scalar Function	Vector Function
Functions which return a scalar quantity when a point on the domain is substituted	Functions which return a vector quantity when a point on the domain is substituted

Scalar Function	Vector Function
Example: $f(x) = x^2$, $g(x, y) = xy$	Example: $\vec{r}(t) = (t)\hat{i} + (t^4)\hat{j} + (3t^2 - 3)\hat{k}$, $\vec{p}(x, y, z) = (xyz)\hat{i} + (x + z^2)\hat{j} + (y^2 z^3)\hat{k}$

Uses

Scalar: Used for real-world measurements and calculations widely. It is also used to calculate physical quantities which are independent of direction

Vector: Used widely in fields where direction matters a lot. Hence, it finds uses in navigation, engineering, data science and computer graphics.