Alternating Series

A series which has alternating signs in their terms is an **alternating seires**. For example:

- $1 - 1 + 1 - 1 + 1 \ldots$
- $1 - 2 + 4 - 8 + 16 \ldots$

They are usually written in the form of a series, with $(-1)^n$ to signify change in sign for consecutive terms. So in general, it is written like this:

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

For example, the above series could be written as:

- $\sum_{n=1}^{\infty}(-1)^{n-1}$
- $\sum_{n=1}^{\infty}(-1)^{n-1} \cdot 2^{n-1}$

Test for Convergence

Now an alternating series keeps changing sign, and terms get added and subtracted, and becomes difficult to know if the series converges or diverges. But there is a way.

The thing is, the terms don't hve to lead to a decrease directly. If the next term subtracted is greater than the current one, it decreases, although gradually. But that still leads to convergence at some point, especially when n is large. Also, the terms should go to 0 at some point, so that the series saturates. It's not just about decreasing, but also about the value tending to 0.

Hence our test is:

$$a_{n+1} > a_n \quad \text{for the series} \quad \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$
$$a_{n+1} < a_n \quad \text{for the series} \quad \sum_{n=1}^{\infty}(-1)^{n}a_n$$

and

$$\lim_{n\to\infty} a_n = 0$$

If any of these test fails, it is divergent.

Absolute and Conditional Convergence

The alternating series is:

- **Absolutely / Completely convergent** if $\sum_{n=1}^{\infty}|u_n|$ is convergent
- **Conditionaly convergent** if $\sum_{n=1}^{\infty}|u_n|$ is divergent and the alternating series is convergent