Differential Operators

Topics covered are:

1. Differential operator
2. Gradient
3. Divergence
4. Curl
5. Directional Derivative

Differential Operator

This is an operator widely used in vector calculus to specify how scalar and vector fields change in space. It is denoted using ∇ (Nabla) where

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Here, $\frac{\partial}{\partial x}$ represents partial differentiation with respect to x, which means differentiation applies only for x, and all the other variables are treated to be constants.

Gradient

Measured on scalar functions, where the gradient of the function gives the maximum increase of the function at a point. The magnitude gives the rate of change at that point.

Consider a scalar function $f(x)$. The gradient of this function is the differential operator used on the function. It is written as $\mathrm{grad}(f)$ or ∇f. So

$$\nabla f = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})(f) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Divergence

Represents the current flow of a vector field at a given point. It is a **scalar** quantity as it only tells “how much” of something is flowing outside at a given point. There are three situations

Situation 1: Source

Vectors move out of a common point or **source**. In this case, divergence is positive as the vectors diverge.

Situation 2: Sink

Vectors move into a common point or **sink**. In this case, divergence is negative as the vectors converge.

Situation 3:

Vector move parallel to each other at every point, indicating a stable flow. In this case, divergence is zero as the vectors neither converge nor diverge.

Divergence of a vector field can be found by doing the dot product of the differential operator and the vector function. Consider the function to be $\vec{F} = a\hat{i} + b\hat{j} + c\hat{k}$. Then divergence is denoted as $\text{div}(\vec{F})$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial a}{\partial x}\hat{i} + \frac{\partial b}{\partial y}\hat{j} + \frac{\partial c}{\partial z}\hat{k}$$

Curl

Represents the rotation of vectors on the vector field. It is a **vector** quantity as it represents the speed of rotation (magnitude) and axis of rotation (direction).

Curl of a vector field can be found by doing the cross product of the differential operator and the vector function. Consider the function to be $\vec{F} = a\hat{i} + b\hat{j} + c\hat{k}$. Then curl is denoted as $\text{curl}(\vec{F})$

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix}$$

Directional Derivative

Represents the rate of change of functions along a direction. Hence, this quantity is a **vector** quantity. It can be found by doing the dot product of the gradient of the function and the unit vector. Only the unit vector is needed because we are concerned only about the direction of the change.

Consider a function f and a point P on the function, where we are concerned about the direction along a vector $\vec{v}$. Hence, we need to find the rate of change of f at point P along $\vec{v}$. This can also be written as $D_{\vec{v}}f$.

$$D_{\vec{v}} f = \frac{\vec{v}}{|\vec{v}|} \cdot \nabla f_P$$

Uses

Gradient: Optimization in ML and other fields, Physical fields (direction of maximum change)

Divergence: Fluid Mechanics, Electromagnetism, Incompressibility

Curl: Fluid Mechanics, Electromagnetism, Irrotational Flows

Directional Derivative: Machine Learning, understanding fields (rate of change in any direction)