Integration

Concepts

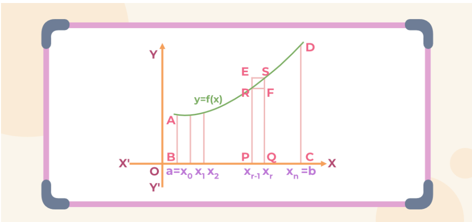
Topics covered are:

1. Antiderivative
2. Fundamental Theorem of Calculus
3. Properties of Definite Integrals
4. Integration by Partial Fractions
5. Integration by Parts
6. Improper Integrals

Antiderivative

Let $f(x)$ be the derivative of a function $F(x)$. Then $F(x) + c$ is said to be the antiderivative of $f(x)$. The constant term c represents the family of the function as differentiating it would result in the same derivative.

Mathematically, $f(x)dx = F(x) + c$



Definite Integral: Let $f(x)$ be a function on $[a, b]$. Split this range into n parts so that each small range $\Delta R = \frac{b-a}{n}$

Let x_i^* be a sample space on the i^{th} interval.

Then the area of the curve can be obtained by adding the area of rectangles in each of the n regions. The larger the value of n, the more approximate the area of the curve becomes. Thus,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

Fundamental Theorem of Calculus

1. Let $f(x)$ be a continuous function on $[a, b]$. If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$
2. Let $f(x)$ be a continuous function on $[a, b]$. If $F(x)$ is the antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Properties of Definite Integrals

1. $\int f(x)\,dx = \int f(u)\,du$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \;\pm\; \int_a^b g(x)\,dx$
4. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$
5. $\int_a^a f(x)\,dx = 0$
6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad ; \quad a < c < b$

Integration by Partial Fractions

Integrand	Partial Fractions
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

It is important to note that if we have an integrand in the form $\frac{f(x)}{g(x)}$, then the degree (highest power the variable is raised to) of $\frac{f(x)}{g(x)}$ so that partial fractions can be taken. In any other case, the integrand cannot be split by using partial fractions.

Integration by Parts

Consider two functions $f(x)$ and $g(x)$, where $f(x)$ is the first function and $g(x)$ is the second function. Then:

$$\int f(x)\cdot g(x)\,dx = f(x)\int g(x)\,dx \;-\; \int \left(f'(x)\int g(x)\,dx\right)dx$$

Or, if $f(x) = u$ and $g(x)dx = dv$, then:

$$\int u\,dv = uv \;-\; \int v\,du$$

Finding first and second function: To find the first and second function, we use ILATE, which stands for:

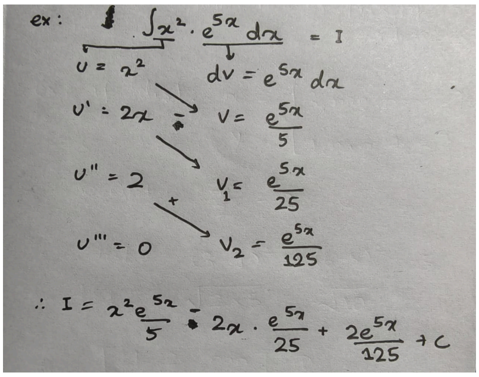
- **Inverse trigonometric functions**
- **Logarithmic**
- **Arithmetic**
- **Trigonometric**
- **Exponential**

The topmost ones get the most priority and it goes down. So in the case of a combination of inverse trigonometric function and an arithmetic expression, we take inverse to be first and arithmetic to be second.

Bernoulli's Formula: This formula does the same as expanding the integral, but it saves a lot of effort when using this technique multiple times in a question. Here is the formula:

$$\int u\,dv = \int u\,dv = uv - u'v_1 + u''v_2 - \cdots + (-1)^n u^{(n)} v_n = uv + \sum_{i=1}^{n} (-1)^i\, u^{(i)} v_i$$

Where, $u^{(n)}$ means to differentiate n times, and v_n means to integrate n times.



This shows an easier way to write it on paper. The arrows denote what terms are to be multiplied, and the sign in between the arrows show whether the terms are to be added or subtracted.

Improper Integral

an integral with either or both of its limits at infinity is an **improper integral**, because it does not have defined limits. There are two types:

Type I	Type II
Either one of the limits is not defined.	Both limits are not defined
Examples: $\int_{-\infty}^{b} f(x)\,dx$, $\int_{a}^{\infty} f(x)\,dx$	Example: $\int_{-\infty}^{\infty} f(x)dx$

Type I

Case 1: $\int_{-\infty}^{b} f(x)\,dx$

Let a variable $a \to -\infty$. Then this can be written as

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx$$

Case 2: $\int_{a}^{\infty} f(x)\,dx$

Let a variable $b \to \infty$. Then this can be written as

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_{a}^{b} f(x)\,dx$$

Type II

$\int_{-\infty}^{\infty} f(x)\,dx$

Let a variable c be in the interval $(-\infty, \infty)$

Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Now, both the cases from Type I can be used

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{c} f(x)\,dx + \lim_{b\to\infty} \int_{c}^{b}$$

Uses

Applications of integrals have been specified in a different PDF.