

Convergence and Divergence

Convergence: It is the phenomenon of a sequence or series saturating at some point. The terms of sequence goes closer to a constant, while the series starts saturating, and barely grows. Think of it like slowing down.

Divergence: It is the phenomenon of a sequence or series growing continuously. The terms of sequence goes further from 0 towards infinity, while the series starts growing much faster, tending towards infinity. Think of it like speeding up.

Test for Convergence of a Sequence

So we know a sequence converges if it's terms tends to a constant as n increases. It can be written as:

$$\lim_{n \rightarrow \infty} u_n = k$$

If $\lim_{n \rightarrow \infty} u_n = \pm\infty$ then it means the sequence diverges.

Test for Convergence of a Series

So we know that a series converges if it's sum starts saturating as n increases. It can be written as:

$$\lim_{n \rightarrow \infty} u_n = k$$

for some real number constant k . If $k = \infty$ then the series diverges.

Convergence Tests

There are many more ways to test if a series converges, and each one of them is helpful for different kinds of series. It is important to know what test to pick for what kinds of series.

Integral Test

Let $\sum u_n$ be a series of positive terms. Let $a_n = f(n)$ be a **continuous, positive and decreasing** function for each $x \in [N, \infty)$. Then the series $\sum_{n=N}^{\infty} u_n$ and the integral $\int_N^{\infty} f(x) dx$ converge or diverge together.

So basically, to do this test:

- Check if the function is:
 - continuous
 - positive
 - decreasing for all values given.
- Write the function in terms of x
- Integrate $f(x)$ with respect to x from N to ∞ , the values that satisfies the above three conditions.
- If the integral results in infinity, the series diverges, otherwise (if a constant value), it converges.

Comparison Test

Let $\sum u_n$ and $\sum v_n$ be two series. Then the series u_n is:

- Convergent if:
 - $u_n < v_n$
 - v_n is convergent
- Divergent if:
 - $u_n > v_n$
 - v_n is divergent

The real problem here lies with choosing the sequence for v_n so that problem becomes easy to solve.

Ratio Test

According to the ratio test, we take two consecutive terms of the sequence, and check the values of their ratio. Consider a sequence u_n . let a constant λ be:

$$\lambda = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n}$$

Now, if:

- $\lambda < 1$ the series converges, because the higher term is lesser than the lower term
- $\lambda > 1$ the series diverges, because the higher term is greater than the lower term
- $\lambda = 1$ the test fails, because it assumes the consecutive terms are equal

Root Test

The root test is the same as ratio test, except that

$$\lambda = \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}}$$