Propositional Logic

By: **Annamalai A**

Propositions and Connectives

- **Proposition**: A declarative statement that is definitely **true (T)** or **False (F)**
    - *It is raining.* — This is a valid proposition (declaration)
    - *Go to your room!* — This is not a valid proposition (command)
    - *Is it raining?* — This is not a valid proposition (interogation)
- **Connective**: A symbol or word used to connect two or more propositions

Word	Symbol
And	$\wedge$
Or	$\vee$
Not	$\neg$ or $\sim$
Conditional	$\implies$
Biconditional	$\iff$

For example:

- Propositions:
    - p be *I want an apple*
    - q be *I want an orange*
- Connective:
    - $p \wedge q$ means *I want an apple and I want an orange*
    - $p \vee q$ means *I want an apple or I want an orange*
    - $\neg p$ means *I do not want an apple*
    - $p \implies q$ means *If I want an apple, then I want an orange*
    - $p \iff q$ means *I want an apple if and only if I want an orange*

Truth Tables

- A truth table is a way to verify the result of a compound statement
- An compound statement is a combination of propositions and connectives

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$p \implies q$	$p \iff q$
F	**F**	F	F	T	T	T
F	**T**	F	T	T	T	F
T	**F**	F	T	F	F	F

p	q	$p \land q$	$p \lor q$	$\neg p$	$p \implies q$	$p \iff q$
T	**T**	T	T	F	T	T

Tautologies and Contradictions

- **Tautology**: A compound statement that is always **true**, no matter the state of the propositions.
    - $p \lor \neg p = T$ for any state of p
    - Forms the base of universal truths
- **Contradiction**: A compound statement that is always **false**, no matter the state of the propositions.
    - $p \land \neg p = F$ for any state of p
    - Forms the base for mathematical proofs

Logical Equivalences

Laws

Important Equivalences

Idempotent Law	$p \lor p = p$	$p \land p = p$
Identity Law	$p \lor F = p$	$p \land T = p$
Dominant Law	$p \lor T = T$	$p \land F = F$
Complement Law	$p \lor \neg p = T$	$p \land \neg p = F$
Commutative Law	$p \lor q = q \lor p$	$p \land q = q \land p$
Associative Law	$(p \lor q) \lor r = p \lor (q \lor r)$	$(p \land q) \land r = p \land (q \land r)$
Distributive Law	$p \lor (q \land r) = (p \lor q) \land (p \lor r)$	$p \land (q \lor r) = (p \land q) \lor (p \land r)$
Adsorption Law	$p \lor (p \land q) = p$	$p \land (p \lor q) = p$
De Morgan Law	$\neg(p \lor q) = \neg p \land \neg q$	$\neg(p \land q) = \neg p \lor \neg q$

- $p \implies q = \neg p \lor q$
- $p \implies q = \neg q \implies \neg p$
- $p \lor q = \neg p \implies q$
- $p \land q = \neg(p \implies \neg q)$
- $(p \implies q) \land (p \implies r) = p \implies (q \land r)$
- $(p \implies r) \land (q \implies r) = (p \lor q) \implies r$
- $(p \implies q) \lor (p \implies r) = p \implies (q \lor r)$
- $(p \implies r) \lor (q \implies r) = (p \land q) \implies r$

Equivalences Involving Biconditional

- $p \iff q = (p \implies q) \land (q \implies p)$

- $p \iff q = \neg p \iff \neg q$
- $p \iff q = (p \wedge q) \vee (\neg p \wedge \neg q)$
- $\neg(p \iff q) = p \iff \neg q$

Principal Normal Forms

Dual of a Compound Proposition

- Replace every $\wedge$ with $\vee$ and every $\vee$ with $\wedge$
- The compound proposition must consist only of $\wedge$, $\vee$ and $\neg$
- If $A\left(p_1, p_2, \ldots, p_n\right)$ denotes the a proposition in n variables, then the dual is denoted by $A^*\left(p_1, p_2, \ldots, p_n\right)$

Normal Forms

- Goal: Reduce compound propositions to a standard form helpin in proofs and comparison
- Two kinds:
  - **Disjunctive Normal Form (DNF)**: Sum of elementary products
  - **Conjunctive Normal Form (CNF)**: Product of elementary sums
- It can be further standardized to:
  - **Principle DNF**: Sum of minterms
  - **Principle CNF**: Product of maxterms

DNF and CNF

- Elementary product: Two or more of the available propositions connected by $\wedge$
- Elementary sum: Two or more of the available propositions connected by $\vee$
- It is derived from logical laws

Steps to obtain:

1. Replace connectors like $\implies$ and $\iff$ with $\neg$, $\vee$, or $\wedge$
2. Use De Morgan's law to remove $\neg$ present on compound statements
3. Use available laws to obtain **sum of elementary products** or **product of elementary sums**

PDNF and PCNF

- Minterm: Products invloving all variables provided. The state comes from the truth table
- Maxterm: Sums involving all variables provided. The state comes from the truth table

Steps to obtain PDNF:

1. Obtain DNF
2. Introduce missing terms in minterms by introducing T

Stepts to obtain PCNF:

1. Obtain PDNF
2. The complement of PDNF

or

1. Obtain CNF
2. Introduce missing terms in maxterms by introducing F

Rules of Inference

- **Rule P**: A premise may be introduced at any step in the derivation
- **Rule T**: A formula S may be introduced in the derivation if S is implied by one or more formulas in the derivation

Rule	Name
$p \land q \therefore p$ or $p \land q \therefore q$	Simplification
$p \therefore p \lor q$ or $q \therefore p \lor q$	Addition
$p, q \therefore p \land q$	Conjunction
$p, p \implies q \therefore q$	Modus Ponens
$p \implies q, \neg q \therefore \neg p$	Modus Tollens
$p \implies q, q \implies r \therefore p \implies r$	Hypothetical Syllogism
$p \lor q, \neg q \therefore p$ or $p \lor q, \neg p \therefore q$	Disjunctive Syllogism
$p \lor q, \neg p \lor r \therefore q \lor r$	Resolution
$p \lor q, p \implies r, q \implies r \therefore r$	Dilemma

- If there are three premises p_1, p_2 and p_3 and conclusion q, then we say:

$$p_1, p_2, p_3 \therefore q \quad \text{which is} \quad (p_1 \land p_2 \land p_3) \implies q$$