Applications of Integration

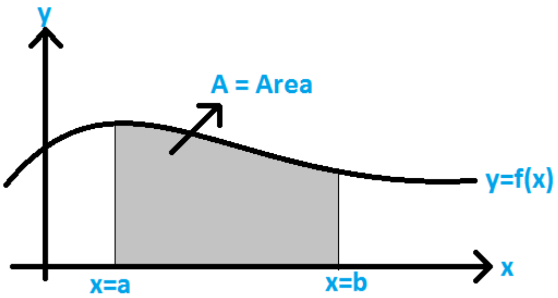
Concepts

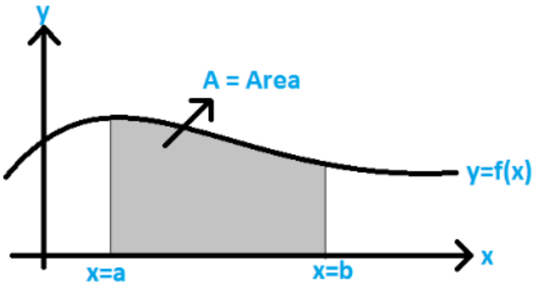
Topics covered are:

1. Area
2. Volume
    a. Cross Section / Slicing Method
    b. Solids of Revolution
    c. Disk Method
    d. Ring / Washer Method
3. Arc Length
4. Surface Area

Area

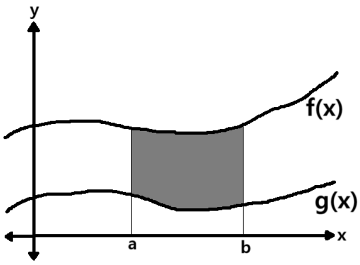
Integration is really useful when finding the area under graphs .i.e. The area between an axis and the curve. It can also be used to find the area bound due to multiple curves.

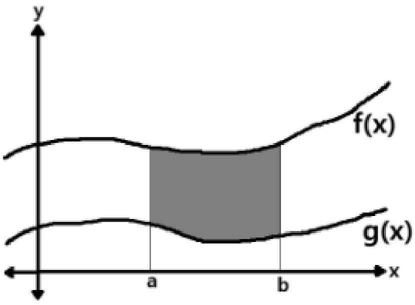




In the following graph, Area is:

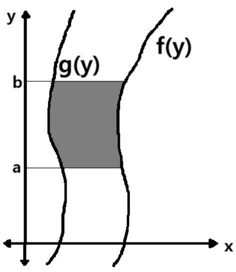
$$A = \int_a^b f(x)\,dx$$

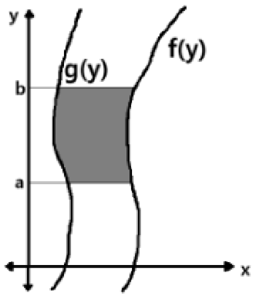




Area bound by two (or more) curves: Consider the functions $f(x)$ and $g(x)$ in $[a, b]$ where $f(x) > g(x)$. Then the area bound by these curves from $x = a$ to $x = b$ is:

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$





In this case, we have two functions $f(y)$ and $g(y)$ in $[a, b]$. If $f(y) > g(y)$, then the area bound by the curves from $y = a$ to $y = b$ is:

$$A = \int_a^b \left(f(y) - g(y)\right) dy$$

Now, let's look at this table with row representing quadrant and columns representing variable of integration and fill in the sign:

	x as variable of integration	y as variable of integration
Q 1	$+$	$+$
Q 2	$+$	$-$
Q 3	$-$	$-$
Q 4	$-$	$+$

When finding the area of an enclosed region, it is advised to take the absolute value of the evaluated integral, so that the areas of the region in each quadrant gets *added*.

Volume

Cross Section / Slicing Method

Similar to finding area of 2D figures using integration, volumes of any 3D shape can also be found using the same method, using only two vairables such as x and y.

Cross Section: The region of intersection of a plane with a solid is called a cross section.

For any solid, the volume V is just the sum of areas of each cross section. If the area of cross section varies with $A(x)$, and the height of the solid is $h = b - a$, where $x = b$ is the right end of the solid and $x = a$ is the left end of the solid, we may take individual cross sections of thickness dx and add them all.

Let the i^{th} cross section be written as x_i^*, then using Riemann sum,

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\, dx = \int_a^b A(x)\, dx$$

We limit $n \to \infty$ because the more cross sections we take with lesser thickness, the more accurate our volume becomes and that is what the integral exactly does.

Solids of Revolution

Certain 2D shapes, when rotated along an axis form a 3D solid, with axis at the centre. All such 3D solids have a circular cross section perpendicular to the axis. Such 3D solids are called solids of revolution.

Disk Method

This method is a way to find the volume of such solids where the solid is filled inside.

Consider a function $f(x)$ to create a 2D shape under it with the axis (x-axis). When this shape is rotated about the axis, we obtain a solid, where the cross section is non-uniform and varies based on the area function $A(x)$. Since the cross sections are circular, the area function definitely represents a circle. Here, the radius varies with x as $f(x)$ due to the shape that forms the solid.

Hence, $A(x) = \pi \cdot \big(f(x)\big)^2$

Now, to find the volume V, just integrate the area function. Hence,

$$V = \int_a^b A(x)\, dx = \int_a^b \pi \cdot \big(f(x)\big)^2\, dx$$

Washer Method

This method is a way to find the volume of such solids where the solid is hollow inside (or not filled completely).

Consider a function $f(x)$ to create a 2D shape under it with the axis (x-axis). When this shape is rotated about the axis, we obtain a solid, where the cross section is non-uniform and varies based on the area function $A_1(x)$. Since the cross sections are circular, the area function definitely represents a circle. Here, the radius varies with x as $f(x)$ due to the shape that forms the solid.

Hence, $A_1(x) = \pi \cdot \big(f(x)\big)^2$

Also, consider a function $g(x)$ that represents the inner curve, and hence the inner portion of the solid. Now, let the area of the curve be $A_2(x)$. This cross section is also circular because the shape is being rotated.

Hence, $A_2(x) = \pi \cdot \big(g(x)\big)^2$

Now the final area of cross section becomes $A(x) = A_1(x) - A_2(x)$ because the volume due to $g(x)$ must be scooped out.

Now, to find the volume V, just integrate the area function. Hence,

$$V = \int_a^b A(x)\,dx = \int_a^b \pi \cdot \Big(\big(f(x)\big)^2 - \big(g(x)\big)^2\Big)\,dx$$

Arc Length

The length of the function curve from the start point to the end point of the curve.

Let a function $y = f(x)$ be a continuous and a differentiable function defined on $[a, b]$. If $f’(x)$ is continuous on $[a, b]$.

Now, the curve can be sliced into very thin pieces. To get accurate results, let $n \to \infty$ where n is the number of segments obtained by slicing. Let i be an arbitrary value specifying a segment on the curve.

In each segment (take the i^{th} segment), the change in x value is $\Delta x_i = x_{i+1} - x_i$ and the change in y value is $\Delta y_i = y_{i+1} - y_i$.
According the pythagorean or distance formula, $D(x_i) = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

Let’s manipulate this function a bit:

$D(x_i) = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = (\Delta x_i)^2 \cdot \sqrt{1 + (\frac{\Delta y_i}{\Delta x_i})^2}$

Now, $(\frac{\Delta y_i}{\Delta x_i})^2$ is nothing but $\frac{dy_i}{dx_i} = f'(x_i)$ at some point on the tangent. But in a smaller scale, it is approximately equal. Hence we can consider this to be true.

Hence, $D(x_i) = \sqrt{1 + \big(f’(x_i)\big)^2}$

Now, this is only for one segment, for n segments, we have to sum all such segments and n is very high for approximation. Hence, length L is

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} D(x_i) dx = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,dx$$

Surface Area

Let $f(x)$ be a function defined on $[a, b]$. Then the surface are of the solid can be obtained by rotating this curve around the x-axis. Hence, this is only applicable for solids that are circular.

Now, at every instant, the radius of the thin circle obtained is $R(x) = f(x)$. Hence circumfirence, $C(x) = 2\pi \cdot f(x)$

Now, to obtain the total surface area, add the circumfirence obtained at every instant. Hence, area A is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} C(x_i)dx = \int_a^b 2\pi \cdot \sqrt{1 + \big(f(x)\big)^2}\, dx$$

where i is the i^{th} circle on the solid.