Introduction to Proof Techniques

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Direct Proofs

This is the most straightforward approach. You start with the premise (P) and use definitions, axioms, and previously proven theorems to reach the conclusion (Q).

- **Strategy:** Assume P is true $\rightarrow$ Logical Steps $\rightarrow$ Therefore, Q is true.
- **Best for:** Statements where the relationship between the hypothesis and conclusion is clear and "forward-moving."
- **Example:** Prove that "If n is an even integer, then n^2 is even."
  1. **Assume P:** Let n be even. By definition, $n = 2k$ for some integer k.
  2. **Steps:** Square it: $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.
  3. **Conclude Q:** Since n^2 is a multiple of 2, it is even.

Proof by Contrapositive

This relies on the logical equivalence: $(P \rightarrow Q) \equiv (\neg Q \rightarrow \neg P)$. Sometimes it is very hard to prove a statement directly, but much easier to prove that "If the conclusion is false, then the premise must have been false."

- **Strategy:** Assume Q is false ($\neg Q$) $\rightarrow$ Logical Steps $\rightarrow$ Therefore, P is false ($\neg P$).
- **Best for:** Statements involving "not equal to" or complex conclusions.
- **Example:** Prove that "If n^2 is even, then n is even."
  1. **Contrapositive:** "If n is **odd**, then n^2 is **odd**."
  2. **Assume $\neg Q$:** Let n be odd ($n = 2k + 1$).
  3. **Steps:** $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
  4. **Conclude $\neg P$:** This is the form of an odd number. Since the contrapositive is true, the original statement is true.

Proof by Contradiction

This is the "nuclear option" of proofs. You assume the statement you are trying to prove is **false** and show that this leads to a total breakdown of logic (a contradiction).

- **Strategy:** Assume P is true AND Q is false $\rightarrow$ Logical Steps $\rightarrow$ Reach a contradiction (e.g., $1 = 0$) $\rightarrow$ Therefore, the assumption was wrong; Q must be true.
- **Best for:** Proving something *cannot* exist or is irrational.
- **Classic Example:** Proving $\sqrt{2}$ is irrational.
  1. **Assume the opposite:** Assume $\sqrt{2}$ is rational ($\frac{a}{b}$ in lowest terms).
  2. **Steps:** Algebraically show that both a and b must be even.
  3. **Contradiction:** If both are even, $\frac{a}{b}$ wasn't in lowest terms. **Contradiction found!**
  4. **Conclude:** $\sqrt{2}$ must be irrational.

Mathematical Induction

Induction is used to prove that a statement $P(n)$ is true for all natural numbers ($n = 1, 2, 3, \ldots$). Think of it like a line of falling dominoes.

- **Step 1: Base Case:** Prove the statement is true for the very first value (usually $n = 1$). This "pushes the first domino."
- **Step 2: Inductive Hypothesis:** Assume the statement is true for some arbitrary integer k.
- **Step 3: Inductive Step:** Use the assumption from Step 2 to prove the statement is true for $k + 1$. This proves that "if any domino falls, the next one must also fall."

Cheatsheet

Technique	Logical Basis	When to use it
Direct	$P \rightarrow Q$	Standard algebraic properties.
Contrapositive	$\neg Q \rightarrow \neg P$	When Q is easier to negate than work with directly.

Technique	Logical Basis	When to use it
Contradiction	$(P \land \neg Q) \to \text{False}$	When proving "uniqueness" or "irrationality."
Induction	$P(1) \land (P(k) \to P(k+1))$	Statements involving sums, series, or n.