Line Integral

By: **Annamalai A**

The concept of line integrals talks about summing something along a curve, rather than along the axis. For this, in general, we take a function depending on two variables x and y, and the curve C written in terms of t.

What is the curve C

The curve C determines along what line we are integrating. It is usually in parametric form, where the variables x and y are written with respect to t. This makes it easy to compute the line integral in both scalar and vector form.

What is the function $f(x, y)$

This function represents the surface that is supposed to be integrated along the curve C. It is written in the form of x and y, which will later be converted in the form of t for easy solving.

Line Integral of Scalar Functions

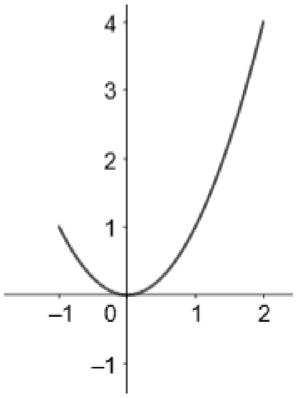
Now, we know we have two functions on the graph. What happens now is, the curve C is also drawn on the surface $f(x, y)$, and now we integrate this curve on the surface with respect to the curve C. So basically, because the curve is also drawn on the surface, it obeys the function f, and hence, the formula for line integral becomes:

$$\text{Line integral} = \int_C f(x, y)\, dr$$

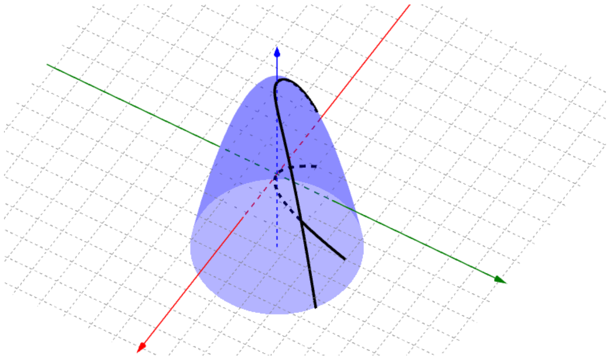
Where $f(x, y)$ is the surface which will also be written in terms of t and dr is a small element along the curve C.

Now, let's explain this with an example:

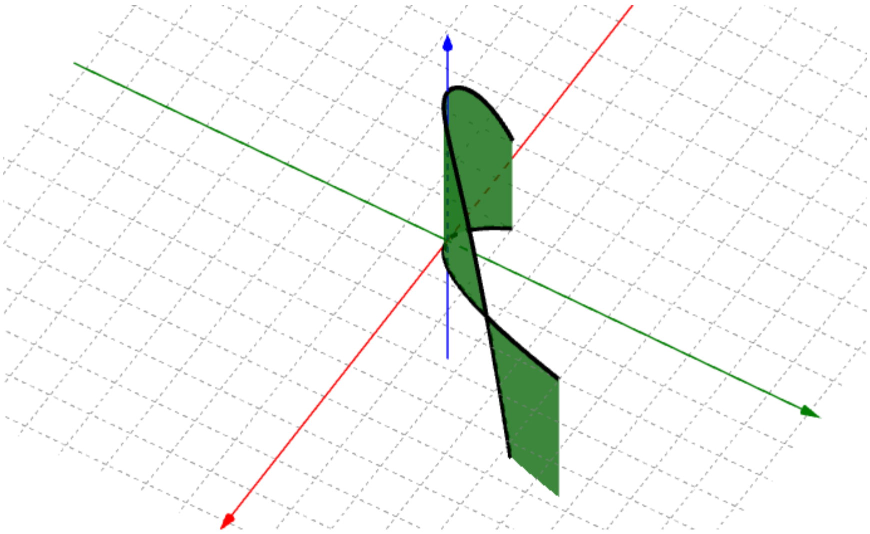
Consider a curve $C = t\hat{i} + t^2\hat{j}$, and the function $f(x, y) = 5 - x^2 - y^2$.



This graph represents the curve C in terms of $x(t)$ and $y(t)$, where $x = t$ and $y = t^2$



This represents the entire picture. The curve on the x axis and y axis (red and green lines) is the curve C, the blue surface is the function $f(x, y)$ and the other black line on the surface is the curve C drawn on the surface.



Here is the line integral to be found visualized. As you can notice, it is the integral of the curve C drawn on the surface (so basically f) and the original curve C.

To experiment and understand more, check out https://www.geogebra.org/m/ma2n5KNH

Parametric Form

In the parametric form we write the curve C using a single variable, say t. This function is usually written as $\vec{r}(t)$ and it traces the curve. Now, we know the original formula is:

$$\text{Line integral} = \int_C f(x, y)\, dr$$

Here, dr is nothing but the change along the direction of the curve at a point. Because our curve is a function of t, we can write $dr = r’(t)dt$.

Now, we do the dot product of the derivative and the function because we want to know the effect of the tangent at any point on the curve. Hence the final formula becomes:

$$\text{Line Integral} = \int_C F\left(\vec{r}(t)\right) r'(t)\, dt$$

Line Integral of Vector Functions

$$\text{Line Integral} = \int_C P\, dx + Q\, dy$$

Now, we have a vector field on our cartesian plane, say F. This vector field is represented as $F = P\hat{i} + Q\hat{j}$, meaning the vector at any point has a magnitude P along the x-axis and Q along the y-axis.

Now the line integral is simply how much we can move on the curve, given the vector field affects our movement. So we take the derivative of the components of the vector at each point, in order to get the instantaneous change of the vector.

This can also be written in parametric form.

Conservative Fields

Conservative field basically talks about if the final work done (The dot product of the direction of vector and our motion at any point) depends only on the initial and final path. It can be found by simply finding the gradient of a scalar function. Let F be our function. F is conservative if there exists a scalar function ϕ such that:

$$\vec{F} = \nabla \phi$$

The gradient points at the direction of maximum increase of the scalar function, hence helping us find if our function is conservative or not.

Tests for Conservative Fields

Considering the region has **no holes, no missing points** and **no missing lines**, the field is conservative if:

- Curl of the vector field is **0**. $\nabla \times \vec{F} = 0$ (Best case for exams)
- Closed path integral is 0. $\int_C \vec{F} \cdot dr = 0$
- Potential function exists, as mentioned above

Surface Integral of a Vector Field

When it comes to surface integral, we do the same thing as a line integral, except we do it over surfaces now. So we have:

$$\int \int_S f(x, y)\, dA = \int \int_S f(x, y)\, dx\, dy$$

How to evaluate this integral? This double integral is nothing but an integral of an integral, and we do partial integration. First, we integrate the function with respect to x, treating the other variable y as a constant, and substitute the limits on the x-axis, and now we have a function of y only, and a normal integral.

Parametric Form

Consider two independent variables u and v that define the points on the surface. Then, the surface can be written as $\vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$. Now the surface integral becomes:

$$\int \int_S f(x, y)\, dx\, dy = \int \int_S f(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Where r_u is the partial derivative of r with respect to u and r_v is the partial derivative of r with respect to v.

Green’s Theorem

$$\int_C P\,dx + Q\,dy = \int\int_S \left(\frac{\partial Q}{\partial x}\right) - \left(\frac{\partial P}{\partial y}\right) dx\,dy$$

According to this theorem, you can convert a line integral to a surface integral and vice versa. How? The theorem tries to say that what happens (or what result we obtain) along the boundaries of a surface is equal to what happens inside the surface.

So we can basically say that the equation on the right hand side is the surface integral version of what is on the left side.

Here, P and Q are the magnitudes of a vector field on the x-axis and y-axis respectively, as we saw previously.

Use Green’s Theorem **only if**:

- C is closed
- C is simple (no self intersection)
- Region has no holes
- P and Q have continuous partial derivatives

Stoke’s Theorem

According to Stoke’s theorem, the result obtained by going along the curve of the surface is equal to the curl of the vector field present in the surface. So basically,

$$\int_C \vec{F} \cdot dr = \int\int_S (\nabla \times F) \cdot dS$$

The **Green’s theorem** is a special version of the **Stoke’s theorem** where Green’s theorem considers a surface on the xy-plane.

Volume Integral

Now, we expand more into volume! Volume integral is just like line or surface integral, but it is for the volume of the region. So we include the third axis, which is the z-axis as well.

$$\int\int\int_V F(x,y,z)\,dx\,dy\,dz$$

Gauss's Divergence Theorem

According to this theorem:

$$\int\int\int_S \operatorname{div}(F)\,dV = \int\int_S F \cdot dr$$