

# Convergence and Divergence

**Convergence:** It is the phenomenon of a sequence or series saturating at some point. The terms of sequence goes closer to a constant, while the series starts saturating, and barely grows. Think of it like slowing down.

**Divergence:** It is the phenomenon of a sequence or series growing continuously. The terms of sequence goes further from 0 towards infinity, while the series starts growing much faster, tending towards infinity. Think of it like speeding up.

## Test for Convergence of a Sequence

So we know a sequence converges if its terms tends to a constant as  $n$  increases. It can be written as:

$$\lim_{n \rightarrow \infty} u_n = k$$

If  $\lim_{n \rightarrow \infty} u_n = \pm\infty$  then it means the sequence diverges.

## Test for Convergence of a Series

So we know that a series converges if its sum starts saturating as  $n$  increases. It can be written as:

$$\lim_{n \rightarrow \infty} u_n = k$$

for some real number constant  $k$ . If  $k = \infty$  then the series diverges.

## Convergence Tests

There are many more ways to test if a series converges, and each one of them is helpful for different kinds of series. It is important to know what test to pick for what kinds of series.

### Integral Test

Let  $\sum u_n$  be a series of positive terms. Let  $a_n = f(n)$  be a **continuous, positive and decreasing** function for each  $x \in [N, \infty)$ . Then the series  $\sum_{n=N}^{\infty} u_n$  and the integral  $\int_N^{\infty} f(x) dx$  converge or diverge together.

So basically, to do this test:

- Check if the function is:
  - continuous
  - positive
  - decreasing for all values given.
- Write the function in terms of  $x$
- Integrate  $f(x)$  with respect to  $x$  from  $N$  to  $\infty$ , the values that satisfies the above three conditions.
- If the integral results in infinity, the series diverges, otherwise (if a constant value), it converges.

## Comparison Test

Let  $\sum u_n$  and  $\sum v_n$  be two series. Then the series  $u_n$  is:

- Convergent if:
  - $u_n < v_n$
  - $v_n$  is convergent
- Divergent if:
  - $u_n > v_n$
  - $v_n$  is divergent

The real problem here lies with choosing the sequence for  $v_n$  so that problem becomes easy to solve.

## Ratio Test

According to the ratio test, we take two consecutive terms of the sequence, and check the values of their ratio. Consider a sequence  $u_n$ . let a constant  $\lambda$  be:

$$\lambda = \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n}$$

Now, if:

- $\lambda < 1$  the series converges, because the higher term is lesser than the lower term
- $\lambda > 1$  the series diverges, because the higher term is greater than the lower term
- $\lambda = 1$  the test fails, because it assumes the consecutive terms are equal

## Root Test

The root test is the same as ratio test, except that

$$\lambda = \lim_{n \rightarrow \infty} (u_n)^{\frac{1}{n}}$$