10.015 Physical World
Week 1

Concept of Vector in Physics and Engineering

Learning Objectives

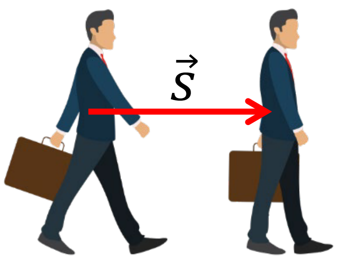
- Define vector and explain the use of vector in kinematics and dynamics in Physics.
- Perform simple vector algebra for vector addition and subtraction.
- Define Cartesian coordinate system and Polar coordinate system and represent vector appropriately in both the systems.
- Define and perform vector dot product.
- Define and perform vector cross product.

Readings

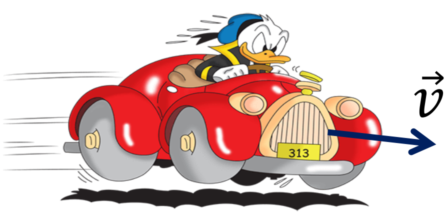
1. Classical Mechanics, Chapter 3.
2. University Physics with Modern Physics, Chapter 1.
3. Physics for Scientist and Engineer, Chapter 3.

Why is Vector Important in Physics?

- A scalar quantity is completely specified by a magnitude with an appropriate unit and has *no direction*. For instance, temperature and air pressure.
- However, many physical quantities possess *both magnitudes and directions*, and so are most conveniently represented as vectors. For example,



The displacement (change of position) of the person at 2 instances can be represented using a vector $\vec{s}$ pointing from the initial to the final position. $|\vec{s}|$ tells us how far the 2 positions are apart, i.e distance.

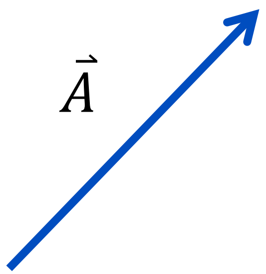


The velocity (rate of change of position) of the car can be represented using a vector $\vec{v}$ pointing in the direction of motion. $|\vec{v}|$ tells us how fast the car is moving at the moment, i.e. speed

The force applied on the foot ball can be represented using a vector $\vec{F}$ pointing in the direction it is kicked. $|\vec{F}|$ tells us how hard the ball is kicked.

Vector

- A vector is a quantity that has both magnitude and direction.
- A vector can be geometrically represented using an arrow, with the length of the arrow for the magnitude and the arrow-head pointing in the direction. (Note: In term 2, you will learn to treat vector as a column matrix, satisfying some abstract properties.)



- The arrow represents a vector denoted $\vec{A}$:
  1. Magnitude is $|\vec{A}| = A$
  2. Direction is $\vec{A}/A = \hat{A}$ (pronounced as "A hat", is a unit vector of magnitude 1, parallel to $\vec{A}$.)
- Thus, mathematically $\vec{A} = A\hat{A}$. i.e. a vector is the product of magnitude and directional unit vector.

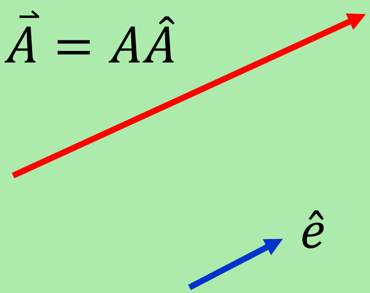
- 2 vectors are identical if and only if they have the same magnitude and direction. i.e. $\vec{A} = \vec{B} \Leftrightarrow |\vec{A}| = |\vec{B}|$ and $\hat{A} = \hat{B}$.

Example : A vector has magnitude of 2, and pointing in the direction of unit vector $\hat{e}$. What is the mathematical expression for the vector?

Ans : The vector is a product of magnitude and unit vector, i.e. $2\hat{e}$.

(note : unit vector is defined with reference to a coordinate system set up, more later.)

$$\vec{A} = A\hat{A}$$



$\hat{e}$

$$A = 2 \; and \; \hat{A} = \hat{e}$$

Concept Question 1: Vector Quantity

Which of the following physical quantities are vectors? (can be more than one correct option)

A. Temperature

B. Mass

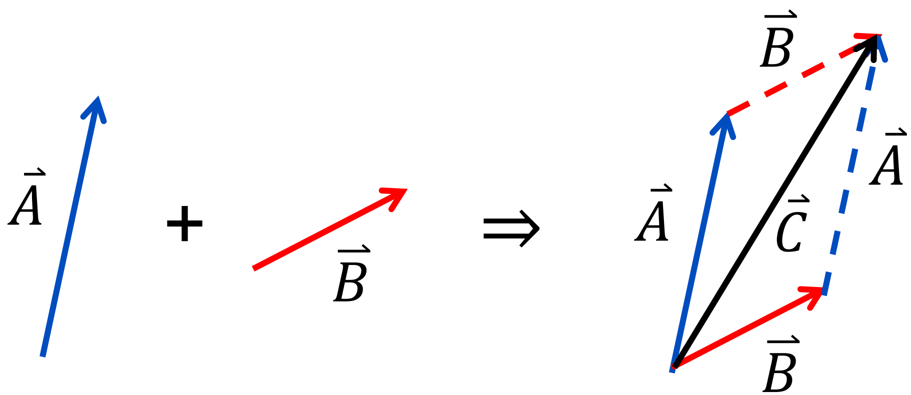
C. Torque

D. Momentum

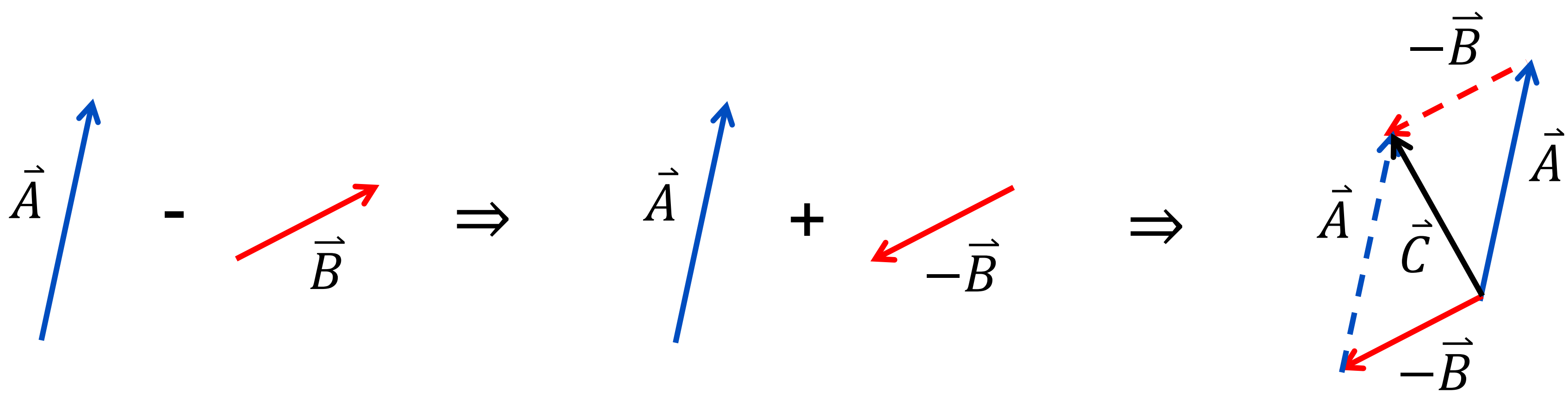
E. Energy

Vector Algebra

- 2 vectors can be added together to form a third vector.
- Vector summation : $\vec{A} + \vec{B} = \vec{C}$
  1. Parallel transport the 2 vectors $\vec{A}$ and $\vec{B}$ until their tails coincide, while keeping the magnitudes and directions of the vectors unchanged.
  2. Construct a parallelogram and the diagonal vector $\vec{C}$ originated from the tail ends is the sum of $\vec{A}$ and $\vec{B}$.

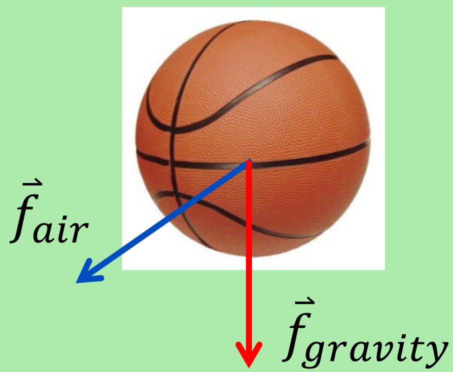


- A vector can be subtracted from another vector too to form a third vector.
- Vector subtraction is just a special case of vector summation.
- Notice that 2 vectors $\vec{A}$ and $-\vec{A}$ differ by a negative sign are just opposite in directions while the magnitudes are the same.
- Vector subtraction : $\vec{A} - \vec{B} = \vec{A} + (-\vec{B}) = \vec{C}$,
  1. Construct new vector $-\vec{B}$ opposite in direction to $\vec{B}$.
  2. Keeping the magnitudes and directions of $\vec{A}$ and $-\vec{B}$ unchanged, parallel transport them until the tails coincide.
  3. Construct a parallelogram and the diagonal vector $\vec{C}$ originated from the tail ends is the sum of $\vec{A}$ and $-\vec{B}$.

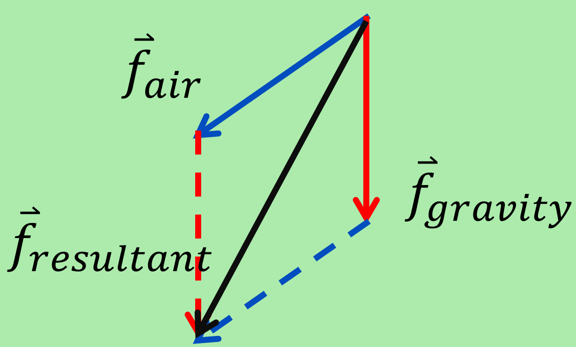


- Vector algebra is important and useful in physics, for example, it helps to find out the total net resultant force on an object when multiple forces are applied simultaneously.

Example : A basketball is launched into the air and experiences air resistance and gravitational force as shown. Determine the resultant force.



Ans : The net resultant force is the vector sum of the 2 forces acting on the ball, $\vec{f}_{resultant} = \vec{f}_{air} + \vec{f}_{gravity}$ and can be constructed as shown.



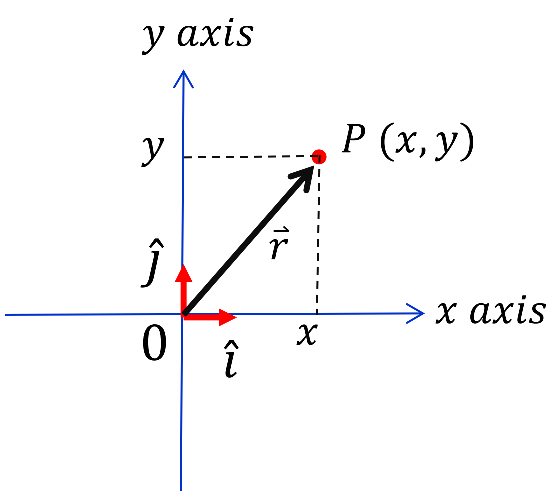
Concept Question 2: Vector Algebra

Is it possible for the sum of 2 vectors of equal length to produce another vector of equal length?

A. Yes, if the 2 vectors are 45˚ apart.
B. Yes, if the 2 vectors are 60˚ apart.
C. Yes, if the 2 vectors are 90˚ apart.
D. Yes, if the 2 vectors are 120˚ apart.
E. No, the sum of 2 vectors is always larger in magnitude than each of the 2 vectors.

Coordinate System – Cartesian

- Coordinate system is a reference that uses a set of numbers (coordinate) to determine (label) position in space. It has a reference position (origin O) from which the displacement to any other position in space is measured.
- A coordinate system also possesses a set of well-defined unit vectors (directions) associated with each of the coordinates.
- One of the most used coordinate system is the Cartesian coordinate system. On a 2D plane, there are 2 perpendicular axes, usually denoted as x-axis and y-axis.



- There are 2-unit vectors associate to each of the axes in the positive directions, usually denoted as $\hat{\imath}$ and $\hat{\jmath}$.
- Coordinate of point P is (x, y), i.e. x units along the x-axis and y units along the y-axis.
- Position vector of P is the displacement vector $\vec{r}$ from the origin O, where

$$\vec{r} = \overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath}$$

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- It is convenient to mathematically express a vector in terms of a set of unit vectors defined in a coordinate system (vectorial representation) for the computation of vector algebra.
- When 2 vectors are summed (subtracted), only the components associated to the same unit vector are summed (subtracted). For example,

$$\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath}$$

$$\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath}$$

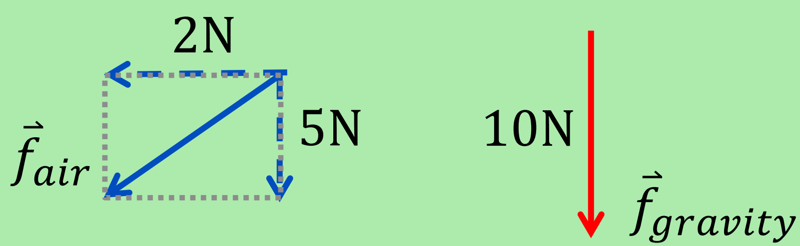
$$\Rightarrow \vec{r}_1 \pm \vec{r}_2 = (x_1 \pm x_2)\hat{\imath} + (y_1 \pm y_2)\hat{\jmath}$$

- The magnitude of a vector expressed in terms of unit vector perpendicular to each other can be calculated easily with Pythagoras rule, i.e.

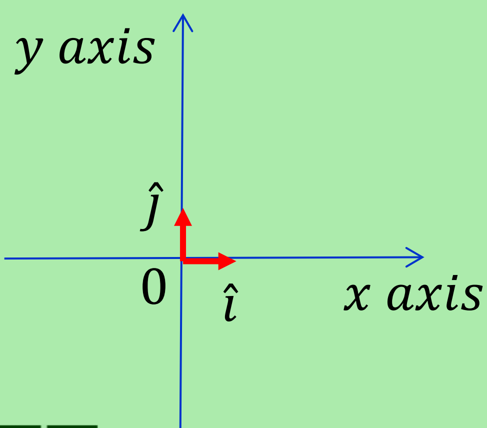
$$\vec{r} = x\hat{\imath} + y\hat{\jmath} \;\Rightarrow\; r = |\vec{r}| = \sqrt{x^2 + y^2}$$

- Cartesian coordinate system is very useful when studying linear motion (motion in a straight line).

Example : A basketball is launched into the air and experiences air resistance and gravitational force. The air resistance has component of 2N leftward and 5N downward, while the gravitational force is 10N downward. Express the forces in an appropriate Cartesian coordinate system and calculate the resultant force accordingly.



Ans : Define a Cartesian coordinate system with positive *x-axis* rightward and positive *y-axis* upward. Then



$$\vec{f}_{air} = -2N\hat{\imath} - 5N\hat{\jmath}$$

$$\vec{f}_{gravity} = -10N\hat{\jmath}$$

$$\vec{f}_{resultant} = \vec{f}_{air} + \vec{f}_{gravity} = -2N\hat{\imath} - 15N\hat{\jmath}$$

$$\left|\vec{f}_{resultant}\right| = \sqrt{(-2)^2 + (-15)^2} \approx 15.13N$$

Concept Question 3: Cartesian Coordinate System

Given $\vec{A} = 3\hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = -2\hat{\imath} - 4\hat{\jmath}$. Determine $\vec{C}$ such that $2\vec{A} - \vec{B} + 4\vec{C} = \vec{0}$ where $\vec{0}$ is a null vector of 0 magnitude.

A. $\vec{C} = -2\hat{\imath} - 2\hat{\jmath}$

B. $\vec{C} = 2\hat{\imath} + 2\hat{\jmath}$

C. $\vec{C} = -8\hat{\imath} - 8\hat{\jmath}$

D. $\vec{C} = 8\hat{\imath} + 8\hat{\jmath}$

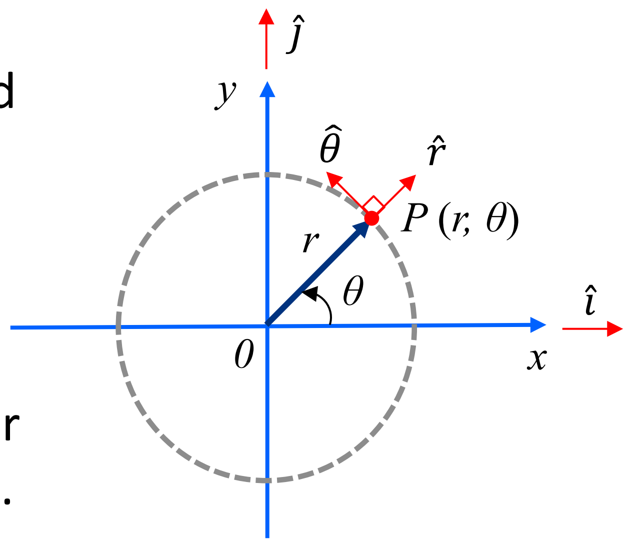
E. $\vec{C} = -2\hat{\imath} + 2\hat{\jmath}$

Case Problem 0: Vector Algebra

The magnitudes of 2 force vectors $\vec{A}$ and $\vec{B}$ are related as $|\vec{B}| = 2|\vec{A}|$. The resultant force $\vec{C} = \vec{A} + \vec{B}$ has magnitude $|\vec{C}| = 10\ N$ and is making an angle 30° with respect to $\vec{A}$. From the given information, find the magnitudes of $\vec{A}$ and $\vec{B}$ and the angle between them. Hint: Draw out the vectors.

Coordinate System - Polar

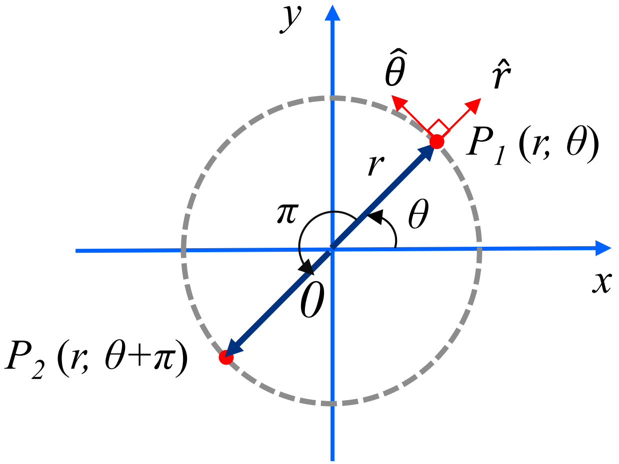
- Cartesian coordinate system is not the only reference system to measure/label position in space.
- The position of point P can also be determined by the distance r from the origin O, and the angle θ measured with respect to *x-axis* (*conventionally anti-clock-wise angle is positive*), i.e. coordinate (r, θ) is equivalent to (x,y).
- The coordinate system using distance r and angle θ is called Polar coordinate system, it is mostly adopted for studying of planar motion, particularly circular motion.



- There are 2-unit vectors $\hat{r}$ and $\hat{\theta}$ associated with the coordinate r and θ. $\hat{r}$ is radially pointing away from the origin, while $\hat{\theta}$ is tangential to the circular circumference (*in the direction of anti-clock-wise sense*) of radius r.
- It is important to note that unit vectors $\hat{r}$ and $\hat{\theta}$ in Polar coordinate system is a function of angle θ. Thus, they are position dependent, unlike $\hat{\imath}$ and $\hat{\jmath}$.

Concept Question 4: Cartesian Coordinate System

The coordinate of point P_1 is (r, θ) and that of P_2 is $(r, \theta + \pi)$. If the 2-unit vectors at P_1 are given by $\hat{r}_1 = \hat{r}$ and $\hat{\theta}_1 = \hat{\theta}$, the corresponding radial and tangential unit vectors at P_2 are? i.e. express $\hat{r}_2$ and $\hat{\theta}_2$ in terms of $\hat{r}$ and $\hat{\theta}$.



A. $\hat{r}$ and $\hat{\theta}$

B. $-\hat{r}$ and $\hat{\theta}$

C. $\hat{r}$ and $-\hat{\theta}$

D. $-\hat{r}$ and $-\hat{\theta}$

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Cartesian and Polar Coordinate System

- The point P coordinate is given equivalently by
  1. Cartesian coordinate (x, y)
  2. Polar coordinate (r, θ)

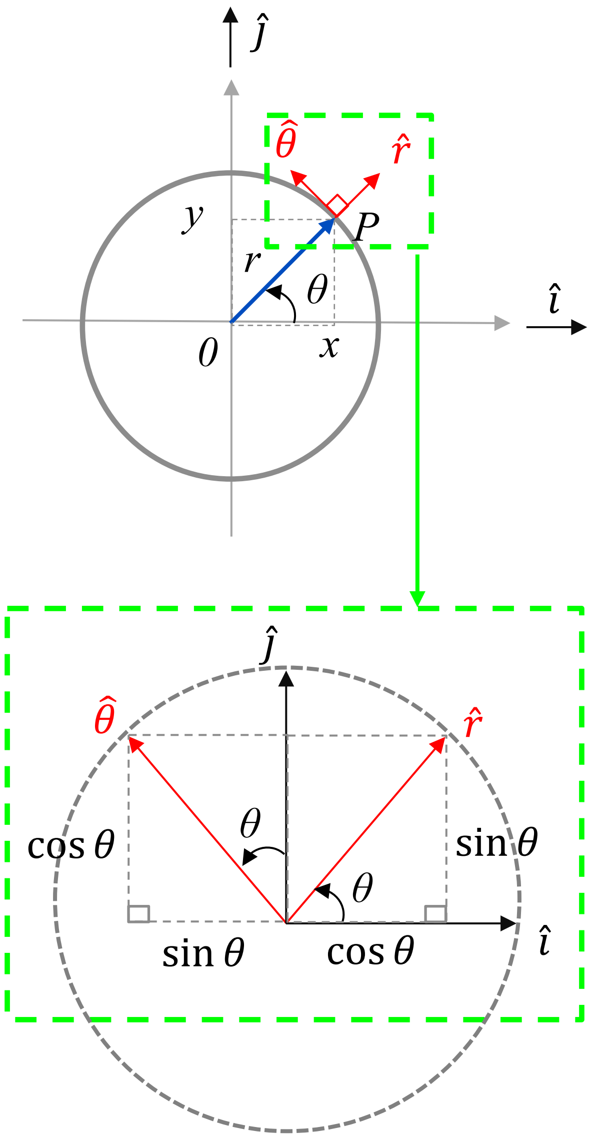
$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

- Thus, the position vector $\vec{r}$ of point P is given by
  1. Cartesian coordinate $\vec{r} = x\hat{\imath} + y\hat{\jmath}$
  2. Polar coordinate $\vec{r} = r\hat{r}$

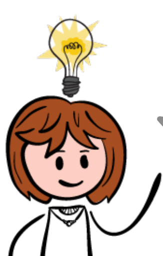
- Since θ is measured with respect to the x-$axis$ as defined, the unit vectors at P of the 2 coordinate systems are related as followed

$$\boxed{\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}}$$

$$\boxed{\hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}}$$



Unit circle of radius 1



Take note $\hat{r}$ and $\hat{\theta}$ are position dependent (functions of θ)!

Case Problem 1: Conversion of Coordinate Systems

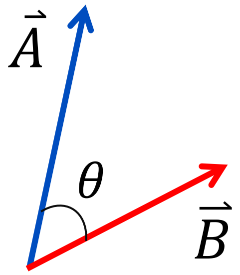
1. Express the following Cartesian coordinate (x,y) in terms of polar coordinate (r,θ). r is the magnitude of displacement from the origin, and θ is the angle measured with respect to *x-axis* (anti-clockwise angular displacement is considered positive).
   a. (3,4)
   b. (3,-4)

2. Express the corresponding polar coordinate system unit vectors $\hat{r}$ and $\hat{\theta}$ in terms of the Cartesian coordinate system unit vectors $\hat{\imath}$ and $\hat{\jmath}$ at the following Cartesian coordinates.
   a. (3,4)
   b. (3,-4)

Vector Dot Product (Scalar Product)

- Vector dot product of 2 vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \cdot \vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = AB \cos \theta$$



where θ is the angle (note: the angle must be no larger than π radian) between the 2 vectors.

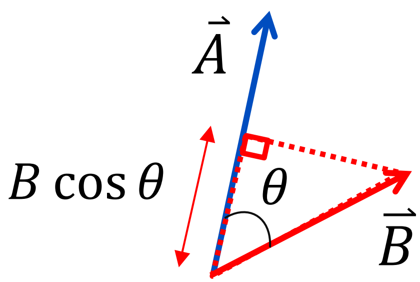
- Dot product of 2 vectors is a pure scalar (no direction).
- Dot product is commutative, i.e. $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Dot product is distributive, i.e.

$$\left(\vec{A} + \vec{B}\right) \cdot \left(\vec{C} + \vec{D}\right) = \vec{A} \cdot \vec{C} + \vec{A} \cdot \vec{D} + \vec{B} \cdot \vec{C} + \vec{B} \cdot \vec{D}$$

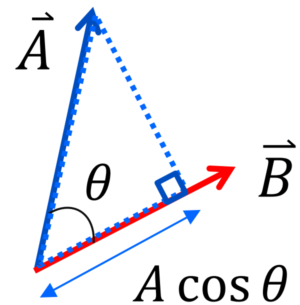
Geometrical Interpretation of Vector Dot Product

- There are 2 equivalent interpretations for the dot product:

1. $\vec{A} \cdot \vec{B} = AB\cos\theta = A\underbrace{(B\cos\theta)}_{\text{Component of } \vec{B} \text{ in the direction of } \vec{A}}$



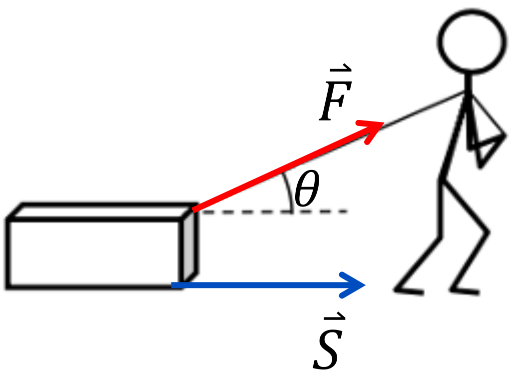
2. $\vec{A} \cdot \vec{B} = AB\cos\theta = BA\cos\theta = B\underbrace{(A\cos\theta)}_{\text{Component of } \vec{A} \text{ in the direction of } \vec{B}}$



- So, dot product $\vec{A} \cdot \vec{B}$ is a mathematical operation that multiplies the magnitude of a vector $\vec{A}$ to the projection of $\vec{B}$ in the direction of $\vec{A}$, and vice versa.

Why is Dot Product Important in Physics

- There are many cases when the dot product between 2 physical vector quantities carries important physical meaning. For example,

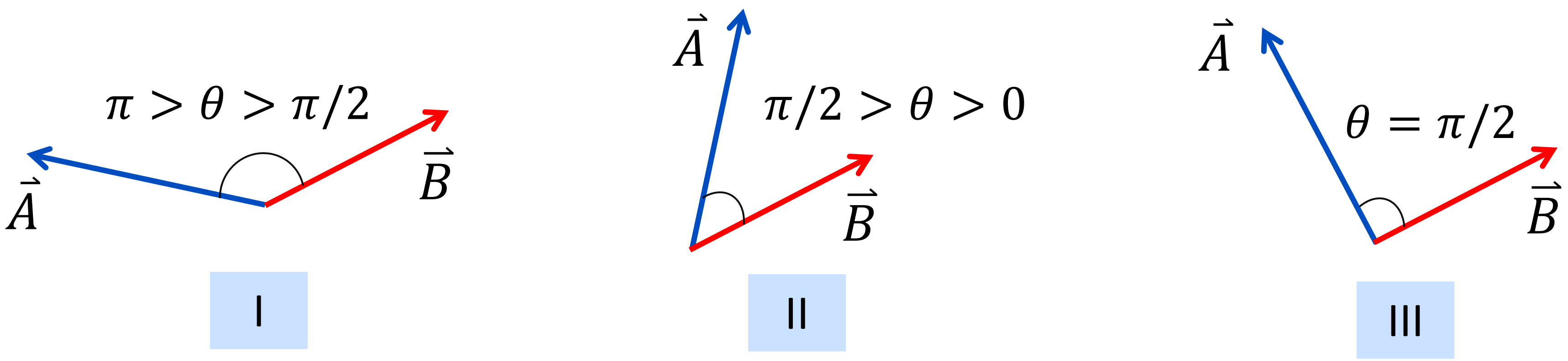


- When determining work done W by a force $\vec{F}$ (amount of mechanical energy converted by the force), only the component of the force in the direction of displacement $\vec{S}$ matters. i.e. $W = \vec{F} \cdot \vec{S}$ (more about energy and work done in week 8).
- Similarly, the power (how fast the work done by a constant force) is determine via a dot product between the force and the velocity of the object, $\vec{v}$, i.e. $P = \vec{F} \cdot \vec{v}$

- The dot product is a very convenient and important mathematical operation to find out how much of a vector quantity is in the direction of interest.

Concept Question 5: Vector Dot Product

Rank the result of the dot product between 2 vectors of the following configurations from large to small.



A. I > II > III

B. I > III > II

C. II > I > III

D. II > III > I

Case Problem 2: Dot Product of Unit Vectors

Write down the answers to the following dot products of the unit vectors in Cartesian coordinate system and Polar coordinate system.

I. $\hat{\imath} \cdot \hat{\imath} =$

II. $\hat{\imath} \cdot \hat{\jmath} =$

III. $\hat{\jmath} \cdot \hat{\imath} =$

IV. $\hat{r} \cdot \hat{r} =$

V. $\hat{r} \cdot \hat{\theta} =$

VI. $\hat{\theta} \cdot \hat{\theta} =$

Dot Product of Any 2 Vectors

- If 2 vectors are expressed in terms of the unit vectors of a coordinate system that are perpendicular to one another, the dot product between 2 vectors is simply given by multiplying the components associated to the same unit vectors together, then sum up all the terms.
- In Cartesian coordinate system,

$$\vec{u}_1 = a_1\hat{\imath} + b_1\hat{\jmath} \qquad \vec{u}_2 = a_2\hat{\imath} + b_2\hat{\jmath}$$

$$\vec{u}_1 \cdot \vec{u}_2 = (a_1\hat{\imath} + b_1\hat{\jmath}) \cdot (a_2\hat{\imath} + b_2\hat{\jmath})$$
$$\Rightarrow \vec{u}_1 \cdot \vec{u}_2 = a_1a_2(\hat{\imath}\cdot\hat{\imath}) + a_1b_2(\hat{\imath}\cdot\hat{\jmath}) + b_1a_2(\hat{\jmath}\cdot\hat{\imath}) + b_1b_2(\hat{\jmath}\cdot\hat{\jmath})$$
$$\Rightarrow \vec{u}_1 \cdot \vec{u}_2 = a_1a_2 + b_1b_2$$

- In Polar coordinate system,

$$\vec{v}_1 = c_1\hat{r} + d_1\hat{\theta} \qquad \vec{v}_2 = c_2\hat{r} + d_2\hat{\theta}$$

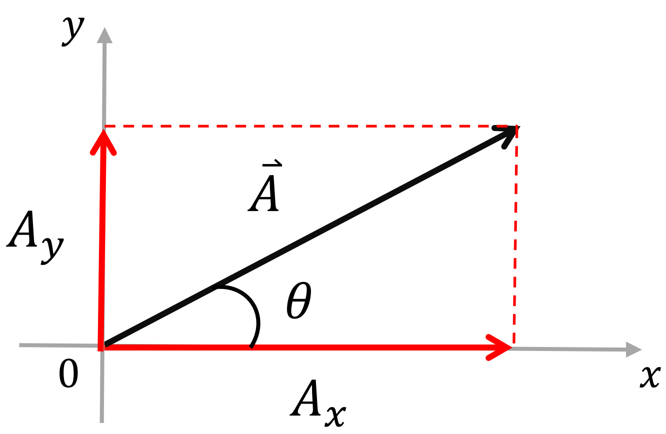
$$\vec{v}_1 \cdot \vec{v}_2 = \left(c_1\hat{r} + d_1\hat{\theta}\right) \cdot \left(c_2\hat{r} + d_2\hat{\theta}\right)$$
$$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = c_1c_2(\hat{r}\cdot\hat{r}) + c_1d_2(\hat{r}\cdot\hat{\theta}) + d_1c_2(\hat{\theta}\cdot\hat{r}) + d_1d_2(\hat{\theta}\cdot\hat{\theta})$$
$$\Rightarrow \vec{v}_1 \cdot \vec{v}_2 = c_1c_2 + d_1d_2$$

Vector Decomposition in Cartesian Coordinate

- Any vector can be decomposed into the sum of its component vectors in the directions of each axis (unit vectors).

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

$$\begin{cases} \Rightarrow A_x = \vec{A}\cdot\hat{\imath} = A\cos\theta \\ \Rightarrow A_y = \vec{A}\cdot\hat{\jmath} = A\cos\left(\dfrac{\pi}{2} - \theta\right) = A\sin\theta \end{cases}$$

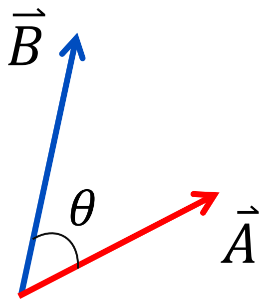


- Recall the interpretation of dot product operation, the component of a vector $\vec{A}$ in the direction of any arbitrary vector $\vec{B}$, is given by $\frac{\vec{A}\cdot\vec{B}}{B} = \vec{A}\cdot\hat{B}$.

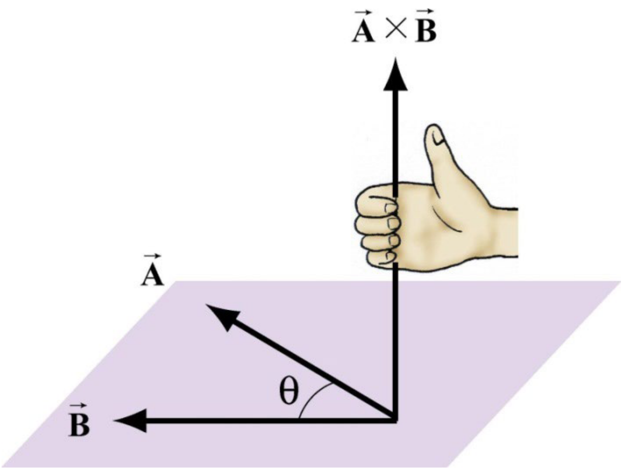
Vector Cross Product

- Vector cross product of 2 vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A} \times \vec{B}$ is defined as

$$\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$$



where θ is the angle (note: the angle must be no larger than π *radian*) between the 2 vectors; and $\hat{n}$ is a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$, determined by applying Right Hand Rule.



Right Hand Rule

1. Rest the edge of stretched right palm vertically on the plane containing both $\vec{A}$ and $\vec{B}$
2. Align the stretched palm in the direction of 1st vector $\vec{A}$
3. Close the palm towards the 2nd vector $\vec{B}$ over the angle θ
4. The thumb is pointing in the direction of $\hat{n}$

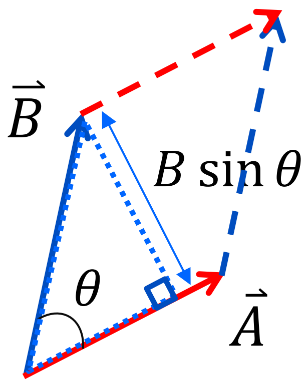
- Cross product of 2 vectors is a vector.
- Cross product is NOT commutative, i.e. $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$, instead $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$.
- Cross product is distributive, i.e.

$$(\vec{A} + \vec{B}) \times (\vec{C} + \vec{D}) = \vec{A} \times \vec{C} + \vec{A} \times \vec{D} + \vec{B} \times \vec{C} + \vec{B} \times \vec{D}$$

Geometrical Interpretation of Vector Cross Product

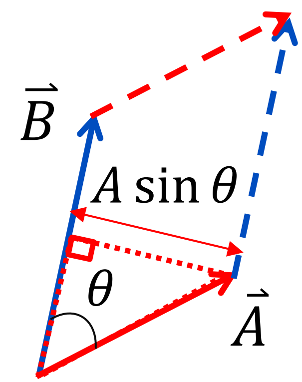
- There are 2 equivalent interpretations for the magnitude of a cross product:

1. $\left|\vec{A}\times\vec{B}\right| = AB\sin\theta = A\underbrace{(B\sin\theta)}_{\text{Component of } \vec{B} \text{ perpendicular to } \vec{A}}$



Component of $\vec{B}$ perpendicular to $\vec{A}$

2. $\left|\vec{A}\times\vec{B}\right| = AB\sin\theta = BA\sin\theta = B\underbrace{(A\sin\theta)}_{\text{Component of } \vec{A} \text{ perpendicular to } \vec{B}}$

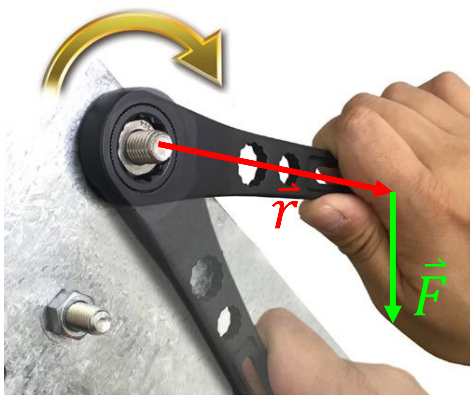


Component of $\vec{A}$ perpendicular to $\vec{B}$

- So, the magnitude of the cross product is the area of the parallelogram formed by the 2 vectors.
- When 2 vectors are parallel (or anti-parallel) to each other, they form a line and there is no area for the parallelogram. Thus, the cross product is 0.

Why is Cross Product Important in Physics

- There are many physical quantities defined by the cross product of 2 vector quantities. For example,

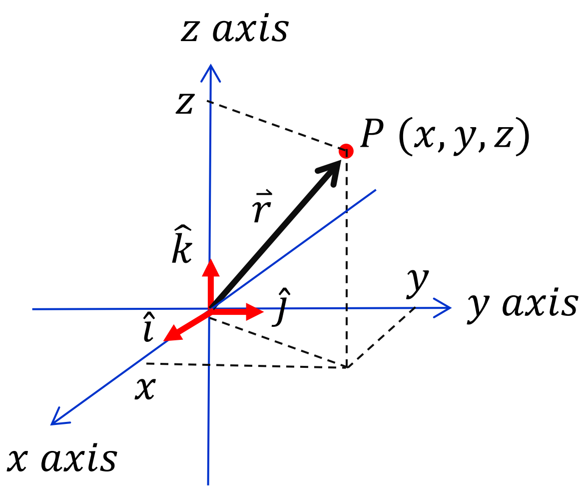


To determine how effective it is to rotate a wrench about a pivot by an applied force $\vec{F}$, we can calculate the torque $\vec{\tau} = \vec{r} \times \vec{F}$ that generates rotational motion (more about torque in week 5).

- There are many other physical quantities defined using cross product such as angular momentum (week 10), Lorentz force on charge particle moving in magnetic field, Bio-Savart's Law of magnetostatic (Technological World, term 2).

Coordinate System – Cartesian Revisit …

- To properly determine the position in 3D space, 1 more axis (called *z-axis*) is added to the *x-axis* and *y-axis*.



- There are 3-unit vectors parallel to each of the axes in the positive directions, usually denoted as $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.
- The 3-unit vectors obey cyclic relation $\hat{\imath} \times \hat{\jmath} = \hat{k}$.
- Coordinate of point P is (x,y,z), i.e. x units along the *x-axis*, y units along the *y-axis*, and z units along the *z-axis*
- Position vector of P is the displacement vector $\vec{r}$ from the origin O, where

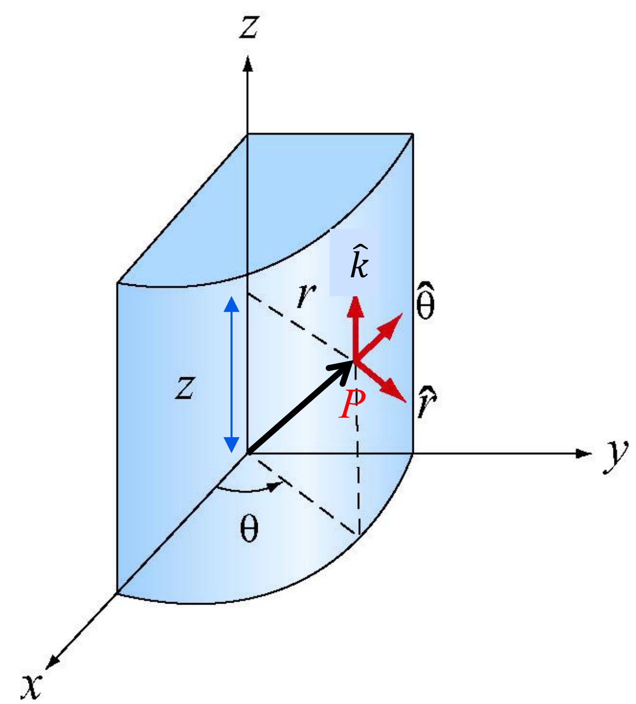
$$\vec{r} = \overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

Cylindrical Coordinate System

- A Polar coordinate system can be extended to 3D space too by including an additional axis (*z-axis*) perpendicular to the horizontal plane. This is the cylindrical coordinate system.
- In a cylindrical coordinate system at the position of a point P,

  ✓ Coordinate = (r, θ, z)

  ✓ Unit vectors = $(\hat{r}, \hat{\theta}, \hat{k})$



- $\hat{r}, \hat{\theta}$ are defined the same as 2D Polar coordinate system. Thus, they are also functions of angular displacement θ.

- $\hat{r}, \hat{\theta}, \hat{k}$ are mutually perpendicular to each other and they obey the cyclic relation $\hat{r} \times \hat{\theta} = \hat{k}$.

Case Problem 3: Cross Product of Unit Vectors

Write down the answers to the following cross products of the unit vectors in Cartesian coordinate system and cylindrical coordinate system.

I. $\hat{\imath} \times \hat{\imath} =$

II. $\hat{\imath} \times \hat{\jmath} =$

III. $\hat{\jmath} \times \hat{k} =$

IV. $\hat{k} \times \hat{\imath} =$

V. $\hat{r} \times \hat{r} =$

VI. $\hat{r} \times \hat{\theta} =$

VII. $\hat{\theta} \times \hat{k} =$

VIII. $\hat{k} \times \hat{r} =$

Cross Product of Any 2 Vectors

- If 2 vectors are expressed in terms of the unit vectors of a coordinate system that are perpendicular to one another, the cross product between 2 vectors is simply given by multiplying the components associated to the unit vectors perpendicular to one another, but the result is still a vector.
- In Cartesian coordinate system,

$$\vec{u}_1 = a_1\hat{\imath} + b_1\hat{\jmath} \qquad \vec{u}_2 = a_2\hat{\imath} + b_2\hat{\jmath}$$

$$\boxed{= -(\hat{\imath} \times \hat{\jmath}) = -\hat{k}}$$

$$\vec{u}_1 \times \vec{u}_2 = (a_1\hat{\imath} + b_1\hat{\jmath}) \times (a_2\hat{\imath} + b_2\hat{\jmath})$$

$$\Rightarrow \vec{u}_1 \times \vec{u}_2 = a_1a_2(\hat{\imath} \times \hat{\imath}) + a_1b_2(\hat{\imath} \times \hat{\jmath}) + b_1a_2(\hat{\jmath} \times \hat{\imath}) + b_1b_2(\hat{\jmath} \times \hat{\jmath})$$

$$\Rightarrow \vec{u}_1 \times \vec{u}_2 = (a_1b_2 - b_1a_2)\hat{k}$$

- In Polar coordinate system,

$$\vec{v}_1 = c_1\hat{r} + d_1\hat{\theta} \qquad \vec{v}_2 = c_2\hat{r} + d_2\hat{\theta}$$

$$\boxed{= -(\hat{r} \times \hat{\theta}) = -\hat{k}}$$

$$\vec{v}_1 \times \vec{v}_2 = \left(c_1\hat{r} + d_1\hat{\theta}\right) \times \left(c_2\hat{r} + d_2\hat{\theta}\right)$$

$$\Rightarrow \vec{v}_1 \times \vec{v}_2 = c_1c_2(\hat{r} \times \hat{r}) + c_1d_2(\hat{r} \times \hat{\theta}) + d_1c_2(\hat{\theta} \times \hat{r}) + d_1d_2(\hat{\theta} \times \hat{\theta})$$

$$\Rightarrow \vec{v}_1 \times \vec{v}_2 = (c_1d_2 - d_1c_2)\hat{k}$$