CSE 215 - Homework 4

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Problem Set 1

(a) **Basis Step**: Prove P(1) is true.

$1^3 = 1$
$[\frac{1(1+1)}{2}]^2 = \frac{2}{2}^2 = 1^2 = 1$

1 = 1, therefore this statement holds for P(1)

Induction Step: Assume P(k) is true for some k≥1. Prove P(k+1) is true.

$(1^3 + 2^3 + ... + k^3) + (k+1)^3$
$= [\frac{k(k+1)}{2}]^2 + (k+1)^3$ (∵ P(k) is true)
$= \frac{k(k+1)}{2}\frac{k(k+1)}{2} + k^3 + 3k^2 + 3k + 1$ (∵ Expansion)
$= \frac{k^2(k^2+2k+1)}{4} + \frac{4k^3+12k^2+12k+4}{4}$
$= \frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4}$
$= \frac{k^4+6k^3+13k^2+12k+4}{4}$
$*$ Find the roots of $k^4 + 6k^3 + 13k^2 + 12k + 4$ with graphing $*$
$= \frac{(k+1)^2(k+2)^2}{4}$ (∵ Factoring the numerator)
$= \frac{(k+1)(k+2)}{2}\frac{(k+1)(k+2)}{2}$ (∵ Factoring Out Perfect Squares)
$= [\frac{(k+1)(k+2)}{4}]^2$ (∵ Combining the 2 Fractions)

∴ P(k + 1) is true.

(b) **Basis Step**: Prove P(0) is true.

$\sum_{i=1}^{0+1} i \times 2^i = 1 \times 2^1 = 2 = 2$
$0 \times 2^{0+2} + 2 = 0 + 2 = 2$

$2 = 2$, therefore this statement holds true fore P(0).

Induction Step: Assume P(k) is true for some k≥0. Prove P(k+1) is true.

$\sum_{i=1}^{k+1} i \times 2^i$
$= (\sum_{i=1}^{k+1} i \times 2^i) + (k+2) \times 2^{k+2}$ (Taking Out the k + 2 term)
$= k \times 2^{k+2} + 2 + (k+2) \times 2^{k+2}$ ($\because$ P(k) is true)
$= (2k+2)2^{k+2} + 2$ ($\because$ Combining Like Terms)
$= (k+1)(2)(2^{k+2}) + 2$ ($\because$ Factoring Out 2)
$= (k+1)(2^{k+1+2}) + 2$ ($\because$ Exponent Rule)

$\therefore$ P(k + 1) is true.

(c) **Basis Step**: Prove P(0) is true.

$\prod_{i=0}^{0} (\frac{1}{2i+1})(\frac{1}{2i+2}) = (\frac{1}{1})(\frac{1}{2}) = \frac{1}{2}$
$\frac{1}{(2(0)+2)!} = \frac{1}{2}$

$\frac{1}{2} = \frac{1}{2}$, therefore this statement holds true for P(0).

Induction Step: Assume P(k) is true for some k≥0. Prove P(k+1) is true.

$\prod_{i=0}^{k+1} (\frac{1}{2i+1})(\frac{1}{2i+2})$
$= (\prod_{i=0}^{k} (\frac{1}{2i+1})(\frac{1}{2i+2}))(\frac{1}{2(k+1)+1})(\frac{1}{2(k+1)+2})$ (Taking out the k + 1 term)
$= \frac{1}{(2k+2)!} \cdot \frac{1}{2k+3} \cdot \frac{1}{2k+4}$ ($\because$ P(k) is true)
$= \frac{1}{(2k+4)!}$ ($\because$ Merging the Denominators)
$= \frac{1}{(2k+2+2)!} = \frac{1}{(2(k+1)+2)!}$
$\therefore$ P(k + 1) is true.

(d) **Basis Step**: Prove P(0) is true.

$0^3 - 7(0) + 3 = 3$

3 is divisible by 3, therefore this statement holds true for P(0).

Induction Step: Assume P(k) is true for some k≥0. Prove P(k+1) is true.

$(k+1)^3 - 7(k+1) + 3$
$= (k+1)(k^2+2k+1) - 7k - 7 + 3$ ($\because$ Expanding $(k+1)^2$)
$= k^3+3k^2+3k+1-7k-7+3$ ($\because$ Expaniding $(k+1)(k^2+2k+1)$)
$= (k^3-7k+3) + (3k^2+3k+1-7)$ ($\because$ Reordering the Terms)
$= (k^3-7k+3) + 3(k^2+k-2)$ ($\because$ Factoring Out a 3)

(k^3-7k+3) is divisible by 3 because we assumed this in the induction step. Therefore it can be written as (3x) where x is an integer.

$= 3x + 3(k^2+k-2)$ ($\because$ P(k) is true)
$= 3(x+k^2+k-2)$ ($\because$ Factored Out a 3)
$= 3 \cdot$ integer
$\therefore$ P(k + 1) is true.

(e) **Basis Step**: Prove P(0) is true.

1 + 3(0) = 1
$4^0 = 1$
1 is $\leq$ 1, therefore the statement holds for P(0).

Induction Step: Assume P(k) is true for some k≥0. Prove P(k+1) is true.

$1 + 3(k+1)$
$= 1 + 3k + 3$ ($\because$ Distributive Property)
$\leq 4^k + 3$ ($\because$ P(k) is true)
$\leq 4^k + 3 \cdot 4^k$ ($\because 3 \leq 3 \cdot 4^k$ for all integer k ≥ 0)
$= 4 \cdot 4^k = 4^{k+1}$ ($\because$ Adding the 2 Terms and Exponent Rules)
$\therefore$ P(k + 1) is true.

(f) **Basis Step**: Prove P(2) is true.

$\sqrt{2} \approx 1.414$
$\frac{1}{\sqrt{1}}\frac{1}{\sqrt{2}} \approx 1.707$

$1.414 > 1.707$, thus this statement holds for P(2).

Induction Step: Assume P(k) is true for some k $\geq$ 2. Prove P(k+1) is true.

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$
$> \sqrt{k} + \frac{1}{\sqrt{k+1}}$ $\qquad$ ($\because$ P(k) is true)
$= \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}$ $\qquad$ ($\because$ Multiply top and bottom by $\frac{\sqrt{k+1}}{\sqrt{k+1}}$)
$= \frac{\sqrt{k^2+k}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}}$ $\qquad$ ($\because$ Merge the two radicals)
$= \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}}$ $\qquad$ ($\because$ Adding Fractions)
$\geq \frac{\sqrt{k^2}+1}{\sqrt{k+1}}$ $\qquad$ ($\because \sqrt{k^2+k} \geq \sqrt{k^2}, k \geq 2$)
$= \frac{k+1}{\sqrt{k+1}}$ $\qquad$ ($\because$ Simplifying $\sqrt{k^2}$)
$= \frac{(k+1)\sqrt{k+1}}{k+1}$ $\qquad$ ($\because$ Multiply top and bottom by $\frac{\sqrt{k+1}}{\sqrt{k+1}}$)
$= \sqrt{k+1}$ $\qquad$ ($\because$ Divide top and bottom by $k+1$)

$\therefore$ P(k + 1) is true.

(g) **Basis Step**: Prove P(2) is true.

$1 + 2x$
$(1+x)^2 = x^2 + 2x + 1$

$2x + 1 \leq x^2 + 2x + 1$ for all $x > -1$, thus the statement holds for P(2).

Induction Step: Assume P(k) is true for some k $\geq$ 2. Prove P(k+1) is true.
$(1+x)^{k+1}$
$= (1+x)(1+x)^k$ $\qquad$ ($\because$ Exponent Rules)

$\geq (1+x)(1+kx)$ ($\because$ P(k) is true)
$= 1 + x + kx + kx^2$ ($\because$ Expanding the Multiplication)
$= 1 + (1+k)x + kx^2$ ($\because$ Factoring Out x)
$\geq 1 + (1+k)x$ ($\because kx^2 \geq 0$)

$\therefore$ P(k + 1) is true.

(h) **Basis Step**: Prove P(2) is true.

$\frac{2(2-1)}{2} = \frac{2}{2} = 1$
Since with 2 people, you can only have 1 handshake, this statement holds for P(2).

Induction Step: Assume P(k) is true for some k $\geq$ 2. Prove P(k+1) is true.

If someone else arrives, he would have to shake k hands, therefore we can just add k to P(k), our induction hypothesis of $\frac{k(k-1)}{2}$.

$\frac{k(k-1)}{2} + k$
$= \frac{k(k-1)+2k}{2}$ ($\because$ Adding k to the fraction)
$= \frac{k^2-k+2k}{2}$ ($\because$ Distributing the k)
$= \frac{k^2+k}{2}$ ($\because$ Simplifying)
$= \frac{k(k+1)}{2}$ ($\because$ Factoring Out the k)

$\therefore$ P(k + 1) is true.

(i) **Basis Step**: Prove P(3) is true.

$\frac{3(3-3)}{2} = 0$
Since a triangle doesn't have any diagonals, this statement holds for P(3).

Induction Step: Assume P(k) is true for some k $\geq$ 3. Prove P(k+1) is true.

If another side is added, it would have add k - 1 more diagonals, therefore we can just add k - 1 to P(k), our induction hypothesis of $\frac{k(k-3)}{2}$.

$\frac{k(k-3)}{2} + k - 1$
$= \frac{k(k-3)+2(k-1)}{2}$ ($\because$ Adding k - 1 to the fraction)
$= \frac{k^2-3k+2k-2}{2}$ ($\because$ Distributing the k and 2)
$= \frac{k^2-k-2}{2}$ ($\because$ Simplifying)
$= \frac{(k-2)(k+1)}{2}$ ($\because$ Factoring Out the k)

$\therefore$ P(k + 1) is true.

(j) **Basis Step**: Prove P(1) is true.

1! = 1

Since 1 number can only be organized in 1 way, and 1! = 1, the statement holds for P(1).
Induction Step: Assume P(k) is true for some k $\geq$ 1. Prove P(k+1) is true.

If we have k object, through the induction hyposthesis, there are $k!$ possible permutations of the objects, now if we add 1 more object, we can find all of the possible permutations of $k+1$ by inserting the number into every possible slot (which is k + 1 possible slots).

$k!(k+1)$
$= (k+1)!$ ($\because$ By Definition)

$\therefore$ P(k + 1) is true.

Problem Set 2

(a) a. **Basis Step**: Prove P(0), P(1), P(2) is true.

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
Since $1 \leq 1$, $2 \leq 3$, and $3 \leq 9$, this statement holds for the basis cases.

Induction Step: Assume P(i) is true for some k $\geq$ 2 and any $i \in [0, k]$. Prove P(k+1) is true.

$h_{k+1} = h_k + h_{k-1} + h_{k-2}$ ($\because$ By Definition)
$\leq 3^k + 3^{k-1} + 3^{k-2}$ ($\because$ P(i) is true)
$= 9 \cdot 3^{k-2} + 3 \cdot 3^{k-2} + 3^{k-2}$ ($\because$ Exponent Rules)
$= (9 + 3 + 1) \cdot 3^{k-2}$ ($\because$ Factor Out 3^{k-3})
$\leq 27 \cdot 3^{k-2}$ ($\because 27 > 13$)
$= 3^k + 1$ ($\because$ Exponent Rules)

$\therefore$ P(k+1) is true.

b. **Basis Step**: Prove P(0), P(1), P(2), P(3) is true.

Since $s > 1.83$, $h_2 \leq s^2$, $h_3 \leq s^3$, and $h_4 \leq s^4$, this statement holds for the basis cases.

Induction Step: Assume P(i) is true for some k $\geq$ 2 and any $i \in [0, k]$. Prove P(k+1) is true.

$h_{k+1} = h_k + h_{k-1} + h_{k-2}$ ($\because$ By Definition)
$\leq s^k + s^{k-1} + s^{k-2}$ ($\because$ P(i) is true)
$= s^2 \cdot s^{k-2} + s \cdot s^{k-2} + s^{k-2}$ ($\because$ Exponent Rules)
$= (s^2 + s + 1) \cdot s^{k-2}$ ($\because$ Factor Out s^{k-3})
$\leq s^3 \cdot s^{k-2}$ ($\because s^3 > s^2 + s + 1$)
$= s^{k+1}$ ($\because$ Exponent Rules)

$\therefore$ P(k+1) is true.

(b) **Basis Step**: Prove P(3) and P(4) is true.

$\frac{7}{4}^3 \approx 5.359$
$a_3 = a_2 + a_1 = 3 + 1 = 4$
$\frac{7}{4}^4 \approx 9.378$
$a_4 = a_3 + a_2 = 4 + 3 = 7$

Since $4 \leq 5.359$ and $7 \leq 9.378$, this statement holds for the basis case.

Induction Step: Assume P(i) is true for some k $\geq$ 3 and any $i \in [0, k]$. Prove P(k+1) is true.

$a_{k+1} = a_k + a_{k-1}$
$\leq \frac{7}{4}^k + \frac{7}{4}^{k-1}$ $\qquad$ ($\because$ P(i) is true)
$= \frac{7}{4} \cdot \frac{7}{4}^{k-1} + \frac{7}{4}^{k-1}$ $\qquad$ ($\because$ Exponent Rules)
$= (\frac{7}{4} + 1)\frac{7}{4}^{k-1}$ $\qquad$ ($\because$ Factoring Out $\frac{7}{4}^{k-2}$)
$\leq (\frac{7}{4})^2 \cdot \frac{7}{4}^{k-1}$ $\qquad$ ($\because (\frac{7}{4})^2 > \frac{7}{4} + 1$)
$= \frac{7}{4}^{k+1}$ $\qquad$ ($\because$ Exponent Rules)

$\therefore$ P(k+1) is true.

(c) **Basis Step**: Prove P(1) is true.

2^1 is a circle with 2 people and if we go clockwise from 1, we eliminate 2 and we're left with 1. Thus, P(1) is true.

Induction Step: Assume P(i) is true for some k $\geq$ 1 and any $i \in [0, k]$. Prove P(k+1) is true.

Case 1: If k + 1 is even, then we would eliminate dots until there are 2 dots left, in which case the second dot would be eliminated.
Case 2: If k + 1 is odd, then we would eliminate dots until there are 2 dots left, in which case the second dot would be eliminated.

Since both cases lead to 1 dot left, P(k + 1) is true.

(d) **Basis Step**: Prove P(1) and P(2) is true.

If r = 1, then it can be written as $c_0 = 1$, and $1 \cdot 3^0 = 1$.
If r = 2, then it can be written as $c_0 = 2$, and $1 \cdot 3^0 = 2$.

Induction Step: Assume P(i) is true for some k $\geq$ 2 and any $i \in [0, k]$. Prove P(k+1) is true.
If we assume P(k) to be true, then every number can be written as a multiple of 3 plus either 0, 1, or 2. Since we can generate 0, 1 or 2, P(k + 1) is true.

(e) **Basis Step**: Prove P(1) is true.

$F_3 F_{k+1} - F_{k+2}^2$

Induction Step: Assume P(i) is true for some k $\geq$ 2 and any $i \in [0, k]$. Prove P(k+1) is true.

$F_{k+3} F_{k+1} - F_{k+2}^2$
$= (F_{k+2} + F_{k+1}) F_{k+1} - F_{k+2} \cdot F_{k+2}$ ($\because$ Definition of Fibonacci Sequence)
$= F_{k+2} F_{k+1} + F_{k+1}^2 - F_{k+2}(F_{k+1} + F_k)$ ($\because$ Definition of Fibonacci Sequence)
$= F_{k+2} F_{k+1} + F_{k+1}^2 - F_{k+2} F_{k+1} - F_{k+2} F_k$ ($\because$ Distributive Property)
$= F_{k+1}^2 - F_{k+2} F_k$ ($\because$ Subtraction)
$= -(F_{k+2} F_k - F_{k+1}^2)$ ($\because$ Factor Out -1)
$= -(-1)^k$ ($\because$ P(k) is true)
$= (-1)^{k+1}$ ($\because$ Exponent Rules)
$\therefore$ P(k + 1) is true.

(f) **Basis Step**: Prove P(0) is true.

Since $0^2 = 0$ and f(0) = 0, the statement holds for P(0).

Induction Step: Assume P(i) is true for some k $\geq$ 0 and any $i \in [0, k]$. Prove P(k+1) is true.

If k is even, then k + 1 would be odd.

$f(k + 1 - 1) + 2k - 1$
$= f(k) + 2k - 1$ ($\because$ Addition)
$= k^2 + 2k - 1$ ($\because$ P(k) is true)
$= (k + 1)^2$ ($\because$ Factoring)

$\therefore$ P(k + 1) is true.

Problem Set 3

(a) 28. $F_{k+1}^2 - F_k^2 - F_{k-1}^2$
$= (F_k + F_{k-1})^2 - F_k^2 - F_{k-1}^2$
$= F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 - F_k^2 - F_{k-1}^2$
$= 2F_kF_{k-1}$
QED

29. $F_{k+1}^2 - F_k^2$
$= (F_k + F_{k-1})^2 - F_k^2$
$= F_k^2 + 2F_kF_{k-1} + F_{k-1}^2 - F_k^2$
$= 2F_kF_{k-1} + F_{k-1}^2$

$F_{k-1}F_{k+2}$
$= F_{k-1}(F_{k+1} + F_k)$
$= F_{k-1}F_{k+1} + F_{k-1}F_k$
$= F_{k-1}(F_k + F_{k-1}) + F_{k-1}F_k$
$= F_{k-1}F_k + F_{k-1}^2 + F_{k-1}F_k$
$= 2F_{k-1}F_k + F_{k-1}^2$
QED

(b) 2b. $1 + 3 + 3^2 + ... + 3^{n-2} + 3^{n-1} = \frac{3^{n+1}-1}{3-1} - 3^n = \frac{3^{n+1}-1}{2} - 3^n$

2d. $2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n = \frac{2^{n+1}-1}{1} - \frac{2^n-1}{1}$

(c) 9. $\frac{1}{2^n-1}$

14. $3 \cdot (n) - 1$

15. $3 \cdot (n-1)^2 - n$