CSE 215 - Homework 6

Vincent Zheng

April 17, 2021

Problem Set 1

28. For a function to be well-defined, it must have one output for every input. Here, $h(\frac{m}{n}) = \frac{m^2}{n}$ is not well defined because:

    Take $\frac{1}{2}$ and $\frac{2}{4}$.

    $\frac{1}{2} = \frac{2}{4}$, however, if we plug these values into $h(\frac{m}{n})$, we get $\frac{1}{2} = \frac{4}{4}$.

    $\frac{1}{2} \neq \frac{4}{4}$ which means equal inputs gives us different outputs, thus, $h(\frac{m}{n})$ is not well defined.

Problem Set 2

35. Suppose there is an $y \in f(A \cap B)$, then there exists $x \in A \cap B$.

    By definition of intersection, $x \in A$ and $x \in B$
    Since $x \in A$, $y = f(x) \in f(A)$
    Since $x \in B$, $y = f(x) \in f(B)$

    Thus, $y \in f(A) \cap (B)$ by definition of intersection.

Problem Set 3

41. Prove $F^{-1}(C-D) \subseteq F^{-1}(C) - F^{-1}(D)$

    Suppose $x \in F^{-1}(C-D)$.
    $y \in C - D$ ($\because$ Definition of function)
    $y \in C$, but $y \notin D$ ($\because$ Definition of difference)
    $x \in F^{-1}(C)$, but $x \notin F^{-1}(D)$ ($\because$ Definition of function)
    $x \in F^{-1}(C) - F^{-1}(D)$ ($\because$ Definition of differnece)
    Thus, $F^{-1}(C-D) \subseteq F^{-1}(C) - F^{-1}(D)$.

Problem Set 4

43a. Case $\chi_{A\cap B}(u) = 0$
    $u \notin A \cap B$ ($\because$ Given by the function)
    $u \notin A$ and $u \notin B$ ($\because$ By definition of intersection)
    $\chi_A(u) = 0$ and $\chi_B(u) = 0$ ($\because$ Given by the function)
    $\chi_A(u) \cdot \chi_B(u) = 0$ ($\because 0 \cdot 0 = 0$)

    Case $\chi_{A\cap B}(u) = 1$
    $u \in A \cap B$ ($\because$ Given by the function)
    $u \in A$ and $u \in B$ ($\because$ By definition of intersection)
    $\chi_A(u) = 1$ and $\chi_B(u) = 1$ ($\because$ Given by the function)
    $\chi_A(u) \cdot \chi_B(u) = 1$ ($\because 1 \cdot 1 = 1$)

    These two properties hold, therefore $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$

43b. Case $u \in A$ and B
    This implies that $\chi_{A\cap B}(u) = 1$ since u can be in A or B.
    $\chi_A(u) = 1$ and $\chi_B(u) = 1$ ($\because$ Defined by the function)
    $\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) = 1+1-1\cdot 1$ ($\because$ Plugging in values)
    $= 1$

    Case $u \in A$ and $u \notin B$
    This implies that $\chi_{A\cap B}(u) = 1$ since u can be in A or B by definition of union.
    $\chi_A(u) = 1$ and $\chi_B(u) = 0$ ($\because$ Defined by the function)

$\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) = 1+0-1\cdot 0$ ($\because$ Plugging in values)
$= 1$

Case $u \notin A$ and $u \in B$
This implies that $\chi_{A\cap B}(u) = 1$ since u can be in A or B by definition of union.
$\chi_A(u) = 0$ and $\chi_B(u) = 1$ ($\because$ Defined by the function)
$\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) = 0+1-0\cdot 1$ ($\because$ Plugging in values)
$= 1$

Case $u \notin A$ and $u \notin B$
This implies that $\chi_{A\cap B}(u) = 0$ since u not in A or B
$\chi_A(u) = 0$ and $\chi_B(u) = 0$ ($\because$ Defined by the function)
$\chi_A(u)+\chi_B(u)-\chi_A(u)\cdot\chi_B(u) = 0+0-0\cdot 0$ ($\because$ Plugging in values)
$= 0$

Since this holds for all possible cases, $\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u)\cdot\chi_B(u)$

Problem Set 5

23a. Suppose (x_1, y_1) and (x_2, y_2) in $\mathbb{R}\times\mathbb{R}$, where $H(x_1, y_1) = H(x_2, y_2)$.

$(x_1 + 1, 2 - y_1) = (x_2 + 1, 2 - y_2)$ ($\because$ Defined by H)
$x_1 + 1 = x_2 + 1$ and $2 - y_1 = 2 - y_2$ ($\because$ Definition of equality of ordered pairs)

$x_1 + 1 = x_2 + 1$
$= x_1 = x_2$ ($\because$ Subtract 1 from both sides)

$2 - y_1 = 2 - y_2$
$= -y_1 = -y_2$ ($\because$ Subtract 2 from both sides)
$= y_1 = y_2$ ($\because$ Divide 2 from both sides)

$(x_1, y_1) = (x_2, y_2)$ ($\because$ Definition of equality of ordered pairs)

Hence, H is one-to-one.

23b. Suppose (x, y) in the domain of H.

Let r = x + 1 and s = 2 - y.

H(r, s)
= H(x - 1, 2 - y) ($\because$ Definition of H)
= (x + 1 - 1, 2 - (2 - y)) ($\because$ Substitution)
= (x, y) ($\because$ Simplify)

Hence, H is onto.

Problem Set 6

29. Suppose real numbers a, b and x are given with b $\neq$ 1.
$log_b(x^a)$
$= log_b(x \cdot x \cdot ... \cdot x)$, where x is multiplied by itself a times.
($\because$ Definition of exponent)
$= log_b(x) + log_b(x) + ... + log_b(x)$, where $log_b(x)$ is added a times
($\because log_b(xy) = log_b(x) + log_b(y)$)
$= a \cdot log_b(x)$ ($\because$ Definition of multiplication)

Problem Set 7

11. Given $H = H^{-1} = \frac{x+1}{x-1}$, find $H \circ H^{-1}$ and $H^{-1} \circ H$

$H \circ H^{-1}$
$= H(\frac{x+1}{x-1})$ ($\because$ Definition of Composition of Functions)
$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$ ($\because$ Substitution)
$= \frac{\frac{x+1}{x-1}+\frac{x-1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x-1}}$ ($\because 1 = \frac{x-1}{x-1}$)
$= \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-(x-1)}{x-1}}$ ($\because$ Adding Fractions)
$= \frac{2x}{2}$ ($\because$ Simplify Fraction)
= x

$H \circ H^{-1}$
$= H(\frac{x+1}{x-1})$ ($\because$ Definition of Composition of Functions)
$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$ ($\because$ Substitution)
$= \frac{\frac{x+1}{x-1}+\frac{x-1}{x-1}}{\frac{x+1}{x-1}-\frac{x-1}{x-1}}$ ($\because 1 = \frac{x-1}{x-1}$)
$= \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-(x-1)}{x-1}}$ ($\because$ Adding Fractions)
$= \frac{2x}{2}$ ($\because$ Simplify Fraction)
= x

Since $H \circ H^{-1}$ and $H \circ H^{-1}$ both result in the identity function, they are inverses of eachother.

Problem Set 8

27. Yes, this is true. Firstly there must be a one to one correspondance between f and g since their inverses are defined.

    If $C \in Z$, and since $g : Y \to Z$, then there exists a y, where $y \in Y$ such that g(y) = C.

    The inverse of this is $g^{-1}(C) = y$

    Also for f, there is similarly also a $y = f(x)$ and $x = f^{-1}(y)$, where $x \in X$ and $y \in Y$

    If we substitute this into $(g \circ f)(x)$, we get:
    g(f(x)) = g(y) = C.
    If we take the inverse of this, we get: $(g \circ f)^{-1}(C) = x$.

    Now we want to prove that $(f^{-1} \circ g^{-1})(C) = x$.
    $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(C)) = f^{-1}(y) = x$.

    Since both $(f^{-1} \circ g^{-1})(C) = x$ and $(g \circ f)^{-1}(C) = x$, this statement is true.

Problem Set 9

12. If S and W have the same cardinality, there needs to be a one-to-one correspondance between S and W. Let us define a function f, where $f : S \to W, f(x) = (b-a)x + a$

    Suppose $f(x_1) = f(x_2)$, where $x_1, x_2 \in S$
    $(b-a)x_1 + a = (b-a)x_2 + a$ ($\because$ Defined by f)
    $= (b-a)x_1 = (b-a)x_2$ ($\because$ Subtract both sides by a)
    $= x_1 = x_2$ ($\because$ Divide both sides by (b - a))
    $\therefore$ f is one=to-one.

    Suppose $y \in W$ and $x = \frac{y-a}{b-a}$.
    Since $a < y < b$, $0 < y - a < b - a$ ($\because$ Subtract by a)
    $0 < \frac{y-a}{b-a} < 1$ ($\because$ Divide by b - a)
    $x \in S$ ($\because 0 < \frac{y-a}{b-a} < 1$)
    Thus, $f(x) = (b-a)(\frac{y-a}{b-a}) - a$.
    = y - a - a ($\because$ Cancel out b - a)
    = y ($\because$ Subtraction)

    Since f is a one to one correspondance between S and W, this means that S and W have the same cardinality.

Problem Set 10

20.
    1. $f(x) = 5x$. Every x value has an integer output, but only multiples of 5 have a corresponding input value.
    2. $f(x) = x^2$. Every x value input has an integer output, but only perfect squares have a corresponding input value.

21.
    1. $f(x) = \frac{x}{5}$. Every output value has a corresponding input value (2 times that number), while not every input value has an integer output (ex: 1).
    2. $f(x) = \sqrt{x}$. Every output value has a corresponding input value (that number squared), while not every input value has an integer output (ex: 2).