Example – Idealized Drone Control

We are given a drone, which has some initial position and velocity, and our problem consists of designing a control sequence that gets the drone "close" to a given position in time T. To describe the dynamics, we use the standard motion model (in the x coordinate):

$$F_x = m\ddot{x},$$

where F is the total force applied on the drone (in the x coordinate). We suppose, that we can directly control the thrust force F_x^t (in the x coordinate) and that the force from F_x^r air resistance is proportional to the velocity through some constant κ :

$$\begin{gathered}
F_x^r = -\kappa\dot{x}, \\
F_x = F_x^t + F_x^r, \\
m\ddot{x} = F_x^t - \kappa\dot{x} \to \ddot{x} = \frac{F_x^t}{m} - \frac{\kappa}{m}\dot{x}.
\end{gathered}$$

For the y coordinate (vertical), we have an additional force – gravity F_g. The corresponding equations are:

$$\begin{gathered}
F_y^r = -\kappa\dot{y}, \quad F_g = -mg \\
F_y = F_y^t + F_y^r + F_g \\
m\ddot{y} = F_y^t - \kappa\dot{y} - mg \to \ddot{y} = \frac{F_y^t}{m} - \frac{\kappa}{m}\dot{y} - g.
\end{gathered}$$

Next we change the notation a bit, let $v^x = \dot{x}$, $v^y = \dot{y}$, $\gamma = \frac{\kappa}{m}$, $u^x = \frac{F_x^t}{m}$, and $u^y = \frac{F_y^t}{m}$. Then we get the following system of differential equations:

$$\begin{aligned}
&\dot{x} = v^x, \\
&\dot{v}^x = -\gamma v^x + u^x, \\
&\dot{y} = v^y, \\
&\dot{v}^y = -\gamma v^y + u^y - g.
\end{aligned}$$

We discretize the time interval $[0, T]$ into $N+1$ steps of size $dt = \frac{T}{N}$ and obtain the following system:

$$\begin{aligned}
&x_{k+1} = x_k + dt \cdot v_k^x \\
&v_{k+1}^x = v_k^x - dt \cdot \gamma v_k^x + dt \cdot u^x \\
&y_{k+1} = y_k + dt \cdot v_k^y \\
&v_{k+1}^y = v_k^y - dt \cdot \gamma v_k^y + dt \cdot u^y - dt \cdot g,
\end{aligned}$$

which is almost a nice linear system, if it were not for the term $dt \cdot g$. This can be addressed by adding an artificial state variable g_k:

$$\begin{aligned}
&x_{k+1} = x_k + dt \cdot v_k^x \\
&v_{k+1}^x = v_k^x - dt \cdot \gamma v_k^x + dt \cdot u^x \\
&y_{k+1} = y_k + dt \cdot v_k^y \\
&v_{k+1}^y = v_k^y - dt \cdot \gamma v_k^y + dt \cdot u^y - dt \cdot g_k, \\
&g_{k+1} = g_k.
\end{aligned}$$

This representation then gives rise to the following linear system (with added noise):

$$\tilde{x}_k = [x_k, v_k^x, y_k, v_k^y, g_k]',\ \tilde{u}_k = [u_k^x, u_k^y]',\ \tilde{w}_k = [w^x, w^{v_x}, w^y, w^{v_y}, 0]'$$
$$\tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k + \tilde{w}_k,$$
$$\tilde{x}_{k+1} = \begin{bmatrix} 1 & dt & 0 & 0 & 0 \\ 0 & 1 - dt \cdot \gamma & 0 & 0 & 0 \\ 0 & 0 & 1 & dt & 0 \\ 0 & 0 & 0 & 1 - dt \cdot \gamma & -dt \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tilde{x}_k + \begin{bmatrix} 0 & 0 \\ dt & 0 \\ 0 & 0 \\ 0 & dt \\ 0 & 0 \end{bmatrix} \tilde{u}_k + \tilde{w}_k,$$

Assignment: For $dt = 0.1, N = 50, \gamma = 0.2, \lambda = 0.005, \Sigma = \text{diag}([0.005, 0.002, 0.005, 0.001, 0])$ and $\tilde{x}_0 = [2, 0, 2, 0, 9.8]'$ find the optimal sequence of controls $\tilde{u}_k$ that minimize:

$$E\Big\{\tilde{x}_N' Q_N \tilde{x}_N + \sum_{k=0}^{N-1} \tilde{x}_k' Q_k \tilde{x}_k + \tilde{u}_k' R_k \tilde{u}_k\Big\},$$

where $Q_k = 0$, $Q_N = (1 - \lambda)I$, $R_k = \lambda I$. Report the optimal value $J_0(\tilde{x}_0)$.