# Example – Lagged Inventory

A modified store inventory management example. The orders $u_k$ are ready with a one-day lag (think of it as a production/delivery time). Additionally, there is no backlog option for orders. One state variable $x_k$ will describe the amount of the product available in the morning (when the store opens). During the day, the demand $w_k$ is realized. In the evening (after the store closes), the order from the previous day arrives and is prepared for the next day.

The system develops through the following equation:

$$x_{k+1} = \max\{x_k - w_k, 0\} + u_{k-1}.$$

In order to apply DP, we need to augment this system – introducing another state variable $s_k = u_{k-1}$, the system equation have the form:

$$\begin{aligned} x_{k+1} &= \max\{x_k - w_k, 0\} + s_k, \\ s_{k+1} &= u_k. \end{aligned}$$

An additional modification will encode the maximum size of the inventory $M$ – any access over that $M$ that is stored over night is simply lost (more complicated variations require a more complex model). The new augmented system has the following form:

$$\begin{aligned} x_{k+1} &= \min\{\max\{x_k - w_k, 0\} + s_k, M\} \\ s_{k+1} &= u_k. \end{aligned}$$

The last restriction is w.r.t. the amount ordered: $0 \leq u_k \leq U$.

The costs will encompass the cost of the inventory $c_1$, the order cost $c_2$, and the profit from selling the items $c_3$ (negative):

$$g(x_k, s_k, u_k, w_k) = c_1(x_k) + c_2(u_k) + c_3(x_k, w_k),$$

and the terminal cost will be just the cost of the inventory

$$g(x_N) = c_1(x_N).$$

A big advantage of DP is that the cost functions can be rather arbitrary (no need for linearity, convexity, or even continuity), e.g.

$$\begin{aligned} c_1(x) &= \sqrt{0.9 \cdot x}, \\ c_2(u) &= \begin{cases} 0, & \text{if } u = 0. \\ 0.3 + u, & \text{otherwise.} \end{cases} \\ c_3(x, w) &= -3 \cdot \min\{x_k, w_k\}. \end{aligned}$$

**Assignment:** Compute the optimal policy (minimizing the expected costs) for

$$N = 15, \quad M = 4, \quad u_k \in [0, \ldots, 3], \quad w_k = \begin{cases} 0, & p(0) = 0.1 \\ 1, & p(1) = 0.3 \\ 2, & p(2) = 0.5 \\ 3, & p(3) = 0.1 \end{cases}$$