Example – Cake eating problem (discounted costs)

Discounted objective:

$$\alpha^N g_N(x_N) + \sum_{k=0}^{N-1} \alpha^k g_k(x_k, u_k, w_k),$$

with discount factor $\alpha \in (0, 1)$. DP algorithm can be written in an alternative form:

$$V_N(x_N) = g_N(x_N)$$

$$V_k(x_k) = \min_{u_k \in U_k(x_k)} E\left\{g_k(x_k, u_k, w_k) + \alpha V_{k+1}(f_k(x_k, u_k, w_k))\right\}$$

Imagine you bought a large cake that you want to eat. Also, you do not want eat it all in one sitting, rather you would like to eat it in N portions/days. However, as the days go by, the cake is getting a bit stale (it is a bit worse each day). What is the best way to divide the cake, so that your enjoyment is maximized?

x_0: Initial cake size,

N: Number of time steps (portions/days),

x_k: Cake size at time $k = 1, \ldots, N$,

u_k: Amount of eaten cake at time $k = 0, \ldots, N-1$,

α : Discount factor – cake getting stale.

System model:

$$x_{k+1} = x_k - u_k,$$

utility (negative because of the minimization) just from eating cake, no uncertainty:

$$g(x_k, u_k, w_k) = -2\sqrt{u_k},$$

terminal state has no value (throw out rest of the cake):

$$g_N(x_N) = 0.$$

Relationship:

$$V_k(x_k) = \min_{0 \le u_k \le x_k} \left[-2\sqrt{u_k} + \alpha V_{k+1}(x_k - u_k)\right]$$

Assignment: Compute the optimal control for $x_0 = 1, \alpha = 0.8, N = 3$.

Stochastic element: a drunk roommate comes in the evening and eats a portion λ of the cake with probability p, or does not eat anything with probability $1 - p$. The modified problem:

$$V_N(x_n) = 0, \quad x_{k+1} = x_k - u_k - w_k(x_k - u_k), \quad w_k = \begin{cases} \lambda, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$