Example – Kalman Filter

Let us recall that for the system with imperfect state information (and without any control):

$$x_{k+1} = A_k x_k + w_k,$$

with available measurements in the form:

$$z_k = C_k x_k + v_k,$$

with $S = E\{(x_0 - E\{x_0\})(x_0 - E\{x_0\})'\}$, $M_k = E\{w_k w_k'\}$, $N_k = E\{v_k v_k'\}$, the Kalman filter has the following form:

Prediction:

$$\hat{x}_{k+1\,|\,k} = A_k \hat{x}_{k\,|\,k}$$

$$\Sigma_{k+1\,|\,k} = A_k \Sigma_{k\,|\,k} A_k' + M_k$$

Update:

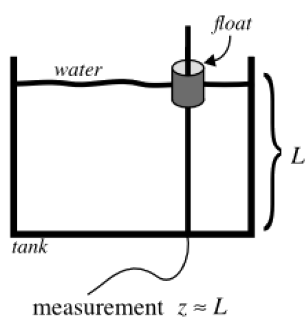
$$\hat{x}_{k\,|\,k} = \hat{x}_{k\,|\,k-1} + \Sigma_{k\,|\,k-1} C_k' (C_k \Sigma_{k\,|\,k-1} C_k' + N_k)^{-1} (z_k - C_k \hat{x}_{k\,|\,k-1})$$

$$\Sigma_{k\,|\,k} = \Sigma_{k\,|\,k-1} - \Sigma_{k\,|\,k-1} C_k' (C_k \Sigma_{k\,|\,k-1} C_k' + N_k)^{-1} C_k \Sigma_{k\,|\,k-1}$$

with the initial conditions:

$$\hat{x}_{0\,|\,-1} = E\{x_0\}, \quad \Sigma_{0\,|\,-1} = S,$$

We consider a simple situation showing a way to measure the level of water in a tank (this is shown in the following figure)



We are trying to estimate the level of water in the tank, which is unknown. The measurements obtained are from the level of the "float". This could be an electronic device, or a simple mechanical device. The water could be:

a) **static** (the average level of the tank is not changing):

- the state is simply the water level and is not changing in time (no w_k), i.e.

$$x_{k+1} = x_k$$

- however, we are going to assume some noise in the measurement:

$$z_k = x_k + v_k$$

b) **filling** (the average level of the tank is increasing):

- in this case, the state has two components: x^1 – water level, x^2 – rate at which the water is pouring in
- we consider a "constant" filling rare and a discretization by dt, and get the following equations:

$$
\begin{aligned}
x^1_{k+1} =& x^1_k + dt \cdot x^2_k + w^1_k \\
x^2_{k+1} =& x^2_k + w^2_k
\end{aligned}
$$

- using $x_k = (x^1_k, x^2_k)'$ and $w_k = (w^1_k, w^2_k)'$, we get the following system:

$$x_{k+1} = \begin{bmatrix} 1 & dt \\ 0 & 1 \end{bmatrix} x_k + w_k$$

- the available measurements still constitute just the information about the current water level (i.e., no direct measurement of the rate):

$$z_k = [1 \quad 0] x_k + v_k$$

Assignment: Compute a simulation of the measurements and, using the Kalman filter, compute the state estimates for

a) the static model with $dt = 1, N_k = 0.1, N = 20, x_0 = 1, \hat{x}_{0\,|\,-1} = 0, S = 1000$

b) the filling model with $dt = 1, N_k = 0.1, N = 100, x_0 = [0, 0.1]', \hat{x}_{0\,|\,-1} = [0, 0]', S = \mathrm{diag}([1000, 1000])$, $M_k = \begin{bmatrix} 0.001 & 0.0005 \\ 0.0005 & 0.0008 \end{bmatrix}$