ECE 102 Homework 2

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January 28, 2022

Problem 1

(a)

$$\begin{aligned} h(t) &= \int_{-\infty}^{t-1} e^t \cos(2\tau + 2 - 2t)\delta(\tau)e^{2-\tau} d\tau \\ &= \int_{-\infty}^{t-1} e^t \cos(2 - 2t)\delta(\tau)e^{2} d\tau \\ &= e^{t+2}\cos(2-2t)u(t-1) \end{aligned}$$

Since $h(t,\tau) = h(t-\tau) = 0$ for $t < \tau$, this system is causal. Furthermore, since e^t is not bounded (it goes to ∞ as $t \to \infty$), this system is not BIBO stable.

(b)

$$
\begin{aligned}
h(t) &= e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)]\delta(\tau)d\tau \\
&= e^{-t}\int_{-\infty}^{t} e^{\tau}\cos(t+\tau)\delta(\tau)d\tau \\
&= e^{-t}\int_{-\infty}^{t} \cos(t)\delta(\tau)d\tau \\
&= e^{-t}\cos(t)u(t)
\end{aligned}
$$

Since $h(t,\tau) = h(t-\tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$
|y(t)| = \left| e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)]x(\tau)d\tau \right|
$$

Since e^t and e^-t are positive for any t we get

$$
\begin{aligned}
|y(t)| &= e^{-t}\int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)x(\tau)|d\tau \\
&= e^{-t}\int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)||x(\tau)|d\tau \\
&\leq e^{-t}\int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)|M d\tau \\
&\leq Me^{-t}\int_{-\infty}^{t} e^{\tau}d\tau \\
&= Me^{-t}e^{t} \\
&= M < +\infty
\end{aligned}
$$

Thus this system is BIBO stable.

(c)

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{t-1} e^{-(t-\tau)}\delta(\tau-2)d\tau \\
&= \int_{-\infty}^{t-1} e^{-(t-2)}\delta(\tau-2)d\tau \\
&= e^{2-t}u(t-3)
\end{aligned}
$$

Since $h(t,\tau) = h(t-\tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$
\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{t-1} e^{-(t-\tau)}x(\tau-2)d\tau \right| \\
&= \int_{-\infty}^{t-1} e^{-(t-\tau)}|x(\tau-2)|d\tau \\
&\le M \int_{-\infty}^{t-1} e^{-(t-\tau)}d\tau \\
&= Me^{(t-1)-t} = Me^{-1} < +\infty
\end{aligned}
$$

Thus this system is BIBO stable.

Problem 2

(a)

$$
\begin{aligned}
h_1(t,\tau) &= \delta(t-\tau)u(t) - \int_{-\infty}^{t-2} e^{-(t-\sigma)}\delta(\sigma-\tau)u(\sigma)d\sigma \\
&= \delta(t-\tau)u(\tau) - \int_{-\infty}^{t-2} e^{-(t-\tau)}\delta(\sigma-\tau)u(\tau)d\sigma \\
&= \delta(t-\tau)u(\tau) - e^{-(t-\tau)}u(\tau)\int_{-\infty}^{t-2} \delta(\sigma-\tau)d\sigma \\
&= \boxed{\delta(t-\tau)u(\tau) - e^{-(t-\tau)}u(\tau)u(t-2-\tau)}
\end{aligned}
$$

$$
\begin{aligned}
h_2(t,\tau) &= \int_{-\infty}^{t} \delta(\sigma-\tau)u(\sigma)d\sigma \\
&= \int_{-\infty}^{t} \delta(\sigma-\tau)u(\tau)d\sigma \\
&= \boxed{u(\tau)u(t-\tau)}
\end{aligned}
$$

(b)

Since these systems as cascading we have

$$
\begin{aligned}
h_{12}(t,\tau) &= \int_{-\infty}^{\infty} h_1(\sigma,\tau)h_2(t,\sigma)d\sigma \\
&= \int_{-\infty}^{\infty} \big(\delta(\sigma-\tau)u(\tau) - e^{-(\sigma-\tau)}u(\tau)u(\sigma-2-\tau)\big)\, u(\sigma)u(t-\sigma)d\sigma
\end{aligned}
$$

Since these systems are linear we can break it up into two integrals $\int_{-\infty}^{\infty}\delta(\sigma-\tau)u(\tau)u(\sigma)u(t-\sigma)d\sigma$ and $\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma-2-\tau)u(t-\sigma)d\sigma$. For the first integral we get

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(\sigma-\tau)u(\tau)u(\sigma)u(t-\sigma)d\sigma &= \int_{-\infty}^{\infty} \delta(\sigma-\tau)u^2(\tau)u(t-\tau)d\sigma \\ &= u(\tau)u(t-\tau)\int_{-\infty}^{\infty} \delta(\sigma-\tau)d\sigma \\ &= u(\tau)u(t-\tau)\end{aligned}$$

To evaluate the second integral let us first evaluate it in two regions, when $t > \tau + 2$, $\tau > 0$, and everywhere else. When $t > \tau + 2$ and $\tau > 0$ we get

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma-2-\tau)u(t-\sigma)d\sigma &= \int_{2+\tau}^{t} e^{-\sigma+\tau}d\sigma \\ &= -e^{-\sigma+\tau}\Big|_{\sigma=2+\tau}^{t} \\ &= e^{-2} - e^{-t+\tau}\end{aligned}$$

And everywhere else we get

$$\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma-2-\tau)u(t-\sigma)d\sigma = 0$$

Thus we get that

$$h_{12}(t,\tau) = \boxed{u(\tau)u(t-\tau) - u(\tau)u(t-2-\tau)(e^{-2} - e^{-t+\tau})}$$

(c)

To find if the system is BIBO, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output $w(t)$ is bounded as well. To find that let us first find if $y(t)$ is stable

$$\begin{aligned}
|y(t)| &= \left|x(t)u(t) - \int_{-\infty}^{t-2} e^{-(t-\tau)}x(\tau)u(\tau)d\tau\right| \\
&\leq |x(t)u(t)| + \left|-\int_{-\infty}^{t-2} e^{-(t-\tau)}x(\tau)u(\tau)d\tau\right| \\
&\leq Mu(t) + \left|\int_{-\infty}^{t-2} e^{-(t-\tau)}x(\tau)u(\tau)d\tau\right| \\
&\leq Mu(t) + M\left|\int_{-\infty}^{t-2} e^{-(t-\tau)}u(\tau)d\tau\right| \\
&= Mu(t) + M\left|\int_{0}^{t-2} e^{-(t-\tau)}d\tau\right| \\
&= Mu(t) + M\left|e^{-(t-\tau)}\Big|_{\tau=0}^{t-2}\right| \\
&= Mu(t) + M\left|e^{-2} - e^{t}\right|
\end{aligned}$$

This is not bounded, since as $t \to \infty$, $y(t) \to \infty$ and thus $y(t)$ is not stable. If $y(t)$ is not stable then $w(t)$ is also not stable since the integral of ∞ from $-\infty$ to ∞ is also ∞.

(d)

Since these systems as cascading we have

$$\begin{aligned}
h_{21}(t,\tau) &= \int_{-\infty}^{\infty} h_2(\sigma,\tau)h_1(t,\sigma)d\sigma \\
&= \int_{-\infty}^{\infty} u(\sigma)u(\sigma-\tau)(\delta(t-\sigma)u(\sigma) - e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma))d\sigma \\
&= \int_{-\infty}^{\infty} \delta(t-\sigma)u(\sigma)u(\sigma-\tau)d\sigma - \int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma \\
&= u(t)u(t-\tau) - \int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma
\end{aligned}$$

$\int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma$ can evaluate to 3 different expression

- When $t < 2$ or when $\tau > t - 2$, $u(\sigma)u(t-2-\sigma)u(\sigma-\tau) = 0$, thus we get
$$\int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma = 0$$
- When $t > 2$ and $\tau < 0$, $u(\sigma)u(t-2-\sigma)u(\sigma-\tau) = 1$ for $0 < \sigma < t-2$ thus we get
$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma &= \int_{0}^{t-2} e^{-(t-\sigma)}d\sigma \\ &= e^{\sigma-t}\Big|_{\sigma=0}^{t-2} \\ &= e^{-2} - e^{-t}\end{aligned}$$
- When $t > 2$ and $0 < \tau < t-2$ we get $u(\sigma)u(t-2-\sigma)u(\sigma-\tau) = 1$ for $\tau < \sigma < t-2$ thus we get
$$\begin{aligned}\int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t-2-\sigma)u(\sigma-\tau)d\sigma &= \int_{\tau}^{t-2} e^{-(t-\sigma)}d\sigma \\ &= e^{\sigma-t}\Big|_{\sigma=\tau}^{t-2} \\ &= e^{-2} - e^{\tau-t}\end{aligned}$$

Thus we get that

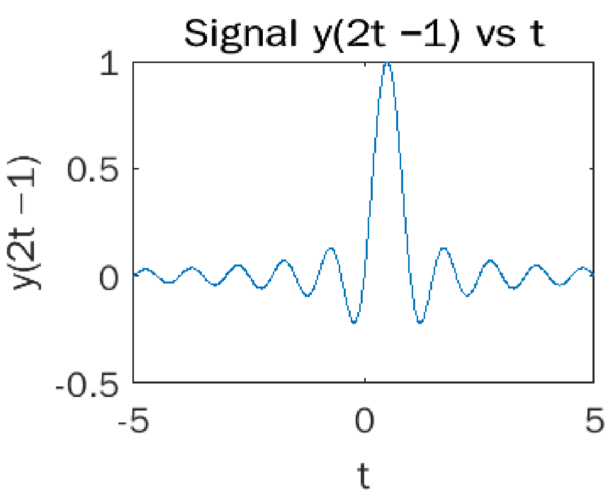
$$h_{21}(t,\tau) = \boxed{\begin{cases} 0 & t < 0 \text{ or } \tau > t \\ 1 & 0 < t < 2 \text{ or } t > \tau > t-2 \\ 1 + e^{-t} - e^{-2} & t > 2 \text{ and } \tau < 0 \\ 1 + e^{\tau-t} - e^{-2} & t > 2 \text{ and } 0 < \tau < t-2 \end{cases}}$$

Problem 3

(a)

```
t=-5:0.01:5;
y=sin(pi*(2*t-1))./(pi*(2*t-1));

figure;
plot(t,y);
xlabel('t');
ylabel('y(2t    1)');
title('Signal y(2t    1) vs t');
```

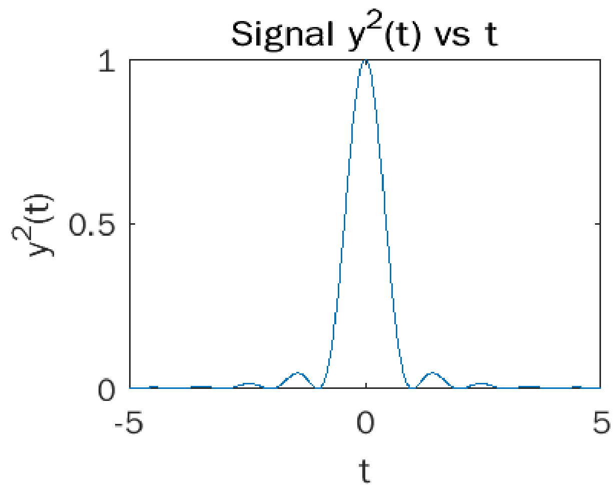


(b)

```
t=-5:0.01:5;
y=sin(pi*t)./(pi*t);

figure;
plot(t,y.^2);
xlabel('t');
ylabel('y^2(t)');
```

```
title('Signal y^2(t) vs t');
```

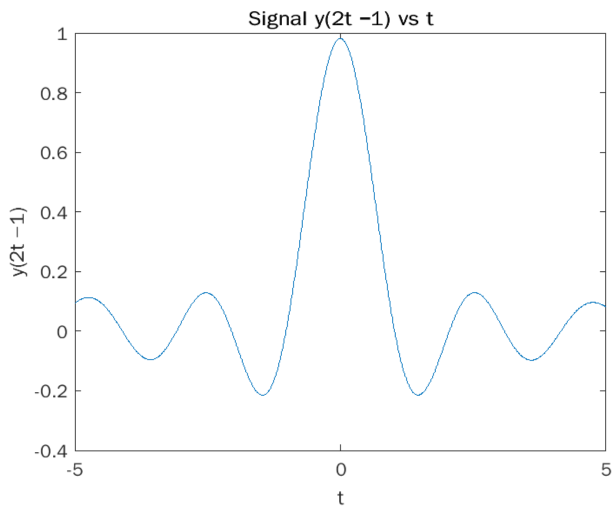


(c)

```
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);

y = sin(pi*t)./(pi*t);
y(51)=1;
z = conv(y,y,'same')*dt;

figure;
plot(t,z);
xlabel('t');
ylabel('z(t)');
title('Signal z(t) vs t');
```



(d)

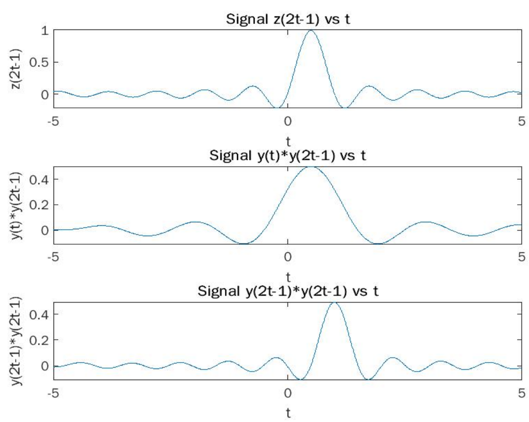
```
fy = @(t) sin(pi*t)./(pi*t);
hfig=figure;
%plot out z(2t-1)
t = linspace(-11, 11, 1000);
dt = t(2) - t(1);
y=fy(t);
%y(51)=1;
z = conv(y,y,'same')*dt;

subplot(3,1,1);
plot((t+1)/2,z);
xlim([-5,5]);
xlabel('t');
ylabel('z(2t-1)');
title('Signal z(2t-1) vs t');

%plot out {y(t)*y(2t-1)}
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);
y=fy(t);
y2=fy(2*t-1);
```

```
subplot(3,1,2);
plot(t,conv(y,y2,'same')*dt);
xlabel('t');
ylabel('{y(t)*y(2t-1)}');
title('Signal {y(t)*y(2t-1)} vs t');

%plot out {y(2t-1)*y(2t-1)}
subplot(3,1,3);
plot(t,conv(y2,y2,'same')*dt);
xlabel('t');
ylabel('{y(2t-1)*y(2t-1)}');
title('Signal {y(2t-1)*y(2t-1)} vs t');
%print figure
print(hfig, '-djpeg', '3d');
```



Thus the answer neither.

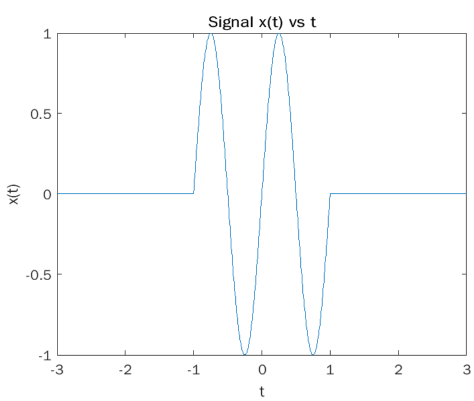
Problem 4

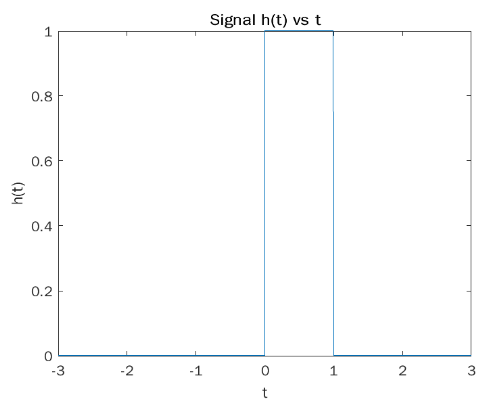
(a)

```
t=linspace(-3,3,1000);
```

```
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
h=@(t) u(t)-u(t-1);

%plot x(t)
figure;
plot(t,x(t));
xlabel('t');
ylabel('x(t)');
title('Signal x(t) vs t');
%plot x(t)
figure;
plot(t,h(t));
xlabel('t');
ylabel('h(t)');
title('Signal h(t) vs t');
```





(b)

$$
\begin{aligned}
y_1(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau
\end{aligned}
$$

Under different values of t, $\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau$ yields different expressions

- When $t \leq -1$ or $t \geq 2$, $u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right) = 0$ for all values of τ, thus we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau = 0$$

- When $-1 < t < 0$, $u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right) = 1$ for $-1 < \tau < t$, thus we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau &= \int_{-1}^{t} \sin(2\pi\tau)d\tau \\
&= -\frac{\cos(2\pi\tau)}{2\pi}\Bigg|_{\tau=-1}^{t} \\
&= \frac{1-\cos(2\pi t)}{2\pi}
\end{aligned}
$$

- When $0 < t < 1$ we get $u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right) = 1$ for $t-1 < \tau < t$, since the period of $\sin(2\pi\tau)$ is 1 we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau = 0$$

- When $1 < t < 2$ we get $u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right) = 1$

for $t-1<\tau<1$, thus we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau)-u(t-\tau-1)\right)d\tau &= \int_{t-1}^{1} \sin(2\pi\tau)d\tau \\
&= -\left.\frac{\cos(2\pi\tau)}{2\pi}\right|_{\tau=t-1}^{1} \\
&= \frac{\cos(2\pi(t-1))-1}{2\pi}
\end{aligned}
$$

Therefore we get that:

$$
y_1(t) = \boxed{\begin{cases} 0 & t \le -1 \\ \frac{1-\cos(2\pi t)}{2\pi} & -1<t<0 \\ 0 & 0 \le t \ge 1 \\ \frac{\cos(2\pi(t-1))-1}{2\pi} & 1<t<2 \\ 0 & t \ge 2 \end{cases}}
$$

(c)

$$
\begin{aligned}
y_2(t) &= x(t)\cdot h(t) \\
&= \boxed{\sin(2\pi t)u(t+1)u(-t+1)(u(t)-u(t-1))}
\end{aligned}
$$

(d)

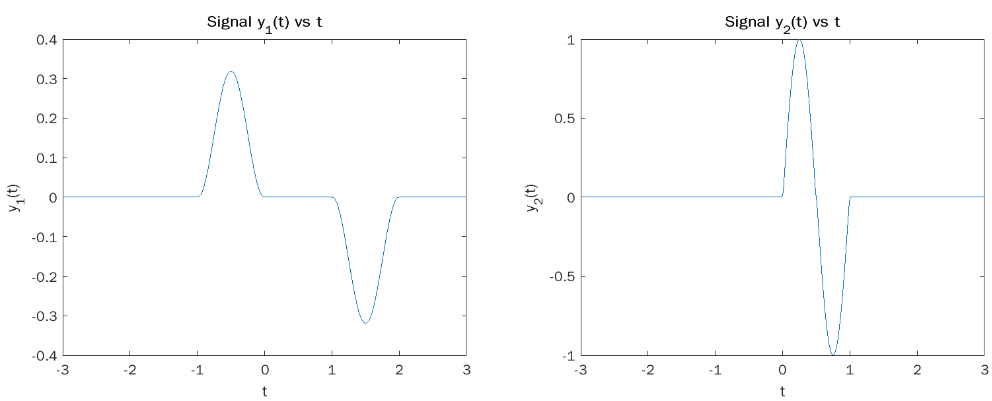
```
t=linspace(-3,3,600);
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
h=@(t) u(t)-u(t-1);

%plot y_1(t)
```

```
y1=[zeros(1,200),...
    (1.-cos(2.*pi.*t(201:300)))./(2*pi),...
    zeros(1,100),...
    (cos(2.*pi.*(t(401:500)-1))-1)./(2*pi),...
    zeros(1,100)];

figure;
plot(t,y1)
xlabel('t');
ylabel('y_1(t)');
title('Signal y_1(t) vs t');

%plot y_2(t)
figure;
plot(t,x(t).*h(t))
xlabel('t');
ylabel('y_2(t)');
title('Signal y_2(t) vs t');
```



(e)

```
t=linspace(-3,3,600);
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
```

```
h=@(t) u(t)-u(t-1);

%plot y1(t)
figure;
plot(t,conv(x(t),h(t),'same')*(t(2)-t(1)));
xlabel('t');
ylabel('y_1(t)');
title('Signal y_1(t) vs t');
```