ECE 102 Homework 5

Lawrence Liu

March 6, 2022

1 Problem 1

(a)

We have

$$\begin{aligned}\int_0^t y(t-\tau)\tau^2 e^{3-\tau}d\tau &= \int_{-\infty}^{\infty}\tau^2 e^{3-\tau}y(t-\tau)u(t-\tau)d\tau\\ &= \left(t^2 e^{3-t}u(t)\right) * y(t)\\ &\to \left(\frac{2e^3}{(s+1)^3}\right)Y(s)\end{aligned}$$

Thus we have

$$sY(s) + \left(\frac{2e^3}{(s+1)^3}\right)Y(s) = sX(s) - X(s)$$

And thus we have

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s+\frac{2e^3}{(s+1)^3}}$$

(b)

The Fourier coefficients of $x(t)$ is

$$a_k = \begin{cases} 1.5 & k = 0 \\ \frac{1}{4} & |k| = 2 \\ 0 & \text{everwhere else} \end{cases}$$

Therefore we have that the fourier series coefficients of $y(t)$, b_k are

$$b_k = H(j)a_k = \begin{cases} 1.5\frac{j-1}{j+\frac{2e^3}{(j+1)^3}} & k = 0 \\ \frac{1}{4}\frac{j-1}{j+\frac{2e^3}{(j+1)^3}} & |k| = 2 \\ 0 & \text{everwhere else} \end{cases}$$

(c)

From the fourier series properties, given that

$$y(t) \to b_k$$

we have

$$\begin{aligned} y(t-3) &\to b_k e^{-3jk} \\ y(2t) &\to b_k \\ y(t-3) * y(2t) &\to b_k^2 e^{-3jk} \end{aligned}$$

Problem 2

(a)

$$
\begin{aligned}
X_1(j\omega) &= \int_{-\infty}^{+\infty} x_1(t)e^{-j\omega t}dt \\
&= \int_0^1 t^2 e^{-j\omega t}dt \\
&= \frac{\left(j\omega^2 + 2\omega - 2j\right)\mathrm{e}^{-j\omega t} + 2j}{\omega^3}
\end{aligned}
$$

(b)

$$
\begin{aligned}
X_2(j\omega) &= \int_{-\infty}^{+\infty} x_2(t)e^{-j\omega t}dt \\
&= 2\pi\delta(\omega) + \int_{-2}^{2} \cos(100t)e^{-j\omega t}dt \\
&= 2\pi\delta(\omega) + \frac{\mathrm{e}^{-2j\omega}\cdot\left(\left(\cos\left(200\right)\, j\omega + 100\sin\left(200\right)\right)\mathrm{e}^{4j\omega} - \cos\left(200\right)\, j\omega + 100\sin\left(200\right)\right)}{10000 - \omega^2}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_{-\infty}^{t} \cos(5(t-\sigma))\delta(\sigma-2)d\sigma &= \cos(5(t-2)) \int_{-\infty}^{t} \delta(\sigma-2)d\sigma \\
&\to \left(\frac{2\pi}{5}\delta\left(\frac{\omega}{5}\right)e^{-j\frac{2\omega}{5}}\right) * \left(\frac{e^{-2j\omega}}{j\omega} + \pi\delta(\omega)\right) \\
&= \left(2\pi\delta(\omega)e^{-j\frac{2\omega}{5}}\right) * \left(\frac{e^{-2j\omega}}{j\omega} + \pi\delta(\omega)\right) \\
&= 2\pi e^{-j\frac{2\omega}{5}} \left(\frac{e^{-2j\omega}}{j\omega} + \pi\delta(\omega)\right)
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
X(0) &= \int_{-\infty}^{\infty} x(t)dt \\
&= 6
\end{aligned}
$$

Likewise, since

$$\int_{-\infty}^{\infty} \frac{1}{j\omega} + \pi\delta(\omega)d\omega = \pi$$

And

$$x(t) = u(t+1) + u(t) - u(t-2) - u(t-3)$$

Thus we have

$$\int_{-\infty}^{\infty} X(j\omega)d\omega = 1.5\pi$$

(b)

We know that
$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = 10$$
Thus from Parseval's Reltaion we have
$$\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega = 20\pi$$

(c)

We have
$$\frac{2\sin(\omega)}{\omega} \rightarrow \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$
$$e^{j2\omega}\frac{2\sin(\omega)}{\omega} \rightarrow y(t) = \begin{cases} 1 & -3 < t < -1 \\ 0 & \text{elsewhere} \end{cases}$$
Thus we have
$$y(-t) = \begin{cases} 1 & 1 < t < 3 \\ 0 & \text{elsewhere} \end{cases}$$
Therefore we have
$$\int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega = 2\pi \int_{-\infty}^{\infty} x(t)y(-t)dt = 10\pi$$

(c)

The inverse Fourier transform of $Re\{X(j\omega)\}$ is the even part of $x(t)$ or
$$u(t+3) + u(t+2) + u(t+1) - u(t-1) - u(t-2) - u(t-3)$$

Problem 4

From condition 1 and 2 we know that $x(t)$ must be of the form

$$x(t) = ce^{-2t}u(t)$$

From condition 3 we have

$$\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = 2\pi$$

Therefore, from parseval's we have

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = 1$$

$$c^2 \int_{0}^{+\infty} e^{-4t} dt = 1 \frac{c^2}{4} = 1$$

Thus we have $c = 1$, $x(t) = e^{-2t}u(t)$ and thus $A = -1$

Problem 5

(a)

$$X(s) = \frac{2\pi}{(s+1)^2 + 4\pi^2} + e^{-2s}$$

$$X(j\omega) = \frac{2\pi}{(j\omega+1)^2 + 4\pi^2} + e^{-2j\omega}$$

(b)

$$|X(j\omega)| = \sqrt{1 + \frac{4\pi^2}{(i\omega-1)^2(i\omega+1)^2 + 4\pi^2(i\omega-1)^2 + 4\pi^2(i\omega+1)^2 + 16\pi^4} + \frac{2\pi e^{2i\omega}}{(i\omega+1)^2 + 4\pi^2} +}$$