

ECE 102 Homework 2

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Problem 1

(a)

$$\begin{aligned}h(t) &= \int_{-\infty}^{t-1} e^t \cos(2\tau + 2 - 2t) \delta(\tau) e^{2-\tau} d\tau \\&= \int_{-\infty}^{t-1} e^t \cos(2 - 2t) \delta(\tau) e^2 d\tau \\&= e^{t+2} \cos(2 - 2t) u(t - 1)\end{aligned}$$

Since $h(t, \tau) = h(t - \tau) = 0$ for $t < \tau$, this system is causal. Furthermore, since e^t is not bounded (it goes to ∞ as $t \rightarrow \infty$), this system is not BIBO stable.

(b)

$$\begin{aligned}h(t) &= e^{-t} \int_{-\infty}^t e^{\tau} [\cos(t) \cos(\tau) - \sin(t) \sin(\tau)] \delta(\tau) d\tau \\&= e^{-t} \int_{-\infty}^t e^{\tau} \cos(t + \tau) \delta(\tau) d\tau \\&= e^{-t} \int_{-\infty}^t \cos(t) \delta(\tau) d\tau \\&= e^{-t} \cos(t) u(t)\end{aligned}$$

Since $h(t, \tau) = h(t - \tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$|y(t)| = \left| e^{-t} \int_{-\infty}^t e^{\tau} [\cos(t) \cos(\tau) - \sin(t) \sin(\tau)] x(\tau) d\tau \right|$$

Since e^t and e^{-t} are positive for any t we get

$$\begin{aligned}|y(t)| &= e^{-t} \int_{-\infty}^t e^{\tau} |\cos(t + \tau) x(\tau)| d\tau \\&= e^{-t} \int_{-\infty}^t e^{\tau} |\cos(t + \tau)| |x(\tau)| d\tau \\&\leq e^{-t} \int_{-\infty}^t e^{\tau} |\cos(t + \tau)| M d\tau \\&\leq M e^{-t} \int_{-\infty}^t e^{\tau} d\tau \\&= M e^{-t} e^t \\&= M < +\infty\end{aligned}$$

Thus this system is BIBO stable.

(c)

$$\begin{aligned}h(t) &= \int_{-\infty}^{t-1} e^{-(t-\tau)} \delta(\tau - 2) d\tau \\&= \int_{-\infty}^{t-1} e^{-(t-2)} \delta(\tau - 2) d\tau \\&= e^{2-t} u(t - 3)\end{aligned}$$

Since $h(t, \tau) = h(t - \tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$\begin{aligned}|y(t)| &= \left| \int_{-\infty}^{t-1} e^{-(t-\tau)} x(\tau - 2) d\tau \right| \\&= \int_{-\infty}^{t-1} e^{-(t-\tau)} |x(\tau - 2)| d\tau \\&\leq M \int_{-\infty}^{t-1} e^{-(t-\tau)} d\tau \\&= M e^{(t-1)-t} = M e^{-1} < +\infty\end{aligned}$$

Thus this system is BIBO stable.

Problem 2

(a)

$$\begin{aligned}
 h_1(t, \tau) &= \delta(t - \tau)u(t) - \int_{-\infty}^{t-2} e^{-(t-\sigma)} \delta(\sigma - \tau)u(\sigma) d\sigma \\
 &= \delta(t - \tau)u(\tau) - \int_{-\infty}^{t-2} e^{-(t-\tau)} \delta(\sigma - \tau)u(\tau) d\sigma \\
 &= \delta(t - \tau)u(\tau) - e^{-(t-\tau)}u(\tau) \int_{-\infty}^{t-2} \delta(\sigma - \tau) d\sigma \\
 &= \boxed{\delta(t - \tau)u(\tau) - e^{-(t-\tau)}u(\tau)u(t - 2 - \tau)}
 \end{aligned}$$

$$\begin{aligned}
 h_2(t, \tau) &= \int_{-\infty}^t \delta(\sigma - \tau)u(\sigma) d\sigma \\
 &= \int_{-\infty}^t \delta(\sigma - \tau)u(\tau) d\sigma \\
 &= \boxed{u(\tau)u(t - \tau)}
 \end{aligned}$$

(b)

Since these systems are cascading we have

$$\begin{aligned}
 h_{12}(t, \tau) &= \int_{-\infty}^{\infty} h_1(\sigma, \tau)h_2(t, \sigma) d\sigma \\
 &= \int_{-\infty}^{\infty} (\delta(\sigma - \tau)u(\tau) - e^{-(\sigma-\tau)}u(\tau)u(\sigma - 2 - \tau)) u(\sigma)u(t - \sigma) d\sigma
 \end{aligned}$$

Since these systems are linear we can break it up into two integrals $\int_{-\infty}^{\infty} \delta(\sigma - \tau)u(\tau)u(\sigma)u(t - \sigma) d\sigma$ and $\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t - \sigma) d\sigma$. For

the first integral we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(\sigma - \tau) u(\tau) u(\sigma) u(t - \sigma) d\sigma &= \int_{-\infty}^{\infty} \delta(\sigma - \tau) u^2(\tau) u(t - \tau) d\sigma \\
&= u(\tau) u(t - \tau) \int_{-\infty}^{\infty} \delta(\sigma - \tau) d\sigma \\
&= u(\tau) u(t - \tau)
\end{aligned}$$

To evaluate the second integral let us first evaluate it in two regions, when $t > \tau + 2$, $\tau > 0$, and everywhere else. When $t > \tau + 2$ and $\tau > 0$ we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-(\sigma - \tau)} u(\tau) u(\sigma) u(\sigma - 2 - \tau) u(t - \sigma) d\sigma &= \int_{2+\tau}^t e^{-\sigma + \tau} d\sigma \\
&= -e^{-\sigma + \tau} \Big|_{\sigma=2+\tau}^t \\
&= e^{-2} - e^{-t+\tau}
\end{aligned}$$

And everywhere else we get

$$\int_{-\infty}^{\infty} e^{-(\sigma - \tau)} u(\tau) u(\sigma) u(\sigma - 2 - \tau) u(t - \sigma) d\sigma = 0$$

Thus we get that

$$h_{12}(t, \tau) = \boxed{u(\tau) u(t - \tau) - u(\tau) u(t - 2 - \tau) (e^{-2} - e^{-t+\tau})}$$

(c)

To find if the system is BIBO, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output $w(t)$ is bounded as well. To find that let us first

find if $y(t)$ is stable

$$\begin{aligned}
|y(t)| &= \left| x(t)u(t) - \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau) d\tau \right| \\
&\leq |x(t)u(t)| + \left| - \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau) d\tau \right| \\
&\leq Mu(t) + \left| \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau) d\tau \right| \\
&\leq Mu(t) + M \left| \int_{-\infty}^{t-2} e^{-(t-\tau)} u(\tau) d\tau \right| \\
&= Mu(t) + M \left| \int_0^{t-2} e^{-(t-\tau)} d\tau \right| \\
&= Mu(t) + M \left| e^{-(t-\tau)} \Big|_{\tau=0}^{t-2} \right| \\
&= Mu(t) + M |e^{-2} - e^t|
\end{aligned}$$

This is not bounded, since as $t \rightarrow \infty$, $y(t) \rightarrow \infty$ and thus $y(t)$ is not stable. If $y(t)$ is not stable then $w(t)$ is also not stable since the integral of ∞ from $-\infty$ to ∞ is also ∞ .

(d)

Since these systems are cascading we have

$$\begin{aligned}
h_{21}(t, \tau) &= \int_{-\infty}^{\infty} h_2(\sigma, \tau) h_1(t, \sigma) d\sigma \\
&= \int_{-\infty}^{\infty} u(\sigma)u(\sigma - \tau) (\delta(t - \sigma)u(\sigma) - e^{-(t-\sigma)}u(\sigma)u(t - 2 - \sigma)) d\sigma \\
&= \int_{-\infty}^{\infty} \delta(t - \sigma)u(\sigma)u(\sigma - \tau) d\sigma - \int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) d\sigma \\
&= u(t)u(t - \tau) - \int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) d\sigma
\end{aligned}$$

$\int_{-\infty}^{\infty} e^{-(t-\sigma)}u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) d\sigma$ can evaluate to 3 different expressions

- When $t < 2$ or when $\tau > t - 2$, $u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) = 0$, thus we get

$$\int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma = 0$$

- When $t > 2$ and $\tau < 0$, $u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) = 1$ for $0 < \sigma < t - 2$ thus we get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma &= \int_0^{t-2} e^{-(t-\sigma)} d\sigma \\ &= e^{\sigma-t} \Big|_{\sigma=0}^{t-2} \\ &= e^{-2} - e^{-t} \end{aligned}$$

- When $t > 2$ and $0 < \tau < t - 2$ we get $u(\sigma)u(t - 2 - \sigma)u(\sigma - \tau) = 1$ for $\tau < \sigma < t - 2$ thus we get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma &= \int_{\tau}^{t-2} e^{-(t-\sigma)} d\sigma \\ &= e^{\sigma-t} \Big|_{\sigma=\tau}^{t-2} \\ &= e^{-2} - e^{\tau-t} \end{aligned}$$

Thus we get that

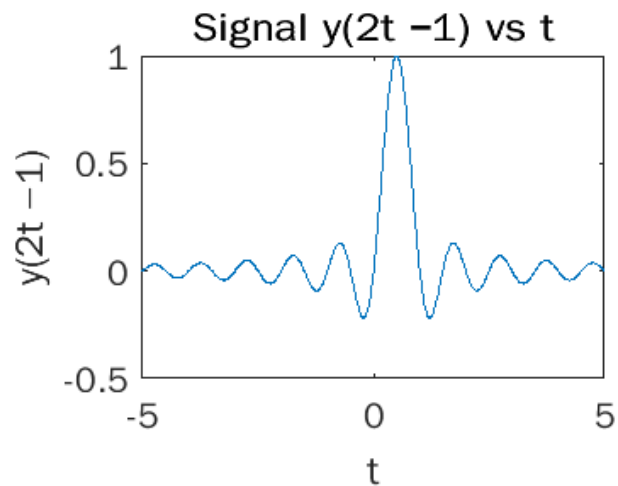
$$h_{21}(t, \tau) = \begin{cases} 0 & t < 0 \text{ or } \tau > t \\ 1 & 0 < t < 2 \text{ or } t > \tau > t - 2 \\ 1 + e^{-t} - e^{-2} & t > 2 \text{ and } \tau < 0 \\ 1 + e^{\tau-t} - e^{-2} & t > 2 \text{ and } 0 < \tau < t - 2 \end{cases}$$

Problem 3

(a)

```
t = -5:0.01:5;  
y = sin(pi*(2*t-1))./(pi*(2*t-1));
```

```
figure;  
plot(t,y);  
xlabel('t');  
ylabel('y(2t-1)');  
title('Signal y(2t-1) vs t');
```



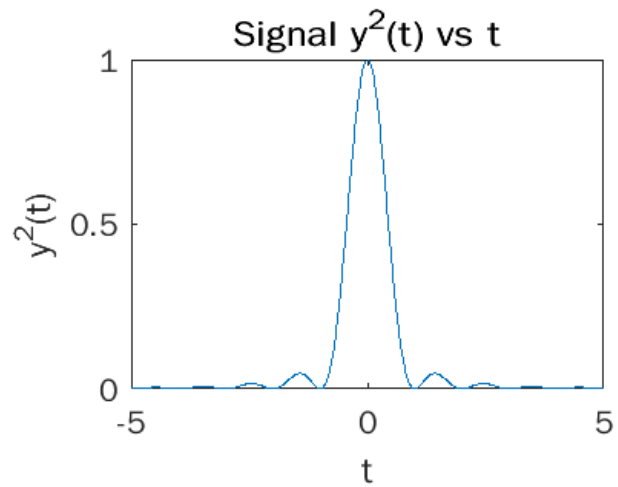
(b)

```
t = -5:0.01:5;  
y = sin(pi*t)./(pi*t);
```

```
figure;  
plot(t,y.^2);  
xlabel('t');  
ylabel('y^2(t)');
```



```
title('Signal  $y^2(t)$  vs  $t$ ');
```

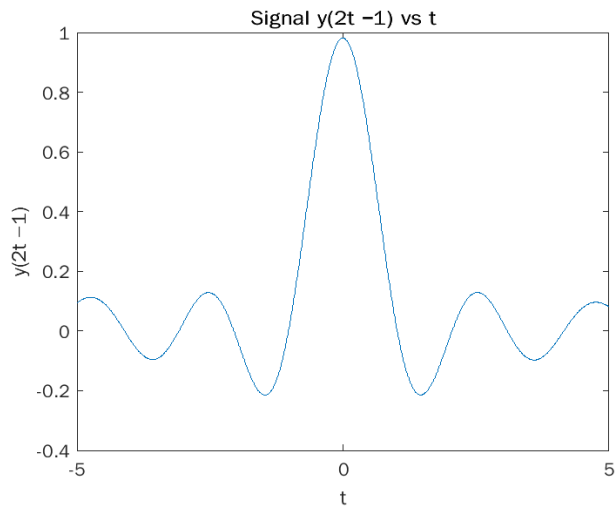


(c)

```
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);
```

```
y = sin(pi*t)./(pi*t);
y(51)=1;
z = conv(y,y,'same')*dt;
```

```
figure;
plot(t,z);
xlabel('t');
ylabel('z(t)');
title('Signal z(t) vs t');
```



(d)

```
fy = @(t) sin(pi*t)./(pi*t);
hfig=figure;
%plot out z(2t-1)
t = linspace(-11, 11, 1000);
dt = t(2) - t(1);
y=fy(t);
%y(51)=1;
z = conv(y,y,'same')*dt;

subplot(3,1,1);
plot((t+1)/2,z);
xlim([-5,5]);
xlabel('t');
ylabel('z(2t-1)');
title('Signal z(2t-1) vs t');

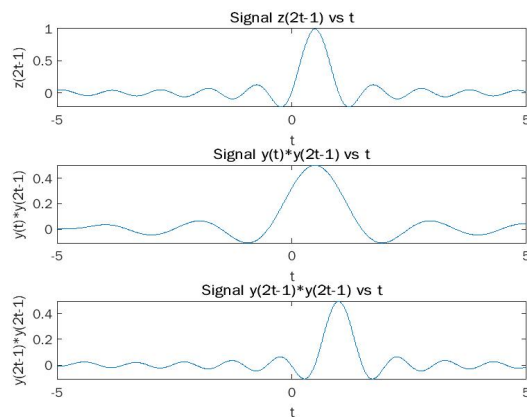
%plot out {y(t)*y(2t-1)}
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);
y=fy(t);
y2=fy(2*t-1);
```

```

subplot(3,1,2);
plot(t,conv(y,y2,'same')*dt);
xlabel('t');
ylabel('{y(t)*y(2t-1)}');
title('Signal {y(t)*y(2t-1)} vs t');

%plot out {y(2t-1)*y(2t-1)}
subplot(3,1,3);
plot(t,conv(y2,y2,'same')*dt);
xlabel('t');
ylabel('{y(2t-1)*y(2t-1)}');
title('Signal {y(2t-1)*y(2t-1)} vs t');
%print figure
print(hfig, '-djpeg', '3d');

```



Thus the answer neither.

Problem 4

(a)

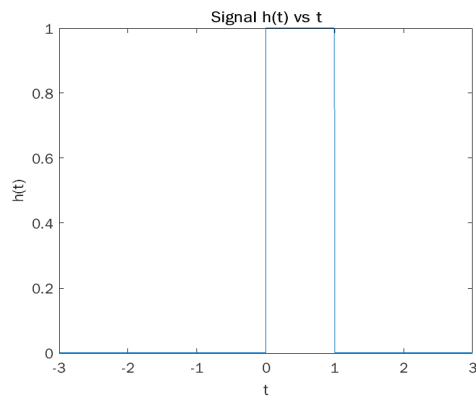
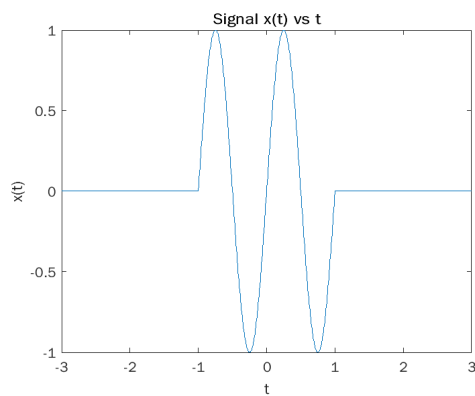
```
t=linspace(-3,3,1000);
```

```

%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
h=@(t) u(t)-u(t-1);

%plot x(t)
figure;
plot(t,x(t));
xlabel('t ');
ylabel('x(t) ');
title('Signal x(t) vs t ');
%plot x(t)
figure;
plot(t,h(t));
xlabel('t ');
ylabel('h(t) ');
title('Signal h(t) vs t ');

```



(b)

$$\begin{aligned}
y_1(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
&= \int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau
\end{aligned}$$

Under different values of t , $\int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau$ yields different expressions

- When $t \leq -1$ or $t \geq 2$, $u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) = 0$ for all values of τ , thus we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau = 0$$

- When $-1 < t < 0$, $u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) = 1$ for $-1 < \tau < t$, thus we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau &= \int_{-1}^t \sin(2\pi\tau) d\tau \\
&= -\frac{\cos(2\pi\tau)}{2\pi} \Big|_{\tau=-1}^t \\
&= \frac{1 - \cos(2\pi t)}{2\pi}
\end{aligned}$$

- When $0 < t < 1$ we get $u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) = 1$ for $t - 1 < \tau < t$, since the period of $\sin(2\pi\tau)$ is 1 we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau = 0$$

- When $1 < t < 2$ we get $u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) = 1$

for $t - 1 < \tau < 1$, thus we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \sin(2\pi\tau) u(\tau + 1) u(-\tau + 1) (u(t - \tau) - u(t - \tau - 1)) d\tau &= \int_{t-1}^1 \sin(2\pi\tau) d\tau \\
&= -\frac{\cos(2\pi\tau)}{2\pi} \Big|_{\tau=t-1}^1 \\
&= \frac{\cos(2\pi(t-1)) - 1}{2\pi}
\end{aligned}$$

Therefore we get that:

$$y_1(t) = \begin{cases} 0 & t \leq -1 \\ \frac{1 - \cos(2\pi t)}{2\pi} & -1 < t < 0 \\ 0 & 0 \leq t \leq 1 \\ \frac{\cos(2\pi(t-1)) - 1}{2\pi} & 1 < t < 2 \\ 0 & t \geq 2 \end{cases}$$