ECE 102 Homework 1

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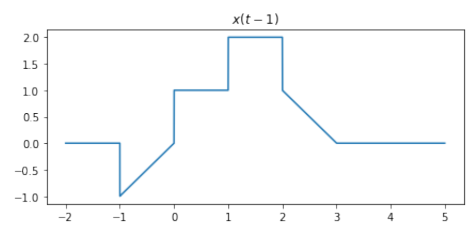
Problem 1

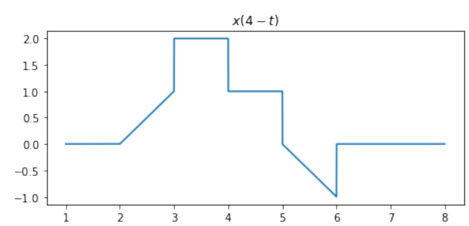
(a)

$x(t-1)$ is simply the signal $x(t)$ delayed by 1. So the plot would look like:

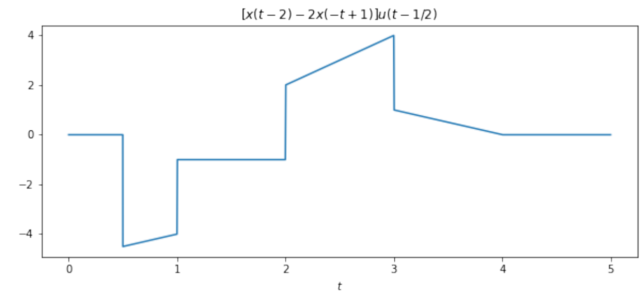


(b)

$x(4-t)$ is simply the signal delayed by 4 and then reversed, so the plot would look like:



(c)



Problem 2

(a)

$$\begin{aligned}
a(t) &= \cos(\theta t)\sin(\psi t) \\
&= \frac{e^{i\theta t} + e^{-i\theta t}}{2}\frac{e^{i\psi t} - e^{-i\psi t}}{2i} \\
&= \frac{1}{4i}(e^{i(\theta+\psi)t} - e^{i(\theta-\psi)t} + e^{i(\psi-\theta)t} - e^{-i(\theta+\psi)t}) \\
&= \frac{1}{4i}(e^{i(\theta+\psi)t} -^{-i(\theta+\psi)t}) + e^{i(\psi-\theta)t} - e^{i(\theta-\psi)t}) \\
&= \frac{2i}{4i}(\sin((\theta+\psi)t) + \sin((\psi-\theta)t)) \\
&= \frac{1}{2}(\sin((\theta+\psi)t) - \sin((\theta-\psi)t))
\end{aligned}$$

(b)

Yes $a(t)$ can be periodic, for instance if we want it to have a period 3 when $\theta = 2\pi$ then $\psi = \frac{4\pi}{3}$.

(c)

The period of $\cos(10t + 1)$ is $\frac{\pi}{5}$ and the period of $\sin(4t - 1)$ is $\frac{\pi}{2}$. The fundamental period of $x(t)$ is the least common multiple of these two periods, which is $\boxed{\pi}$.

Problem 3

(a)

The period of $\cos(3\Omega_0 t)$ is $\frac{2\pi}{3\Omega_0}$ and the period if $\cos(\Omega_0 t)$ is $\frac{2\pi}{\Omega_0}$. So the period of $x(t) = \cos(3\Omega_0 t) + 5\cos(\Omega_0 t)$ is $\frac{2\pi}{\Omega_0}$. When $\Omega_0 = \pi$ the period is $\boxed{2}$

(b)

The power of a signal is

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

Since $x(t)$ is real we have

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\cos(3\Omega_0 t) + 5\cos(\Omega_0 t)\right)^2 dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T} \left(\cos(3\pi t) + 5\cos(\pi t)\right)^2 dt \end{aligned}$$

We know that the period of $x(t)$ is 2 when $\Omega_0 = \pi$. Thus let $T = 2N$. Since $T \to \infty$ leads to $N \to \infty$ we get

$$P_x = \lim_{N\to\infty} \frac{1}{4N} \int_{-2N}^{2N} (\cos(3\pi t) + 5\cos(\pi t)^2 dt$$

Since $\left(\cos(3\pi t) + 5\cos(\pi t)\right)^2$ is even:

$$P_x = \lim_{N\to\infty} \frac{1}{2N} \int_{0}^{2N} \left(\cos(3\pi t) + 5\cos(\pi t)\right)^2 dt$$

Since $\left(\cos(3\pi t) + 5\cos(\pi t)\right)^2$ is periodic with period 2

$$\begin{aligned} P_x &= \lim_{N\to\infty} \frac{1}{2} \int_{0}^{2} \left(\cos(3\pi t) + 5\cos(\pi t)\right)^2 dt \\ &= \lim_{N\to\infty} \frac{1}{2} \int_{0}^{2} \left(\cos^2(3\pi t) + 10\cos(3\pi t)\cos(\pi t) + 25\cos^2(\pi t)\right)^2 dt \end{aligned}$$

Therefore we need to solve for $\int_0^2 \cos^2(3\pi t)dt$, $\int_0^2 \cos(3\pi t)\cos(\pi t)dt$ and $\int_0^2 \cos^2(\pi t)dt$

$$
\begin{aligned}
\int_0^2 \cos^2(3\pi t)dt &= 3\int_0^{2/3} \cos^2(3\pi t)dt \\
&= 3\int_0^{2/3} \frac{1+\cos(6\pi t)}{2}dt \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\int_0^2 \cos(3\pi t)\cos(\pi t)dt &= \frac{1}{4}\int_0^2 (e^{3\pi t}+e^{-3\pi t})(e^{\pi t}+e^{-\pi t})dt \\
&= \frac{1}{4}\left(\int_0^2 (e^{4\pi t}+e^{-4\pi t})dt + \int_0^2 (e^{2\pi t}+e^{-2\pi t})dt\right) \\
&= \frac{1}{2}\left(\int_0^2 \cos(4\pi t)dt + \int_0^2 \cos(2\pi t)dt\right) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\int_0^2 \cos^2(\pi t)dt &= \int_0^2 \frac{1+\cos(2\pi t)}{2}dt \\
&= 1
\end{aligned}
$$

Therefore we have that

$$
\begin{aligned}
P_x &= \frac{1}{2}(1+25) \\
&= \boxed{13}
\end{aligned}
$$

(c)

$$
P_1 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(3\Omega_0 t)dt
$$

Let $T = NT_0$ where $T_0 = \frac{2\pi}{3\Omega_0} = \frac{2}{3}$ is the period of $\cos(3\Omega_0 t)$. We have $T \to \infty$ leads to $N \to \infty$

$$\begin{aligned}
P_1 &= \lim_{N\to\infty} \frac{1}{2NT_0} \int_{-NT_0}^{NT_0} \cos^2(3\Omega_0 t) dt \\
&= \lim_{N\to\infty} \frac{1}{NT_0} \int_{0}^{NT_0} \cos^2(3\Omega_0 t) dt \\
&= \lim_{N\to\infty} \frac{1}{T_0} \int_{0}^{T_0} \cos^2(3\Omega_0 t) dt \\
&= \lim_{N\to\infty} \frac{1}{T_0} \int_{0}^{T_0} \frac{1 + \cos(6\Omega_0 t)}{2} dt \\
&= \lim_{N\to\infty} \frac{1}{T_0} \frac{T_0}{2} \\
&= \frac{1}{2}
\end{aligned}$$

We can do the same for P_2

$$P_2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 25\cos^2(\Omega_0 t) dt$$

Let $T = NT_0$ where $T_0 = \frac{2\pi}{\Omega_0} = 2$ is the period of $\cos(\Omega_0 t)$. We have $T \to \infty$ leads to $N \to \infty$

$$\begin{aligned}
P_2 &= 25 \lim_{N\to\infty} \frac{1}{2NT_0} \int_{-NT_0}^{NT_0} \cos^2(\Omega_0 t) dt \\
&= 25 \lim_{N\to\infty} \frac{1}{NT_0} \int_{0}^{NT_0} \cos^2(\Omega_0 t) dt \\
&= 25 \lim_{N\to\infty} \frac{1}{T_0} \int_{0}^{T_0} \cos^2(\Omega_0 t) dt \\
&= 25 \lim_{N\to\infty} \frac{1}{T_0} \int_{0}^{T_0} \frac{1 + \cos(2\Omega_0 t)}{2} dt \\
&= 25 \lim_{N\to\infty} \frac{1}{T_0} \frac{T_0}{2} \\
&= \frac{25}{2}
\end{aligned}$$

So therefore we get that indeed $P = P_1 + P_2$

(d)

No $\gamma(t)$ is not periodic. A signal made up of the summation of two signals are periodic if and only if $\frac{T_0}{T_1} = \frac{N}{M}$ where T_0 and T_1 are the periods of the two signals and N and M are integers. The period of $\cos(t)$ is 2π and the period of $\cos(\frac{\pi}{2}t)$ is 4. It is impossible for $\frac{2\pi}{4}$ to be equal to one integer divided by another.
From the definition, the power of $\gamma(t)$ is:

$$P_\gamma = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\gamma(t)|^2 dt$$

However since $\gamma(t)$ isn't periodic, we cannot use the technique in the lecture where we rewrite T as NT_0. Therefore the only way is to solve the integral and the limit:

$$\begin{aligned}
P_\gamma &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\cos(t) + \cos(\frac{\pi}{2}t))^2 dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (\cos^2(t) + \cos^2(\frac{\pi}{2}t) + 2\cos(t)\cos(\frac{\pi}{2}t))dt \\
&= P_1 + P_2 + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2\cos(t)\cos(\frac{\pi}{2}t)dt
\end{aligned}$$

Therefore, in order for $P_\gamma = P_1 + P_2$ we need $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2\cos(t)\cos(\frac{\pi}{2}t)dt = 0$. The first thing to note is that $\cos(t)\cos(\frac{\pi}{2}t)$ is even, therefore we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 2\cos(t)\cos(\frac{\pi}{2}t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} 2\cos(t)\cos(\frac{\pi}{2}t)dt$$

Next we rewrite $\cos(t)\cos(\frac{\pi}{2}t)$ in terms of complex exponentials and simplify

$$\begin{aligned}
\cos(t)\cos(\frac{\pi}{2}t) &= \frac{e^{it} + e^{-it}}{2} \frac{e^{i\frac{\pi}{2}t} + e^{-i\frac{\pi}{2}t}}{2} \\
&= \frac{1}{4}(e^{i(1+\frac{\pi}{2})t} + e^{-i(1+\frac{\pi}{2})t} + e^{i(\frac{\pi}{2}-1)t} + e^{-i(\frac{\pi}{2}-1)t}) \\
&= \frac{1}{2}(\cos((1+\frac{\pi}{2})t) + \cos((\frac{\pi}{2}-1)t))
\end{aligned}$$

Thus we get:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T 2\cos(t)\cos(\frac{\pi}{2}t)dt &= \lim_{T\to\infty}\frac{1}{T}\int_0^T (\cos((1+\frac{\pi}{2})t)+\cos((\frac{\pi}{2}-1)t))dt \\ &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((1+\frac{\pi}{2})t)dt + \lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((\frac{\pi}{2}-1)t)dt\end{aligned}$$

Therefore we will need to calculate $\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((1+\frac{\pi}{2})t)dt$ and $\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((\frac{\pi}{2}-1)t)dt$. Let us calculate $\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((1+\frac{\pi}{2})t)dt$ first by letting $T = NT_1$ where T_1 is the period of $\cos((1+\frac{\pi}{2})t)$. As $T\to\infty$, $N\to\infty$ therefore we get:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((1+\frac{\pi}{2})t)dt &= \lim_{N\to\infty}\frac{1}{NT_1}\int_0^{NT_1} \cos((1+\frac{\pi}{2})t)dt \\ &= \lim_{N\to\infty}\frac{1}{T_1}\int_0^{T_1} \cos((1+\frac{\pi}{2})t)dt \\ &= 0\end{aligned}$$

Likewise for $\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((\frac{\pi}{2}-1)t)dt$, let $T = NT_2$ where T_2 is the period of $\cos((\frac{\pi}{2}-1)t)$. As $T\to\infty$, $N\to\infty$ therefore we get:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \cos((\frac{\pi}{2}-1)t)dt &= \lim_{N\to\infty}\frac{1}{NT_2}\int_0^{NT_2} \cos((\frac{\pi}{2}-1)t)dt \\ &= \lim_{N\to\infty}\frac{1}{T_2}\int_0^{T_2} \cos((\frac{\pi}{2}-1)t)dt \\ &= 0\end{aligned}$$

Thus we get $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^T 2\cos(t)\cos(\frac{\pi}{2}t)dt = 0$, and thus $P = P_1 + P_2$

Problem 4

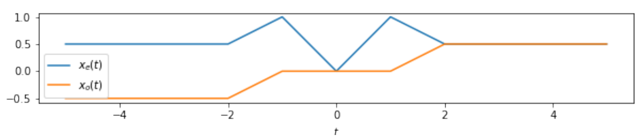
(a)

The even and odd parts of the signal are:

$$x_e(t) = \begin{cases} |t| & |t| \le 1 \\ -\frac{1}{2}|t| + \frac{3}{2} & 1 < |x| \le 2 \\ \frac{1}{2} & |t| > 2 \end{cases}$$

$$x_o(t) = \begin{cases} -\frac{1}{2} & t \leq -2 \\ \frac{1}{2}t + \frac{1}{2} & -2 < t \leq -1 \\ 0 & -1 < t \leq 1 \\ \frac{1}{2}t - \frac{1}{2} & 1 < t \leq 2 \\ \frac{1}{2} & 2 < t \end{cases}$$

Plotted out, it looks like:



(b)

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} (x_o(t) + x_e(t))^2 dt \\ &= \int_{-\infty}^{\infty} (x_o^2(t) + 2x_e(t)x_o(t) + x_e^2(t)) dt \\ &= \int_{-\infty}^{\infty} |x_o(t)|^2 dt + \int_{-\infty}^{\infty} |x_e(t)|^2 dt + \int_{-\infty}^{\infty} 2x_e(t)x_o(t) dt \end{aligned}$$

$x_e(t)x_o(t)$ is odd. Therefore $\int_{-\infty}^{\infty} 2x_e(t)x_o(t)dt = 0$, and thus we get:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x_o(t)|^2 dt + \int_{-\infty}^{\infty} |x_e(t)|^2 dt$$