ECE 102 Homework 2

Lawrence Liu

January 28, 2022

Problem 1

(a)

$$h(t) = \int_{-\infty}^{t-1} e^t \cos(2\tau + 2 - 2t)\delta(\tau)e^{2-\tau}d\tau$$
$$= \int_{-\infty}^{t-1} e^t \cos(2 - 2t)\delta(\tau)e^2 d\tau$$
$$= e^{t+2}\cos(2 - 2t)u(t - 1)$$

Since $h(t, \tau) = h(t - \tau) = 0$ for $t < \tau$, this system is causal. Furthermore, since e^t is not bounded (it goes to ∞ as $t \to \infty$), this system is not BIBO stable.

(b)

$$h(t) = e^{-t} \int_{-\infty}^{t} e^{\tau}[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)]\delta(\tau)d\tau$$

$$= e^{-t} \int_{-\infty}^{t} e^{\tau}\cos(t+\tau)\delta(\tau)d\tau$$

$$= e^{-t} \int_{-\infty}^{t} \cos(t)\delta(\tau)d\tau$$

$$= e^{-t}\cos(t)u(t)$$

Since $h(t,\tau) = h(t-\tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$|y(t)| = \left| e^{-t} \int_{-\infty}^{t} e^{\tau}[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)]x(\tau)d\tau \right|$$

Since e^t and e^{-t} are positive for any t we get

$$|y(t)| = e^{-t} \int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)x(\tau)|d\tau$$

$$= e^{-t} \int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)||x(\tau)|d\tau$$

$$\leq e^{-t} \int_{-\infty}^{t} e^{\tau}|\cos(t+\tau)|Md\tau$$

$$\leq Me^{-t} \int_{-\infty}^{t} e^{\tau}d\tau$$

$$= Me^{-t}e^{t}$$

$$= M < +\infty$$

Thus this system is BIBO stable.

(c)

$$h(t) = \int_{-\infty}^{t-1} e^{-(t-\tau)}\delta(\tau - 2)d\tau$$

$$= \int_{-\infty}^{t-1} e^{-(t-2)}\delta(\tau - 2)d\tau$$

$$= e^{2-t}u(t - 3)$$

Since $h(t, \tau) = h(t - \tau) = 0$ for $t < \tau$, this system is causal. To find if the system is BIBO stable, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output is bounded as well

$$|y(t)| = \left| \int_{-\infty}^{t-1} e^{-(t-\tau)}x(\tau - 2)d\tau \right|$$

$$= \int_{-\infty}^{t-1} e^{-(t-\tau)}|x(\tau - 2)|d\tau$$

$$\leq M \int_{-\infty}^{t-1} e^{-(t-\tau)}d\tau$$

$$= Me^{(t-1)-t} = Me^{-1} < +\infty$$

Thus this system is BIBO stable.

Problem 2

(a)

$$h_1(t, \tau) = \delta(t - \tau)u(t) - \int_{-\infty}^{t-2} e^{-(t-\sigma)}\delta(\sigma - \tau)u(\sigma)d\sigma$$

$$= \delta(t - \tau)u(\tau) - \int_{-\infty}^{t-2} e^{-(t-\tau)}\delta(\sigma - \tau)u(\tau)d\sigma$$

$$= \delta(t - \tau)u(\tau) - e^{-(t-\tau)}u(\tau)\int_{-\infty}^{t-2} \delta(\sigma - \tau)d\sigma$$

$$= \boxed{\delta(t - \tau)u(\tau) - e^{-(t-\tau)}u(\tau)u(t - 2 - \tau)}$$

$$h_2(t, \tau) = \int_{-\infty}^{t} \delta(\sigma - \tau)u(\sigma)d\sigma$$

$$= \int_{-\infty}^{t} \delta(\sigma - \tau)u(\tau)d\sigma$$

$$= \boxed{u(\tau)u(t - \tau)}$$

(b)

Since these systems as cascading we have

$$h_{12}(t, \tau) = \int_{-\infty}^{\infty} h_1(\sigma, \tau)h_2(t, \sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} \left(\delta(\sigma - \tau)u(\tau) - e^{-(\sigma-\tau)}u(\tau)u(\sigma - 2 - \tau) \right) u(\sigma)u(t - \sigma)d\sigma$$

Since these systems are linear we can break it up into two integrals $\int_{-\infty}^{\infty} \delta(\sigma - \tau)u(\tau)u(\sigma)u(t - \sigma)d\sigma$ and $\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t - \sigma)d\sigma$. For

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the first integral we get

$$\int_{-\infty}^{\infty} \delta(\sigma-\tau)u(\tau)u(\sigma)u(t-\sigma)d\sigma = \int_{-\infty}^{\infty} \delta(\sigma-\tau)u^2(\tau)u(t-\tau)d\sigma$$

$$= u(\tau)u(t-\tau)\int_{-\infty}^{\infty} \delta(\sigma-\tau)d\sigma$$

$$= u(\tau)u(t-\tau)$$

The second integral can be evaluated to 3 different expression:

- When $t < \tau + 2$ or when $t < 0$ we get that $u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)$ for all σ, thus we get

$$\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)d\sigma = 0$$

- $t > 0$ and $\tau < 0$ we get that $u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)$ for $0 < \sigma < t$, thus we get

$$\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)d\sigma = \int_{0}^{t} e^{-(\sigma-\tau)}u(\tau)d\sigma$$

$$= -u(\tau)\,e^{\tau-\sigma}\big|_{\sigma=0}^{t}$$

$$= u(\tau)(e^{\tau} - e^{\tau-t})$$

However since $\tau < 0$ we get $u(\tau) = 0$ and thus

$$\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)d\sigma = 0$$

- When $t > \tau + 2$ and $\tau > 0$ we get that $u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)$ for $\tau + 2 < \sigma < t$, thus we get

$$\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}u(\tau)u(\sigma)u(\sigma - 2 - \tau)u(t-\sigma)d\sigma = \int_{2+\tau}^{t} e^{-\sigma+\tau}d\sigma$$

$$= -e^{-\sigma+\tau}\big|_{\sigma=2+\tau}^{t}$$

$$= (e^{-2} - e^{-t+\tau})$$

Thus we get that

$$h_{12}(t,\tau) = \boxed{u(\tau)u(t-\tau) - u(\tau)u(t - 2 - \tau)(e^{-2} - e^{-t+\tau})}$$

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(c)

To find if the system is BIBO, let us input a bounded signal $|x(t)| < M < +\infty$ and determine if the output $w(t)$ is bounded as well. To find that let us first find if $y(t)$ is stable

$$
\begin{aligned}
|y(t)| &= \left| x(t)u(t) - \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau)d\tau \right| \\
&\leq |x(t)u(t)| + \left| -\int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau)d\tau \right| \\
&\leq Mu(t) + \left| \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau)d\tau \right| \\
&\leq Mu(t) + M \left| \int_{-\infty}^{t-2} e^{-(t-\tau)} u(\tau)d\tau \right| \\
&= Mu(t) + M \left| \int_{0}^{t-2} e^{-(t-\tau)} d\tau \right| \\
&= Mu(t) + M \left| e^{-(t-\tau)} \big|_{\tau=0}^{t-2} \right| \\
&= Mu(t) + M \left| e^{-2} - e^{t} \right|
\end{aligned}
$$

This is not bounded, since as $t \to \infty$, $y(t) \to \infty$ and thus $y(t)$ is not stable. If $y(t)$ is not stable then $w(t)$ is also not stable since the integral of ∞ from $-\infty$ to ∞ is also ∞.

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(d)

Since these systems as cascading we have

$$h_{21}(t, \tau) = \int_{-\infty}^{\infty} h_2(\sigma, \tau) h_1(t, \sigma) d\sigma$$

$$= \int_{-\infty}^{\infty} u(\sigma) u(\sigma - \tau)(\delta(t - \sigma) u(\sigma) - e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma)) d\sigma$$

$$= \int_{-\infty}^{\infty} \delta(t - \sigma) u(\sigma) u(\sigma - \tau) d\sigma - \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma$$

$$= u(t) u(t - \tau) - \int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma$$

$\int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma$ can evaluate to 3 different expression

- When $t < 2$ or when $\tau > t - 2$, $u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) = 0$, thus we get

$$\int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma = 0$$

- When $t > 2$ and $\tau < 0$, $u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) = 1$ for $0 < \sigma < t - 2$ thus we get

$$\int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma = \int_0^{t-2} e^{-(t-\sigma)} d\sigma$$

$$= e^{\sigma - t} \big|_{\sigma=0}^{t-2}$$

$$= e^{-2} - e^{-t}$$

- When $t > 2$ and $0 < \tau < t - 2$ we get $u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) = 1$ for $\tau < \sigma < t - 2$ thus we get

$$\int_{-\infty}^{\infty} e^{-(t-\sigma)} u(\sigma) u(t - 2 - \sigma) u(\sigma - \tau) d\sigma = \int_{\tau}^{t-2} e^{-(t-\sigma)} d\sigma$$

$$= e^{\sigma - t} \big|_{\sigma=\tau}^{t-2}$$

$$= e^{-2} - e^{\tau - t}$$

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Thus we get that

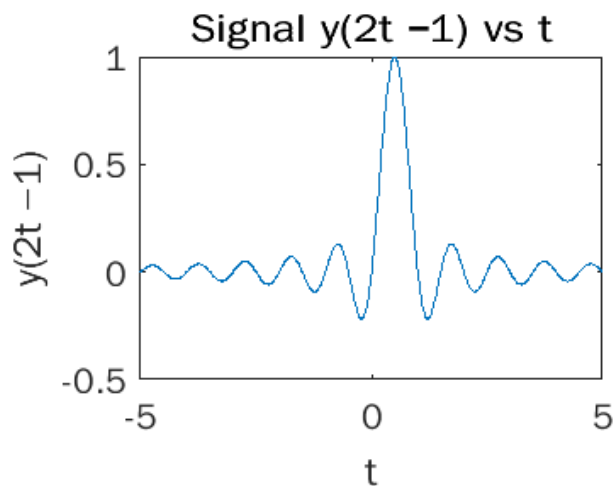$$h_{21}(t,\tau) = \begin{cases} 0 & t < 0 \text{ or } \tau > t \\ 1 & 0 < t < 2 \text{ or } t > \tau > t-2 \\ 1 + e^{-t} - e^{-2} & t > 2 \text{ and } \tau < 0 \\ 1 + e^{\tau-t} - e^{-2} & t > 2 \text{ and } 0 < \tau < t-2 \end{cases}$$
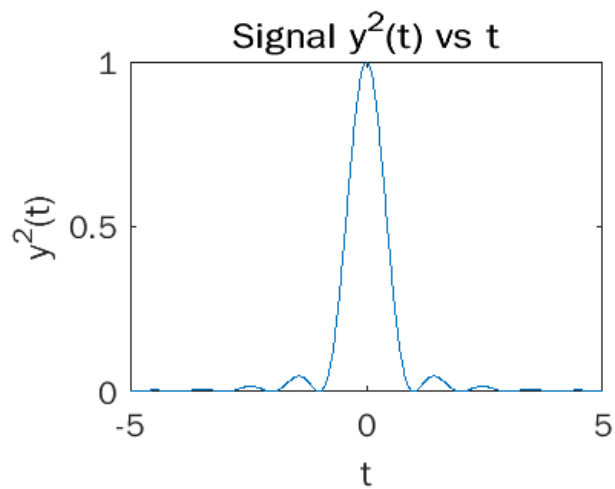
Problem 3

(a)

```
t = −5:0.01:5;
y=sin ( pi ∗(2∗t −1))./( pi ∗(2∗t −1));

figure;
plot (t ,y);
xlabel ('t');
ylabel ('y(2t    1 )');
title ('Signal y(2t    1 ) vs t');
```


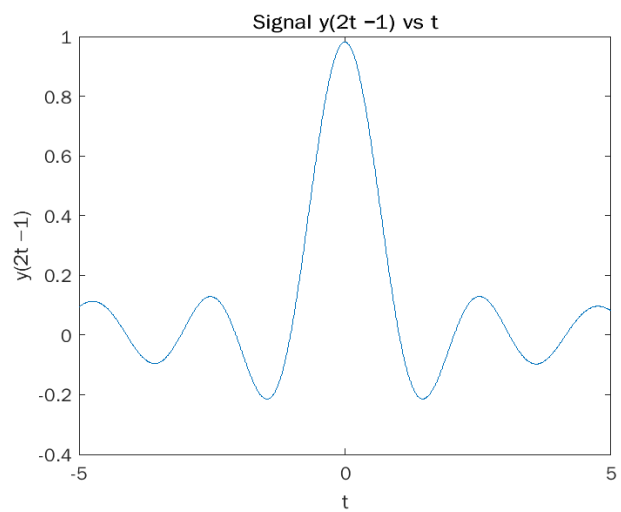Signal y(2t −1) vs t

(b)

```
t=-5:0.01:5;
y=sin(pi*t)./(pi*t);

figure;
plot(t,y.^2);
xlabel('t');
ylabel('y^2(t)');
title('Signal y^2(t) vs t');
```


(c)

```
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);

y = sin(pi*t)./(pi*t);
y(51)=1;
z = conv(y,y,'same')*dt;

figure;
plot(t,z);
xlabel('t');
```

```
ylabel('z(t)');
title('Signal z(t) vs t');
```



Signal y(2t −1) vs t

**(d)**

```
fy = @(t) sin(pi*t)./(pi*t);
hfig=figure;
%plot out z(2t-1)
t = linspace(-11, 11, 1000);
dt = t(2) - t(1);
y=fy(t);
%y(51)=1;
z = conv(y,y,'same')*dt;

subplot(3,1,1);
plot((t+1)/2,z);
xlim([-5,5]);
xlabel('t');
ylabel('z(2t-1)');
title('Signal z(2t-1) vs t');

%plot out {y(t)*y(2t-1)}
t = linspace(-5, 5, 1000);
dt = t(2) - t(1);
y=fy(t);
y2=fy(2*t-1);
subplot(3,1,2);
plot(t,conv(y,y2,'same')*dt);
xlabel('t');
ylabel('{y(t)*y(2t-1)}');
title('Signal {y(t)*y(2t-1)} vs t');

%plot out {y(2t-1)*y(2t-1)}
subplot(3,1,3);
plot(t,conv(y2,y2,'same')*dt);
xlabel('t');
ylabel('{y(2t-1)*y(2t-1)}');
title('Signal {y(2t-1)*y(2t-1)} vs t');
%print figure
print(hfig, '-djpeg', '3d');
```
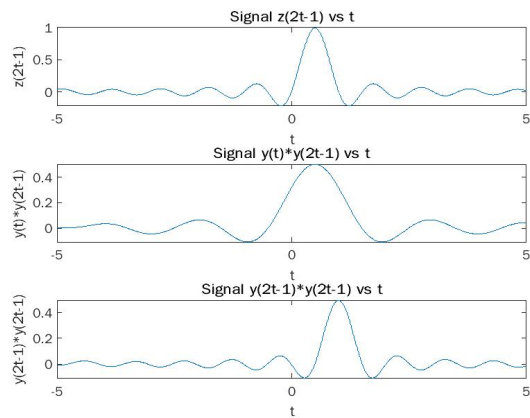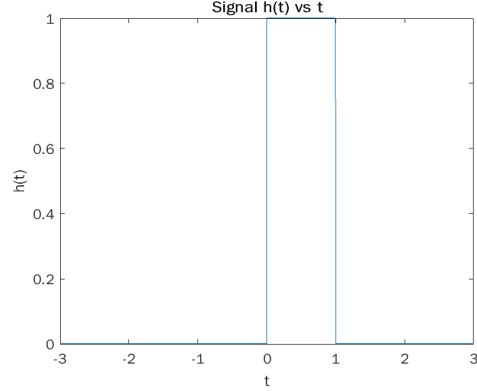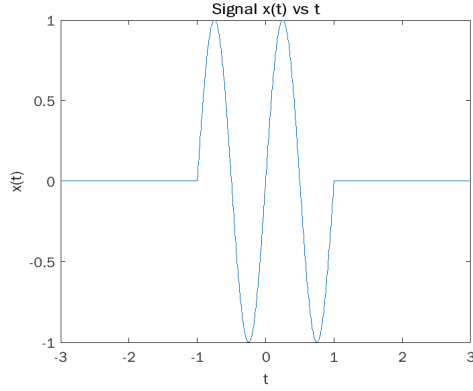
Thus the answer neither.

# Problem 4

## (a)

```
t=linspace(−3,3,1000);
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.∗pi.∗t).∗u(t + 1).∗u(−t + 1);
h=@(t) u(t)−u(t−1);

%plot x(t)
figure;
plot(t,x(t));
xlabel('t');
ylabel('x(t)');
title('Signal x(t) vs t');
%plot x(t)
figure;
plot(t,h(t));
xlabel('t');
```

```
ylabel('h(t)');
title('Signal h(t) vs t');
```


(b)

$$y_1(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau$$

Under different values of t, $\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau$ yields different expressions

- When $t \leq -1$ or $t \geq 2$, $u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right) = 0$ for all values of τ, thus we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau = 0$$

- When $-1 < t < 0$, $u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right) = 1$ for

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$-1 < \tau < t$, thus we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau = \int_{-1}^{t} \sin(2\pi\tau)d\tau$$

$$= -\frac{\cos(2\pi\tau)}{2\pi}\Big|_{\tau=-1}^{t}$$

$$= \frac{1 - \cos(2\pi t)}{2\pi}$$

- When $0 < t < 1$ we get $u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right) = 1$ for $t-1 < \tau < t$, since the period of $\sin(2\pi\tau)$ is 1 we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau = 0$$

- When $1 < t < 2$ we get $u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right) = 1$ for $t-1 < \tau < 1$, thus we get

$$\int_{-\infty}^{\infty} \sin(2\pi\tau)u(\tau+1)u(-\tau+1)\left(u(t-\tau) - u(t-\tau-1)\right)d\tau = \int_{t-1}^{1} \sin(2\pi\tau)d\tau$$

$$= -\frac{\cos(2\pi\tau)}{2\pi}\Big|_{\tau=t-1}^{1}$$

$$= \frac{\cos(2\pi(t-1)) - 1}{2\pi}$$

Therefore we get that:

$$y_1(t) = \begin{cases} 0 & t \leq -1 \\ \frac{1-\cos(2\pi t)}{2\pi} & -1 < t < 0 \\ 0 & 0 \leq t \geq 1 \\ \frac{\cos(2\pi(t-1))-1}{2\pi} & 1 < t < 2 \\ 0 & t \geq 2 \end{cases}$$

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(c)

$$y_2(t) = x(t) \cdot h(t)$$
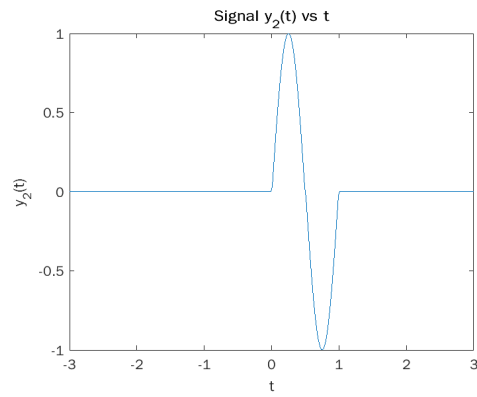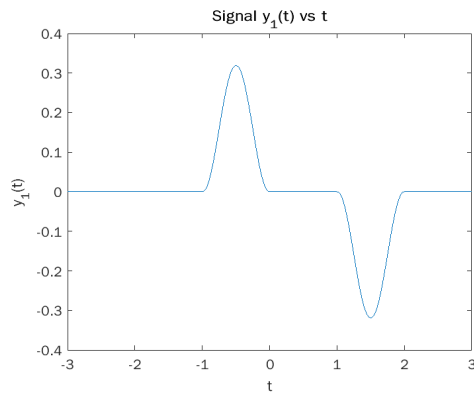$$= \boxed{\sin(2\pi t)u(t+1)u(-t+1)(u(t)-u(t-1))}$$

(d)

```
t=linspace(-3,3,600);
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
h=@(t) u(t)-u(t-1);

%plot y_1(t)
y1=[zeros(1,200),...
    (1.-cos(2.*pi.*t(201:300)))./(2*pi),...
    zeros(1,100),...
    (cos(2.*pi.*(t(401:500)-1))-1)./(2*pi),...
    zeros(1,100)];

figure;
plot(t,y1)
xlabel('t');
ylabel('y_1(t)');
title('Signal y_1(t) vs t');

%plot y_2(t)
figure;
plot(t,x(t).*h(t))
xlabel('t');
ylabel('y_2(t)');
title('Signal y_2(t) vs t');
```

Signal $y_1(t)$ vs t

Signal $y_2(t)$ vs t

(e)

```
t=linspace(-3,3,600);
%define functions
u=@(t) heaviside(t);
x = @(t) sin(2.*pi.*t).*u(t + 1).*u(-t + 1);
h=@(t) u(t)-u(t-1);

%plot y1(t)
figure;
plot(t,conv(x(t),h(t),'same')*(t(2)-t(1)));
xlabel('t');
ylabel('y_1(t)');
title('Signal y_1(t) vs t');
```


Signal $y_1(t)$ vs t