ECE 102 Homework 5

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1 Problem 1

(a)

We have that

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Therefore we have

$$\begin{aligned}
y(t) &= x(2t-3) + 4\frac{d^2x(t)}{dt^2} \\
&= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0(2t-3)} + 4\frac{d^2}{dt^2}\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} a_k e^{-3jk\omega_0} e^{j2k\omega_0 t} + 4\sum_{k=-\infty}^{+\infty} a_k \frac{d^2}{dt^2} e^{jk\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} a_k e^{-3jk\omega_0} e^{j2k\omega_0 t} - 4\sum_{k=-\infty}^{+\infty} a_k k^2\omega_0^2 e^{jk\omega_0 t} \\
b_k &= \boxed{\begin{cases} -4a_k k^2\omega_0^2 & \text{if k is odd} \\ a_{k/2} e^{\frac{-3jk\omega_0}{2}} - 4a_k k^2\omega_0^2 & \text{if k is even} \end{cases}}
\end{aligned}$$

(b)

if

$$x(t) \rightarrow a_k$$

we have

$$\begin{aligned}
x(t+1) &\rightarrow a_k e^{jk\omega_0} \\
e^{j\omega_0 t} x(t+1) &\rightarrow a_{k-1} e^{j(k-1)\omega_0} \\
\int_{-\infty}^{t} e^{-j\omega_0 t} x(t+1) &\rightarrow \frac{1}{jk\omega_0} a_{k-1} e^{j(k-1)\omega_0} \\
\int_{-\infty}^{t+2\alpha} e^{-j\omega_0 t} x(t+1) &\rightarrow \frac{1}{jk\omega_0} a_{k-1} e^{j(k-1)\omega_0} e^{2jk\omega_0 \alpha}
\end{aligned}$$

Thus we get

$$b_k = \boxed{\frac{1}{jk\omega_0} a_{k-1} e^{j(k-1)\omega_0} e^{2jk\omega_0 \alpha}}$$

(c)

$$\begin{aligned}
y(t) &= \frac{dx^3(t)}{dt} \\
&= 3x^2(t) \frac{dx(t)}{dt}
\end{aligned}$$

Thus we get that

$$b_k = 3 \int_{T_0} x^2(t) x'(t) e^{-jk\omega_0 t}$$

Since

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x'(t) = \sum_{k=-\infty}^{+\infty} jk\omega_0 a_k e^{jk\omega_0 t}$$

Thus we get

$$b_k = \boxed{3 \sum_{n+m+p=k} a_n a_m jp\omega_0 a_p}$$

Problem 2

(a)

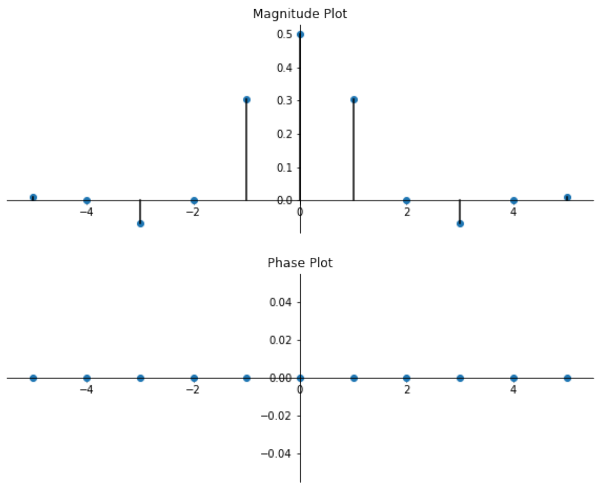
for $k \neq 0$

$$\begin{aligned} a_k &= \frac{1}{6}\int_{-3}^{3} x(t)e^{-jk\frac{2\pi}{6}t}dt \\ &= \frac{1}{6}\left(\int_{-2}^{-1}(t+2)e^{-jk\frac{2\pi}{6}t}dt + \int_{-1}^{1} e^{-jk\frac{2\pi}{6}t}dt + \int_{1}^{2}(-t-2)e^{-jk\frac{2\pi}{6}t}dt\right) \\ &= \frac{3(\cos(\frac{\pi k}{3}) - \cos(\frac{2\pi k}{3}))}{\pi^2 k^2} \end{aligned}$$

for $k = 0$

$$a_0 = \frac{1}{2}$$

(b)



(c)

$$x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)$$
$$x(t-2) = r(t) - r(t-1) - r(t-3) + r(t-4)$$
$$e^{-2s}X(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}$$
$$X(s) = \frac{e^{2s} - e^{s} - e^{-s} + e^{-2s}}{s^2}$$
$$a_k = \frac{1}{6}\left. X(s)\right|_{s=\frac{jk\pi}{3}} = \frac{3(\cos(\frac{\pi k}{3}) - \cos(\frac{2\pi k}{3}))}{\pi^2 k^2}$$

(d)

$$\begin{aligned} x(t) &= \frac{1}{2} + \sum_{k=1}^{\infty} a_k e^{k\Omega_0 ti} + a_{-k} e^{-k\Omega_0 ti} \\ &= \frac{1}{2} + \sum_{k=1}^{\infty} a_k (e^{k\Omega_0 ti} + e^{-k\Omega_0 ti}) \\ &= \frac{1}{2} + 2\sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) \end{aligned}$$

Thus we get

$$a_k = \frac{3(\cos(\frac{\pi k}{3}) - \cos(\frac{2\pi k}{3}))}{\pi^2 k^2}$$

and

$$b_k = 0$$

Problem 3

(a)

$$x(t) = \sum_{k=-\infty}^{\infty} e^{-2jk\pi} e^{2jk\pi t}$$

thus we get

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} e^{-2jk\pi} H(jk2\pi) e^{2jk\pi t} \\ &= \sum_{k=-\infty}^{\infty} e^{-2jk\pi} \frac{1}{jk2\pi + 4} e^{2jk\pi t} \\ &= \sum_{k=-\infty}^{\infty} \frac{e^{2jk\pi(t-1)}}{jk2\pi + 4} \end{aligned}$$

(b)

$$x(t) = \sum_{k=-\infty}^{\infty} \left(e^{-2jk\pi} - e^{-jk\pi}\right) e^{jk\pi t}$$

thus we get

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} \left(e^{-2jk\pi} - e^{-jkpi}\right) H(jk\pi) e^{jk\pi t} \\ &= \sum_{k=-\infty}^{\infty} \left(e^{-2jk\pi} - e^{-jkpi}\right) \frac{1}{jk\pi + 4} e^{jk\pi t} \\ &= \sum_{k=-\infty}^{\infty} \frac{e^{jk\pi(t-2)} - e^{jk\pi(t-1)}}{jk\pi + 4} \end{aligned}$$

Problem 4

(a)

From conditions (i,ii,iv,v) we get that $x(t)$ must be of the

$$x(t) = 2X_1 \cos(t\frac{\pi}{3}) + X_2 e^{t\frac{2\pi}{3}} + X_{-2} e^{-t\frac{2\pi}{3}}$$

From condition iii we get

$$\begin{aligned} x(t) &= -x(t-3) \\ 2X_1 \cos(t\frac{\pi}{3}) + X_2 e^{t\frac{2\pi}{3}} + X_{-2} e^{-t\frac{2\pi}{3}} &= -2X_1 \cos((t-3)\frac{\pi}{3}) - X_2 e^{(t-3)\frac{2\pi}{3}} - X_{-2} e^{-(t-3)\frac{2\pi}{3}} \\ &= -2X_1 \left(\cos(t)\cos(\pi) + \sin(t)\sin(\pi)\right) - X_2 e^{t\frac{2\pi}{3}} - X_{-2} e^{-t\frac{2\pi}{3}} \\ &= 2X_1 \cos(t) - X_2 e^{t\frac{2\pi}{3}} - X_{-2} e^{-t\frac{2\pi}{3}} \end{aligned}$$

Since this must hold for all t we get that $X_2 = 0$ and $X_{-2} = 0$ Similarly from condition (vi) we get that $X_1 = \frac{1}{2}$, thus

$$x(t) = \cos(\frac{\pi t}{3})$$

(b)

(i)

$x(t)$ is not real

(ii)

$x(t)$ is not even

(iii)

Since $x(t)$ is odd, its derivative is odd.

Problem 5

We can find the coefficients of the Fourier series by calculating the Laplace transform of $x(t)$.

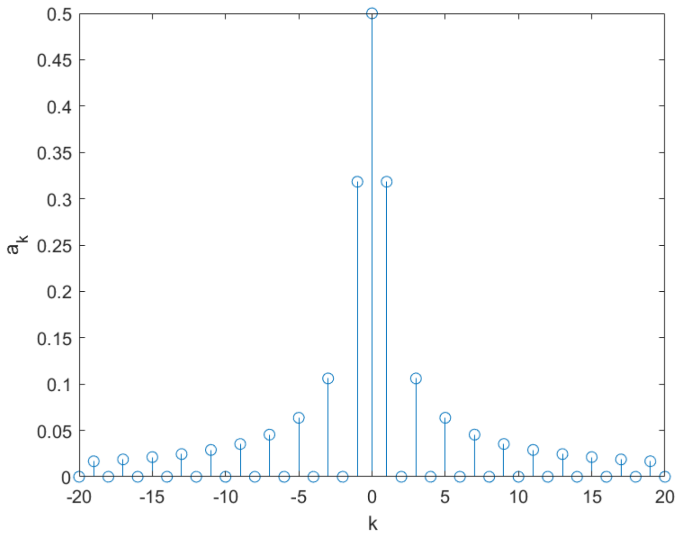
For $x_1(t) = u(t) - u(t-1)$

$$X(s) = \frac{1 - e^{-s}}{s}$$

thus

$$a_k = \frac{1}{2} X(s)\big|_{s=jk\pi} = \frac{1 - e^{-jk\pi}}{2jk\pi}$$

Therefore the plot of the magnitudes of $x_1(t)$ is



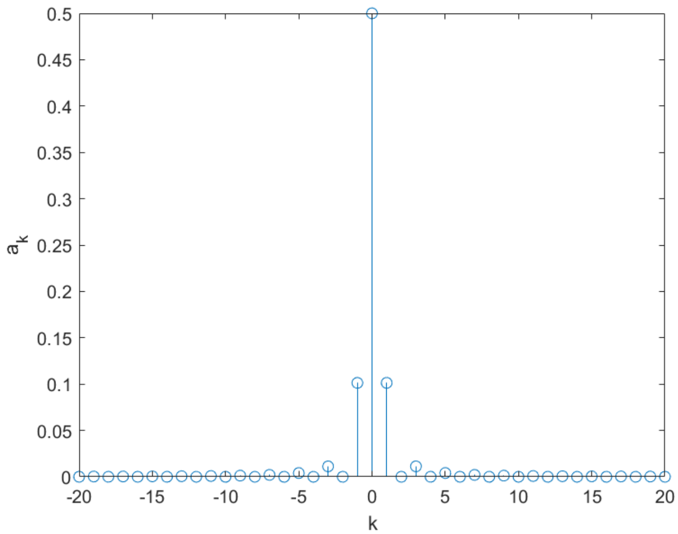
For $x_1(t) = r(t) - 2r(t-1) + r(t-2)$

$$X(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$$

thus

$$a_k = \frac{1}{2} \left. X(s) \right|_{s=jk\pi} = -\frac{1}{2} \frac{1 - 2e^{-jk\pi} + e^{-2jk\pi}}{k^2\pi^2}$$

Therefore the plot of the magnitudes of $x_2(t)$ is



The Fourier series coefficents for $x_2(t)$ decayed faster than those for $x_1(t)$ as $|k|$ increased. This means that $x_2(t)$ is "smoother" since for of the signals is distributed in the lower frequencies