ECE 102 Homework 6

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1 Problem 1

(a)

We have

$$\begin{aligned}
\int_0^t y(t-\tau)\tau^2 e^{3-\tau} d\tau &= \int_{-\infty}^{\infty} \tau^2 e^{3-\tau} y(t-\tau)u(t-\tau)d\tau \\
&= \left(t^2 e^{3-t}u(t)\right) * y(t) \\
&\to \left(\frac{2e^3}{(s+1)^3}\right) Y(s)
\end{aligned}$$

Thus we have

$$sY(s) + \left(\frac{2e^3}{(s+1)^3}\right) Y(s) = sX(s) - X(s)$$

And thus we have

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s+\frac{2e^3}{(s+1)^3}}$$

(b)

The Fourier coefficients of $x(t)$ is

$$a_k = \begin{cases} 1.5 & k = 0 \\ \frac{1}{4} & |k| = 2 \\ 0 & \text{everwhere else} \end{cases}$$

Therefore we have that the fourier series coefficients of $y(t)$, b_k are

$$b_k = H(j)a_k = \begin{cases} 1.5\frac{j-1}{j+\frac{2e^3}{(j+1)^3}} & k = 0 \\ \frac{1}{4}\frac{j-1}{j+\frac{2e^3}{(j+1)^3}} & |k| = 2 \\ 0 & \text{everwhere else} \end{cases}$$

(c)

From the fourier series properties, given that

$$y(t) \to b_k$$

we have

$$\begin{aligned} y(t-3) &\to b_k e^{-6jk} \\ y(2t) &\to b_k \\ y(t-3) * y(2t) &\to b_k^2 e^{-6jk} \end{aligned}$$

Problem 2

(a)

$$\begin{aligned}
X_1(j\omega) &= \int_{-\infty}^{+\infty} x_1(t)e^{-j\omega t}dt \\
&= \int_0^1 t^2 e^{-j\omega t}dt \\
&= \frac{\left(j\omega^2 + 2\omega - 2j\right)\mathrm{e}^{-j\omega t} + 2j}{\omega^3}
\end{aligned}$$

(b)

$$\begin{aligned}
X_2(j\omega) &= \int_{-\infty}^{+\infty} x_2(t)e^{-j\omega t}dt \\
&= 2\pi\delta(\omega) + \int_{-2}^{2} \cos(100t)e^{-j\omega t}dt \\
&= 2\pi\delta(\omega) + \frac{\mathrm{e}^{-2j\omega}\cdot\left(\left(\cos\left(200\right)\ j\omega + 100\sin\left(200\right)\right)\mathrm{e}^{4j\omega} - \cos\left(200\right)\ j\omega + 100\sin\left(200\right)\right)}{10000-\omega^2}
\end{aligned}$$

(c)

$$\int_{-\infty}^{t} \cos(5(t-\sigma))\delta(\sigma-2)d\sigma = \cos(5(t-2))\int_{-\infty}^{t} \delta(\sigma-2)d\sigma$$

We have that

$$
\begin{aligned}
\cos(t) &\to \pi(\delta(\omega-1)+\delta(\omega+1)) \\
\cos(t-2) &\to \pi e^{-2j\omega}(\delta(\omega-1)+\delta(\omega+1)) \\
\cos(5(t-2)) &\to \frac{\pi}{5}e^{-2j\frac{\omega}{5}}(\delta(\frac{\omega}{5}-1)+\delta(\frac{\omega}{5}+1)) \\
&= \pi e^{-2j}\delta(\omega-5)+\pi e^{2j}\delta(\omega+5)
\end{aligned}
$$

$$
\begin{aligned}
\cos(5(t-2))\int_{-\infty}^{t}\delta(\sigma-2)d\sigma &\to \left(\pi e^{-2j}\delta(\omega-5)+\pi e^{2j}\delta(\omega+5)\right) * \left(\frac{e^{-2j\omega}}{j\omega}+\pi\delta(\omega)\right) \\
&= \pi e^{-2j}\left(\frac{e^{-2j(\omega-5)}}{j(\omega-5)}+\pi\right)\delta(\omega-5)+\pi e^{2j}\left(\frac{e^{-2j(\omega+5)}}{j(\omega+5)}+\pi\right)\delta(\omega+5)
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
X(0) &= \int_{-\infty}^{\infty} x(t)dt \\
&= 6
\end{aligned}
$$

Likewise, since

$$\int_{-\infty}^{\infty}\frac{1}{j\omega}+\pi\delta(\omega)d\omega=\pi$$

And

$$x(t)=u(t+1)+u(t)-u(t-2)-u(t-3)$$

Thus we have

$$\int_{-\infty}^{\infty}X(j\omega)d\omega=3\pi$$

(b)

We know that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = 10$$

Thus from Parseval's Reltaion we have

$$\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 20\pi$$

(c)

We have

$$\frac{2\sin(\omega)}{\omega} \rightarrow \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

$$e^{j2\omega}\frac{2\sin(\omega)}{\omega} \rightarrow y(t) = \begin{cases} 1 & -3 < t < -1 \\ 0 & \text{elsewhere} \end{cases}$$

Thus we have

$$y(-t) = \begin{cases} 1 & 1 < t < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore we have

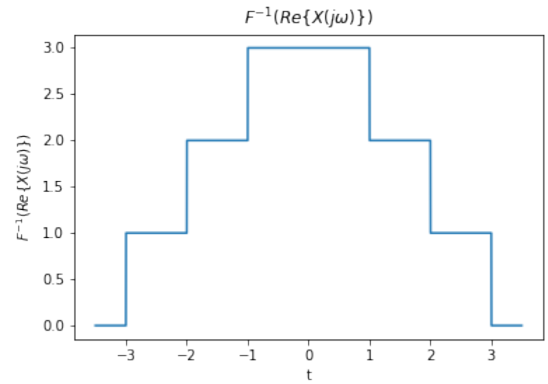
$$\int_{-\infty}^{\infty} X(\omega)Y(\omega)d\omega = 2\pi \int_{-\infty}^{\infty} x(t)y(-t)dt = 6\pi$$

(d)

The inverse Fourier transform of $Re\{X(j\omega)\}$ is the even part of $x(t)$ or

$$u(t+3) + u(t+2) + u(t+1) - u(t-1) - u(t-2) - u(t-3)$$

The plot of which looks like



Problem 4

From condition 1 and 2 we know that $x(t)$ must be of the form

$$x(t) = -ce^{-2t}u(t) + ce^{-t}u(t)$$

From condition 3 we have

$$\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = 2\pi$$

Therefore, from parseval's we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |x(t)|^2 dt &= 1 \\ c^2 \int_0^{+\infty} e^{-4t} dt + 2c^2 \int_0^{+\infty} e^{-3t} dt + c^2 \int_0^{+\infty} e^{-2t} dt &= 1 \\ \frac{c^2}{4} - \frac{2c^2}{3} + \frac{c^2}{2} &= 1 \end{aligned}$$

Thus we have $c = 2\sqrt{3}$, $x(t) = 2\sqrt{3}(e^{-t} - e^{-2t})u(t)$ and thus $A = -2\sqrt{3}$

Problem 5

(a)

$$X(s) = \frac{2\pi}{(s+1)^2 + 4\pi^2} + e^{-2s}$$
$$X(j\omega) = \frac{2\pi}{(j\omega+1)^2 + 4\pi^2} + e^{-2j\omega}$$

(b)

We have that
$$|X(j\omega)|^2 = \sqrt{X(j\omega)X^*(j\omega)}$$
Since $x(t)$ is real we have
$$X*(j\omega) = X(-j\omega)$$
Therefore we have
$$|X(j\omega)|^2 = \left(\frac{2\pi}{(j\omega+1)^2 + 4\pi^2} + e^{-2j\omega}\right)\left(\frac{2\pi}{(-j\omega+1)^2 + 4\pi^2} + e^{2j\omega}\right)$$
$$|X(j\omega)| = \sqrt{\left(\frac{2\pi}{(j\omega+1)^2 + 4\pi^2} + e^{-2j\omega}\right)\left(\frac{2\pi}{(-j\omega+1)^2 + 4\pi^2} + e^{2j\omega}\right)}$$
if $X(j\omega) + X(-j\omega) \geq 0$ we have
$$\measuredangle X(j\omega) = \arctan\left(\frac{X(j\omega) - X(-j\omega)}{i(X(j\omega) + X(-j\omega))}\right)$$
$$= \arctan\left(\frac{-2\sin(2\omega)(\omega^4 - 8\pi^2\omega^2 + 2\omega^2 + 1 + 8\pi^2 + 16\pi^4) - 8\pi\omega}{2\cos(2\omega)(\omega^4 - 8\pi^2\omega^2 + 2\omega^2 + 1 + 8\pi^2 + 16\pi^4) + 4\pi\left(-\omega^2 + 1 + 4\pi^2\right)}\right)$$
And if $X(j\omega) + X(-j\omega) < 0$ we have
$$\measuredangle X(j\omega) = \arctan\left(\frac{-2\sin(2\omega)(\omega^4 - 8\pi^2\omega^2 + 2\omega^2 + 1 + 8\pi^2 + 16\pi^4) - 8\pi\omega}{2\cos(2\omega)(\omega^4 - 8\pi^2\omega^2 + 2\omega^2 + 1 + 8\pi^2 + 16\pi^4) + 4\pi\left(-\omega^2 + 1 + 4\pi^2\right)}\right) + \pi$$

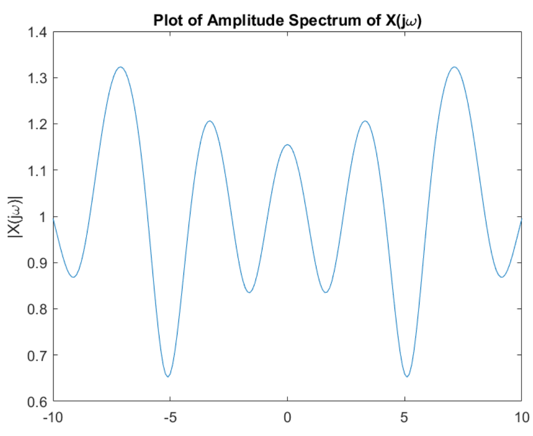
(c)

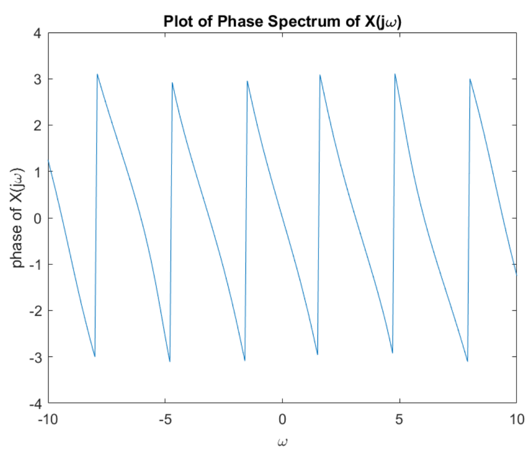
using this code, we get

```
omega=-10:0.1:10;

X=(2*pi)./((1i*omega+1).^2+4*pi.^2)+exp(-2i*omega);

plot(omega,abs(X))
xlabel '\omega'
ylabel '|X(j\omega)|'
title('Plot of Amplitude Spectrum of X(j\omega)')
figure;
plot(omega,angle(X))
xlabel '\omega'
ylabel 'phase of X(j\omega)'
title('Plot of Phase Spectrum of X(j\omega)')
```




Plot of Phase Spectrum of X(jω)
phase of X(jω)
ω
4
3
2
1
0
-1
-2
-3
-4
-10
-5
0
5
10