ECE 131A HW 3

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Problem 1

Then this must be a geometric distribution, so the pmf would be

$$p(Y = y) = (1-p)^{y-1}p$$

Problem 2

(a)

$$\frac{\frac{1}{1}}{\sum_{n=1}^{10} \frac{1}{n^s}} = \boxed{0.341417152147}$$

(b)

$$\frac{\sum_{n=6}^{10} \frac{1}{n^s}}{\sum_{n=1}^{10} \frac{1}{n^s}} = \boxed{0.22043083593}$$

Problem 3

(a)

we have

$$k \int_0^1 x^2(1-x)^2 dx = 1$$
$$k \int_0^1 x^2 - 2x^3 + x^4 dx = 1$$
$$k \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1 = 1$$
$$k \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5} \right) = 1$$
$$\boxed{k = 30}$$

(b)

$$\begin{aligned} P(X \geq \frac{3}{4}) &= 30 \int_{\frac{3}{4}}^1 x^2(1-x)^2 dx \\ &= 30 \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_{\frac{3}{4}}^1 \\ &= \boxed{0.103515625} \end{aligned}$$

Problem 4

(a)

Let the price of gas be a random variable X, then we have that the total cost $Y = 12X + 1$, thus

$$E[Y] = E[12X + 1] = 12E[X] + 1 = 12 \cdot 4.40 + 1 = \boxed{53.8}$$

(b)

We have

$$\text{Var}(X) = 12^2 \text{Var}(Y) = 12^2 \left(\frac{1}{12}(0.2)^2 \right) = 0.48$$

Problem 5

(a)

We have that as a function of the distance d of the shorter part in porportion to the entire rod, so evenly distributed from 0 to 0.5. then the ratio of the lengths is

$$r(d) = \frac{d}{1-d}$$

Since this function is invertible for the range of $d \in [0, 0.5]$, we have

$$d(r) = \frac{r}{r+1}$$

thus we have that the probability density function is

$$f_R(r) = f_d(d(r)) \left| \frac{dd(r)}{dr} \right|$$

$$f_R(r) = \boxed{\frac{2}{(r+1)^2}}$$

(b)

$$E[R] = \int_0^1 \frac{2r}{(r+1)^2} dr = 0.38629436112$$

(c)

$$E[R^2] = \int_0^1 \frac{2r^2}{(r+1)^2} dr = 0.22741127776$$
$$Var(R) = E[R^2] - E[R]^2 = \boxed{0.07818794432}$$

Problem 6

This is an erlang distibution, with $k = 7$, this's cdf $P(X \leq 10) = 1 - \sum_{n=0}^{6} \frac{(10\lambda)^n}{n!} e^{-10\lambda}$, so we have that the probability of wait for more than 10 minutes is, since $\lambda = 1$

$$1 - P(X \leq 10) = \boxed{\sum_{n=0}^{6} \frac{(10)^n}{n!} e^{-10} = 0.130141420882}$$

Problem 7

(a)

Since the probability fo getting $X = 1$ is 0 for a countinuous random variable, we have that

$$P(Y \geq 1) = P(X > 1) = \boxed{e^{-3}}$$

(b)

$$P(Y \geq 10) = P(X > 9) = \boxed{e^{-30}}$$

(c)

$$P(Y \geq x) = P(X > x) = \boxed{e^{-3(x)}}$$

Problem 8

The standard z score for a standard normal Z to have $P(Z \leq z) = 0.1492$ is -1.03987, so we have that

$$x = \mu + \sigma z = -1.03987 \cdot 2\sqrt{2} + 22 = \boxed{19.0588034858}$$

Problem 9

(a)

Let $X = \int_{-\infty}^{\infty} e^{-x^2} dx$, then we have that

$$\begin{aligned} X^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy \\ &= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta \\ &= 2\pi \int_{0}^{\infty} e^{-r^2} r dr \\ &= \pi \end{aligned}$$

Thus we have that $X = \sqrt{\pi}$, then we have that

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{1}$$

Let $y = x - \mu$, then we have that

$$\int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy$$

Let $z = \frac{y}{\sigma\sqrt{2}}$, then we have that $dz = \frac{dy}{\sigma\sqrt{2}}$, thus we have that

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \\ &= 1 \end{aligned}$$

Thus $f(x)$ could be a density function

(b)

we have through integration by parts

$$\begin{aligned}
\sqrt{2\pi}(1-\Phi(x)) &= \int_x^{+\infty} e^{-\frac{y^2}{2}}\,dy \\
&= -y^{-1}e^{-\frac{y^2}{2}}\Big|_x^{\infty} - \int_x^{\infty} y^{-2}e^{-\frac{y^2}{2}}\,dy \\
&= x^{-1}e^{-\frac{x^2}{2}} - \int_x^{\infty} y^{-2}e^{-\frac{y^2}{2}}\,dy \\
&= x^{-1}e^{-\frac{x^2}{2}} + y^{-3}e^{-\frac{y^2}{2}}\Big|_x^{\infty} + \int_x^{\infty} y^{-3}e^{-\frac{y^2}{2}}\,dy \\
&= (x^{-1}-x^{-3})e^{-\frac{x^2}{2}} + \int_x^{\infty} y^{-3}e^{-\frac{y^2}{2}}\,dy
\end{aligned}$$

Therefore since $\int_x^{\infty} y^{-3}e^{-\frac{y^2}{2}}\,dy > 0$ and $\int_x^{\infty} y^{-2}e^{-\frac{y^2}{2}}\,dy > 0$ we have

$$(x^{-1}-x^{-3})e^{-\frac{x^2}{2}} < \sqrt{2\pi}(1-\Phi(x)) < x^{-1}e^{-\frac{x^2}{2}}$$

(c)

For $x > 0$ we have that

$$P(X > x + \frac{a}{x}|X > x) = \frac{P(X > x + \frac{a}{x})}{P(X > x)}$$

Now we use the squeeze theorem, we have that

$$\frac{\left(x+\frac{a}{x}\right)^{-1} e^{-\frac{1}{2}(x+\frac{a}{x})^2}}{(x^{-1}-x^{-3})e^{-\frac{x^2}{2}}} > \frac{P(X > x + \frac{a}{x})}{P(X > x)} > \frac{\left(\left(x+\frac{a}{x}\right)^{-1} - \left(x+\frac{a}{x}\right)^{-3}\right) e^{-\frac{1}{2}(x+\frac{a}{x})^2}}{x^{-1}e^{-\frac{x^2}{2}}}$$

since $x + \frac{a}{x} \to x$ as $x \to \infty$ we have that

$$\lim_{x\to\infty} \frac{\left(x+\frac{a}{x}\right)^{-1} e^{-\frac{1}{2}(x+\frac{a}{x})^2}}{(x^{-1}-x^{-3})e^{-\frac{x^2}{2}}} = e^{-a}$$

and

$$\lim_{x\to\infty} \frac{\left(\left(x+\frac{a}{x}\right)^{-1} - \left(x+\frac{a}{x}\right)^{-3}\right) e^{-\frac{1}{2}(x+\frac{a}{x})^2}}{x^{-1}e^{-\frac{x^2}{2}}} = e^{-a}$$

Thus we have that

$$P\left(X > x + \frac{a}{x} | X > x\right) \to e^{-a}$$

As $x \to \infty$