ECE 131A, Fall 2022 **Homework #5**

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Due Friday, 18 November 2022, uploaded to Gradescope.
Covers material up to Lecture 12.
100 points total.

1. (10 points) **Normal random variable**

   (a) (5 points) Consider a standard Normal random variable Z. Find a constant c with the property that $P(-c < Z < c) = 0.8324$.

   (b) (5 points) Now consider a Normal random variable Y with $E(Y) = 4.2$ and $Var(Y) = 2.3$. Find a constant c with the property that $P(4.2 - c < Y < 4.2 + c) = 0.8324$.

2. (10 points) **Heavy books**

   Suppose that the books published by a certain book publisher have weights that (roughly) have a Normal distribution with mean 14.2 ounces and standard deviation 1.7 ounces.

   (a) (3 points) What is the probability that such a book weighs less than 1 pound?

   (b) (3 points) What is the probability that such a book weighs in the range 13 to 15 ounces?

   (c) (4 points) Suppose that we select ten books from this publisher, and that their weights are independent. A book is considered "heavy" if it weighs 16 ounces or more. What is the probability that exactly three of the ten selected books are considered "heavy"?

3. (10 points) **Constant joint probability density function**

   Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices located at the points $(0, 0), (3, 0), (5, 2), (0, 2)$.

   (a) (3 points) Find $P(X \geq 3)$.

   (b) (3 points) Find $P(Y \geq 1)$.

   (c) (4 points) Find $P(\max(X, Y) \leq 1)$.

4. (10 points) **Group Me messaging**

   Suppose that the time (in seconds) until the next message arrives in Group Me is a continuous random variable X, and the time until the reply is denoted by Y. For this reason, we always have $Y > X$. Suppose that the joint probability density function of X and Y is

   $$f_{X,Y}(x, y) = \frac{1}{750} e^{-(\frac{x}{150} + \frac{y}{30})}$$

   for $y > x > 0$, and $f_{X,Y}(x, y) = 0$ otherwise. Find $E[X]$.

5. (10 points) **Functions of random variables**

   Consider a pair of random variables X, Y with constant joint density on the triangle with vertices at $(0, 0), (3, 0)$, and $(0, 3)$.

(a) (5 points) Find the expected value of the sum of X and Y.

(b) (5 points) Find the variance of X.

6. (10 points) **Tray of drinks**

   Consider a tray with 8 lemonades and 3 raspberry juices. Alice and Bob each take 1 drink from the tray, without replacement. Assume that all of their choices are equally likely. Let X be the number of lemonades that Alice and Bob get. Find the variance of X.

7. (10 points) **Happy bear pairs**

   Consider a collection of 6 bears. There is a pair of red bears consisting of one father bear and one mother bear. There is a similar green bear pair, and a similar blue bear pair. These 6 bears are all placed in a straight line, and all arrangements in such a line are equally likely. A bear pair is happy if it is sitting together. Let X denote the number of happy bear pairs. Let $X_i = 1$ if the i^{th} bear pair is sitting together, and $X_i = 0$ otherwise.

   (a) (4 points) Find $Cov(X_i, X_i) = E(X_iX_i) - E(X_i)E(X_i), \quad \text{for } 1 \leq i \leq 3$.

   (b) (4 points) Find $Cov(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j), \quad \text{for } 1 \leq i, j \leq 3 \text{ and } i \neq j$.

   (c) (2 points) Find $Var(X)$.

8. (10 points) **Rare bird flying over UCLA**

   Suppose that the number of rare birds that fly over UCLA in 1 hour has a Poisson distribution with mean 2. Also suppose that the number of birds flying is independent from hour to hour (e.g., the number of birds between noon and 1 PM does not affect the number of birds between 1 PM and 2 PM, etc.). During 40 hours of observation, what is the approximate probability that 75 or more birds are seen?

9. (10 points) **Sum of geometric random variables**

   Suppose that $X_1, \cdots, X_{500}$ are independent Geometric random variables, each of which have $E(X_j) = \frac{8}{5}$.

   (a) (2 points) Find the expected value and the variance of the sum of the 500 random variables.

   (b) (4 points) Find a good approximation for $P(780 < X_1 + \cdots + X_{500} < 820)$.

   (c) (4 points) Suppose that Y is a Negative Binomial random variable, independent from the X_j's, with parameters $r = 250$ and $p = \frac{1}{3}$. Calculate a good estimate for $P(Y < X_1 + \cdots + X_{500})$.

10. (10 points) **Moment generating function**

    (a) (4 points) Suppose that X is an Exponential random variable with $E(X) = \frac{1}{3}$. Find the moment generating function $g(t) = M_X(t) = E(e^{tX})$ of X. (It is OK to assume $t < 3$.)

    (b) (4 points) Compute the derivative of the moment generating function with respect to t.

    (c) (2 points) Compute $M_X^{'}(0)$.