ECE 131A, Fall 2022 **Homework #4**

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Due Monday, 7 November 2022, uploaded to Gradescope.
Covers material up to Lecture 10.
100 points total.

1. (10 points) **Memoryless property**

   Let Y be a discrete random variable that assumes only non-negative integer values and that satisfies the memoryless property. Find the probability mass function (pmf) of Y.

2. (10 points) **Frequency of requests to websites**

   Ten news websites are ranked in terms of popularity, and the frequency of requests to these sites are known to follow a Zipf distribution.

   (a) (5 points) What is the probability that a request is for the top-ranked site?

   (b) (5 points) What is the probability that a request is for one of the bottom five sites?

3. (10 points) **Probability density function**

   Let X have probability density $f_X(x) = kx^2(1-x)^2$ for $0 \leq x \leq 1$, and $f_X(x) = 0$ otherwise, where k is constant.

   (a) (5 points) Find the value of k.

   (b) (5 points) Find $P(X \geq \frac{3}{4})$.

4. (10 points) **Uniform random variables**

   Suppose that, when you buy gas at the gas station, the price is uniformly distributed between \$4.30 and \$4.50 per gallon. You plan to buy 12 gallons of gasoline, plus a candy bar for an extra \$1.00. (Assume that there is no tax on your purchase.)

   (a) (5 points) Find the expected value of the cost of your purchase.

   (b) (5 points) Find the variance of the cost of your purchase.

5. (15 points) **Breaking a rod at random**

   You break a rod at random into two pieces. Let R be the ratio of the lengths of the shorter to the longer piece.

   (a) (5 points) Find the probability density function $f_R(r)$.

   (b) (5 points) Compute $E[R]$.

   (c) (5 points) Compute $Var(R)$.

6. (10 points) **Passengers at a taxi stand**

   Passengers arrive at a taxi stand at an airport at a rate of one passenger per minute. The taxi driver will not leave until seven passengers arrive to fill his van. Suppose that passenger inter-arrival times are exponential random variable, and let X be the time to fill a van. Find the probability that more than 10 minutes elapse until the van is full.

7. (15 points) **Rounding down**

   Let $Y = \lfloor X \rfloor$ denote the largest integer that is less than or equal to X. For instance: $\lfloor 7.2 \rfloor = 7$, $\lfloor 2.99 \rfloor = 2$ and $\lfloor 4 \rfloor = 4$. Now suppose that X is an exponential random variable with $E(X) = \frac{1}{3}$.

   (a) (5 points) Find $P(Y \geq 1)$.

   (b) (5 points) Find $P(Y \geq 10)$.

   (c) (5 points) Can you generalize? What is $P(Y \geq x)$, when x is a (non-negative) integer?

8. (5 points) **Measuring the amount of sugar in candies**

   The quantity of sugar X (measured in grams) in a randomly-selected piece of candy is normally distributed, with $E(X) = 22$ and $Var(X) = 8$. Find the quantity x of sugar, so that exactly 14.92% of the candy has less than x grams of sugar.

9. (15 points) **Inequalities satisfied by a normal distribution**

   (a) (5 points) Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, and deduce that

   $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \qquad -\infty < x < \infty,$$

   is a density function if $\sigma > 0$.

   (b) (5 points) Show that the $\mathcal{N}(0,1)$ cumulative distribution function Φ satisfies

   $$(x^{-1} - x^{-3})e^{-\frac{1}{2}x^2} < \sqrt{2\pi}[1 - \Phi(x)] < x^{-1}e^{-\frac{1}{2}x^2}, \qquad x > 0.$$

   These bounds are of interest because Φ has no closed form.

   (c) (5 points) Let $X \sim \mathcal{N}(0,1)$, and $a > 0$. Show that $P(X > x + \frac{a}{x} | X > x) \to e^{-a}$ as $x \to \infty$.