ECE 131A, Fall 2022 **Homework #6**

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Due Friday, 2 December 2022, uploaded to Gradescope.
Covers material up to Lecture 14.
100 points total.

1. (5 points) **Lifetime of a pen**

   A student uses pens whose lifetime is an exponential random variable with mean 1 week. Find an estimate for the minimum number of pens he should buy at the beginning of a 15-week semester, so that with probability 0.99 he does not run out of pens during the semester.

2. (5 points) **Binary transmission channels**

   A binary transmission channel introduces bit errors with probability 0.15 . Estimate the probability that there are 20 or fewer errors in 100 bit transmissions.

3. (5 points) **Linear transformation of gaussian random vectors**

   Let $X = [x_1, x_2]^T$ be a real-valued gaussian random vector with the following mean vector and covariance matrix:

$$\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

   (a) (1 point) Find the probability density $f_X(x_1, x_2)$ explicitly in terms of x_1 and x_2

   (b) (2 points) Suppose we define the following linear transformation of the gaussian random vector

$$Y = AX + B,$$

   where

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}$$
$$B = 1.$$

   Find the probability density function of Y.

   (c) (2 points) Find $P(Y \geq 2)$

4. (10 points) **Independent gaussian random variables**

   Let X_1,X_2,X_3,X_4 be independent zero-mean, unit-variance Gaussian random variables that are processed as follows:

$$Y_1 = X_1 + X_2,$$
$$Y_2 = X_2 + X_3,$$
$$Y_3 = X_3 + X_4.$$

(a) (2 points)Find the covariance matrix of $Y = [Y_1, Y_2, Y_3]^T$.

(b) (1 point) Find the joint probability density function of Y.

(c) (4 points) Find the joint probability density function of Y_1 and Y_2; Y_1 and Y_3.

(d) (3 points)Find a transformation A such that the vector $Z = AY$ consists of independent Gaussian random variables

5. (5 points) **Expected value of jointly gaussian random variables**

   Let $X = [X_1, X_2, X_3, X_4]^T$ be zero-mean jointly gaussian random variables. Show that

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$

6. (5 points) **Error in rounding**

   The sum of a list of 64 real numbers is to be computed. Suppose that numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Estimate the probability that the total error in the sum of the 64 numbers exceeds 4.

7. (10 points) **Symmetric Binary transmission system**

   A binary transmission system sends a "0" bit using a signal of -1V voltage and a "1" bit by transmitting a $+1$V signal, The received signal is corrupted by noise N that has a Laplacian distribution with parameter α. Assume that "0" bits and "1" bits are equiprobable. Let the received signal $Y = X + N$, where X is the transmitted signal.

   (a) (2 points) Find the probability density function of Y given that a "0" was transmitted

   (b) (2 points) Find the probability density function of Y given that a "1" was transmitted

   (c) (1 point) Find the probability density function of Y

   (d) (3 points) Suppose that the receiver decides a "0" was sent if $Y < 0$, and a "1" was sent if $Y \geq 0$. What is the probability that the receiver makes an error given that a $X = +1$? What is the error probability given that a $X = -1$?

   (e) (2 points) What is the overall probability of error?

8. (10 points) **Asymmetric Binary transmission system**

   A binary transmission system transmits a signal X (-1 to send a "0" bit; $+1$ to send a "1" bit). The received signal is $Y = X + N$ where noise N has a zero-mean Gaussian distribution with variance σ^2. Assume that "0" bits are three times as likely as "1" bits.

   (a) (2 points) Find the conditional pdf of Y given the input value: $f_Y(y|X = +1)$ and $f_Y(y|X = -1)$.

   (b) (3 points) The receiver decides a "0" was transmitted if the observed value of y satisfies

$$f_Y(y|X = -1)P[X = -1] > f_Y(y|X = +1)P[X = +1]$$

   and it decides a "1" was transmitted otherwise. Use the results from part (a) to show that this decision rule is equivalent to: If $Y < T$ decide "0"; if $Y \geq T$ decide "1".

(c) (3 points) What is the probability that the receiver makes an error given that +1 was transmitted? −1 was transmitted? Assume $\sigma^2 = 1/16$.

(d) (2 points) What is the overall probability of error?

9. (10 points) **Characteristic function of sum of random variables**

   Let X and Y be independent exponential random variables with parameters 2 and 10, respectively. Let $Z = X + Y$.

   (a) (3 points) Find the characteristic function of Z.

   (b) (2 points) Find the pdf of Z from the characteristic function found in part a.

10. (5 points) **Trapped miner**

    A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours of travel. If we assume the miner is all times equally likely to choose any one of the door, what is the expected length of time he reaches safety?

11. (5 points) **Painting bears**

    Suppose that, at the start of the day, a jar contains a random number of bears that has a Poisson distribution with mean 18. During the day, each such bear is randomly painted red, blue, or yellow, with each of these three possibilities being equally likely, and with the painting of bears being independent. Suppose that, at the end of the day, we discover that there are 7 red bears and 3 blue bears.

    (a) (3 points) Given all of the information above, what kind of random variable is the number of yellow bears at the end of the day?

    (b) (2 points) How many yellow bears do we expect?

12. (5 points) **Conditional expectation**

    Consider a pair of random variables X, Y with constant joint density on the quadrilateral with vertices located at the points $(0, 0), (3, 0), (5, 2), (0, 2)$.

    (a) (1 point) Find the probability density function of Y.

    (b) (2 points) Find the conditional probability density function $f_{X|Y}(x|y)$ of X, given $Y = y$.

    (c) (1 point) Verify that $f_{X|Y}(x|y)$ is a valid density function

    (d) (1 point) Find $E(X|Y = y)$, for a fixed y with $0 \leq y \leq 2$.

13. (5 points) **Radioactive emissions**

    Suppose the number of particle emissions by a radioactive mass in t seconds is a Poisson random variable with mean λt. Obtain a bound for the probability that $|\frac{N(t)}{t} - \lambda|$ exceeds ϵ, where $N(t)$ is the number of particle emissions by a radioactive mass in t seconds.

14. (5 points) **Moments of a sum**

    Let $X_1, X_2, \cdots, X_n$ be random variables with the same mean and with covariance function

    $$Cov(X_i, X_j) = \sigma^2 \rho^{|i-j|},$$

    where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + X_2 + \cdots + X_n$.

15. (5 points) **Eating habits**

    In a study on eating habits, a particular participant averages $750cm^3$ of food per meal.

    (a) (3 points) It is extraordinarily rare for this participant to eat more than $1000cm^3$ of food at once. Find a bound on the probability of such an event.

    (b) (2 points) If the standard deviation of a meal size is $100cm^3$, then find a bound on the event that the meal is either too much food, i.e., more than $1000cm^3$, or an insufficient amount of food, namely, less than $500cm^3$.

16. (5 points) **Height of corns**

    The height at a certain point of summer for a particular species of corn is randomly distributed, with mean height 5.2 feet, and standard deviation of 0.05 feet.

    (a) (2 points) Find a bound on the probability that a randomly chosen stalk of corn is 6 feet or higher at that point in the summertime.

    (b) (3 points) Find a bound on the probability that a randomly chosen stalk of corn is outside the range between 5 feet and 5.4 feet high at that point in the summertime.