

ECE 131A HW 1

Lawrence Liu

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Problem 1

(a)

$$\begin{aligned} P[B_0] &= P[A_0]P[B_0|A_0] + P[A_1]P[B_0|A_1] \\ &= \boxed{\frac{1}{2} \cdot (1 - \epsilon_1 + \epsilon_2)} \end{aligned}$$

(b)

We have

$$P[B_1] = 1 - P[B_0] = \frac{1}{2} \cdot (\epsilon_1 + 1 - \epsilon_2)$$

Therefore from Bayes law we have

$$\begin{aligned} P[A_1|B_1] &= \frac{P[B_1|A_1]P[A_1]}{P[B_1]} \\ &= \frac{1 - \epsilon_2}{\epsilon_1 + 1 - \epsilon_2} \end{aligned}$$

$$\begin{aligned}
 P[A_0|B_1] &= \frac{P[B_1|A_0]P[A_0]}{P[B_1]} \\
 &= \frac{\epsilon_1}{\epsilon_1 + 1 - \epsilon_2}
 \end{aligned}$$

Therefore we have for $\epsilon_1 = 0.25$ and $\epsilon_2 = 0.5$: we will have

$$\begin{aligned}
 P[A_1|B_1] &= \frac{1 - 0.5}{0.25 + 1 - 0.5} \\
 &= \frac{2}{3} \\
 P[A_0|B_1] &= \frac{0.25}{0.25 + 1 - 0.5} \\
 &= \frac{1}{3}
 \end{aligned}$$

Therefore A_1 will be more likely.

Problem 2

(a)

(i)

$$\boxed{\binom{19}{15}}$$

(ii)

$$\boxed{\binom{15}{4}}$$

(b)

If we care about the order in which we place the balls in the buckets, the probability that we place all 5 balls in different buckets is $\frac{1}{5^5}$. There are $5!$ ways to order the balls to place into the buckets, so since we do not care about the order in which we place the balls in the buckets, the probability that we place all 5 balls in different buckets is $\frac{1}{5^5} \cdot 5! = 0.0384$.

(c)

This is a multinomial permutation so we have

$$\frac{9!}{4!2!3!} = \boxed{1260}$$

Problem 3

(a)

There are $\frac{n!}{(n-r)!}$ ways to choose r numbers from the first n positive integers. Of these, there are $r!$ ways to order the r numbers such that their order is increasing. Therefore the probability the numbers are chosen in increasing order is just $\boxed{\frac{1}{r!}}$.

(b)

Problem 4

(a)

Let p be the probability that A is not hit, then we have

$$\begin{aligned} p &= p_A(1 - p_B) + (1 - p_B)(1 - p_A)p \\ p(1 - (1 - p_B)(1 - p_A)) &= (1 - p_A)p_B \\ p &= \boxed{\frac{(1 - p_B)p_A}{1 - (1 - p_B)(1 - p_A)}} \end{aligned}$$

(b)

Let probability that the duel ends with both being hit be p_{AB} We have:

$$\begin{aligned} p_{AB} &= p_A p_B + (1 - p_A)(1 - p_B)p_{AB} \\ p_{AB}(1 - (1 - p_A)(1 - p_B)) &= p_A p_B \\ p_{AB} &= \boxed{\frac{p_A p_B}{1 - (1 - p_A)(1 - p_B)}} \end{aligned}$$

(c)

The probability that the duel ends after n rounds, is the probability that the two dualists do not hit each other for $n-1$ rounds, so $((1 - p_A)(1 - p_B))^{n-1}$, multiplied by the probability that one or both get hit at the n th round so $1 - (1 - p_A)(1 - p_B)$. Therefore the total probability is

$$\boxed{((1 - p_A)(1 - p_B))^{n-1} \cdot (1 - (1 - p_A)(1 - p_B))}$$

(d)

Let N be the random variable representing the number of rounds of shots until the duel ends. Furthermore let H_A be the event that A is not hit. From part (a) we have $P(H_A) = \frac{(1-p_B)p_A}{1-(1-p_B)(1-p_A)}$ we have

$$\begin{aligned} P(N = n|H_A) &= \frac{P(N = n, H_A)}{P(H_A)} \\ &= \frac{((1-p_A)(1-p_B))^{n-1} p_A(1-p_B)}{\frac{(1-p_B)p_A}{1-(1-p_B)(1-p_A)}} \\ &= \boxed{((1-p_A)(1-p_B))^{n-1} (1 - (1-p_B)(1-p_A))} \end{aligned}$$

(e)

Let N be the random variable that representing the number of rounds of shots until the duel ends. Furthermore let H_{AB} be the event that A and B are hit, we have

$$\begin{aligned} P(H_{AB}|N = n) &= \frac{((1-p_A)(1-p_B))^{n-1} p_A p_B}{((1-p_A)(1-p_B))^{n-1} \cdot (1 - (1-p_A)(1-p_B))} \\ &= \boxed{\frac{p_A p_B}{1 - (1-p_B)(1-p_A)}} \end{aligned}$$

Problem 5

(a)

There are $\binom{10}{3}$ to choose 3 socks from 10 if we do not care about the color of the socks we pick. If we do care about not selecting a pair of socks with the same colors we have $2^3 \binom{5}{3}$ ways to choose 3 socks from 10. therefore the probability is $\frac{2^3 \binom{5}{3}}{\binom{10}{3}} = \boxed{0.66666}$.

(b)

There are $\binom{2s}{r}$ ways to choose r socks from $2s$ if we do not care about the color of the socks we pick. If we do care about selecting the socks such that no two have the same color we have that we can do that in $2^r \binom{s}{r}$ ways.

Therefore the probability is $\boxed{\frac{2^r \binom{s}{r}}{\binom{2s}{r}}}$.

Problem 6

(a)

Let the random event denoting whether we received the correct answer be C and the random event denoting whether we asked a Tourist be T . Then we have

$$P(C) = P(C|T)P(T) + P(C|T^c)P(T^c)$$

$$P(C) = \frac{2}{3} \frac{3}{4} = \boxed{\frac{1}{2}}$$

(b)

Let the random event A_1 denote our first response, and A_2 denote our second response we have that

$$P(C|A_1 = A_2) = P(T|A_1 = A_2)P(C|A_1 = A_2, T) + P(T^c|A_1 = A_2)P(C|A_1 = A_2, T^c)$$

Since

$$\begin{aligned}
P(T|A_1 = A_2) &= \frac{P(T)P(A_1 = A_2|T)}{P(A_1 = A_2)} \\
&= \frac{P(T)P(A_1 = A_2|T)}{P(A_1 = A_2|T)P(T) + P(A_1 = A_2|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^2}{4^2} + \frac{1}{4^2}\right) \cdot \frac{2}{3}}{\left(\frac{3^2}{4^2} + \frac{1}{4^2}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{20}{36}
\end{aligned}$$

And

$$P(C|A_1 = A_2, T) = \frac{\left(\frac{3}{4}\right)^2}{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{9}{10}$$

and

$$P(C|A_1 = A_2, T^c) = 0$$

Therefore we have

$$\begin{aligned}
P(C|A_1 = A_2) &= \frac{20}{36} \cdot \frac{9}{10} + 0 \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(c)

Let the random event A_3 denote the third response we get

$$P(C|A_1 = A_2 = A_3) = P(T|A_1 = A_2 = A_3)P(C|A_1 = A_2 = A_3, T) + P(T^c|A_1 = A_2 = A_3)P(C|A_1 = A_2 = A_3, T^c)$$

Since

$$\begin{aligned}
P(T|A_1 = A_2 = A_3) &= \frac{P(T)P(A_1 = A_2 = A_3|T)}{P(A_1 = A_2 = A_3)} \\
&= \frac{P(T)P(A_1 = A_2 = A_3|T)}{P(A_1 = A_2 = A_3|T)P(T) + P(A_1 = A_2 = A_3|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^3}{4^3} + \frac{1}{4^3}\right) \cdot \frac{2}{3}}{\left(\frac{3^3}{4^3} + \frac{1}{4^3}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{56}{120}
\end{aligned}$$

And

$$P(C|A_1 = A_2 = A_3, T) = \frac{\left(\frac{3}{4}\right)^3}{\left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3} = \frac{27}{28}$$

and

$$P(C|A_1 = A_2, T^c) = 0$$

Therefore we have

$$\begin{aligned}
P(C|A_1 = A_2) &= \frac{56}{120} \cdot \frac{27}{28} + 0 \\
&= \boxed{\frac{9}{20}}
\end{aligned}$$

(d)

Let the random event A_4 denote the fourth response we get

$$P(C|A_1 = A_2 = A_3 = A_4) = P(T|A_1 = A_2 = A_3 = A_4)P(C|A_1 = A_2 = A_3 = A_4, T) + P(T^c|A_1 = A_2 = A_3 = A_4)P(C|A_1 = A_2 = A_3 = A_4, T^c)$$

Since

$$\begin{aligned}
P(T|A_1 = A_2 = A_3 = A_4) &= \frac{P(T)P(A_1 = A_2 = A_3 = A_4|T)}{P(A_1 = A_2 = A_3 = A_4)} \\
&= \frac{P(T)P(A_1 = A_2 = A_3 = A_4|T)}{P(A_1 = A_2 = A_3 = A_4|T)P(T) + P(A_1 = A_2 = A_3 = A_4|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^4}{4^4} + \frac{1}{4^4}\right) \cdot \frac{2}{3}}{\left(\frac{3^4}{4^4} + \frac{1}{4^4}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{82}{210}
\end{aligned}$$

And

$$P(C|A_1 = A_2 = A_3 = A + 4, T) = \frac{\left(\frac{3}{4}\right)^4}{\left(\frac{3}{4}\right)^4 + \left(\frac{1}{4}\right)^4} = \frac{81}{82}$$

and

$$P(C|A_1 = A_2 = A_3 = A_4, T^c) = 0$$

Therefore we have

$$\begin{aligned}
P(C|A_1 = A_2 = A_3 = A_4) &= \frac{82}{210} \cdot \frac{81}{82} + 0 \\
&= \boxed{\frac{27}{70}}
\end{aligned}$$