ECE 131A HW 1

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Problem 1

(a)

$$\begin{aligned} P[B_0] &= P[A_0]P[B_0|A_0] + P[A_1]P[B_0|A_1] \\ &= \boxed{\frac{1}{2} \cdot (1 - \epsilon_1 + \epsilon_2)} \end{aligned}$$

(b)

We have

$$P[B_1] = 1 - P[B_0] = \frac{1}{2} \cdot (\epsilon_1 + 1 - \epsilon_2)$$

Therefore from Bayes law we have

$$\begin{aligned} P[A_1|B_1] &= \frac{P[B_1|A_1]P[A_1]}{P[B_1]} \\ &= \frac{1 - \epsilon_2}{\epsilon_1 + 1 - \epsilon_2} \end{aligned}$$

$$
\begin{aligned}
P[A_0|B_1] &= \frac{P[B_1|A_0]P[A_0]}{P[B_1]} \\
&= \frac{\epsilon_1}{\epsilon_1 + 1 - \epsilon_2}
\end{aligned}
$$

Therefore we have for $\epsilon_1 = 0.25$ and $\epsilon_2 = 0.5$: we will have

$$
\begin{aligned}
P[A_1|B_1] &= \frac{1 - 0.5}{0.25 + 1 - 0.5} \\
&= \frac{2}{3} \\
P[A_0|B_1] &= \frac{0.25}{0.25 + 1 - 0.5} \\
&= \frac{1}{3}
\end{aligned}
$$

Therefore A_1 will be more likely.

Problem 2

(a)

(i)

$$\boxed{\tbinom{19}{15}}$$

(ii)

$$\boxed{\tbinom{15}{4}}$$

(b)

If we care about the order in which we place the balls in the buckets, the probability that we place all 5 balls in diffrent buckets is $\frac{1}{5^5}$. There are 5! ways to order the balls to place into the buckets, so since we do not care about the order in which we place the balls in the buckets, the probability that we place all 5 balls in diffrent buckets is $\frac{1}{5^5} \cdot 5! = 0.0384$.

(c)

This is a multnomial permutation so we have

$$\frac{9!}{4!2!3!} = \boxed{1260}$$

Problem 3

(a)

Let N be the number of ways we can choose r from n postive integers such that the order is increasing we have:

$$\begin{aligned} N &= \sum_{i_1=1}^{n-r+1} \sum_{i_2=i_1+1}^{n-r+2} \cdots \sum_{i_r=i_{r-1}+1}^{n} 1 \\ &= \sum_{i_1=1}^{n-r+1} \sum_{i_2=i_1+1}^{n-r+2} \cdots \sum_{i_{r-1}=i_{r-2}+1}^{n-1} (n - i_{r-1}) \end{aligned}$$

Problem 4

(a)

Let p be the probability that A is not hit, then we have

$$p = p_A(1 - p_B) + (1 - p_B)(1 - p_A)p$$

$$p(1 - (1 - p_B)(1 - p_A)) = (1 - p_A)p_B$$

$$p = \boxed{\frac{(1 - p_B)p_A}{1 - (1 - p_B)(1 - p_A)}}$$

(b)

Let probability that the duel ends with both being hit be p_{AB} We have:

$$p_{AB} = p_A p_B + (1 - p_A)(1 - p_B)p_{AB}$$
$$p_{AB}\,(1 - (1 - p_A)(1 - p_B)) = p_A p_B$$
$$p_{AB} = \boxed{\frac{p_A p_B}{1 - (1 - p_A)(1 - p_B)}}$$

(c)

The probabiblity is that the dual ends after n rounds, is the probability that the two dualists do not hit each other for $n-1$ rounds, so $((1 - p_A)(1 - p_B))^{n-1}$, multiplied by the probabiblity that one or both get hit at the n th round so $1 - (1 - p_A)(1 - p_B)$. Therefore the total probabiblity is

$$\boxed{((1 - p_A)(1 - p_B))^{n-1} \cdot (1 - (1 - p_A)(1 - p_B))}$$

(d)

Let N be the random variable representing the number of rounds of shots until the duel ends. Furthermore let H_A be the event that A is not hit. From part (a) we have $P(H_A) = \frac{(1-p_B)p_A}{1-(1-p_B)(1-p_A)}$ we have

$$\begin{aligned} P(N = n|H_A) &= \frac{P(N = n, H_A)}{P(H_A)} \\ &= \frac{((1-p_A)(1-p_B))^{n-1}\, p_A(1-p_B)}{\frac{(1-p_B)p_A}{1-(1-p_B)(1-p_A)}} \\ &= \boxed{((1-p_A)(1-p_B))^{n-1}\,(1-(1-p_B)(1-p_A))} \end{aligned}$$

(e)

Let N be the random variable that representing the number of rounds of shots until the duel ends. Furthermore let H_{AB} be the event that A and B are hit, we have

$$\begin{aligned} P(H_{AB}|N = n) &= \frac{((1-p_A)(1-p_B))^{n-1}\, p_A p_B}{((1-p_A)(1-p_B))^{n-1} \cdot (1-(1-p_A)(1-p_B))} \\ &= \boxed{\frac{p_A p_B}{1-(1-p_B)(1-p_A)}} \end{aligned}$$

Problem 5

(a)

There are $\binom{10}{3}$ to choose 3 socks from 10 if we do not care about the color of the socks we pick. If we do care about not selecting a pair of socks with the same colors we have $\frac{1}{3!} \cdot 10 \cdot 8 \cdot 7$ ways to choose 3 socks from 10. therefore the probability is $\frac{\frac{1}{3!}\cdot 10\cdot 8\cdot 7}{\binom{10}{3}} = \boxed{0.77777777777}$.

(b)

There are $\binom{2s}{r}$ ways to choose r socks from $2s$ if we do not care about the color of the socks we pick. If we do care about selecting the socks such that no two have the same color we have that we can do that in $\frac{2s}{r!}\frac{(2s-2)!}{(2s-r-1)!}$ ways. Therefore the probability is

$$\begin{aligned}\frac{\frac{2s}{r!}\frac{(2s-2)!}{(2s-r-1)!}}{\binom{2s}{r}} &= \frac{(2s)\frac{(2s-2)!}{(r!)(2s-r-1)!}}{\frac{2s!}{r!(2s-r)!}} \\ &= \boxed{\frac{2s-r}{2s-1}}\end{aligned}$$

Problem 6

(a)

Let the random event denoting whether we recived the correct answer be C and the random event denoting whether we asked a Tourist be T. Then we have

$$P(C) = P(C|T)P(T) + P(C|T^c)P(T^c)$$
$$P(C) = \frac{2}{3}\frac{3}{4} = \boxed{\frac{1}{2}}$$

(b)

Let the random event A_1 denote our first response, and A_2 denote our second response we have that

$$P(C|A_1 = A_2) = P(T|A_1 = A_2)P(C|A_1 = A_2, T) + P(T^c|A_1 = A_2)P(C|A_1 = A_2, T^c)$$

Since

$$
\begin{aligned}
P(T|A_1 = A_2) &= \frac{P(T)P(A_1 = A_2|T)}{P(A_1 = A_2)} \\
&= \frac{P(T)P(A_1 = A_2|T)}{P(A_1 = A_2|T)P(T) + P(A_1 = A_2|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^2}{4^2} + \frac{1}{4^2}\right) \cdot \frac{2}{3}}{\left(\frac{3^2}{4^2} + \frac{1}{4^2}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{20}{36}
\end{aligned}
$$

And

$$P(C|A_1 = A_2, T) = \frac{\left(\frac{3}{4}\right)^2}{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{9}{10}$$

and

$$P(C|A_1 = A_2, T^c) = 0$$

Therefore we have

$$
\begin{aligned}
P(C|A_1 = A_2) &= \frac{20}{36} \cdot \frac{9}{10} + 0 \\
&= \boxed{\frac{1}{2}}
\end{aligned}
$$

(c)

Let the random event A_3 denote the third response we get

$P(C|A_1 = A_2 = A_3) = P(T|A_1 = A_2 = A_3)P(C|A_1 = A_2 = A_3, T) + P(T^c|A_1 = A_2 = A_3)P(C|A_1$

Since

$$
\begin{aligned}
P(T|A_1 = A_2 = A_3) &= \frac{P(T)P(A_1 = A_2 = A_3|T)}{P(A_1 = A_2 = A_3)} \\
&= \frac{P(T)P(A_1 = A_2 = A_3|T)}{P(A_1 = A_2 = A_3|T)P(T) + P(A_1 = A_2 = A_3|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^3}{4^3} + \frac{1}{4^3}\right) \cdot \frac{2}{3}}{\left(\frac{3^3}{4^3} + \frac{1}{4^3}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{56}{120}
\end{aligned}
$$

And

$$P(C|A_1 = A_2 = A_3, T) = \frac{\left(\frac{3}{4}\right)^3}{\left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3} = \frac{27}{28}$$

and

$$P(C|A_1 = A_2, T^c) = 0$$

Therefore we have

$$
\begin{aligned}
P(C|A_1 = A_2) &= \frac{56}{120} \cdot \frac{27}{28} + 0 \\
&= \boxed{\frac{9}{20}}
\end{aligned}
$$

(d)

Let the random event A_4 denote the fourth response we get

$$P(C|A_1 = A_2 = A_3 = A_4) = P(T|A_1 = A_2 = A_3 = A_4)P(C|A_1 = A_2 = A_3 = A_4, T) + P(T^c|A_1 =$$

Since

$$\begin{aligned}
P(T|A_1 = A_2 = A_3 = A_4) &= \frac{P(T)P(A_1 = A_2 = A_3 = A_4|T)}{P(A_1 = A_2 = A_3 = A_4)} \\
&= \frac{P(T)P(A_1 = A_2 = A_3 = A_4|T)}{P(A_1 = A_2 = A_3 = A_4|T)P(T) + P(A_1 = A_2 = A_3 = A_4|T^c)P(T^c)} \\
&= \frac{\left(\frac{3^4}{4^4} + \frac{1}{4^4}\right) \cdot \frac{2}{3}}{\left(\frac{3^4}{4^4} + \frac{1}{4^4}\right) \cdot \frac{2}{3} + \frac{1}{3}} \\
&= \frac{82}{210}
\end{aligned}$$

And

$$P(C|A_1 = A_2 = A_3 = A + 4, T) = \frac{\left(\frac{3}{4}\right)^4}{\left(\frac{3}{4}\right)^4 + \left(\frac{1}{4}\right)^4} = \frac{81}{82}$$

and

$$P(C|A_1 = A_2 = A_3 = A_4, T^c) = 0$$

Therefore we have

$$\begin{aligned}
P(C|A_1 = A_2 = A_3 = A_4) &= \frac{82}{210} \cdot \frac{81}{82} + 0 \\
&= \boxed{\frac{27}{70}}
\end{aligned}$$