# ECE 131A HW 6

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## Problem 1

We have that assuming the number of pens is large, then the total lifetime of the pens is distributed according to a normal with mean $n$ and variance $n$ therefore we have that we want to find a $n$ such that

$$-2.32635 = \frac{15-n}{\sqrt{n}}$$

solving this we get that $n = 27.11$ therefore the student would need $\boxed{28}$ pens in order for the probability of having a pen to run out during a semester to be greater than 0.99

## Problem 2

The number of errors in 100 transmission can be estimated as a normal distribution with mean of 15 and variance of $100 \cdot 0.15 \cdot 0.85 = 12.75$, therefore the probability of having more than 20 errors has a corresponding $Z$ value of 1.4 and therefore a corresponding probability of $\boxed{0.91924}$.

# Problem 3

## (a)

We have that

$$\Sigma^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

And

$$det(\Sigma) = 5$$

Therefore we have that

$$\Sigma^{-1}(\mu - X) = \frac{1}{5}\begin{bmatrix} 3(\mu_1 - x_1) + (\mu_2 - x_2) \\ (\mu_1 - x_1) + 2(\mu_2 - x_2) \end{bmatrix}$$

and therefore we have that

$$(\mu - X)^T\Sigma^{-1}(\mu - X) = \frac{3}{5}(\mu_1 - x_1)^2 + \frac{2}{5}(\mu_2 - x_2)(\mu_1 - x_1) + \frac{2}{5}(\mu_2 - x_2)^2$$

Therefore we have that

$$f(x_1, x_2) = (2\pi)^{-1}\frac{1}{\sqrt{5}}e^{-\frac{3}{10}(1-x_1)^2 - \frac{1}{5}(2-x_2)(1-x_1) - \frac{1}{5}(2-x_2)^2}$$

## (b)

We have that Y is distributed as an normal distribution with mean $A\mu + B = 4$ and variance $A\Sigma A^T = 3$ therefore we have that the pdf of $Y$ is

$$f(y) = \frac{1}{\sqrt{2\pi 3}}e^{-\frac{(y-4)^2}{6}}$$

## (c)

Therefore we have that given that $Z$ is a standard normal distribution

$$P(Y \geq 2) = P(Z \geq \frac{2-4}{\sqrt{3}}) = 1 - 0.12425 = \boxed{0.87575}$$

# Problem 4

## (a)

Therefore we have that the covariance matrix is

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

## (b)

We have that the pdf of $Y$ is

$$f(y_1, y_2, y_3) = (2\pi)^{-\frac{3}{2}} \frac{1}{2} e^{\frac{1}{4}(y_1 y_2 + y_2 y_3) - \frac{3}{8}(y_1^2 + y_2^2 + y_3^2)}$$

## (c)

We have that the distirbution for $Y_1$ and $Y_2$ is a given by a linear transformation of $[Y_1, Y_2, Y_3]$ with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Therefore we have that this mu is $A\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the covariance matrix is $A\Sigma A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Therefore we have that

$$f(y_1, y_2) = \frac{1}{2\pi} \frac{1}{\sqrt{3}} e^{-\frac{1}{3}(y_1^2 + y_2^2 - y_1 y_2)}$$

Likewise we have that the distribution for $Y_1$ and $Y_3$ is a given by a linear transformation of $[Y_1, Y_2, Y_3]$ with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore we have that this mu is $A\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the covariance matrix is $A\Sigma A^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Therefore we have that

$$f(y_1, y_3) = \frac{1}{2\pi} \frac{1}{2} e^{-\frac{1}{4}(y_1^2 + y_3^2)}$$

### (d)

we can simply use a linear transformation

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

## Problem 6

We can approximate the sum of the 64 numbers as a gaussian with mean 0 and variance $64\frac{1}{12} = \frac{16}{3}$ therefore we have that the probability of the sum being greater than 4 is the same as the probability that a $Z$, a standard normal distribution, is greater than $\frac{4}{\sqrt{\frac{16}{3}}} = \sqrt{3}$, which is $1 - 0.95837 = \boxed{0.04163}$.

## Problem 7

### (a)

We have that

$$f(y|0) = \frac{1}{2\alpha}e^{-\frac{|y+1|}{\alpha}}$$

### (b)

We have that

$$f(y|1) = \frac{1}{2\alpha}e^{-\frac{|y-1|}{\alpha}}$$

### (c)

We have that

$$f(y) = 0.5\frac{1}{2\alpha}e^{-\frac{|y-1|}{\alpha}} + 0.5\frac{1}{2\alpha}e^{-\frac{|y+1|}{\alpha}}$$

### (d)

We have that the probability of error given that "0" was sent is

$$\int_0^{\infty} \frac{1}{2\alpha}e^{-\frac{y+1}{\alpha}}dy = \frac{e^{-\frac{1}{\alpha}}}{2}$$

and the probability of error given that "1" was sent is

$$\int_{-\infty}^{0} \frac{1}{2\alpha}e^{-\frac{1-y}{\alpha}}dy = \frac{e^{-\frac{1}{\alpha}}}{2}$$

### (e)

Therefore we have that the probability of error is

$$\frac{1}{2}\frac{e^{-\frac{1}{\alpha}}}{2} + \frac{1}{2}\frac{e^{-\frac{1}{\alpha}}}{2} = \frac{e^{-\frac{1}{\alpha}}}{2}$$

## Problem 8

### (a)

$$f_Y(y|X = +1) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-1)^2}{2\sigma^2}}$$
$$f_Y(y|X = -1) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y+1)^2}{2\sigma^2}}$$

## (b)

We have that for $y < 0$, $f_Y(y|X = +1) < f_Y(y|X = -1)$ and for $y > 0$, $f_Y(y|X = +1) > f_Y(y|X = -1)$ and at $y = 0$ we have that $f_Y(y|X = +1) = f_Y(y|X = -1)$. Therefore we have this is equivalent to saying that when $Y < 0$ we decide "0" and when $Y \geq 0$ we decide "1".

## (c)

The probability of error given that "0" was sent is

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}} dy = 0.00003$$

and the probability of error given that "1" was sent is

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}} dy = 0.00003$$

## (d)

Therefore we have that the probability of error is

$$\frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}} = 0.00003$$

# Problem 9

## (a)

$$\phi_z(t) = \frac{2}{2 - it} \cdot \frac{10}{10 - it}$$

### (b)

$$\phi_z(t) = \frac{20}{8}\frac{1}{2-it} - \frac{20}{8}\frac{1}{10-it}$$

Computing the inverse fourier transform we get

$$f(z) = \frac{20}{8}e^{-2z} - \frac{20}{8}e^{-10z} \text{ when } z \geq 0 \text{ and } f(z) = 0 \text{ when } z < 0$$

# Problem 10

$$E[T] = \frac{1}{3} \cdot 3 + \frac{2}{3}E[T] + \frac{1}{3}(5+7)$$
$$E[T] = 15$$

# Problem 11

## (a)

We have that the number of bears that are yellow $Y$ have the following probability

$$\begin{aligned}
P(Y=n) &= \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \binom{k}{n} p^n (1-p)^{k-n} \\
&= p^n e^{-\lambda} \sum_{k=n}^{\infty} \lambda^k \frac{1}{n!(k-n)!} (1-p)^{k-n} \\
&= \frac{p^n e^{-\lambda}}{n!} \sum_{k=n}^{\infty} \lambda^k \frac{1}{(k-n)!} (1-p)^{k-n} \\
&= \frac{p^n \lambda^n e^{-\lambda}}{n!} e^{\lambda(1-p)} \\
&= \frac{p^n \lambda^n e^{-p\lambda}}{n!}
\end{aligned}$$

Therefore it is distributed as a possion with parameter $p\lambda$.

## (b)

We have that

$$E[Y|B=7, R=3] = E[Y] = \frac{18}{3} = \boxed{6}$$

# Problem 12

## 0.1 (a)

We have that the $f(x,y) = \frac{1}{8}$ for $0 \leq x \leq 5$ and $0 \leq y \leq 2$ and $f(x,y) = 0$ otherwise. Therefore we have that

$$f_y(y) = \int_0^{3+y} \frac{1}{8} dx = \frac{3+y}{8}$$

for $0 \leq y \leq 2$ and $f_y(y) = 0$ otherwise.

## 0.2 (b)

We have

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{\frac{1}{8}}{\frac{3+y}{8}} = \frac{1}{3+y}$$

for $0 \leq x \leq 3+y$ and $f_{X|Y}(x|y) = 0$ otherwise.

## 0.3 (c)

$$\int_0^{3+y} \frac{1}{3+y} dx = 1$$

Therefore it is a valid density function

## 0.4 (d)

We have that

$$E[X|Y=y] = \int_0^{3+y} \frac{x}{3+y} dx = \frac{3+y}{2}$$

# Probelem 13

We have that from chebshev's inequality we have that

$$P(|X - \mu| > \epsilon) \leq \frac{\lambda}{t^2 \epsilon^2}$$

# Problem 14

Let $E[X_i] = \mu$ then we have

$$E[S_n] = n\mu$$

And

$$Var(S_n) = n\sigma^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \sigma^2 \rho^{j-i}$$

# Problem 15

## (a)

We have from markov's inequality

$$P(X > 1000) \leq \boxed{\frac{750}{1000}}$$

## (b)

Using chebshev we have that

$$P(|X - 750| > 250) \leq \boxed{\frac{100^2}{250^2}}$$

## Problem 16

### (a)

We have that the probability that the height of the corn $H$ is bounded by the markov inequality:

$$P(H \geq 6) \leq \frac{5.2}{6}$$

### (b)

by chebshev's inequality we have that

$$P(|H - 5.2| \geq 0.2) \leq \frac{1}{16}$$