ECE 131A HW 6

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Problem 1

We have that assuming the number of pens is large, then the total lifetime of the pens is distributed according to a normal with mean n and variance n therefore we have that we want to find a n such that

$$-2.32635 = \frac{15 - n}{\sqrt{n}}$$

solving this we get that $n = 27.11$ therefore the student would need $\boxed{28}$ pens in order for the probability of having a pen to run out during a semester to be greater than 0.99

Problem 2

The number of errors in 100 transmission can be estimated as a normal distribution with mean of 15 and variance of $100 \cdot 0.15 \cdot 0.85 = 12.75$, therefore the probability of having more than 20 errors has a corresponding Z value of 1.4 and therefore a corresponding probability of $\boxed{0.91924}$.

Problem 3

(a)

We have that

$$\Sigma^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

And

$$det(\Sigma) = 5$$

Therefore we have that

$$\Sigma^{-1}(\mu - X) = \frac{1}{5}\begin{bmatrix} 3(\mu_1 - x_1) + (\mu_2 - x_2) \\ (\mu_1 - x_1) + 2(\mu_2 - x_2) \end{bmatrix}$$

and therefore we have that

$$(\mu - X)^T \Sigma^{-1}(\mu - X) = \frac{3}{5}(\mu_1 - x_1)^2 + \frac{2}{5}(\mu_2 - x_2)(\mu_1 - x_1) + \frac{2}{5}(\mu_2 - x_2)^2$$

Therefore we have that

$$f(x_1, x_2) = (2\pi)^{-1}\frac{1}{\sqrt{5}}e^{-\frac{3}{10}(1-x_1)^2 - \frac{1}{5}(2-x_2)(1-x_1) - \frac{1}{5}(2-x_2)^2}$$

(b)

We have that Y is distributed as an normal distribution with mean $A\mu + B = 4$ and variance $A\Sigma A^T = 3$ therefore we have that the pdf of Y is

$$f(y) = \frac{1}{\sqrt{2\pi 3}}e^{-\frac{(y-4)^2}{6}}$$

(c)

Therefore we have that given that Z is a standard normal distribution

$$P(Y \geq 2) = P(Z \geq \frac{2-4}{\sqrt{3}}) = 1 - 0.12425 = \boxed{0.87575}$$

Problem 4

(a)

Therefore we have that the covariance matrix is

$$\Sigma = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(b)

We have that the pdf of Y is

$$f(y_1, y_2, y_3) = (2\pi)^{-\frac{3}{2}} \frac{1}{2} e^{\frac{1}{4}(y_1 y_2 + y_2 y_3) - \frac{3}{8}(y_1^2 + y_2^2 + y_3^2)}$$

(c)

We have that the distirbution for Y_1 and Y_2 is a given by a linear transformation of $[Y_1, Y_2, Y_3]$ with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Therefore we have that this mu is $A\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the covariance matrix is $A\Sigma A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Therefore we have that

$$f(y_1, y_2) = \frac{1}{2\pi} \frac{1}{\sqrt{3}} e^{-\frac{1}{3}(y_1^2 + y_2^2 - y_1 y_2)}$$

Likewise we have that the distribution for Y_1 and Y_3 is a given by a linear transformation of $[Y_1, Y_2, Y_3]$ with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Therefore we have that this mu is $A\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and the covariance matrix is $A\Sigma A^T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Therefore we have that

$$f(y_1, y_3) = \frac{1}{2\pi} \frac{1}{2} e^{-\frac{1}{4}(y_1^2 + y_3^2)}$$

(d)

we can simply use a linear transformation

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Problem 6

We can approximate the sum of the 64 numbers as a gaussian with mean 0 and variance $64\frac{1}{12} = \frac{16}{3}$ therefore we have that the probability of the sum being greater than 4 is the same as the probability that a Z, a standard normal distribution, is greater than $\frac{4}{\sqrt{\frac{16}{3}}} = \sqrt{3}$, which is $1 - 0.95837 = \boxed{0.04163}$.

Problem 7

(a)

We have that

$$f(y|0) = \frac{1}{2\alpha} e^{-\frac{|y+1|}{\alpha}}$$

(b)

We have that

$$f(y|1) = \frac{1}{2\alpha} e^{-\frac{|y-1|}{\alpha}}$$

(c)

We have that

$$f(y) = 0.5\frac{1}{2\alpha}e^{-\frac{|y-1|}{\alpha}} + 0.5\frac{1}{2\alpha}e^{-\frac{|y+1|}{\alpha}}$$

(d)

We have that the probability of error given that "0" was sent is

$$\int_0^{\infty} \frac{1}{2\alpha}e^{-\frac{y+1}{\alpha}}dy = \frac{e^{-\frac{1}{\alpha}}}{2}$$

and the probability of error given that "1" was sent is

$$\int_{-\infty}^{0} \frac{1}{2\alpha}e^{-\frac{1-y}{\alpha}}dy = \frac{e^{-\frac{1}{\alpha}}}{2}$$

(e)

Therefore we have that the probability of error is

$$\frac{1}{2}\frac{e^{-\frac{1}{\alpha}}}{2} + \frac{1}{2}\frac{e^{-\frac{1}{\alpha}}}{2} = \frac{e^{-\frac{1}{\alpha}}}{2}$$

Problem 8

(a)

$$f_Y(y|X=+1) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-1)^2}{2\sigma^2}}$$
$$f_Y(y|X=-1) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y+1)^2}{2\sigma^2}}$$

(b)

We have that for $y < 0$, $f_Y(y|X = +1) < f_Y(y|X = -1)$ and for $y > 0$, $f_Y(y|X = +1) > f_Y(y|X = -1)$ and at $y = 0$ we have that $f_Y(y|X = +1) = f_Y(y|X = -1)$. Therefore we have this is equivalent to saying that when $Y < 0$ we decide "0" and when $Y \geq 0$ we decide "1".

(c)

The probability of error given that "0" was sent is

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}} dy = 0.00003$$

and the probability of error given that "1" was sent is

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}} dy = 0.00003$$

(d)

Therefore we have that the probability of error is

$$\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-1)^2}{2\sigma^2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+1)^2}{2\sigma^2}} = 0.00003$$

Problem 9

(a)

$$\phi_z(t) = \frac{2}{2 - it} \cdot \frac{10}{10 - it}$$

0.1 (b)

$$\phi_z(t) = -\frac{20}{8}\frac{1}{10-it} + \frac{20}{8}\frac{1}{2-it}$$

Computing the inverse fourier transform we get

$$f(z) = \frac{20}{8}e^{-2z} - \frac{20}{8}e^{-10z} \text{ when } z \geq 0 and f(z) = 0 \text{ when } z < 0$$

Problem 10

$$E[T] = \frac{1}{3} \cdot 3 + \frac{2}{3}E[T] + \frac{1}{3}(5+7)$$
$$E[T] = 15$$

Problem 11

(a)