ECE 133A HW 6

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Exercise A9.4

Since $A = T(B)$, $D_v = T(E)$, and $D_h = T(E^T)$, and letting b, e, and f being the column major ordering of B, E and We have that A, D_v, and D_h are all real matrices. Thus, we have

$$A^T = A^H = \frac{1}{n^2}\tilde{W}\mathbf{diag}(\overline{\tilde{W}b})\tilde{W}^H$$

Where $\circ$ denotes element wise multiplication, the same can be done for D_v^T and D_h^T. Thus we have that

$$A^TA = \frac{1}{n^2}\tilde{W}\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right)\tilde{W}^H x$$

And likewise for $D_v^T D_v$ and $D_h^T D_h$. Thus we have that

$$(A^TA + \lambda D_v^T D_v + \lambda D_h^T D_h)x = A^T y$$

becomes:

$$\frac{1}{n^2}\tilde{W}\left(\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right) + \lambda\mathbf{diag}\left(\tilde{W}e \circ \overline{\tilde{W}e}\right) + \lambda\mathbf{diag}\left(\tilde{W}f \circ \overline{\tilde{W}f}\right)\right)\tilde{W}^H x = \frac{1}{n^2}\tilde{W}\mathbf{diag}(\tilde{W}b)\tilde{W}^H y$$
$$\frac{1}{n^2}\left(\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right) + \lambda\mathbf{diag}\left(\tilde{W}e \circ \overline{\tilde{W}e}\right) + \lambda\mathbf{diag}\left(\tilde{W}f \circ \overline{\tilde{W}f}\right)\right)\tilde{W}^H x = \frac{1}{n^2}\mathbf{diag}(\tilde{W}b)\tilde{W}^H y$$

let $z = \frac{1}{n^2}\tilde{W}^H x$ Then we get

$$\left(\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right) + \lambda\mathbf{diag}\left(\tilde{W}e \circ \overline{\tilde{W}e}\right) + \lambda\mathbf{diag}\left(\tilde{W}f \circ \overline{\tilde{W}f}\right)\right) z = \frac{1}{n^2}\mathbf{diag}(\tilde{W}b)\tilde{W}^H y$$

Then solving for $\tilde{W}b$, $\tilde{W}e$, and $\tilde{W}f$ and $\frac{1}{n^2}\tilde{W}^H y$ will cost $n^2\log(n)$ flops each, and then solving for $\left(\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right) + \lambda\mathbf{diag}\left(\tilde{W}e \circ \overline{\tilde{W}e}\right) + \lambda\mathbf{diag}\left(\tilde{W}f \circ \overline{\tilde{W}f}\right)\right)$ will cost us $10n$ flops. Likewise solving for $\frac{1}{n^2}\mathbf{diag}(\tilde{W}b)\tilde{W}^H y$ is just element wise multiplication of $\frac{1}{n^2}\tilde{W}^H y$ and $\tilde{W}b$ which is n^2 flops. Then we can solve for z by dividing $\frac{1}{n^2}\mathbf{diag}(\tilde{W}b)\tilde{W}^H y$ by $\left(\mathbf{diag}\left(\tilde{W}b \circ \overline{\tilde{W}b}\right) + \lambda\mathbf{diag}\left(\tilde{W}e \circ \overline{\tilde{W}e}\right) + \lambda\mathbf{diag}\left(\tilde{W}f \circ \overline{\tilde{W}f}\right)\right)$ element wise which will cost us n^2 flops. Then we can solve for x by just multiplying z with $\tilde{W}$ or in other words, dong the FFT on z, which will cost us $n^2\log(n)$ flops. Therefore in total our algorithm will cost us $5n^2\log(n) + 12n^2$ flops. This can be implemented with the following algorithm

```
function x=deblur(Y,B,lambda)
    %convert Y to column major order
    y = Y(:);
    %convert B to column major order
    b = B(:);
    %make the E matrix where all values are 0 except for
    %the top left bottom left values
    %which are 1 and -1 respectively
    E=zeros(size(B));
    E(1,1)=1;
    E(end,1)=-1;
    %convert E to column major order
    e = E(:);
    %convert the transpose of E to column major order
    f = E'(:);

    n=size(B,1);

    fft_b=reshape( fft2( reshape( b, n, n ) ), n^2, 1);
    fft_e=reshape( fft2( reshape( e, n, n ) ), n^2, 1);
    fft_f=reshape( fft2( reshape( f, n, n ) ), n^2, 1);
    ifft_y=reshape( ifft2( reshape( y, n, n ) ), n^2, 1);
```

```
    z=(fft_b.*ifft_y)./(fft_b.*conj(fft_b)+
            lambda*(fft_e.*conj(fft_e)+fft_f.*conj(fft_f)));
    x=reshape( fft2( reshape( z, n, n ) ), n^2, 1);
end
```

(b)

With the function above and the code below, I generated the following deblurred images for different values of λ.

```
load HW6/deblur.mat
gcf=figure;
Lambdas=10.^(-6:-2);
x0=10;
y0=10;
width=500*2;
height=500*(length(Lambdas)+1)/2;
set(gcf,'position',[x0,y0,width,height])
subplot((length(Lambdas)+1)/2,2,1)
imshow(Y,[])
title('Original Image');
hold on
for i=1:length(Lambdas)
    i
    subplot((length(Lambdas)+1)/2,2,i+1);
    lambda=Lambdas(i)
    x=deblur(Y,B,lambda);
    imshow(reshape(x,1024,1024),[]);
    title(sprintf('lambda=%f',lambda));
end
print -dpng problem1.png
hold off
```

To get the following images

Original Image
lambda=0.000001
lambda=0.000010
lambda=0.000100
lambda=0.001000
lambda=0.010000

Visually it seems like $\lambda = 10^{-3}$ produces the best debluring.

Exercise A10.1

(a)

Let $u = [u(0), u(1), \cdots, u(N-1)]^T$, then we have that $x = u$ we want to minimize the energy or

$$||u||^2$$

Furthermore we can express

$$s_1(N) = 0.1u(N-2) + (0.95+1)\cdot 0.1u(N-3) + (0.95^2+0.95+1)\cdot 0.1u(N-4) + \cdots + (\sum_{i=0}^{N-2} 0.95^i)\cdot 0.1u(0)$$

$$s_2(N) = 0.1u(N-1) + (0.95)\cdot 0.1u(N-2) + (0.95^2)\cdot 0.1u(N-3) + \cdots + (0.95^{N-1})\cdot 0.1u(0)$$

So therefore we have that

$$C = \begin{bmatrix} 0.1\sum_{i=0}^{N-2} 0.95^i & \cdots & 0.1(0.95+1) & 0.1 & 0 \\ 0.1(0.95^{N-1}) & \cdots & 0.1(0.95^2) & 0.1(0.95) & 0.1 \end{bmatrix}$$

And

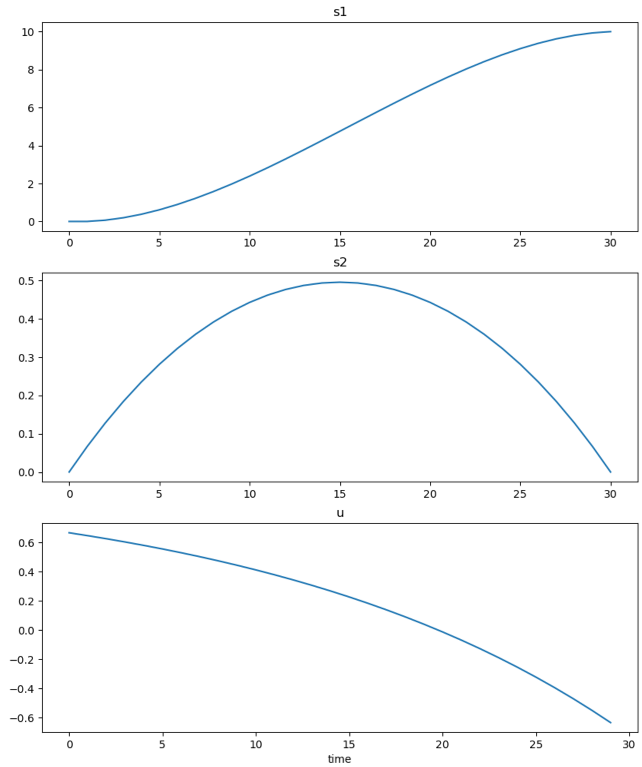
$$d = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

(b)

```
using PyPlot

function create_C(N)
    C = zeros(2,N)

    #make the first row of C
    for i=1:N-1
```

```
        for j=1:N-i
            C[1,i]+=0.1*(0.95^(j-1))
        end
    end
    # C[1,N-1]=0.1

    #make the second row of C
    for i=1:N
        C[2,i] = 0.1*(0.95^(N-i))
    end
    return C
end

# calculate u
N=30
C = create_C(N)
d=[10,0]
u = C\d
s1=zeros(N+1)
s2=zeros(N+1)
for i=1:N
    s1[i+1]=s1[i]+s2[i]
    s2[i+1]=0.95*s2[i]+0.1*u[i]
end

fig,axs=subplots(3,1,figsize=(10,12))
axs[1].plot(s1)
axs[1][:set_title]("s1")
axs[2][:plot](s2)
axs[2][:set_title]("s2")
axs[3][:plot](u)
axs[3][:set_title]("u")
axs[3][:set_xlabel]("time")

savefig("problem2a.png")
close()
```

We get the following plot



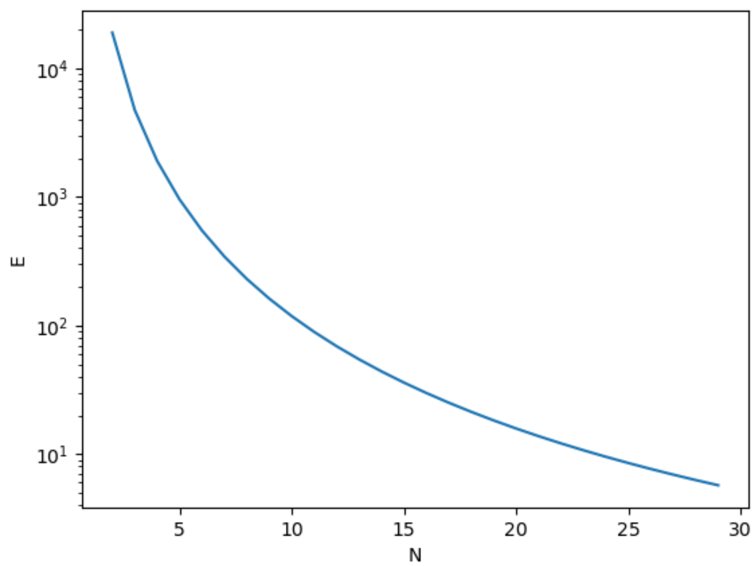
(c)

With the following code:

```
using PyPlot

function create_C(N)
    C = zeros(2,N)
```

```
    #make the first row of C
    for i=1:N-1
        for j=1:N-i
            C[1,i]+=0.1*(0.95^(j-1))
        end
    end
    # C[1,N-1]=0.1

    #make the second row of C
    for i=1:N
        C[2,i] = 0.1*(0.95^(N-i))
    end
    return C
end

N=2:29
println(N)
E=zeros(length(N))
for i=1:length(N)
    C = create_C(N[i])
    d=[10,0]
    u = C\d
    E[i]=sum(u.^2)
end
plot(N,E)
xlabel("N")
ylabel("E")
yscale("log")
savefig("problem2b.png")
close()
```

We get the following plot



Exercise A10.9

(a)

We use Langrage multipliers to solve this problem.

$$L(x) = ||Ax - b||^2 + \lambda e_i^T x$$

$$
\begin{aligned}
\nabla L(x) &= 0 \\
2A^T(Ax - b) + \lambda e_i &= 0 \\
2A^TAx &= 2A^Tb - \lambda e_i \\
A^TAx &= A^Tb - \frac{\lambda}{2} e_i \\
x &= (A^TA)^{-1}A^Tb - \frac{\lambda}{2}(A^TA)^{-1}e_i \\
x &= \hat{x} - \frac{\lambda}{2}(A^TA)^{-1}e_i
\end{aligned}
$$

subsituting this back into the constraint we get

$$
\begin{aligned}
e_i^T x &= 0 \\
e_i^T \left(\hat{x} - \frac{\lambda}{2}(A^TA)^{-1}e_i \right) &= 0 \\
\hat{x}_i - \frac{\lambda}{2}(A^TA)^{-1}_{ii} &= 0 \\
\hat{x}_i &= \frac{\lambda}{2}(A^TA)^{-1}_{ii} \lambda = \frac{2\hat{x}_i}{(A^TA)^{-1}_{ii}}
\end{aligned}
$$

And thus we get that

$$
x = \hat{x} - \frac{\hat{x}_i}{(A^TA)^{-1}_{ii}}(A^TA)^{-1}e_i
$$

(b)

Calculating the QR factorization of A costs $2mn^2$ flops. And solving $\hat{x}$ costs an additional $2mn + n^2$ flops. Then we can solve for $(A^TA)^{-1}e_i$ using backwards and forwards substitution which costs $2n^2$ flops, and then finding $(A^TA)^{-1}_{ii}$ is just finding the value for the ith index in the vector $(A^TA)^{-1}e_i$ which costs 0 flops. This is the same for $\hat{x}_i$. Then calculating $\frac{\hat{x}_i}{(A^TA)^{-1}_{ii}}$ costs 1 flops, and then multiplying that to every value of $(A^TA)^{-1}e_i$ costs n flops. And then subtracting the resulting vector from $\hat{x}$ will cost n flops. So the total cost is $\boxed{2mn^2 + 2mn + 3n^2 + 2n + 1}$ flops.