ECE 133A HW 1

Lawrence Liu

October 20, 2022

Exercise T8.8

For a specific t_i we have

$$f(t_i) = y_i$$
$$\frac{c_1 + c_2 t_i + c_3 t_i^2}{1 + d_1 t_i + d - 2t_i^2} = y_i$$
$$c_1 + c_2 t_i + c_3 t_i^2 = y_i(1 + d_1 t_i + d_2 t_i^2)$$
$$c_1 + c_2 t_i + c_3 t_i^2 - y_i d_1 t_i - y_i d_2 t_i^2 = y_i$$

Therefore we can construct a matrix A and a vector b, such that $A\theta = b$, where $\theta = [c_1, c_2, c_3, d_1, d_2]^T$. We have that for 5 values of t_i and the corresponding 5 values of y_i,

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & -y_1 t_1 & -y_1 t_1^2 \\ 1 & t_2 & t_2^2 & -y_2 t_2 & -y_2 t_2^2 \\ 1 & t_3 & t_3^2 & -y_3 t_3 & -y_3 t_3^2 \\ 1 & t_4 & t_4^2 & -y_4 t_4 & -y_4 t_4^2 \\ 1 & t_5 & t_5^2 & -y_5 t_5 & -y_5 t_5^2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Therefore we can then just solve for θ with $\theta = A^{-1}b$.

Thus for the values of t and y given, we can get the values of *theta* with the following code:

```
t=[1,2,3,4,5]
y=[-1,12,10,1,-3]

A=zeros(5,5)

for i=1:5
    A[i,:]=[1,t[i],t[i]^2,-y[i]*t[i],-y[i]*t[i]^2]
end
print("theta= ")
println(round.(A\y,digits=3))

println("A=")
println(A)
```

We get that

$$\theta = \boxed{(-6.117, 6.99, -1.322, -0.709, 0.158)}$$

Exercise T8.11

Squaring these, we get that for $1 \leq i \leq 4$:

$$\begin{aligned}\rho_i^2 &= ||x - a_i||^2 \\ &= (x_1 - a_{i1})^2 + (x_2 - a_{i2})^2 + (x_3 - a_{i3})^2\end{aligned}$$

Subtracting ρ_4^2 from the others, we get that for $1 \leq i \leq 3$

$$\begin{aligned}\rho_i^2 - \rho_4^2 =&(2a_{41} - 2a_{i1})x_1 + a_{i1}^2 - a_{41}^2 \\ &+ (2a_{42} - 2a_{i2})x_2 + a_{i2}^2 - a_{42}^2 \\ &+ (2a_{43} - 2a_{i3})x_3 + a_{i3}^2 - a_{43}^2\end{aligned}$$

Therefore for the values of ρ_i and a_i we get that:

$$x = (0.605, 0.405, -0.503)$$

Which can be found with the following code in Julia

```
using LinearAlgebra

A=transpose([[-10, 10, 10] [0, 10, 0] [-10, 10, 0] [-20, -10, -10]])
rho=[17.7518, 9.6417, 14.3198, 24.9654]
```

```
M=zeros(3,3)
b=zeros(3)
# println(size(A))
for i=1:3
    # println(b)
    b[i]=rho[i].^2-rho[4].^2
    for j=1:3
        M[i,j]=2*(A[4,j]-A[i,j])
        b[i]+=A[4,j].^2-A[i,j].^2
    end
end

x=M\b
println(b)
print("x= ")
println(round.(x,digits=3))

for i=1:4
    println(norm(x-A[i,:]))
end
```

Exercise A3.8

(a)

We have that $f(s_k, t_k) = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i j s_k^{i-1} t_k^{j-1} = y_k$, then we will have

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & s_1 & s_1 t_1 & s_1 t_1^2 & s_1^2 & s_1^2 t_1 & s_1^2 t_1^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_9 & t_9^2 & s_9 & s_9 t_9 & s_9 t_9^2 & s_9^2 & s_9^2 t_9 & s_9^2 t_9^2 \end{bmatrix}$$

$$x = [c_1 1, c_1 2, c_1 3, c_2 1, c_2 2, c_2 3, c_3 1, c_3 2, c_3 3]^T$$

$$b = \begin{bmatrix} y_1 \\ \vdots \\ y_9 \end{bmatrix}$$

(b)

Using the code below:

```
S=[0,0,0,1,1,1,2,2,2]
T=[0,1,2,0,1,2,0,1,2]
Y=[4,0,5,7,4,9,0,3,4]
```

```
M=zeros(9,9)

for k in 1:9
    for i in 1:3
        for j in 1:3
            M[k,3*(i-1)+j]=S[k]^(i-1)*T[k]^(j-1)
        end
    end
end

println(M)
println(M\Y)
```

We get that

$$
\begin{aligned}
c_{11} &= 4.0 & c_{12} &= -8.5 & c_{13} &= 4.5 \\
c_{21} &= 8.0 & c_{22} &= -3.25 & c_{23} &= 1.75 \\
c_{31} &= -5.0 & c_{32} &= 4.75 & c_{33} &= -2.25
\end{aligned}
$$

Exercise A4.3

we have, that in order for $(A + uv^T)$ to be nonsingular, there must exist a matrix B such that $(A + uv^T)B = I$. Therefore need to prove that

$$(A + uv^T)\left(A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} uv^T A^{-1}\right) = I$$

$$
\begin{aligned}
(A + uv^T)\left(A^{-1} - \frac{1}{1 + v^T A^{-1} u} A^{-1} uv^T A^{-1}\right) =& I - \frac{1}{1 + v^T A^{-1} u} uv^T A^{-1} \\
& + uv^T A^{-1} - \frac{1}{1 + v^T A^{-1} u} uv^T A^{-1} uv^T A^{-1} \\
=& I - \frac{1}{1 + v^T A^{-1} u} uv^T A^{-1} \\
& + uv^T A^{-1} - \frac{(v^T A^{-1} u)}{1 + v^T A^{-1} u} uv^T A^{-1} \\
=& I - \frac{1 + v^T A^{-1} u}{1 + v^T A^{-1} u} uv^T A^{-1} + uv^T A^{-1} \\
=& I - uv^T A^{-1} + uv^T A^{-1} \\
=& I
\end{aligned}
$$

Therefore we have that A is singular if $1 + v^T A^{-1} u \neq 0$, ie if $v^T A^{-1} u \neq -1$, and that $(A + uv^T)^{-1} = A^{-1} - \frac{1}{1+v^T A^{-1} u} A^{-1} uv^T A^{-1}$.

Exercise A4.12

(a)

When $a = \pm 1$, because this would make the columns not linearly independent.

(b)

We will have that

$$A^{-1} = \begin{bmatrix} c_1 & 0 & \cdots & 0 & c_2 \\ 0 & c_1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & c_2 & \cdots & c_1 & 0 \\ c_2 & 0 & \cdots & 0 & c_1 \end{bmatrix}$$

Therefore we must have that

$$c_1 + \alpha c_2 = 1$$

$$c_2 + \alpha c_1 = 0$$

thus we get that $c_2 = -\alpha c_1$ and thus

$$c_1 = \frac{1}{1 - \alpha^2}$$

$$c_2 = -\frac{\alpha}{1 - \alpha^2}$$

Exercise A4.9

(a)

We have

$$\begin{aligned}
(YX)^T &= (AB)^T \left(B^\dagger A^\dagger\right)^T \\
&= B^T A^T \left(A^\dagger\right)^T \left(B^\dagger\right)^T \\
&= B^T A^T \left(A^T\right)^\dagger \left(B^\dagger\right)^T \\
&= B^T \left(B^\dagger\right)^T \\
&= B^T \left(B^T (BB^T)^{-1}\right)^T \\
&= B^T \left((BB^T)^{-1} B\right) \\
&= \left(B^T (BB^T)^{-1}\right) B \\
&= B^\dagger B \\
&= B^\dagger A^\dagger A B \\
&= YX
\end{aligned}$$

Therefore YX is symmetric.

(b)

We have that

$$\begin{aligned}
(XY)^T &= Y^T X^T \\
&= \left(B^\dagger A^\dagger\right)^T (AB)^T \\
&= \left(A^\dagger\right)^T \left(B^\dagger\right)^T B^T A^T \\
&= \left(A^\dagger\right)^T A^T \\
&= \left(A(A^T A)^{-1}\right) A^T \\
&= A\left((A^T A)^{-1} A^T\right) \\
&= AA^\dagger \\
&= ABB^\dagger A^\dagger \\
&= XY
\end{aligned}$$

Therefore XY is symmetric

(c)

We have that

$$\begin{aligned}
YXY &= B^\dagger A^\dagger ABY \\
&= B^\dagger IBY \\
&= B^\dagger BY \\
&= IY \\
&= Y
\end{aligned}$$

(d)

We have

$$\begin{aligned} XYX &= ABB^\dagger A^\dagger AB \\ &= ABB^\dagger IB \\ &= ABB^\dagger BY \\ &= AB \\ &= X \end{aligned}$$