

ECE 133A HW 4

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Exercise A5.6

(a)

Let $DX + XD = A$, we have that $A_{ij} = (D_{ii} + D_{jj})X_{ij}$, Since $DX + XD = B$ we have that

$$\begin{aligned} A_{ij} &= B_{ij} \\ B_{ij} &= (D_{ii} + D_{jj})X_{ij} \end{aligned}$$

Therefore we get that

$$X_{ij} = \frac{B_{ij}}{D_{ii} + D_{jj}}$$

for any i, j , this will exist since $D_{ii} + D_{jj} \neq 0$ for all i and j this computation will cost us 2 flops, 1 for addition and one for division so in total solving for all X_{ij} will cost us $2n^2$ flops.

(b)

Let

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}$$

Then we have that

$$LX = \begin{bmatrix} L_{11}X_{11} & L_{11}X_{12} & \cdots & L_{11}X_{1n} \\ L_{21}X_{11} + L_{22}X_{21} & L_{21}X_{12} + L_{22}X_{22} & \cdots & L_{21}X_{1n} + L_{22}X_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{n1}X_{11} + \cdots + L_{nn}X_{n1} & L_{n1}X_{12} + \cdots + L_{nn}X_{n2} & \cdots & L_{n1}X_{1n} + \cdots + L_{nn}X_{nn} \end{bmatrix}$$

And

$$XL^T = \begin{bmatrix} L_{11}X_{11} & L_{21}X_{11} + L_{22}X_{12} & \cdots & L_{n1}X_{11} + \cdots + L_{nn}X_{1n} \\ L_{11}X_{21} & L_{21}X_{21} + L_{22}X_{22} & \cdots & L_{n1}X_{21} + \cdots + L_{nn}X_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{11}X_{n1} & L_{21}X_{n1} + L_{22}X_{n2} & \cdots & L_{n1}X_{n1} + \cdots + L_{nn}X_{nn} \end{bmatrix}$$

therefore we have that

$$B_{ij} = \sum_{k=1}^i L_{ik}X_{kj} + \sum_{k=1}^j L_{jk}X_{ik}$$

Therefore if we know X_{lm} for all $0 \leq l \leq i$ and $0 \leq m \leq j$ Except for X_{ij} we can express

$$X_{ij} = \frac{B_{ij} - \sum_{k=1}^{i-1} L_{ik}X_{kj} - \sum_{k=1}^{j-1} L_{jk}X_{ik}}{L_{ii} + L_{jj}}$$

Since $L_{ii} + L_{jj} \neq 0$ for all i, j this will exist. The two summations will cost us $2(i-1 + j-1) - 2$ flops, and the subtractions will cost us 2 flops, and the division will cost us 2 flops since we need to first compute the sum $L_{ii} + L_{jj}$ so in total this computation will cost us $2(i+j)$ flops. Therefore to solve for all X_{ij} it will cost us

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n 2(i+j) &= 2 \sum_{i=1}^n \sum_{j=1}^n (i+j) \\
&= 2 \sum_{i=1}^n ni + \sum_{j=1}^n nj \\
&= 2 \left(\frac{n^2(n+1)}{2} + \frac{n^2(n+1)}{2} \right) \\
&= \boxed{2n^2(n+1)}
\end{aligned}$$

Exercise A6.3

(a)

Exercise A6.9

(a)

We have that

$$S^{k-1} = \begin{bmatrix} 0 & I_{k-1} \\ I_{n-k+1} & 0 \end{bmatrix}$$

Therefore S^{k-1} will circularly shift W when we multiple W by it, in other words, the i th row of WS^{k-1} , $(WS^{k-1})_i$ will be

$$(WS^{k-1})_i = [\omega^{-i(k-1)} \quad \omega^{-ik} \quad \dots \quad \omega^{-i(n-1)} \quad \omega^0 \quad \omega^i \quad \dots \quad \omega^{i(k-2)}]$$

Likewise for $\mathbf{diag}(We_k)W$, we have that the i th row of $\mathbf{diag}(We_k)W$, $(\mathbf{diag}(We_k)W)_i$ will be

$$\begin{aligned} (\mathbf{diag}(We_k)W)_i &= [\omega^{-i(k-1)} \quad \omega^{-i}\omega^{-i(k-1)} \quad \dots \quad \omega^i n - 1 \omega^{-i(k-1)}] \\ &= [\omega^{-i(k-1)} \quad \omega^{-ik} \quad \dots \quad \omega^{-i(n+k-2)}] \end{aligned}$$

Therefore for $k > 1$, (since it is obvious for $k = 1$ the equality holds)

$$(\mathbf{diag}(We_k)W)_i = [\omega^{-i(k-1)} \quad \omega^{-ik} \quad \dots \quad \omega^{-i(n-1)} \quad \omega^0 \quad \omega^i \quad \dots \quad \omega^{i(k-2)}]$$

Since $\omega = e^{2\pi j/n}$.

Therefore we have that

$$WS^{k-1} = \mathbf{diag}(We_k)W$$

(b)

Let A_i be the matrix with the i th column being a and the rest of the matrix being 0, then we have that

$$\begin{aligned} T(a) &= \sum_{i=1}^n s^{i-1} A_i \\ &= \sum_{i=1}^n \frac{1}{n} W^H \mathbf{diag}(We_i) W A_i \\ &= \frac{1}{n} W^H \sum_{i=1}^n \mathbf{diag}(We_i) W A_i \end{aligned}$$

We have that $W A_i$ is a matrix with the i th column being $[\sum_{j=1}^n a_j, \sum_{j=1}^n a_j \omega^{-(j-1)}, \dots, \sum_{j=1}^n a_j \omega^{-(j-1)}]$ and the rest of the matrix being 0, therefore we have that,

$$W A_i = A_i^T W^T = A_i^T W$$

Therefore we have

$$\begin{aligned}
T(a) &= \sum_{i=1}^n \frac{1}{n} W^H \mathbf{diag}(W e_k) W A_i \\
&= \sum_{i=1}^n \frac{1}{n} W^H \mathbf{diag}(W e_k) A_i^T W \\
&= \frac{1}{n} W^H \left(\sum_{i=1}^n \mathbf{diag}(W e_k) A_i^T \right) W
\end{aligned}$$

We have that QQ