ECE 133A HW 4

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Exercise T13.3

(a)

With the following julia code we get:

```
using MAT
using LinearAlgebra
using PyPlot
# using Statistics

include("mooreslaw.m")
# println(T)
Years,Transistors=T[:,1],T[:,2]
# println(Years)
# println(Transistors)
A=transpose([reshape(ones(size(Years)),1,:);reshape(Years.-1970,1,:)])
# println(A)
log_Transistors=log10.(Transistors)
theta=A\log_Transistors
println("theta_1=",theta[1])
println("theta_2=",theta[2])
#plot out
plot(Years,log_Transistors,"o")
plot(Years,A*theta)
xlabel("Years")
ylabel("Transistors (log10)")
title("Moore's Law")
legend(["Data","Fit"])
savefig("Moore's Law.png")
close()
```

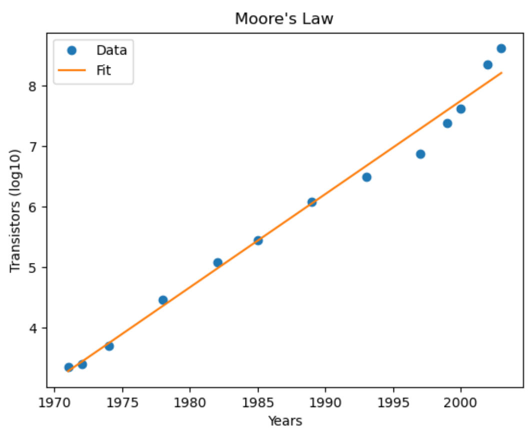
that

$$\theta_1 = 3.125592633829346$$

and

$$\theta_2 = 0.1540181798438225$$

which results in the following fit:



(b)

From our fit we expect the number of transistors to be:

$$10^{\theta_1+\theta_2(2015-1970)} \approx 10^{10}$$

Which is more than the acutally number of $4 \cdot 10^9$ transistors:

(c)

This is in line with Moore's law since $2\theta_2 = 0.30803635968$ which is close to $\log_{10}(2) = 0.30102999566$

Exercise T12.12

(a)

Let $p_{ik} = [u_{ik}, v_{ik}]$ then we have that

$$||p_{i_1} - pj_1||^2 + \cdots + ||p_{i_L} - p_{j_L}||^2 = (u_{i_1} - u_{j_1})^2 + \cdots + (u_{i_L} - u_{j_L})^2 + (v_{i_1} - v_{j_1})^2 + \cdots + (v_{i_L} - v_{j_L})^2$$

$$||p_{i_1} - pj_1||^2 + \cdots + ||p_{i_L} - p_{j_L}||^2 = D(u) + D(v)$$

(b)

Let C be the incident matrix of the graph we have that we want to minimize

$$||C^T u||^2 + ||C^T v||^2$$

Letting the matrix made up of the of the first $N - K$ rows of C be the matrix A and the last K rows of C, likewise let u_1 and v_1 be the first $N - K$ rows of u and v and u_2 and v_2 be the last K rows of u and v respectively, then we have

$$\begin{aligned} ||B^T u||^2 + ||B^T v||^2 &= ||A^T u_1 + B^T u_2||^2 + ||A^T v_1 + B^T v_2||^2 \\ &= \left|\left| \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} + \begin{bmatrix} B^T u_2 \\ B^T v_2 \end{bmatrix} \right|\right|^2 \end{aligned}$$

So then we have that in term of a least square problem we have $||Ax - b||^2$

$$A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix}$$

And

$$b = \begin{bmatrix} B^T u_2 \\ B^T v_2 \end{bmatrix}$$

And

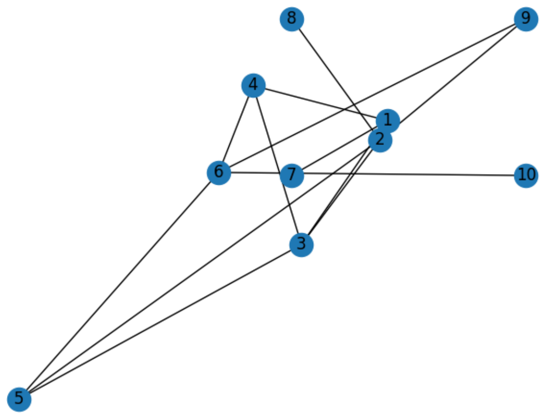
$$x = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

(c)

With the following code in python:

```
import networkx as nx
import matplotlib.pyplot as plt
import numpy as np

# Create a graph
G = nx.Graph()
#add nodes
G.add_nodes_from([1,2,3,4,5,6,7,8,9,10])
#add edges
start_end=[(1, 3), (1, 4), (1, 7), (2, 3), (2, 5), (2, 8), (2, 9),
(3, 4), (3, 5), (4, 6), (5, 6), (6, 9), (6, 10)]
for i in start_end:
    G.add_edge(i[0],i[1])

#get transistion matrix
C = nx.to_numpy_matrix(G).T
# print(C)
u_2=np.array([0,0,1,1])
v_2=np.array([0,1,1,0])

B=C[-u_2.shape[0]:]
# print((B.T@u_2).shape)
b=np.concatenate((B.T@u_2,B.T@v_2),axis=1).T
# print(b.shape)

A=C[:-u_2.shape[0]]
M=np.zeros((2*A.shape[1],2*A.shape[0]))
M[:A.shape[1],:A.shape[0]]=A.T
M[A.shape[1]:,A.shape[0]:]=A.T
# print(M)
# print(M.shape)
x=np.linalg.lstsq(M,-b)[0][:,0]
# print(x[:6])
# print(v_2.reshape((-1,1)).shape)
u=np.concatenate([x[:6],u_2.reshape((-1,1))])
v=np.concatenate([x[6:],v_2.reshape((-1,1))])
pos={}
for i in range(1,11):
    pos[i]=(u[i-1,0],v[i-1,0])
#plot graph
fig, ax = plt.subplots()
nx.draw(G,pos=pos, with_labels=True, ax=ax)
# plt.sca(ax)
# plt.xticks([-2,-1,0,1,2])
# plt.yticks([-2,-1,0,1,2])
# plt.grid()
plt.savefig("fig1.png")
plt.close()
```

We get the following plot:



Exercise A8.3

We can get that

$$\alpha t_i + \beta = \ln(\frac{y_i}{1-y_i})$$

So therefore we can have a least squares problem, with

$$A = \begin{bmatrix} t_1 & 1 \\ t_2 & 1 \\ \vdots & \vdots \\ t_n & 1 \end{bmatrix}$$

and

$$b = [\ln(\frac{y_1}{1-y_1}), \ln(\frac{y_2}{1-y_2}), \ldots, \ln(\frac{y_n}{1-y_n})]^T$$

and

$$x = [\alpha, \beta]^T$$

Then we have a least squares problem of

$$||Ax - b||^2$$

To find α and β I used the following code:

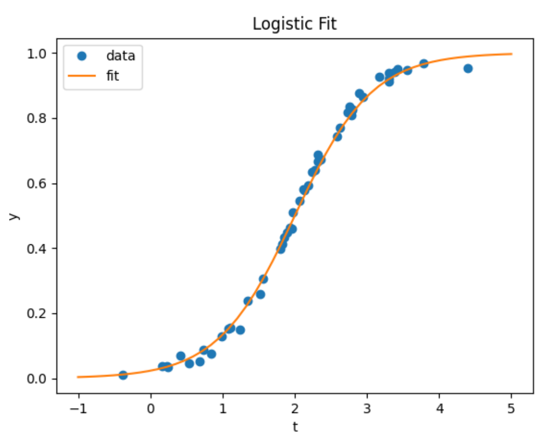
```
using PyPlot
include("logistic_fit.jl")

t,y=logistic_fit()

A=ones(length(t),2)
A[:,1]=t
b=log.(y./(ones(length(t)).-y))

x=A\b
# println(x)
# println(x[1].*t)
# println(x[2])
t1=LinRange(-1,5,50)
y1=exp.(x[2].+(x[1].*t1))./(ones(length(t1)).+exp.(x[2].+(x[1]*t1)))
println("alpha=",x[1])
println("beta=",x[2])
plot(t,y,"o",label="data")
plot(t1,y1,label="fit")
xlabel("t")
ylabel("y")
title("Logistic Fit")
legend()
savefig("fig2.png")
close()
```

which results in

$$\alpha = 1.8676293241597044$$

$$\beta = -3.739673238861126$$

and the following fit:



Exercise A5.11

(a)

we have that

$$
\begin{aligned}
x &= A^{\dagger} b \\
x &= (A^T A)^{-1} A^T b \\
x &= \left(\begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 10^{-k} & 0 \\ 0 & 10^{-k} \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 10^{-k} & 0 \\ 1 & 0 & 10^{-k} \end{bmatrix} \begin{bmatrix} -10^{-k} \\ 1+10^{-k} \\ 1-10^{-k} \end{bmatrix} \\
x &= \left(\begin{bmatrix} 1+10^{-2k} & 1 \\ 1 & 1+10^{-2k} \end{bmatrix} \right)^{-1} \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix} \\
x &= \begin{bmatrix} \frac{1+10^{-2k}}{10^{-4k}+2\cdot 10^{-2k}} & -\frac{1}{10^{-4k}+2\cdot 10^{-2k}} \\ -\frac{1}{10^{-4k}+2\cdot 10^{-2k}} & \frac{1+10^{-2k}}{10^{-4k}+2\cdot 10^{-2k}} \end{bmatrix} \begin{bmatrix} 10^{-2k} \\ -10^{-2k} \end{bmatrix} \\
x &= \frac{10^{-2k}}{10^{-4k}+2\cdot 10^{-2k}} \begin{bmatrix} 2+10^{-2k} \\ -2-10^{-2k} \end{bmatrix}
\end{aligned}
$$

thus we have for $k = 6$:

$$x = \frac{10^{-12}}{10^{-24}+2\cdot 10^{-12}} \begin{bmatrix} 2+10^{-12} \\ -2-10^{-12} \end{bmatrix}$$

And for $k = 7$ we have

$$x = \frac{10^{-14}}{10^{-28}+2\cdot 10^{-14}} \begin{bmatrix} 2+10^{-14} \\ -2-10^{-14} \end{bmatrix}$$

And for $k = 8$ we have

$$x = \frac{10^{-16}}{10^{-32}+2\cdot 10^{-16}} \begin{bmatrix} 2+10^{-16} \\ -2-10^{-16} \end{bmatrix}$$

(b)

We have that with the following code:

```
for k = [6,7,8]

    A=transpose([[1,1] [10.0^-k,0] [0,10.0^-k]])
    b=[-(10.0^-k),1+10.0^-k, 1-10.0^-k]
    println("k=",k)
    println("x=",A\b)
    println("--------------------")
end
```

we get that

```
k=6
x=[0.9999999999176988, -0.999999999917699]
--------------------
k=7
x=[1.0000000027436178, -1.0000000027436176]
--------------------
k=8
x=[1.0000000043137076, -1.0000000043137076]
--------------------
```

(c)

We have that with the following code:

```
for k = [6,7,8]

    A=transpose([[1,1] [10.0^-k,0] [0,10.0^-k]])
    b=[-(10.0^-k),1+10.0^-k, 1-10.0^-k]
    println("k=",k)
    println("x=",(A'*A) \ (A'*b))
    println("--------------------")
end
```

we get that

```
k=7
x=[1.0000000027436178, -1.0000000027436176]
--------------------
k=8
x=[1.0000000043137076, -1.0000000043137076]
--------------------
```

And for $k = 8$ we get a singular exception error since `A'*A` gets rounded to a matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ that is singular

(a)

$$f(y) = ||Ay - b||^2 + (c^T y - d)^2$$

To minimize we take the derivative of it with respect to y_i for all $1 \leq n \leq N$ and set it to zero have

$$\frac{\partial}{\partial y_i} f(y) = 2(A^T(Ay - b))_i + 2(c^T y - d)c_i = 0$$

Thus we have

$$\nabla f(y) = 2(A^T(Ay - b) + c(c^T y - d)) = 0$$

which gives us

$$\begin{aligned} \nabla f(y) &= 0 \\ 2(A^T(Ay - b) + c(c^T y - d)) &= 0 \\ A^T(Ay - b) + c(c^T y - d) &= 0 \end{aligned}$$

if $\hat{y}$ is a solution then we must have that

$$A^T(A\hat{y} - b) + c(c^T \hat{y} - d) = 0$$

we can confirm this, since

$$\hat{y} = \hat{x} + \frac{d - c^T \hat{x}}{1 + c^T (A^T A)^{-1} c} (A^T A)^{-1} c$$

we have:

$$\begin{aligned}
A^T(A\hat{y}-b)+c(c^T\hat{y}-d)&=0\\
A^TA\frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}(A^TA)^{-1}c+cc^T\hat{x}+c(c^T\frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}(A^TA)^{-1}c-d)&=0\\
dc-c^T\hat{x}c+cc^T(d-c^T\hat{x})(A^TA)^{-1}c-c(d-c^T\hat{x})(1+c^T(A^TA)^{-1}c)&=0\\
cc^T(d-c^T\hat{x})(A^TA)^{-1}c-c(d-c^T\hat{x})(c^T(A^TA)^{-1}c)&=0\\
cc^T(d-c^T\hat{x})(A^TA)^{-1}c-cc^T(d-c^T\hat{x})(A^TA)^{-1}c&=0
\end{aligned}$$

(b)

We first compute the QR factorization fo A, which will cost us $2mn^2$ flops, then we can compute $\hat{x}$ with an additonal $2mn+n^2$ flops. Likewise, since we can rewrite $(A^TA)^{-1}c$ as $(R^TQ^TQR)^{-1}c=(R^TR)^{-1}c$, which we can solve in $2n^2$ flops. then computing $c^T\hat{x}$ and $c^T(A^TA)^{-1}c$ will each cost us an additional $2n-1$ flops, then computing $\frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}$ will cost us 3 flops.Then computing $\hat{x}+\frac{d-c^T\hat{x}}{1+c^T(A^TA)^{-1}c}(A^TA)^{-1}c$ will cost us $2n$ flops, so in total this algorithm will cost us $\boxed{2mn^2+2mn+3n^2+6n-1}$ flops.