ECE 133A HW 4

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Exercise A5.6

(a)

Let $DX + XD = A$, we have that $A_{ij} = (D_{ii} + D_{jj})X_{ij}$, Since $DX + XD = B$ we have that

$$A_{ij} = B_{ij}$$
$$B_{ij} = (D_{ii} + D_{jj})X_{ij}$$

Therefore we get that

$$X_{ij} = \frac{B_{ij}}{D_{ii} + D_{jj}}$$

for any i, j, this will exist since $D_{ii} + D_{jj} \neq 0$ for all i and j this computation will cost us 2 flops, 1 for addition and one for division so in total solving for all X_{ij} will cost us $2n^2$ flops.

(b)

Let

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix}$$

$$X = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{bmatrix}$$

Then we have that

$$LX = \begin{bmatrix} L_{11}X_{11} & L_{11}X_{12} & \cdots & L_{11}X_{1n} \\ L_{21}X_{11} + L_{22}X_{21} & L_{21}X_{12} + L_{22}X_{22} & \cdots & L_{21}X_{1n} + L_{22}X_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{n1}X_{11} + \cdots + L_{nn}X_{n1} & L_{n1}X_{12} + \cdots + L_{nn}X_{n2} & \cdots & L_{n1}X_{1n} + \cdots + L_{nn}X_{nn} \end{bmatrix}$$

And

$$XL^T = \begin{bmatrix} L_{11}X_{11} & L21X_{11} + L22X_{12} & \cdots & L_{n1}X_{11} + \cdots + L_{nn}X_{1n} \\ L_{11}X_{21} & L21X_{21} + L22X_{22} & \cdots & L_{n1}X_{21} + \cdots + L_{nn}X_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ L_{11}X_{n1} & L21X_{n1} + L22X_{n2} & \cdots & L_{n1}X_{n1} + \cdots + L_{nn}X_{nn} \end{bmatrix}$$

therefore we have that

$$B_{ij} = \sum_{k=1}^{i} L_{ik}X_{kj} + \sum_{k=1}^{j} L_{jk}X_{ik}$$

Therefore if we know X_{lm} for all $0 \le l \le i$ and $0 \le m \le j$ Except for X_{ij} we can express

$$X_{ij} = \frac{B_{ij} - \sum_{k=1}^{i-1} L_{ik}X_{kj} - \sum_{k=1}^{j-1} L_{jk}X_{ik}}{L_{ii} + L_{jj}}$$

Since $L_{ii} + L_{jj} \neq 0$ for all i, j this will exist. The two summations will cost us $2(i - 1 + j - 1) - 2$ flops, and the subtractions will cost us 2 flops, and the division will cost us 2 flops since we need to first compute the sum $L_{ii} + L_{jj}$ so in total this computation will cost us $2(i + j)$ flops. Therefore to solve for all X_{ij} it will cost us

$$\sum_{i=1}^{n} \sum_{j=1}^{n} 2(i + j) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (i + j)$$

$$= 2 \sum_{i=1}^{n} ni + \sum_{j=1}^{n} nj$$

$$= 2 \left(\frac{n^2(n + 1)}{2} + \frac{n^2(n + 1)}{2} \right)$$

$$= \boxed{2n^2(n + 1)}$$

Exercise A6.3

(a)

Since $S^T = -S$ we have that

$$S = \begin{bmatrix} 0 & c_{12} & \cdots & c_{1n} \\ -c_{12} & 0 & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \cdots & 0 \end{bmatrix}$$

therefore we have that for any $X = [x_1, x_2, \cdots, x_n]^T$

$$Sx = \begin{bmatrix} 0 + c_{12}x_2 + \cdots + c_{1n}x_n \\ -c_{12}x_1 + 0 + \cdots + c_{2n}x_n \\ \vdots \\ -c_{1n}x_1 - c_{2n}x_2 + \cdots + 0 \end{bmatrix}$$

Therefore we have:

$$x^T S x = \sum_{i=1}^{n} \sum_{j=i+1}^{n} c_{ij} x_i x_j - \sum_{i=1}^{n} \sum_{j=i+1}^{n} c_{ij} x_i x_j = 0$$

In order for $(I - S)x = 0$ will only happen if $x = 0$,

$$(I - S)x = 0$$
$$x^T(I - S)x = 0$$
$$x^T I x - x^T S x = 0$$
$$x^T I x = 0 x^T x \qquad\qquad = 0 x \cdot x = 0$$

Therefore $(I - S)x = 0$ will only happen if $x = 0$ so we have that $I - S$ is nonsingular.

(b)

Similarly as how $I - S$ is nonsingular we can show that $I + S$ is nonsingular. Since

$$(I + S)x = 0$$
$$x^T(I + S)x = 0$$
$$x^T I x + x^T S x = 0$$
$$x^T I x = 0 x^T x \qquad\qquad = 0 x \cdot x = 0$$

Which once again leads to the result that $(I + S)x = 0$ only when $x = 0$, and thus $I + S$ is nonsingular. Therefore we have that $(I - S)^{-1}$ and $(I + S)^{-1}$ exist. Thus we get

$$(I + S)(I - S)^{-1} = (2I - (I - S))(I - S)^{-1}$$
$$= 2(I - S)^{-1} - I$$
$$= (I - S)^{-1}(2I - (I - S))$$
$$= (I - S)^{-1}(I + S)$$

(c)

In order for a matrix A to be symmetric we must have that $A^T A = I$ we have

$$
\begin{aligned}
\left((I+S)(I-S)^{-1}\right)^T \left((I+S)(I-S)^{-1}\right) &= \left((I-S)^{-1}\right)^T (I-S)(I+S)(I-S)^{-1} \\
&= \left((I-S)^{-1}\right)^T (I-S)(I-S)^{-1}(I+S) \\
&= \left((I-S)^{-1}\right)^T (I+S) \\
&= \left((I-S)^T\right)^{-1} (I-S)^{-1}(I+S) \\
&= \left((I+S)\right)^{-1} (I+S) \\
&= I
\end{aligned}
$$

Therefore we have that $A = (I+S)(I-S)^{-1}$ is orthogonal.

Exercise A6.9

(a)

We have that
$$
S^{k-1} = \begin{bmatrix} 0 & I_{k-1} \\ I_{n-k+1} & 0 \end{bmatrix}
$$

Therefore S^{k-1} will circularly shift W when we multiple W by it, in other words, the ith row of WS^{k-1}, $\left(WS^{k-1}\right)_i$ will be

$$
\left(WS^{k-1}\right)_i = \begin{bmatrix} \omega^{-i(k-1)} & \omega^{-ik} & \cdots & \omega^{-i(n-1)} & \omega^0 & \omega^i & \cdots & \omega^{i(k-2)} \end{bmatrix}
$$

Likewise for $\mathbf{diag}(We_k)W$, we have that the ith row of $\mathbf{diag}(We_k)W$, $\left(\mathbf{diag}(W_{e_k})W\right)_i$ will be

$$
\begin{aligned}
\left(\mathbf{diag}(We_k)W\right)_i &= \begin{bmatrix} \omega^{-i(k-1)} & \omega^{-i}\omega^{-i(k-1)} & \cdots & \omega^i n - 1\omega^{-i(k-1)} \end{bmatrix} \\
&= \begin{bmatrix} \omega^{-i(k-1)} & \omega^{-ik} & \cdots & \omega^{-i(n+k-2)} \end{bmatrix}
\end{aligned}
$$

Therefore for $k > 1$, (since it is obvious for $k = 1$ the equality holds)

$$(\mathbf{diag}(We_k)W)_i = \begin{bmatrix} \omega^{-i(k-1)} & \omega^{-ik} & \cdots & \omega^{-i(n-1)} & \omega^0 & \omega^i & \cdots & \omega^{i(k-2)} \end{bmatrix}$$

Since $\omega = e^{2\pi j/n}$.

Therefore we have that

$$WS^{k-1} = \mathbf{diag}(We_k)W$$

(b)

Let A_i be the matrix with the ith column being a and the rest of the matrix being 0, then we have that

$$T(a) = \sum_{i=1}^{n} s^{i-1} A_i$$

$$= \sum_{i=1}^{n} \frac{1}{n} W^H \mathbf{diag}(We_i)W A_i$$

$$= \frac{1}{n} W^H \sum_{i=1}^{n} text\mathbf{diag}(We_i)W A_i$$

We have that WA_i is a matrix with the ith column being $[\sum_{j=1}^{n} a_j, \sum_{j=1}^{n} a_j \omega^{-(j-1)}, \cdots \sum_{j=1}^{n} a_j \omega^{-(j-1)}]$ and the rest of the matrix being 0, therefore we have that,

$$WA_i = A_i^T W^T \qquad\qquad = A_i^T W$$

Therefore we have

$$T(a) = \sum_{i=1}^{n} \frac{1}{n} W^H \mathbf{diag}(We_k)W A_i$$

$$= \sum_{i=1}^{n} \frac{1}{n} W^H \mathbf{diag}(We_k) A_i^T W$$

$$= \frac{1}{n} W^H \left(\sum_{i=1}^{n} \mathbf{diag}(We_k) A_i^T \right) W$$

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We have that

Exercise A6.15

(a)

We have

$$Q \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} Q^T = \frac{1}{2} \begin{bmatrix} I_n & I_n \\ J_n & -J_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n & J_n \\ I_n & -J_n \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} I_n & I_n \\ J_n & -J_n \end{bmatrix} \begin{bmatrix} I_n & J_n \\ -I_n & J_n \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 2J_n \\ 2J_n & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix}$$

If a matrix is orthogonal then we have that $Q^T Q = I_{2n}$, therefore we have that

$$Q^T Q = \frac{1}{2} \begin{bmatrix} I_n & J_n \\ I_n & -J_n \end{bmatrix} \begin{bmatrix} I_n & I_n \\ J_n & -J_n \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2I_n & 0 \\ 0 & 2I_n \end{bmatrix}$$

$$= I_{2n}$$

So Q is orthogonal.

(b)

We have

$$J_{2n}A = AJ_{2n}$$

$$Q^T J_{2n} A = Q^T A J_{2n}$$

$$Q^T J_{2n} AQ = Q^T A J_{2n} Q$$

$$Q^T J_{2n} AQ = Q^T AQ \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} Q^T Q$$

$$Q^T J_{2n} AQ \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} = Q^T AQ \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

$$Q^T J_{2n} AQ \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} = Q^T AQ$$

$$Q^T AQ = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} Q^T AQ \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}$$

$$Q^T AQ = \frac{1}{2} \begin{bmatrix} I_n & J_n \\ -I_n & J_n \end{bmatrix} A \begin{bmatrix} I_n & -I_n \\ J_n & J_n \end{bmatrix}$$

Let us express A in terms of n x n submatricies, B, C, D, and E. We have
$A = \begin{bmatrix} B & D \\ E & C \end{bmatrix}$, and

$$J_{2n}A = AJ_{2n}$$

$$\begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix} \begin{bmatrix} B & D \\ E & C \end{bmatrix} = \begin{bmatrix} B & D \\ E & C \end{bmatrix} \begin{bmatrix} 0 & J_n \\ J_n & 0 \end{bmatrix}$$

$$\begin{bmatrix} J_n E & J_n C \\ J_n B & J_n D \end{bmatrix} = \begin{bmatrix} D J_n & B J_n \\ C J_n & E J_n \end{bmatrix}$$

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Therefore we have:

$$
\begin{aligned}
Q^T A Q &= \frac{1}{2} \begin{bmatrix} I_n & J_n \\ -I_n & J_n \end{bmatrix} A \begin{bmatrix} I_n & -I_n \\ J_n & J_n \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} I_n & J_n \\ -I_n & J_n \end{bmatrix} \begin{bmatrix} B & D \\ E & C \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ J_n & J_n \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} I_n & J_n \\ -I_n & J_n \end{bmatrix} \begin{bmatrix} B + DJ_n & -B + DJ_n \\ E + CJ_n & -E + CJ_n \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} B + DJ_n + J_n E + J_n C J_n & -B + DJ_n - J_n E + J_n C J_n \\ -B - DJ_n + J_n E + J_n C J_n & B - DJ_n - J_n E + J_n C J_n \end{bmatrix}
\end{aligned}
$$

From $J_{2n} A = A J_{2n}$ we get that $DJ_n - J_n E = 0$ and $C J_n - J_n B = 0$. Therefore we have that $-B + DJ_n - J_n E + J_n C J_n = 0$ and $-B - DJ_n + J_n E + J_n C J_n = 0$ and thus we get:

$$
Q^T A Q = \frac{1}{2} \begin{bmatrix} B + DJ_n + J_n E + J_n C J_n & 0 \\ 0 & B - DJ_n - J_n E + J_n C J_n \end{bmatrix}
$$

(c)

Multiplying both sides of $Ax = b$ by Q^T we get:

$$
Q^T A Q x = Q^T b
$$

Then letting $x = Qy$ we get

$$
Q^T A Q y = Q^T b
$$

Since $Q^T A Q$ is block diagonal, this is just solving two n x n systems of equations. which will take $2 \cdot \frac{2}{3} n^3$ operations. Since multiplying b by Q^T takes $2n^2$ operations, and multiplying y by Q takes $2n^2$ operations, we have that the leading term of our imporoved solution is $\frac{4}{3} n^3$ So we have been able to reduce the number of flops by a factor if 4.

Exercise A6.16

We have that

$$A_{ij} = \sum_{k=1}^{n} Q_{ik} R_{kj}$$

Since R is upper triangular we have that

$$A_{ij} = \sum_{k=i}^{j} Q_{ik} R_{kj}$$

Exercise A7.5

We first LU factorize A, which will cost us $\frac{2}{3}n^3$ flops. Then we solve for $x' = A^{-1}b$, which will cost us $2n^2$ flops, then with this we can solve for $x'' = A^{-2}b = A^{-1}x'$, which will cost us $2n^2$ flops. Then once again, solving for $x''' = A^{-3}b = A^{-1}x''$, which will cost us $2n^2$ flops. then sum up b and x', x'', x''' to get x, which will cost us $3n$ flops. So in total we will take $\boxed{\frac{2}{3}n^3 + 6n^2 + 3n}$ flops.

Exercise A7.10

First we LU factorize A, which will cost us $\frac{2}{3}n^3$ flops. Then we solve for $x = A^{-1}b$, which will cost us $2n^2$ flops, and we solve for $x_u = A^{-1}u$, which will cost us $2n^2$ flops, then we have

$$y = (A + uv^T)^{-1}b = x - \frac{1}{1 + v^T x_u} x_u v^T x$$

Then we calculate $v^T x_u$, which will cost us $2n - 1$ flops and $v^T x$ which will likewise cost $2n - 1$ flops. Then we have

$$y = (A + uv^T)^{-1}b = x - \frac{v^T x}{1 + v^T x_u} x_u$$

Solving for x consists of first of all finding $\frac{x_u v^T}{1+v^T x_u}$ given $v^T x_u$ and $v^T x$, which will cost 2 flops, and then we multiply that value to every value in x_u, which will cost us n flops, and then subtracting the resulting vector from x will once again cost us n flops, thus we have that the total flops needed would be:

$$\frac{2}{3}n^3 + 4n^2 + 4n + 2n = \boxed{\frac{2}{3}n^3 + 4n^2 + 6n}$$