ECE 133A HW 1

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Exercise A1.7

For the optimal coefficents we have:

$$
\begin{aligned}
J &= \frac{1}{n}||c_1\mathbf{1} + c_2 a - b||^2 \\
&= \frac{1}{n}||(m_b - m_a c_2)\mathbf{1} + c_2 a - b||^2 \\
&= \frac{1}{n}\sum_{k=1}^{n}((m_b - m_a c_2) + c_2 a_k - b_k)^2 \\
&= \frac{1}{n}\sum_{k=1}^{n}(c_2(a_k - m_a) - (b_k - m_b))^2 \\
&= \frac{1}{n}\left(c_2^2\sum_{k=1}^{n}(a_k - m_a)^2 - c_2\sum_{k=1}^{n}(a_k - m_a)(b_k - m_b) + \sum_{k=1}^{n}(b_k - m_b)^2\right) \\
&= c_2^2 s_a^2 - \frac{2c_2}{n}(a - m_a\mathbf{1})^T(b - m_b\mathbf{1}) + s_b^2 \\
&= \rho^2 s_b^2 - 2\frac{\rho s_b}{s_a}\rho s_a s_b + s_b^2 \\
&= s_b^2 - \rho^2 s_b^2 \\
&= \boxed{(1-\rho^2)s_b^2}
\end{aligned}
$$

Exercise A1.8

(a)

We want to maximize J with respect to c_1, so we take the derivative of J with respect to c_1 and find the value of c_1 that makes the derivative equal to zero:

$$\begin{aligned}\frac{\partial J}{\partial c_1} = \frac{2}{n}\sum_{k=1}^{n}\frac{c_1 + c_2 a_k - b_k}{1+c^2} &= 0\\ \frac{1}{n}\sum_{k=1}^{n}(c_1 + c_2 a_k - b_k) &= 0\\ c_1 + \frac{1}{n}\sum_{k=1}^{n}(c_2 a_k - b_k) &= 0\\ c_1 + c_2 m_a - m_b &= 0\\ c_1 &= m_b - c_2 m_a\end{aligned}$$

(b)

We have for the optimal c_1,

$$\begin{aligned}J &= \frac{||c_2(a - m_a\mathbf{1}) - (b - m_b\mathbf{1})||^2}{n(1+c_2^2)}\\ &= \frac{1}{n(1+c_2^2)}\left(\sum_{k=1}^{n}(c_2 a_k - m_a c_2 - b_k + m_b)^2\right)\\ &= \frac{1}{(1+c_2^2)}\left(c_2^2 s_a^2 + s_b^2 - 2\rho s_a s_b c_2\right)\end{aligned}$$

Taking the derivative with respect to c_2, we get

$$\frac{\partial J}{\partial c_2} = \frac{1}{(1+c_2^2)^2}\left((2c_2 s_a^2 - 2\rho s_a s_b)(1+c_2^2) - 2c_2(c_2^2 s_a^2 + s_b^2 - 2\rho s_a s_b c_2)\right)$$

Setting this equal to 0 to mimimize J with respect to c_2 we get

$$\begin{aligned}
\frac{1}{(1+c_2^2)^2}\left((2c_2 s_a^2 - 2\rho s_a s_b)(1+c_2^2) - 2c_2(c_2^2 s_a^2 + s_b^2 - 2\rho s_a s_b c_2)\right) &= 0 \\
2c_2 s_a^2 - 2\rho s_a s_b - 2\rho s_a s_b c_2^2 - 2c_2 s_b^2 + 4\rho s_a s_b c_2^2 &= 0 \\
\rho s_a s_b c_2^2 + 2c_2(s_a^2 - s_b^2) - 2\rho s_a s_b &= 0 \\
\rho c_2^2 + \left(\frac{s_a}{s_b} - \frac{s_b}{s_a}\right) c_2 - \rho &= 0
\end{aligned}$$

When $\rho \neq 0$ the optimal solution is one with the same sign as ρ, since that makes $-2\rho s_a s_b c_2$ negative, which will minimize J, since all the other terms of J involves squares, which are always nonnegative

(c)

With the following code in Julia

```
using MAT
using LinearAlgebra
using PyPlot
using Statistics

function calculate_mean(v)
        n=size(v,1)
        #println(n)
        #println(mean(v))
        #println(sum(v)/n)
        return sum(v)/n
end

function calculate_std(v,m)
        n=size(v,1)
        #println(std(v,correct=false))
        return norm(v.-m)/sqrt(n)
end

function calculate_rho(a,m_a,s_a,b,m_b,s_b)
        n=size(a,1)
        #println("testing")
        #println(cov(a,b,corrected=false)/(s_a*s_b))
        #println(dot(a.-m_a,b.-m_b)/n)
        return (1/(s_a*s_b*n))*(transpose(a.-m_a)*(b.-m_b))
end

function ortho_c2(rho,s_a,s_b)
        if rho==0
```

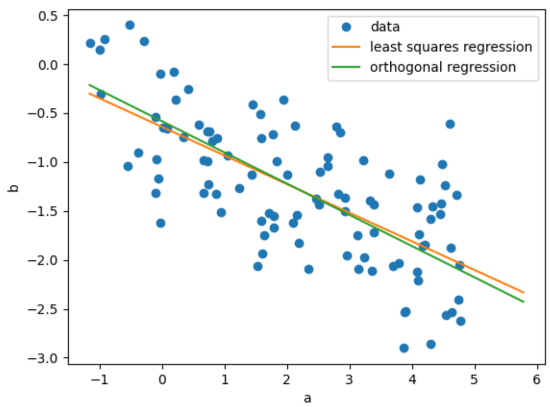
```
                return 0
        else
                b=(s_a/s_b-s_b/s_a)
                c_21=(-b+sqrt(b^2+4*(rho^2)))/(2*rho)
                c_22=(-b-sqrt(b^2+4*(rho^2)))/(2*rho)
                if sign(c_21)==sign(rho)
                        return c_21
                else
                        return c_22
                end
        end
end


function main()
        include("orthregdata.m")
        #print(size(a))
        m_a=calculate_mean(a)
        m_b=calculate_mean(b)
        s_a=calculate_std(a,m_a)
        #println(s_a)
        #println("---")
        s_b=calculate_std(b,m_b)
        rho=calculate_rho(a,m_a,s_a,b,m_b,s_b)
        #println(rho)
        #println(cor(a,b))
        #plot out scatter
        #println("plotting")
        plot(a,b,"o",label="data")
        #least squares solution
        c_2=rho*s_b/s_a
        #println(c_2)
        c_1=m_b-m_a*c_2
        #println(c_1)
        x=LinRange(minimum(a),maximum(a)+1,4)
        plot(x,x.*c_2.+c_1,label="least squares regression")
        #orthogonal solution
        c_2=ortho_c2(rho,s_a,s_b)
        c_1=m_b-m_a*c_2
        #println(rho*(c_2^2)+(s_a/s_b-s_b/s_a)*c_2-rho)
        #println(c_2)
        #println(rho)
        plot(x,x.*c_2.+c_1,label="orthogonal regression")
        legend()
        xlabel("a")
        ylabel("b")
        savefig("test.png")

end

main()
```

We get the following regression:



Exercise A2.4

Expanding $(1+A)^{n-1}$, we get

$$(1+A)^{n-1} = \sum_{k=1}^{n-1} A^k$$

Since $A^k_{i,j}$ represents whether a path of length exists from j to i, (with $A^k_{i,j} > 0$ representing that such a path exitst) we can see that the sum of the A^k matrices for $1 \leq k \leq n-1$ represents all the paths that can possibly exists. Therefore A is irreducible if it possible to travel to one node to any other node, ie that G_A is strongly connected.

Exercise A2.8

We can start by multiplying each x_i subvectors with B, each of these operations takes $(2n-1)n$ flops, so for the n subvectors it will cost us $(2n-1)n^2$ flops.

Now we get that for the ith output subvector $y_i = \sum_{j=1}^{n} A_{i,j}Bx_j$. Multiplying Ai, j with Bx_j costs us n flops, therefore repeating this n times for $1 \leq i \leq n$ results in n^2 flops. Summing these vectors $A_{i,j}Bx_j$ costs $n(n-1)$ flops. So in total computing this subvector costs us $2n(n-1)$ flops. We will need to repeat this for each of the n output subvectors, so it will cost use $2n^2(n-1)$.

Therefore in total, this algorithm will cost us $2n^2(n-1)+(2n-1)n^2$ flops, so approximatley $4n^3$ flops for large n. In comparison the algorithm for computing (n^2, n^2) matrix multiplied with an n^2 vector will take $(2n^2-1)n^2$ flops, or $2n^4$ flops approximatley for large n.

Exercise A2.10

(a)

For any k that satisfy $i \leq k < j$, we have that we can rearange the multiplication of $A_iA_{i+1}...A_j$ to $(A_i..A_k)(A_{k+1}..A_j)$.

We have that multiplying the matrices from i to k will cost us $c_{i,k}$ flops and result in a matrix of size (n_{i-1}, n_k), likewise multiplying the matrices from $k+1$ to j will cost us $c_{k+1,j}$ flops and result in a matrix of size (n_k, n_j).

Multiplying these two matrices together will cost us $2n_{i-1}n_kn_j$. Therefore if we split up the computation of $A_iA_{i+1}...A_j$ into two parts, one from i to k the other from $k+1$ to j, we will have thatv the total cost of computing $A_iA_{i+1}...A_j$ is:

$$c_{i,k} + c_{k+1,j} + 2n_{i-1}n_kn_j$$

Therefore the minimum of the cost of computing $A_iA_{i+1}...A_j$ is when k is the value that minimizes the above expression. In otherwords:

$$c_{i,j} = \min_{k=i,i+1,\ldots,j-1}(c_{i,k} + c_{k+1,j} + 2n_{i-1}n_kn_j)$$

(b)

We have that the triangular table of $c_{i,j}$ for $A_1A_2A_3A_4$ is:

$$\begin{array}{cccc} 0 & 1.00 \cdot 10^{10} & 1.20 \cdot 10^{10} & 1.21 \cdot 10^{9} \\ & 0 & 1.00 \cdot 10^{11} & 1.20 \cdot 10^{9} \\ & & 0 & 2.00 \cdot 10^{8} \\ & & & 0 \end{array}$$

Therefore the optimal cost of computing $A_1A_2A_3A_4$ is multiply A_1 with the product of A_2 multiplied with the product of A_3 and A_4, which takes us $\boxed{1.21 \cdot 10^9}$ flops.

Likewise since the triangular table of $c_{i,j}$ for $A_1A_2A_3$ is:

$$\begin{array}{ccc} 0 & 1.00 \cdot 10^{10} & 1.20 \cdot 10^{10} \\ & 0 & 1.00 \cdot 10^{11} \\ & & 0 \end{array}$$

The optimal cost of $A_1A_2A_3$ is to multiply (A_1A_2) with A_3, which takes us $1.20 \cdot 10^{10}$ flops. So interestingly the optimal flops to compute $A_1A_2A_3A_4$ less than the optimal flops to compute $A_1A_2A_3$, despite the fact that $A_1A_2A_3A_4$ involves one more matrix multiplication. Likewise the optimal order for the multiplication also changes when we try to compute $A_1A_2A_3A_4$ compared to $A_1A_2A_3$.