Homework 2

1. Verify the four properties on slide 1.44 for the pseudo-inverse defined on slide 1.39.

2. The componentwise product or Hadamard product $C = A \circ B$ of two $n \times n$ matrices A, B is the $n \times n$ matrix C with elements $C_{ij} = A_{ij}B_{ij}$. (In MATLAB: `C = A .* B`.)

   (a) Suppose A is $n \times n$ and d is an n-vector. Verify that
   $$A \circ (dd^T) = \mathbf{diag}(d) A \, \mathbf{diag}(d)$$
   where $\mathbf{diag}(d)$ is the diagonal matrix with the vector d on its diagonal. Use this observation to show that if A is positive semidefnite, then the matrix $A \circ (dd^T)$ is positive semidefinite.

   (b) Suppose A and B are positive semidefinite. Show that $A \circ B$ is positive semidefinite. *Hint.* Every positive semidefinite matrix B can be factored as $B = DD^T$; see slide 2.27.

   (c) Suppose A and B are positive definite. Show that $A \circ B$ is positive definite.

   (d) As an application, let
   $$f(x) = c_0 + c_1 x + \cdots + c_d x^d$$
   be a polynomial with nonnegative coefficients $c_0, \ldots, c_d$. Suppose X is a positive semidefinite $n \times n$ matrix, and define Y as the $n \times n$ matrix with elements
   $$Y_{ij} = f(X_{ij}), \quad i, j = 1, \ldots, n.$$
   We obtain Y by applying the polynomial f to each element of X. Show that Y is positive semidefinite.

3. Let U be an $n \times m$ matrix with orthonormal columns and $m < n$. Give an eigendecomposition of the following matrices.

   (a) $A = UU^T$, (b) $A = I - UU^T$, (c) $A = I - 2UU^T$.

   Recall that $UU^T x$ is the projection of x on the range of U. The figure shows the geometrical meaning of the three matrices for $n = 2$, $m = 1$.