Homework 3

1. Suppose A is a symmetric $n \times n$ matrix. Explain how to find the solution of the following problems from the eigendecomposition of A. In each problem the optimization variable is an $n \times k$ matrix X.

   (a)
   $$\begin{array}{ll} \text{maximize} & \mathrm{trace}(X^T A X) \\ \text{subject to} & X^T X = I. \end{array}$$

   (b)
   $$\begin{array}{ll} \text{minimize} & \mathrm{trace}(X^T A X) \\ \text{subject to} & X^T X = I. \end{array}$$

   (c) Assume A is positive definite.
   $$\begin{array}{ll} \text{maximize} & \det(X^T A X) \\ \text{subject to} & X^T X = I. \end{array}$$

   (d)
   $$\begin{array}{ll} \text{maximize} & \|X^T A X\|_F \\ \text{subject to} & X^T X = I. \end{array}$$

2. Suppose A is a symmetric $n \times n$ matrix.

   (a) What is the value of t that minimizes $\|A - tI\|_F$?

   (b) What is the value of t that minimizes $\|A - tI\|_2$?

   Express the optimal t in terms of the eigenvalues of A.

3. Suppose A and B are symmetric $n \times n$ matrices. Show that
   $$\lambda_{\max}(A) + \lambda_{\min}(B) \leq \lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B).$$
   Here $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalue of the matrix.

4. Let A be an $m \times n$ matrix with $m \geq n$ and define
   $$B = \left[\begin{array}{cc} 0 & A \\ A^T & 0 \end{array}\right].$$

(a) Suppose $A = U\Sigma V^T$ is a full SVD of A. Verify that

$$B = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & \Sigma \\ \Sigma^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^T.$$

(b) Derive from this an eigendecomposition of B. *Hint:* if Σ_1 is square, then

$$\begin{bmatrix} 0 & \Sigma_1 \\ \Sigma_1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & -\Sigma_1 \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}.$$

(c) What are the $m + n$ eigenvalues of B?