ECE 133B HW2

Lawrence Liu

April 26, 2023

1 Problem 1

We have that

$$A^\dagger = C^T(CC^T)^{-1}(B^TB)^{-1}B^T$$

Therefore we have that

$$\begin{aligned}
AA^\dagger A &= BCC^T(CC^T)^{-1}(B^TB)^{-1}B^TBC \\
&= BIIC \\
&= BC \\
&= A
\end{aligned}$$

And:

$$\begin{aligned}
A^\dagger AA^\dagger &= C^T(CC^T)^{-1}(B^TB)^{-1}B^TBCC^T(CC^T)^{-1}(B^TB)^{-1}B^T \\
&= C^T(CC^T)^{-1}\left((B^TB)^{-1}(B^TB)\right)\left((CC^T)(CC^T)^{-1}\right)(B^TB)^{-1}B^T \\
&= C^T(CC^T)^{-1}(B^TB)^{-1}B^T \\
&= A^\dagger
\end{aligned}$$

And

$$\begin{aligned}
AA^\dagger &= BCC^T(CC^T)^{-1}(B^TB)^{-1}B^T \\
&= B(B^TB)^{-1}B^T \\
(AA^\dagger)^T &= B(B^TB)^{-1}(CC^T)^{-1}CC^TB^T \\
&= B(B^TB)^{-1}B^T
\end{aligned}$$

Therefore we have that $AA^\dagger$ is symmetric. Likewise we have

$$\begin{aligned}
A^\dagger A &= C^T(CC^T)^{-1}(B^TB)^{-1}B^TBC \\
&= C^T(CC^T)^{-1}C \\
(A^\dagger A)^T = C^TB^TB(B^TB)^{-1}(CC^T)^{-1}C \\
&= C^T(CC^T)^{-1}C
\end{aligned}$$

Therefore we have that $A^\dagger A$ is symmetric.

2 Problem 2

(a)

We have

$$\begin{aligned}
A \circ (dd^T) &= \begin{bmatrix} A_{11}d_1d_1 & A_{12}d_1d_2 & \cdots & A_{1n}d_1d_n \\ A_{21}d_2d_1 & A_{22}d_2d_2 & \cdots & A_{2n}d_2d_n \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}d_nd_1 & A_{n2}d_nd_2 & \cdots & A_{nn}d_nd_n \end{bmatrix} \\
&= \mathrm{diag}(d) \begin{bmatrix} A_{11}d_1 & A_{12}d_2 & \cdots & A_{1n}d_n \\ A_{21}d_1 & A_{22}d_2 & \cdots & A_{2n}d_n \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}d_1 & A_{n2}d_2 & \cdots & A_{nn}d_n \end{bmatrix} \\
&= \mathrm{diag}(d)A\mathrm{diag}(d)
\end{aligned}$$

If we have that A is positive semi definite, then we have that

$$A = BB^T$$

Therefore we have that

$$A \circ (dd^T) = \mathrm{diag}(d)BB^T\mathrm{diag}(d)$$

Let $C = \mathrm{diag}(d)B$, then we have that

$$A \circ (dd^T) = CC^T$$

Therefore we have that $A \circ (dd^T)$ is positive semi definite.

(b)

We have that

$$
\begin{aligned}
A \circ B &= \begin{bmatrix} A_{11}B_{11} & A_{12}B_{12} & \cdots & A_{1n}B_{1n} \\ A_{21}B_{21} & A_{22}B_{22} & \cdots & A_{2n}B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B_{n1} & A_{n2}B_{n2} & \cdots & A_{nn}B_{nn} \end{bmatrix} \\
&= \begin{bmatrix} A_{11}D_1D_1^T & A_{12}D_1D_2^T & \cdots & A_{1n}D_1D_n^T \\ A_{21}D_2D_1^T & A_{22}D_2D_2^T & \cdots & A_{2n}D_2D_n^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}D_nD_1^T & A_{n2}D_nD_2^T & \cdots & A_{nn}D_nD_n^T \end{bmatrix} \\
&= \begin{bmatrix} C_1C_1^TD_1D_1^T & C_1C_2^TD_1D_2^T & \cdots & C_1C_n^TD_1D_n^T \\ C_2C_1^TD_2D_1^T & C_2C_2^TD_2D_2^T & \cdots & C_2C_n^TD_2D_n^T \\ \vdots & \vdots & \ddots & \vdots \\ C_nC_1^TD_nD_1^T & C_nC_2^TD_nD_2^T & \cdots & C_nC_n^TD_nD_n^T \end{bmatrix} \\
&= (C \circ D)(C \circ D)^T
\end{aligned}
$$

Where $A = CC^T$ and $B = DD^T$. And C_i and D_i are the ith row of C and D respectively. Therefore we have that $A \circ B$ is positive semi definite.

(c)

We have that

$$
\begin{aligned}
v^T(A \circ B)v = v^T &= \begin{bmatrix} A_{11}B_{11} & A_{12}B_{12} & \cdots & A_{1n}B_{1n} \\ A_{21}B_{21} & A_{22}B_{22} & \cdots & A_{2n}B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B_{n1} & A_{n2}B_{n2} & \cdots & A_{nn}B_{nn} \end{bmatrix} v \\
&= v^T \begin{bmatrix} \sum_{j=1}^n A_{1j}B_{1j}v_j \\ \sum_{j=1}^n A_{2j}B_{2j}v_j \\ \vdots \\ \sum_{j=1}^n A_{nj}B_{nj}v_j \end{bmatrix} \\
&= \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ij}v_i v_j \\
&= \mathrm{tr}(\mathrm{diag}(v) A \mathrm{diag}(v) B) \\
&= \mathrm{tr}(\mathrm{diag}(v)^2 AB)
\end{aligned}
$$

We have that if $v \neq 0$ every element of $\mathrm{diag}(v)^2$ along the diagonal is positive, likewise we have that $\mathrm{tr}(AB) > 0$, therefore we have that $v^T(A \circ B)v > 0$ for all $v \neq 0$ and therefore $A \circ B$ is positive definite.

(d)

We have that

$$X_2 = (X \circ X)$$

Is positive semi definite from part b, therefore from induction if we assume

$$X_n = (X \circ X \circ \cdots \circ X)$$

is positive semi definite, then we have that

$$X_{n+1} = (X_n \circ X)$$

is positive semi definite from part b. Therefore we have that X_n is positive semi definite for all n. Therefore we have that

$$Y = c_0 I + \sum_{i=1}^n c_i X_i$$

is positive semi definite since c_i is nonnegative for all i and X_i is positive semi definite for all i.

Problem 3

(a)

We have that

$$A = UU^T$$

Therefore we have that $Q = U$ and $\Lambda = I_m$.

(b)

We have that

$$A = VIV^T$$

Where V is a matrix whose columns form the span of range$(U)^\perp$ Therefore we have that $Q = V$ and $\Lambda = I_{n-m}$.

(c)

We have that

$$A = [UV] \begin{bmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{bmatrix} [UV]^T$$

Therefore we have that $Q = [UV]$ and $\Lambda = \begin{bmatrix} -I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}$, and thus the eigenvalues are 1 and -1.