ECE 133B HW1

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1 Problem 1

Let A_i denote the ith column of A, and C_i' denote the ith column of C that is not in the null space of A, and B_i' denote the ith column of B that is not in the null space of A. We have that for any $w \in W + v$.

$$w = \sum_{i=1}^{n} A_i x_i + \sum_{i=1}^{m} B_i' y_i + \sum_{i=1}^{p} C_i' z_i$$

Therefore we can see that $[A, B, C]$ is a basis for $W + V$.

Problem 2

(a)

If we want a path to be between any two nodes, we would need at least $n-1$ edges with one connected to each node. With these we could form a path between any two nodes. Therefore there will be $n-1$ linearly independent vectors in the node incidence matrix A, and the other $m-n+1$ vectors will be linear combinations of these. Therefore the rank of A is $n-1$.

(b)

Then the rank would be $\sum_{i=1}^{n'}(n_i - 1)$ where n' is the number of clusters and n_i is the number of nodes in the ith cluster this is because each cluster has at least $n_i - 1$ linearly independent vectors in the node incidence matrix A, and since these cluster's are not connected, their vectors are linearly independent. Therefore the rank of A is $\sum_{i=1}^{n'}(n_i - 1)$.

Problem 3

(a)

Let $AB = C$, then we have that each column of C is a linear combinations of the coloumns of A, ie, the ith column of C, C_i is:

$$C_i = \sum_{j=1}^{n} A_j B_{ij}$$

where A_j is the jth column of A and B_{ij} is the element in the ith row and jth column of B. Thus we have that:

$$\text{rank}(AB) \leq \text{rank}(A)$$

Likewise if we transpose we get that

$$\text{rank}((AB)^T) = \text{rank}(B^T A^T) \leq \text{rank}(B^T)$$

Since row rank is the same as column rank, we have that:

$$\text{rank}(AB) \leq \text{rank}(B)$$

Therefore we have that:

$$\text{rank}(AB) \leq \min\left[\text{rank}(A), \text{rank}(B)\right]$$

(b)

We have that the columns of $A + B$ are linear combinations of the columns of A and B, therefore they are drawn from the combined vector space of the span of A and B. Therefore we have that:

$$\text{rank}(A + B) \le \text{rank}(A) + \text{rank}(B)$$

Problem 4

We have that

$$\begin{aligned} A^2 &= A \\ BCBC &= BC \\ B^\dagger BCBCC^\dagger &= B^\dagger BCC^\dagger \\ IBCI &= I \\ BC &= I \end{aligned}$$

We can write the trace of BC as

$$\text{tr}(BC) = \sum_{i=1}^{n} \sum_{i=1}^{m} B_{ij} C_{ij}$$

if B is a (n x m) matrix and C is a (m x n) matrix. We have that

$$\begin{aligned} \text{tr}(BC) &= \sum_{i=1}^{n} \sum_{i=1}^{m} B_{ij} C_{ij} \\ &= \sum_{i=1}^{m} \sum_{i=1}^{n} C_{ij} B_{ij} \\ &= \text{tr}(CB) \end{aligned}$$

Therefore we have that

$$\text{tr}(BC) = \text{tr}(CB) = \text{tr}(I) = n$$

And thus

$$\text{tr}(A) = \text{tr}(BC) = n$$