

ECE 231A HW 3

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November 20, 2022

Problem 1

(a)

Then we have

$$I(X; Y|Z) = H(Y|Z) - H(Y|X, Z) = H(Y|Z) = \alpha H(p)$$

Thus we have that the capacity is α

(b)

We have

$$\begin{aligned} I(X; Y) &= H(Y) - H(Y|X) \\ &= -p\alpha \log_2(p\alpha) - (1 - p\alpha) \log_2((1 - p\alpha)) - H(Y|X) \\ &= -p\alpha \log_2(p\alpha) - (1 - p\alpha) \log_2((1 - p\alpha)) - pH(\alpha) \end{aligned}$$

To find its maximum, we take its derivative and set it to 0

$$\begin{aligned}
\frac{\partial I(X;Y)}{\partial p} &= 0 \\
-\alpha \log_2(p\alpha) - \frac{\alpha}{\ln(2)} + \alpha \log_2(1 - p\alpha) + \frac{\alpha}{\ln(2)} - H(\alpha) &= 0 \\
-H(\alpha) &= \alpha \log_2(p\alpha) - \alpha \log_2(1 - p\alpha) \\
2^{\frac{-H(\alpha)}{\alpha}} &= \frac{p\alpha}{1 - p\alpha} \\
p &= \boxed{\frac{1}{\alpha(2^{\frac{H(\alpha)}{\alpha}} + 1)}}
\end{aligned}$$

Problem 2

(a)

We have that,

$$\begin{aligned}
I(X;Y|S) &= H(Y|S) - H(Y|X, S) \\
&= H(Y, S) - H(S) - H(Y, X, S) - H(S, X) \\
&= I(X, S; Y, S) - H(S)
\end{aligned}$$

And thus

$$I(X, S; Y, S) = I(X; Y|S) + H(S)$$

(b)

We have that

$$\begin{aligned}
C &= \max_{p_S(s)} \left(\max_{p_X(x|s)} (I(X, S; Y, S)) \right) \\
&= \max_{p_S(s)} \left(\max_{p_X(x|s)} (I(X; Y|S) + H(S)) \right) \\
&= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} (I(X; Y|S)) \right) \\
&= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} (H(Y|S) + H(X|S) - H(Y, X|S)) \right) \\
&= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} \left(\sum_{s=1}^K p_S(s) (H(Y|S=s) + H(X|S=s) - H(Y, X|S=s)) \right) \right) \\
&= \max_{p_S(s)} \left(H(S) + \sum_{s=1}^K p_S(s) C_s \right)
\end{aligned}$$

(c)

We have that we want the following constraint to be true

$$\sum_{s=1}^K p_S(s) = 1$$

So then with a lagrangian we have

$$\frac{\partial}{\partial p_S(s)} \left(H(S) + \sum_{s=1}^K p_S(s) C_s + \lambda \sum_{s=1}^K p_S(s) - 1 \right) = -\log_2(p_S(s)) - \frac{1}{\ln(2)} + C_s + \lambda$$

Setting this equal to 0 we get that

$$p_S * (s) = 2^{-\frac{1}{\log 2(e)}} 2^\lambda 2^{C_s}$$

since $\sum_{s=1}^K p_S * (s) = 1$, we have

$$2^\lambda = \frac{1}{2^{-\frac{1}{\log 2(e)}} \sum_{s=1}^K 2^{C_s}}$$

And thus we have

$$p_S * (s) = \boxed{\frac{2^{C_s}}{\sum_{s=1}^K 2^{C_s}}}$$

(d)

Then we have that

$$\begin{aligned} H(S) &= - \sum_{s=1}^K p_S(s) \log_2(p_S(s)) \\ &= \log_2 \left(\sum_{s=1}^K 2^{C_s} \right) - \sum_{s=1}^K p_S(s) C_s \end{aligned}$$

Thus we have that

$$C = H(S) + \sum_{s=1}^K p_S(s) C_s = \log_2 \left(\sum_{s=1}^K 2^{C_s} \right)$$

Problem 3