ECE 231A HW 3

Lawrence Liu

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Problem 1

(a)

Then we have

$$I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = H(Y|Z) = \alpha H(p)$$

Thus we have that the capacity is α

(b)

We have

$$\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= -p\alpha \log_2(p\alpha) - (1-p\alpha)\log_2((1-p\alpha)) - H(Y|X) \\
&= -p\alpha \log_2(p\alpha) - (1-p\alpha)\log_2((1-p\alpha)) - pH(\alpha)
\end{aligned}$$

To find its maximum, we take its derivative and set it to 0

$$
\begin{aligned}
\frac{\partial I(X;Y)}{\partial p} &= 0 \\
-\alpha \log_2(p\alpha) - \frac{\alpha}{\ln(2)} + \alpha \log_2(1-p\alpha) + \frac{\alpha}{\ln(2)} - H(\alpha) &= 0 \\
-H(\alpha) &= \alpha \log_2(p\alpha) - \alpha \log_2(1-p\alpha) \\
2^{\frac{-H(\alpha)}{\alpha}} &= \frac{p\alpha}{1-p\alpha} \\
p &= \boxed{\frac{1}{\alpha(2^{\frac{H(\alpha)}{\alpha}}+1)}}
\end{aligned}
$$

Problem 2

(a)

We have that,

$$
\begin{aligned}
I(X;Y|S) &= H(Y|S) - H(Y|X,S) \\
&= H(Y,S) - H(S) - H(Y,X,S) - H(S,X) \\
&= I(X,S;Y,S) - H(S)
\end{aligned}
$$

And thus

$$
I(X,S;Y,S) = I(X;Y|S) + H(S)
$$

(b)

We have that

$$\begin{aligned}
C &= \max_{p_S(s)}\left(\max_{p_X(x|s)}\left(I(X,S;Y,S)\right)\right)\\
&= \max_{p_S(s)}\left(\max_{p_X(x|s)}\left(I(X;Y|S)+H(S)\right)\right)\\
&= \max_{p_S(s)}\left(H(S)+\max_{p_X(x|s)}\left(I(X;Y|S)\right)\right)\\
&= \max_{p_S(s)}\left(H(S)+\max_{p_X(x|s)}\left(H(Y|S)+H(X|S)-H(Y,X|S)\right)\right)\\
&= \max_{p_S(s)}\left(H(S)+\max_{p_X(x|s)}\left(\sum_{s=1}^{K}p_S(s)(H(Y|S=s)+H(X|S=s)-H(Y,X|S=s))\right)\right)\\
&= \max_{p_S(s)}\left(H(S)+\sum_{s=1}^{K}p_S(s)C_s\right)
\end{aligned}$$

(c)

We have that we want the following constraint to be true

$$\sum_{s=1}^{K}p_S(s)=1$$

So then with a lagrangian we have

$$\frac{\partial}{\partial p_S(s)}\left(H(S)+\sum_{s=1}^{K}p_S(s)C_s+\lambda\sum_{s=1}^{K}p_S(s)-1\right)=-\log_2(p_S(s))-\frac{1}{\ln(2)}+C_s+\lambda$$

Setting this equal to 0 we get that

$$p_S*(s)=2^{-\frac{1}{\log 2(e)}}2^{\lambda}2^{C_s}$$

since $\sum_{s=1}^{K}p_S*(s)=1$, we have

$$2^{\lambda}=\frac{1}{2^{-\frac{1}{\log 2(e)}}\sum_{s=1}^{K}2^{C_s}}$$

And thus we have

$$p_S * (s) = \boxed{\frac{2^{C_s}}{\sum_{s=1}^{K} 2^{C_s}}}$$

(d)

Then we have that

$$\begin{aligned} H(S) &= -\sum_{s=1}^{K} p_S(s) \log_2(p_S(s)) \\ &= \log_2\left(\sum_{s=1}^{K} 2^{C_s}\right) - \sum_{s=1}^{K} p_S(s) C_s \end{aligned}$$

Thus we have that

$$C = H(S) + \sum_{s=1}^{K} p_S(s) C_s = \log_2\left(\sum_{s=1}^{K} 2^{C_s}\right)$$

Problem 3

(a)

We want the outputs to Y_1 and Y_2 to be totally dependent on the input X. Thus we have that

$$I(S, Y_1, Y_2) = H(S) - H(S|Y_1, Y_2) = H(S)$$

(b)

Since X_1 is a BEC channel, the maximum distribution is just $P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2}$. This will give us that the capacity of this channel is $1 - \alpha$. Thus we have that the minimum capacity of the noiseless channel is α.

(c)

Send one symbol through the erasure channel and one through the noiseless channel, then if the fraction of the symbols through the noiseless is greater than α, we send the next symbol through the noisy channel, and if it is less than α, we send the next symbol through the noiseless channel. This will converge to α with time.

(d)

We have that

$$\begin{aligned} I(X_1, Y_1) &= H(X_1) - H(X_1|Y_1) \\ &= (1-\alpha)H(X_1) \end{aligned}$$

We want to maximize this subject to the constrainIts on $P(X_1 = 1) = p$

$$p \geq 0$$

$$1 - p \leq 1$$

$$E[q(X_1)] = 3p + 2(1-p) = 2 + p \leq Q$$

Then with the first two conditions will be inactive constraints since entropy gets bigger as p gets closer to $\frac{1}{2}$, but the third one will be active. Thus we have that

$$f = (1-\alpha)H(p) + \lambda(2 + p - Q)$$

$$\frac{\partial f}{\partial p} = (1-\alpha)H'(p) + \lambda = 0$$

$$\frac{\partial f}{\partial \lambda} = 2 + p - Q = 0$$

Thus we have that

$$\log_2\left(\frac{p}{1-p}\right) = \frac{\lambda}{1-\alpha}$$

$$p = \frac{1}{1 + 2^{-\frac{\lambda}{1-\alpha}}}$$

plugging this back into the third constraint we get that

$$p = Q - 2$$

$$1 = (Q-2)(1 + 2^{-\frac{\lambda}{1-\alpha}})$$
$$\frac{3-Q}{Q-2} = 2^{-\frac{\lambda}{1-\alpha}}$$
$$\lambda = (1-\alpha)\log_2\left(\frac{Q-2}{3-Q}\right)$$

Thus we get that

$$p = 3 - Q$$

and thus the maximum $I(X_1, Y_1)$ is $(1-\alpha)H_2(3-Q)$, and thus the maximum of $I(S; Y_1)$ is $\boxed{(1-\alpha)H_2(3-Q)}$ as well.

(e)

Then we must have the rate of the other channel must be $1-(1-\alpha)H_2(3-Q)$

Problem 4

(a)

We can intuitively see that the stationary distribution is $\frac{1}{3}$ for all, thus we have that the rate is

$$\mathcal{H}(S) = H_3(q, q, 1-2q)$$

Thus in order for lossless coding, we must have that q must satisfy the following

$$R = 1 - \alpha > \mathcal{H}(S)$$

$$\boxed{1 - \alpha > H_2(2q, 1-2q) + 2q}$$

(b)

From the source coding theorem we have that

$$1 - \alpha > \beta \mathcal{H}(S)$$

Thus the maximum beta is

$$\beta = \boxed{\frac{2(1-\alpha)}{3}}$$

Problem 5

(a)

We have that

$$I(X;Y) = H(X) - H(X|Y) = H(X) - pH(X|Y=E) - (1-p)H(X|Y=X)$$

$$I(X;Y) = H(X) - pH(X|Y=E) = (1-p)H(X)$$

thus to maximize we just have all the values of X be equally likely thus we will have

$$\boxed{C = (1-p)\log_2(L)}$$

(b)

Since feedback cannot increase the capacity of the channel, we have that the capacity still is $\boxed{(1-p)\log_2(L)}$

(c)

Once again since feedback cannot increase the capacity of the channel, we have that the capacity still is $\boxed{(1-p)\log_2(L)}$