ECE 231A HW 3

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Problem 1

(a)

Then we have

$$I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = H(Y|Z) = \alpha H(p)$$

Thus we have that the capacity is α

(b)

We have

$$
\begin{aligned}
I(X;Y) &= H(Y) - H(Y|X) \\
&= -p\alpha \log_2(p\alpha) - (1 - p\alpha) \log_2((1 - p\alpha)) - H(Y|X) \\
&= -p\alpha \log_2(p\alpha) - (1 - p\alpha) \log_2((1 - p\alpha)) - pH(\alpha)
\end{aligned}
$$

To find its maximum, we take its derivative and set it to 0

$$\frac{\partial I(X;Y)}{\partial p} = 0$$

$$-\alpha \log_2(p\alpha) - \frac{\alpha}{\ln(2)} + \alpha \log_2(1 - p\alpha) + \frac{\alpha}{\ln(2)} - H(\alpha) = 0$$

$$-H(\alpha) = \alpha \log_2(p\alpha) - \alpha \log_2(1 - p\alpha)$$

$$2^{\frac{-H(\alpha)}{\alpha}} = \frac{p\alpha}{1 - p\alpha}$$

$$p = \boxed{\frac{1}{\alpha(2^{\frac{H(\alpha)}{\alpha}} + 1)}}$$

Problem 2

(a)

We have that,

$$\begin{aligned} I(X;Y|S) &= H(Y|S) - H(Y|X,S) \\ &= H(Y,S) - H(S) - H(Y,X,S) - H(S,X) \\ &= I(X,S;Y,S) - H(S) \end{aligned}$$

And thus

$$I(X,S;Y,S) = I(X;Y|S) + H(S)$$

(b)

We have that

$$C = \max_{p_S(s)} \left(\max_{p_X(x|s)} \left(I(X, S; Y, S) \right) \right)$$

$$= \max_{p_S(s)} \left(\max_{p_X(x|s)} \left(I(X; Y|S) + H(S) \right) \right)$$

$$= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} \left(I(X; Y|S) \right) \right)$$

$$= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} \left(H(Y|S) + H(X|S) - H(Y, X|S) \right) \right)$$

$$= \max_{p_S(s)} \left(H(S) + \max_{p_X(x|s)} \left(\sum_{s=1}^{K} p_S(s)(H(Y|S=s) + H(X|S=s) - H(Y, X|S=s)) \right) \right)$$

$$= \max_{p_S(s)} \left(H(S) + \sum_{s=1}^{K} p_S(s) C_s \right)$$

(c)

We have that we want the following constraint to be true

$$\sum_{s=1}^{K} p_S(s) = 1$$

So then with a lagrangian we have

$$\frac{\partial}{\partial p_S(s)} \left(H(S) + \sum_{s=1}^{K} p_S(s) C_s + \lambda \sum_{s=1}^{K} p_S(s) - 1 \right) = -\log_2(p_S(s)) - \frac{1}{\ln(2)} + C_s + \lambda$$

Setting this equal to 0 we get that

$$p_S * (s) = 2^{-\frac{1}{\log 2(e)}} 2^\lambda 2^{C_s}$$

since $\sum_{s=1}^{K} p_S * (s) = 1$, we have

$$2^\lambda = \frac{1}{2^{-\frac{1}{\log 2(e)}} \sum_{s=1}^{K} 2^{C_s}}$$

And thus we have

$$p_S * (s) = \boxed{\frac{2^{C_s}}{\sum_{s=1}^{K} 2^{C_s}}}$$

(d)

Then we have that

$$H(S) = -\sum_{s=1}^{K} p_S(s) \log_2(p_S(s))$$

$$= \log_2\left(\sum_{s=1}^{K} 2^{C_s}\right) - \sum_{s=1}^{K} p_S(s) C_s$$

Thus we have that

$$C = H(S) + \sum_{s=1}^{K} p_S(s) C_s = \log_2\left(\sum_{s=1}^{K} 2^{C_s}\right)$$

Problem 3

(a)

We want the outputs to Y_1 and Y_2 to be totally dependent on the input X. Thus we have that

$$I(S, Y_1, Y_2) = H(S) - H(S|Y_1, Y_2) = H(S)$$

(b)

Since X_1 is a BEC channel, the maximum distribution is just $P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2}$. This will give us that the capacity of this channel is $1 - \alpha$. Thus we have that the minimum capacity of the noiseless channel is α.

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(c)

(d)

We have that

$$I(X_1, Y_1) = H(X_1) - H(X_1|Y_1)$$
$$= (1 - \alpha)H(X_1)$$

We want to maximize this subject to the constrainIts on $P(X_1 = 1) = p$

$$p \geq 0$$

$$1 - p \leq 1$$

$$E[q(X_1)] = 3p + 2(1 - p) = 2 + p \leq Q$$

Then with the first two conditions will be inactive constraints since entropy gets bigger as p gets closer to $\frac{1}{2}$, but the third one will be active. Thus we have that

$$f = (1 - \alpha)H(p) + \lambda(2 + p - Q)$$

$$\frac{\partial f}{\partial p} = (1 - \alpha)H'(p) + \lambda = 0$$

$$\frac{\partial f}{\partial \lambda} = 2 + p - Q = 0$$

Thus we have that

$$\log_2\left(\frac{p}{1 - p}\right) = \frac{\lambda}{1 - \alpha}$$

$$p = \frac{1}{1 + 2^{-\frac{\lambda}{1-\alpha}}}$$

plugging this back into the third constraint we get that

$$p = Q - 2$$

$$1 = (Q - 2)(1 + 2^{-\frac{\lambda}{1-\alpha}})$$

$$\frac{3 - Q}{Q - 2} = 2^{-\frac{\lambda}{1-\alpha}}$$

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$$\lambda = (1 - \alpha) \log_2 \left(\frac{Q - 2}{3 - Q} \right)$$

Thus we get that

$$p = 3 - Q$$

and thus the maximum $I(X_1, Y_1)$ is $(1-\alpha)H_2(3-Q)$, and thus the maximum of $I(S; Y_1)$ is $\boxed{(1 - \alpha)H_2(3 - Q)}$ as well.

(e)

Then we must have the rate of the other channel must be $1-(1-\alpha)H_2(3-Q)$

Problem 4

(a)

We can intuitively see that the stationary distribution is $\frac{1}{3}$ for all, thus we have that the rate is

$$\mathcal{H}(S) = H_3(q, q, 1 - 2q)$$

Thus in order for lossless coding, we must have that q must satisfy the following

$$R = 1 - \alpha > \mathcal{H}(S)$$

$$\boxed{1 - \alpha > H_2(2q, 1 - 2q) + 2q}$$

(b)

From the source coding theorem we have that

$$1 - \alpha > \beta \mathcal{H}(S)$$

Thus the maximum beta is

$$\beta = \boxed{\frac{2(1-\alpha)}{3}}$$

Problem 5

(a)

We have that

$$I(X;Y) = H(X) - H(X|Y) = H(X) - pH(X|Y=E) - (1-p)H(X|Y=X)$$

$$I(X;Y) = H(X) - pH(X|Y=E) = (1-p)H(X)$$

thus to maximize we just have all the values of X be equally likely thus we will have

$$\boxed{C = (1-p)\log_2(L)}$$

(b)

Since feedback cannot increase the capacity of the channel, we have that the capacity still is $\boxed{(1-p)\log_2(L)}$

(c)

Once again since feedback cannot increase the capacity of the channel, we have that the capacity still is $\boxed{(1-p)\log_2(L)}$