ECE 231A HW 2

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Problem 1

(a)

No this is not necessqarily true. Consider the following example:

(b)

We have that

$$\begin{aligned}
I(X_1;X_2;X_3) =& I(X_1;X_2) - I(X_1;X_2|X_3) \\
=& H(X_1) + H(X_2) - H(X_1,X_2) - (H(X_1|X_3) - H(X_1|X_2,X_3)) \\
=& H(X_1) + H(X_2) + H(X_3) - H(X_1,X_2) - H(X_1,X_3) \\
& - H(X_2,X_3) + H(X_1,X_2,X_3)
\end{aligned}$$

Since that

$$\begin{aligned}
I(X_1;X_2|X_3) &= H(X_1|X_3) - H(X_1|X_2,X_3) \\
&= H(X_1,X_3) - H(X_3) - H(X_1,X_2,X_3) + H(X_2,X_3)
\end{aligned}$$

$$
\begin{aligned}
I(X_2;X_3|X_1) &= H(X_2|X_1) - H(X_2|X_3,X_1) \\
&= H(X_2,X_1) - H(X_1) - H(X_1,X_2,X_3) + H(X_3,X_1)
\end{aligned}
$$

$$
\begin{aligned}
I(X_1;X_3|X_2) &= H(X_1|X_2) - H(X_1|X_3,X_2) \\
&= H(X_2,X_1) - H(X_2) - H(X_1,X_2,X_3) + H(X_3,X_2)
\end{aligned}
$$

Therefore we have that

$$I(X_1;X_2;X_3) = I(X_1;X_2) - I(X_1;X_2|X_3)$$

$$I(X_1;X_2;X_3) = I(X_2;X_3) - I(X_2;X_3|X_1)$$

$$I(X_1;X_2;X_3) = I(X_1;X_3) - I(X_1;X_3|X_2)$$

Since $I(X_1;X_2) \geq 0$, $I(X_2;X_3) \geq 0$, and $I(X_1;X_3) \geq 0$, we have that

$$I(X_1;X_2;X_3) \geq -I(X_1;X_2|X_3)$$

$$I(X_1;X_2;X_3) \geq -I(X_2;X_3|X_1)$$

$$I(X_1;X_2;X_3) \geq -I(X_1;X_3|X_2)$$

Therefore we have that

$$I(X_1;X_2;X_3) \geq -\min(I(X_1;X_2|X_3), I(X_2;X_3|X_1), I(X_1;X_3|X_2))$$

Problem 2

Once again from

$$I(X_1;X_2;X_3) = I(X_1;X_2) - I(X_1;X_2|X_3)$$

$$I(X_1;X_2;X_3) = I(X_2;X_3) - I(X_2;X_3|X_1)$$

$$I(X_1;X_2;X_3) = I(X_1;X_3) - I(X_1;X_3|X_2)$$

Since $I(X_1;X_2|X_3) \geq 0$, $I(X_2;X_3|X_1) \geq 0$, and $I(X_1;X_3|X_2) \geq 0$, we have that

$$I(X_1;X_2;X_3) \leq I(X_1;X_2)$$

$$I(X_1;X_2;X_3) \leq I(X_2;X_3)$$

$$I(X_1;X_2;X_3) \leq I(X_1;X_3)$$

Therefore we have that

$$I(X_1;X_2;X_3) \leq \min(I(X_1;X_2), I(X_2;X_3), I(X_1;X_3))$$

Problem 3

(a)

Let the increase in Alice's score after the ith round be represented by the random variable Z_A^i we have that

$$Z_A^i = \begin{cases} 4 & \text{w.p. } 1/15 \\ 7 & \text{w.p. } 5/15 \\ 0 & \text{w.p. } 9/15 \end{cases}$$

And the increase in Bob's score after the ith round be represented by the random variable Z_B^i we have that

$$Z_B^i = \begin{cases} 3 & \text{w.p. } 1/15 \\ 5 & \text{w.p. } 2/15 \\ 6 & \text{w.p. } 6/15 \\ 0 & \text{w.p. } 6/15 \end{cases}$$

Then we have that

$$S_A^n = \sum_{i=1}^{n} Z_A^i$$

and

$$S_B^n = \sum_{i=1}^{n} Z_B^i$$

As $n \to \infty$ we have that

$$\lim_{n\to\infty} S_A^n = \lim_{n\to\infty} \sum_{i=1}^{n} Z_A^i = E[Z_A^i] = \boxed{2.6}$$

and

$$\lim_{n\to\infty} S_B^n = \lim_{n\to\infty} \sum_{i=1}^{n} Z_B^i = E[Z_B^i] = \boxed{3.225}$$

(b)

We will need to make $\alpha > 7$ since we need to increase the expected value, therefore the probabilites for Z_A would not change, however instead of being 7 with the probability of 5/15 we would have α with the probability of 5/15. Likewise Z_B would not change. Therefore we would have that our new

$$E[Z_A^i] = \frac{4}{15} + \frac{5}{15}\alpha$$

To make this greater than 3.225 we would have that $\alpha > 8.875$, so the least value of alpha to make Alice's score greater than Bob's after n rounds is $\boxed{\alpha = 9}$ if α has to be an integer.