

# ECE 231A HW 2

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## Problem 1

(a)

No this is not necessarily true. Consider the following example:

(b)

We have that

$$\begin{aligned} I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\ &= H(X_1) + H(X_2) - H(X_1, X_2) - (H(X_1|X_3) - H(X_1|X_2, X_3)) \\ &= H(X_1) + H(X_2) + H(X_3) - H(X_1, X_2) - H(X_1, X_3) \\ &\quad - H(X_2, X_3) + H(X_1, X_2, X_3) \end{aligned}$$

Since that

$$\begin{aligned} I(X_1; X_2|X_3) &= H(X_1|X_3) - H(X_1|X_2, X_3) \\ &= H(X_1, X_3) - H(X_3) - H(X_1, X_2, X_3) + H(X_2, X_3) \end{aligned}$$

$$\begin{aligned}
I(X_2; X_3|X_1) &= H(X_2|X_1) - H(X_2|X_3, X_1) \\
&= H(X_2, X_1) - H(X_1) - H(X_1, X_2, X_3) + H(X_3, X_1)
\end{aligned}$$

$$\begin{aligned}
I(X_1; X_3|X_2) &= H(X_1|X_2) - H(X_1|X_3, X_2) \\
&= H(X_2, X_1) - H(X_2) - H(X_1, X_2, X_3) + H(X_3, X_2)
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\
I(X_1; X_2; X_3) &= I(X_2; X_3) - I(X_2; X_3|X_1) \\
I(X_1; X_2; X_3) &= I(X_1; X_3) - I(X_1; X_3|X_2)
\end{aligned}$$

Since  $I(X_1; X_2) \geq 0$ ,  $I(X_2; X_3) \geq 0$ , and  $I(X_1; X_3) \geq 0$ , we have that

$$\begin{aligned}
I(X_1; X_2; X_3) &\geq -I(X_1; X_2|X_3) \\
I(X_1; X_2; X_3) &\geq -I(X_2; X_3|X_1) \\
I(X_1; X_2; X_3) &\geq -I(X_1; X_3|X_2)
\end{aligned}$$

Therefore we have that

$$I(X_1; X_2; X_3) \geq -\min(I(X_1; X_2|X_3), I(X_2; X_3|X_1), I(X_1; X_3|X_2))$$

## Problem 2

Once again from

$$\begin{aligned}
I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\
I(X_1; X_2; X_3) &= I(X_2; X_3) - I(X_2; X_3|X_1) \\
I(X_1; X_2; X_3) &= I(X_1; X_3) - I(X_1; X_3|X_2)
\end{aligned}$$

Since  $I(X_1; X_2|X_3) \geq 0$ ,  $I(X_2; X_3|X_1) \geq 0$ , and  $I(X_1; X_3|X_2) \geq 0$ , we have that

$$\begin{aligned}
I(X_1; X_2; X_3) &\leq I(X_1; X_2) \\
I(X_1; X_2; X_3) &\leq I(X_2; X_3) \\
I(X_1; X_2; X_3) &\leq I(X_1; X_3)
\end{aligned}$$

Therefore we have that

$$I(X_1; X_2; X_3) \leq \min(I(X_1; X_2), I(X_2; X_3), I(X_1; X_3))$$

## Problem 2

We have that

$$I(X;Y|U) = H(X|U) - H(X|Y,U)$$

since  $X$  and  $U$  are independent we have

$$I(X;Y|U) = H(X) - H(X|Y,U)$$

Since  $I(X;Y,U)$  we get:

$$I(X;Y,U) = I(X;Y|U)$$

## Problem 3

(a)

Let the increase in Alice's score after the  $i$ th round be represented by the random variable  $Z_A^i$  we have that

$$Z_A^i = \begin{cases} 4 & \text{w.p. } 1/15 \\ 7 & \text{w.p. } 5/15 \\ 0 & \text{w.p. } 9/15 \end{cases}$$

And the increase in Bob's score after the  $i$ th round be represented by the random variable  $Z_B^i$  we have that

$$Z_B^i = \begin{cases} 3 & \text{w.p. } 1/15 \\ 5 & \text{w.p. } 2/15 \\ 6 & \text{w.p. } 6/15 \\ 0 & \text{w.p. } 6/15 \end{cases}$$

Then we have that

$$S_A^n = \sum_{i=1}^n Z_A^i$$

and

$$S_B^n = \sum_{i=1}^n Z_B^i$$

As  $n \rightarrow \infty$  we have that

$$\lim_{n \rightarrow \infty} S_A^n = \lim_{n \rightarrow \infty} \sum_{i=1}^n Z_A^i = E[Z_A] = \boxed{2.6}$$

and

$$\lim_{n \rightarrow \infty} S_B^n = \lim_{n \rightarrow \infty} \sum_{i=1}^n Z_B^i = E[Z_B] = \boxed{3.225}$$

(b)

We will need to make  $\alpha > 7$  since we need to increase the expected value, therefore the probabilities for  $Z_A$  would not change, however instead of being 7 with the probability of 5/15 we would have  $\alpha$  with the probability of 5/15. Likewise  $Z_B$  would not change. Therefore we would have that our new

$$E[Z_A] = \frac{4}{15} + \frac{5}{15}\alpha$$

To make this greater than 3.225 we would have that  $\alpha > 8.875$ , so the least value of alpha to make Alice's score greater than Bob's after n rounds is  $\boxed{\alpha = 9}$  if  $\alpha$  has to be an integer.