Homework Set #4

Due 22nd November 2022, before 11:59pm.

Submit your solutions to Gradescope with Entry Code:**57DN5B**

Problem 1 (Binary multiplier channel)

Consider the channel $Y = XZ$, where X and Z are independent binary random variables that take on values 0 and 1. The random variable Z is Bernoulli(α) [*i.e.*, $\mathbb{P}[Z = 1] = \alpha$].

(a) Suppose that the receiver can observe Z as well as Y. What is the capacity of this channel in this case?
Hint: Have you seen this channel before?

(b) Now assume that the receiver has only access to Y. Find the capacity of the channel in this case and find the maximizing distribution on X. Assuming $\mathbb{P}[X = 1] = p$ you should find the value of p such that maximizes the mutual information $I(X; Y)$.
Hint:

1. *Expand the mutual information as $I(X; Y) = H(Y) - H(Y|X)$.*
2. *Find the probability $\mathbb{P}[Y = 1]$ and write the entropy of Y.*
3. *Then you have to find an expression for $H(Y|X)$. To this end, you may use the following expansion*

$$H(Y|X) = H(Y|X = 0)\mathbb{P}[X = 0] + H(Y|X = 1)\mathbb{P}[X = 1].$$

4. *Now you have an expression for the mutual information with respect to the parameters α and p. You have to maximize this expression with respect to p.*

Problem 2 (Channels with state)

Consider K discrete memoryless channels $P_1, \ldots, P_K$ with capacities $C_1, \ldots, C_K$ respectively. Consider a communication system that has access to all these K channels but is restricted to use exactly one channel at each time instant. Note that the input to this system can be viewed as (X, S), where S denotes the channel used for communication and X is the input to this selected channel. Similarly the output of this system is (Y, S) where S is again the channel used and Y is the output received on this channel.

(a) Show that,

$$I(X, S; Y, S) = H(S) + I(X; Y|S).$$

(b) Conclude that the capacity of the above communication system is equal to,

$$C = \max_{p_S(s)} \left(H(S) + \sum_{s=1}^{K} p_S(s) C_s \right) \tag{1}$$

(c) Observe that in (1), $H(S)$ is concave in $p_S(s)$ and the second term is linear. Using Kuhn-Tucker conditions show that in the maximizing distribution $p_S(s)$ is proportional to 2^{C_s}. Give an expression for the maximizing distribution $p^*_S(s)$.

(d) Conclude that,

$$C = \log\left(2^{C_1} + \ldots + 2^{C_K}\right)$$

Problem 3 (Capacity with costs and physical degradation)

Alice observes a binary random variable $S \in \{0,1\}$, where $S \sim Bernoulli(\frac{1}{2})$. She wants to send S perfectly to her friend Bob. Alice is connected with Bob through two channels (See Figure 1):

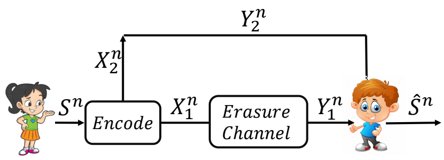

Figure 1: Degraded Channel

Figure 2: Problem 3: Memoryless Channel with costs and physical degradation

- The first channel (Y_1) is an erasure channel with erasure probability α. See Figure 3. However, Alice pays a cost $q(0) = 2$ when $X_1 = 0$ and $q(1) = 3$ when $X_1 = 1$. Alice can only afford an expected cost at most Q, i.e., $\mathbb{E}[q(X_1)] \leq Q$ through the erasure channel.

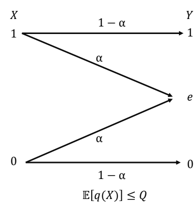

Figure 3: Erasure Channel

- The second channel (Y_2) is a noiseless channel, with input X_2 and output $Y_2 = X_2$.

Alice encodes her input S into two parts X_1 and X_2, where X_1 is transmitted through the erasure channel and X_2 is transmitted through the noiseless channel. Suppose that Alice wants to send most of the information through the erasure channel to minimize the transmission rate through the noiseless channel.
Note that Alice needs to encode n-length sequences in order to make everything work out, *i.e.,* it uses $S^n = (S_1, \ldots, S_n)$ i.i.d symbols from the source and encodes them to X_1^n, X_2^n sequences through the erasure and noiseless channels, respectively.

(a) Show that we need $I(S; Y_1, Y_2) = H(S) = 1$ if we need $\hat{S}^n$ to be perfect representation of S^n.

(b) Neglect the cost constraint $\mathbb{E}[q(X_1)] \leq Q$ for this part. What is the distribution on X_1 to achieve the maximum rate on the first channel (Y_1)? Find a non-trivial lower bound on the rate R of the second (noiseless) channel, such that Bob can exactly recover S from observing Y_1, Y_2, i.e., $I(S; Y_1, Y_2) = H(S) = 1$, *i.e.,* find the minimum R to achieve this.

(c) Suggest a simple scheme to achieve the maximum rate in erasure channel as well as the minimum rate in the noiseless channel obtained in part (b).
Hint: Start with the case α is a rational number.

(d) Find the maximum information that Bob can know about S as a function of Q from observing Y_1 under the cost constraint $\mathbb{E}[q(X_1)] \leq Q$. What is the distribution on X_1 to get the maximum rate?
Hint: Note that $S \leftrightarrow X_1 \leftrightarrow Y_1$ and hence $I(S; Y_1) \leq I(X_1; Y_1)$.

(e) Suppose that Alice sends the maximum rate on the first channel Y_1 given in part (d). Find a non-trivial lower bound on the rate of the second channel $(I(S; Y_2))$ as a function of Q such that Bob can exactly recover S from observing Y_1, Y_2, i.e., $I(S; Y_1, Y_2) = H(S) = 1$.

Problem 4 (Markov Source-channel coding)

Figure 4: Setup for transmitting source S, which is a stationary Markov source, through as BEC with parameter α. Note that $\beta = \frac{L}{n}$, which is the ratio of the number of source symbols to the channel symbols.

In the Fig. 4, consider a stationary Markov source $S \in \mathcal{S} = \{0, 1, 2\}$ with the transition probabilities:

$$\Pi = \begin{bmatrix} 1-2q & q & q \\ q & 1-2q & q \\ q & q & 1-2q \end{bmatrix},$$

where $0 < q < \frac{1}{2}$. We want to send this source over a Binary Erasure Channel with erasure probability of α, *i.e.*,

$$Y = \begin{cases} X, & \text{with probability } 1-\alpha \\ E, & \text{with probability } \alpha \end{cases}$$

(a) If we want to reconstruct the source losslessly, find the condition on q as a function of α when $\beta = 1$, *i.e.*, there is no bandwidth mismatch, where β is the number of source symbols per channel symbol.
Hint: You do not need to compute the stationary distribution for this problem. Also, leave your solution in form of $H_3(.)$ which is the entropy of a ternary random variable in terms of its distribution, or if possible simplify it in terms of $H_2(\cdot)$.

(b) Now, consider a case of bandwidth mismatch, *i.e.*, $\beta \neq 1$. Find the maximum value of β, when $q = \frac{1}{4}$ such that we can reconstruct the source losslessly.
Hint: Recall source-channel coding theorem with bandwidth mismatch.

Problem 5 (Automatic repeat request)

Suppose we have a packet erasure channel, where the input has alphabet $\mathcal{X}$, where $|\mathcal{X}| = L$, and the output is either received perfectly or is erased. That is, if we send X_i, we receive:

$$Y_i = \begin{cases} X_i & \text{with probability } 1-p \\ E & \text{with probability } p \end{cases}$$

where E indicates that the packet was dropped (erased).

(a) Find the capacity of this packet erasure channel.

(b) Suppose we are allowed causal feedback in the form of ACK/NACK, *i.e.,* the receiver informs the transmitter if the packet was received or erased. What is the capacity of such a channel?

(c) Suppose we use a simple feedback scheme, where the transmitter repeats any packet that is dropped until it is successfully received. What is the rate achieved with this scheme?