

ECE 231A HW 3

Lawrence Liu

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Problem 1

(a)

$$\begin{aligned}\lim_{n \rightarrow \infty} [p(X_1, \dots, X_n)]^{\frac{1}{n}} &= 2^{\lim_{n \rightarrow \infty} \frac{1}{n} \log_2[p(X_1, \dots, X_n)]} \\ &= 2^{\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log_2[p(X_i)]} \\ &= 2^{E[\ln[p(X_i)]]} \\ &= \boxed{2^{-H(x)}}$$

(b)

$$\begin{aligned} E \left[\left(\prod_{i=1}^n f(X_i) \right)^{\frac{1}{n}} \right] &= \left(\left(E \left[\left(\prod_{i=1}^n f(X_i) \right)^{\frac{1}{n}} \right] \right)^n \right)^{\frac{1}{n}} \\ &\leq \left(E \left[\prod_{i=1}^n f(X_i) \right] \right)^{\frac{1}{n}} \\ &= (E^n[f(X_1)])^{\frac{1}{n}} \\ &= E[f(X_1)] \end{aligned}$$

Therefore we have that

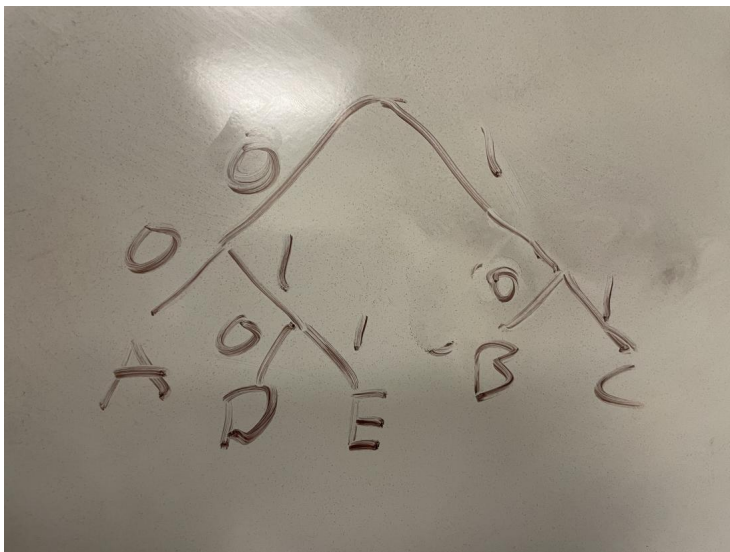
$$E \left[\left(\prod_{i=1}^n f(X_i) \right)^{\frac{1}{n}} \right] \leq E[f(X_i)]$$

Problem 2

(a)

$$H(X) = \boxed{2.246 \text{ Shannons}}$$

(b)



So we have that the average length is $\boxed{2.3}$ bits.

(c)

codeword A is 001

codeword B is 0110

codeword C is 1001

codeword D is 1100

codeword E is 11110

Therefore the SFE codeword average length is $\boxed{3.8}$ bits.

(d)

We have that the CDF of BAC is

$$P(X_1 = A) + P(X_1 = B, X_2 = A)(P(X_3 = A) + P(X_3 = B) + P(X_3 = C)) = \boxed{0.342}$$

Therefore for the SFE code, we have that

$$\bar{F} = P(X_1 = A) + P(X_1 = B, X_2 = A)(P(X_3 = A) + P(X_3 = B) + P(X_3 = C)) - \frac{1}{2}P(X_1 = B, X_2 = A)$$

and

$$l = -\lceil \log_2(P(X_1 = B, X_2 = A, X_3 = c)) \rceil + 1 = 8$$

Thus we have that the SFE encoding is 01010110

Problem 3

(a)

$$F(X^1) = \begin{cases} 0.2 & \text{if } X^1 = A \\ 0.5 & \text{if } X^1 = B \\ 1 & \text{if } X^1 = C \end{cases}$$

The interval for the first symbol is $[0, 0.2)$.

(b)

$$F(X^1 X^2) = \begin{cases} 0.04 & \text{if } X^1 X^2 = AA \\ 0.1 & \text{if } X^1 X^2 = AB \\ 0.2 & \text{if } X^1 X^2 = AC \\ 0.26 & \text{if } X^1 X^2 = BA \\ 0.35 & \text{if } X^1 X^2 = BB \\ 0.5 & \text{if } X^1 X^2 = BC \\ 0.6 & \text{if } X^1 X^2 = CA \\ 0.75 & \text{if } X^1 X^2 = CB \\ 1 & \text{if } X^1 X^2 = CC \end{cases}$$

Therefore the interval corresponding to AC is $[0.1, 0.2)$

(c)

For the interval $[0, 0.2)$, we have that the midpoint is 0.1 and the length is $-\lceil \log_2(0.2) \rceil + 1 = 4$ bits. Therefore the code word is 0001

For the interval $[0.1, 0.2)$, we have that the midpoint is 0.15 and the length is $-\lceil \log_2(0.01) \rceil + 1 = 8$ bits, so the code word is 00100

(d)

Therefore the cdf for

$$F(X^1X^2X^3X^4 = ACCB) = 0.1 + 0.1 * 0.6 = 0.175$$

And thus we have that interval for the ACCB is $[0.15, 0.16)$ so the midpoint is

$$\bar{F} = \frac{0.16 + 0.175}{2} = 0.1675$$

And since $p = 0.2 \cdot 0.5^2 \cdot 0.3 = 0.015$, we have that

$$l = -\lceil \log_2(0.015) \rceil + 1 = 8$$

Thus we have that the SFE code for ACCB is 00101010

(e)

The binary encoding for 0.16 is 0.001010001111010111 and the binary encoding for 0.1675 is 0.001010101110000101 and the binary encoding for 0.175 is 0.00101100110011001101. Therefore 5 bits can be know if we do not know how ACCB continues.

Problem 4

ABBBACBBBCAAABB