ECE 231 Project 2

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December 11, 2022

1 Problem 1 (Polarization)

1.1 Part (a)

$I(U_1, U_2; Y_1, Y_2) =$
$I(X_1, X_2; Y_1, Y_2) =$ (U_1, U_2 can be determined from X_1, X_2 and vice versa).
$H(X_1, X_2) - H(X_1, X_2|Y_1, Y_2) =$

$H(X_1) - H(X_1|Y_1, Y_2) + H(X_2) - H(X_2|Y_1, Y_2) =$ (X_1, X_2 are indepedendent and X_1, X_2 are also conditionally independent given Y_1, Y_2)

$H(X_1) - H(X_1|Y_1) + H(X_2) - H(X_2|Y_2) =$ (X_1 is conditionally independent from Y_2 given Y_1, and X_2 is conditionally independent from Y_1 given Y_2).

$I(X_1; Y_1) + I(X_2; Y_2)$, as desired.

We also have:
$I(U_1, U_2; Y_1, Y_2) =$
$H(U_1, U_2) - H(U_1, U_2|Y_1, Y_2) =$
$H(U_1) + H(U_2) - H(U_1, U_2|Y_1, Y_2) =$ (U_1, U_2 are independent)
$H(U_1) + H(U_2) - H(U_1|Y_1, Y_2) - H(U_2|Y_1, Y_2, U_1) =$
$H(U_1) - H(U_1|Y_1, Y_2) + H(U_2) - H(U_2|Y_1, Y_2, U_1) =$
$I(U_1; Y_1, Y_2) + I(U_2; Y_1, Y_2, U_1)$, as desired.

1.2 Part (b)

Because X_1, X_2 go through identitical channels with identical capacities to become Y_1, Y_2, we must have $I(X_1; Y_1) = I(X_2; Y_2)$
To prove the right hand inequality:
$I(X_2; Y_2) =$
$H(X_2) - H(X_2|Y_2) \leq$
$H(X_2) - H(X_2|Y_1, Y_2, U_1) =$
$I(X_2; Y_1, Y_2, U_1) =$
$I(U_2; Y_1, Y_2, U_1)$, as desired.
To prove the left hand inequality:
$I(U_1; Y_1, Y_2) =$
$I(U_1; Y_1, Y_2) =$
$I(U_1; Y_1|Y_2) + I(U_1; Y_2)$

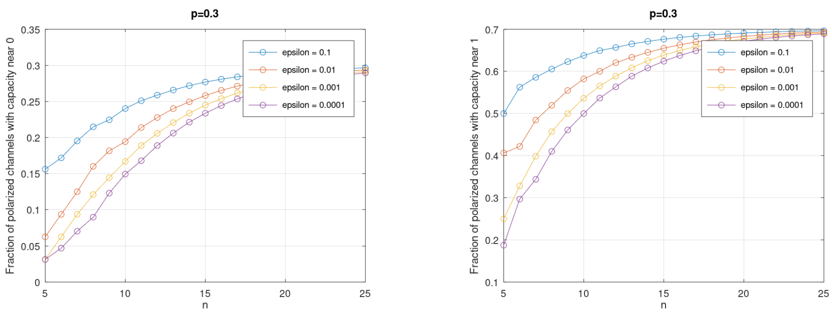
The second term is 0 because Y_2 is independent from U_1.
We have:
$I(U_1; Y_1|Y_2) =$
$I(X_1 \oplus X_2; Y_1|Y_2) \leq$
$I(X_1, X_2; Y_1|Y_2) =$ (as X_1, X_2 uniquely determines $X_1 \oplus X_2$)
$I(X_1; Y_1|Y_2) =$ (X_2 is independent from X_1, Y_1 even given Y_2)
$I(X_1; Y_1)$ (Y_2 is independent from X_1, X_2)
This is what we wanted to prove.

2 Problem 2 (Polarization of BEC)

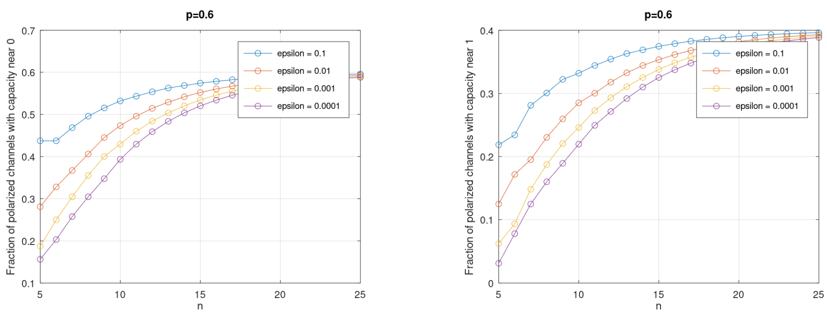
Using the fact that $BEC(p)$ has capacity $1-p$, we have:
The capacity of W^- is $1 - 2p + p^2 = (1-p)^2$
The capacity of W is $1-p$
The capacity of W^+ is $1-p^2$
It is clear that $(1-p)^2 \leq 1-p$ as $0 \leq 1-p \leq 1$.
It is also clear that $1-p \leq 1-p^2$ as $0 \leq p \leq 1$.
Therefore channel W^- is worse than W, and W^+ is better than W.

Problem 3 (Coding Problem: Polarization of BEC)

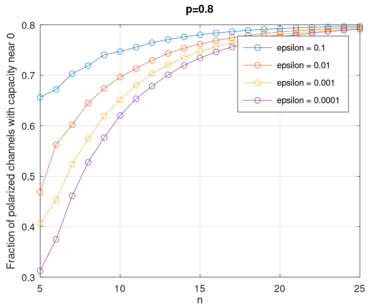
We get the following plots for $p = 0.3$

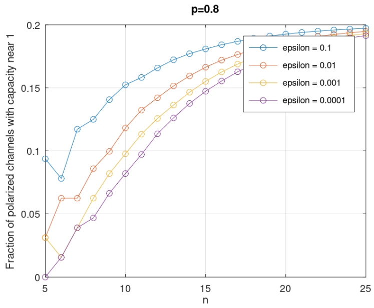


And the following plots for $p = 0.6$



And the following plots for $p = 0.8$





This was generated with the following code:

```
%% Code snippets for ECE 231A: Information Theory: Porject Module #3
% Problem 3 code snippets
clear all;
clc;
close all;

p = [0.3,0.6,0.8]; % Different erasure probabilities for BEC
epsilon = [0.1,0.01,0.001,0.0001];
n = 5:25; % log2 block length

% output for capacity near 0
% rows: corresponding to different n and
% columns: corresponding to different epsilon
output0 = zeros(length(n),length(epsilon));
% output for capacity
output1 = zeros(length(n),length(epsilon));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%% Enter your code here %%%%%%
for i=1:length(p)
  channel_ps=[p(i)^2,2*p(i)-p(i)^2];
  for n_=2:n(length(n))
    % disp(n_)
    new_channel_ps=zeros(1,2*length(channel_ps));
    % disp(size(new_channel_ps))
    for j=1:length(channel_ps)
      new_channel_ps(2*j-1)=channel_ps(j)^2;
      new_channel_ps(2*j)=2*channel_ps(j)-channel_ps(j)^2;
    end
    channel_capacities=ones(1,length(new_channel_ps))-new_channel_ps;
    if any(n==n_)
      for j=1:length(epsilon)
        % size(channel_capacities)
        % sum(channel_capacities<epsilon(j))
```

```
          % channel_capacities<epsilon(j)
          output0(n_-4,j)=sum(channel_capacities<=epsilon(j))/length(channel_cap
        end
        for j=1:length(epsilon)
          output1(n_-4,j)=sum(channel_capacities>=1-epsilon(j))/length(channel_ca
        end
      end
      channel_ps=new_channel_ps;
      % disp("————————————————————————————————")
    end

    %I move the plotting inside of the loop because its better that way, so then
    %you can see the plots for each p
    f1 = figure;
    for j=1:length(epsilon)
        plot(n,output1(:,j),'-o','DisplayName',['epsilon = ' num2str(epsilon(j))]
        hold on;
    end
    grid on;
    legend;
    title(['p=' num2str(p(i))]    );
    xlabel('n');
    ylabel('Fraction of polarized channels with capacity near 1');
    saveas(f1,['p=' num2str(p(i))  ' near 1.png']);

    f2 = figure;
    for j=1:length(epsilon)
        plot(n,output0(:,j),'-o','DisplayName',['epsilon = ' num2str(epsilon(j))]
        hold on;
    end
    grid on;
    legend;
    xlabel('n');
    title(['p=' num2str(p(i))]    );
    ylabel('Fraction of polarized channels with capacity near 0');
    saveas(f2,['p=' num2str(p(i))  ' near 0.png']);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

f1 = figure;
for i=1:length(epsilon)
    plot(n,output1(:,i),'-o','DisplayName',['epsilon = ' num2str(epsilon(i))]);
    hold on;
end
grid on;
legend;
xlabel('n');
ylabel('Fraction of polarized channels with capacity near 1');

f2 = figure;
```

```
for i=1:length(epsilon)
    plot(n,output0(:,i),'-o','DisplayName',['epsilon = ' num2str(epsilon(i))]);
    hold on;
end
grid on;
legend;
xlabel('n');
ylabel('Fraction of polarized channels with capacity near 0');
```

Note that because the plotting functions provided would only plot for 1 value of p, I moved them inside the for loop. However because we were instructed to not touch the code beyond the parts we could touch, I left the original plotting functions in the code.