ECE 231A HW 1

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Problem 3

(a)

The entropy of $H(X)$ is

$$H(X) = -\sum_{x\in X} p(x)\log\left(p(x)\right)$$

and we have that

$$\begin{aligned}
H(X|Y) &= \sum_{x\in X, y\in Y} p(y)H(X|Y=y) \\
&= \left(\sum_{x\in S} p(x)\right) H(X|Y=1) + \left(\sum_{x\notin S} p(x)\right) H(X|Y=0)
\end{aligned}$$

We have

$$\begin{aligned}
H(X|Y=1) &= -\sum_{x\in S} p(x|Y=1)\log(p(x|Y=1)) \\
&= -\sum_{x\in S} \frac{p(x)}{\sum_{x\in S} p(x)} \log\left(\frac{p(x)}{\sum_{x\in S} p(x)}\right)
\end{aligned}$$

likewise we have

$$
\begin{aligned}
H(X|Y=0) &= -\sum_{x\notin S} p(x|Y=0)\log(p(x|Y=0)) \\
&= -\sum_{x\notin S} \frac{p(x)}{\sum_{x\notin S} p(x)} \log\left(\frac{p(x)}{\sum_{x\notin S} p(x)}\right)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
H(X|Y) &= -\sum_{x\in S} p(x)\left(\log(p(x)) - \log\left(\sum_{x\in S} p(x)\right)\right) - \sum_{x\notin S} p(x)\left(\log(p(x)) - \log\left(\sum_{x\notin S} p(x)\right)\right) \\
&= H(X) + \sum_{x\in S} p(x)\log\left(\sum_{x\in S} p(x)\right) + \sum_{x\notin S} p(x)\log\left(\sum_{x\notin S} p(x)\right)
\end{aligned}
$$

Therefore we have

$$
H(X) - H(X|Y) = \boxed{-\sum_{x\in S} p(x)\log\left(\sum_{x\in S} p(x)\right) - \sum_{x\notin S} p(x)\log\left(\sum_{x\notin S} p(x)\right)}
$$

(b)

$H(X) - H(X|Y)$ is maximized when $\sum_{x\in S} p(x) = \sum_{x\notin S} p(x) = \frac{1}{2}$, this is possible when $S = \boxed{2, 5}$ or $S = \boxed{1, 2, 4}$

Problem 6

(a)

$$
\begin{aligned}
H(x) &= -\sum_{x \in X} p(x)\log(p(x)) \\
&= p(A)\log(p(A)) + p(B)\log(p(B)) + p(C)\log(p(C)) \\
&+ p(D)\log(p(D)) + p(E)\log(p(E)) + p(F)\log(p(F)) \\
&= \frac{1}{2}\log(4) + \frac{1}{4}\log(8) + \frac{3}{16}\log(\frac{16}{3}) + \frac{1}{16}\log(16) \\
&= \boxed{2.2452 \text{ shannons}}
\end{aligned}
$$