ECE 231A HW 2

Lawrence Liu

October 22, 2022

Problem 1

(a)

No this is not necessqarily true. Consider the following example:

(b)

We have that

$$\begin{aligned}
I(X_1;X_2;X_3) =& I(X_1;X_2) - I(X_1;X_2|X_3) \\
=& H(X_1) + H(X_2) - H(X_1,X_2) - (H(X_1|X_3) - H(X_1|X_2,X_3)) \\
=& H(X_1) + H(X_2) + H(X_3) - H(X_1,X_2) - H(X_1,X_3) \\
& - H(X_2,X_3) + H(X_1,X_2,X_3)
\end{aligned}$$

Since that

$$\begin{aligned}
I(X_1;X_2|X_3) &= H(X_1|X_3) - H(X_1|X_2,X_3) \\
&= H(X_1,X_3) - H(X_3) - H(X_1,X_2,X_3) + H(X_2,X_3)
\end{aligned}$$

$$\begin{aligned} I(X_2; X_3|X_1) &= H(X_2|X_1) - H(X_2|X_3, X_1) \\ &= H(X_2, X_1) - H(X_1) - H(X_1, X_2, X_3) + H(X_3, X_1) \end{aligned}$$

$$\begin{aligned} I(X_1; X_3|X_2) &= H(X_1|X_2) - H(X_1|X_3, X_2) \\ &= H(X_2, X_1) - H(X_2) - H(X_1, X_2, X_3) + H(X_3, X_2) \end{aligned}$$

Therefore we have that

$$I(X_1; X_2; X_3) = I(X_1; X_2) - I(X_1; X_2|X_3)$$
$$I(X_1; X_2; X_3) = I(X_2; X_3) - I(X_2; X_3|X_1)$$
$$I(X_1; X_2; X_3) = I(X_1; X_3) - I(X_1; X_3|X_2)$$

Since $I(X_1; X_2) \geq 0$, $I(X_2; X_3) \geq 0$, and $I(X_1; X_3) \geq 0$, we have that

$$I(X_1; X_2; X_3) \geq -I(X_1; X_2|X_3)$$
$$I(X_1; X_2; X_3) \geq -I(X_2; X_3|X_1)$$
$$I(X_1; X_2; X_3) \geq -I(X_1; X_3|X_2)$$

Therefore we have that

$$I(X_1; X_2; X_3) \geq -\min(I(X_1; X_2|X_3), I(X_2; X_3|X_1), I(X_1; X_3|X_2))$$

(c)

Once again from

$$I(X_1; X_2; X_3) = I(X_1; X_2) - I(X_1; X_2|X_3)$$
$$I(X_1; X_2; X_3) = I(X_2; X_3) - I(X_2; X_3|X_1)$$
$$I(X_1; X_2; X_3) = I(X_1; X_3) - I(X_1; X_3|X_2)$$

Since $I(X_1; X_2|X_3) \geq 0$, $I(X_2; X_3|X_1) \geq 0$, and $I(X_1; X_3|X_2) \geq 0$, we have that

$$I(X_1; X_2; X_3) \leq I(X_1; X_2)$$
$$I(X_1; X_2; X_3) \leq I(X_2; X_3)$$
$$I(X_1; X_2; X_3) \leq I(X_1; X_3)$$

Therefore we have that

$$I(X_1; X_2; X_3) \leq \min(I(X_1; X_2), I(X_2; X_3), I(X_1; X_3))$$

Problem 2

We have that

$$I(X;Y|U) = H(X|U) - H(X|Y,U)$$

since X and U are indepndent we have

$$I(X;Y|U) = H(X) - H(X|Y,U)$$

Since $I(X;Y,U)$ we get:

$$I(X;Y,U) = I(X;Y|U)$$

Problem 3

(a)

Let the increase in Alice's score after the ith round be represented by the random variable Z_A^i we have that

$$Z_A^i = \begin{cases} 4 & \text{w.p. } 1/15 \\ 7 & \text{w.p. } 5/15 \\ 0 & \text{w.p. } 9/15 \end{cases}$$

And the increase in Bob's score after the ith round be represented by the random variable Z_B^i we have that

$$Z_B^i = \begin{cases} 3 & \text{w.p. } 1/15 \\ 5 & \text{w.p. } 2/15 \\ 6 & \text{w.p. } 6/15 \\ 0 & \text{w.p. } 6/15 \end{cases}$$

Then we have that

$$S_A^n = \sum_{i=1}^{n} Z_A^i$$

and

$$S_B^n = \sum_{i=1}^{n} Z_B^i$$

As $n \to \infty$ we have that

$$\lim_{n\to\infty} S_A^n = \lim_{n\to\infty} \sum_{i=1}^{n} Z_A^i = E[Z_A^i] = \boxed{2.6}$$

and

$$\lim_{n\to\infty} S_B^n = \lim_{n\to\infty} \sum_{i=1}^{n} Z_B^i = E[Z_B^i] = \boxed{3.2666}$$

(b)

We will need to make $\alpha > 7$ since we need to increase the expected value, therefore the probabilites for Z_A would not change, however instead of being 7 with the probability of 5/15 we would have α with the probability of 5/15. Likewise Z_B would not change. Therefore we would have that our new

$$E[Z_A^i] = \frac{4}{15} + \frac{5}{15}\alpha$$

To make this greater than 3.2666 we would have that $\boxed{\alpha > 9}$,

Problem 4

(a)

We have that $H(\hat{p}(Y_m)|Y_m) = 0$ and $H(\hat{p}(Y_m)|Y_m, p) = 0$, therefore we have that $I(\hat{p}(Y_m); p|Y_m) = 0$

$$\begin{aligned} I(\hat{p}(Y_m); p|Y_m) + I(\hat{p}(Y_m); p) = I(\hat{p}(Y_m); p, Y_m) &= I(p; Y_m|\hat{p}(Y_m)) + I(p; Y_m) \\ I(\hat{p}(Y_m); p) &= I(p; Y_m|\hat{p}(Y_m)) + I(p; Y_m) \\ I(\hat{p}(Y_m); p) &\leq I(p; Y_m) \end{aligned}$$

Problem 5

(a)

In order for $\mu = [\mu_1, \mu_2]^T$ to be a stationary distribution for a Markov chain we must have $\mu^T \Pi = \mu^T$, therefore we have the following two series of equations

$$\frac{1}{4}\mu_1 + \frac{2}{3}\mu_2 = \mu_1$$
$$\frac{3}{4}\mu_1 + \frac{1}{3}\mu_2 = \mu_2$$

Solving these we get, and applying the condition that $\mu_1 + \mu_2 = 1$, we get $\mu_1 = \frac{\frac{2}{3}}{\frac{3}{4}+\frac{2}{3}} = \boxed{0.470}$ and $\mu_2 = \frac{\frac{3}{4}}{\frac{3}{4}+\frac{2}{3}} = \boxed{0.529}$ Therefore the entropy of this $H(X) = -0.470\log_2(0.470) - 0.529\log_2(0.529) = \boxed{0.9975 \text{ shannons}}$

(b)

We have that

$$\begin{aligned}
\mathcal{H}(x) &= -\sum_i \mu_i \sum_j P_{ij}\log_2(P_{ij}) \\
&= -0.470\left(\frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{3}{4}\log_2\left(\frac{3}{4}\right)\right) - 0.529\left(\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right) \\
&= \boxed{0.8679345589507923}
\end{aligned}$$

This is less than $H(X)$ since $\mathcal{H}(x) = \lim_{n\to\infty} H(X_n|X_{n-1},\ldots,X_1)$, since conditioning reduces entropy, thus we have that $H(X_n|X_{n-1},\ldots,X_1) < H(X)$.

(c)

Since we can determine X_n given $Y_n, ..., Y_1$ and since Y_n is a deterministic function we have

$$H(Y_n|Y_{n-1}, \ldots, Y_1) = H(X_n|X_{n-1}, ..., X_1)$$

And therefore we have that $\mathcal{H}(Y) = \mathcal{H}(X) = \boxed{0.8679345589507923}$

(d)

We have that

$$H(Y_n, X_n, X_{n-1}) = H(Y_n|X_n, X_{n-1}) + H(X_n, X_{n-1}) = H(Y_n) + H(X_n, X_{n-1}|Y_n)$$

Therefore we get

$$H(X_n, X_{n-1}) = H(Y_n) + H(X_n, X_{n-1}|Y_n)$$

Since XOR if given one of the values, we can imediately determine the other value we have

$$H(X_n, X_{n-1}) = H(Y_n) + H(X_{n-1})$$
$$H(X_n|X_{n-1}) = H(Y_n)H(Y_n) = \boxed{0.8679345589507923}$$