ECE 231A Project 3 Part 2

November 14, 2022

Problem 1

We can rewrite $P(l(x) < l'(x)) > P(l(x) > l'(x))$ as

$$\sum_x p(x)sign(l(x) - l'(x)) \leq 0$$

using the identity $sign(a - b) \leq D^t - 1$, we get

$$\begin{aligned}
\sum_x p(x)sign(l(x) - l'(x)) &\leq \sum_x p(x)\left(D^{l(x)-l'(x)} + 1\right) \\
&= \sum_x D^{l'(x)} - 1
\end{aligned}$$

From the kraft inequality, we then get that $\sum_x D^{l'(x)} \leq 1$. and thus we get that $\sum_x D^{l'(x)} - 1 \leq 0$.

Problem 2

(a)

let $l'(x) = -\log(q(x))$, then we have

$$\begin{aligned}
p(l'(x) \leq l^*(x) - \gamma) &= p(-\log_D(q(x)) \leq -\log_D(p(x)) - \gamma) \\
&= p(q(x) \geq p(x)D^{\gamma}) \\
&= \sum_{x:p(x)D^{\gamma} \leq q(x)} p(x) \\
&\leq \sum_{x:p(x)D^{\gamma} \leq q(x)} q(x)D^{-\gamma} \\
&= \sum_{x} q(x)D^{-\gamma} \\
&= D^{-\gamma}
\end{aligned}$$

Thus we get

$$p(l'(x) \leq l^*(x) - \gamma) \leq |D|^{-\gamma}$$

(b)

We have

$$\begin{aligned}
p(l(x) > l'(x) + 1) &= p\left(\lceil \log_D\left(\frac{1}{p(x)}\right)\rceil > l'(x) + 1\right) \\
&\leq p(1 - log_D(p(x)) > l'(x) + 1) \\
&=\leq p(-\log_D(p(x)) < l'(x))
\end{aligned}$$

Since $l(x)$ must be an integer we have that this is equal to

$$p(-\log_D(p(x)) < l'(x)) = p(l'(x) \leq -\log_D(p(x)) - 1)$$

from part (a) we know that

$$p(l'(x) \leq -\log_D(p(x)) - 1) \leq D^{-1} \leq \frac{1}{2}$$

since $D \geq 2$. thus we have that

$$p(l(x) > l'(x) + 1) \leq \frac{1}{2}$$

and thus

$$p(l(x) > l'(x) + 1) \leq p(l(x) \leq l'(x) + 1)$$

(c)

$$P[l'(x) < l(x) - 1] \quad < \quad P[l'(x) > l(x) - 1]$$

$$\mathbb{E}[l'(x)] \leq H(x) + 1$$

$$\mathbb{E}[l(x) - 1] \leq H(x) \leq \mathbb{E}[l'(x)]$$

$$\sum_{i \in X-1} [p(x_i) \log \frac{1}{p(x_i)}] < \sum_{i \in X} p(x_i) \log \frac{1}{p(x_i)} \leq \sum_{i \in X} p(x_i) \log \left(\frac{1}{p(x_i)}\right)$$

$$\mathbb{P}[l(x) - 1 > l'(x)] < \mathbb{P}[l'(x) \geq H(x) > l(x) - 1]$$

$$\mathbb{P}[l(x) - 1 > l'(x)] < \mathbb{P}[l'(x) > l(x) - 1] \quad \square$$