ECE 231A HW 1

Lawrence Liu

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Problem 3

(a)

The entropy of $H(X)$ is

$$H(X) = -\sum_{x \in X} p(x) \log\left(p(x)\right)$$

and we have that

$$\begin{aligned} H(X|Y) &= \sum_{x \in X, y \in Y} p(y) H(X|Y = y) \\ &= \left(\sum_{x \in S} p(x)\right) H(X|Y = 1) + \left(\sum_{x \notin S} p(x)\right) H(X|Y = 0) \end{aligned}$$

We have

$$\begin{aligned} H(X|Y = 1) &= -\sum_{x \in S} p(x|Y = 1) \log(p(x|Y = 1)) \\ &= -\sum_{x \in S} \frac{p(x)}{\sum_{x \in S} p(x)} \log\left(\frac{p(x)}{\sum_{x \in S} p(x)}\right) \end{aligned}$$

likewise we have

$$
\begin{aligned}
H(X|Y=0) &= -\sum_{x\notin S} p(x|Y=0)\log(p(x|Y=0)) \\
&= -\sum_{x\notin S} \frac{p(x)}{\sum_{x\notin S} p(x)} \log\left(\frac{p(x)}{\sum_{x\notin S} p(x)}\right)
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
H(X|Y) &= -\sum_{x\in S} p(x)\left(\log(p(x)) - \log\left(\sum_{x\in S} p(x)\right)\right) - \sum_{x\notin S} p(x)\left(\log(p(x)) - \log\left(\sum_{x\notin S} p(x)\right)\right) \\
&= H(X) + \sum_{x\in S} p(x)\log\left(\sum_{x\in S} p(x)\right) + \sum_{x\notin S} p(x)\log\left(\sum_{x\notin S} p(x)\right)
\end{aligned}
$$

Therefore we have

$$
H(X) - H(X|Y) = \boxed{-\sum_{x\in S} p(x)\log\left(\sum_{x\in S} p(x)\right) - \sum_{x\notin S} p(x)\log\left(\sum_{x\notin S} p(x)\right)}
$$

(b)

$H(X) - H(X|Y)$ is maximized when $\sum_{x\in S} p(x) = \sum_{x\notin S} p(x) = \frac{1}{2}$, this is possible when $S = \boxed{2, 5}$ or $S = \boxed{1, 2, 4}$

Problem 5

(a)

We have that

$$
H(X) = -\sum_{i\in\chi_1} (1-\gamma)p(i)\log\left((1-\gamma)p(i)\right) - \sum_{i\in\chi_2} \gamma q(i)\log\left(\gamma q(i)\right)
$$

Likewise for $H(X,Y)$ we have

$$H(X,Y) = -\sum_{y\in\{1,2\}} \sum_{x\in\{1,2,..m\}} P(x,y)\log(P(x,y))$$

we have that

$$p(x,1) = \begin{cases} (1-\gamma)p(x) & \text{if } x \in \chi_1 \\ 0 & \text{if } x \notin \chi_2 \end{cases}$$

$$p(x,2) = \begin{cases} 0 & \text{if } x \in \chi_1 \\ \gamma q(x) & \text{if } x \notin \chi_2 \end{cases}$$

Therefore we have

$$H(X,Y) = -\sum_{i\in\chi_1}(1-\gamma)p(i)\log\left((1-\gamma)p(i)\right) - \sum_{i\in\chi_2}\gamma q(i)\log\left(\gamma q(i)\right)$$

And thus we have

$$\boxed{H(X,Y) = H(X)}$$

Problem 6

(a)

$$\begin{aligned} 1, 2H(x) &= -\sum_{x\in X} p(x)\log(p(x)) \\ &= p(A)\log(p(A)) + p(B)\log(p(B)) + p(C)\log(p(C)) \\ &+ p(D)\log(p(D)) + p(E)\log(p(E)) + p(F)\log(p(F)) \\ &= \frac{1}{2}\log(4) + \frac{1}{4}\log(8) + \frac{3}{16}\log(\frac{16}{3}) + \frac{1}{16}\log(16) \\ &= \boxed{2.2452 \text{ shannons}} \end{aligned}$$