

ECE 231A HW 2

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Problem 1

(a)

No this is not necessarily true. Consider the following example:

(b)

We have that

$$\begin{aligned} I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\ &= H(X_1) + H(X_2) - H(X_1, X_2) - (H(X_1|X_3) - H(X_1|X_2, X_3)) \\ &= H(X_1) + H(X_2) + H(X_3) - H(X_1, X_2) - H(X_1, X_3) \\ &\quad - H(X_2, X_3) + H(X_1, X_2, X_3) \end{aligned}$$

Since that

$$\begin{aligned} I(X_1; X_2|X_3) &= H(X_1|X_3) - H(X_1|X_2, X_3) \\ &= H(X_1, X_3) - H(X_3) - H(X_1, X_2, X_3) + H(X_2, X_3) \end{aligned}$$

$$\begin{aligned} I(X_2; X_3|X_1) &= H(X_2|X_1) - H(X_2|X_3, X_1) \\ &= H(X_2, X_1) - H(X_1) - H(X_1, X_2, X_3) + H(X_3, X_1) \end{aligned}$$

$$\begin{aligned} I(X_1; X_3|X_2) &= H(X_1|X_2) - H(X_1|X_3, X_2) \\ &= H(X_2, X_1) - H(X_2) - H(X_1, X_2, X_3) + H(X_3, X_2) \end{aligned}$$

Therefore we have that

$$\begin{aligned} I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\ I(X_1; X_2; X_3) &= I(X_2; X_3) - I(X_2; X_3|X_1) \\ I(X_1; X_2; X_3) &= I(X_1; X_3) - I(X_1; X_3|X_2) \end{aligned}$$

Since $I(X_1; X_2) \geq 0$, $I(X_2; X_3) \geq 0$, and $I(X_1; X_3) \geq 0$, we have that

$$\begin{aligned} I(X_1; X_2; X_3) &\geq -I(X_1; X_2|X_3) \\ I(X_1; X_2; X_3) &\geq -I(X_2; X_3|X_1) \\ I(X_1; X_2; X_3) &\geq -I(X_1; X_3|X_2) \end{aligned}$$

Therefore we have that

$$I(X_1; X_2; X_3) \geq -\min(I(X_1; X_2|X_3), I(X_2; X_3|X_1), I(X_1; X_3|X_2))$$

(c)

Once again from

$$\begin{aligned} I(X_1; X_2; X_3) &= I(X_1; X_2) - I(X_1; X_2|X_3) \\ I(X_1; X_2; X_3) &= I(X_2; X_3) - I(X_2; X_3|X_1) \\ I(X_1; X_2; X_3) &= I(X_1; X_3) - I(X_1; X_3|X_2) \end{aligned}$$

Since $I(X_1; X_2|X_3) \geq 0$, $I(X_2; X_3|X_1) \geq 0$, and $I(X_1; X_3|X_2) \geq 0$, we have that

$$\begin{aligned} I(X_1; X_2; X_3) &\leq I(X_1; X_2) \\ I(X_1; X_2; X_3) &\leq I(X_2; X_3) \\ I(X_1; X_2; X_3) &\leq I(X_1; X_3) \end{aligned}$$

Therefore we have that

$$I(X_1; X_2; X_3) \leq \min(I(X_1; X_2), I(X_2; X_3), I(X_1; X_3))$$

Problem 2

We have that

$$I(X;Y|U) = H(X|U) - H(X|Y,U)$$

since X and U are independent we have

$$I(X;Y|U) = H(X) - H(X|Y,U)$$

Since $I(X;Y,U)$ we get:

$$I(X;Y,U) = I(X;Y|U)$$

Problem 3

(a)

Let the increase in Alice's score after the i th round be represented by the random variable Z_A^i we have that

$$Z_A^i = \begin{cases} 4 & \text{w.p. } 1/15 \\ 7 & \text{w.p. } 5/15 \\ 0 & \text{w.p. } 9/15 \end{cases}$$

And the increase in Bob's score after the i th round be represented by the random variable Z_B^i we have that

$$Z_B^i = \begin{cases} 3 & \text{w.p. } 1/15 \\ 5 & \text{w.p. } 2/15 \\ 6 & \text{w.p. } 6/15 \\ 0 & \text{w.p. } 6/15 \end{cases}$$

Then we have that

$$S_A^n = \sum_{i=1}^n Z_A^i$$

and

$$S_B^n = \sum_{i=1}^n Z_B^i$$

As $n \rightarrow \infty$ we have that

$$\lim_{n \rightarrow \infty} S_A^n = \lim_{n \rightarrow \infty} \sum_{i=1}^n Z_A^i = E[Z_A^i] = \boxed{2.6}$$

and

$$\lim_{n \rightarrow \infty} S_B^n = \lim_{n \rightarrow \infty} \sum_{i=1}^n Z_B^i = E[Z_B^i] = \boxed{3.2666}$$

(b)

We will need to make $\alpha > 7$ since we need to increase the expected value, therefore the probabilities for Z_A would not change, however instead of being 7 with the probability of 5/15 we would have α with the probability of 5/15. Likewise Z_B would not change. Therefore we would have that our new

$$E[Z_A^i] = \frac{4}{15} + \frac{5}{15}\alpha$$

To make this greater than 3.2666 we would have that $\boxed{\alpha > 9}$,

Problem 4

Problem 5

(a)

In order for $\psi = [\psi_1, \psi_2]^T$ to be a stationary distribution for a Markov chain we must have $\psi\Pi = \psi$, therefore we have the following two series of equations

$$\frac{1}{4}\psi_1 + \frac{3}{4}\psi_2 = \psi_1$$

$$\psi_1 + \psi_2 = 1$$

Solving these we get $\psi_1 = \psi_2 = \frac{1}{2}$. Therefore the entropy of this $H(X) = \log_2(2) = \boxed{1 \text{ shannons}}$