

ECE 231A HW 5

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December 1, 2022

Problem 1

(a)

First let us prove that

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \max_i \frac{a_i}{b_i}$$

We can prove this through induction, we already have that the base case when $n = 2$ is true. Now we consider the case the case of $n + 1$, let us arrange the a_i and b_i in such a way such that $\frac{a_{n+1}}{b_{n+1}}$ be the minimum of the $n + 1$ $\frac{a_i}{b_i}$'s. Then we have that

$$\frac{a_{n+1}}{b_{n+1}} \leq \min_{1 \leq i \leq n} \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$$

Therefore we have that

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \leq \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \max_i \frac{a_i}{b_i}$$

Thus we have proven that $\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}$ is bounded by the minimum and maximum of the $\frac{a_i}{b_i}$'s. Using this property we get that

$$\frac{\sum_{x \in \mathcal{X}} \tilde{P}_X(x) D(W_{Y|X} || P_Y)}{\sum_{x \in \mathcal{X}} \tilde{P}_X(x) c(x)} \leq \max_{x \in \mathcal{X}} \frac{\tilde{P}_X(x) D(W_{Y|X} || P_Y)}{\tilde{P}_X(x) c(x)} = \max_{x \in \mathcal{X}} \frac{D(W_{Y|X} || P_Y)}{c(x)}$$

(b)

We have that

Problem 2

(a)

We have that in order to maximize the differential entropy of Z we want to maximize the variance of Z , this is given by

$$\text{Var}(Z) = E[(Z_1 + Z_2)^2] - (E[Z_1 + Z_2])^2$$

Since Z_1 is independent of Z_2 we have that $E[Z_1 Z_2] = E[Z_1] E[Z_2]$, thus we get that

$$\text{Var}(Z) = E[Z_1^2] + E[Z_2^2] - E^2[Z_1] - E^2[Z_2] \leq 2\sigma^2 - E^2[Z_1] - E^2[Z_2]$$

Therefore in order for this to be maximized, we want $E[Z_1] = E[Z_2] = 0$. Thus we get that both Z_1 and Z_2 are Gaussian with mean 0. And that the maximal differential entropy of Z is given by $\frac{1}{2} \log(4\pi\sigma^2) + \frac{1}{2}$.

(b)

Once again we want to maximize the differential entropy of Z , and thus we want to maximize the variance of Z , this is given by

$$\text{Var}(Z) = E[(\sum_{i=1}^n Z_i)^2] - (E[\sum_{i=1}^n Z_i])^2$$

We have that

$$\left(\sum_{i=1}^n Z_i\right)^2 = \sum_{i=1}^n Z_i^2 + \sum_{i \neq j} Z_i Z_j$$

And

$$E^2[\sum_{i=1}^n Z_i] = \sum_{i=1}^n E^2[Z_i] + \sum_{i \neq j} E[Z_i]E[Z_j]$$

Since Z_i 's are independent we have that

$$E[Z_i Z_j] - E[Z_i]E[Z_j] = 0$$

For all $i \neq j$, thus we get that

$$\text{Var}(Z) = \sum_{i=1}^n E[Z_i^2] - \sum_{i=1}^n E^2[Z_i] = n\sigma^2 - \sum_{i=1}^n E^2[Z_i]$$

Therefore we must have that $E[Z_i] = 0$ for all i , thus we get that the maximum differential entropy of Z is given by $\boxed{\frac{1}{2} \log(2\pi n\sigma^2) + \frac{1}{2}}$.

(c)

We have that

$$\text{Var}(Z) = E[(\sum_{i=1}^n Z_i)^2] - (E[\sum_{i=1}^n Z_i])^2$$

We can rewrite this as

$$\begin{aligned}\text{Var}(Z) &= E\left[\sum_{i=1}^n Z_i^2\right] - E\left[\sum_{i=1}^n Z_i\right]^2 + \sum_{i \neq j} E[Z_i Z_j] - E[Z_i]E[Z_j] \\ &= \sum_{i=1}^n \text{Var}(Z_i) + \sum_{i \neq j} \text{Cov}(Z_i, Z_j)\end{aligned}$$

We have that $\text{Cov}(Z_i, Z_j) \leq \sqrt{\text{Var}(Z_i)\text{Var}(Z_j)}$, and since $\text{Var}(Z_i) \leq \sigma^2$ we get that

$$\text{Var}(z) \leq n^2 \sigma^2$$

With equality only happening when $E[Z_i] = 0$, therefore we get that the joint distribution of Z_i 's as expressed by a vector $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ is distributed as a multivariate Gaussian with mean $\mu = [0, \dots, 0]$ and covariance matrix Σ where $\Sigma_{ij} = \sigma^2$ for all $1 \leq i, j \leq n$. And with this the maximum

differential entropy of Z is given by $\boxed{\frac{1}{2} \log(2\pi n^2 \sigma^2) + \frac{1}{2}}$.