ECE 231 Project 2

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December 12, 2022

1 Problem 1 (Polarization)

1.1 Part (a)

$I(U_1, U_2; Y_1, Y_2) =$
$I(X_1, X_2; Y_1, Y_2) =$ (U_1, U_2 can be determined from X_1, X_2 and vice versa).
$H(X_1, X_2) - H(X_1, X_2|Y_1, Y_2) =$

$H(X_1) - H(X_1|Y_1, Y_2) + H(X_2) - H(X_2|Y_1, Y_2) =$ (X_1, X_2 are indepedendent and X_1, X_2 are also conditionally independent given Y_1, Y_2)

$H(X_1) - H(X_1|Y_1) + H(X_2) - H(X_2|Y_2) =$ (X_1 is conditionally independent from Y_2 given Y_1, and X_2 is conditionally independent from Y_1 given Y_2).

$I(X_1; Y_1) + I(X_2; Y_2)$, as desired.

We also have:
$I(U_1, U_2; Y_1, Y_2) =$
$H(U_1, U_2) - H(U_1, U_2|Y_1, Y_2) =$
$H(U_1) + H(U_2) - H(U_1, U_2|Y_1, Y_2) =$ (U_1, U_2 are independent)
$H(U_1) + H(U_2) - H(U_1|Y_1, Y_2) - H(U_2|Y_1, Y_2, U_1) =$
$H(U_1) - H(U_1|Y_1, Y_2) + H(U_2) - H(U_2|Y_1, Y_2, U_1) =$
$I(U_1; Y_1, Y_2) + I(U_2; Y_1, Y_2, U_1)$, as desired.

1.2 Part (b)

Because X_1, X_2 go through identitical channels with identical capacities to become Y_1, Y_2, we must have $I(X_1; Y_1) = I(X_2; Y_2)$
To prove the right hand inequality:
$I(X_2; Y_2) =$
$H(X_2) - H(X_2|Y_2) \leq$
$H(X_2) - H(X_2|Y_1, Y_2, U_1) =$
$I(X_2; Y_1, Y_2, U_1) =$
$I(U_2; Y_1, Y_2, U_1)$, as desired.
To prove the left hand inequality:
$I(U_1; Y_1, Y_2) =$
$I(U_1; Y_1, Y_2) =$
$I(U_1; Y_1|Y_2) + I(U_1; Y_2)$

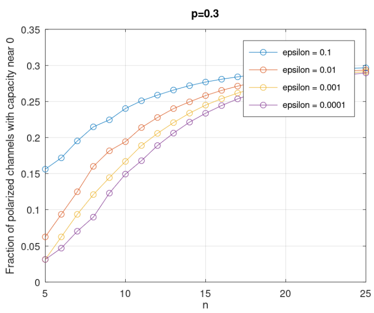
The second term is 0 because Y_2 is independent from U_1.
We have:
$I(U_1; Y_1|Y_2) =$
$I(X_1 \oplus X_2; Y_1|Y_2) \leq$
$I(X_1, X_2; Y_1|Y_2) =$ (as X_1, X_2 uniquely determines $X_1 \oplus X_2$)
$I(X_1; Y_1|Y_2) =$ (X_2 is independent from X_1, Y_1 even given Y_2)
$I(X_1; Y_1)$ (Y_2 is independent from X_1, X_2)
This is what we wanted to prove.

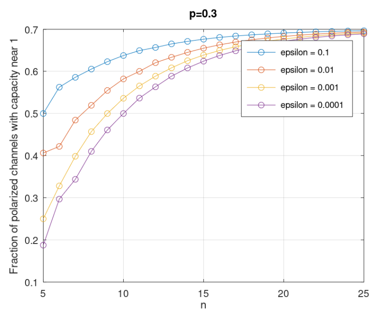
2 Problem 2 (Polarization of BEC)

Using the fact that $BEC(p)$ has capacity $1-p$, we have:
The capacity of W^- is $1 - 2p + p^2 = (1-p)^2$
The capacity of W is $1-p$
The capacity of W^+ is $1-p^2$
It is clear that $(1-p)^2 \leq 1-p$ as $0 \leq 1-p \leq 1$.
It is also clear that $1-p \leq 1-p^2$ as $0 \leq p \leq 1$.
Therefore channel W^- is worse than W, and W^+ is better than W.

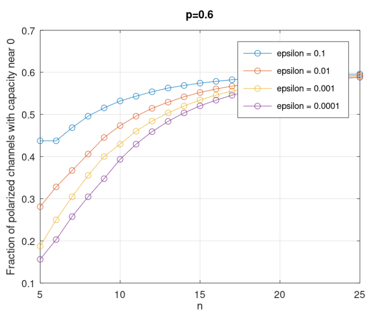
Problem 3 (Coding Problem: Polarization of BEC)

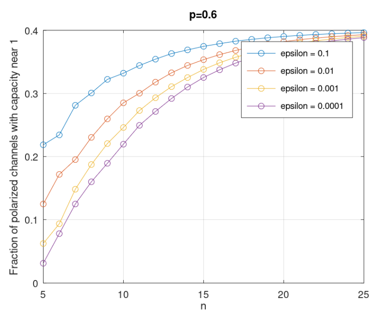
We get the following plots for $p = 0.3$



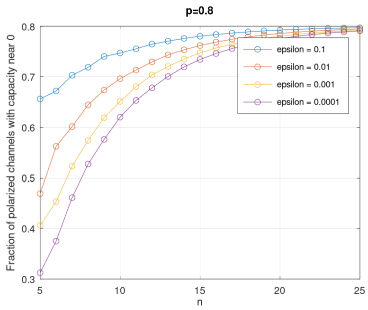


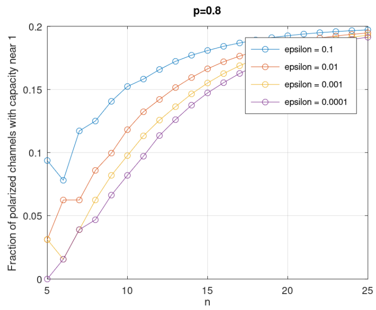
And the following plots for $p = 0.6$





And the following plots for $p = 0.8$





This was generated with the following code:

```
%% Code snippets for ECE 231A: Information Theory: Porject Module #3
% Problem 3 code snippets
clear all;
clc;
close all;

p = [0.3,0.6,0.8]; % Different erasure probabilities for BEC
epsilon = [0.1,0.01,0.001,0.0001];
n = 5:25; % log2 block length

% output for capacity near 0
% rows: corresponding to different n and
% columns: corresponding to different epsilon
output0 = zeros(length(n),length(epsilon));
% output for capacity
output1 = zeros(length(n),length(epsilon));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%% Enter your code here %%%%%%
for i=1:length(p)
  channel_ps=[p(i)^2,2*p(i)-p(i)^2];
  for n_=2:n(length(n))
    % disp(n_)
    new_channel_ps=zeros(1,2*length(channel_ps));
    % disp(size(new_channel_ps))
    for j=1:length(channel_ps)
      new_channel_ps(2*j-1)=channel_ps(j)^2;
      new_channel_ps(2*j)=2*channel_ps(j)-channel_ps(j)^2;
    end
    channel_capacities=ones(1,length(new_channel_ps))-new_channel_ps;
    if any(n==n_)
      for j=1:length(epsilon)
        % size(channel_capacities)
        % sum(channel_capacities<epsilon(j))
```

```
            % channel_capacities<epsilon(j)
            output0(n_-4,j)=sum(channel_capacities<=epsilon(j))/length(channel_cap
        end
        for j=1:length(epsilon)
            output1(n_-4,j)=sum(channel_capacities>=1-epsilon(j))/length(channel_ca
        end
    end
    channel_ps=new_channel_ps;
    % disp("--------------------------------------------")
end

%I move the plotting inside of the loop because its better that way, so then
%you can see the plots for each p
f1 = figure;
for j=1:length(epsilon)
    plot(n,output1(:,j),'-o','DisplayName',['epsilon = ' num2str(epsilon(j))]
    hold on;
end
grid on;
legend;
title(['p=' num2str(p(i))]    );
xlabel('n');
ylabel('Fraction of polarized channels with capacity near 1');
saveas(f1,['p=' num2str(p(i)) ' near 1.png']);

f2 = figure;
for j=1:length(epsilon)
    plot(n,output0(:,j),'-o','DisplayName',['epsilon = ' num2str(epsilon(j))]
    hold on;
end
grid on;
legend;
xlabel('n');
title(['p=' num2str(p(i))]    );
ylabel('Fraction of polarized channels with capacity near 0');
saveas(f2,['p=' num2str(p(i)) ' near 0.png']);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

f1 = figure;
for i=1:length(epsilon)
    plot(n,output1(:,i),'-o','DisplayName',['epsilon = ' num2str(epsilon(i))]);
    hold on;
end
grid on;
legend;
xlabel('n');
ylabel('Fraction of polarized channels with capacity near 1');

f2 = figure;
```

```
for i=1:length(epsilon)
    plot(n,output0(:,i),'-o','DisplayName',['epsilon = ' num2str(epsilon(i))]);
    hold on;
end
grid on;
legend;
xlabel('n');
ylabel('Fraction of polarized channels with capacity near 0');
```

Note that because the plotting functions provided would only plot for 1 value of p, I moved them inside the for loop. However because we were instructed to not touch the code beyond the parts we could touch, I left the original plotting functions in the code.

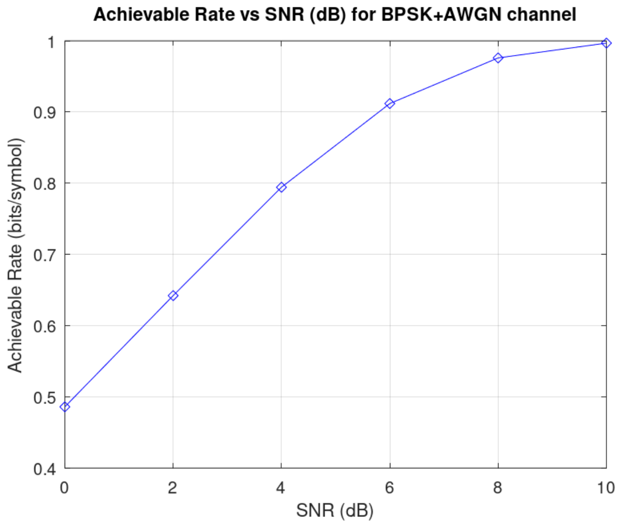
3 Problem 4 (Using Polar codes for BPSK+AWGN channel)

3.1 (a)

Using the formula for the capacity of the BPSK+AWGN channel found in the paper "Analysis and Design of Power-Efficient Coding Schemes With Parallel Concatenated Convolutional Codes" by Simon Huettinger and J. Huber, we have that the capcity is given by

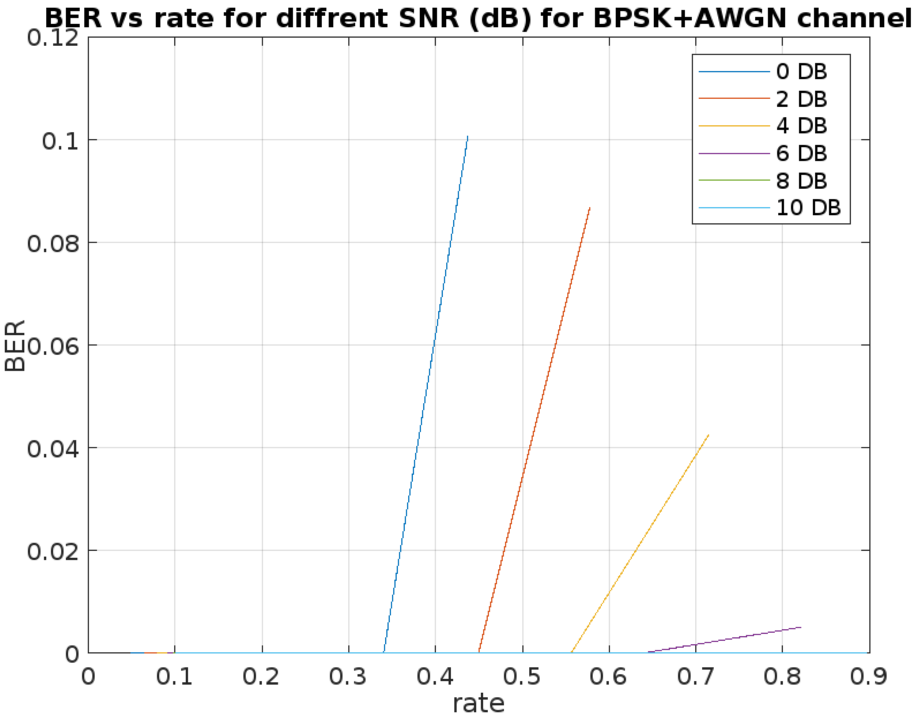
$$C = 1 - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-1)^2}{2\sigma^2}} \log_2(1 + e^{-\frac{2x}{\sigma^2}})dx \tag{1}$$

When we computed this integral we only did it from -20 to 20 since values beyond that result in NaN values and thus results in an incorrect integral. The resulting plot looks like



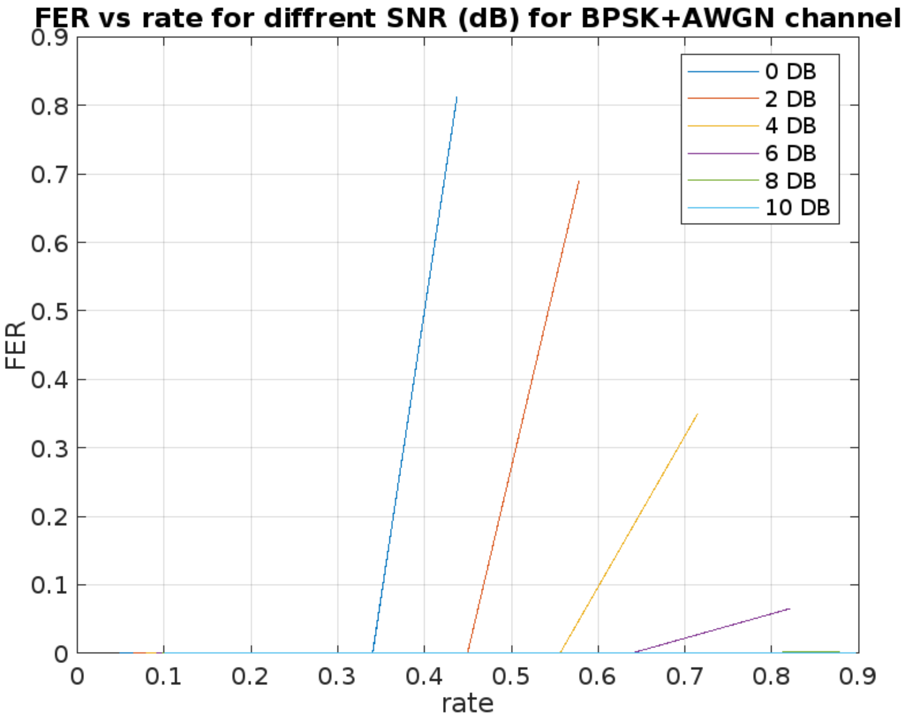
3.2 (b)

We get the following plot for BER vs encoding rate for diffrerent SNR values



3.3 (c)

We get the following plot for FER vs encoding rate for diffrerent SNR values



note: I modified the code outside of the comments saying to modify the code to have a plotting function, and have the rates be iterated over as a for loop nested inside a for loop for the SNR values. Also I reduced the number of trials to make the code run faster.

The code to do everything for problem 4 is included below:

```
%% Code snippets for ECE 231A: Information Theory: Porject Module #3
% Problem 4 code snippets
clear all;
close all;
clc;
warning off;

% scenario
    SNR_db = 0:2:10;
    size(SNR_db)
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    %%% Write your code here that computes the achievable %%
    %%% rates for BPSK+AWGN channel at different SNR(in dB)%
    %%% Plot the figure: achievable rate vs SNR (in dB) %%%%

    capacities=zeros(1,length(SNR_db));
```

```
    for i=1:length(SNR_db)
        sigma = sqrt(10^(-SNR_db(i)/10));
        f = @(t) 1/(sigma*sqrt(2*pi))*exp(-(t-1).^2/(2*sigma^2)).*log2(1+exp(-2
        capcity = 1-integral(f,-20,20);
        capacities(i) = capcity;
    end

    figure(1)
    plot(SNR_db,capacities ,'-db')
    xlabel('SNR (dB)')
    ylabel('Achievable Rate (bits/symbol)')
    title('Achievable Rate vs SNR (dB) for BPSK+AWGN channel')
    grid on;
    saveas(gcf,'Achievable Rate vs SNR (dB) for BPSK+AWGN channel.png')

FERs=zeros(5,length(SNR_db));
BERs=zeros(5,length(SNR_db));
Rs=0.1:0.2:0.9;
max_trials = 2E3;
seed1 = 0;
seed2 = 0;
n = 10; % log2 block length
for i=1:length(Rs)
    for snrpoint = 1:length(SNR_db)
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        R=Rs(i)*capacities(snrpoint);
        N = 2^n;
        K = round(N*R);
        % Construct code by using Binary Erasure Channel heuristic
        z = 1-R; % erasure rate (z is a real number)

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        %%%%% Enter your code here from Problem 3 here %%%%%%
        %%%%% that computes the erasure probabilities  %%%%%%
        %%%%% of N = 2^n polarized channels            %%%%%%

        channel_ps=[2*z - z^2,z^2];
        for n_=2:n(length(n))
            % disp(n_)
            new_channel_ps=zeros(1,2*length(channel_ps));
            % disp(size(new_channel_ps))
            for j=1:length(channel_ps)
                new_channel_ps(2*j-1)=2*channel_ps(j)-channel_ps(j)^2;
                          new_channel_ps(2*j)=channel_ps(j)^2;

            end
            channel_ps=new_channel_ps;
            % disp("------------------------------------")
        end
```

```
z = channel_ps;
for n_ = 0:N-1
  z(n_+1) = channel_ps(bin2dec(fliplr(dec2bin(n_, n)))+1);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%% Sorting the N = 2^n erasure probabilities %%%%
[vals, inds] = sort(z,'ascend'); % z is a vector of length N=2^n here
freeInds = sort(inds(1:K),'ascend'); % Choosing first K small erasure p
frozInds = sort(inds(K+1:N),'ascend');
FI = zeros(1,N);
FV = zeros(1,N);
FI(frozInds) = 1;

%% initialize statistics
FER = zeros(1,length(SNR_db)); % block error rate
BER = zeros(1,length(SNR_db)); % bit error rate
%% begin test
fprintf('\nN: %d, Rate: %4.3f, SNR_db = %3.2f\n', N, R, SNR_db(snrpoint
%%% Write your code here %%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

sigma = sqrt(10^(-SNR_db(snrpoint)/10));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
rand('seed',seed1);
randn('seed',seed2);
% statistics
fer =0;
ber = 0;
u = zeros(1,N);
for trial = 1:max_trials
    % Put information bits in the K coordinates of u
    u(freeInds) = round(rand(1,K));
    % Encodes the message bits u to input data to the
    % channel x
    x = polar_encode(u);
    % Modulate and transmit the data through the channel
    [y,Ly] = bpsk_awgn_channel(x,sigma);
    % Decode the polar code from the observed y
    uh = polar_decode(Ly,FI,FV);

    %%%% Enter your code here %%%%%
    %%%% Compute the BER and FER %%%

    BERs(i,snrpoint) = BERs(i,snrpoint)+sum(uh ~= u);
    FERs(i,snrpoint) = FERs(i,snrpoint)+any(uh ~= u);



    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
            if mod(trial,1000) == 0
                fprintf('N: %d, K: %d, SNR = %4.2f, FER: %5.3e, BER: %5.3e, Tr
            end
        end
    end
    BERs=BERs/(N*max_trials);
    FERs=FERs/(max_trials);
    % plot results
end
figure(2)
for snrpoint = 1:length(SNR_db)
    plot(Rs*capacities(snrpoint),BERs(:,snrpoint))
    hold on;
end
xlabel('rate')
ylabel('BER')
title('BER vs rate for diffrent SNR (dB) for BPSK+AWGN channel')
legend(string(SNR_db)+" DB")
grid on;
saveas(gcf,'BER vs rate for diffrent SNR (dB) for BPSK+AWGN channel.png')
figure(3)
for snrpoint = 1:length(SNR_db)
    plot(Rs*capacities(snrpoint),FERs(:,snrpoint))
    hold on;
end
hold on;
xlabel('rate')
ylabel('FER')
title('FER vs rate for diffrent SNR (dB) for BPSK+AWGN channel')
legend(string(SNR_db)+" DB")
grid on;
saveas(gcf,'FER vs rate for diffrent SNR (dB) for BPSK+AWGN channel.png')
```

4 How we divided up the work

David and Rohit did problem 1 and 2, Lawrence did problem 3, and Lawrence and David did problem 4.