ECE C143A Homework 4

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Problem 1

(a)

$P(N = 0) = 1 - 0.25 = \boxed{0.75}$

(b)

We want $P(N = 0|R = 0)$, we know that

$P(R = 1|N = 0) = P(E = 0)P(R = 1|E = 0, N = 0) + P(E = 1)P(R = 1|E = 1, N = 0) = 0.01$

$P(R = 0|N = 0) = 0.99$ and thus from bayes law we have

$$\begin{aligned} P(N = 0|R = 0) &= P(R = 0|N = 0)\frac{P(N = 0)}{P(R = 0)} \\ &= 0.9\frac{0.1 \cdot 0.75}{P(R = 0)} \end{aligned}$$

To find $P(R=0)$ we must find $P(R=1)$,

$$\begin{aligned}P(R=1) = P(E=1)P(N=1)P(R=1|E=1,N=1) \\ + P(E=1)P(N=0)P(R=1|E=1,N=0) \\ + P(E=0)P(N=1)P(R=1|E=0,N=1) \\ + P(E=0)P(N=0)P(R=1|E=0,N=0)\end{aligned}$$

$$P(R=1) = 0.9 \cdot 0.25 \cdot 1 + 0.1 \cdot 0.75 \cdot 0.1 + 0.1 \cdot 0.25 \cdot 0.1 = 0.235$$

Therefore $P(R=0) = 1 - P(R=1) = 0.765$, thus

$$P(N=0|R=0) = 0.99\frac{0.75}{P(R=0)} = \boxed{0.97}$$

(c)

We have

$$\begin{aligned}P(E=0,N=0|R=0) &= \frac{P(E=0,N=0,R=0)}{P(R=0)} \\ &= \frac{P(R=0|E=0,N=0)P(E=0)P(N=0)}{P(R=0)} \\ &= \frac{(1-P(R=1|E=0,N=0))P(E=0)P(N=0)}{P(R=0)} \\ &= \frac{0.9 \cdot 0.1 \cdot 0.75}{0.765} \\ &= \boxed{0.088}\end{aligned}$$

This intuitively makes sense because we need two conidtions to occur, equipment broken and neuron spike not occuring.

(d)

Let us consider the case $E=1$, $N=0$ given $R=1$ we have

$$P(E=1,N=0|R=1) = \frac{P(R=1|E=1,N=0)P(E=1)P(N=0)}{P(R=1)}$$

Since $P(R=1|E=1,N=0)=0$, we thus have $P(E=1,N=0|R=1)=0$. However

$$\begin{aligned} P(E=1|R=1) &= P(R=1|E=1)\frac{P(E=1)}{P(R=1)} \\ &= (P(R=1|E=1,N=0)P(N=0)+ \\ &P(R=1|E=1,N=1)P(N=1))\frac{P(E=1)}{P(R=1)} \end{aligned}$$

This therefore $P(E=1|R=1)>1$, likewise

$$\begin{aligned} P(N=0|R=1) &= P(R=1|N=0)\frac{P(N=0)}{P(R=1)} \\ &= (P(R=1|E=1,N=0)P(E=1)+ \\ &P(R=1|E=0,N=0)P(E=0))\frac{P(N=0)}{P(R=1)} \end{aligned}$$

This therefore $P(E=1|R=1)>1$, therfore, $P(E=1|R=1)P(N=0|R=1)>0$ and is not equal to $P(E=1,N=0|R=1)$ therefore they are conditionally dependent. Ie they are not independent given R

Problem 2

(a)

$$\boxed{P(a,b,c,d)=P(c)P(a|c)P(d)P(b|a,d)}$$

(b)

$$\begin{aligned}
P(C,D) &= \sum_A \sum_B P(C,D,A,B) \\
&= \sum_A \sum_B P(C)P(D)P(A|C)P(B|A,D) \\
&= P(C)P(D) \sum_A \sum_B P(A|C)P(B|A,D) \\
&= P(C)P(D) \sum_A P(A|C) \sum_B P(B|A) \\
&= P(C)P(D)
\end{aligned}$$

Therefore C and D are independent.

(c)

$$\begin{aligned}
P(C,D|A,B) &= \frac{P(C,D,A,B)}{P(A,B)} \\
&= \frac{P(C)P(D)P(A|C)P(B|A,D)}{P(A,B)} \\
&= \frac{P(D)P(C|A)P(A)P(B|A,D)}{P(A,B)} \\
&= \frac{P(D)P(C|A)P(D|B,A)}{P(D|A)} \\
&= \frac{P(D)}{P(D|A)} P(C|A)P(D|B,A) \\
&= \frac{P(C|A)P(D|B,A)}{P(D,A)}
\end{aligned}$$

Therefore C and D are not independent given A and B.

(d)

$$
\begin{aligned}
P(a,d) &= \sum_B P(a,d,b) \\
&= \sum_a P(d)P(b)P(b|d,a) \\
&= P(d)P(b)\sum_B P(b|d,a) \\
&= P(d)P(b)
\end{aligned}
$$

Therefore a and d are independent

(e)

$$
\begin{aligned}
P(a,d|b) &= \frac{P(a,d,b)}{P(b)} \\
&= \frac{P(a)P(d)P(b|a,d)}{P(b)} \\
&= \frac{P(a)P(d)}{P(b)}\frac{P(b|d)P(a|b,d)}{P(a|d)} \\
&= boxedP(a|b,d)P(d|b)
\end{aligned}
$$

Therefore a and d are not independent given b

(f)

$$
\begin{aligned}
P(c,b) &= \sum_A P(c,b,a) \\
&= \sum_A P(c)P(a|c)P(b|a) \\
&= P(c)\sum_A P(a|c)P(b|a)
\end{aligned}
$$

Since $\sum_A P(a|c)P(b|a) \neq P(b)$ in general, c and b are not independent.

(g)

$$
\begin{aligned}
P(c, b|a) &= \frac{P(c, b, a)}{a} \\
&= \frac{P(c)P(a|c)P(b|a)}{P(a)} \\
&= P(c|a)P(b|a)
\end{aligned}
$$

Therefore c and b are independent given a.

Problem 3

(a)

TODO

(b)

$$
P(x_1, ..., x_D, z) = \boxed{P(z)\Pi_{i=1}^{D} P(x_1|z)}
$$

(c)

Yes they are independent, intuitively thinking for two any two dimensions x_i and x_j, this becomes a graphical model with one parent and two children, which was proved in lecture to be independent.

Problem 4

(a)

$$P(x_1, x_2, x_3, x_4) = \boxed{P(x_1)P(x_2|x_1)P(x_3|x_2)P(x_4|x_3)}$$

(b)

for x_i we have

$$\begin{aligned}
Var(x_i) &= E[x_i^2] - E^2[x_i] \\
&= E[E[x_i^2|x_{i-1}]] \\
&= E[\sigma^2 + x_{i-1}^2] \\
&= \sigma^2 + E[E[x_{i-1}^2|x_{i-2}]] \\
&\vdots \\
&= i\sigma^2
\end{aligned}$$

And for any i and j such that $i < j$

$$\begin{aligned}
Cov(x_i, x_j) &= E[(x_i - E[x_j])(x_i - E[x_j])] \\
&= E[x_i x_j] \\
&= E[E[x_i x_j|x_{j-1}]] \\
&= E[x_i x_{j-1}] \\
&\vdots \\
&= E[x_i^2] \\
&= i\sigma^2
\end{aligned}$$

Therefore, the covariance matrix Σ is

$$\Sigma = \boxed{\begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2 \end{bmatrix}}$$

(c)

From python the inverse of the precision matrix is

$$\Sigma^{-1} = \boxed{\frac{1}{\sigma^2}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}$$

(d)

The zeros occur only when the nodes have at least one node in between, therefore these nodes are conditionally independent.