ECE C143A Homework 4

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Problem 1

(a)

$P(N = 0) = 1 - 0.25 = \boxed{0.75}$

(b)

We want $P(N = 0|R = 0)$, we know that

$P(R = 1|N = 0) = P(E = 0)P(R = 1|E = 0, N = 0) + P(E = 1)P(R = 1|E = 1, N = 0) = 0.01$

$P(R = 0|N = 0) = 0.99$ and thus from bayes law we have

$$P(N = 0|R = 0) = P(R = 0|N = 0)\frac{P(N = 0)}{P(R = 0)}$$
$$= 0.9\frac{0.1 \cdot 0.75}{P(R = 0)}$$

To find $P(R = 0)$ we must find $P(R = 1)$,

$$P(R = 1) = P(E = 1)P(N = 1)P(R = 1|E = 1, N = 1)$$
$$+ P(E = 1)P(N = 0)P(R = 1|E = 1, N = 0)$$
$$+ P(E = 0)P(N = 1)P(R = 1|E = 0, N = 1)$$
$$+ P(E = 0)P(N = 0)P(R = 1|E = 0, N = 0)$$

$$P(R = 1) = 0.9 \cdot 0.25 \cdot 1 + 0.1 \cdot 0.75 \cdot 0.1 + 0.1 \cdot 0.25 \cdot 0.1 = 0.235$$

Therefore $P(R = 0) = 1 - P(R = 1) = 0.765$, thus

$$P(N = 0|R = 0) = 0.99 \frac{0.75}{P(R = 0)} = \boxed{0.97}$$

(c)

We have

$$P(N = 0|R = 0, E = 0) = \frac{P(E = 0, N = 0, R = 0)}{P(R = 0, E = 0)}$$
$$= \frac{P(R = 0|E = 0, N = 0)P(E = 0, N = 0)}{P(R = 0, E = 0)}$$
$$= \frac{(1 - P(R = 1|E = 0, N = 0))P(E = 0)P(N = 0)}{P(R = 0|E = 0)P(E = 0)}$$
$$= \frac{(1 - P(R = 1|E = 0, N = 0))P(N = 0)}{(P(R = 0|E = 0, N = 0)P(N = 0) + P(R = 0|E = 0, N = 1)P(N = 1))}$$
$$= \frac{0.9 \cdot 0.75}{0.9 \cdot 0.75 + 0.9 \cdot 0.25}$$
$$= \boxed{0.75}$$

This intuitively makes sense because the equiment being broken means that a recorded spike provides no aditional information.

(d)

Let us consider the case $E = 1$, $N = 0$ given $R = 1$ we have

$$P(E = 1, N = 0|R = 1) = \frac{P(R = 1|E = 1, N = 0)P(E = 1)P(N = 0)}{P(R = 1)}$$

Since $P(R = 1|E = 1, N = 0) = 0$, we thus have $P(E = 1, N = 0|R = 1) = 0$. However

$$P(E = 1|R = 1) = P(R = 1|E = 1)\frac{P(E = 1)}{P(R = 1)}$$

$$= (P(R = 1|E = 1, N = 0)P(N = 0)+$$

$$P(R = 1|E = 1, N = 1)P(N = 1))\frac{P(E = 1)}{P(R = 1)}$$

This therefore $P(E = 1|R = 1) > 1$, likewise

$$P(N = 0|R = 1) = P(R = 1|N = 0)\frac{P(N = 0)}{P(R = 1)}$$

$$= (P(R = 1|E = 1, N = 0)P(E = 1)+$$

$$P(R = 1|E = 0, N = 0)P(E = 0))\frac{P(N = 0)}{P(R = 1)}$$

This therefore $P(E = 1|R = 1) > 1$, therfore, $P(E = 1|R = 1)P(N = 0|R = 1) > 0$ and is not equal to $P(E = 1, N = 0|R = 1)$ therefore they are conditionally dependent. Ie they are not independent given R

Problem 2

(a)

$$\boxed{P(a, b, c, d) = P(c)P(a|c)P(d)P(b|a, d)}$$

3

(b)

$$P(C, D) = \sum_{A} \sum_{B} P(C, D, A, B)$$

$$= \sum_{A} \sum_{B} P(C)P(D)P(A|C)P(B|A, D)$$

$$= P(C)P(D) \sum_{A} \sum_{B} P(A|C)P(B|A, D)$$

$$= P(C)P(D) \sum_{A} P(A|C) \sum_{B} P(B|A)$$

$$= P(C)P(D)$$

Therefore C and D are independent.

(c)

$$P(C, D|A, B) = \frac{P(C, D, A, B)}{P(A, B)}$$

$$= \frac{P(C)P(D)P(A|C)P(B|A, D)}{P(A, B)}$$

$$= \frac{P(D)P(C|A)P(A)P(B|A, D)}{P(A, B)}$$

$$= \frac{P(D)P(C|A)P(D|B, A)}{P(D|A)}$$

$$= \frac{P(D)}{P(D|A)}P(C|A)P(D|B, A)$$

$$= \frac{P(C|A)P(D|B, A)}{P(D, A)}$$

Therefore C and D are not independent given A and B.

(d)

$$P(a,d) = \sum_B P(a,d,b)$$
$$= \sum_a P(d)P(b)P(b|d,a)$$
$$= P(d)P(b) \sum_B P(b|d,a)$$
$$= P(d)P(b)$$

Therefore a and d are independent

(e)

$$P(a,d|b) = \frac{P(a,d,b)}{P(b)}$$
$$= \frac{P(a)P(d)P(b|a,d)}{P(b)}$$
$$= \frac{P(a)P(d)}{P(b)} \frac{P(b|d)P(a|b,d)}{P(a|d)}$$
$$= boxed P(a|b,d)P(d|b)$$

Therefore a and d are not independent given b

(f)

$$P(c,b) = \sum_A P(c,b,a)$$
$$= \sum_A P(c)P(a|c)P(b|a)$$
$$= P(c) \sum_A P(a|c)P(b|a)$$

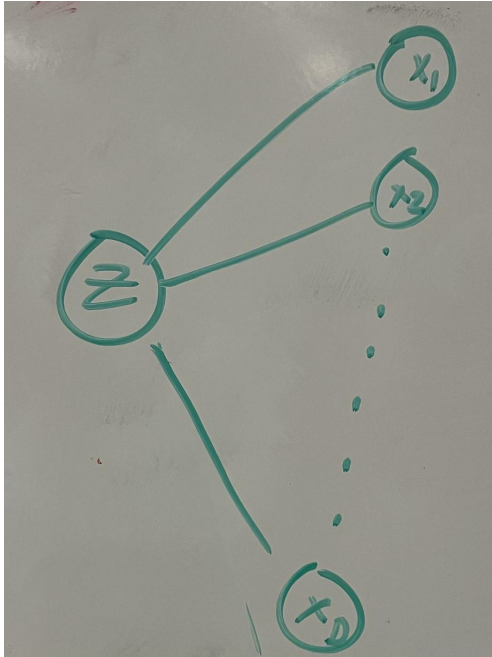Since $\sum_A P(a|c)P(b|a) \neq P(b)$ in general, c and b are not independent.

(g)

$$\begin{aligned}
P(c, b|a) &= \frac{P(c, b, a)}{a} \\
&= \frac{P(c)P(a|c)P(b|a)}{P(a)} \\
&= P(c|a)P(b|a)
\end{aligned}$$

Therefore c and b are independent given a.

Problem 3

(a)

(b)

$$P(x_1, ..., x_D, z) = \boxed{P(z)\Pi_{i=1}^{D}P(x_1|z)}$$

(c)

Yes they are independent, intuitively thinking for two any two dimensions x_i and x_j, this becomes a graphical model with one parent and two children, which was proved in lecture to be independent.

Problem 4

(a)

$$P(x_1, x_2, x_3, x_4) = \boxed{P(x_1)P(x_2|x_1)P(x_3|x_2)P(x_4|x_3)}$$

(b)

for x_i we have

$$\begin{aligned}
Var(x_i) &= E[x_i^2] - E^2[x_i] \\
&= E[E[x_i^2|x_{i-1}]] \\
&= E[\sigma^2 + x_{i-1}^2] \\
&= \sigma^2 + E[E[x_{i-1}^2|x_{i-2}]] \\
&\;\;\vdots \\
&= i\sigma^2
\end{aligned}$$

And for any i and j such that $i < j$

$$\begin{aligned}
Cov(x_i, x_j) &= E[(x_i - E[x_j])(x_i - E[x_j])] \\
&= E[x_i x_j] \\
&= E[E[x_i x_j|x_{j-1}]] \\
&= E[x_i x_{j-1}] \\
&\;\;\vdots \\
&= E[x_i^2] \\
&= i\sigma^2
\end{aligned}$$

Therefore, the covariance matrix Σ is

$$\Sigma = \boxed{\begin{bmatrix}
\sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\
\sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\
\sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\
\sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2
\end{bmatrix}}$$

(c)

From python the inverse of the precision matrix is

$$\Sigma^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(d)

The zeros occur only when the nodes have at least one node in between, therefore these nodes are conditionally independent.