Poisson Processes

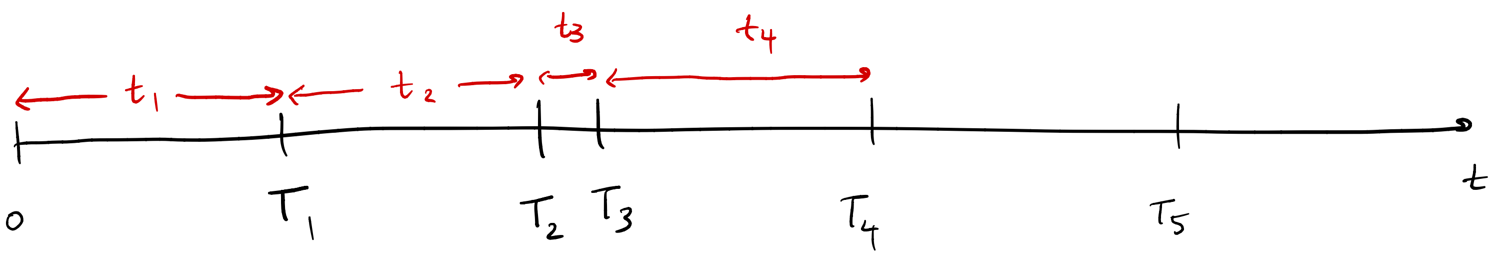
Model events that occur through time in a probabilistic framework.

$\downarrow$

spikes

$$T_1 = t_1$$

$$T_2 = t_1 + t_2$$



"Interspike interval" (ISI)

$$t_i \sim \exp(\lambda)$$

t_i i.i.d.

Exponential Distribution

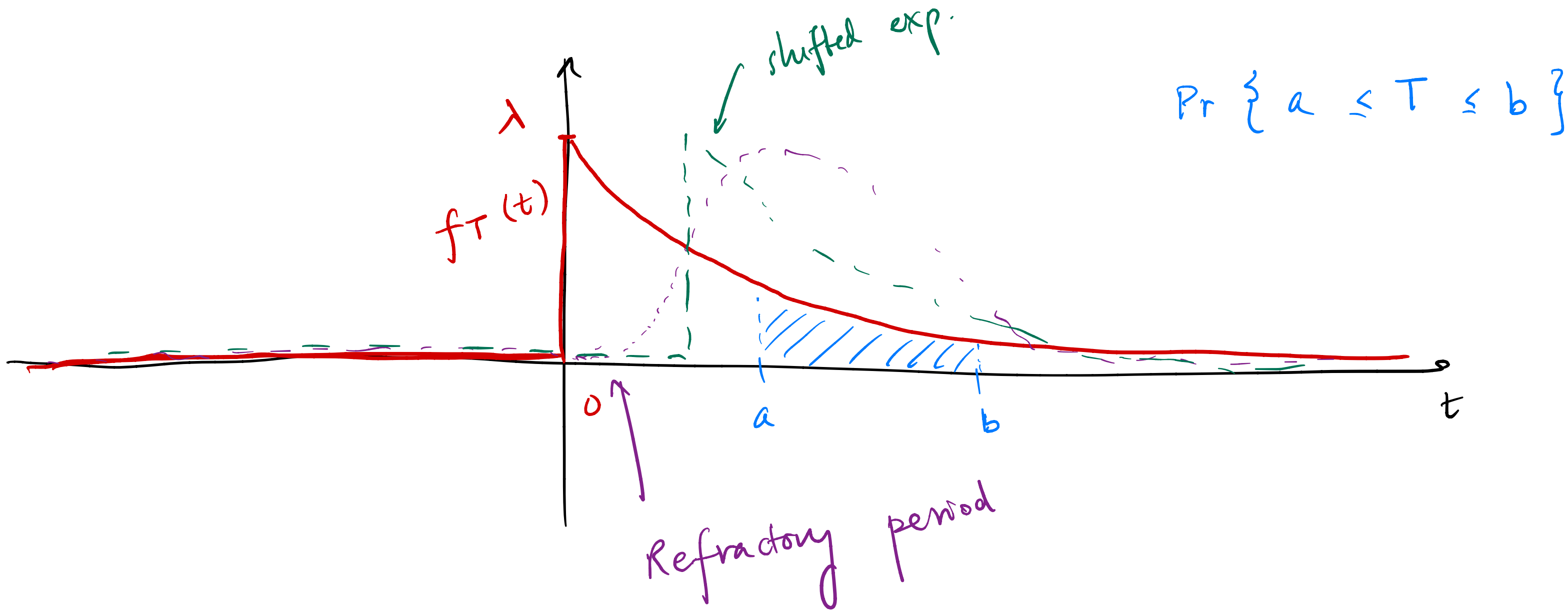
(convenient, leads to good properties)

$[=]$ spikes/s

A R.V., T, is exp. distributed with rate $\lambda \geq 0$

if its probability density function (pdf) is:

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases}$$

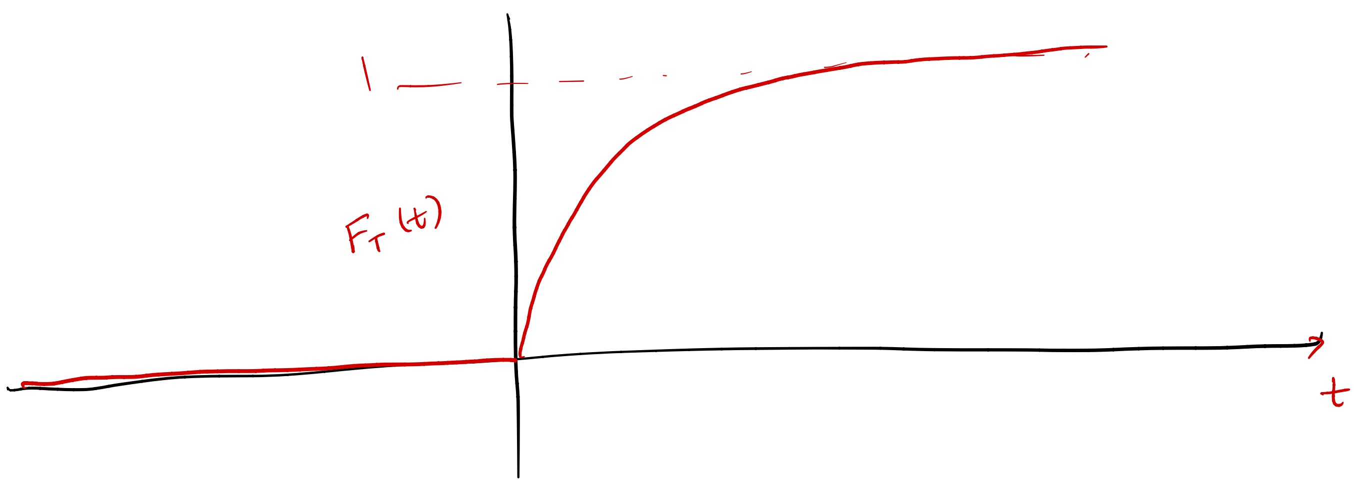


T can also be described by its cdf

$$F_T(t) = \Pr\{T \le t\}$$

$$= \int_{-\infty}^{t} f_T(\tau)\, d\tau$$

$$= \begin{cases} 1 - e^{-\lambda t}, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases}$$



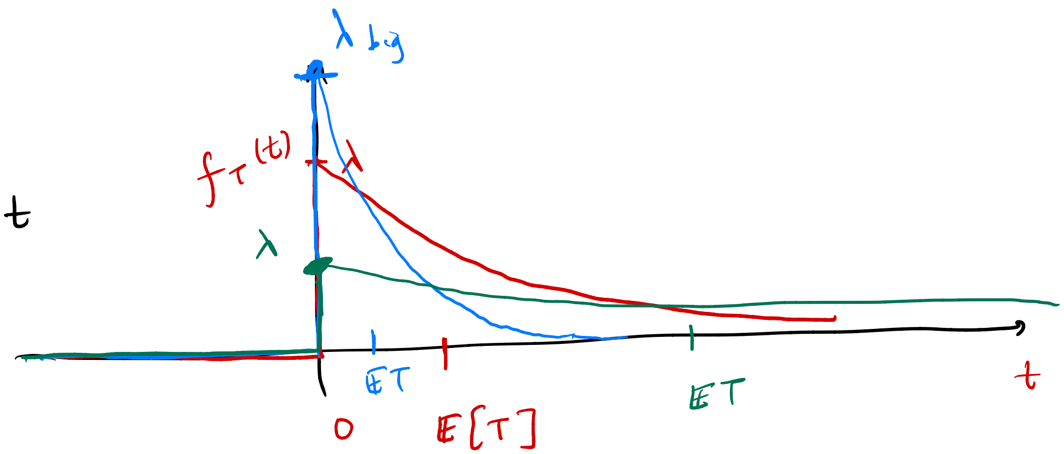
$$\Pr\{T > t\} = 1 - \Pr\{T \le t\}$$

$$= \begin{cases} e^{-\lambda t}, & t \geq 0 \\ 1, & \text{else} \end{cases}$$

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else.} \end{cases}$$

Mean, variance:

$$\mathbb{E}[T] = \int_{-\infty}^{\infty} t \cdot f_T(t)\, dt$$



see notes (integration by parts)

$$= \frac{1}{\lambda}$$

In words, $\mathbb{E}[T]$ is the average time between spikes

λ is big

λ is small

Variance: $$\mathrm{var}(T) = \underbrace{\mathbb{E}[T^2]}_{\int_{-\infty}^{\infty} t^2 f_T(t)\,dt} - (\mathbb{E}(T))^2$$

$$= \mathbb{E}\left[(T - \mathbb{E}(T))^2 \right]$$

$$= \frac{1}{\lambda^2}$$

The exp. distribution is memoryless

Say I've been waiting for a bus. The bus inter-arrival times, T (a R.V.), is exponentially distributed. If I've been waiting for t seconds, the probability that I must wait at least s more seconds is the same as if I had not waited at all.

$$\Pr\{T > t+s \mid T > t\} = \Pr\{T > s\} \rightarrow e^{-\lambda s}$$

$\uparrow$

Mathematical statement of the memoryless prop.

$$p(a,b) = p(a)\,p(b|a) = p(b)\,p(a|b)$$

Memoryless property: $\Pr\{T > t+s \mid T > t\} = \Pr\{T > s\}$ $= e^{-\lambda s}$

$$\Pr\{T > t+s \mid T > t\} = \frac{\Pr\{T > t+s,\ T > t\}}{\Pr\{T > t\}}$$

$$= \frac{\Pr\{T > t+s\} \cdot \cancel{\Pr\{T > t \mid T > t+s\}}^{=1}}{\Pr\{T > t\}}$$

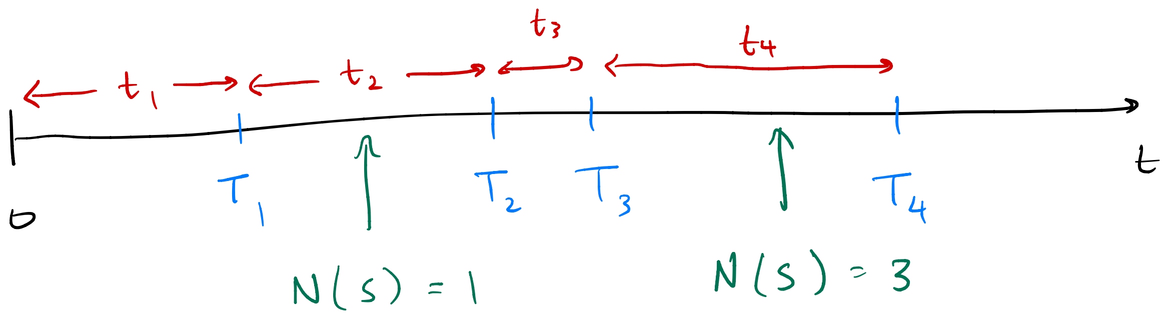
$$\boxed{\Pr\{T > t+s,\ T > t\} = \Pr(T > t+s)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s}$$

$$= \Pr\{T > s\}$$

Define the Poisson Process (PP)



Let $t_1, t_2, \ldots$ be independent exp. distr. R.V.'s w/ parameter λ

$$t_i \text{ iid } \sim \exp(\lambda)$$

Let $T_n = t_1 + t_2 + \ldots + t_n$, for all $n \geq 1$, and let $T_0 = 0$

T_n : absolute time of the nth spike.

We define $N(s) = \max\{n : T_n \leq s\}$

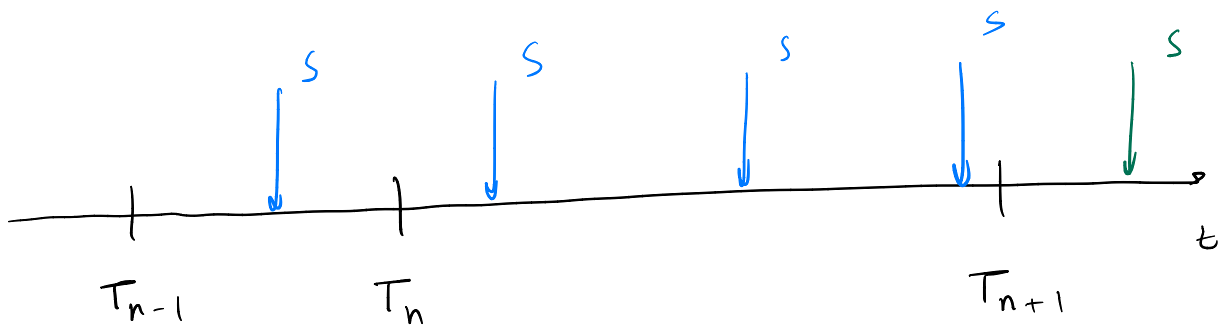
↳ $N(s)$ accepts a time, s, and returns the # of spikes that have occured up to and including time s.

Why is $N(s)$ called a PP rather than an exponential process?

Property 1: $N(s)$ is a Poisson distributed R.V. with mean λs.

$$\Pr\{N(s)=n\} = \frac{(\lambda s)^n e^{-\lambda s}}{n!}$$

How do we show this?



$$N(s)=n \iff T_n \le s < T_{n+1}$$

$$\Pr(N(s)=n) = \Pr(T_n \le s,\ T_{n+1} > s)$$

$$= \Pr(T_n \le s)\ \Pr(T_{n+1} > s \mid T_n \le s)$$

[abstraction to simplify / for intuition, consider T_n is discrete]

$$= \Pr(T_n = s)\ \Pr(T_{n+1} > s \mid T_n = s)$$

$$+ \Pr(T_n = s-1)\ \Pr(T_{n+1} > s \mid T_n = s-1)$$

$$+ \Pr(T_n = s-2)\ \Pr(T_{n+1} > s \mid T_n = s-2)$$

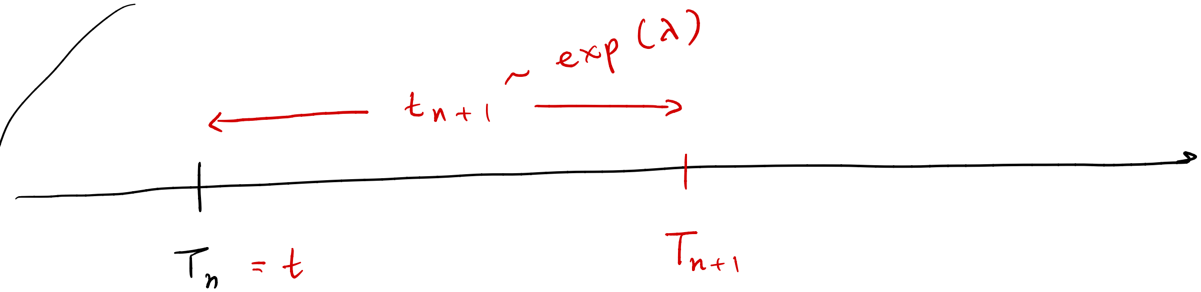
$$+ \quad \vdots$$

$$= \sum_{t=0}^{s} \Pr(T_n = t)\ \Pr(T_{n+1} > s \mid T_n = t)$$

[remove abstraction]

$$= \int_0^s f_{T_n}(t) \cdot \Pr(T_{n+1} > s \mid T_n = t)\, dt$$

$$\Pr(N(s)=n) = \int_0^s \Pr(T_{n+1} > s \mid T_n = t)\, f_{T_n}(t)\, dt$$



$$= \int_0^s \Pr(T_n + t_{n+1} > s \mid T_n = t)\, f_{T_n}(t)\, dt$$

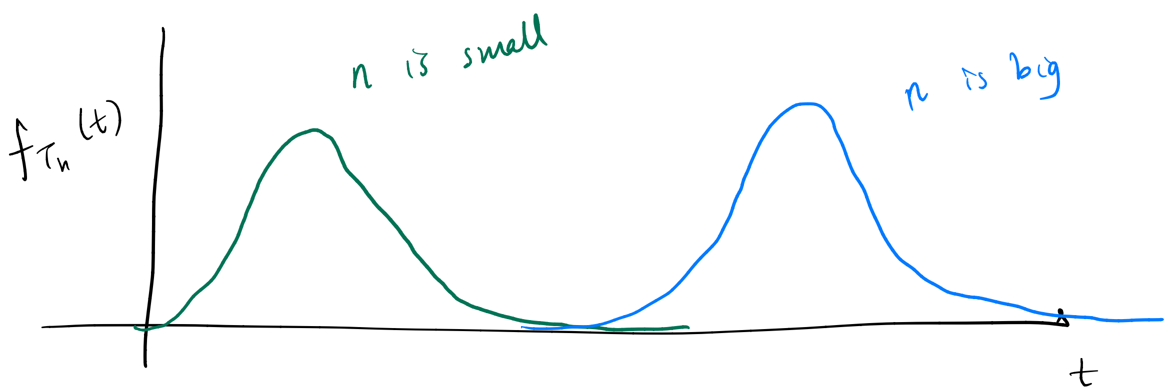
Plug in $T_n = t$

$\Rightarrow t + t_{n+1} > s$

$\Rightarrow t_{n+1} > s - t$

$$= \int_0^s \Pr(t_{n+1} > s - t)\, f_{T_n}(t)\, dt$$

$$= \int_0^s e^{-\lambda(s-t)}\, f_{T_n}(t)\, dt$$



$T_n = t_1 + t_2 + \dots + t_n$ $\qquad$ $t_i \sim \exp(\lambda)$, independent

if I know $f_{t_i} = \exp(\lambda)$, how do I calc f_{T_n}?

$$f_{T_n} = f_{t_1} * f_{t_2} * \dots * f_{t_n}$$

$A \quad B$, $\quad f_A$, f_B $\qquad A \perp\!\!\!\perp B$

$$C = A + B$$

$$\Pr(C = c) = \Pr(A + B = c)$$

$$= \int_{-\infty}^{\infty} \Pr(A = a) \Pr(B = c - A \mid A = a)\, da$$

$$= \int_{-\infty}^{\infty} \Pr(A = a) \Pr(B = c - a)\, da$$

$$\Rightarrow \quad \underline{f_C(c) = \int_{-\infty}^{\infty} f_A(a) f_B(c - a)\, da}$$

$$f_{T_n}(t) = f_{t_1} * f_{t_2} * \cdots * f_{t_n}$$

$$u(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases}$$

$$f_{t_i} = \lambda e^{-\lambda t} u(t)$$

$$e^{-\lambda t} u(t) \Longleftrightarrow \frac{1}{\lambda + j\omega}$$

$$t^n e^{-\lambda t} u(t) \Longleftrightarrow \frac{n!}{(\lambda + j\omega)^{n+1}}$$

FT of both sides

$$\mathcal{F}[f_{T_n}] = \prod_{i=1}^{n} \mathcal{F}[f_{t_i}]$$

$$= \prod_{i=1}^{n} \lambda \cdot \frac{1}{\lambda + j\omega}$$

$$= \frac{\lambda^n}{(\lambda + j\omega)^n}$$

$$f_{T_n} = \mathcal{F}^{-1}\left[\frac{\lambda^n}{(\lambda + j\omega)^n}\right] = \frac{\lambda^n}{(n-1)!} \mathcal{F}^{-1}\left[\frac{(n-1)!}{(\lambda + j\omega)^n}\right]$$

$$= \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} u(t)$$

$$\boxed{f_{T_n} = \lambda e^{-\lambda t}\left[\frac{(\lambda t)^{n-1}}{(n-1)!}\right], \quad \text{if } t \geq 0}$$

Gamma distribution

[if n is an integer, Erlang distribution]

$$\Pr(N(s) = n) = \int_0^s e^{-\lambda(s-t)} f_{T_n}(t)\, dt$$

$$= \int_0^s e^{-\lambda(s-t)} \lambda e^{-\lambda t} \left[\frac{(\lambda t)^{n-1}}{(n-1)!} \right] dt$$

$$= \frac{\lambda^n e^{-\lambda s}}{(n-1)!} \int_0^s t^{n-1}\, dt$$

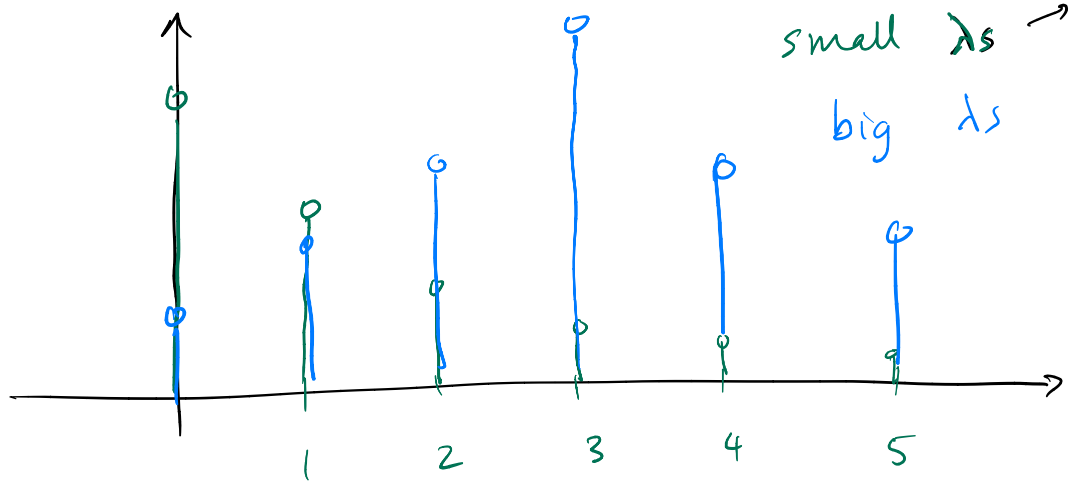
$$= \frac{\lambda^n e^{-\lambda s}}{(n-1)!} \left. \frac{t^n}{n} \right|_{t=0}^{t=s}$$

$$= \frac{\lambda^n e^{-\lambda s} \cdot s^n}{n!}$$

$$\boxed{\Pr(N(s) = n) = \frac{(\lambda s)^n e^{-\lambda s}}{n!}}$$

$$N(s) \sim \text{Poisson}(\lambda s)$$

$$\Pr(N(s) = n) = \frac{(\lambda s)^n e^{-\lambda s}}{n!}$$

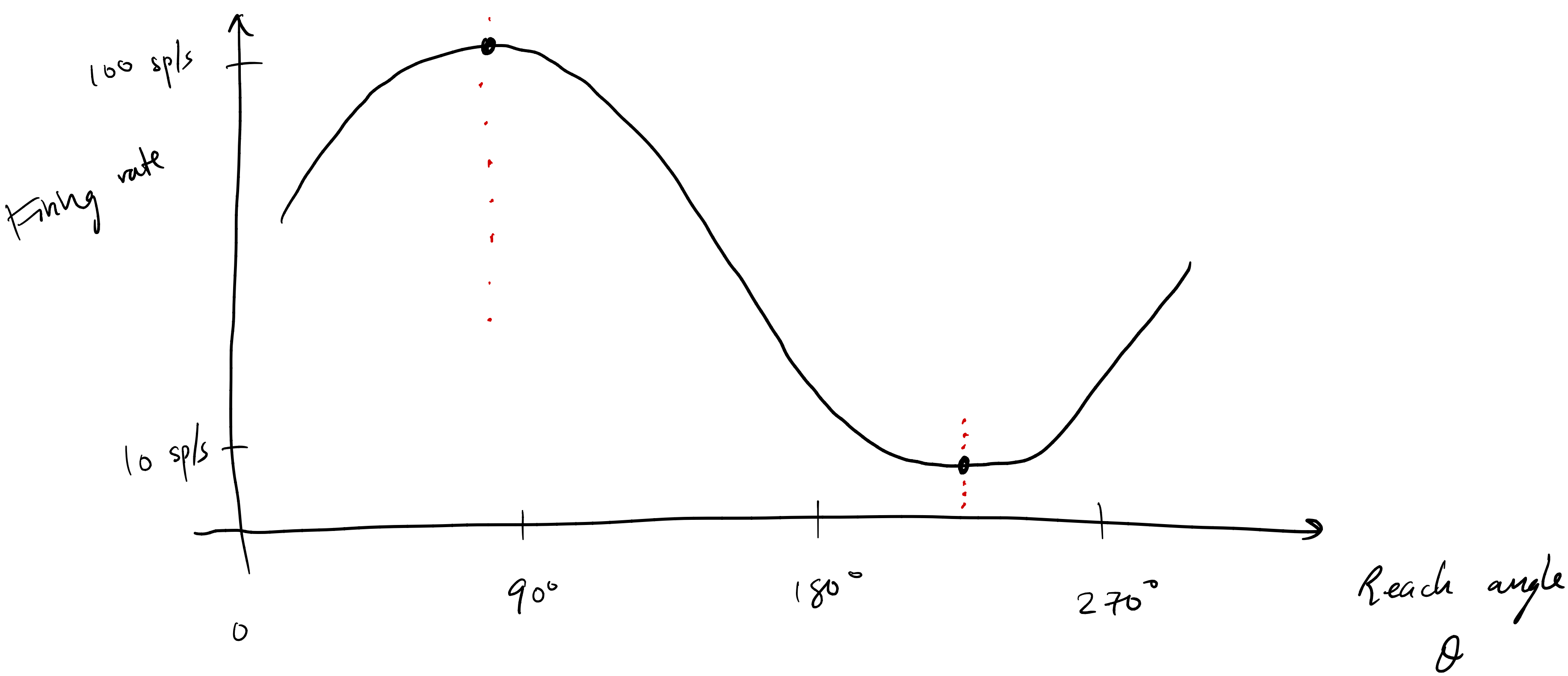


λ [=] spikes/s
s [=] s

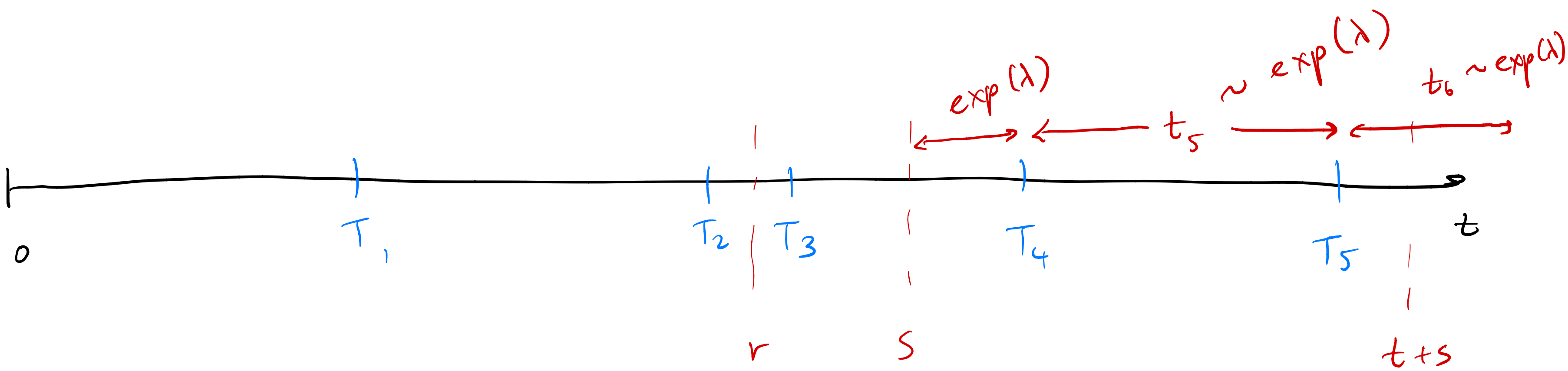
$$\mathbb{E}[N(s)] = \lambda s = \sum_{n=0}^{\infty} n \cdot \Pr(N(s) = n)$$

$$\mathrm{Var}(N(s)) = \lambda s$$

$$\text{Fano factor} = \frac{\text{Var}}{\text{mean}} = \frac{\lambda s}{\lambda s} = 1$$



Property 2: $N(t+s) - N(s)$, for $t \geq 0$, is a PP and is independent of $N(r)$ for $0 \leq r < s$.



$N(r) = 2$

$N(s) = 3$

$N(s+t) = 5$

$N(t+s) - N(s) = 2$

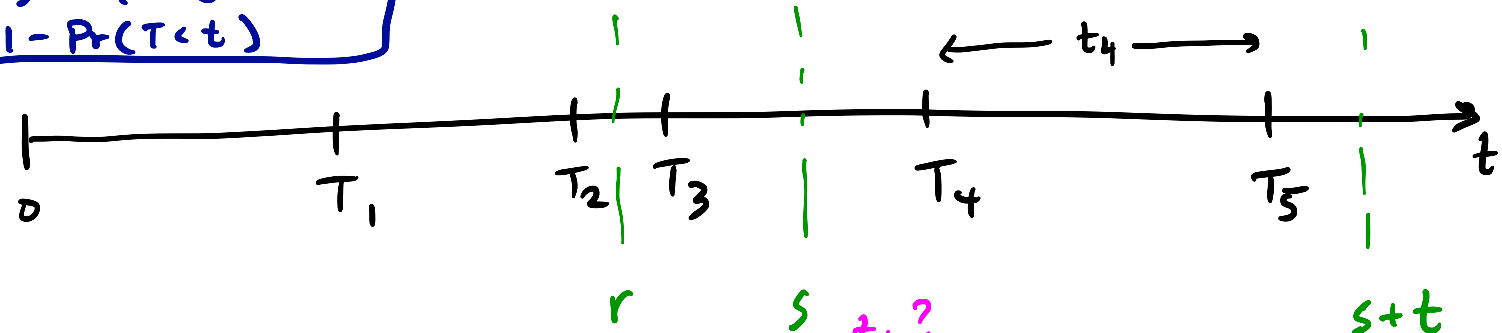
$=$ "Restart property"

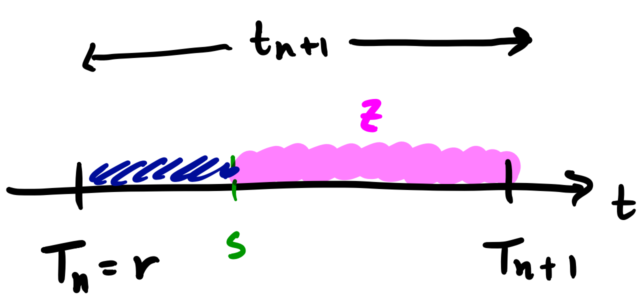
$$N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$$

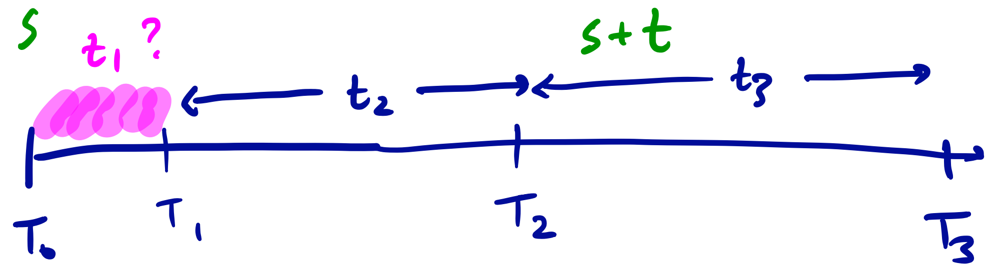
$Pr(T < t) = 1 - e^{-\lambda t}$

$Pr(T > t) = 1 - Pr(T < t)$

$= e^{-\lambda t}$







Is Z exp. distr. ?

$$Pr\left(Z > z \mid T_n = r,\ N(s) = n\right) \overset{?}{=} e^{-\lambda z}$$

$$= Pr\left(t_{n+1} > z + \underline{(s-r)} \mid \underline{T_n = r},\ t_{n+1} > s-r\right)$$

ind of t_i's

$T_n = t_1 + t_2 + \dots + t_n$

$t_{n+1} \perp\!\!\!\perp t_i$

$$= Pr\left(t_{n+1} > z + s - r \mid t_{n+1} > s - r\right)$$

memoryless

$$= Pr\left(t_{n+1} > z\right)$$

$$= e^{-\lambda z}$$