Lecture 8, Poisson Processes 3

1. HW #2 now due Monday, April 25, 2022.

Poisson Processes

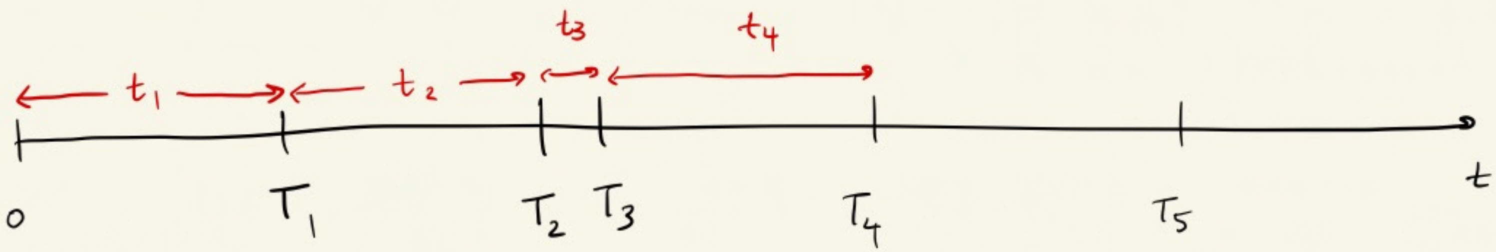
Model events that occur through time in a probabilistic framework.

↓

spikes

$T_1 = t_1$

$T_2 = t_1 + t_2$



"Interspike interval" (ISI)

$t_i \sim \exp(\lambda)$

t_i i.i.d.

Exponential Distribution

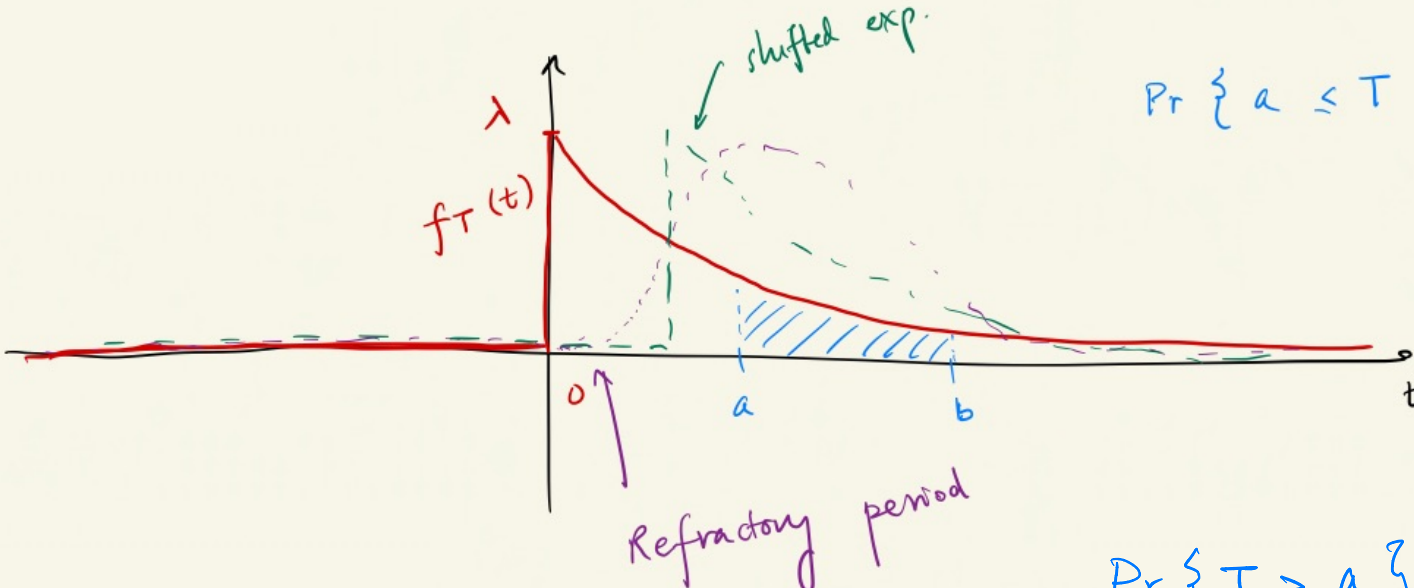
(convenient, leads to good properties)

$[=]$ spikes/s

A R.V., T, is exp. distributed with rate $\lambda \geq 0$

if its probability density function (pdf) is:

$$\underline{f_T(t)} = \begin{cases} \lambda e^{-\lambda t} & , \text{ if } t \geq 0 \\ 0 & , \text{ else} \end{cases}$$



$Pr\{a \leq T \leq b\}$

$Pr\{T > a\} \neq Pr\{T > b\}$

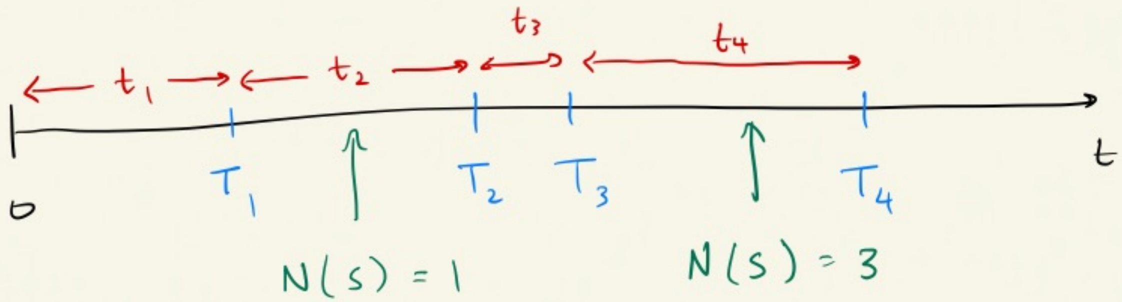
The exp. distribution is memoryless

Say I've been waiting for a bus. The bus inter-arrival times, T (a R.V.), is exponentially distributed. If I've been waiting for t seconds, the probability that I must wait at least s more seconds is the same as if I had not waited at all.

$$\Pr\{T > t+s \mid T > t\} = \Pr\{T > s\} \rightarrow e^{-\lambda s}$$

Mathematical statement of the memoryless prop.

Define the Poisson Process (PP)



Let $t_1, t_2, \ldots$ be independent exp. distr. R.V.'s w/ parameter λ

$$t_i \text{ iid } \sim \exp(\lambda)$$

Let $T_n = t_1 + t_2 + \ldots + t_n$, for all $n \geq 1$, and let $T_0 = 0$

T_n : absolute time of the n^{th} spike.

We define $N(s) = \max\{n : T_n \leq s\}$

↳ $N(s)$ accepts a time, s, and returns the # of spikes that have occurred up to and including time s.

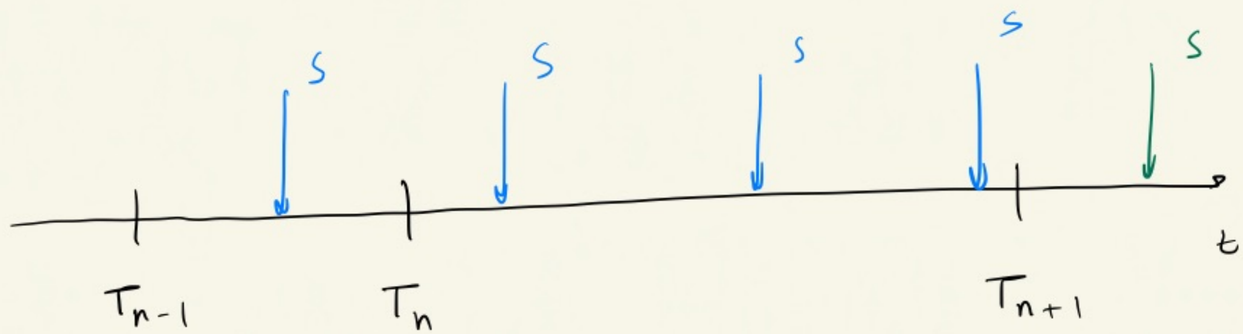
"Discrete-value Continuous Time Processes"

Why is $N(s)$ called a PP rather than an exponential process?

Property 1: $N(s)$ is a Poisson distributed R.V. with mean λs.

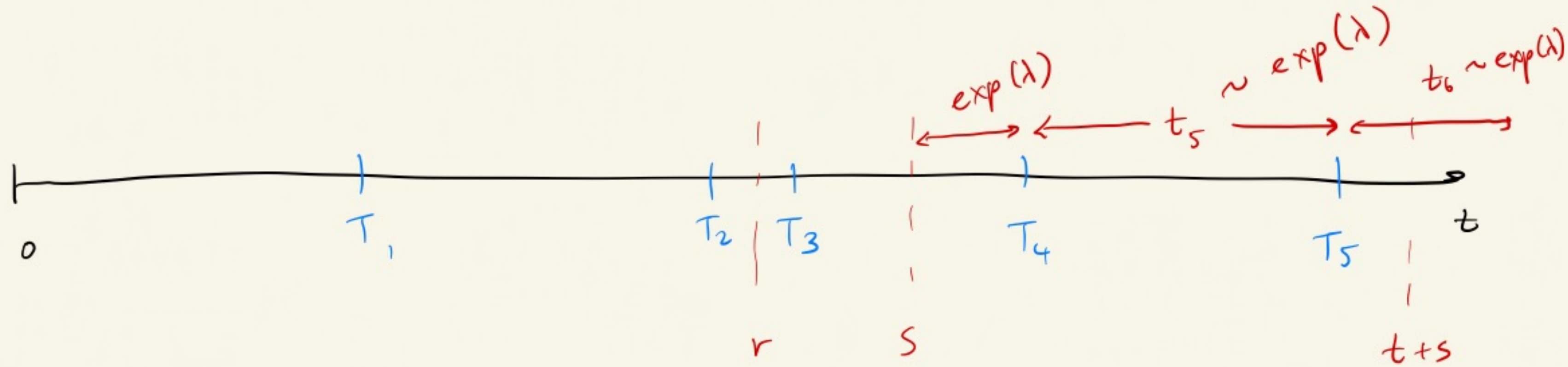
$$\Pr\{N(s) = n\} = \frac{(\lambda s)^n e^{-\lambda s}}{n!}$$

How do we show this?



$$\Pr\{N(s) = n\} \Longleftrightarrow \Pr\{T_n \le s < T_{n+1}\}$$

Property 2: $N(t+s) - N(s)$, for $t \geq 0$, is a PP and is independent of $N(r)$ for $0 \leq r \leq s$.



$N(r) = 2$

$N(s) = 3$

$N(s+t) = 5$

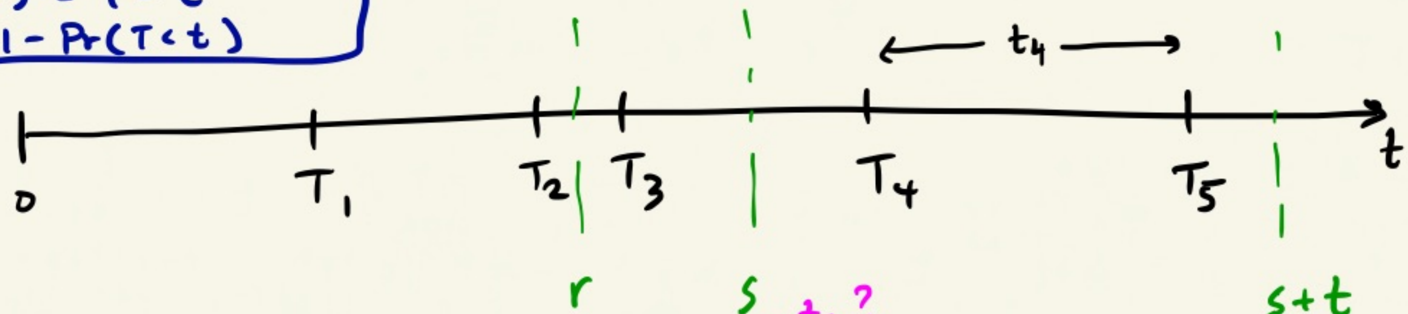
$N(t+s) - N(s) = 2$

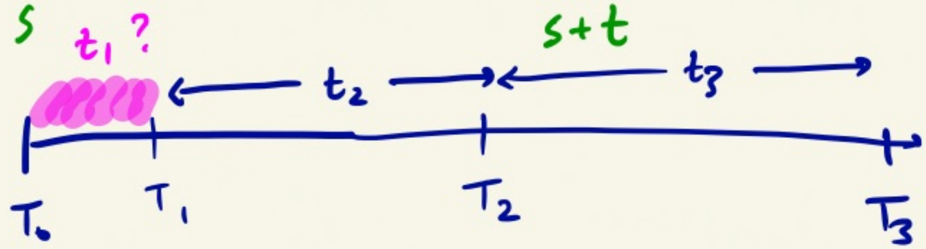
$=$ "Restart property"

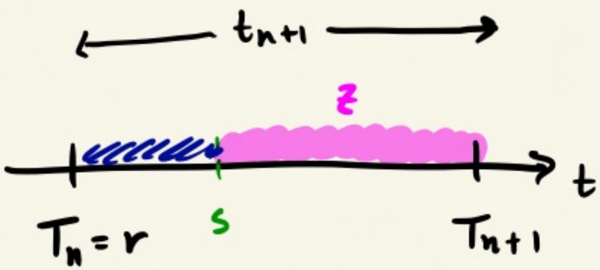
$$N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$$

NOT TESTED; rigorous proof of exponential ISI

$$\Pr(T < t) = 1 - e^{-\lambda t}$$
$$\Pr(T > t) = 1 - \Pr(T < t) = e^{-\lambda t}$$







Is Z exp. distr.?

$$\Pr(Z > z \mid T_n = r, N(s) = n) \overset{?}{=} e^{-\lambda z}$$

$$= \Pr(t_{n+1} > z + (s-r) \mid T_n = r, t_{n+1} > s-r)$$

ind of t_i's

$T_n = t_1 + t_2 + \dots + t_n$

$t_{n+1} \perp t_i$

$$= \Pr(t_{n+1} > z + s - r \mid t_{n+1} > s - r)$$

memoryless

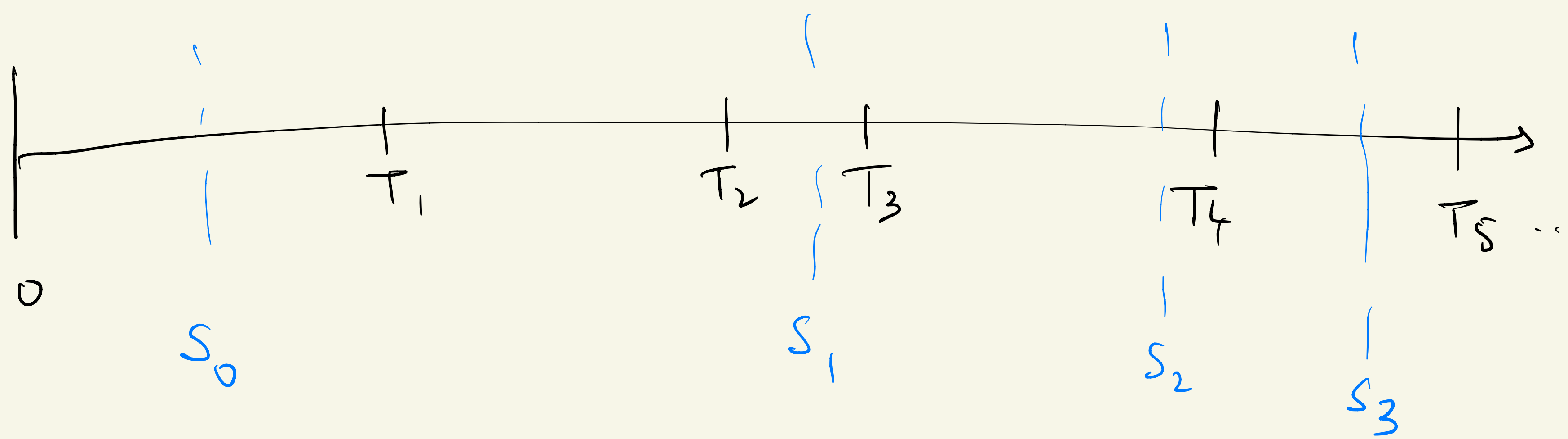
$$= \Pr(t_{n+1} > z)$$

$$= e^{-\lambda z}$$

Property 3: $N(s)$ has independent increments.

If $s_0 < s_1 < s_2 < \cdots < s_n$, then

$$N(s_1) - N(s_0) \perp\!\!\!\perp N(s_2) - N(s_1) \perp\!\!\!\perp N(s_3) - N(s_2) \cdots$$



$$N(s_1) - N(s_0) \sim \text{Poisson}(\lambda(s_1 - s_0))$$

If $\{N(s),\ s \geq 0\}$ is a PP w/ rate λ, then:

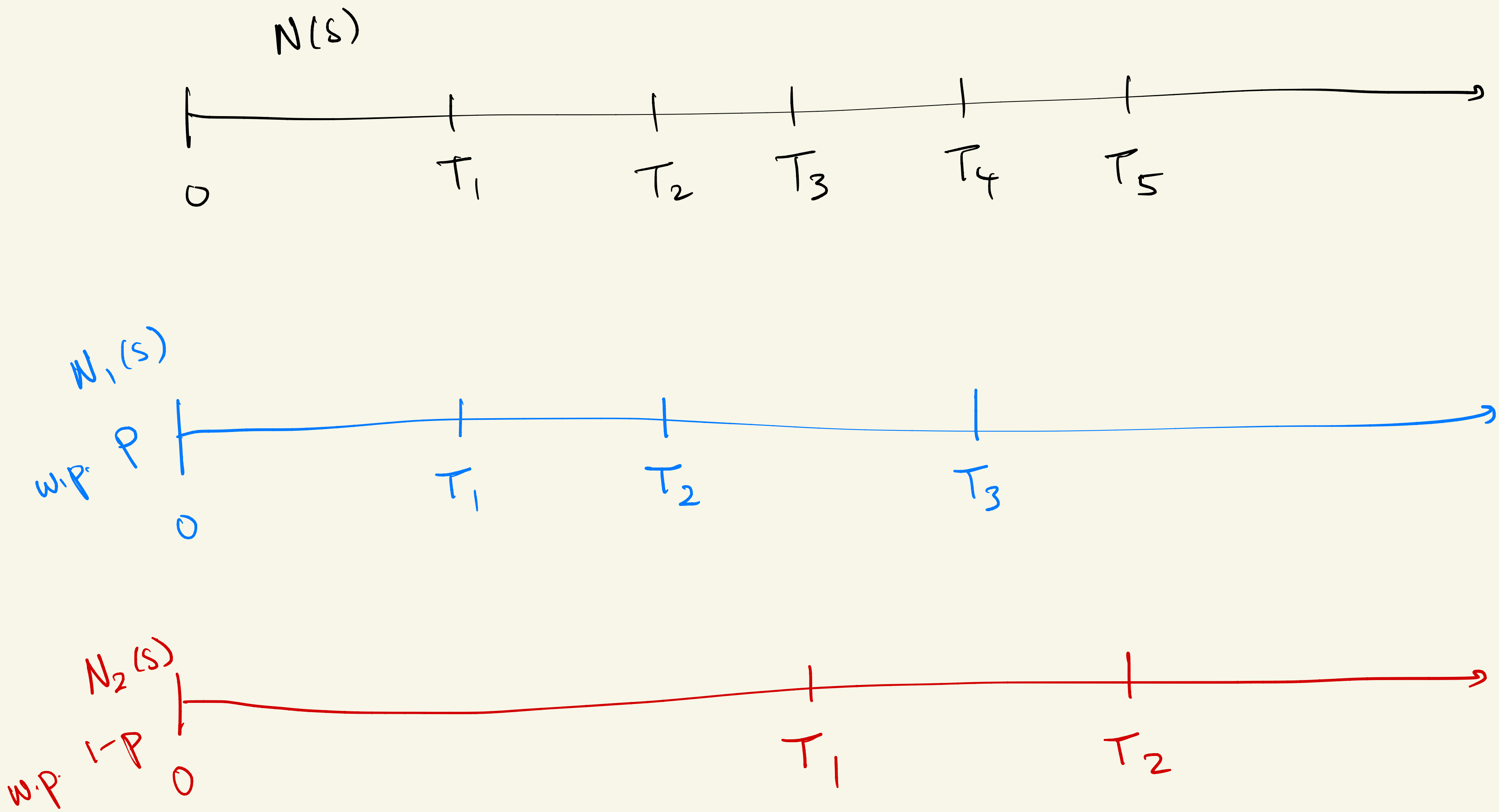
(i) $N(0) = 0$

(ii) $N(t+s) - N(s) \sim \text{Poisson}(\lambda t)$

(iii) $N(s)$ has independent increments

Thinning of a PP

$N(s)$ w/ rate λ



$N_1(s)$ is a PP w/ rate λp

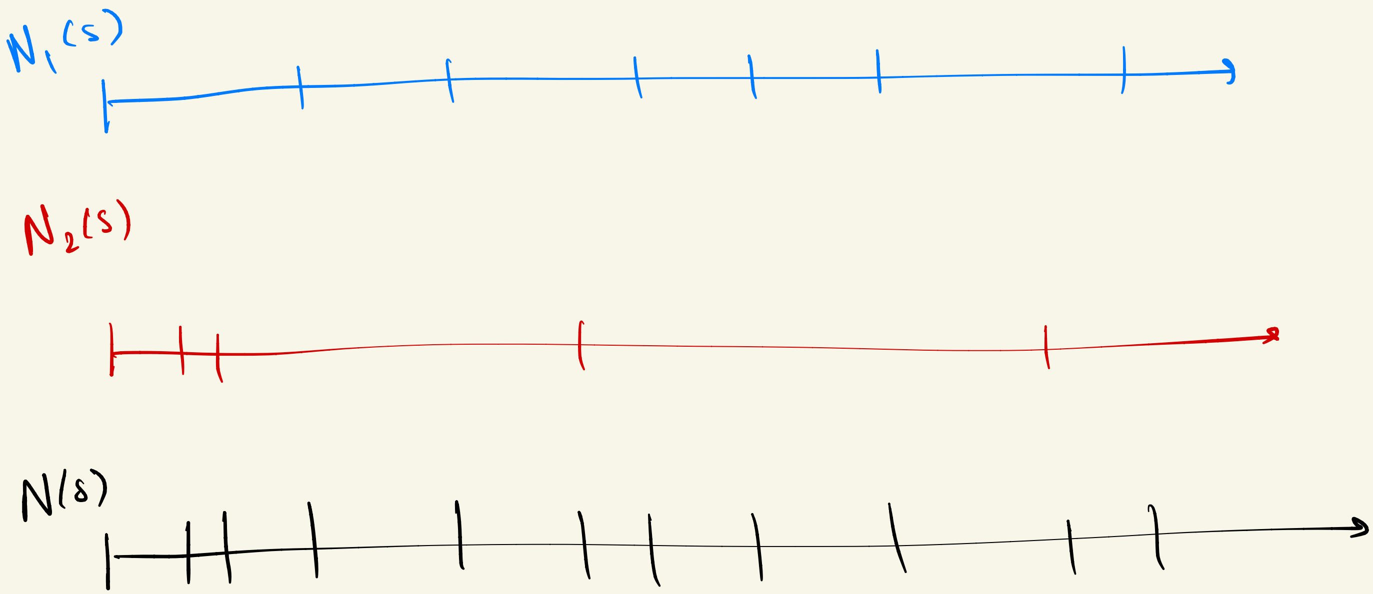
$N_2(s)$ " " " " " $\lambda(1-p)$

Superposition

$N_1(s)$ is a PP w/ rate λ_1

$N_2(s)$ is a " " " λ_2

$N(s) = N_1(s) + N_2(s)$ is a PP w/ rate $\lambda_1 + \lambda_2$

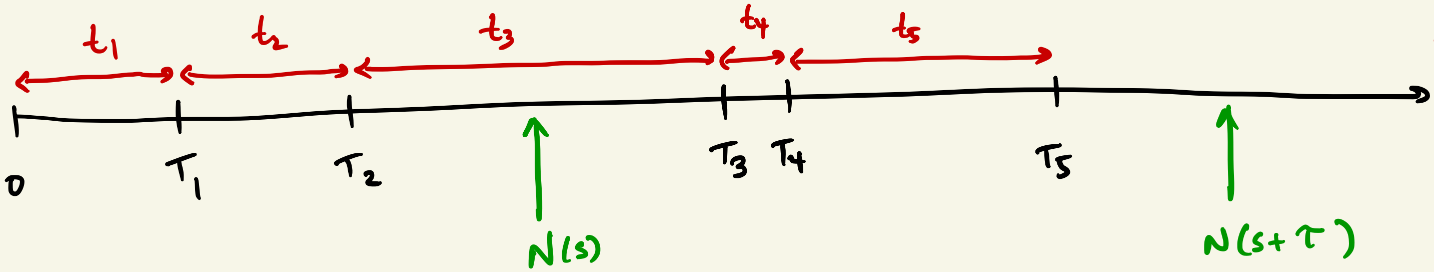


Poisson process summary:

$t_i \sim \exp(\lambda)$

$t_4 \perp\!\!\!\perp t_1$

$t_4 \mid t_1 \overset{(d)}{=} t_4$



1. What are the distr. of …

(a) t_1? $\exp(\lambda)$ (b) T_4? $\sim \text{Gamma}\left(\begin{smallmatrix} n=4, \\ \lambda \end{smallmatrix}\right)$ (c) T_1? $\exp(\lambda)$ (d) $t_4 \mid t_1$? $\exp(\lambda)$

(e) $N(s)$? $\text{Poisson}(\lambda s)$ (f) $N(s+\tau)$? $\text{Poisson}(\lambda(s+\tau))$

2. Are the following statements true?

(a) $t_2 \perp\!\!\!\perp t_3$? Yes

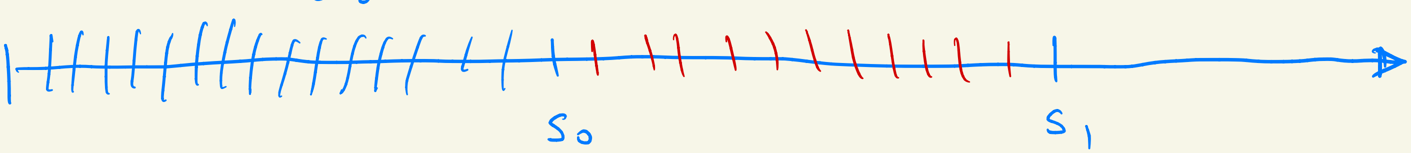
(b) $N(s+\tau) \perp\!\!\!\perp \underbrace{N(s)}^{=2}$? No.

(b) $T_2 \perp\!\!\!\perp T_3$? No.

(d) $N(s+\tau) - N(s) \perp\!\!\!\perp N(s)$ Yes.

$\lambda = 100\text{ Hz}$

$\mathbb{E}[\tau] = \frac{1}{\lambda} = 10\text{ms}$



PP w/ rate λ

$t_i \sim \exp(\lambda)$

$\mathbb{E}[\tau] = \frac{1}{\lambda}$

Inhomogeneous PP's

Key diff: $\lambda(r)$ is a function of time.

Define: $\{N(s),\ s \geq 0\}$ is a inhomogeneous PP w/ rate $\lambda(r)$ if:

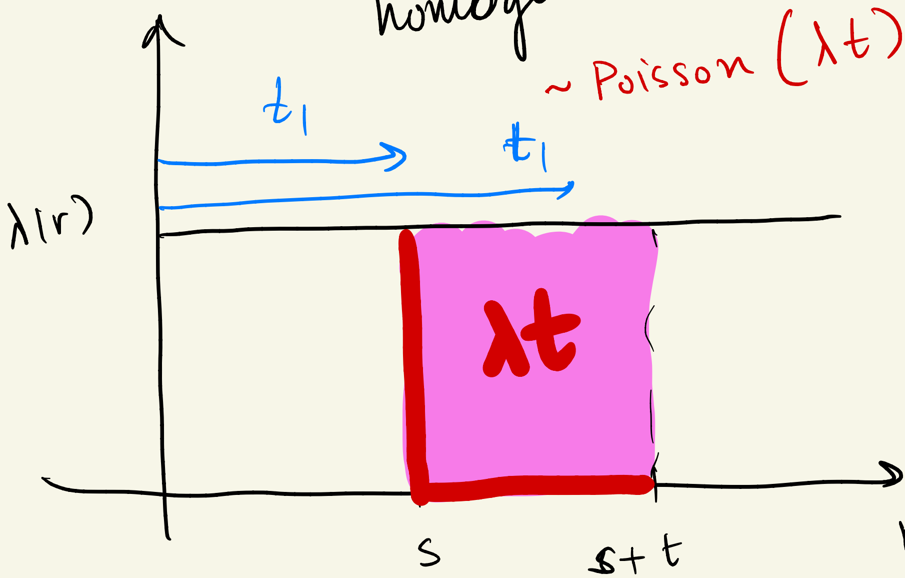
(i) $N(0) = 0$

(ii) $N(t+s) - N(s) \sim \text{Poisson}\left(\int_s^{s+t} \lambda(r)\, dr\right)$

(iii) $N(s)$ has independent increments.

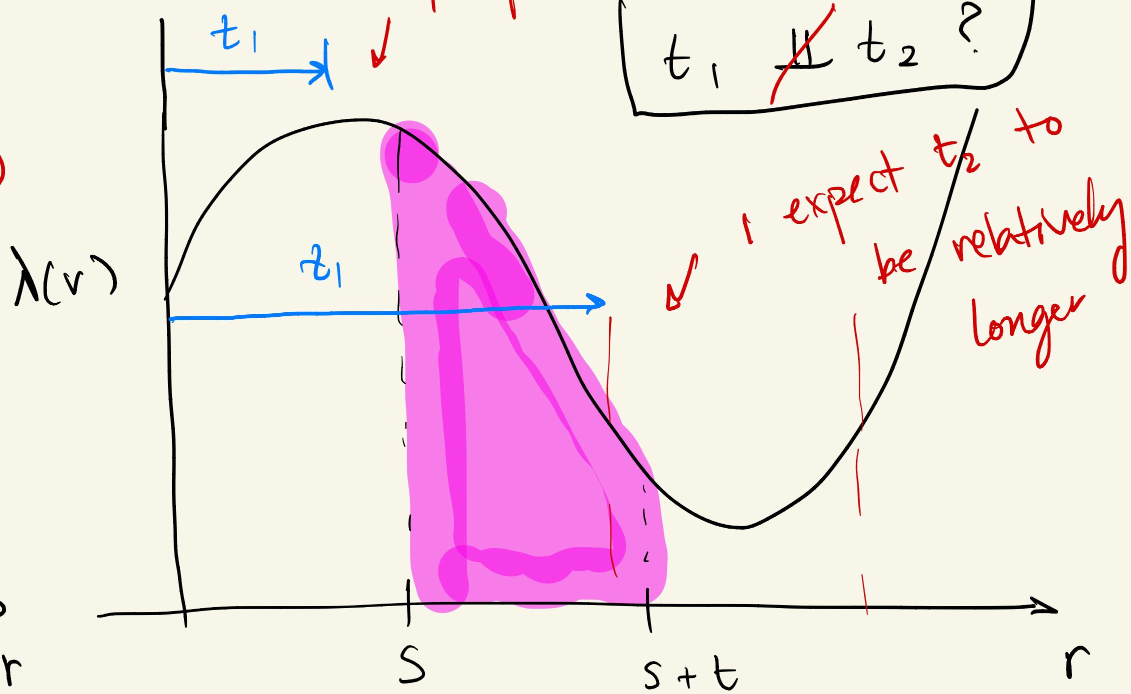
homogeneous PP

$\sim \text{Poisson}(\lambda t)$



I expect t_2 to be short.

$\boxed{t_1 \perp\!\!\!\perp t_2\ ?}$



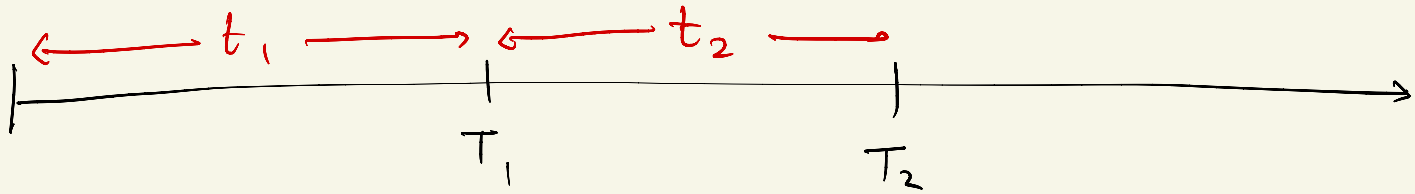
1. Distribution of ISI's ?

(Are they exponential ?)

2. Memoryless ISIs ?

3. Are the ISI's independent ?

(Intuition : No.)



$$f_{t_1} = \frac{d}{dt} F_{t_1} = -\frac{d}{dt}\left(1 - F_{t_1}\right)$$

$F_{t_1}(t) = \Pr(t_1 < t)$

$1 - F_{t_1}(t) = \Pr(t_1 > t)$

if this is true, then $t_1 \sim \exp$ $\;= e^{-\lambda t}$

$$\Pr(t_1 > t) = \Pr(N(t) = 0)$$

$$= \frac{(\mu(t))^0 \cdot e^{-\mu(t)}}{0!}$$

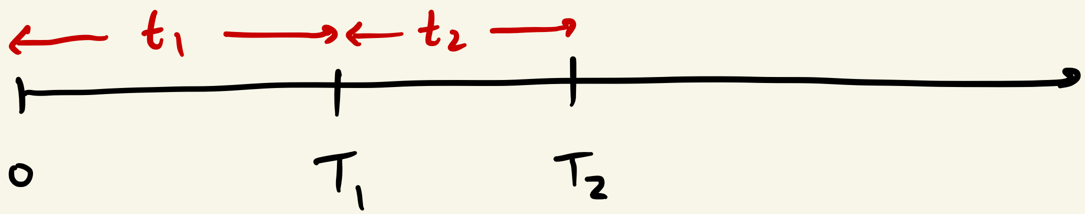
$$\mu(t) = \int_0^t \lambda(r)\,dr$$

$$= e^{-\mu(t)}$$

$$f_{t_1} = -\frac{d}{dt} e^{-\mu(t)} = e^{-\mu(t)} \cdot \lambda(t)$$

exp: pdf is $\lambda e^{-\lambda t}$

$\hookrightarrow$ t_1 is NOT EXPONENTIAL
(NOT MEMORYLESS)



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pw: neuron

$$\Pr(t_2 > s \mid t_1 = t) = ?$$

Is equivalent to which probabilities ?

(a) $\Pr(N(t+s) = 2)$

(b) $\Pr(N(t+s) = 1)$

(c) $\Pr(N(t+s) - N(t) = 0)$

(d) $\Pr(N(t+s) = 0)$

(e) $\Pr(N(t+s) - N(t) = 1)$