

ECE C143A Homework 4

Lawrence Liu

May 20, 2022

Problem 1

(a)

$$P(N = 0) = 1 - 0.25 = \boxed{0.75}$$

(b)

We want $P(N = 0|R = 0)$, we know that

$$P(R = 1|N = 0) = P(E = 0)P(R = 1|E = 0, N = 0) + P(E = 1)P(R = 1|E = 1, N = 0) = 0.01$$

$P(R = 0|N = 0) = 0.99$ and thus from bayes law we have

$$\begin{aligned} P(N = 0|R = 0) &= P(R = 0|N = 0) \frac{P(N = 0)}{P(R = 0)} \\ &= 0.9 \frac{0.1 \cdot 0.75}{P(R = 0)} \end{aligned}$$

To find $P(R = 0)$ we must find $P(R = 1)$,

$$\begin{aligned}
P(R = 1) &= P(E = 1)P(N = 1)P(R = 1|E = 1, N = 1) \\
&\quad + P(E = 1)P(N = 0)P(R = 1|E = 1, N = 0) \\
&\quad + P(E = 0)P(N = 1)P(R = 1|E = 0, N = 1) \\
&\quad + P(E = 0)P(N = 0)P(R = 1|E = 0, N = 0)
\end{aligned}$$

$$P(R = 1) = 0.9 \cdot 0.25 \cdot 1 + 0.1 \cdot 0.75 \cdot 0.1 + 0.1 \cdot 0.25 \cdot 0.1 = 0.235$$

Therefore $P(R = 0) = 1 - P(R = 1) = 0.765$, thus

$$P(N = 0|R = 0) = 0.99 \frac{0.75}{P(R = 0)} = \boxed{0.97}$$

(c)

We have

$$\begin{aligned}
P(N = 0|R = 0, E = 0) &= \frac{P(E = 0, N = 0, R = 0)}{P(R = 0, E = 0)} \\
&= \frac{P(R = 0|E = 0, N = 0)P(E = 0, N = 0)}{P(R = 0, E = 0)} \\
&= \frac{(1 - P(R = 1|E = 0, N = 0))P(E = 0)P(N = 0)}{P(R = 0|E = 0)P(E = 0)} \\
&= \frac{(1 - P(R = 1|E = 0, N = 0))P(N = 0)}{(P(R = 0|E = 0, N = 0)P(N = 0) + P(R = 0|E = 0, N = 1)P(N = 1))} \\
&= \frac{0.9 \cdot 0.75}{0.9 \cdot 0.75 + 0.9 \cdot 0.25} \\
&= \boxed{0.75}
\end{aligned}$$

This intuitively makes sense because the equipment being broken means that a recorded spike provides no additional information.

(d)

Let us consider the case $E = 1, N = 0$ given $R = 1$ we have

$$P(E = 1, N = 0 | R = 1) = \frac{P(R = 1 | E = 1, N = 0)P(E = 1)P(N = 0)}{P(R = 1)}$$

Since $P(R = 1 | E = 1, N = 0) = 0$, we thus have $P(E = 1, N = 0 | R = 1) = 0$. However

$$\begin{aligned} P(E = 1 | R = 1) &= P(R = 1 | E = 1) \frac{P(E = 1)}{P(R = 1)} \\ &= (P(R = 1 | E = 1, N = 0)P(N = 0) + \\ &\quad P(R = 1 | E = 1, N = 1)P(N = 1)) \frac{P(E = 1)}{P(R = 1)} \end{aligned}$$

This therefore $P(E = 1 | R = 1) > 0$, likewise

$$\begin{aligned} P(N = 0 | R = 1) &= P(R = 1 | N = 0) \frac{P(N = 0)}{P(R = 1)} \\ &= (P(R = 1 | E = 1, N = 0)P(E = 1) + \\ &\quad P(R = 1 | E = 0, N = 0)P(E = 0)) \frac{P(N = 0)}{P(R = 1)} \end{aligned}$$

This therefore $P(E = 1 | R = 1) > 0$, therefore, $P(E = 1 | R = 1)P(N = 0 | R = 1) > 0$ and is not equal to $P(E = 1, N = 0 | R = 1)$ therefore they are conditionally dependent. Ie they are not independent given R

Problem 2

(a)

$$P(a, b, c, d) = P(c)P(a|c)P(d)P(b|a, d)$$

(b)

$$\begin{aligned} P(C, D) &= \sum_A \sum_B P(C, D, A, B) \\ &= \sum_A \sum_B P(C)P(D)P(A|C)P(B|A, D) \\ &= P(C)P(D) \sum_A \sum_B P(A|C)P(B|A, D) \\ &= P(C)P(D) \sum_A P(A|C) \sum_B P(B|A) \\ &= P(C)P(D) \end{aligned}$$

Therefore C and D are independent.

(c)

$$\begin{aligned} P(C, D|A, B) &= \frac{P(C, D, A, B)}{P(A, B)} \\ &= \frac{P(C)P(D)P(A|C)P(B|A, D)}{P(A, B)} \\ &= \frac{P(C)P(D)P(A|C)P(B, A, D)}{P(A, B)P(A, D)} \\ &= \frac{P(C)P(D)P(A|C)P(D|B, A)}{P(A, D)} \\ &= \frac{P(C)P(A|C)P(D|B, A)}{P(A)} \\ &= P(C|A)P(D|B, A) \qquad \qquad \qquad = P(C|B, A)P(D|B, A) \end{aligned}$$

Therefore C and D are independent given A and B .

(d)

$$\begin{aligned}P(a, d) &= \sum_B P(a, d, b) \\&= \sum_a P(d)P(b)P(b|d, a) \\&= P(d)P(b) \sum_B P(b|d, a) \\&= P(d)P(b)\end{aligned}$$

Therefore a and d are independent

(e)

$$\begin{aligned}P(a, d|b) &= \frac{P(a, d, b)}{P(b)} \\&= \frac{P(a)P(d)P(b|a, d)}{P(b)} \\&= \frac{P(a)P(d)}{P(b)} \frac{P(b|d)P(a|b, d)}{P(a|d)} \\&= boxed{P(a|b, d)P(d|b)}\end{aligned}$$

Therefore a and d are not independent given b

(f)

$$\begin{aligned}P(c, b) &= \sum_A P(c, b, a) \\&= \sum_A P(c)P(a|c)P(b|a) \\&= P(c) \sum_A P(a|c)P(b|a)\end{aligned}$$

Since $\sum_A P(a|c)P(b|a) \neq P(b)$ in general, c and b are not independent.

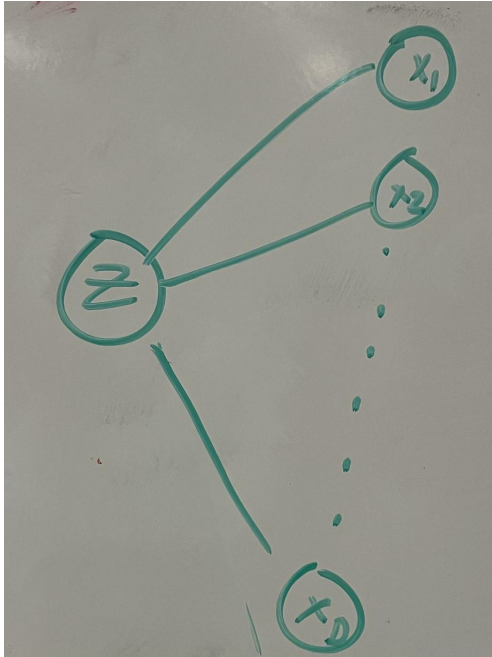
(g)

$$\begin{aligned} P(c, b|a) &= \frac{P(c, b, a)}{a} \\ &= \frac{P(c)P(a|c)P(b|a)}{P(a)} \\ &= P(c|a)P(b|a) \end{aligned}$$

Therefore c and b are independent given a .

Problem 3

(a)



(b)

$$P(x_1, \dots, x_D, z) = \boxed{P(z) \prod_{i=1}^D P(x_i | z)}$$

(c)

Yes they are independent, intuitively thinking for two any two dimensions x_i and x_j , this becomes a graphical model with one parent and two children, which was proved in lecture to be not independent.

Problem 4

(a)

$$P(x_1, x_2, x_3, x_4) = \boxed{P(x_1)P(x_2|x_1)P(x_3|x_2)P(x_4|x_3)}$$

(b)

for x_i we have

$$\begin{aligned} \text{Var}(x_i) &= E[x_i^2] - E^2[x_i] \\ &= E[E[x_i^2|x_{i-1}]] \\ &= E[\sigma^2 + x_{i-1}^2] \\ &= \sigma^2 + E[E[x_{i-1}^2|x_{i-2}]] \\ &\vdots \\ &= i\sigma^2 \end{aligned}$$

And for any i and j such that $i < j$

$$\begin{aligned} \text{Cov}(x_i, x_j) &= E[(x_i - E[x_j])(x_i - E[x_j])] \\ &= E[x_i x_j] \\ &= E[E[x_i x_j|x_{j-1}]] \\ &= E[x_i x_{j-1}] \\ &\vdots \\ &= E[x_i^2] \\ &= i\sigma^2 \end{aligned}$$

Therefore, the covariance matrix Σ is

$$\Sigma = \boxed{\begin{bmatrix} \sigma^2 & \sigma^2 & \sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 & 2\sigma^2 & 2\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 3\sigma^2 \\ \sigma^2 & 2\sigma^2 & 3\sigma^2 & 4\sigma^2 \end{bmatrix}}$$

(c)

From python the inverse of the precision matrix is

$$\Sigma^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(d)

The zeros occur only when the nodes have at least one node in between, therefore these nodes are conditionally independent.