Quantum Mechanics 115C: Homework 1

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1.1 Outer product

In class, we talked about the definition of an inner product in Hilbert space. For this problem, you will examine the outer product in Euclidean space and in a two dimensional Hilbert space.

(a) Suppose we have three Euclidean vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ that span the space. Suppose further that the vectors have the following form:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

We can define the inner product between two vectors as $\vec{w}^T \vec{u} = \sum_i w_i u_i$, where $\vec{w}$ and $\vec{u}$ are two arbitrary vectors. We can also define an outer product by $\vec{w}\vec{u}^T = w_i u_j$. Use this definition of the outer product to show that:

$$\sum_n \vec{v}_n \vec{v}_n^T = \vec{v}_1 \vec{v}_1^T + \vec{v}_2 \vec{v}_2^T + \vec{v}_3 \vec{v}_3^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}_3 \tag{2}$$

where $\mathbf{1}_3$ represents the three-dimensional identity matrix.

(b) Now let's look at the analogous relation in a two-dimensional Hilbert space. By using the vector representations or otherwise, show that for the x-spins:

$$\sum_{n=\uparrow_x,\downarrow_x} |n\rangle \langle n| = \mathbf{1}_2, \tag{3}$$

where $\mathbf{1}_2$ represents the two-dimensional identity matrix.

(c) By using the vector representations or otherwise, show that for the y-spins:

$$\sum_{n=\uparrow_y,\downarrow_y} |n\rangle \langle n| = \mathbf{1}_2, \tag{4}$$

where $\mathbf{1}_2$ represents the two-dimensional identity matrix.

(d) Suppose a spin-1/2 particle has a state vector $|\uparrow_z\rangle$. Apply $\sum_{n=\uparrow_y,\downarrow_y} |n\rangle \langle n|$ to the state vector. What is the result? Express your answer in terms of $|\uparrow_y\rangle$ and $|\downarrow_y\rangle$ only. How do you interpret this result?

(e) The inner product in Hilbert space can be written as $\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \psi^*(x)\phi(x)\, dx$. What is the outer product relation, suitably generalized to the continuous variable "x" in this case, that allows us to go from the left-hand side to the right-hand of the equality?

1.2 Applying the quantum rules to a spin-1 particle

Consider a quantum mechanical system with a single spin-1 degree of freedom. Let $|\epsilon\rangle$ (where $\epsilon = 0, \pm 1$) label the normalized eigenvectors of the z-component of spin, $\hat{S}_z$, You may use without proof the following matrix representation of $\hat{S}_z$ in the $|\epsilon\rangle$ basis:

$$\langle \epsilon' | \hat{S}_z | \epsilon \rangle = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5}$$

Suppose the Hamiltonian of the system is given by:

$$H = \frac{\Omega}{\hbar} (\hat{S}_z)^2 \tag{6}$$

(a) List the possible results that can be obtained from a measurement of the z-component of the spin ($\hat{S}_z$) and of the energy (H) of the system.

(b) Suppose that at $t = 0$, the system is prepared in the state $|\Psi\rangle$ given by:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |-1\rangle) \tag{7}$$

Calculate both the amplitude and probability that at a subsequent time, the system is again found to be in the state $|\Psi\rangle$.

(c) Suppose now that the system is prepared in the state:

$$|\Theta\rangle = \frac{1}{\sqrt{14}} (3\,|1\rangle + 2\,|0\rangle + |-1\rangle) \tag{8}$$

Imagine that a measurement is made of the energy, and the largest possible value is found. What is the normalized state vector immediately after measurement?

(d) Now, immediately after the measurement in part (c), assume that $\hat{S}_z$ is measured. What is the probability that $\hbar$ is obtained?

(e) Had the energy not been measured before the measurement of $\hat{S}_z$, what is the probability that $\hbar$ was obtained?

1.3 Matrix representation of the harmonic oscillator

The purpose of this problem is to illustrate the matrix representation of the harmonic oscillator problem. These are infinite dimensional matrices. Thus, it is sufficient to write down the initial 5×5 matrix in order to see the pattern and put "..." for the remainder of the matrix.

(a) Write out the matrices $\langle n | a_+ | m \rangle$ and $\langle n | a_- | m \rangle$ in the basis where $|n\rangle$ and $|m\rangle$ are stationary states (i.e. in the energy eigenbasis). Are these matrices Hermitean? Why or why not?

(b) Use the matrices computed in part (a) to write out the matrices $\langle n | \hat{x} | m \rangle$ and $\langle n | \hat{p} | m \rangle$. Are these matrices Hermitean? Why or why not?

(c) By squaring the matrices in part (b) or otherwise, write out the matrices $\langle n|\,\hat{x}^2\,|m\rangle$ and $\langle n|\,\hat{p}^2\,|m\rangle$. Are these matrices Hermitean? Why or why not?

(d) Using the results obtained in part (a), write out the matrix $\langle n|\,\hat{H}\,|m\rangle$.

(e) Using the results obtained in part (c), write out the matrix $\langle n|\,\hat{H}\,|m\rangle$, and show that the matrix is the same as that obtained in part (d). Is this matrix Hermitean?

(f) Suppose we represent the 2^{nd} excited of the harmonic oscillator as:

$$|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

What is the outcome when applying the operators, a_+, a_-, $\hat{x}$, $\hat{p}$, $\hat{x}^2$, $\hat{p}^2$ and $\hat{H}$ to $|2\rangle$?

1.4 Dipole moment of a stationary state

The dipole moment is defined as $\mathbf{p} = q\mathbf{r}$. Show that for a system with definite parity (for which the stationary states are either odd or even), that the dipole moment in a stationary state must be zero, i.e. that $\langle n|\,\mathbf{p}\,|n\rangle = 0$, where $|n\rangle$ is a non-degenerate stationary state.

Hints: (1) It may be helpful to think about the one-dimensional case first, and then generalize to three dimensions. (2) The proof should take you no more than a few lines.