Physics 115C HW1

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1 Problem 1

(a)

We have that

$$\vec{v_1}\vec{v_1}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And:

$$\vec{v_2}\vec{v_2}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And:

$$\vec{v_3}\vec{v_3}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore we have that

$$\sum_n \vec{v_n}\vec{v_n}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{1}_3$$

(b)

We have that $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, therefore we have that

$$|\uparrow_x\rangle\langle\uparrow_x| = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

And:

$$|\downarrow_x\rangle\langle\downarrow_x| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Therefore we have that

$$\sum_{n=\uparrow_x,\downarrow_x} |n\rangle\langle n| = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{1}_2$$

(c)

We have that $|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ and $|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$, therefore we have that

$$|\uparrow_y\rangle\langle\uparrow_y| = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix}$$

And:

$$|\downarrow_y\rangle\langle\downarrow_y| = \frac{1}{2}\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}$$

Therefore we have that

$$\sum_{n=\uparrow_y,\downarrow_y} |n\rangle\langle n| = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{1}_2$$

(d)

We have that

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}|\uparrow_y\rangle + \frac{1}{\sqrt{2}}|\downarrow_y\rangle$$

Since $|\uparrow_y\rangle$ is orthogonal to $|\downarrow_y\rangle$, we have that $\langle\uparrow_y|\downarrow_y\rangle = \langle\downarrow_y|\uparrow_y\rangle = 0$. Therefore we have that

$$(|\uparrow_y\rangle\langle\uparrow_y| + |\downarrow_y\rangle\langle\downarrow_y|)\,|\uparrow_z\rangle = \frac{1}{\sqrt{2}}|\uparrow_y\rangle + \frac{1}{\sqrt{2}}|\downarrow_y\rangle$$

This state is a superposition of $|\uparrow_y\rangle$ and $|\downarrow_y\rangle$, with probability $\frac{1}{2}$ for each.

(e)

We have that for a n euclidean vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$, that span a space, we have that any $|\psi\rangle$, can we written as:

$$|\psi\rangle = \sum_{i=1}^{N} |v_i\rangle\langle v_i|\psi\rangle$$

Since $\sum_{i=1}^{N} |v_i\rangle\langle v_i| = \mathbf{1}_n$, therefore

$$|\psi\rangle = \mathbf{1}_n|\psi\rangle = \sum_{i=1}^{N} |v_i\rangle\langle v_i|\psi\rangle$$

If we generalize this to an infinite dimension Hilbert space, with continous variable x, we have that

$$\mathbf{1} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|$$

Therefore we have that

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|\psi\rangle$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle\,\psi(x)$$

And

$$|\phi\rangle = \int_{-\infty}^{\infty} dx\,|x\rangle\,\phi(x)$$

And thus

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\phi(x)dx$$

Problem 2

TODO

Problem 3

(a)

We have that

$$a_+ |n\rangle = \sqrt{n+1}\, |n+1\rangle$$
$$a_- |n\rangle = \sqrt{n}\, |n-1\rangle$$

Therefore

$$\langle m| a_+ |n\rangle = \sqrt{n+1}\, \langle m|n+1\rangle = \sqrt{n+1}\delta_{m,n+1}$$

And

$$\langle m| a_- |n\rangle = \sqrt{n}\, \langle m|n-1\rangle = \sqrt{n}\delta_{m,n-1}$$

Therefore

$$\langle m| a_+ |n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and:

$$\langle m| a_- |n\rangle = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b)

We have that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)$$

Therefore we have that

$$\langle m|\,\hat{x}\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right)$$

$$\langle m|\,\hat{x}\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

And:

$$\langle m|\,\hat{p}\,|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right)$$

$$\langle m|\,\hat{p}\,|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\begin{bmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

These matrices are hermitian since $\langle m|\,\hat{x}\,|n\rangle$ is real and symmetric, and since $\langle m|\,\hat{p}\,|n\rangle$ is purely imaginary and skew-symmetric.

(c)

We have that

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_-^2 + a_+a_- + a_-a_+\right)$$

Thus:

$$\hat{x}^2 |n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\,|n+2\rangle + \sqrt{n(n-1)}\,|n-2\rangle + (n+1)\,|n\rangle + n\,|n\rangle\right)$$

Therefore

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} + \sqrt{n(n-1)}\delta_{m,n-2} + (n+1)\delta_{m,n} + n\delta_{m,n}\right)$$

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{\hbar}{2m\omega}\begin{bmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Likewise we have that:

$$\hat{p}^2 = \frac{\hbar m\omega}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right)$$

$$\hat{p}^2 |n\rangle = \frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\,|n+2\rangle - \sqrt{n(n-1)}\,|n-2\rangle - (n+1)\,|n\rangle - n\,|n\rangle\right)$$

$$\langle m|\,\hat{p}^2\,|n\rangle = -\frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2} - (n+1)\delta_{m,n} - n\delta_{m,n}\right)$$

$$\langle m|\,\hat{p}^2\,|n\rangle = -\frac{\hbar m\omega}{2}\begin{bmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

These matrices are hermitian since they are real and symmetric.

(d)

We have that

$$H = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)$$

Therefore we have that

$$\langle m|\,\hat{H}\,|n\rangle = \hbar\omega\left(\langle m|\,a_+a_-\,|n\rangle + \frac{1}{2}\delta_{m,n}\right)$$

From part (a) we have that:

$$\langle m|\,a_+a_-\,|n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\langle m|\,a_+a_-\,|n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Therefore

$$\langle m|\,\hat{H}\,|n\rangle = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1+\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2+\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3+\frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4+\frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(e)

We have that

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Therefore we have that from part (c):

$$\langle m|\,\hat{H}\,|n\rangle = \frac{1}{2m}\bar{m}\hat{p}^2\,|n\rangle + \frac{1}{2}m\omega^2\bar{n}\hat{x}^2\,|n\rangle$$

$$\langle m|\hat{H}|n\rangle = -\frac{\omega\hbar}{4}\begin{bmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\begin{bmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus we get

$$\langle m|\hat{H}|n\rangle = \frac{\omega\hbar}{4}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 10 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 14 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Which is the same matrix we found in part (d). Since this matrix only has nonzero elements on the diagonal and the values are real, this matrix is hermitian.

(f)

We have that

$$a_+|2\rangle = \sqrt{3}|3\rangle$$
$$a_-|2\rangle = \sqrt{2}|1\rangle$$

Therefore we have that

$$\hat{x}|2\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{3}|3\rangle + \sqrt{2}|1\rangle)$$

$$\hat{p}|2\rangle = i\sqrt{\frac{\hbar m\omega}{2}}(\sqrt{3}|3\rangle - \sqrt{2}|1\rangle)$$

And:

$$\hat{x}^2|2\rangle = \frac{\hbar}{2m\omega}(\sqrt{12}|4\rangle + \sqrt{2}|0\rangle + 5|2\rangle)$$
$$\hat{p}^2|2\rangle = -\frac{\hbar m\omega}{2}(\sqrt{12}|4\rangle + \sqrt{2}|0\rangle - 5|2\rangle)$$

Therefore we have that

$$\hat{H}|2\rangle = \hbar\omega\left(2 + \frac{1}{2}\right)|2\rangle$$

Problem 4