Quantum Mechanics 115C: Homework 2

Spring 2023
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2.1 Infinite square well with delta function perturbation

Consider the infinite square well problem with walls at 0 and L. Imagine that we place a delta function perturbation at the center of the well of the form:

$$H' = \alpha\delta(x - L/2) \tag{1}$$

(a) Use first-order non-degenerate perturbation theory to compute the first order corrections to the energies to all energy levels.

(b) In part **(a)**, you should have found that some states do not shift in energy to first order. Draw the probability density, $|\psi_n^2(x)|$, for the $n = 1, n = 2, n = 3, n = 4$ and $n = 5$ energy levels without the perturbation. From these sketches, explain why, physically, some states did not exhibit an energy shift to first order.

(c) Similarly, can you explain why the delta function potential does induce an energy shift for the other states?

(d) Where would you put the delta function to produce the maximum first order energy correction for the first three states that exhibit zero energy shift?

(e) How does the first-order correction vary as you move the delta function across the well starting at $x = 0$ and moving to $x = L$? Sketch the behavior of the first order correction versus the position of the delta function for the $n = 2$ state.

2.2 Two level system

Consider the Hamiltonian:

$$H = H^0 + H' = \omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda\Omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

(a) What are the energy eigenvalues and eigenstates of the unperturbed Hamiltonian? (*You should be able to just write this down!*)

(b) What are the energy eigenvalues of the full Hamiltonian?

(c) The energy eigenstates of the unperturbed Hamiltonian can be written as $|0\rangle$ and $|1\rangle$. Show that the energy eigenstates of the full Hamiltonian are:

$$|\psi_1\rangle = \cos(\varphi/2)\,|0\rangle + \sin(\varphi/2)\,|1\rangle \qquad \text{and} \qquad |\psi_2\rangle = -\sin(\varphi/2)\,|0\rangle + \cos(\varphi/2)\,|1\rangle \tag{3}$$

where $\varphi = \arctan(\lambda\Omega/\omega)$.

(d) Draw the energy eigenvalues as a function of ω for $\lambda = 0$ and for $\lambda \neq 0$. Make sure to include positive and negative values of ω on your graph. What happens around $\omega \approx 0$? Why does this happen?

(e) For $\lambda \ll 1$, you can expand the results from **(b)** and **(c)** to obtain the lowest order correction to your answers to **(a)**. What are the lowest order corrections to the energy eigenvalues and the energy eigenstates using this procedure?

(f) Now, we will use perturbation theory. To first order in perturbation theory, what is the shift of the energy eigenvalues?

(g) Does your answer to **(f)** make sense when comparing your answer to **(e)**? Why or why not?

(h) What is the first order correction to the energy eigenstates?

(i) Does your answer to **(h)** make sense when comparing your answer to **(e)**? Why or why not?

(j) Calculate the energy shift to second order in perturbation theory. How does your answer compare to your results from **(e)**? Does this make sense? Why or why not?

2.3 Most general two level system

In this problem, we will consider convergence of the most general two-level system. The unperturbed Hamiltonian is:

$$H^0 = \begin{pmatrix} E_a^0 & 0 \\ 0 & E_b^0 \end{pmatrix} \tag{4}$$

and the perturbation is:

$$H' = \lambda \begin{pmatrix} V_{aa} & V_{ab} \\ V_{ba} & V_{bb} \end{pmatrix} \tag{5}$$

with $V_{ab}^* = V_{ba}$, V_{aa} and V_{bb} real, so that H is Hermitian.

(a) Find the exact energies for this two-level system.

(b) Expand your result from **(a)** to second order in λ (and then set $\lambda = 1$).

(c) Setting $V_{aa} = V_{bb} = 0$, show that the series in **(b)** only converges if:

$$\left| \frac{V_{ab}}{E_b^0 - E_a^0} \right| < \frac{1}{2} \tag{6}$$