Physics 115C HW 3

Lawrence Liu

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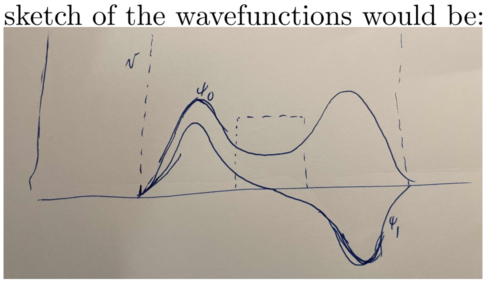
Problem 1

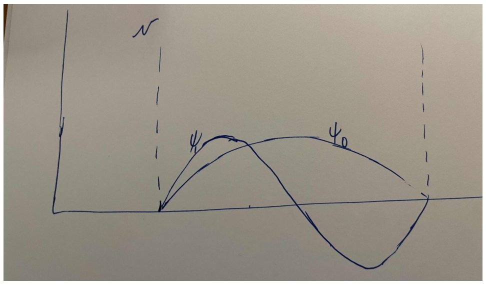
(a)

Since the ground state should have no nodes, we would expect it to be symmetric. And since the first excited state should have be orthogonal to the ground state, we would expect it to be antisymmetric. Therefore a rough sketch of the wavefunctions would be:

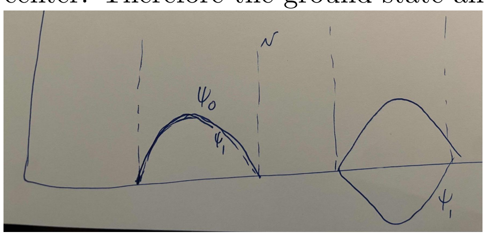


(b)

We have that if the central barrier height is 0, then the problem becomes the infinite square well problem. Therefore the ground state and first excited state are:

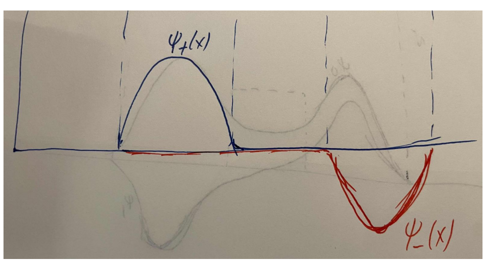


And if the central barrier height is ∞, then the particle cannot exist in the center. Therefore the ground state and first excited state are:



Thus we can see that the splitting will be large when the central barrier is close to 0, and large when the central barrier is close to ∞.

(c)



(d)

The $\psi_\pm(x)$ wavefunctions are not parity symmetric. However the Hamiltonian is parity symmetric.

(e)

The expectation value of $\langle\psi_+|\, x\, |\psi_+\rangle$ and $\langle\psi_-|\, x\, |\psi_-\rangle$ are bot not 0 since $\psi_\pm(x)$ is not parity symmetric. However the expectation value of $\langle\psi_0|\, x\, |\psi_0\rangle$ and $\langle\psi_1|\, x\, |\psi_1\rangle$ are 0 since $\psi_0(x)$ and $\psi_1(x)$ are parity symmetric.

(f)

No in the case of finite barrier height we will have that $|\psi_0\rangle$ and $|\psi_1\rangle$ will no longer be degenerate. Therefore they will evolve at different rates, and thus $|\pm\rangle$ will no longer be stationary states.

(g)

$$|+(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_0t/\hbar}\,|0\rangle + e^{-iE_1t/\hbar}\,|1\rangle\right)$$

(h)

The probability that the particle is in state $|-\rangle$ is given by

$$\begin{aligned}
|\,\langle -|+(t)\rangle\,|^2 &= \frac{1}{4}\left|e^{-iE_0t/\hbar} + e^{-iE_1t/\hbar}\right|^2 \\
&= \frac{1}{4}\left((\cos\left(\frac{E_0t}{\hbar}\right) + \cos\left(\frac{E_1t}{\hbar}\right))^2 + (\sin\left(\frac{E_0t}{\hbar}\right) + \sin\left(\frac{E_1t}{\hbar}\right))^2\right) \\
&= \frac{1}{4}\left(2 + 2\cos\left(\frac{(E_0 - E_1)t}{\hbar}\right)\right)
\end{aligned}$$

Therefore we can see that the first time particle will turn into state $|-\rangle$ is when $t = \frac{\pi\hbar}{E_1 - E_0}$.

(i)

As we raise the barrier height, the splitting between the ground state and the first excited state will decrease therefore the time it takes for the particle to turn into state $|-\rangle$ will increase. And when the barrier height becomes infinite, the particle will never turn into state $|-\rangle$.

(j)

$$\frac{2\pi\hbar}{E_1 - E_0} = \frac{1}{24 \cdot 10^9}$$
$$E_1 - E_0 = 1.590 \cdot 10^{-23} J$$

The wavelength of the light emited is given by

$$\lambda = \frac{hc}{E_1 - E_0} = 0.012m$$

Therefore we can see that the wavelength of the light emited is in the microwave range.

(k)

The time it would take to tunnel is

$$\frac{1}{2 \cdot 160\mu\text{Hz}} = 3125s$$

This tunneling time is much longer than Ammonia, therefore we can conclude that the barrier height for AsH_3 is much higher than that of Ammonia.

(l)

It would depend how long the measurement is taken. Both atoms have an "instantenous" dipole, ie if we measure instantenously, we will find that the dipole is non-zero, since the As or N atom will be on one side of hydrogen plane. But when the As or N atom oscillates to the other side of the hydrogen plane, the dipole will flip. So the net dipole if we measure for a long time (on the order of days) will be 0.

Problem 2

(a)

From the periodic boundary condition we must have that:

$$\psi(x) = \psi(x+L)$$
$$Ne^{ikx} = Ne^{ik(x+L)}$$
$$e^{ikL} = 1$$
$$kL = 2\pi n$$
$$\hbar k = \frac{2\pi\hbar n}{L}$$

(b)

We must have that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$
$$\int_{0}^{L} |\psi(x)|^2 dx = 1$$
$$\int_{0}^{L} N^2 dx = 1$$
$$N = \frac{1}{\sqrt{L}}$$

(c)

The energy eigenvalues are

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (2\pi n)^2}{2mL^2}$$

Therefore we can see that the $+n$ and $-n$ energy eigenvalues are the same, and thus are degenerate.

(d)

The perturbation can be written as

$$H' = -\beta p$$

$$H' = -\beta \frac{\hbar}{i} \frac{d}{dx}$$

We can see that this commutes with the hamiltonian, $H = \frac{1}{2m} p^2$. Therefore the eigenstates we would use would be the energy eigenstates of the original unperturbed hamiltonian.

(e)

We have that the first order correction to the energy is given by

$$E_n^1 = \langle n| H' |n\rangle$$

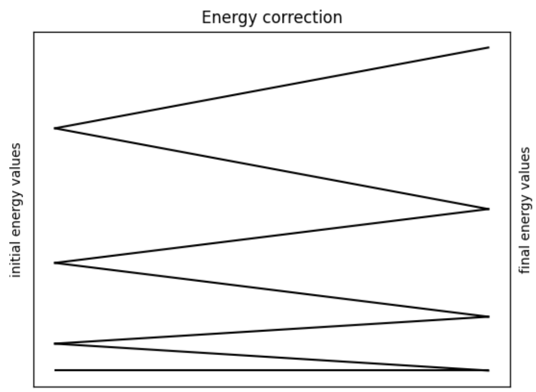
$$E_n^1 = -\beta \langle n| \frac{\hbar}{i} \frac{d}{dx} |n\rangle$$

$$E_n^1 = -\beta \frac{\hbar}{i} \int_0^L \psi_n^*(x) \frac{d}{dx} \psi_n(x) dx$$

$$E_n^1 = -\beta \frac{\hbar}{L} k_n$$

$$E_n^1 = \boxed{-\beta \frac{2\pi\hbar n}{L}}$$

(f)



As we can see the lifting of degeneracy does not affect each pair in the same way, the higher n states have a larger energy shift than the lower n states.

(g)

Yes there is a degeneracy in the system, when $\beta_z = \frac{2\pi\hbar z}{2mL}$ the zth excited state will be degenerate with the unperturbed ground state.

(h)

No because the perturbation is too big, so we cannot use perturbation theory.

(i)

Yes because the perturbation has the same eigenstates to the unperturbed hamiltonian, therefore we would have that the new energy eigenvalues E_n would just be given by

$$E_n = \langle n| H + H' |n\rangle$$

$$E_n = \langle n| H |n\rangle + \langle n| H' |n\rangle$$

THerefore we can see the correction to the energy eigenvalue:

$$E_n - \langle n| H |n\rangle = \langle n| H' |n\rangle$$

Which is the same value we derived with perturbation theory.

(j)

We have that:

$$\left|\psi_n^1\right\rangle = \sum_{m \neq n} \frac{\langle m| H' |n\rangle}{E_n - E_m} |m\rangle$$

Since $|n\rangle$ for all n are eigenstates of H', we have that $\langle m| H' |n\rangle = 0$ for all $m \neq n$. Therefore we have that

$$\left|\psi_n^1\right\rangle = 0$$

Which is what we expected from the fact that the perturbation commutes with the hamiltonian.

(k)

We have that the second order correction to the energy is given by

$$E_n^2 = \sum_{m \neq n} \frac{|\langle m| H' |n\rangle|^2}{E_n - E_m}$$

As we noted before $\langle m| H' |n\rangle = 0$ for all $m \neq n$. Therefore we have that

$$E_n^2 = 0$$

This is what we expected from the fact that the perturbation commutes with the hamiltonian and thus how the energy correction from first order perturbation theory is the exactly what the energy correction is when we calculate the energy eigenvalue directly. Likewise, we would expect the higher order corrections to be 0 as well for the same reason.

Problem 3

(a)

We have that

$$\begin{aligned}\langle 1| H' |1\rangle &= \frac{1}{L}\int_0^L V\sin\left(\frac{4\pi x}{L}\right)dx \\ &= 0\end{aligned}$$

Likewise

$$\begin{aligned}\langle -1| H' |-1\rangle &= \frac{1}{L}\int_0^L V\sin\left(\frac{4\pi x}{L}\right)dx \\ &= 0\end{aligned}$$

We also have that

$$\begin{aligned}\langle 1| H' |-1\rangle &= \frac{1}{L}\int_0^L e^{-ik_1x}V\sin\left(\frac{4\pi x}{L}\right)e^{-ik_1x}dx \\ &= \frac{V}{2Li}\int_0^L (e^{\frac{i4\pi x}{L}} - e^{-\frac{i4\pi x}{L}})e^{-i\frac{4\pi}{L}x}dx \\ &= \frac{V}{2Li}\int_0^L (1 - e^{-\frac{i8\pi x}{L}})dx \\ &= \frac{V}{2Li}\left(x + \frac{L}{8i\pi}e^{-\frac{i8\pi x}{L}}\right)\Big|_0^L \\ &= \frac{V}{2Li}\left(L + \frac{L}{8i\pi} - \frac{L}{8i\pi}\right) \\ &= \frac{V}{2i}\end{aligned}$$

Likewise we have that

$$\begin{aligned}
\langle -1| H' |1\rangle &= \frac{1}{L}\int_0^L e^{ik_1x} V \sin\left(\frac{4\pi x}{L}\right) e^{ik_1x} dx \\
&= \frac{V}{2Li}\int_0^L (e^{\frac{i4\pi x}{L}} - e^{-\frac{i4\pi x}{L}}) e^{i\frac{4\pi}{L}x} dx \\
&= \frac{V}{2Li}\int_0^L (e^{\frac{i8\pi x}{L}} - 1) dx \\
&= \frac{V}{2Li}\left(\frac{L}{8i\pi} e^{\frac{i8\pi x}{L}} - x\right)\Bigg|_0^L \\
&= -\frac{V}{2i}
\end{aligned}$$

Therefore we have that

$$\begin{bmatrix} 0 & \frac{V}{2i} \\ -\frac{V}{2i} & 0 \end{bmatrix} = \frac{V}{2i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\pm i$. Therefore we have that the first order correction to the energy is $\pm\frac{V}{2}$.

(b)

The eigenvectors of the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ are $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$. Therefore we have that the normalized good states are:

$$\frac{1}{\sqrt{2}}(|-1\rangle \pm i\,|1\rangle)$$

(c)

Because $\langle 2|2\rangle = \langle -2|-2\rangle = 1$ we still have that

$$\langle 2| H' |2\rangle = \langle -2| H' |-2\rangle = 0$$

However we have that

$$\begin{aligned}
\langle -2| H' |2\rangle &= \frac{1}{L}\int_0^L e^{ik_2x} V \sin\left(\frac{4\pi x}{L}\right) e^{-ik_2x} dx \\
&= \frac{V}{2Li}\int_0^L (e^{\frac{i4\pi x}{L}} - e^{-\frac{i4\pi x}{L}}) e^{-i\frac{8\pi}{L}x} dx \\
&= 0
\end{aligned}$$

And since $(\langle -2| H' |2\rangle)^* = \langle 2| H' |-2\rangle$ we have that $\langle 2| H' |-2\rangle = 0$ as well. Therefore we have that the first order shift to the second lowest energy level is 0.

(d)

The energy split by V for the lowest energy degenerate state and 0 for second highest energy degenerate state.

(e)

The highest energy degenerate states should not be lifted as well, since the we would have that

$$\langle -n| H' |n\rangle = \frac{V}{2iL}\int_0^L e^{i2k_nx}(e^{i4\pi x/L} - e^{-i4\pi x/L}) dx = 0$$

if $n > 1$ then we will have that this integral becomes the integral of

$$e^{i\pi\frac{x}{L}(4-2n)} - e^{i\pi\frac{x}{L}(-4-2n)}$$

Which for $n > 1$ is periodic with L and thus the integral is 0. And as we noted before since $(\langle -n| H' |n\rangle)^* = \langle n| H' |-n\rangle$ we have that $\langle n| H' |-n\rangle = 0$ as well. And we will have that $\langle -n| H' |-n\rangle = \langle n| H' |n\rangle = 0$ as well. Therefore we have that the first order correction to the energy is 0 for all n.

(f)

If we changed the wavelength of the perturbation to $\lambda = \frac{L}{2^n}$ then we would split the n degenerate state instead.