Quantum Mechanics 115C: Homework 8

Spring 2023
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8.1 A quantum description of photon experiments

We talked about the Mach-Zehnder interferometer in class, and how a single photon can seemingly exhibit interference. In this problem, we will use a more quantum mechanical language to describe this interferometer, so that it applies to a single photon. Let us write the incoming states of the photons as $|i_1\rangle$ and $|i_2\rangle$. In addition, let us write the output photon states as $|o_1\rangle$ and $|o_2\rangle$, as shown Fig. 1. The beamsplitter is a 50/50 beamsplitter.

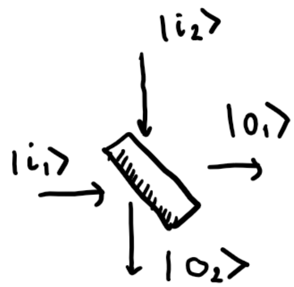

Figure 1: States of the beam splitter in quantum mechanical language.

(a) Using the beam splitter matrix we obtained in class, write down the analogous matrix equation using the quantum state formalism. Write out the $|i_1\rangle$ and $|i_2\rangle$ in terms of $|o_1\rangle$ and $|o_2\rangle$. Note that the hatched part of the beam splitter is half-silvered. Imagine a single photon in state $|i_1\rangle$ before the beam splitter. In words, how do you describe the state after the beam splitter?

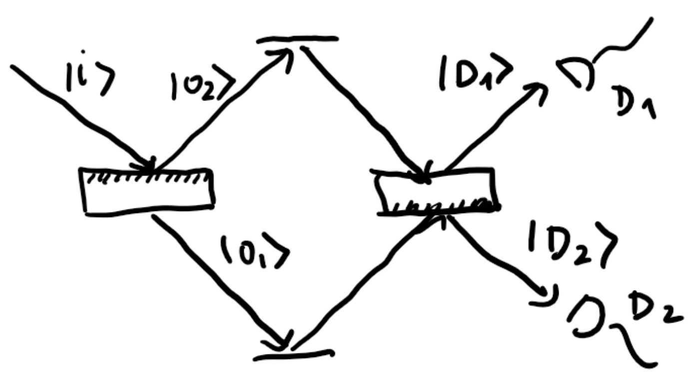

Figure 2: Mach-Zehnder interferometer in quantum mechanical language.

(b) Let us now turn to the Mach-Zehnder interferometer, as drawn schematically in Fig. 2, where the horizontal lines represent mirrors. Here, $|D_1\rangle$ and $|D_2\rangle$ are the states entering detectors D_1 and D_2 respectively. Assume that the upper and lower paths have the same length. Write $|D_1\rangle$ and $|D_2\rangle$ in terms of $|o_1\rangle$ and $|o_2\rangle$. Then write $|D_1\rangle$ and $|D_2\rangle$ in terms of $|i\rangle$.

(c) By taking $|\langle i|D_1\rangle|^2$ and $|\langle i|D_2\rangle|^2$, write down the probability that detector D_1 will click and the probability that detector D_2 will click.

(d) Suppose instead that the initial photon was shone from below at the other input port, as depicted in Fig. 3. In this case, what are the probabilities of D_1 and D_2 clicking?

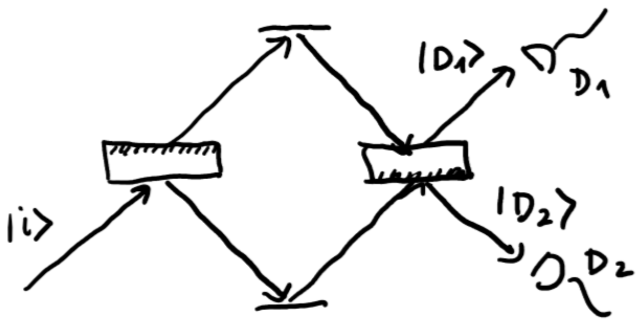

Figure 3: Mach-Zehnder interferometer with initial photon coming in from the bottom port.

8.2 Decay rate of the hydrogen atom

In this problem, you will calculate one of the most famous results using Fermi's Golden Rule, the lifetimes (the inverse of the decay rate) of the various $n = 2$ states of the hydrogen atom back to the $n = 1$ ground state in the dipole approximation. To do this, you will have to calculate several matrix elements, but many of these will be zero. You can use problems 6.2 and 6.3 as a guide here.

(a) What are all the matrix elements that need to be calculated in this problem? How many are there? (Consider each component of $\mathbf{r}$ as a different matrix element).

(b) Use the result from a previous homework, where you calculated a set of these matrix elements, to simplify the z matrix elements. Which matrix element(s) are non-zero?

(c) Calculate the remaining matrix elements. It may help to note that you can calculate the matrix elements like $\langle 100| x |21 \pm 1\rangle$ simultaneously, as the matrix elements only differ in sign.

(d) Use the expression derived in class for the spontaneous decay rate to obtain a lifetime for each of the $n = 2$ states to the $n = 1$ state (i.e. $|200\rangle \to |100\rangle$, $|210\rangle \to |100\rangle$, $|211\rangle \to |100\rangle$ and $|21-1\rangle \to |100\rangle$).

(e) The experimentally obtained lifetimes of the following decay processes $|210\rangle \to |100\rangle$, $|211\rangle \to |100\rangle$ and $|21-1\rangle \to |100\rangle$, are roughly 1.6 ns, whereas the $|200\rangle \to |100\rangle$ decay process takes 2.4 ms. How close are your answers to the experimentally obtained lifetimes? If you obtained any significant discrepancies between your calculations and experimentally obtained values, can you account for them? Why might the theoretical and experimental values be different?

8.3 Center of mass and relative mass wavefunctions

Imagine a two-particle Hamiltonian of the following form:

$$H = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V(|\mathbf{x}_1 - \mathbf{x}_2|)$$

which is a generalized form of the hydrogen atom Hamiltonian. Notice that the potential is only dependent on the distance between the two particles, as would be for the Coulomb interaction between the proton and electron. In this problem we are going to perform a canonical transformation to the center of mass and relative mass coordinate system.

(a) By defining the following quantities,

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \qquad \text{and} \qquad \mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2},$$

and

$$\mathbf{p} = \frac{m_2}{m_1 + m_2}\mathbf{p}_1 - \frac{m_1}{m_1 + m_2}\mathbf{p}_2 \qquad \text{and} \qquad \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2,$$

Show that the Hamiltonian can be written as:

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} + V(\mathbf{x}),$$

where

$$M = m_1 + m_2 \qquad \text{and} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

(b) Show that $[\mathbf{X}_i, \mathbf{P}_j] = i\hbar\delta_{ij}$ and that $[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij}$, where $i, j = x, y, z$. Since the commutation relations are identical after performing this transformation, we call this transformation a canonical transformation.

(c) Because the Hamiltonian can be separated into a center of mass degrees of freedom and its relative degrees of freedom, we can use separation of variables to solve the Hamiltonian. Using the wavefunction

$$\Psi(\mathbf{X}, \mathbf{x}) = \Psi_{CM}(\mathbf{X})\Psi_{rel}(\mathbf{x})$$

write down the Schrodinger equation for this problem.

(d) Now divide the Schrodinger equation by $\Psi_{CM}(\mathbf{X})\Psi_{rel}(\mathbf{x})$ and show that one term only depends on $\Psi_{CM}(\mathbf{X})$ and the other only depends on $\Psi_{rel}(\mathbf{x})$.

(e) Because these terms must add to give the total energy E, these terms must both be separately equal to constants. By equating the first term to a constant E_{CM} and the second term to a constant E_{rel}, show that we get two independent Schrodinger equations for the center of mass and the relative degrees of freedom of the form:

$$\frac{\mathbf{P}^2}{2M}\Psi_{CM}(\mathbf{X}) = E_{CM}\Psi_{CM}(\mathbf{X})$$

and

$$\left[\frac{\mathbf{p}^2}{2\mu} + V(\mathbf{x})\right]\Psi_{rel}(\mathbf{x}) = E_{rel}\Psi_{rel}(\mathbf{x})$$

with $E = E_{CM} + E_{rel}$.

(f) By taking into account the mass of the proton and electron, what is the reduced mass for the hydrogen atom? Are we justified in ignoring the proton mass in the Schrodinger equation for the relative coordinates?

Optional: In principle, what we did here for two particles could be performed with many more particles, as long as the potential only depends on the relative distance between particles. If this can always be done, we can always write down a Schrodinger equation for the center of mass of the many-particle system. Why then do we not see wave-like properties at the level of macroscopic systems? (There is no right answer here; you are free to just write down your thoughts.)

8.4 Extra Credit: Hong-Ou-Mandel effect

Imagine two photons incident on a beam splitter, as schematically depicted in Fig. 1. In this problem we will construct two-photon states using the prescription $|\psi\rangle = |i_1\rangle\,|i_2\rangle$, as we did for the orbital part of the helium atom (refer to those notes if necessary).

(a) Consider two *distinguishable* photons incident on the beam splitter. How many possible outcomes are there in this experiment? Write down the two-photon state after the beam splitter in terms of $|r_1\rangle$, $|t_1\rangle$, $|r_2\rangle$ and $|t_2\rangle$, where these states label the reflection and transmission states of photons 1 and 2 respectively. Make sure the state is appropriately normalized.

(*Hint: If the photons are distinguishable, the reflected and transmitted states will also be distinguishable. This is why we label them $|r_1\rangle$, $|t_1\rangle$, $|r_2\rangle$ and $|t_2\rangle$.*)

(b) Suppose we now place detectors at the output ports of the beam splitter, as depicted in Fig. 4. Using the state you wrote down in part **(a)**, calculate the probability that D_1 will click once *and* D_2 will click once in a particular measurement. Calculate the probability that D_1 will click twice. Make sure you write down the inner product you used to calculate these probabilities.

(c) Now consider the case where the photons are *indistinguishable.* For this to be the case, the photons have to arrive at the same location and simultaneously on the beam splitter with the same polarization, wavelength, etc. The two photons must be truly indistinguishable. In this case, the distinction between, say, $|r_1\rangle$ and $|t_2\rangle$ disappears. Write down the two-photon state after the beam splitter in terms of $|D_1\rangle$ and $|D_2\rangle$. Make sure the state is properly normalized.

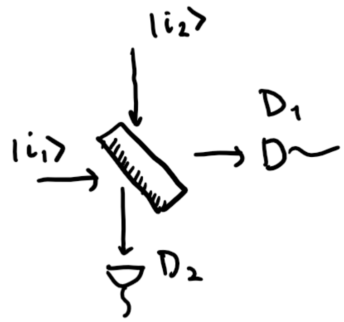

Figure 4: Two photons incident on the beam splitter with detectors.

(d) What is the probability now of obtaining one photon at D_1 and one photon at D_2? What is the probability of both photons arriving at D_1? What is the probability of both photons arriving at D_2?

(e) Think about what your answer means: does a single photon always interfere with itself, or can it interfere with another photon? Can the photons interfere in this experiment if they are distinguishable?

(f) Suppose we had set this experiment up in a way similar to the Aspect, Roger, Grangier experiment of 1986, where we connected the detectors D_1 and D_2 to a coincidence counter and only two indistinguishable photons arrive at the beam splitter (one from each input port) simultaneously. What would we obtain for the value of P_c, where P_c is the probability that both detectors click simultaneously? What value does this imply for the value of $\alpha \propto P_c$? Note that α is defined for this problem as:

$$\alpha = \frac{\text{Probability of } D_1 \text{ and } D_2 \text{ clicking simultaneously}}{(\text{Probability of } D_1 \text{ clicking}) \times (\text{Probability of } D_2 \text{ clicking})} = \frac{Pc}{P_{D1}P_{D2}} \tag{1}$$