Quantum Mechanics 115C: Homework 6

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6.1 Rabi oscillations from last homework re-visited

For this problem, please refer to problems (5.3) and (5.4) from the previous homework set.

In problem (5.3), we obtained a solution for the Rabi oscillation problem at slight detuning (i.e. not exactly at resonance). Remember that we got that:

$$|c_2(t)|^2 = \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2(\Omega t/2)$$

where $\Omega^2 = \delta\omega^2 + \gamma^2/\hbar^2$.

In problem (5.4)(b), on the other hand, you obtained a very similar-looking expression for the probability of measuring the system in the excited state at time t, given that the system was in the ground state at $t = 0$, $P_{0\to 1}(t) = (\lambda^2/\Delta^2)\sin^2\Delta t$.

(a) Can you map some variables to others when comparing the two problems? For instance, what are the relationships between γ, $\delta\omega$, and Ω from problem (5.3) and λ, ω and Δ from problem (5.4)?

Hint: Be careful of factors of two.

(b) Using the mapping you obtained in part (a), write down the corresponding time-independent $\tilde{H}_0$ and $\tilde{H}_1$ for problem (5.3) in analogy to problem (5.4). What would $\tilde{H}_0$ look like were the radiation perfectly at resonance? What are the eigenvalues and eigenvectors for $\tilde{H} = \tilde{H}_0 + \tilde{H}_1$ at resonance? Notice that in problem (5.3), we solved a manifestly time-dependent Hamiltonian, whereas in problem (5.4), we were solving a time-independent problem (apart from turning on H' at $t = 0$, which could have been on for the whole time without any change in result).

Optional: Can you think about why these problems end up looking so similar? It may be useful to think about the concept of "dressed states", which we briefly touched on in class, or to looks up this concept elsewhere. How do you then interpret the eigenvectors of part (b)?

(c) Plot the maximum value of $|c_2(t)|^2$ given above as a function of ω for $t = \pi/\Omega$. Remember that $\delta\omega = \omega_{21} - \omega$. Is this a functional form that you recognize? What is its name?

6.2 Hydrogen atom between capacitor plates

A hydrogen atom is in the ground state ($n = 1$, $l = m = 0$) for $t < 0$. For this problem, you can neglect spin. Suppose the atom is placed between the plates of a capacitor, and a weak, spatially uniform but time-dependent decaying field is applied at $t = 0$. The field (for $t > 0$) is:

$$\mathbf{E} = \mathbf{E}_0 e^{-\gamma t}$$

for some $\gamma > 0$. Take $\mathbf{E}_0$ along the z-axis, so that $\mathbf{E}_0 = |\mathbf{E}_0|\hat{z}$. What is the probability, to first order in $\mathbf{E}_0$ that the atom will be in each of the four $n = 2$ states as $t \to \infty$? You can express your answer in terms of the electric field strength, $|\mathbf{E}_0|$, the decay time, γ, the electron charge, e, fundamental constants like ϵ_0 and $\hbar$, and the Bohr radius, a.

Hint (1): Many of the integrals will be zero, so you don't have to perform unnecessary ones.

Hint (2): Only take the $t \to \infty$ limit at the end of the problem.

6.3 Electric dipole selection rules

As you've probably noticed, matrix elements like $\langle n'l'm'| x |nlm\rangle$, $\langle n'l'm'| y |nlm\rangle$ and $\langle n'l'm'| z |nlm\rangle$ show up a lot when using time-dependent perturbation theory. This is because the dipole operator is proportional to $\mathbf{r}$. In this problem, you will figure out a general rule of thumb to immediately tell whether these matrix elements are zero.

(a) Calculate the following commutation relations: $[L_z, x]$, $[L_z, y]$, and $[L_z, z]$.

(b) Insert the $[L_z, z]$ commutation relation between the two states $|n'l'm'\rangle$ and $|nlm\rangle$. By noting that $L_z |nlm\rangle = \hbar m |nlm\rangle$, prove that either the matrix element is zero or $m = m'$. Explain in words what this means physically when it comes to transitions between states $|n'l'm'\rangle$ and $|nlm\rangle$ with an electric field polarized in the z-direction. With light polarized in the z-direction, is a transition from $|100\rangle \to |211\rangle$ possible for the hydrogen atom within the electric dipole approximation?

(c) Insert the $[L_z, x]$ commutation relation between the two states $|n'l'm'\rangle$ and $|nlm\rangle$ and obtain a relationship between $\langle n'l'm'| x |nlm\rangle$ and $\langle n'l'm'| y |nlm\rangle$. Notice that by calculating the x matrix element, you can automatically obtain the y matrix element, so you never actually have to do the extra calculation!

(d) Using the commutation relation $[L_z, x \pm iy]$, prove that either the matrix element $\langle n'l'm'| x \pm iy |nlm\rangle$ is zero or $m' = m \pm 1$. How do we interpret the operator $x \pm iy$? Does this operator remind you of any operators you've seen previously?

(e) Insert the $[L_z, y]$ commutation relation between the two states $|n'l'm'\rangle$ and $|nlm\rangle$. Using your answer to part **(c)**, show that either $(m - m')^2 = 1$ or $\langle n'l'm'| x |nlm\rangle = \langle n'l'm'| y |nlm\rangle = 0$. In words, explain what these relations imply about the matrix elements of x and y.