Physics 115C HW 8

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Problem 1

(a)

We have that the beam splitter matrix is:

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

We note that the beam splitter matrix is unitary, therefore we have that

$$|i_1\rangle = \frac{1}{\sqrt{2}}\left(|O_1\rangle - |O_2\rangle\right)$$

$$|i_2\rangle = \frac{1}{\sqrt{2}}\left(|O_1\rangle + |O_2\rangle\right)$$

We also have that the state after the beamsplitter of a photon in state is a superposition of the two output states ie we have that the output state is:

$$\frac{1}{\sqrt{2}}\left(|O_1\rangle + |O_2\rangle\right)$$

(b)

We have that

$$|D_1\rangle = -\frac{1}{\sqrt{2}}\left(|O_1\rangle + |O_2\rangle\right)$$

$$|D_2\rangle = -\frac{1}{\sqrt{2}}\left(|O_2\rangle - |O_1\rangle\right)$$

Thus we have

$$|D_1\rangle = -\frac{1}{2}\left(|i\rangle - |i\rangle\right)$$

$$|D_2\rangle = -\frac{1}{2}\left(|i\rangle + |i\rangle\right)$$

(c)

Thus we have that

$$|brai\,|D_1\rangle\,|^2 = 0$$

and

$$|brai\,|D_2\rangle\,|^2 = 1$$

Thus we can see that all the photons are detected by detector 2.

(d)

We have in this case:

$$|D_1\rangle = -\frac{1}{2}\left(|i\rangle + |i\rangle\right)$$

$$|D_2\rangle = -\frac{1}{2}\left(|i\rangle - |i\rangle\right)$$

Thus we have that

$$|\,\langle i|D_1\rangle\,|^2 = 1$$

and

$$|\,\langle i|D_2\rangle\,|^2 = 0$$

Thus we can see that all the photons are detected by detector 1.

Problem 2

(a)

For each state, we can decompose r into x, y, and z. And there are 4 states we are intrested in, so we need to calculate 12 matrix elements.

(b)

We have that from hw 6:

$$\langle 100| z |200\rangle = 0$$

$$\langle 100| z |211\rangle = 0$$

$$\langle 100| z |21-1\rangle = 0$$

Likewise we have that

$$\langle 100| x |200\rangle = 0$$

$$\langle 100| x |210\rangle = 0$$

and

$$\langle 100| y |200\rangle = 0$$

$$\langle 100| y |210\rangle = 0$$

(c)

We have that from HW 6:

$$\langle 100| z |210\rangle = \frac{2^{\frac{15}{2}} a_0}{3^5}$$

And:

$$\begin{aligned}
\langle 100|\, x \,|21\pm 1\rangle &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \phi_{100}^* x \phi_{21\pm1} r^2 \sin(\theta) dr d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \phi_{100}^* \phi_{21\pm1} r^3 \sin^2(\theta)\cos(\phi) dr d\theta d\phi \\
&= \frac{1}{8a_0^4\pi}\int_0^{2\pi} \cos(\phi) e^{\pm i\phi} \int_0^{\pi} \sin^3(\theta) \int_0^{\infty} r^4 e^{-3r/(2a_0)} dr d\theta d\phi \\
&= \frac{1}{8a_0^4\pi}\frac{256a_0^5}{81}\frac{4}{3}\pi \\
&= \frac{128}{243}a_0
\end{aligned}$$

Likewise we have that:

$$\begin{aligned}
\langle 100|\, y \,|21\pm 1\rangle &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \phi_{100}^* y \phi_{21\pm1} r^2 \sin(\theta) dr d\theta d\phi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \phi_{100}^* \phi_{21\pm1} r^3 \sin^2(\theta)\sin(\phi) dr d\theta d\phi \\
&= \frac{1}{8a_0^4\pi\sqrt{3}}\int_0^{2\pi} \sin(\phi) e^{\pm i\phi} \int_0^{\pi} \sin^3(\theta) \int_0^{\infty} r^4 e^{-3r/(2a_0)} dr d\theta d\phi \\
&= \pm\frac{1}{8a_0^4\pi\sqrt{3}}\frac{256a_0^5}{81}\frac{4}{3}i\pi \qquad = \pm\frac{i128}{243}a_0
\end{aligned}$$

(d)

We have that the transistion rates are:

$$\Gamma_{210\to100} = \frac{e^2\omega^3}{3\pi\epsilon_0\hbar c^3}\left|\langle 100|\, z \,|210\rangle\right|^2 = 6.260\times10^8 s^{-1}$$

$$\Gamma_{211\to100} = \frac{e^2\omega^3}{3\pi\epsilon_0\hbar c^3}\left(\left|\langle 100|\, x \,|211\rangle\right|^2 + \left|\langle 100|\, y \,|211\rangle\right|^2\right) = 6.260\times10^8 s^{-1}$$

$$\Gamma_{21-1\to100} = \frac{e^2\omega^3}{3\pi\epsilon_0\hbar c^3}\left(\left|\langle 100|\, x \,|21-1\rangle\right|^2 + \left|\langle 100|\, y \,|21-1\rangle\right|^2\right) = 6.260\times10^8 s^{-1}$$

Therefore we get that lifetime for each are 1.597ns. Likewise since $\langle 100|\, x \,|200\rangle = \langle 100|\, y \,|200\rangle = \langle 100|\, z \,|200\rangle = 0$ we have that

$$\Gamma_{200\to100} = 0$$

Therefore the lifetime for this state is infinite.

(e)

The values we calculated for the $|210\rangle$ and $|21\pm\rangle$ state is very close to what we theoretically calculated. However the the transition lifetime for the $|200\rangle$ state is not infinite, which is not what we theoretically calculated, however it is several orders of magnitude larger than the other states, so it is still a very long lifetime. The reason why the transistion lifetime is not infinite is because of intermediate states.

Problem 3

(a)

We have

$$\begin{aligned}
H &= \frac{P^2}{2M} + \frac{p^2}{2m} + V(x) \\
&= \frac{p_1^2 + 2p_1p_2 + p_2^2}{2(m_1+m_2)} + \frac{m_2^2 p_1 - 2m_1m_2p_1 + m_1^2 p_2}{2m_1m_2(m_1+m_2)} + V(x) \\
&= \frac{m_1m_2p_1^2 + m_1m_2p_2^2}{2m_1m_2(m_1+m_2)} + \frac{m_2^2 p_1 + m_1^2 p_2}{2m_1m_2(m_1+m_2)} + V(x) \\
&= \frac{m_2(m_1+m_2)p_2^2 + m_1(m_1+m_2)p_1^2}{2m_1m_2(m_1+m_2)} + V(x) \\
&= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(|x_1 - x_2|)
\end{aligned}$$

(b)

We have that:

$$[X_i, P_i] = \frac{m_1[p_{1i}, x_{1i}] + m_2[p_{2i}, x_{2i}]}{m_1+m_2} = i\hbar$$

and for $i \neq j$:

$$[X_i, P_j] = \frac{m_1[p_{1i}, x_{1j}] + m_2[p_{2i}, x_{2j}]}{m_1+m_2} = 0$$

Likewise:

$$[x_i, p_i] = \frac{m_2}{m_1 + m_2}[x_{1i}, p_{1i}] + \frac{m_1}{m_1 + m_2}[x_{2i}, p_{2i}] = i\hbar$$

and for $i \neq j$:

$$[x_i, p_j] = \frac{m_2}{m_1 + m_2}[x_{1i}, p_{1j}] + \frac{m_1}{m_1 + m_2}[x_{2i}, p_{2j}] = 0$$

(c)

We have that thus the Schrodinger equation is:

$$\left(\frac{P^2}{2M} + \frac{p^2}{2m} + V(x)\right)\psi_{CM}(X)\psi_{rel}(x) = E\psi_{CM}(X)\psi_{rel}(x)$$

(d)

Dividing by $\psi_{CM}(X)\psi_{rel}(x)$ we get:

$$\frac{1}{\psi_{CM}(X)}\frac{P^2}{2M}\psi_{CM}(X) + \frac{1}{\psi_{rel}(x)}\frac{p^2}{2m}\psi_{rel}(x) + \frac{1}{\psi_{rel}(x)}V(x)\psi_{rel}(x) = E$$

therefore we can see that we have a term $\frac{1}{\psi_{CM}(X)}\frac{P^2}{2M}\psi_{CM}(X)$ that only depends on X and a term $\frac{1}{\psi_{rel}(x)}\frac{p^2}{2m}\psi_{rel}(x) + \frac{1}{\psi_{rel}(x)}V(x)\psi_{rel}(x)$ that only depends on x.

(e)

If we have:

$$\frac{P^2}{2M}\psi_{CM}(X) = E_{CM}\psi_{CM}(X)$$

and

$$\frac{p^2}{2m}\psi_{rel}(x) + V(x)\psi_{rel}(x) = E_{rel}\psi_{rel}(x)$$

then we have that substituting back into the equation we derived above:

$$E_{CM}\frac{1}{\psi_{CM}(X)}\psi_{CM}(X) + E_{rel}\frac{1}{\psi_{rel}(x)}\psi_{rel}(x) = E_{CM} + E_{rel} = E$$

(f)

By taking into account the mass of the proton and the electron we have the reduced mass is:

$$\mu = 9.104 \times 10^{-28} kg$$

The mass of the proton far outweighs the mass of the electron. Therefore we can see that the relative momentum is approximately the momentum of the electron.

Problem 4

(a)

We have

$$\frac{1}{2} \left(|r_1\rangle |r_2\rangle + ketr_1 |t_2\rangle + |t_1\rangle |r_2\rangle + |t_1\rangle |t_2\rangle \right)$$

(b)

Therefore the probability that Detector 1 and Detector 2 both click is 0.5. And the probability that Detector 1 clicks twice is 0.25 and the probability that Detector 2 clicks twice is 0.25.

(c)

If the photons are indistringuishable then we have that the state is

$$\frac{1}{2} \left(-|D_2\rangle |D_1\rangle - |D_2\rangle |D_2\rangle + |D_1\rangle |D_2\rangle + |D_1\rangle |D_1\rangle \right)$$

Then if renormalizing then we have:

$$\frac{1}{\sqrt{2}} \left(|D_1\rangle |D_1\rangle - |D_2\rangle |D_2\rangle \right)$$

(d)

Therefore we can see that the probability that Detector 1 and Detector 2 both click is 0 and the probability that Detector 1 clicks twice is 0.5 and the probability that Detector 2 clicks twice is 0.5.

(e)

A single photon can interfere with other photon only if they are indistinguishable. If they are distinguishable then they will not interfere.

(f)

We have that $P_C = 0$ thus

$$\alpha = \frac{P_C}{P_{D1}P_{D2}} = 0$$