Quantum Mechanics 115C: Homework 7

Spring 2023
Instructor: Anshul Kogar
Posted: 05/19/2023
Due Date: 05/29/2023

7.1 Fermi's golden rule for a constant perturbation

In class, we obtained Fermi's golden rule specifically for a harmonic perturbation. In this problem, we will derive Fermi's golden rule for a perturbation that is "constant" in time. While this may seem trivial since we are doing time-dependent perturbation theory, it is actually one of the most important cases.

For instance, consider an atom or nucleus in an excited state. This is not a stable state – eventually the system will decay, emitting a photon or other particle. The excited state isn't really an eigenstate of the Hamiltonian, otherwise it would be completely stable (i.e. a stationary state). Nevertheless, it makes sense to discuss the energy levels of the hydrogen atom, for example, as if they were stable. The perturbation that causes the decay is the "rest" of the true Hamiltonian. What we mean is that somehow, never mind how, at time t_0, we prepared the atom in an excited state that is not exactly an eigenstate of the whole Hamiltonian. Now we include the rest of the Hamiltonian as a small constant piece that we "forgot" when calculating the energy levels to determine the decay rate.

(a) Consider a perturbation of the form:

$$H'(t) = \Theta(t)V \tag{1}$$

where $\Theta(t)$ is the step function (i.e. it is zero for $t < 0$ and is one for $t \geq 0$), and V is an operator. Write down the first order perturbation probability for transitioning from the initial state, $|i\rangle$ to final state, $|f\rangle$.

(b) Assume now that the final state, $|f\rangle$ is part of a group of final states. What is the total probability of transitioning from the initial state to the final state?

(c) Assume now that this group of final states form a continuum. Express your answer from part **(b)** in terms of an integral. You may use the definition $\rho(E_f)\,dE_f$ as the number of final states within an energy range dE_f.

(d) Use the two approximations that the matrix elements and the density of final states, $\rho(E_f)$, are roughly constant over the width of the central peak. What do these approximations allow you to do? How do you justify these approximations?

(e) Perform the integral and write down Fermi's golden rule for the case of a constant perturbation.

7.2 Fermi's golden rule in 1d with harmonic oscillator-like potential

In this problem we are going to calculate the transition rate of a spin-1/2 electron confined to harmonic oscillator-type potential to a free-particle final state. Imagine that the harmonic oscillator potential looks like the one plotted in Fig. 1. This potential is not a true harmonic oscillator potential, because the parabola does not extend all the way to infinity. Nonetheless, we can approximate the ground state wavefunction by that of the harmonic oscillator ground state wavefunction. The electron is then subject to an oscillatory time-dependent electric field.

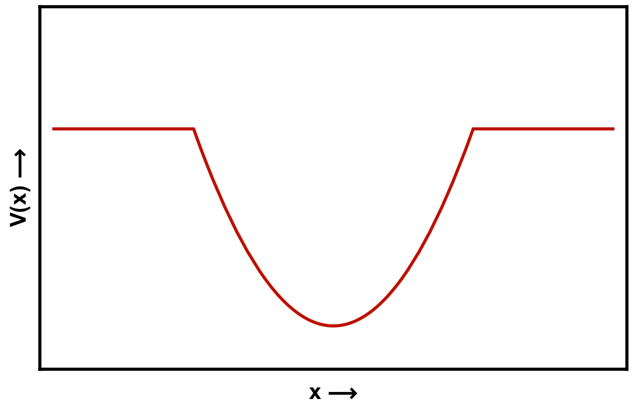

Figure 1: Harmonic oscillator-type potential.

Imagine that the perturbation has the form:

$$H'(t) = E_0 x(e^{-i\omega t} + e^{i\omega t}) \tag{2}$$

Like in class, we are going to be concerned mostly with absorption. Therefore, we can neglect the second term in parentheses since that represents the stimulated emission process. Imagine that frequency of the perturbation is high enough to eject the electron out of the potential well so that it ends up in a free particle state. Assume periodic boundary conditions for the free particle state.

(a) Assume that the particle is initially in the ground state of the harmonic oscillator. Write down the initial state wavefunction.

(b) Write down the final state wavefunction, assuming that the free particle is confined to a box of length L. Assume the box is one-dimensional.

(c) Write down Fermi's Golden Rule.

(d) Calculate the matrix element (by applying the trick we used for the photoelectric effect problem or otherwise).

(e) Calculate the density of final states in one dimension for the free particle.

(f) Write down the transition rate out of the ground state into a free particle state in terms of ω, the frequency of the harmonic perturbation.

7.3 Fermi's golden rule in four dimensions

(a) State Fermi's golden rule for a perturbation of the form:

$$H_1(t) = U\cos(\omega t) \tag{3}$$

Write down (a separate) Fermi's Golden Rule for both the absorption and stimulated emission terms, and write out the energy conservation conditions for each term.

(b) A spinless particle of mass m moves in a large four-dimensional hypercube of volume V with periodic boundary conditions. It experiences a weak oscillating perturbation,

$$H'(t) = \frac{\hbar^2 \lambda}{2m} \delta(\mathbf{r}) \cos(\omega t) \tag{4}$$

acting at the center of the box. Sketch a 1D cut of the potential, $H'(t)$ at $t = 0, \pi/2\omega$ and π/ω, (i.e. plot the curve only along, say, x).

(c) Calculate, using Fermi's golden rule, the transition rate out of the ground state. (You may use without proof the result that the surface *volume* of a four-dimensional hypersphere of unit radius is $2\pi^2$.)