Physics 115C HW1

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April 10, 2023

1 Problem 1

(a)

We have that

$$\vec{v_1}\vec{v_1}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And:

$$\vec{v_2}\vec{v_2}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And:

$$\vec{v_3}\vec{v_3}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore we have that

$$\sum_n \vec{v_n}\vec{v_n}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{1}_3$$

(b)

We have that $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ and $|\downarrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, therefore we have that

$$|\uparrow_x\rangle\langle\uparrow_x| = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

And:

$$|\downarrow_x\rangle\langle\downarrow_x| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Therefore we have that

$$\sum_{n=\uparrow_x,\downarrow_x} |n\rangle\langle n| = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{1}_2$$

(c)

We have that $|\uparrow_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ and $|\downarrow_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$, therefore we have that

$$|\uparrow_y\rangle\langle\uparrow_y| = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -i\\i & 1\end{bmatrix}$$

And:

$$|\downarrow_y\rangle\langle\downarrow_y| = \frac{1}{2}\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i\\-i & 1\end{bmatrix}$$

Therefore we have that

$$\sum_{n=\uparrow_y,\downarrow_y} |n\rangle\langle n| = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \mathbf{1}_2$$

(d)

We have that

$$|\uparrow_z\rangle = \frac{1}{\sqrt{2}}|\uparrow_y\rangle + \frac{1}{\sqrt{2}}|\downarrow_y\rangle$$

Since $|{\uparrow_y}\rangle$ is orthogonal to $|{\downarrow_y}\rangle$, we have that $\langle{\uparrow_y}|{\downarrow_y}\rangle = \langle{\downarrow_y}|{\uparrow_y}\rangle = 0$. Therefore we have that

$$(|{\uparrow_y}\rangle\langle{\uparrow_y}| + |{\downarrow_y}\rangle\langle{\downarrow_y}|)\,|{\uparrow_z}\rangle = \frac{1}{\sqrt{2}}|{\uparrow_y}\rangle + \frac{1}{\sqrt{2}}|{\downarrow_y}\rangle$$

This state is a superposition of $|{\uparrow_y}\rangle$ and $|{\downarrow_y}\rangle$, with probability $\frac{1}{2}$ for each.

(e)

We have that for a n euclidean vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$, that span a space, we have that any $|\psi\rangle$, can we written as:

$$|\psi\rangle = \sum_{i=1}^{N} |v_i\rangle\langle v_i|\psi\rangle$$

Since $\sum_{i=1}^{N} |v_i\rangle\langle v_i| = \mathbf{1}_n$, therefore

$$|\psi\rangle = \mathbf{1}_n|\psi\rangle = \sum_{i=1}^{N} |v_i\rangle\langle v_i|\psi\rangle$$

If we generalize this to an infinite dimension Hilbert space, with continous variable x, we have that

$$\mathbf{1} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|$$

Therefore we have that

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi\rangle$$

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\,\psi(x)$$

And

$$|\phi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\,\phi(x)$$

And thus

$$\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\phi(x)dx$$

Problem 2

TODO

Problem 3

(a)

We have that

$$a_+ |n\rangle = \sqrt{n+1}\, |n+1\rangle$$
$$a_- |n\rangle = \sqrt{n}\, |n-1\rangle$$

Therefore

$$\langle m| a_+ |n\rangle = \sqrt{n+1}\, \langle m|n+1\rangle = \sqrt{n+1}\delta_{m,n+1}$$

And

$$\langle m| a_- |n\rangle = \sqrt{n}\, \langle m|n-1\rangle = \sqrt{n}\delta_{m,n-1}$$

Therefore

$$\langle m| a_+ |n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and:

$$\langle m| a_- |n\rangle = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b)

We have that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}} (a_+ - a_-)$$

Therefore we have that

$$\langle m| \hat{x} |n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right)$$

$$\langle m| \hat{x} |n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

And:

$$\langle m| \hat{p} |n\rangle = i\sqrt{\frac{\hbar m\omega}{2}} \left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right)$$

$$\langle m| \hat{p} |n\rangle = i\sqrt{\frac{\hbar m\omega}{2}} \begin{bmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

These matrices are hermitian since $\langle m| \hat{x} |n\rangle$ is real and symmetric, and since $\langle m| \hat{p} |n\rangle$ is purely imaginary and skew-symmetric.

(c)

We have that

$$\hat{x}^2 = \frac{\hbar}{2m\omega} \left(a_+^2 + a_-^2 + a_+a_- + a_-a_+\right)$$

Thus:

$$\hat{x}^2 |n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\,|n+2\rangle + \sqrt{n(n-1)}\,|n-2\rangle + (n+1)\,|n\rangle + n\,|n\rangle\right)$$

Therefore

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} + \sqrt{n(n-1)}\delta_{m,n-2} + (n+1)\delta_{m,n} + n\delta_{m,n}\right)$$

$$\langle m|\,\hat{x}^2\,|n\rangle = \frac{\hbar}{2m\omega}\begin{bmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Likewise we have that:

$$\hat{p}^2 = \frac{\hbar m\omega}{2}\left(a_+^2 + a_-^2 - a_+a_- - a_-a_+\right)$$

$$\hat{p}^2 |n\rangle = \frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\,|n+2\rangle - \sqrt{n(n-1)}\,|n-2\rangle - (n+1)\,|n\rangle - n\,|n\rangle\right)$$

$$\langle m|\,\hat{p}^2\,|n\rangle = -\frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2} - (n+1)\delta_{m,n} - n\delta_{m,n}\right)$$

$$\langle m|\,\hat{p}^2\,|n\rangle = -\frac{\hbar m\omega}{2}\begin{bmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

These matrices are hermitian since they are real and symmetric.

(d)

We have that

$$H = \hbar\omega\left(a_+a_- + \frac{1}{2}\right)$$

Therefore we have that

$$\langle m|\,\hat{H}\,|n\rangle = \hbar\omega\left(\langle m|\,a_+a_-\,|n\rangle + \frac{1}{2}\delta_{m,n}\right)$$

From part (a) we have that:

$$\langle m|\,a_+a_-\,|n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\langle m|\,a_+a_-\,|n\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Therefore

$$\langle m|\,\hat{H}\,|n\rangle = \hbar\omega \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1+\frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2+\frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3+\frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4+\frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(e)

We have that

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Therefore we have that from part (c):

$$\langle m|\,\hat{H}\,|n\rangle = \frac{1}{2m}\bar{m}\hat{p}^2\,|n\rangle + \frac{1}{2}m\omega^2\bar{n}\hat{x}^2\,|n\rangle$$

$$\langle m|\hat{H}|n\rangle = -\frac{\omega\hbar}{4}\begin{bmatrix} -1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & -3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & -5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & -7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}\begin{bmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & \sqrt{10} & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 0 & 0 & \sqrt{10} & 0 & 9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Thus we get

$$\langle m|\hat{H}|n\rangle = \frac{\omega\hbar}{4}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 6 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 10 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 14 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 18 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Which is the same matrix we found in part (d). Since this matrix only has nonzero elements on the diagonal and the values are real, this matrix is hermitian.

(f)

We have that

$$a_+|2\rangle = \sqrt{3}|3\rangle$$
$$a_-|2\rangle = \sqrt{2}|1\rangle$$

Therefore we have that

$$\hat{x}|2\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{3}|3\rangle + \sqrt{2}|1\rangle)$$

$$\hat{p}|2\rangle = i\sqrt{\frac{\hbar m\omega}{2}}(\sqrt{3}|3\rangle - \sqrt{2}|1\rangle)$$

And:

$$\hat{x}^2|2\rangle = \frac{\hbar}{2m\omega}(\sqrt{12}|4\rangle + \sqrt{2}|0\rangle + 5|2\rangle)$$
$$\hat{p}^2|2\rangle = -\frac{\hbar m\omega}{2}(\sqrt{12}|4\rangle + \sqrt{2}|0\rangle - 5|2\rangle)$$

Therefore we have that

$$\hat{H}|2\rangle = \hbar\omega\left(2 + \frac{1}{2}\right)|2\rangle$$

Problem 4

Let us consider the one dimensional case first. If a state has definite parity then we have that $|\langle x|n\rangle|^2$ is even. Therefore we have that

$$\langle n| x |n\rangle = \int x|\langle x|n\rangle|^2 dx = 0$$

Therefore the dipole moment of a stationary state is 0 in one dimension. Generalizing to 3 dimensions we have that $\langle r|n\rangle$ is even along all 3 dimensions, ie that $\langle r|n\rangle = \psi(x, y, z) = \psi(-x, y, z) = \psi(x, -y, z) = \psi(x, y, -z)$. Therefore we have that

$$\langle n| r |n\rangle = \int r|\langle r|n\rangle|^2 d^3r = 0$$

Therefore we have that

$$\langle n| (qr) |n\rangle = \langle n| \mathbf{p} |n\rangle = 0$$