Physics 115C HW 3

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Problem 1

(a)

It would take $(-13.6\text{eV} - (-14.36\text{eV})) = 0.76\text{eV}$ to remove one of the two electrons from the Hydrogen anion.

(b)

We would expect that the ground state energy of the Hydrogen anion without electron electron repulsion would be -27.2eV. And thus it would take $(-13.6\text{eV} - (-27.2\text{eV})) = 13.6\text{eV}$ to remove one of the electrons.

(c)

If the helium atom had 3 electrons, we would expect two to fill up the $1s$ orbital and one to fill up the $2s$ orbital. If we ignore electron electron repulsion, we would expect that the 1s energy of the helium atom would be -54.4eV, and the 2s electron would have an energy of -13.6eV. Therefore we would expect the ground state energy to be -68eV.

(d)

The shielding effect of 3 electrons on a two proton nucleus would be higher than the shielding effect of 2 electrons on a one proton nucleus.

Problem 2

(a)

We have that from the Virial theorem for a harmonic oscillator that

$$\langle T \rangle = \langle V \rangle$$

Therefore:

$$\langle T \rangle = \frac{E_n}{2}$$

We have that our trial wavefunction is effectively the equal to the wavefunction for a harmonic oscillator with $\omega = \lambda$. And since the energy of the ground state of a harmonic oscillator is given by

$$E_n = \hbar\omega\frac{1}{2}$$

We have that

$$\langle \psi_{trial} | T | \psi_{trial} \rangle = \boxed{\frac{\hbar\lambda}{4}}$$

.

(b)

We shall prove that

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{\prod_{k=1}^{n}(2k-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}$$

Through induction, for $n = 1$ we have that

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} x^2 e^{-\frac{1}{2}\frac{x^2}{\frac{1}{2a}}} dx$$

As we can see the integral is now the integral for the second moment of a gaussian with variance $\frac{1}{2a}$ centered around 0. Therefore we have that

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

Now we assume that the integral holds for $n = k$ and we shall prove that it holds for $n = k + 1$. We have that

$$\begin{aligned}
-\int_{-\infty}^{\infty} x^{2k+2} e^{-ax^2} dx &= \frac{d}{da}\int_{-\infty}^{\infty} x^{2k} e^{-ax^2} dx \\
&= \frac{d}{da}\frac{\sqrt{\pi}\prod_{k=1}^{n}(2k-1)}{2^n a^{n+\frac{1}{2}}} \\
&= -\frac{\sqrt{\pi}(n+\frac{1}{2})\prod_{k=1}^{n}(2k-1)}{2^n a^{n+\frac{1}{2}+1}} \\
\int_{-\infty}^{\infty} x^{2k+2} e^{-ax^2} dx &= \boxed{\frac{\prod_{k=1}^{n+1}(2k-1)}{(2a)^{n+1}}\sqrt{\frac{\pi}{a}}}
\end{aligned}$$

We have that

$$\begin{aligned}
\langle\psi_{trial}|\, V\,|\psi_{trial}\rangle &= \left(\frac{m\lambda}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} kx^4 e^{-\frac{m\lambda}{\hbar}x^2} dx \\
&= \boxed{\frac{3k}{2^2\left(\frac{m\lambda}{\hbar}\right)^2}}
\end{aligned}$$

(c)

We have that

$$\bar{H} = \frac{\hbar\lambda}{4} + \frac{3k}{4\left(\frac{m\lambda}{\hbar}\right)^2}$$

Taking the derivative and setting it equal to 0 we have that:

$$\frac{\hbar}{4} - \frac{2 \cdot 3k\hbar^2}{4m^2\lambda^3} = 0$$
$$\hbar = \frac{2 \cdot 3k\hbar^2}{m^2\lambda^3}$$
$$\lambda^3 = \frac{2 \cdot 3k\hbar}{m^2}$$
$$\lambda = \boxed{\left(\frac{2 \cdot 3k\hbar}{m^2}\right)^{\frac{1}{3}}}$$

(d)

We thus have that for $\hbar = m = 1$ and $k = \frac{1}{2}$:

$$\lambda = 3^{\frac{1}{3}}$$

Therefore we have that

$$\bar{H} = \frac{3^{\frac{1}{3}}}{4} + \frac{3}{8 \cdot 3^{\frac{2}{3}}} = \boxed{0.540}$$

Which is $\boxed{2\%}$ away from the numerical value of 0.53.

Problem 3

(a)

We have that

$$\begin{pmatrix} \dot{c_1} \\ \dot{c_2} \end{pmatrix} = \frac{1}{i\hbar} \begin{pmatrix} 0 & \frac{\gamma}{2}e^{-i(\omega_{21}-\omega)t} \\ \frac{\gamma}{2}e^{i(\omega_{21}-\omega)t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

When $\delta\omega = \omega_{21} - \omega$ we have that

$$\begin{pmatrix} \dot{c_1} \\ \dot{c_2} \end{pmatrix} = \frac{1}{i\hbar} \begin{pmatrix} 0 & \frac{\gamma}{2}e^{-i\delta\omega t} \\ \frac{\gamma}{2}e^{i\delta\omega t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(b)

We have that

$$\dot{c_2} = \frac{1}{i\hbar}\frac{\gamma}{2}e^{i\delta\omega t}c_1$$

$$\ddot{c_2} = \frac{1}{i\hbar}\frac{\gamma}{2}e^{i\delta\omega t}\dot{c_1} + \frac{\delta\omega\gamma}{2\hbar}e^{i\delta\omega t}c_1$$

$$= -\frac{\gamma^2}{4\hbar^2}c_2 + i\delta\omega\dot{c_2}$$

$$\ddot{c_2} - i\delta\omega\dot{c_2} + \frac{\gamma^2}{4\hbar^2}c_2 = 0$$

(c)

We have that the corresponding characteristic equation is

$$\lambda^2 - i\delta\omega\lambda + \frac{\gamma^2}{4\hbar^2} = 0$$

Therefore we have

$$\lambda = \frac{i\delta\omega \pm \sqrt{-\delta\omega^2 - \frac{\gamma^2}{\hbar^2}}}{2}$$

$$\lambda = i\frac{\delta\omega + \Omega}{2}$$

Where $\Omega = \sqrt{\delta\omega^2 + \frac{\gamma^2}{\hbar^2}}$ Therefore we have that the solution is off the form

$$c_2(t) = Ae^{i\frac{\delta\omega+\Omega}{2}t} + Be^{i\frac{\delta\omega-\Omega}{2}t}$$

Which we can rearage to:

$$c_2(t) = C_+e^{i\xi_+t} + C_-e^{i\xi_-t}$$

Where $\xi_\pm = \frac{\delta\omega\pm\Omega}{2}$.

(d)

We have that:

$$\begin{aligned} c_2(0) &= 0 \\ C_+ + C_- &= 0 \\ C_- &= -C_+ \end{aligned}$$

Likewise we have that

$$\begin{aligned} c_1(0) &= 1 \\ i\hbar\frac{2}{\gamma}\dot{c_2}(0) &= 1 \\ i\hbar\frac{2}{\gamma}\left(-i\xi_+ C_- + i\xi_- C_-\right) &= 1 \\ \frac{2i\hbar}{\gamma}\left(\xi_- - \xi_+\right) C_- &= 1 \\ \frac{2i\hbar}{\gamma}\left(-\Omega\right) C_- &= 1 \\ C_- &= -\frac{\gamma}{2i\hbar\Omega} \end{aligned}$$

Therefore we have that

$$c_2(t) = \frac{\gamma}{2i\hbar\Omega}\left(e^{i\xi_+ t} - e^{i\xi_- t}\right)$$

And thus we have that

$$c_2^*(t) = \frac{-\gamma}{2i\hbar\Omega}\left(e^{-i\xi_+ t} - e^{-i\xi_- t}\right)$$

$$\begin{aligned} |c_2(t)|^2 &= \frac{\gamma^2}{4\hbar^2\Omega^2}\left(e^{i\xi_+ t} - e^{i\xi_- t}\right)\left(e^{-i\xi_+ t} - e^{-i\xi_- t}\right) \\ &= \frac{\gamma^2}{4\hbar^2\Omega^2}\left(2 - 2\cos\left(\xi_+ - \xi_-\right) t\right) \\ &= \frac{\gamma^2}{2\hbar^2\Omega^2}\left(1 - \cos\left(\Omega t\right)\right) \\ &= \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right) \end{aligned}$$

(e)

The maximum value of $|c_2(t)|^2$ is $\frac{\gamma^2}{\hbar^2\Omega^2}$ which is reached when $\sin^2\left(\frac{\Omega t}{2}\right) = 1$. We have that

$$\frac{\gamma^2}{\hbar^2\Omega^2} = \frac{\gamma^2}{\hbar^2(\delta\omega^2 + \frac{\gamma^2}{\hbar^2})}$$

Therefore we can see that when $\delta\omega = 0$, we recover back the result for the rabbi oscialltion at resonance, and for values of $\delta\omega \neq 0$ we have that $\delta\omega^2 > 0$ and thus $\frac{\gamma^2}{\hbar^2(\delta\omega^2+\frac{\gamma^2}{\hbar^2})} < 1$.

(f)

Since the particle can only take two possible states, we have that at equal superposition, we must have

$$\begin{aligned}
|c_2(t)|^2 &= \frac{1}{2} \\
\frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right) &= \frac{1}{2} \\
\sin\left(\frac{\Omega t}{2}\right) &= \frac{\hbar\Omega}{\gamma\sqrt{2}} \\
\frac{\Omega t}{2} &= \arcsin\left(\frac{\hbar\Omega}{\gamma\sqrt{2}}\right) \\
t &= \frac{2}{\Omega}\arcsin\left(\frac{\hbar\Omega}{\gamma\sqrt{2}}\right)
\end{aligned}$$

Thus we can see that the time it takes for the particle to reach equal superposition increases as $|\delta\omega|$ increases and it takes longer than at resonance.

Problem 4

(a)

We have that

$$H_0 + H_1 = \begin{pmatrix} 2\hbar\omega & \hbar\lambda \\ \hbar\lambda & 0 \end{pmatrix}$$

In order for $E_\pm$ to be eigenvalue and $v_\pm$ to be eigenvectors, we must have that

$$\begin{aligned}
(H_0 + H_1)v_+ &= E_+ v_+ \\
d\begin{pmatrix} 2\hbar\omega(\omega+\Delta) + \hbar\lambda^2 \\ \hbar\lambda(\omega+\Delta) \end{pmatrix} &= \hbar(\omega+\Delta)d\begin{pmatrix} \omega+\Delta \\ \lambda \end{pmatrix} \\
\begin{pmatrix} 2\hbar\omega^2 + 2\hbar\omega\Delta + \hbar\lambda^2 \\ \hbar\lambda(\omega+\Delta) \end{pmatrix} &= \begin{pmatrix} \hbar(\omega+\Delta)^2 \\ \hbar\lambda(\omega+\Delta) \end{pmatrix} \\
\begin{pmatrix} \hbar\omega^2 + 2\hbar\omega\Delta + \hbar\Delta^2 \\ \hbar\lambda(\omega+\Delta) \end{pmatrix} &= \begin{pmatrix} \hbar\omega^2 + 2\hbar\omega\Delta + \hbar\Delta^2 \\ \hbar\lambda(\omega+\Delta) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
(H_0 + H_1)v_- &= E_- v_- \\
d\begin{pmatrix} -2\hbar\omega\lambda + \hbar\lambda(\omega+\Delta) \\ -\hbar\lambda^2 \end{pmatrix} &= \hbar(\omega-\Delta)d\begin{pmatrix} -\lambda \\ \omega+\Delta \end{pmatrix} \\
\begin{pmatrix} -\hbar\omega\lambda + \hbar\lambda\Delta \\ -\hbar\lambda^2 \end{pmatrix} &= \begin{pmatrix} -\hbar\omega\lambda + \hbar\lambda\Delta \\ \hbar(\omega^2 - \Delta^2) \end{pmatrix} \\
\begin{pmatrix} -\hbar\omega\lambda + \hbar\lambda\Delta \\ -\hbar\lambda^2 \end{pmatrix} &= \begin{pmatrix} -\hbar\omega\lambda + \hbar\lambda\Delta \\ -\hbar\lambda^2 \end{pmatrix}
\end{aligned}$$

Therefore we can see that v_+ and v_- are the eigenvectors of $H_0 + H_1$ and E_+ and E_- are the eigenvalues of $H_0 + H_1$. We can see that the magnitudes of the eigenvalues are 1 as well since $(\omega+\Delta)^2 + \lambda^2 = 2\omega\Delta + \Delta^2 + \omega^2 + \lambda^2 = 2\Delta(\omega+\Delta) = d^{-2}$.

(b)

We have that

$$
\begin{aligned}
|0\rangle &= c((\omega+\Delta)v_+ - \lambda v_-) \\
&= cd \begin{pmatrix} (\omega+\Delta)^2 + \lambda^2 \\ 0 \end{pmatrix}
\end{aligned}
$$

Therefore we have that $c = d$ and thus we have

$$|0\rangle = d((\omega+\Delta)v_+ - \lambda v_-)$$

Likewise we have that

$$|1\rangle = d(\lambda v_+ + (\omega+\Delta)v_-)$$

Let us denote $v_\pm = |\pm\rangle$. We have that the time evolution of $|0,t\rangle$ is given by

$$|0,t\rangle = d((\omega+\Delta)e^{-\frac{iE_+t}{\hbar}}|+\rangle - \lambda e^{-\frac{iE_-t}{\hbar}}|-\rangle)$$

Therefore we have that the probability that the system is in state $|1\rangle$ is given by

$$
\begin{aligned}
|\langle 1|0,t\rangle|^2 &= (\omega+\Delta)^2\lambda^2 d^4 \left| e^{-\frac{iE_+t}{\hbar}} - e^{-\frac{iE_-t}{\hbar}} \right|^2 \\
&= (\omega+\Delta)^2\lambda^2 d^4 (2 - 2\cos\left(\frac{E_+ - E_-}{\hbar}t\right)) \\
&= 4(\omega+\Delta)^2\lambda^2 \frac{1}{4\Delta^2(\omega+\Delta)^2}\sin(\Delta t)^2 \\
&= \frac{\lambda^2}{\Delta^2}\sin(\Delta t)^2
\end{aligned}
$$

(c)

We have that the first order correction is:

$$
\begin{aligned}
c_1^{(1)}(t) &= \frac{1}{i\hbar}\int_0^t e^{-2\omega i t'}\hbar\lambda dt \\
&= \frac{\lambda}{2\omega}\left(e^{-2\omega i t} - 1\right)
\end{aligned}
$$

Therefore the approximate probability that the system is in state $|1\rangle$ is given by

$$
\begin{aligned}
|c_0^{(1)}(t)|^2 &= \frac{\lambda^2}{4\omega^2}\left(e^{-2\omega it}-1\right)\left(e^{2\omega it}-1\right)\\
&= \frac{\lambda^2}{4\omega^2}\left(2-2\cos(2\omega t)\right)\\
&= \frac{\lambda^2}{\omega^2}\sin^2(\omega t)
\end{aligned}
$$

(d)

If $\omega >> \lambda$ then we have that we can write our transition probability as

$$
\begin{aligned}
\frac{\lambda^2}{\Delta^2}\sin(\Delta t)^2 &= \frac{\lambda^2}{\omega^2\left(1+\frac{\lambda^2}{\omega^2}\right)}\sin\left(\omega\sqrt{1+\frac{\lambda^2}{\omega^2}t}\right)\\
&\approx \frac{\lambda^2}{\omega^2}\sin^2(\omega t)
\end{aligned}
$$