Physics 115C HW 3

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Problem 1

(a)

It would take $(-13.6\text{eV} - (-14.36\text{eV})) = 0.76\text{eV}$ to remove one of the two electrons from the Hydrogen anion.

(b)

Problem 2

(a)

We have that from the Virial theorem for a harmonic oscillator that

$$\langle T \rangle = \langle V \rangle$$

Therefore:

$$\langle T \rangle = \frac{E_n}{2}$$

We have that our trial wavefunction is effectively the equal to the wavefunction for a harmonic oscillator with $\omega = \lambda$. And since the energy of the ground state of a harmonic oscillator is given by

$$E_n = \hbar\omega\frac{1}{2}$$

We have that

$$\langle\psi_{trial}|\, T\, |\psi_{trial}\rangle = \boxed{\frac{\hbar\lambda}{4}}$$

.

(b)

We shall prove that

$$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2}\, dx = \frac{\prod_{k=1}^{n}(2k-1)}{(2a)^n}\sqrt{\frac{\pi}{a}}$$

Through induction, for $n = 1$ we have that

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = \sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}\sqrt{\frac{a}{\pi}}x^2 e^{-\frac{1}{2}\frac{x^2}{\frac{1}{2a}}}\, dx$$

As we can see the integral is now the integral for the second moment of a gaussian with variance $\frac{1}{2a}$ centered around 0. Therefore we have that

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

Now we assume that the integral holds for $n = k$ and we shall prove that it holds for $n = k + 1$. We have that

$$\begin{aligned}
-\int_{-\infty}^{\infty} x^{2k+2} e^{-ax^2}\, dx &= \frac{d}{da}\int_{-\infty}^{\infty} x^{2k} e^{-ax^2}\, dx \\
&= \frac{d}{da}\frac{\sqrt{\pi}\prod_{k=1}^{n}(2k-1)}{2^n a^{n+\frac{1}{2}}} \\
&= -\frac{\sqrt{\pi}(n+\frac{1}{2})\prod_{k=1}^{n}(2k-1)}{2^n a^{n+\frac{1}{2}+1}} \\
\int_{-\infty}^{\infty} x^{2k+2} e^{-ax^2}\, dx &= \boxed{\frac{\prod_{k=1}^{n+1}(2k-1)}{(2a)^{n+1}}\sqrt{\frac{\pi}{a}}}
\end{aligned}$$

We have that

$$\begin{aligned}
\langle\psi_{trial}| V |\psi_{trial}\rangle &= \left(\frac{m\lambda}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} kx^4 e^{-\frac{m\lambda}{\hbar}x^2} dx \\
&= \boxed{\frac{3k}{2^2\left(\frac{m\lambda}{\hbar}\right)^2}}
\end{aligned}$$

(c)

We have that

$$\bar{H} = \frac{\hbar\lambda}{4} + \frac{3k}{4\left(\frac{m\lambda}{\hbar}\right)^2}$$

Taking the derivative and setting it equal to 0 we have that:

$$\begin{aligned}
\frac{\hbar}{4} - \frac{2\cdot 3k\hbar^2}{4m^2\lambda^3} &= 0 \\
\hbar &= \frac{2\cdot 3k\hbar^2}{m^2\lambda^3} \\
\lambda^3 &= \frac{2\cdot 3k\hbar}{m^2} \\
\lambda &= \boxed{\left(\frac{2\cdot 3k\hbar}{m^2}\right)^{\frac{1}{3}}}
\end{aligned}$$

(d)

We thus have that for $\hbar = m = 1$ and $k = \frac{1}{2}$:

$$\lambda = 3^{\frac{1}{3}}$$

Therefore we have that

$$\bar{H} = \frac{3^{\frac{1}{3}}}{4} + \frac{3}{8\cdot 3^{\frac{2}{3}}} = \boxed{0.540}$$

Which is $\boxed{2\%}$ away from the numerical value of 0.53.

Problem 3

(a)

We have that

$$\begin{pmatrix} \dot{c_1} \\ \dot{c_2} \end{pmatrix} = \frac{1}{i\hbar} \begin{pmatrix} 0 & \frac{\gamma}{2} e^{-i(\omega_{21}-\omega)t} \\ \frac{\gamma}{2} e^{i(\omega_{21}-\omega)t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

When $\delta\omega = \omega_{21} - \omega$ we have that

$$\begin{pmatrix} \dot{c_1} \\ \dot{c_2} \end{pmatrix} = \frac{1}{i\hbar} \begin{pmatrix} 0 & \frac{\gamma}{2} e^{-i\delta\omega t} \\ \frac{\gamma}{2} e^{i\delta\omega t} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(b)

We have that

$$\begin{aligned} \dot{c_2} &= \frac{1}{i\hbar} \frac{\gamma}{2} e^{i\delta\omega t} c_1 \\ \ddot{c_2} &= \frac{1}{i\hbar} \frac{\gamma}{2} e^{i\delta\omega t} \dot{c_1} + \frac{\delta\omega\gamma}{2\hbar} e^{i\delta\omega t} c_1 \\ &= -\frac{\gamma^2}{4\hbar^2} c_2 + i\delta\omega \dot{c_2} \end{aligned}$$

$$\ddot{c_2} - i\delta\omega\dot{c_2} + \frac{\gamma^2}{4\hbar^2} c_2 = 0$$

(c)

We have that the corresponding characteristic equation is

$$\lambda^2 - i\delta\omega\lambda + \frac{\gamma^2}{4\hbar^2} = 0$$

Therefore we have

$$\lambda = \frac{i\delta\omega \pm \sqrt{-\delta\omega^2 - \frac{\gamma^2}{\hbar^2}}}{2}$$

$$\lambda = i\frac{\delta\omega + \Omega}{2}$$

Where $\Omega = \sqrt{\delta\omega^2 + \frac{\gamma^2}{\hbar^2}}$ Therefore we have that the solution is off the form

$$c_2(t) = Ae^{i\frac{\delta\omega+\Omega}{2}t} + Be^{i\frac{\delta\omega-\Omega}{2}t}$$

Which we can rearage to:

$$c_2(t) = C_+e^{i\xi_+t} + C_-e^{i\xi_-t}$$

Where $\xi_\pm = \frac{\delta\omega\pm\Omega}{2}$.

(d)

We have that:

$$\begin{aligned}
c_2(0) &= 0 \\
C_+ + C_- &= 0 \\
C_- &= -C_+
\end{aligned}$$

Likewise we have that

$$\begin{aligned}
c_1(0) &= 1 \\
i\hbar\frac{2}{\gamma}\dot{c_2}(0) &= 1 \\
i\hbar\frac{2}{\gamma}\left(-i\xi_+C_- + i\xi_-C_-\right) &= 1 \\
\frac{2i\hbar}{\gamma}\left(\xi_- - \xi_+\right)C_- &= 1 \\
\frac{2i\hbar}{\gamma}\left(-\Omega\right)C_- &= 1 \\
C_- &= -\frac{\gamma}{2i\hbar\Omega}
\end{aligned}$$

Therefore we have that

$$c_2(t) = \frac{\gamma}{2i\hbar\Omega}\left(e^{i\xi_+t} - e^{i\xi_-t}\right)$$

And thus we have that

$$c_2^*(t) = \frac{-\gamma}{2i\hbar\Omega}\left(e^{-i\xi_+ t} - e^{-i\xi_- t}\right)$$

$$\begin{aligned}
|c_2(t)|^2 &= \frac{\gamma^2}{4\hbar^2\Omega^2}\left(e^{i\xi_+ t} - e^{i\xi_- t}\right)\left(e^{-i\xi_+ t} - e^{-i\xi_- t}\right) \\
&= \frac{\gamma^2}{4\hbar^2\Omega^2}\left(2 - 2\cos\left(\xi_+ - \xi_-\right)t\right) \\
&= \frac{\gamma^2}{2\hbar^2\Omega^2}\left(1 - \cos\left(\Omega t\right)\right) \\
&= \frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right)
\end{aligned}$$

(e)

The maximum value of $|c_2(t)|^2$ is $\frac{\gamma^2}{\hbar^2\Omega^2}$ which is reached when $\sin^2\left(\frac{\Omega t}{2}\right) = 1$. We have that

$$\frac{\gamma^2}{\hbar^2\Omega^2} = \frac{\gamma^2}{\hbar^2(\delta\omega^2 + \frac{\gamma^2}{\hbar^2})}$$

Therefore we can see that when $\delta\omega = 0$, we recover back the result for the rabbi oscialltion at resonance, and for values of $\delta\omega \neq 0$ we have that $\delta\omega^2 > 0$ and thus $\frac{\gamma^2}{\hbar^2(\delta\omega^2+\frac{\gamma^2}{\hbar^2})} < 1$.

(f)

Since the particle can only take two possible states, we have that at equal superposition, we must have

$$\begin{aligned}
|c_2(t)|^2 &= \frac{1}{2} \\
\frac{\gamma^2}{\hbar^2\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right) &= \frac{1}{2} \\
\sin\left(\frac{\Omega t}{2}\right) &= \frac{\hbar\Omega}{\gamma\sqrt{2}} \\
\frac{\Omega t}{2} &= \arcsin\left(\frac{\hbar\Omega}{\gamma\sqrt{2}}\right) \\
t &= \frac{2}{\Omega}\arcsin\left(\frac{\hbar\Omega}{\gamma\sqrt{2}}\right)
\end{aligned}$$

Thus we can see that the time it takes for the particle to reach equal superposition increases as $|\delta\omega|$ increases and it takes longer than at resonance.

Problem 4

(a)

We have