Introduction

Traditionally, the hedge ratio is assumed to be constant over the relevant time. However, recent empirical financial research reveals that asset returns volatility is time-varying, Park and Switzer (1995) suggest that consistent hedge ratios may not be appropriate as the joint distribution of cash and futures prices is changing over time, while Baillie and Myers (1991) also prove that the conditional distribution between cash and futures price changes then hedge ratios will vary over time. To obtain time-varying hedge ratios, accurate volatility forecasting is necessary. Volatility is directly related to the uncertainty and risk of the market and is the most concise and effective indicator to reflect the quality and efficiency of the financial market. Thus, it is of great importance in the fields of securities investment, asset pricing and risk management. Volatility forecasting models can be used to derive hedged ratios. Incorporating time-variation into the hedge ratio improves the performance of the hedge in terms of risk reduction (McMillan, 2005). This paper supports the view that an accurate estimation of the volatility process plays an essential role in the construction of hedging strategies and improving the hedging effectiveness.

The main task of this report is to act as a risk manager for a large American company that has a target of hedging a long position in the S & P 500 index with a short position in the FTSE 100 index. The hedged portfolio is expected to perform better than the unhedged portfolio in terms of risk. Risk characteristics are presented by variance in this paper. To construct this hedge portfolio, it is necessary to estimate the time-varying minimum-variance hedge ratios, which can improve the risk-reduction capacity better than the constant hedged ratio. This paper adopts the rolling window model and the EWMA model to obtain estimates needed for the time-varying minimum-variance hedge ratios.

The motivation for the analysis is to examine the important role volatility forecasting plays in the construction of the hedged portfolio and how to use different models to estimate volatility to get time-varying minimum-variance hedge ratios. The accurate volatility forecasting is important not only because the unobservable variance of returns make it difficult to make an investment decision, but also because the effects of volatility can be offset through hedging. Hence, accurate estimation of volatility is of value in efficiently and effectively implementing. Since volatility tends to aggregate, the natural way to estimate time-varying volatility is to use estimates based on recent returns. Through the utilizing of the proper models, the change of volatility could be calculated with more precision and accuracy (Kumar,2006). However, there is no definite answer to the question of which model should be used to estimate the volatility. In order to predict volatility efficiently and practically, this paper mainly uses the rolling window model and the EWMA model to estimate the volatility. Therefore, we use these two models and compute the optimal hedge ratio. The methodologies used in this research are discussed in detail in terms of their principles, features, advantages and disadvantages in the part of the method. Also, the efficiency of our hedge ratios is verified by the reduction of Variance after using hedging strategies.

The main findings are as follow: we summarize the features of the sample, which can provide overall stock performance in S & P500 index and FPSE 100 index. Then, we obtain hedge ratios by implementing two models to predict volatility. We find that using hedging a long position in S & P500 index with a short position in FPSE 100 index could effectively reduce the variance of returns. This is due to the fact that variance reflects the magnitude of volatility. The reduction in volatility indicates a reduction in risk (Wang, 2015). Last but not least, different volatility estimates resulting from these two models, and the main findings are showed in the following section of results.

The outline of this report is as follows: Section two introduces the principal, process of data selection and summary statistics. Section three describes the theories and methods used in the practical analysis of this article. With data obtained from online recourse, the rolling window and EWMA volatility forecasting models are utilized in the process of estimating hedge ratios. In the next section, we draw the results of the two models' volatility predictions and the hedged portfolio we find. And in the last part, we summarized the research findings and suggestions for future study.

In summary, this paper estimates the time-varying minimum-variance hedge ratio through two models and examines the capacity of risk reduction through hedging strategies constructed by two volatility-forecasting models.

Data

The data file used in our report was collected from Yahoo Finance (https://uk.finance.yahoo.com/). It contains the historical data of the S&P 500 index and the FTSE 100 index for the last six years, from April 1st, 2014 to March 31st, 2020. Observations related to public holidays either in the United Kingdom and the United States was deleted, to ensure the continuity and consistency of the data for the two indexes, leaving 1488 trading days in our data set. Through the adjusted close prices of these 1488 observations, we calculated the daily simple returns for these two indices over the six years. Summary statistics for the data are given in Table 1.

Table 1 Summary statistics

	^GSPC	^FTSE
MEAN	0.027%	-0.006%
VARIANCE	1.21×10^{-4}	1.02×10^{-4}
SKEWNESS	-0.793	-0.950
KURTOSIS	24.636	17.099
JB STATISTICS	37759.456**	12540.452**

Notes: The Bera-Jarque statistics is distributed asymptotically as a $\chi^2(2)$ under the null of normality. * and ** indicate significance at the 5% and 1% levels respectively.

It is evident that the GSPC returns series is slightly more volatile than FTSE. GSPC performs better than FTSE in terms of mean returns in this period. Besides, both share indexes are strongly non-normal as their JB statistics are extremely high. Also, they are leptokurtic and significantly skewed to the left.

Method

In this part, the definition of volatility and the reason why there is a need for estimating volatility is explained in detail. According to the case study, Volatility is defined as the variability of security returns, and normally is measured by variance. In practice, the variance of return is unobserved. Therefore there is a need for estimating volatility. In addition, because of the assumption that the true variance is constant over time, we are able to estimate variance by using the sample variance of past returns.

There are mainly two widely used methodologies in this research to estimate volatility; Rolling window method and Exponentially weighted moving average model (EWMA). In the following paragraph, the principles and characteristics of the two methods will be introduced, and their limitations and advantages would also be compared and discussed.

The first approach is rolling window or moving average volatility model, which is the sample variance estimated. It is based on the principle that time-varying volatility can be estimated based on returns from the recent past, as volatility is normally clustered. The general form of the model is as follows:

$$\hat{\sigma}_{t+2}^2 = \frac{1}{M-1}\sum_{i=0}^{M-1}(r_{t-i} - \bar{r}_t)^2$$

M is the length of the time period of the obtained data, which is also named as window length. To get the optimal window length which can bring maximum risk reduction, data table is used in EXCEL to repeat the process hundreds of times with different values of M, and find the optimal window length. The advantages of this estimator are that it is very simple to calculate and that except for the window size it does not involve any kind of estimation (Suganuma, 2000). The window length is one of the most essential factors of the rolling window model, it literally means how long the period of the date we utilise to forecast future volatility. This factor is of great significance in the process of estimation, and two considerations should be balanced in terms of choice of window length. From one perspective, a long enough time period is needed to make sure the precision with which we estimate the return variance. From the other perspective, the time period or window length should be short enough to make the estimation to be more relative with the recent past data. This is because of the fact that the everyday returns have the same amount of influence on the estimated volatility in rolling window model. If the window length is too long, the impact of data from a fairly long period of time ago could be overestimated. Moreover, according to the practice experience, decreasing the window length increases the sensitivity of the rolling variance estimator to observations that lie within the window, and consequently increases the volatility of the volatility estimator. Therefore, in principle, rolling window model includes only the most recent observations to estimate conditional volatility.

A noticeable drawback of rolling window model is ghost feature, which means that the lag out of the observation window poses an impact on the estimates. From the fundamental point of view, rolling window approach gives past observations either full weight (if they are recent enough to lie within the window) or zero weight (if they fall out of the window). It may to some extent influence the accuracy of the estimates. By comparison, exponentially weighted moving average (EWMA) estimate can effectively overcome this problem.

The EWMA estimator yields a dynamic process for the conditional variance that is not stationary, EWMA estimators regarding to the variance of a set of returns and the covariance between two sets of returns can be written as

$$\hat{\sigma}_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)r_t^2$$
$$\hat{\sigma}_{ij,t+1}^2 = \lambda\sigma_{ij,t}^2 + (1-\lambda)r_{i,t}r_{j,t}$$

Because of the fact that short horizon returns in practice could normally be zero, we can assume that the mean return is equal to zero. Under this assumption, the principle behind this expression is to weight observations in the sample in a such way that their importance declines smoothly into the past. According to Suganuma, in this method, the weights are geometrically declining, so the most recent observation has more weight compared to older ones. Therefore, this weighting scheme helps to capture the dynamic properties of the data.

The prerequisite for using EWMA estimator is that σ_1^2 should be specified and so the estimated variances are not unique; however, because of the exponential weighting, the choice of σ_1^2 rapidly becomes irrelevant as we move through the sample. For instance, we could set σ_1^2 to zero. Correspondingly, it is common to ignore the first 100 variance estimates, so as to remove the effect of our specification of σ_1^2.

As the only unknown parameter of the function, decay factor λ can be set as the value that optimises a particular economic or statistical criterion. In former research in 2000, Bollen found that the optimal value of lambda is time varying and should be based upon recent historical data. In financial practice, λ value of daily returns is usually estimated to be 0.94, and for monthly return is 0.97. This result is proved to be infinitely close to the real cases and therefore is reliable. This result is proved to be infinitely close to the real cases and therefore is reliable. In our analysis, we first set it to 0.94, and then calculated the hedging effect under this value. In addition, we calculated the value of λ under the optimal hedging effect through the solver.

EWMA model is nested by Generalised autoregressive conditional heteroscedasticity (GARCH) model. As a special case of GARCH, EWMA is more widely used in practice than GARCH. This is due to the fact that GARCH model has two more parameters that are difficult to estimate in reality, although EWMA replaces them with two estimated constant, the results are proved to be Infinitely close to the real cases. In other words, EWMA model could achieve near perfect results more efficiently.

In the case study, after obtaining the variance by meanings of Rolling window or EWMA models, it is not difficult to calculate the time varying hedge ratio of the portfolio. The return in hedging a long position in the S & P 500 index with a short position in the FTSE 100 index is given by

$$R_{p,t} = R_{GSPC,t} - h_t R_{FTSE,t}$$

where h_t is the time-varying hedge ratio. Thus, it is necessary to estimate the time-varying minimum-variance hedge ratio, given by

$$h_t = \frac{\sigma_{GSPC,FTSE,t}}{\sigma^2_{FTSE,t}}$$

Thus, we need to estimate the conditional variance of FTSE 100 index returns, and the conditional covariance between S & P 500 index returns and FTSE 100 index returns. Meanwhile, by meanings of estimating the reduction of variance, we could also estimate the time varying hedge ratio constructed by the two models. By doing so, we are able to measure the effectiveness of our rolling window and the EWMA hedge ratio, by comparing the difference of variance between unhedged portfolio and hedged portfolio. Thereby, we can adjust the proportion of investment in a timely and correct way.

The research process contains the following preconditions. In the EWMA model, the size of the hedge ratio is calculated in two cases that the delay factor equal to 0.94 and not equal to 0.94, so as to derive the change of volatility. The rolling window model calculates the size of the hedge ratio and changes in volatility when the calculation window is 250 days. When the rolling window is changed, the risk reduction capacity of hedged portfolios will change as well.

In addition to the two models used in the case study, implied volatility also worth mentioning. It is usually used in option pricing and is different from the former estimators in principle. As the Rolling Window estimator and EWMA estimator are all backward looking, but the Implied volatility estimating is forward forecast. Under the efficient market hypothesis, the fair value of an option should equal to its market price, Given certain assumptions, option price is given by a function of risk free rate, time to maturity, strike price and volatility of the underlying asset price. In this case, all the other three parameters are known, and the fair value of option is equal to the market price, which is also known, By inverting the appropriate option pricing formula, we can infer the volatility of the asset price that is consistent with the observed option price.

Results

This report selects the rolling window model and EWMA model, as mentioned before, to analyze the possibility of reducing volatility through hedging a long position in S&P500 index with a short position in FPSE 100 index and, more specifically, estimate how much reduction the hedging portfolio does. The results of these two different models are as follow:

Considering the rolling window model, the window length this research chooses is 250 days, in order to contain as much data or situation as possible to increase the accuracy of estimating return variance. After calculating simple returns of S&P 500 index and FPSE 100 index from April 1st 2014 to March 30th 2020, a series of conditional variance and conditional covariance for two index has been estimated. Then, it is accessible to construct the time-varying hedge ratio and calculate hedge portfolio return and its variance which is an essential metric to evaluate the feasibility of this hedge. Therefore, through model operation, the variance of returns of S&P 500 index is at 0.013367%, while the variance of returns of S&P 500 index after hedging by FPSE 100 index is at 0.007903%. Comparing the magnitude of hedge and unhedged variance, the percentage of volatility of S&P 500 index has been reduced at 40.88% across the hedge portfolio with 250 days window length. This implies that the hedge portfolio has a positive impact on reducing the risk of investing in S&P 500 index alone.

Additionally, in order to calculate the optimal window length, the Data table is involved as an indispensable data analysis tool in the analysis. It repeats the process of estimation with a wide range of window lengths between 10 and 250 days and plots all the percentage of declining of the variance of returns in figure 1. We find that, as a measure of hedging effectiveness, the average risk reduction when using hedging strategies instead of no hedging at all is almost stable at 40.00% (minimum 34.00%; maximum 40.90%).

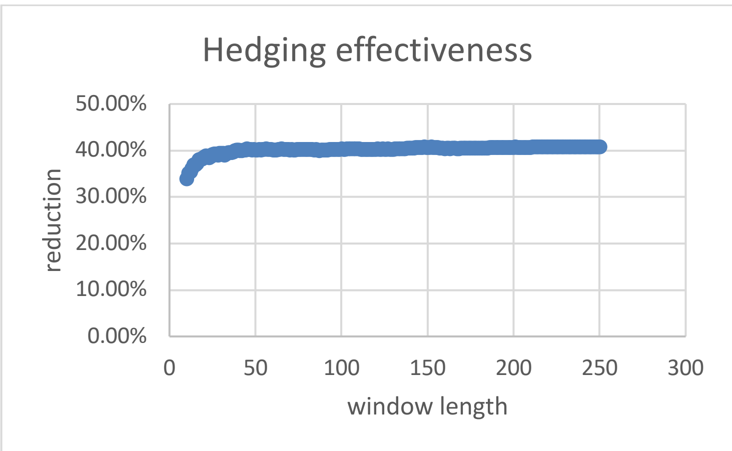

Figure 1 Hedging effectiveness

Comparing these results to it that is estimated with 250 days window length, we find 246 days window length is the most effective one. In this situation, we compare the variance of returns of hedged portfolio and S&P 500 index, and it peaks at 40.90%, which illustrates the greatest degree of decreasing volatility of S&P 500 index using the rolling window model.

The other model is EWMA estimator which uses an alternative way to weight observation. This way is introducing a new variable defined as decay factor,λ, to weight observation. At the beginning of this approach, 0.94 is selected as a decay factor and σ_1^2 is set to 0. Then, based on historical data from April 1st 2014 to March 30th 2020, we calculate simple returns of S&P 500 index and FPSE 100 index. After that, this study estimates a series of conditional variance and conditional covariance about two indexes across EWMA approach with σ_1^2 of 0 and λ of 0.94 and calculates a series of time-varying hedge ratios according to these two variables estimated before. Finally, using the time-varying hedge ratio constructed, the variance of returns of S&P 500 index without hedging is at 0.0133372% and the variance of returns of S&P 500 index after hedging by FPSE 100 index is at 0. 0081801%. Comparing these two variance using same method as rolling window model previously, the proportion of reduction of the volatility of S&P 500 index hedged by FPSE 100 index is at 38.67%. This means that the hedge portfolio is efficient to reduce the volatility of S&P 500 index, which is consistent with results from the former model. In addition, although the decay factor is set to 0.94, it is not the optimal one. 0.9905 is the excellent decay factor through the solver approach. When λ is equal to 0.9905 and other conditions remain consistent, the maximum percentage of reduction of the variance of returns of S&P 500 index hedged by FPSE 100 index is at 40.68%. Therefore, under EWMA estimator, the maximum volatility that a hedged portfolio can reduce is 40.68%.

Overall, through hedged by FPSE 100 index, the reduction of the volatility of S&P 500 index is at 40.88% using the rolling window model with a window length of 250 days while at 38.67% using EWMA model with the decay factor of 0.94 and σ_1^2 of 0. The optimal reduction these two models calculate is at 40.90% and 40.68% respectively, which is achieved at a window length of 246 days and the decay factor of 0.9905. According to these results, the hedge portfolio may be achievable and appropriate, and the validity of these two models can be proved. In other words, the risk of investing in a long position in S&P 500 index can be reduced by hedging with a short position in FPSE 100 index. The results support the view that the two volatility forecasting models we used can effectively derive the time-varying hedged ratios and reduce the investment risk.

Conclusion

The results from these two models, rolling window model, and EWMA model, are similar, which implies that it is possible to hedge a long position in S&P 500 index with a short position in FPSE 100 index. More detailed, the hedge portfolio can decline 40.88% and 38.67% of the volatility of S&P 500 index respectively, considering the rolling window model with a window length of 250 days and EWMA model with a decay factor of 0.94. After optimization, the decrease of variance can reach 40.90% and 40.68% respectively, when the window length is 246 days and the decay factor is 0.9905. Although the result from the rolling window model is not completely equal to that from the EWMA model, it is effective to reduce the risk of holding a long position in S&P 500 index through hedging this index with FPSE 100 index. In other words, holding this hedge portfolio is more reasonable and reliable than holding S&P 500 index alone since the portfolio can significantly reduce the volatility of investment on this index by around 40%.

However, although these two models are feasible in this study, there are several problems in these approaches. In the rolling window model, ghost features in the estimation of volatility exist continually and are hard to avoid. This problem is the result of that the rolling window estimator gives observations either full weight or zero weight. It means that observation may be ignored if it is so old or out of sample selected by window length. Therefore, there may be an error in the estimation of volatility due to the absence of some important shock that may impact heavily on volatility. In the EWMA model, similarly, there is an inevitable problem that the EWMA model is not possible to represent the true process to generate volatility. It is because that the process EWMA model established for the conditional variance is not stationary. Specifically, this model forecasts the volatility that cannot converge to a long value as the increase of the forecast horizon, which implies that the unconditional variance is infinite.

Using more advanced and scientific models is an efficient method to solve these problems. Considering the problem, full or zero weight of observation in the rolling window model, a reasonable way is to give different weights for different period observations and make the weight of more recent observation larger. Approaches the same as this way to weight observation are various, including linearly declining weights, hyperbolically declining weights, etc. The most common one is the exponentially weighted moving average which is the second method used in this report. Similarly, the other problem mentioned before can be solved by using an alternative model that nests the EWMA model, known as generalized autoregressive conditional heteroscedasticity, or GARCH. Therefore, in the future research, GARCH model might be a more effective model to estimate the volatility so that the results from research are likely to be more accurate and correct.

Overall, although problems exist constantly in the rolling window model and EWMA model, data collection and calculation of essential variables are corrected and successful, which leads to the results from this research reasonable and reliable. Eventually, the effect of this hedge portfolio can efficiently reduce the risk of holding a long position in S&P 500 index is rational and acceptable. In the future study, a more advanced model like GARCH model that solves problems mentioned before can be adopted to estimate a more accurate estimation.

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