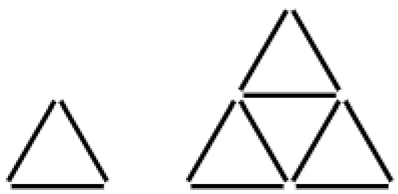
Final Portfolio

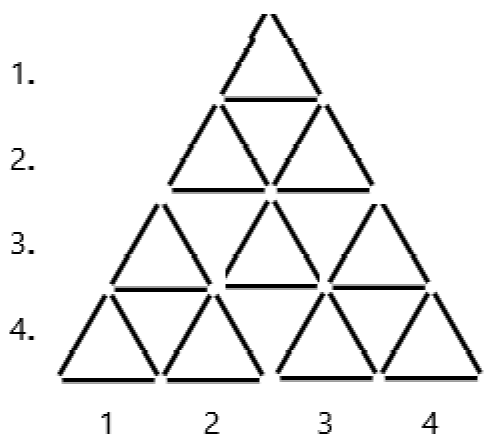
Introduction
The following seven specimens of my work are some of my best examples of my work throughout the term. My own assessment of my letter grade follows the problems.

Section 2.2, Question 13
Set theory
If you have enough toothpicks, you can make a large triangular grid. Below, are the triangular grids of size 1 and of size 2. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks.



a) Let t_n be the number of toothpicks required to make a size n triangular grid. Write out the first 5 terms of the sequence $t_1, t_2, ...$

I first began by drawing out the pictures that come next, and I immediately saw a pattern emerge. That pattern is that, when drawn out, the n value of t_n is equal to the number of triangles in the bottom row. Not counting the "upside down" triangles, since this exercise is to determine the number of toothpicks.



The first 5 terms are:
$t_1 = 3$
$t_2 = 9$
$t_3 = 18$
$t_4 = 30$
$t_5 = 45$

therefore:
{3, 9, 18, 30, 45} are the first 5 elements of the sequence.

b). Find a recursive definition for the sequence. Explain why you are correct.

Referring to the picture above makes it easy to see that we are adding the previous number of toothpicks to the value of ”n” multiplied by three. We have to multiply n by 3 because there are 3 toothpicks per triangle and we have to add the previous entry’s toothpick value because that is how this pattern iterates, as evidenced by the first two entries.

therefore:

My recursive definition is: $t_{n-1} + 3n$

c). Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence? Explain why your answer is correct.

This sequence is arithmetic in respect to partial sums because $t_n = 3(1 + 2 + 3 + ... + n)$

d). Use your results from part (c) to find a closed formula for the sequence. Show your work.

I found the closed definition through trial and error. I had plugged in several values to 3n(x), where x was my variable value. I eventually noticed that x = n+1 gave me double the value of the first five entries. My solution was to divide that by two. Arriving at:

$$\frac{3n(n+1)}{2}$$

This makes sense, because 3n(n+1) would result in a rectangular ”area” and dividing it by 2 would give the triangular ”area”. This is when looking at the drawing and realizing that each row is an index, and each column could be thought of as the value stored at that index.

therefore:

My closed definition is $\frac{3n(n+1)}{2}$

Problem 11
Propositional logic
Tommy Flanagan was telling you what he ate yesterday afternoon. He tells you, “I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn’t drink soda or tea.” Of course you know that Tommy is the World’s worst liar, and everything he says is false. What did Tommy eat?

Justify your answer by writing all of Tommy’s statements using sentence variables (P, Q, R, S, T), taking their negations, and using these to deduce what Tommy actually ate.

Answer:
We are told that Tommy Flanagan is the World’s biggest liar, and everything he says is false. One way to find out what Tommy truly ate was to take the negations of what he claims, and to treat those negations as truths.

let P = *Tommy had Popcorn*
let Q = *Tommy had Cucumber sandwich*
let R = *Tommy had Raisins*

let S = *Tommy had Soda*
let T = *Tommy had Tea*

Tommy made the following statements:
statement 1: *"I had either popcorn or raisins"* -or- $P \vee R$
statement 2: *"If I had cucumber sandwich, then I had soda"* -or- $Q \rightarrow S$
statement 3: *"I don't drink soda or tea"* -or- $\neg S \wedge \neg T$

Here are the negations of Tommy's respective statements, which I found by using De Morgan's Laws:
statement 4: $\neg P \wedge \neg R$ -or- *"not popcorn, and not raisins"*
statement 5: $Q \wedge \neg S$ -or- *"cucumber and not soda"*
statement 6: $S \vee T$ -or- *"soda or tea"*

Since Tommy Flanagan is the worst liar ever, we know that the opposite of what he says is actually the truth. So the negations of the statements he made can be treated as true statements.
We know that Tommy did not eat popcorn or raisins because of statement 4. We know that he had cucumber and did not have soda because of statement 5, and since he did not have soda, he must have had tea, in order to make statement 6 true, because an "or" statement requires one or both sides of the statement to be true in order to return a truthy value. We know that statements 4-6 are all true, so he must have had either soda or tea.

Final result: Tommmy ate a cucumber sandwich and supped tea.

Problem 7a:
direct/indirect proofs
Consider the statement: for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.
a. Prove the statement. What sort of proof are you using?

Claim:
For all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6.

Proof:
let a be an even integer, so can be written as a = 2k where k is some integer.
let b be a multiple of 3, so it can be written as b = 3n where n is some integer
let m be some integer that is a multiple of 6
so, the product ab = (2k)(3n)
Our original statement claims that ab = 6m where m is some integer if a is even
So if (2k)(3n) = 6m, then that means 6(kn) = 6(m) then that means kn = m

Therefore, The statement that for all integers a and b, if a is even and b is a multiple of 3, then ab is a multiple of 6, is a true statement.

This is a direct proof.

Problem 9
Prove the statement: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

Claim:
For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even.

Proof:
let us suppose that the original statement is not true.
in this case: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then a and b are something other than even
if a and b are both odd, then a = 2k + 1 and b = 2n + 1, where k and n are some integers.
Substituting into the original statement: $(2k + 1)^2 + (2n + 1)^2 = c^2$
I am using the hint that c^2 is a multiple of 4. I understand that must be true, as long as c is even, because we define and even number as 2n, so $(2n)^2 = 4n^2$, which will always be divisible by 4.

Continuing algebraically:

$$
\begin{aligned}
c^2 &= (2k+1)^2 + (2n+1)^2 \\
&= (2k+1)(2k+1) + (2n+1)(2n+1) \\
&= (4k^2 + 2k + 2k + 1) + (4n^2 + 2n + 2n + 1) \\
&= (4k^2 + 4k + 1) + (4n^2 + 4n + 1) \\
&= (4k^2 + 4k) + (4n^2 + 4n) + 2 \\
&= 4(k^2 + k) + 4(n^2 + n) + 2
\end{aligned}
$$

$4(k^2 + k) + 4(n^2 + n) + 2$ represents a number that is even, but not divisible by four.
I know this because all numbers multiplied by 4 will be both even and divisible by four. We know this from the closure of integers
But then adding two makes the resultant integer not divisible by four, but still even.
This creates a contradiction, because I know c^2 must be divisible by four, but the result of the algebra was an integer divisible by four plus 2. this makes division of c by four an impossibility.
C cannot be 1 because we are using integers and a + b must result in at least 2.

Therefore, the original statement that for all integers a, b, and c, if $a^2 + b^2 = c^2$, then a or b is even must be true! We can say that because we have found a contradiction in the negation of the original statement.

Problem 17
Now give a valid proof (by induction, even though you might be able to do so without using induction) of the statement, "for all $n \epsilon \mathbb{N}$, the number $n^2 + n$ is even."

(hint) For the inductive case, you will need to show that $(k + 1)^2 + (k + 1)$ is even. Factor this out and locate the part of it that is $k^2 + k$. What have you assumed about that quantity?

claim:
The number $n^2 + n$, where n is a natural number, is an even number.
We can prove this statement through induction.

basis:
We want to prove that the number $n^2 + n$ is even for all natural numbers, up to the nth integer. This process is begun by proving the base case.

Let P(n) be the statement "The number $n^2 + n$ is an even number"
For our base case, let n = 1

$$
\begin{aligned}
&= (1)^2 + (1) \text{ plugging the base case value into formula} \\
&= 1 + 1 \\
&= 2 \text{ this proves our base case is even}
\end{aligned}
$$

Because the result of the arithmetic is an even integer we know that P(n) is a true statement when n = 1. This proves the base case holds true for the given statement.

Since our base case is proven to be true, we get to assume that it's true up until the nth number, then we get to use induction, which is the assumption that if the statement is true for the nth number, it must also be true for the kth number, and then we need to prove it is also true for the number proceeding k, a.k.a. $(k+1)$.

induction:
Let P(k) be the statement "The number $k^2 + k$ is an even number, where n is a natural number"
We get to assume P(k) is true, because that is the foundational step in inductive reasoning.
Then we want to show that P(K+1) is also true
WTS: P(k) $\rightarrow$ P(k+1), or that $(k + 1)^2 + (k + 1)$ is an even number

Consider the predicate P in relation to the number $k^2 + k$

$$
\begin{aligned}
&= (k + 1)^2 + (k + 1) \\
&= (k + 1)(k + 1) + (k + 1) \text{ expand} \\
&= k^2 + 2k + 1 + (k + 1) \text{ algebra} \\
&= k^2 + k + 2k + 2 \text{ this shows } k^2 + k \text{ Which we know is even via the assumption}
\end{aligned}
$$

We have already proven via the definition of an even integer and the closure of integers that 2k+2 is even. So we know that P(n) is always true while n is equal to 2k+2

therefore:
We can reason by induction that the number $n^2 + n$ is an even number for all natural numbers, n. We can safely say this because we have shown through the principles of induction and the closure of integers that

the statement is true.

Section 1.1, Question 14
Combinatorics
The number 735000 factors as $2^3 * 3 * 5^4 * 7^2$. How many divisors does it have? Explain your answer using the multiplicative principle.

Solving:
I will begin by re-writing the original factorization so I can see all of the exponents:

$2^3 * 3^1 * 5^4 * 7^2$

The multiplicative principle states that the total number of possible outcomes is the product of A and B, where A and B are separate independent choices. Knowing this, we can start by determining the number of possible outcomes within each particular factor.

To accomplish this, I will analyze each factor below:

2^3 represents one factor of 735000. This could also be written as $2 * 2 * 2$, but for this exercise we need to include the 2^0 because that represents the number 1, which will always be a factor of any integer. This gives us 4 possible ways to count the "2" factor as a factor of 735000.

3^1 is the next factor we are given, which has two factors, in itself: 3 and 1. Has 2 possible factors.

5^4 is the next factor and has 5 possible factors of 735000.

7^2 is the final prime factor and gives us 3 more possible factors of 735000.

So according to the multiplicative principle, the possible number of factors for 735000 can be found by multiplying the number of possible outcomes together. This would be $4 * 2 * 5 * 3$, because 2^3 has 4 possible outcomes, 3^1 has 2 possible outcomes, 5^4 has 5 possible outcomes, and 7^2 has 3 possible outcomes.

So, we have that $4 * 2 * 5 * 3 = 120$. This tells us that there are 120 possible factors to the number 735000.

therefore:
There are 120 possible factors to the number 735000, determined using the multiplicative principle.

SQ-11
Graph Theory
Suppose G is a connected graph and T is a cycle-free subgraph of G. Suppose also that if any edge 'e' of G that is not in T is added to T, the resulting graph contains a cycle. Prove that T is a spanning tree for G.

WTS:
We want to show that T is a spanning tree for G.

A spanning tree connects all vertices, and has no cycles. We must show that T has both of these attributes. We need to show that T is a subgraph that touches all vertices.

Given:
G is a connected graph
T is a subgraph of G
T has no cycles - this checks one of our spanning tree boxes

Proof:
A spanning tree has n-1 edges, where n is the number of vertices. If we assume that T is a spanning tree:
If we add 1 edge, then T would no longer meet the definition of a spanning tree.

If we assume it's not a spanning tree, then why does adding one edge always create a cycle?
I think this is the answer, proof by contradiction.

Therefore:
Because there is a contradiction in the negation of the supposition, the original statement is true.
T is a spanning tree of G.

My own grade
This is very difficult. I am stuck between a high B and a low A. I know I worked hard and did my best to understand the material, but I also know that a few students have a deeper understanding of the principles. If this class was being graded on a curve, I know I would not be in the top 5%, possibly not in the top 10%. But when I ignore the "competition," I think I earned a low A. I feel comfortable saying so because I know that I learned how to use some very valuable tools in the field of logic and reason.
As a reflective note, I have noticed that talking about the principles and using the new vocabulary helps me remember which tools to use in which scenarios.
Like I said in my final homework submission, these types of mathematics are what I was expecting when I signed up for computer science! Thank you so much!